Phase Plane

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The general form of a vector field on the phase plane is

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

This system can be written more compactly in vector notation as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where $\begin{cases} \mathbf{x} = (x_1, x_2) \\ \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})) \end{cases}$

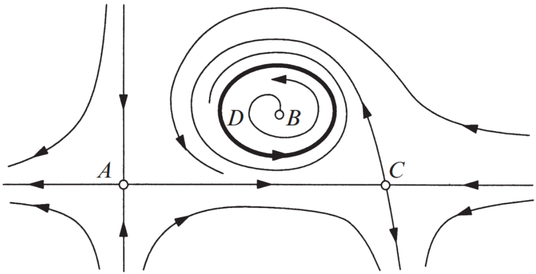

Figure 6.1.2

- Fixed Points: $\dot{\mathbf{x}} = 0$, A, B and C
- Closed orbits: $\mathbf{x}(t + T) = \mathbf{x}(t)$, D
- The arrangement of trajectories near the fixed points and closed orbits.
- The stability or instability of the fixed points and closed orbits.

$$\dot{x} = x + e^{-y}$$
$$\dot{y} = -y$$

- Fixed Points: $(x^*, y^*) = (-1, 0)$
- Stability: since the solution to y is $y(t) = y_0 e^{-t}$, $y(t) \to 0$ when $t \to \infty$. Hence $e^{-y} \to 1$, now $\dot{x} \approx x + 1$, this has exponentially growing solutions, unstable.

nullclines: the curves where $\dot{x} = 0$ either or $\dot{y} = 0$.

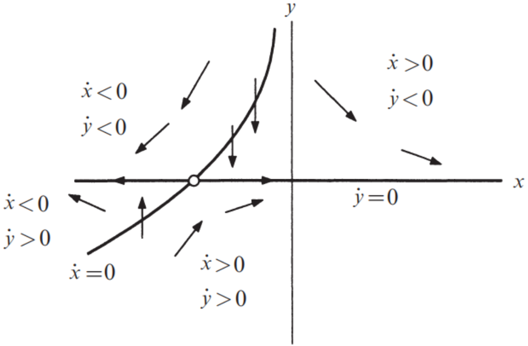

Figure 6.1.3

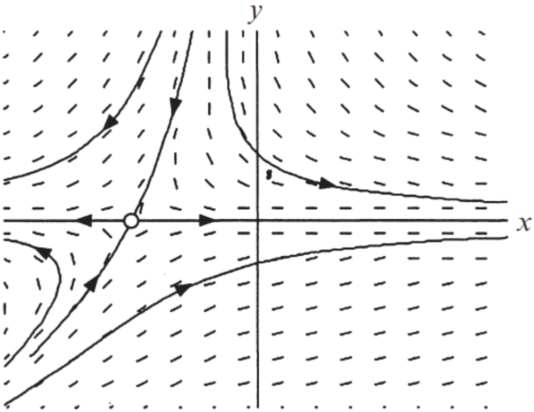

Figure 6.1.4

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Existence and Uniqueness Theorem: Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Suppose that $\mathbf{f}$ is continuous and that all its partial derivatives $\partial f_i / \partial x_j$, $i, j = 1, \ldots, n$, are continuous for $\mathbf{x}$ in some open connected set $D \subset \mathbf{R}^n$. Then for $\mathbf{x}_0 \in D$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

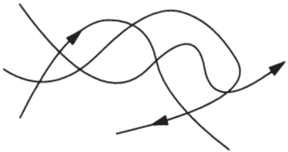

Figure 6.2.1

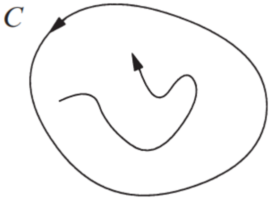

Figure 6.2.2

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$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Suppose that (x^*, y^*) is a fixed point, i.e.,

$$f(x^*, y^*) = 0, g(x^*, y^*) = 0$$

Small disturbance:

$$u = x - x^*, v = y - y^*$$

$$\begin{aligned}
\dot{u} &= \dot{x} && \text{(since } x^* \text{ is a constant)} \\
&= f(x^* + u, y^* + v) && \text{(by substitution)} \\
&= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(Taylor series expansion)} \\
&= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) && \text{(since } f(x^*, y^*) = 0\text{).}
\end{aligned}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + O\left(u^2, v^2, uv\right)$$

Jacobian matrix at the fixed point (x^*, y^*):

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

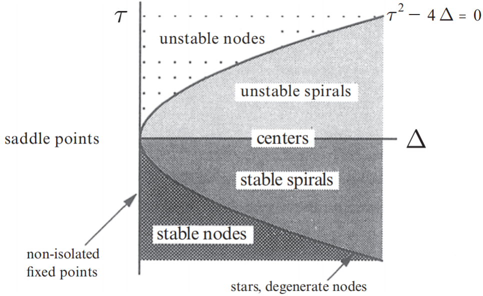
linearized system:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The Effect of Small Nonlinear Terms

Is it really safe to neglect the quadratic terms?

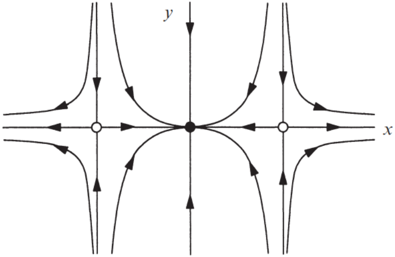
Andronov et al.(1973) showed that the answer is yes:
as long as the fixed point for the linearized system is not one of the borderline cases (centers, degenerate nodes, stars, or non-isolated fixed points)



$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y$$
$$A = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

At $(0,0)$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, stable node

At $(\pm 1, 0)$, $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, both saddle points



$$\dot{x} = -y + ax\left(x^2 + y^2\right)$$
$$\dot{y} = x + ay\left(x^2 + y^2\right)$$

According to Linearization: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the origin is a center

Polar coordinates : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r\dot{r} &= x\dot{x} + y\dot{y} \\ &= x\left(-y + ax\left(x^2 + y^2\right)\right) + y\left(x + ay\left(x^2 + y^2\right)\right) \\ &= a\left(x^2 + y^2\right)^2 \\ &= ar^4 \end{aligned}$$

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}$$

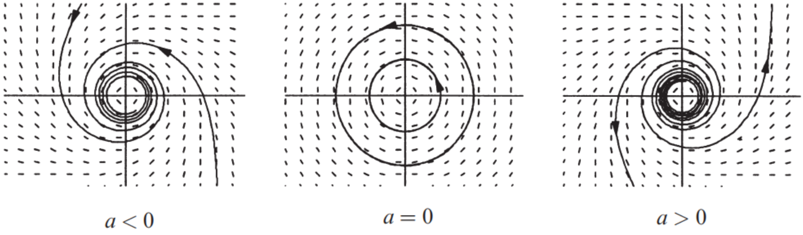


Figure 6.3.2

A phase portrait is **structurally stable** if its topology cannot be changed by an arbitrarily small perturbation to the vector field.

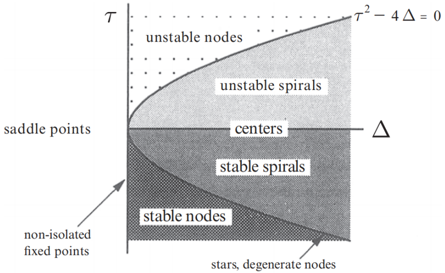


Figure 5.2.8

Robust cases:
Repellers (also called *sources*): both eigenvalues have positive real part.
Attractors (also called *sinks*): both eigenvalues have negative real part.
Saddles: one eigenvalue is positive and one is negative.

Marginal cases:
Centers: both eigenvalues are pure imaginary.
Higher-order and non-isolated fixed points: at least one eigenvalue is zero.