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Diverse behaviors in non-uniform chiral and non-chiral swarmalators

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We study the emergent behaviors of a population of swarming coupled oscillators, dubbed swarmalators. Previous work considered the simplest, idealized case: identical swarmalators with global coupling. Here we expand this work by adding more realistic features: local coupling, non-identical natural frequencies, and chirality. This more realistic model generates a variety of new behaviors including lattices of vortices, beating clusters, and interacting phase waves. Similar behaviors are found across natural and artificial micro-scale collective systems, including social slime mold, spermatozoa vortex arrays, and Quincke rollers. Our results indicate a wide range of future use cases, both to aid characterization and understanding of natural swarms, and to design complex interactions in collective systems from soft and active matter to micro-robotics.

Contents

In the original model:

$$|\mathbf{v}_i| = v_0 = 0$$

In this paper:

$$\mathbf{v}_i = c_i \mathbf{n}_i$$

$$c_i = \omega_i R_i$$

$$\mathbf{n}_i = \begin{bmatrix} \cos\left(\theta_i + \frac{\pi}{2}\right) \\ \sin\left(\theta_i + \frac{\pi}{2}\right) \end{bmatrix}$$

New phase offset terms, $Q_{\dot{x}}, Q_{\dot{\theta}}$, which enable 'frequency coupling'.

$$Q_{\dot{x}} = \frac{\pi}{2}\left|\frac{\omega_j}{|\omega_j|} - \frac{\omega_i}{|\omega_i|}\right|$$
$$Q_{\dot{\theta}} = \frac{\pi}{4}\left|\frac{\omega_j}{|\omega_j|} - \frac{\omega_i}{|\omega_i|}\right|$$

My understanding:

$$Q_{\dot{x}} = \frac{\pi}{2}\left|\operatorname{sgn}\omega_j - \operatorname{sgn}\omega_i\right| = \begin{cases} \pi, & \operatorname{sgn}\omega_j \neq \operatorname{sgn}\omega_i \\ 0, & \operatorname{sgn}\omega_j = \operatorname{sgn}\omega_i \end{cases}$$
$$J\cos\left(\theta_j - \theta_i - Q_{\dot{x}}\right) = \begin{cases} -J\cos\left(\theta_j - \theta_i\right), & Q_{\dot{x}} = \pi \\ J\cos\left(\theta_j - \theta_i\right), & Q_{\dot{x}} = 0 \end{cases}$$

$$\dot{\mathrm{x}}_i = \mathrm{v}_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathrm{x}_j - \mathrm{x}_i}{|\mathrm{x}_j - \mathrm{x}_i|}\left(A + J\cos\left(\theta_j - \theta_i - Q_{\dot{x}}\right)\right) - B\frac{\mathrm{x}_j - \mathrm{x}_i}{|\mathrm{x}_j - \mathrm{x}_i|^2}\right]$$

$$J\cos\left(\theta_j - \theta_i - Q_{\dot{x}}\right) = \begin{cases} -J\cos\left(\theta_j - \theta_i\right), & \operatorname{sgn}\omega_j \neq \operatorname{sgn}\omega_i \\ J\cos\left(\theta_j - \theta_i\right), & \operatorname{sgn}\omega_j = \operatorname{sgn}\omega_i \end{cases}$$

How can I ensure that $J\cos\left(\theta_j - \theta_i\right) > 0$, which means $\theta_j - \theta_i > \pi$ when $\operatorname{sgn}\omega_j \neq \operatorname{sgn}\omega_i$?

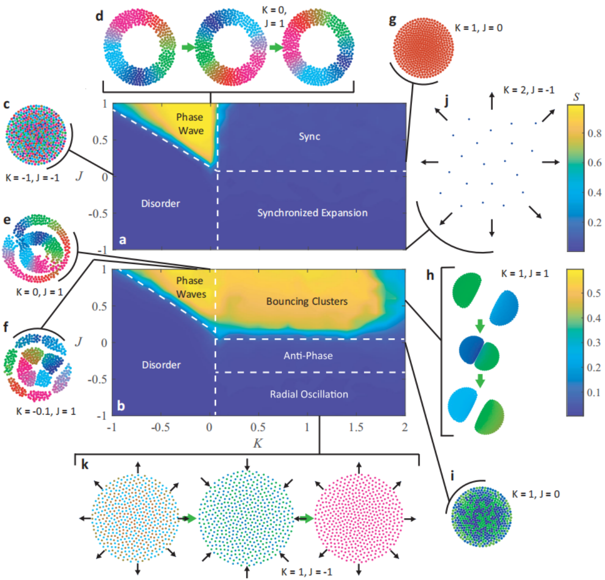
Model equations:

$$\dot{\mathrm{x}}_i = \mathrm{v}_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathrm{x}_j-\mathrm{x}_i}{|\mathrm{x}_j-\mathrm{x}_i|}\left(A + J\cos\left(\theta_j-\theta_i-Q_{\dot{x}}\right)\right) - B\frac{\mathrm{x}_j-\mathrm{x}_i}{|\mathrm{x}_j-\mathrm{x}_i|^2}\right]$$

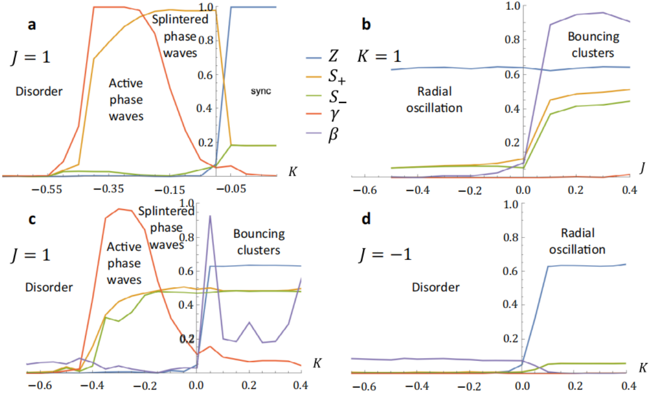
$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin\left(\theta_j-\theta_i-Q_{\dot{\theta}}\right)}{|\mathrm{x}_j-\mathrm{x}_i|}$$

Several different cases of natural frequencies ω in this paper:

1. Single frequency ($F1$): $\omega_i = 1$for all swarmalators.
2. Two frequencies ($F2$): Exactly half of the swarmalators have $\omega_i = 1$ and the other half have $\omega_i = -1$.
3. Single uniform distribution ($F3$): All swarmalators have their natural frequency randomly selected from a single uniform distribution, such that $\omega_i \sim U(1, \Omega)$.
4. Double uniform distribution ($F4$): Exactly half of the swarmalators have their natural frequency randomly selected from one uniform distribution ($\omega_i \sim U(1, \Omega)$) and the second half have their natural frequency selected from another uniform distribution ($\omega_i \sim U(-\Omega, -1)$).


d
K = 0, J = 1
g
K = 1, J = 0
c
K = -1, J = -1
Phase Wave
Sync
Disorder
Synchronized Expansion
a
j
K = 2, J = -1
S
e
K = 0, J = 1
Phase Waves
Bouncing Clusters
Anti-Phase
Disorder
Radial Oscillation
b
h
K = 1, J = 1
f
K = -0.1, J = 1
J
K
k
K = 1, J = -1
i
K = 1, J = 0

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j)}, \quad \beta = \left| \frac{1}{n_\omega > 0} \sum_{j=1}^{n_{\omega>0}} \mathbf{x}_{\omega>0} - \frac{1}{n_{\omega<0}} \sum_{j=1}^{n_{\omega<0}} \mathbf{x}_{\omega<0} \right|$$

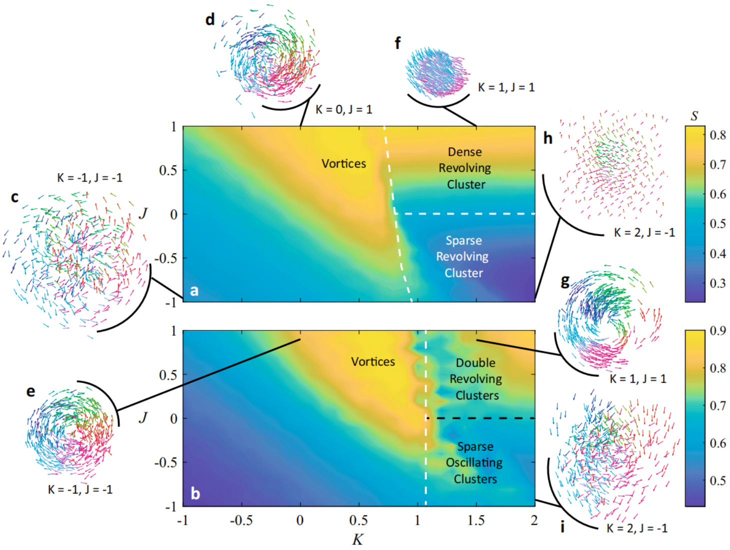


Contents

Revolving swarmalators

Revolving swarmalators' motion and phase coupling behavior is defined when $c_i \neq 0$ for all agents and $Q_{\dot{x}}, Q_{\dot{\theta}} = 0$

$$
\begin{cases}
\dot{\mathbf{x}}_i = \mathbf{v}_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\left(A + J\cos\left(\theta_j - \theta_i\right)\right) - B\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|^2}\right] \\
\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin(\theta_j-\theta_i)}{|\mathbf{x}_j-\mathbf{x}_i|} \\
\mathbf{v}_i = \omega_i\begin{bmatrix}\cos\left(\theta_i + \frac{\pi}{2}\right) \\ \sin\left(\theta_i + \frac{\pi}{2}\right)\end{bmatrix}
\end{cases}
$$


d
K = 0, J = 1
f
K = 1, J = 1
K = -1, J = -1
c
h
K = 2, J = -1
g
K = 1, J = 1
e
K = -1, J = -1
i
K = 2, J = -1
Vortices
Dense
Revolving
Cluster
Sparse
Revolving
Cluster
a
Vortices
Double
Revolving
Clusters
Sparse
Oscillating
Clusters
b
J
K
S
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2
0.8
0.7
0.6
0.5
0.4
0.3
0.9
0.8
0.7
0.6
0.5

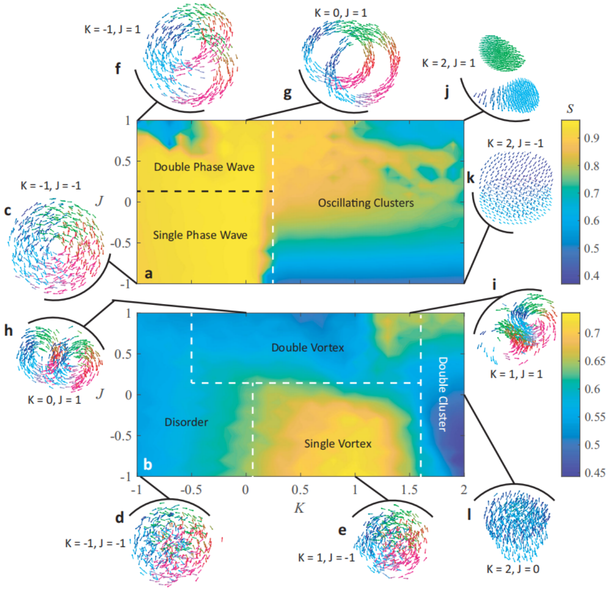
Contents

Frequency-coupled chiral swarmalators

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\left(A + J\cos\left(\theta_j - \theta_i - Q_{\dot{x}}\right)\right) - B\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|^2}\right] \\ \dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin(\theta_j-\theta_i-Q_{\dot{\theta}})}{|\mathbf{x}_j-\mathbf{x}_i|} \\ \mathbf{v}_i = \omega_i\begin{bmatrix}\cos\left(\theta_i + \frac{\pi}{2}\right) \\ \sin\left(\theta_i + \frac{\pi}{2}\right)\end{bmatrix} \end{cases}$$

$$Q_{\dot{x}} = \frac{\pi}{2}\left|\frac{\omega_j}{|\omega_j|} - \frac{\omega_i}{|\omega_i|}\right|$$

$$Q_{\dot{\theta}} = \frac{\pi}{4}\left|\frac{\omega_j}{|\omega_j|} - \frac{\omega_i}{|\omega_i|}\right|$$


K = -1, J = 1
f
K = 0, J = 1
g
K = 2, J = 1
j
S
0.9
0.8
0.7
0.6
0.5
0.4
1
0.5
0
-0.5
-1
J
Double Phase Wave
Single Phase Wave
Oscillating Clusters
a
K = -1, J = -1
c
K = 2, J = -1
k
i
K = 1, J = 1
h
K = 0, J = 1
Double Vortex
Disorder
Single Vortex
Double Cluster
b
0.7
0.65
0.6
0.55
0.5
0.45
-1
-0.5
0
0.5
1
1.5
2
K
d
K = -1, J = -1
e
K = 1, J = -1
l
K = 2, J = 0

Contents

3D Revolving swarmalators

$$
\begin{cases}
\dot{\mathrm{x}}_i = \mathrm{v}_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathrm{x}_j-\mathrm{x}_i}{|\mathrm{x}_j-\mathrm{x}_i|}\left(A + J\cos\left(\theta_j - \theta_i\right)\right) - B\frac{\mathrm{x}_j-\mathrm{x}_i}{|\mathrm{x}_j-\mathrm{x}_i|^2}\right] \\
\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin(\theta_j-\theta_i)}{|\mathrm{x}_j-\mathrm{x}_i|} \\
\mathrm{v}_i = \omega_i\begin{bmatrix}\cos\left(\theta_i + \frac{\pi}{2}\right) \\ \sin\left(\theta_i + \frac{\pi}{2}\right) \\ z_i\end{bmatrix}
\end{cases}
$$

- $$z_i = \begin{cases}\cos\theta_i, & \omega_i > 0 \\ \sin\theta_i, & \omega_i < 0\end{cases}$$

- $$z_i \sim U(0,1)$$

$$\dot{\mathbf{x}}_i = \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|}\left(1 + J\cos(\theta_j - \theta_i)\right) - \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|^3}\right], \quad (21)$$

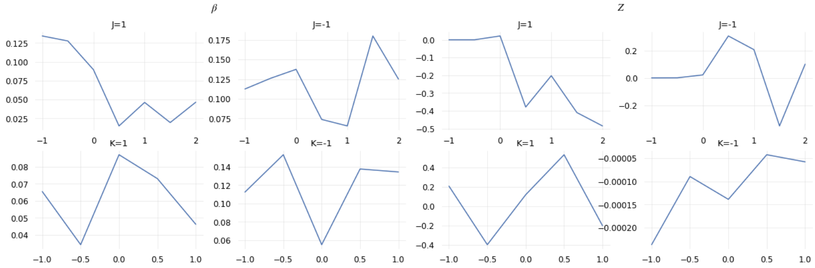
$$\dot{\theta}_i = \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin(\theta_j - \theta_i)}{|\mathbf{x}_j - \mathbf{x}_i|}, \quad (22)$$

where $\mathbf{x}_i = (x_i, y_i, z_i)$. These are the same as Eqs. (3) and (4), except the exponent of the hard shell repulsion is now 3 (we choose this because it yields simple formulas for the radii of certain states).

Order parameter

$$z_i = \begin{cases} \cos\theta_i, & \omega_i > 0 \\ \sin\theta_i, & \omega_i < 0 \end{cases}$$

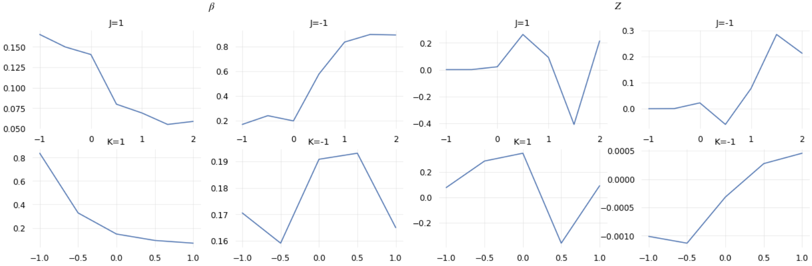
$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j)}, \ \beta = \left| \frac{1}{n_{\omega>0}} \sum_{j=1}^{n_{\omega>0}} \mathrm{x}_{\omega>0} - \frac{1}{n_{\omega<0}} \sum_{j=1}^{n_{\omega<0}} \mathrm{x}_{\omega<0} \right|$$



Order parameter

$$z_i \sim U(0,1)$$

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j)},\ \beta = \left|\frac{1}{n_{\omega>0}}\sum_{j=1}^{n_{\omega>0}} \mathsf{x}_{\omega>0} - \frac{1}{n_{\omega<0}}\sum_{j=1}^{n_{\omega<0}} \mathsf{x}_{\omega<0}\right|$$



3D Frequency-coupled chiral swarmalators

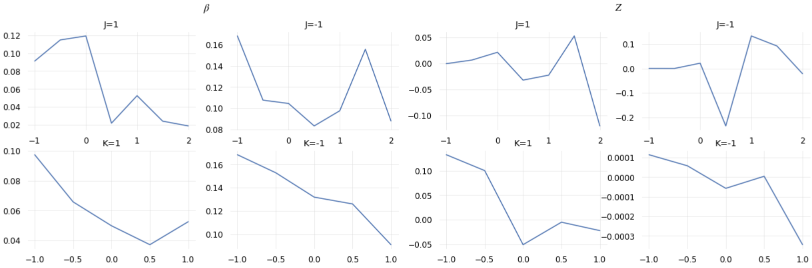
$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i + \frac{1}{N}\sum_{j\neq i}^{N}\left[\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\left(A+J\cos\left(\theta_j-\theta_i-Q_{\dot{x}}\right)\right)-B\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|^2}\right] \\ \dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin(\theta_j-\theta_i-Q_{\dot{\theta}})}{|\mathbf{x}_j-\mathbf{x}_i|} \\ \mathbf{v}_i = \omega_i\begin{bmatrix}\cos\left(\theta_i+\frac{\pi}{2}\right) \\ \sin\left(\theta_i+\frac{\pi}{2}\right) \\ z_i\end{bmatrix} \end{cases}$$

$$Q_{\dot{x}} = \frac{\pi}{2}\left|\frac{\omega_j}{|\omega_j|}-\frac{\omega_i}{|\omega_i|}\right|$$
$$Q_{\dot{\theta}} = \frac{\pi}{4}\left|\frac{\omega_j}{|\omega_j|}-\frac{\omega_i}{|\omega_i|}\right|$$

Order parameter

$$z_i = \begin{cases} \cos\theta_i, & \omega_i > 0 \\ \sin\theta_i, & \omega_i < 0 \end{cases}$$

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j)},\ \beta = \left| \frac{1}{n_{\omega>0}} \sum_{j=1}^{n_{\omega>0}} \mathrm{x}_{\omega>0} - \frac{1}{n_{\omega<0}} \sum_{j=1}^{n_{\omega<0}} \mathrm{x}_{\omega<0} \right|$$



Order parameter

$$z_i \sim U(0,1)$$

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j)},\ \beta = \left|\frac{1}{n_{\omega>0}}\sum_{j=1}^{n_{\omega>0}} \mathbf{x}_{\omega>0} - \frac{1}{n_{\omega<0}}\sum_{j=1}^{n_{\omega<0}} \mathbf{x}_{\omega<0}\right|$$