Conservative Systems & Reversible Systems

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Contents

1 Conservative Systems

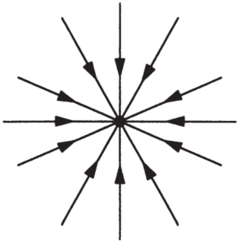
2 Reversible Systems

Conservative Systems

$$
\begin{aligned}
m\ddot{x} &= F(x) \\
m\ddot{x} + \frac{dV}{dx} &= 0 \quad \left(F(x) = -\frac{dV}{dx} \right) \\
m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} &= 0 \quad (\text{multiply both sides by } \dot{x}) \\
\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x) \right] &= 0 \quad \left(\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\frac{dx}{dt} \right) \\
\frac{dE}{dt} &= 0 \quad \left(E = \frac{1}{2}m\dot{x}^2 + V(x) \right)
\end{aligned}
$$

Systems for which a conserved quantity exists are called **conservative systems**.

- A conserved quantity is a real-valued continuous function $E(x)$ that is constant on trajectories, i.e. $dE/dt = 0$
- To avoid trivial examples, we also require that $E(x)$ be nonconstant on every open set.
- A conservative system cannot have any attracting fixed points.

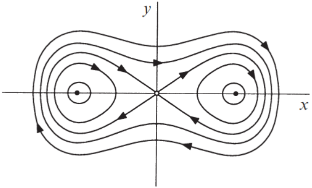


$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$

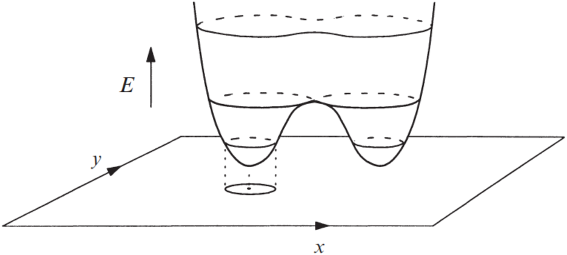
Since $F = -\frac{dV}{dx} = x - x^3$, $\ddot{x} = x - x^3$

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}$$

Fix points: Saddle point: $(0,0)$, Centers: $(\pm 1, 0)$



Trajectories that start and end at the same fixed point are called homoclinic orbits.



The energy E is plotted above each point (x, y) of the phase plane. The resulting surface is often called the energy surface for the system.

Theorem 6.5.1: (Nonlinear centers for conservative systems) Consider the system $\dot{\mathrm{x}} = f(\mathrm{x})$, where $\mathrm{x} = (x, y) \in R^2$, and f is continuously differentiable. Suppose there exists a conserved quantity $E(x)$ and suppose that x^* is an isolated fixed point (i.e., there are no other fixed points in a small neighborhood surrounding x^*). If x^* is a local minimum of E, then all trajectories sufficiently close to x^* are closed.

1. The theorem is valid for local maxima of E also.
2. We need to assume that x^* is isolated.

Contents

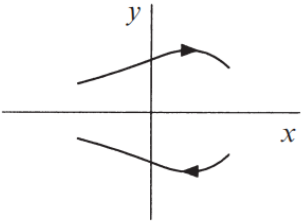
1 Conservative Systems

2 Reversible Systems

Reversible Systems

$$\dot{x} = y$$
$$\dot{y} = \frac{1}{m} F(x)$$

Make the change of variables $t \to -t$ and $y \to -y$: if $(x(t), y(t))$ is a solution, then so is $(x(-t), -y(-t))$



More generally, let's define a reversible system to be any second-order system that is invariant under $t \to -t$ and $y \to -y$

$$\dot{x} = f\left(x, y\right)$$
$$\dot{y} = g\left(x, y\right)$$

where f is odd in y and g is even in y $\left(\text{i.e., } f(x, -y) = -f(x, y) \text{ and } g(x, -y) = g(x, y)\right)$ is reversible

Theorem 6.6.1: (Nonlinear centers for reversible systems) Suppose the origin $x^* = 0$ is a linear center for the continuously differentiable system and suppose that the system is reversible. Then sufficiently close to the origin, all trajectories are closed curves.

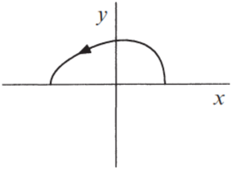


Figure 6.6.2

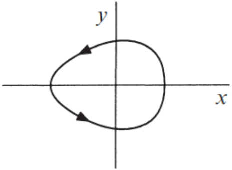


Figure 6.6.3

$$\dot{x} = y - y^3$$
$$\dot{y} = -x - y^2$$
$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- $\tau = 0$, $\Delta > 0$, so the origin is a linear center.
- the system is reversible, since the equations are invariant under the transformation $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$

The other fixed points of the system are $(-1, 1)$ and $(-1, -1)$, they are saddle points.

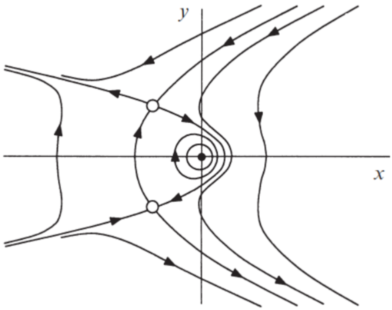


Figure 6.6.4

The twin saddle points are joined by a pair of trajectories, they are called **heteroclinic trajectories** or **saddle connections**.

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^2\end{aligned} \quad (x > 0)$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Fixed points: $(0,0)$, saddle point One of eigendirection is $v_1 = (1,1)$, the other is $v_2 = (1,-1)$
when $0 < x < 1$, $\dot{y} = x - x^2 > 0$, y is increasing, when $x > 1$, $\dot{y} = x - x^2 < 0$, y is decreasing. $\dot{x} = y > 0$. Eventually reaching $y = 0$.

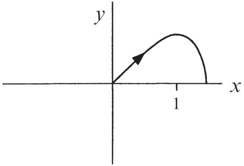


Figure 6.6.5

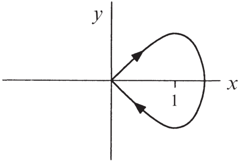


Figure 6.6.6

$$\dot{x} = -2\cos x - \cos y$$
$$\dot{y} = -2\cos y - \cos x$$

The system is invariant under the change of variables $x \to -x$, $y \to -y$ and $t \to -t$.

$$\cos x^* = \cos y^* = 0, (x^*, y^*) = \left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$$

$$A = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}_{\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

$\tau = -4, \Delta = 3, \tau^2 - 4\Delta = 4$, the fixed point is a stable node.