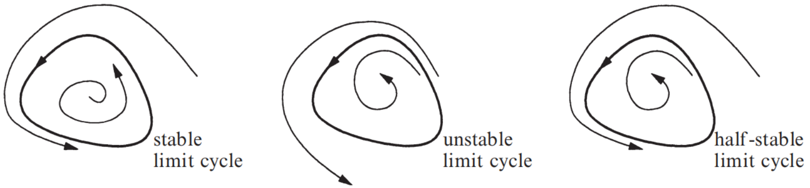
Limit Cycles

Speaker: Yichen Lu

August 3, 2023

Contents

A **limit cycle** is an isolated closed trajectory.



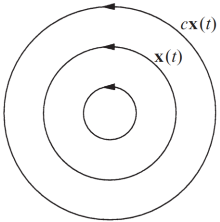


Figure 7.0.2

Consider the system

$$\dot{r} = r\left(1 - r^2\right), \dot{\theta} = 1 \ (r \geq 0)$$

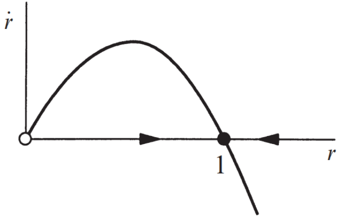


Figure 7.1.1

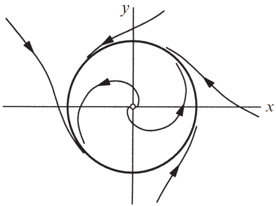
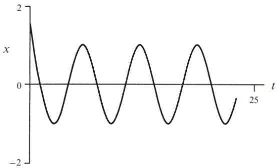


Figure 7.1.2

we plot $x(t) = r(t)\cos\theta(t)$



so $x(t) = \cos(t + \theta_0)$ is a solution of the system.

van der Pol equation

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$$

$\mu\left(x^2 - 1\right)$ is a **nonlinear damping**, This term acts like ordinary positive damping for $|x| < 1$, but like negative damping for $|x| > 1$.

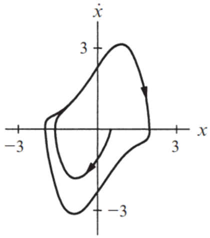


Figure 7.1.4

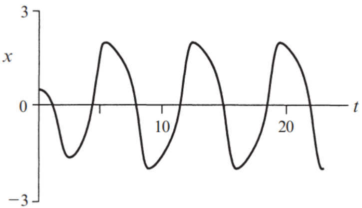


Figure 7.1.5

Contents

Gradient Systems

A **gradient system** with **potential function** V, if the system can be written in $\dot{x} = -\nabla V(x)$.

Theorem 7.2.1: Closed orbits are impossible in gradient systems.

Proof:

$$
\begin{aligned}
\Delta V &= \int_0^T \frac{dV}{dt} \mathrm{dt} \\
&= \int_0^T (\nabla V \cdot \dot{x})\, dt \\
&= -\int_0^T \|\dot{x}\|^2\, dt \\
&< 0
\end{aligned}
$$

Example 7.2.1

$$\dot{x} = \sin y$$
$$\dot{y} = x \cos y$$

We can find a potential function $V(x, y) = -x \sin y$, satisfy $\dot{x} = -\partial V/\partial x$ and $\dot{y} = -\partial V/\partial y$, so the system is a gradient system.

Example 7.2.2 The nonlinearly damped oscillator $\ddot{x} + (\dot{x})^3 + x = 0$ has no periodic solutions.

$$E(x, \dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$$
$$\dot{E} = \dot{x}(x + \ddot{x}) = \dot{x}(-\dot{x}^3) = -\dot{x}^4 \leq 0$$
$$\Delta E = \int_0^T \dot{E} dt = -\int_0^T \dot{x}^4 dt \leq 0$$

Contents

glossary

- **Limit cycle** is an isolated closed trajectory.
- **Nonlinear damping item** This term acts like ordinary positive damping for some situations, but like negative damping for other situations.
- **Gradient system** is a system that can be written in $\dot{x} = -\nabla V(x)$.