Chemotactic Chiral Active Matter

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1.1 Definitions

1.1.1 Self-propelled dynamics

$$\dot{x}_i = v \cos \theta_i \ , \tag{1a}$$
$$\dot{y}_i = v \sin \theta_i \ , \tag{1b}$$

1.1.2 Polar alignment dynamics

- Additive coupling:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} f(r_{ij}) \sin(\theta_j - \theta_i + \alpha) \ , \tag{2}$$

- Mean-field coupling by oscillator number:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} f(r_{ij}) \sin(\theta_j - \theta_i + \alpha) \ , \tag{3}$$

which is similar to the swarmalator model.

Here, $f(r_{ij})$ is a function of $r = |\mathbf{r}_i - \mathbf{r}_j|$, and K is the coupling strength. The function $f(r)$ can be defined as

1. $f_H(r) = H(d_0 - r)$, $r_0 > 0$;

2. $f_E(r) = e^{-\frac{r}{d_0}}$, $r_0 > 0$.

The natural frequencies ω_i are distributed with following two cases:

1. **Single-chiral swarmalators:** The natural frequencies ω_i are distributed in $U(\omega_{\min}, \omega_{\max})$ for all swarmalators and $\omega_{\min}\omega_{\max} > 0$.

2. **Double-chiral swarmalators:** The frequencies are distributed in two symmetric uniform distributions, representing two types of chirality. Exactly half of the swarmalators have natural frequencies $\omega_i \sim U(\omega_{\min}, \omega_{\max})$ and the other half have natural frequencies $\omega_i \sim U(-\omega_{\max}, -\omega_{\min})$.

1.1.3 Chemotactic dynamics

Consider two chemical fields $u(\mathbf{r}, t), v(\mathbf{r}, t)$ that are produced by the ensemble of two symmetrically chiral swarmalators. Swarmalators interact with the chemical field and move towards/against the regions with higher concentration, which can be described by the following equation ($i = 1, 2, \ldots, N$):

$$\dot{\mathbf{r}}_i^s = v\mathbf{p}(\theta_i^s) \tag{4a}$$
$$\dot{\theta}_i^s = \omega_i^s + \alpha^s \mathbf{p}_i^s \times \nabla u + \beta^s \mathbf{p}_i^s \times \nabla v \tag{4b}$$

where α, β_i^s denote the 'chemotactic' coupling strength and $\mathbf{p}(\theta) = (\cos\theta, \sin\theta)$ is the unit vector pointing in the direction of the i-th swarmalator, $s \in \{p, n\}$ denotes the two chiral species. Here, we used the notation $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$.

These two fields evolve as

$$\dot{u} = k_0 \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j^p) - k_d u + D_u \nabla^2 u \ , \tag{5a}$$

$$\dot{v} = k_0 \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j^n) - k_d v + D_v \nabla^2 v \ , \tag{5b}$$

where S_+ and S_- are the sets of two chiral swarmalators, k_0 is the production rate, k_d is the decay rate, $D_{u,v}$ are the diffusion coefficients.

1.1.4 Mixed phase dynamics

$$\dot{\mathbf{r}}_i = v\mathbf{p}_i \tag{6a}$$

$$\dot{\theta}_i = \omega_i + \beta_i^u \mathbf{p}_i \times \nabla u + \beta_i^v \mathbf{p}_i \times \nabla v + \frac{K}{N} \sum_{j=1}^{N} f\left(|\mathbf{r}_j - \mathbf{r}_i|\right) \sin\left(\theta_j - \theta_i\right) , \tag{6b}$$

1.1.5 General Chemotactic Model For Two Species

Type 1:

$$\dot{\mathbf{r}}_i^{1,2} = v\mathbf{p}\left(\theta_i^{1,2}\right) - \sum_{j \in A_i^{1,2}} \frac{1}{|\mathbf{r}_j - \mathbf{r}_i^{1,2}|^\beta} , \tag{7a}$$

$$\dot{\theta}_i^{1,2} = \alpha_{1,2} |\nabla c_{1,2}| \sin\left(\varphi_{c_{1,2}} - \theta_i^{1,2}\right) , \tag{7b}$$

$$\dot{c}_1 = D_1 \nabla^2 c_1 + F_1\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^1\right) , \tag{7c}$$

$$\dot{c}_2 = D_2 \nabla^2 c_2 + F_2\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^2\right) , \tag{7d}$$

Type 2:

$$\dot{\mathbf{r}}_i^{1,2} = v\mathbf{p}\left(\theta_i^{1,2}\right) + \alpha_{1,2} \nabla c_{1,2} - \sum_{j \in A_i^{1,2}} \frac{1}{|\mathbf{r}_j - \mathbf{r}_i^{1,2}|^\beta} , \tag{8a}$$

$$\dot{\theta}_i^{1,2} = KX\left(\theta, \mathbf{r}\right) , \tag{8b}$$

$$\dot{c}_1 = D_1 \nabla^2 c_1 + F_1\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^1\right) , \tag{8c}$$

$$\dot{c}_2 = D_2 \nabla^2 c_2 + F_2\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^2\right) , \tag{8d}$$

1.1.6 Chemotactic Model with Lotka-Volterra Functions

Let $F_1\left(c_1, c_2\right) = c_1\left(k_1 - k_2 c_2\right)$ and $F_2\left(c_1, c_2\right) = c_2\left(k_3 c_1 - k_4\right)$, where k_1, k_2, k_3, k_4 are constants.

$$\dot{\mathbf{r}}_i^{1,2} = v\mathbf{p}\left(\theta_i^{1,2}\right) - \sum_{j \in A_i^{1,2}} \frac{1}{|\mathbf{r}_j - \mathbf{r}_i^{1,2}|^\beta} , \tag{9a}$$

$$\dot{\theta}_i^{1,2} = \alpha_{1,2} |\nabla c_{1,2}| \sin\left(\varphi_{c_{1,2}} - \theta_i^{1,2}\right) , \tag{9b}$$

$$\dot{c}_1 = D_1 \nabla^2 c_1 + c_1\left(k_1 - k_2 c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^1\right) , \tag{9c}$$

$$\dot{c}_2 = D_2 \nabla^2 c_2 + c_2\left(k_3 c_1 - k_4\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^2\right) , \tag{9d}$$

where $\varphi_{c_{1,2}} = \arctan\left(\frac{\partial_y c_{1,2}}{\partial_x c_{1,2}}\right)$ and $A_i^{1,2} = \left\{j \mid r_c \geq |\mathbf{r}_j - \mathbf{r}_i^{1,2}|\right\}$.

2 Short format coarse graining

We begin with Eq. (2), replacing the finite coupling distance alignment interaction with a pseudopotential (the 'δ'-interaction). This substitution is justified when the interaction is sufficiently short-ranged, making the specific shape of the associated interaction potential irrelevant to the dynamics of many swarmalators. The pseudopotential is defined as:

$$\dot{\mathbf{r}}_i^c = v\mathbf{p}\left(\theta_i^c\right) , \tag{10a}$$

$$\dot{\theta}_i^c = \omega_i^c + K\sum_{j=1} \delta\left(\mathbf{r}_j^c - \mathbf{r}_i^c\right)\sin\left(\theta_j^c - \theta_i^c\right)$$

$$+ K\sum_{j=1} \delta\left(\mathbf{r}_j^b - \mathbf{r}_i^b\right)\left[\sin\left(\theta_j^b - \theta_i^b + \alpha_0\right) - \sin\alpha_0\right] , \tag{10b}$$

where $c \in \{+, -\}$ is the chirality of the swarmalator i and $b = +$ if $c = -$ and vice versa. Then following [1] we derive a continuum equation of motion for the combined N-swarmalator probability density

$$\rho^c\left(\mathbf{r}, \theta, t\right) = \sum_{i=1} \rho_i^c\left(\mathbf{r}, \theta, t\right) , \tag{11}$$

where $\rho_i^c\left(\mathbf{r}, \theta, t\right) = \delta\left(\mathbf{r}_i^c\left(t\right) - \mathbf{r}\right)\delta\left(\theta_i^c\left(t\right) - \theta\right)$ is the probability density of finding i-th swarmalator at position $\mathbf{r}$ with phase θ and chirality c at time t. Since the deterministic dynamical equation Eq. (??) conserves the number of oscillators with a given natural frequency over time, the distribution function evolves according to a continuity equation of the following form:

$$\frac{\partial \rho_i^c}{\partial t} = -\nabla \cdot \left(\rho_i^c v_{\mathbf{r}}\right) - \frac{\partial}{\partial \theta}\left(\rho_i^c v_\theta^{c,i}\right) . \tag{12}$$

Here, the velocity fields read

$$v_{\mathbf{r}}\left(\mathbf{r}, \theta, t\right) = v\mathbf{p}\left(\theta\right) , \tag{13a}$$

$$v_\theta^{c,i}\left(\mathbf{r}, \theta, t\right) = \omega_i^c + K\int \mathrm{d}\phi\rho^c\left(\mathbf{r}, \phi, t\right)\sin\left(\phi - \theta\right)$$

$$+ K\int \mathrm{d}\phi\rho^b\left(\mathbf{r}, \phi, t\right)\left[\sin\left(\phi - \theta + \alpha_0\right) - \sin\alpha_0\right] . \tag{13b}$$

Summing Eq. (12) over the i and c indices, and using the definition of the density ρ^c in Eq. (11), we obtain

$$\frac{\partial \rho^c\left(\mathbf{r}, \theta, t\right)}{\partial t} = -v\mathbf{p}\left(\theta\right) \cdot \nabla\rho^c\left(\mathbf{r}, \theta, t\right) - \frac{\partial}{\partial \theta}\Omega\left(\mathbf{r}, \theta, t\right)$$

$$+ K\frac{\partial}{\partial \theta}\rho^c\int \mathrm{d}\phi\rho^c\left(\mathbf{r}, \phi, t\right)\sin\left(\phi - \theta\right) \tag{14}$$

$$+ K\frac{\partial}{\partial \theta}\rho^c\int \mathrm{d}\phi\rho^b\left(\mathbf{r}, \phi, t\right)\left[\sin\left(\phi - \theta + \alpha_0\right) - \sin\alpha_0\right] ,$$

where $\Omega\left(\mathbf{r}, \theta, t\right) = \sum_{i=1}\rho_i^c\left(\mathbf{r}, \theta, t\right)\omega_i^c$. Spatiotemporal dynamics of the ISS indicates $\forall i, c$, $\rho_i^c\left(\mathbf{r}, \theta, t\right) \equiv \rho_{\mathrm{ISS}}\left(\mathbf{r}, \theta, t\right)$, which yields

$$\Omega\left(\mathbf{r}, \theta, t\right) = \rho^c\left(\mathbf{r}, \theta, t\right)\frac{\left(\omega_{\max} + \omega_{\min}\right)}{2} . \tag{15}$$

Transforming Eq. (14) to Fourier space, yields an equation of motion for the Fourier modes $\varrho_k^c\left(\mathbf{r}, t\right) = \int \rho^c\left(\mathbf{r}, \theta, t\right)\mathrm{e}^{\mathrm{i}k\theta}\mathrm{d}\theta$ of ρ^c:

$$\frac{\partial \varrho_k^c}{\partial t} = -\frac{v}{2}\left[\frac{\partial}{\partial x}\left(\varrho_{k+1}^c + \varrho_{k-1}^c\right) - \mathrm{i}\frac{\partial}{\partial y}\left(\varrho_{k+1}^c - \varrho_{k-1}^c\right)\right]$$

$$- \left[\frac{\mathrm{i}k\left(\omega_{\max} + \omega_{\min}\right)}{2}\varrho_k^c - k^2\right]\varrho_k^c \tag{16}$$

$$+ \frac{\mathrm{i}Kk}{2\pi}\sum_{m=-\infty}^{\infty}\varrho_{k-m}^c F_{-m}\varrho_m^c$$

3 Coarse graining

We now follow the strategy in [1] to consider the evolution of the density function for a single particle

$$\rho_i(\mathbf{r}, \theta, \omega, t) = \delta(\mathbf{r}_i(t) - \mathbf{r}) \delta(\theta_i(t) - \theta) g(\omega) \,, \tag{17}$$

which denotes the probability of finding a particle at position $\mathbf{r}$, with orientation θ and natural frequency ω, where $g(\omega)$ is the time independent swarmalator frequency distribution. The density function ρ_i satisfies the continuity equation, and we shall then demonstrate how one may write a closed equation for the global density

$$\rho(\mathbf{r}, \theta, \omega, t) = \sum_{i=1}^{N} \rho_i(\mathbf{r}, \theta, \omega, t) \,, \tag{18a}$$

$$\varrho(\mathbf{r}, \theta, t) = \int_{-\infty}^{+\infty} \rho(\mathbf{r}, \theta, \omega, t) \, \mathrm{d}\omega \,. \tag{18b}$$

The derivation follows a well known argument. Consider an arbitrary function f defined on the coordinate space of the system. Using the definition of the density it is a tautology that

$$f(\mathbf{r}_i(t), \theta_i(t), \omega_i) = \iiint \mathrm{d}\mathbf{r}\mathrm{d}\theta\mathrm{d}\omega \rho_i(\mathbf{r}, \theta, \omega, t) f(\mathbf{r}, \theta, \omega) \,. \tag{19}$$

Expanding the differential equation over the next time step δt one obtains

$$\begin{aligned}
\frac{\mathrm{d}f(\mathbf{r}_i, \theta_i, \omega_i)}{\mathrm{d}t} &= \frac{\partial f(\mathbf{r}_i, \theta_i, \omega_i)}{\partial \mathbf{r}_i} \cdot \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} + \frac{\partial f(\mathbf{r}_i, \theta_i, \omega_i)}{\partial \theta_i} \frac{\mathrm{d}\theta_i}{\mathrm{d}t} + \frac{\partial f(\mathbf{r}_i, \theta_i, \omega_i)}{\partial \omega_i} \frac{\partial \omega_i}{\partial t} \\
&= \iiint \mathrm{d}\mathbf{r}\mathrm{d}\theta\mathrm{d}\omega \rho_i(\mathbf{r}, \theta, \omega, t) \left(\frac{\partial f(\mathbf{r}, \theta, \omega)}{\partial \mathbf{r}} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \frac{\partial f(\mathbf{r}, \theta, \omega)}{\partial \theta} \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial f(\mathbf{r}, \theta, \omega)}{\partial \omega} \frac{\mathrm{d}\omega}{\mathrm{d}t} \right) \\
&= \iiint \mathrm{d}\mathbf{r}\mathrm{d}\theta\mathrm{d}\omega \left(\rho_i(\mathbf{r}, \theta, \omega, t) \dot{\mathbf{r}} \cdot \nabla f(\mathbf{r}, \theta, \omega) + \rho_i(\mathbf{r}, \theta, \omega, t) \dot{\theta} \frac{\partial}{\partial \theta} f(\mathbf{r}, \theta, \omega) \right) \,.
\end{aligned} \tag{20}$$

Re-arranging the above integral by integration by parts we obtain

$$\frac{\mathrm{d}f(\mathbf{r}_i, \theta_i, \omega_i)}{\mathrm{d}t} = \iiint \mathrm{d}\mathbf{r}\mathrm{d}\theta\mathrm{d}\omega f(\mathbf{r}, \theta, \omega) \left(-\nabla \cdot (\rho_i(\mathbf{r}, \theta, \omega, t) \dot{\mathbf{r}}) - \frac{\partial}{\partial \theta} \left(\rho_i(\mathbf{r}, \theta, \omega, t) \dot{\theta} \right) \right) \,. \tag{21}$$

However, from (19) we may also deduce

$$\frac{\mathrm{d}f(\mathbf{r}_i, \theta_i, \omega_i)}{\mathrm{d}t} = \iiint \mathrm{d}\mathbf{r}\mathrm{d}\theta\mathrm{d}\omega f(\mathbf{r}, \theta, \omega) \frac{\partial \rho_i(\mathbf{r}, \theta, \omega, t)}{\partial t} \,. \tag{22}$$

Comparing equations (21) and (22) we find (using the fact that f is an arbitrary function) that

$$\frac{\partial \rho_i(\mathbf{r}, \theta, \omega, t)}{\partial t} = -\nabla \cdot (\rho_i(\mathbf{r}, \theta, \omega, t) \dot{\mathbf{r}}) - \frac{\partial}{\partial \theta} \left(\rho_i(\mathbf{r}, \theta, \omega, t) \dot{\theta} \right) \,. \tag{23}$$

We emphasize that this argument is standard, and the only subtlety is that we have not carried out any thermal averaging at this point. Summing equation (23) over the i and using the definition of the density ρ we obtain

$$\frac{\partial \rho(\mathbf{r}, \theta, \omega, t)}{\partial t} = -\nabla \cdot (\rho(\mathbf{r}, \theta, \omega, t) \dot{\mathbf{r}}) - \frac{\partial}{\partial \theta} \left(\rho(\mathbf{r}, \theta, \omega, t) \dot{\theta} \right) \,. \tag{24}$$

(1) For the case of the phase coupling dynamics, the equation for the density ρ is

$$\begin{aligned}
\frac{\partial \rho(\mathbf{r}, \theta, \omega, t)}{\partial t} &= -\nabla \cdot (\rho(\mathbf{r}, \theta, \omega, t) v\mathbf{p}(\theta)) - \frac{\partial}{\partial \theta} \left(\rho(\mathbf{r}, \theta, \omega, t) \left(\omega + G\sum_{j=1}^{N} \sin(\theta_j - \theta) \delta(\mathbf{r}_j - \mathbf{r}) \right) \right) \\
&= -v\mathbf{p}(\theta) \cdot \nabla \rho(\mathbf{r}, \theta, \omega, t) - \omega \frac{\partial}{\partial \theta} \rho(\mathbf{r}, \theta, \omega, t) \\
&\quad - G\frac{\partial}{\partial \theta} \rho(\mathbf{r}, \theta, \omega, t) \iiint \mathrm{d}\mathbf{r}'\mathrm{d}\theta'\mathrm{d}\omega' \rho(\mathbf{r}', \theta', \omega', t) \sin(\theta' - \theta) \delta(\mathbf{r}' - \mathbf{r}) \,,
\end{aligned} \tag{25}$$

where $\mathbf{p}\left(\theta\right) = \left(\cos\theta, \sin\theta\right)$. Then for the density ϱ we have

$$\frac{\partial \varrho\left(\mathbf{r},\theta,t\right)}{\partial t} = -v\mathbf{p}\left(\theta\right) \cdot \nabla \varrho\left(\mathbf{r},\theta,t\right) - \frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty} \omega\rho\left(\mathbf{r},\theta,\omega,t\right)\mathrm{d}\omega$$
$$- G\frac{\partial}{\partial\theta}\varrho\left(\mathbf{r},\theta,t\right)\iint \mathrm{d}\mathbf{r}'\mathrm{d}\theta'\varrho\left(\mathbf{r}',\theta',t\right)\sin\left(\theta'-\theta\right)\delta\left(\mathbf{r}'-\mathbf{r}\right) \tag{26}$$

(2) For the case of the chemotactic dynamics, the equation for the density ρ^s is

$$\frac{\partial \rho^s\left(\mathbf{r},\theta,\omega,t\right)}{\partial t} = -\nabla\cdot\left(\rho^s\left(\mathbf{r},\theta,\omega,t\right)v\mathbf{p}\left(\theta\right)\right) - \frac{\partial}{\partial\theta}\left(\rho^s\left(\mathbf{r},\theta,\omega,t\right)\left(\omega + \alpha^s\mathbf{p}\left(\theta\right)\times\nabla u + \beta^s\mathbf{p}\left(\theta\right)\times\nabla v\right)\right)$$
$$= -v\mathbf{p}\left(\theta\right)\cdot\nabla\rho^s\left(\mathbf{r},\theta,\omega,t\right) - \omega\frac{\partial}{\partial\theta}\rho^s\left(\mathbf{r},\theta,\omega,t\right)$$
$$- \frac{\partial}{\partial\theta}\rho^s\left(\mathbf{r},\theta,\omega,t\right)\alpha^s\left[|\nabla u|\sin\left(\theta+\varphi_u\right) + |\nabla v|\sin\left(\theta+\varphi_v\right)\right] \tag{27}$$

where $\varphi_c = \arg\left(-\partial_y c + \mathrm{i}\partial_x c\right), c = u, v$. Then for the density ϱ^s we have

$$\frac{\partial \varrho^s\left(\mathbf{r},\theta,t\right)}{\partial t} = -v\mathbf{p}\left(\theta\right)\cdot\nabla\varrho^s\left(\mathbf{r},\theta,t\right) - \frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty}\omega\rho^s\left(\mathbf{r},\theta,\omega,t\right)\mathrm{d}\omega$$
$$- \frac{\partial}{\partial\theta}\varrho^s\left(\mathbf{r},\theta,t\right)\alpha^s\left[|\nabla u|\sin\left(\theta+\varphi_u\right) + |\nabla v|\sin\left(\theta+\varphi_v\right)\right] \tag{28}$$

Next, let's determine the value of item

$$\int_{-\infty}^{+\infty}\omega\rho^s\left(\mathbf{r},\theta,\omega,t\right)\mathrm{d}\omega. \tag{29}$$

The uniform distribution of disorder state indicates $g\left(\omega\right) = \left[2\left(\omega_{\max} - \omega_{\min}\right)\right]^{-1}$, which is an ω-independent constant. Then we have

$$\int_{-\infty}^{+\infty}\omega\rho^s\left(\mathbf{r},\theta,\omega,t\right)\mathrm{d}\omega = \begin{cases}\frac{1}{2}\rho^s\left(\mathbf{r},\theta,\omega,t\right)\left(\omega_{\max}^2 - \omega_{\min}^2\right), & \text{SingleChirality} \\ 0, & \text{DoubleChirality}\end{cases}. \tag{30}$$

Similarly, equation (18b) can be rewritten as

$$\varrho^s\left(\mathbf{r},\theta,t\right) = \rho^s\left(\mathbf{r},\theta,\omega,t\right)\int_{-\infty}^{+\infty}\mathrm{d}\omega = \begin{cases}2\left(\omega_{\max} - \omega_{\min}\right)\rho^s\left(\mathbf{r},\theta,\omega,t\right), & \text{SingleChirality} \\ 0, & \text{DoubleChirality}\end{cases}. \tag{31}$$

Substituting equations (31) into (30), we obtain

3.0.1 Angular Fourier expansion of the phase-space distribution

As $\varrho(\mathbf{r},\theta,t)$ is a periodic function of θ, it can be expanded in a Fourier series, defined as

$$\hat{\varrho}_k(\mathbf{r},t) = \int_{-\pi}^{\pi}\varrho(\mathbf{r},\theta,t)\mathrm{e}^{\mathrm{i}k\theta}\mathrm{d}\theta. \tag{32}$$

The inverse Fourier transform is

$$\varrho(\mathbf{r},\theta,t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\hat{\varrho}_k(\mathbf{r},t)\mathrm{e}^{\mathrm{i}k\theta}. \tag{33}$$

In this framework, the uniform distribution $\varrho_0(\mathbf{r},\theta,t) = \left(2\pi\right)^{-1}\varrho_0^*$ corresponds to $\hat{\varrho}_k(\mathbf{r},\omega,t) = \left(2\pi\right)^{-1}\varrho_0^*$ for $k = 0$.

Let us use as a basis of the plane the two orthogonal vectors $\mathbf{p}_1 = (1, 0)$ and $\mathbf{p}_2 = (0, 1)$. In order to obtain an evolution equation for the velocity field, we multiply equations (26) and (28) by $\mathbf{e}(\theta)$ and integrate over θ from $-\pi$ to π. For equation (28), we obtain ($j = 1, 2$)

$$\frac{\partial}{\partial t} \int_{-\pi}^{\pi} \mathbf{e}_j(\theta) \, \varrho(\mathbf{r}, \theta, t) \, \mathrm{d}\theta + v \sum_{l=1}^{2} \frac{\partial}{\partial \mathbf{r}_l} \int_{-\pi}^{\pi} \mathbf{e}_j(\theta) \, \mathbf{e}_l(\theta) \, \varrho(\mathbf{r}, \theta, t) \, \mathrm{d}\theta = \int_{-\pi}^{\pi} \mathbf{e}_j(\theta) \, (I_{\text{freq}} + I_{\text{chem}}) \, \mathrm{d}\theta \,, \quad (34)$$

where

$$I_{\text{freq}} = -\frac{\partial}{\partial \theta} \int_{-\infty}^{+\infty} \omega \rho^s(\mathbf{r}, \theta, \omega, t) \, \mathrm{d}\omega \,, \tag{35a}$$

$$I_{\text{chem}} = -\frac{\partial}{\partial \theta} \varrho^s(\mathbf{r}, \theta, t) \, \alpha^s \left[|\nabla u| \sin(\theta + \varphi_u) + |\nabla v| \sin(\theta + \varphi_v) \right] \,. \tag{35b}$$

To proceed further, it is convenient to identify complex numbers with two-dimensional vectors, in such a way that $\mathbf{e}(\theta)$ is mapped onto $\mathrm{e}^{\mathrm{i}\theta}$. Then, in the same way, $v\hat{\varrho}_1(\mathbf{r}, t)$ is associated with the momentum field $\mathbf{w}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \, \mathbf{u}(\mathbf{r}, t)$. Hence, we wish to rewrite equation (34) in such complex notations. For later use, we shall write it in a slightly more general form, replacing $\mathrm{e}^{\mathrm{i}\theta}$ by $\mathrm{e}^{\mathrm{i}k\theta}$:

$$\frac{\partial}{\partial t} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} \varrho(\mathbf{r}, \theta, t) \, \mathrm{d}\theta + v \sum_{l=1}^{2} \frac{\partial}{\partial \mathbf{r}_l} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} \mathbf{e}_l(\theta) \, \varrho(\mathbf{r}, \theta, t) \, \mathrm{d}\theta = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} (I_{\text{freq}} + I_{\text{chem}}) \, \mathrm{d}\theta \,. \tag{36}$$

Equation (34) is recovered for $k = 1$, up to the mapping between complex numbers and two-dimensional vectors. The first term on the left-hand side is simply $\partial\hat{\varrho}_k / \partial t$. The second term on the left-hand side can be evaluated as follows: For $l = 1, 2$ and k integer, let us define the complex quantity $Q_l^{(k)}(\mathbf{r}, t)$ as

$$Q_l^{(k)}(\mathbf{r}, t) = \int_{-\pi}^{\pi} \mathrm{d}\theta \mathrm{e}^{\mathrm{i}k\theta} \mathbf{e}_l(\theta) f(\mathbf{r}, \theta, t). \tag{37}$$

The following relations are then easily obtained:

$$\begin{aligned} Q_1^{(k)}(\mathbf{r}, t) &= \frac{1}{2} [\hat{f}_{k+1}(\mathbf{r}, t) + \hat{f}_{k-1}(\mathbf{r}, t)], \\ Q_2^{(k)}(\mathbf{r}, t) &= \frac{1}{2\mathrm{i}} [\hat{f}_{k+1}(\mathbf{r}, t) - \hat{f}_{k-1}(\mathbf{r}, t)]. \end{aligned} \tag{38}$$

The right-hand side of equation (36) is computed by inserting the Fourier series expansion (33) into equations (35). After a rather straightforward calculation, one finds

References

[1] David S Dean. Langevin equation for the density of a system of interacting langevin processes. *Journal of Physics A: Mathematical and General*, 29(24):L613, dec 1996.