Chemotactic Chiral Active Matter

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1 Models

1.1 Definitions

1.1.1 Self-propelled dynamics

$$\dot{x}_i = v\cos\theta_i \ , \tag{1a}$$
$$\dot{y}_i = v\sin\theta_i \ , \tag{1b}$$

1.1.2 Polar alignment dynamics

- Additive coupling:

$$\dot{\theta}_i = \omega_i + K\sum_{j=1}^{N} f\left(r_{ij}\right)\sin\left(\theta_j - \theta_i + \alpha\right) \ , \tag{2}$$

- Mean-field coupling by oscillator number:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f\left(r_{ij}\right)\sin\left(\theta_j - \theta_i + \alpha\right) \ , \tag{3}$$

  which is similar to the swarmalator model.

Here, $f\left(r_{ij}\right)$ is a function of $r = |\mathbf{r}_i - \mathbf{r}_j|$, and K is the coupling strength. The function $f\left(r\right)$ can be defined as

1. $f_H\left(r\right) = H\left(d_0 - r\right), \ r_0 > 0$;
2. $f_E\left(r\right) = e^{-\frac{r}{d_0}}, \ r_0 > 0$.

The natural frequencies ω_i are distributed with following two cases:

1. **Single-chiral swarmalators:** The natural frequencies ω_i are distributed in $U\left(\omega_{\min}, \omega_{\max}\right)$ for all swarmalators and $\omega_{\min}\omega_{\max} > 0$.
2. **Double-chiral swarmalators:** The frequencies are distributed in two symmetric uniform distributions, representing two types of chirality. Exactly half of the swarmalators have natural frequencies $\omega_i \sim U\left(\omega_{\min}, \omega_{\max}\right)$ and the other half have natural frequencies $\omega_i \sim U\left(-\omega_{\max}, -\omega_{\min}\right)$.

1.1.3 Chemotactic dynamics

Consider two chemical fields $u\left(\mathbf{r}, t\right), v\left(\mathbf{r}, t\right)$ that are produced by the ensemble of two symmetrically chiral swarmalators. Swarmalators interact with the chemical field and move towards/against the regions with higher concentration, which can be described by the following equation ($i = 1, 2, \ldots, N$):

$$\dot{\mathbf{r}}_i^s = v\mathbf{p}\left(\theta_i^s\right) \tag{4a}$$
$$\dot{\theta}_i^s = \omega_i^s + \alpha^s\mathbf{p}_i^s \times \nabla u + \beta^s\mathbf{p}_i^s \times \nabla v \tag{4b}$$

where α, β_i^s denote the 'chemotactic' coupling strength and $\mathbf{p}\left(\theta\right) = \left(\cos\theta, \sin\theta\right)$ is the unit vector pointing in the direction of the i-th swarmalator, $s \in \{p, n\}$ denotes the two chiral species. Here, we used the notation $\mathbf{a} \times \mathbf{b} = a_1b_2 - a_2b_1$.

These two fields evolve as

$$\dot{u} = k_0\sum_{j=1}^{N}\delta\left(\mathbf{r} - \mathbf{r}_j^p\right) - k_d u + D_u\nabla^2 u \ , \tag{5a}$$

$$\dot{v} = k_0\sum_{j=1}^{N}\delta\left(\mathbf{r} - \mathbf{r}_j^n\right) - k_d v + D_v\nabla^2 v \ , \tag{5b}$$

where S_+ and S_- are the sets of two chiral swarmalators, k_0 is the production rate, k_d is the decay rate, $D_{u,v}$ are the diffusion coefficients.

1.1.4 Mixed phase dynamics

$$\dot{\mathbf{r}}_i = v\mathbf{p}_i \tag{6a}$$

$$\dot{\theta}_i = \omega_i + \beta_i^u \mathbf{p}_i \times \nabla u + \beta_i^v \mathbf{p}_i \times \nabla v + \frac{K}{N}\sum_{j=1}^{N} f\left(|\mathbf{r}_j - \mathbf{r}_i|\right)\sin\left(\theta_j - \theta_i\right) \ , \tag{6b}$$

1.1.5 General Chemotactic Model For Two Species

Type 1:

$$\dot{\mathbf{r}}_i^{1,2} = v\mathbf{p}\left(\theta_i^{1,2}\right) - \sum_{j\in A_i^{1,2}} \mathbf{I}_{ij}^{1,2} \ , \tag{7a}$$

$$\dot{\theta}_i^{1,2} = |\nabla c_{1,2}| \sin\left(\varphi_{c_{1,2}} - \theta_i^{1,2}\right) + F\left(\theta, \mathbf{r}\right) \ , \tag{7b}$$

$$\dot{c}_1 = D_1 \nabla^2 c_1 + F_1\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^1\right) \ , \tag{7c}$$

$$\dot{c}_2 = D_2 \nabla^2 c_2 + F_2\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^2\right) \ , \tag{7d}$$

where $\mathbf{I}_{ij}^{1,2} = \frac{\mathbf{r}_j - \mathbf{r}_i^{1,2}}{|\mathbf{r}_j - \mathbf{r}_i^{1,2}|^2}$, $\varphi_{c_{1,2}} = \arctan\left(\frac{\partial_y c_{1,2}}{\partial_x c_{1,2}}\right)$ and $A_i^{1,2} = \left\{ j \mid r_c \geqslant |\mathbf{r}_j - \mathbf{r}_i^{1,2}| \right\}$.
Type 2:

$$\dot{\mathbf{r}}_i^{1,2} = v\mathbf{p}\left(\theta_i^{1,2}\right) + \alpha_{1,2}\nabla c_{1,2} - \sum_{j\in A_i^{1,2}} \mathbf{I}_{ij}^{1,2} \ , \tag{8a}$$

$$\dot{\theta}_i^{1,2} = F\left(\theta, \mathbf{r}\right) \ , \tag{8b}$$

$$\dot{c}_1 = D_1 \nabla^2 c_1 + F_1\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^1\right) \ , \tag{8c}$$

$$\dot{c}_2 = D_2 \nabla^2 c_2 + F_2\left(c_1, c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^2\right) \ , \tag{8d}$$

1.1.6 Chemotactic Model with Lotka-Volterra Functions

Let $F_1\left(c_1, c_2\right) = c_1\left(k_1 - k_2 c_2\right)$ and $F_2\left(c_1, c_2\right) = c_2\left(k_3 c_1 - k_4\right)$, where k_1, k_2, k_3, k_4 are constants.

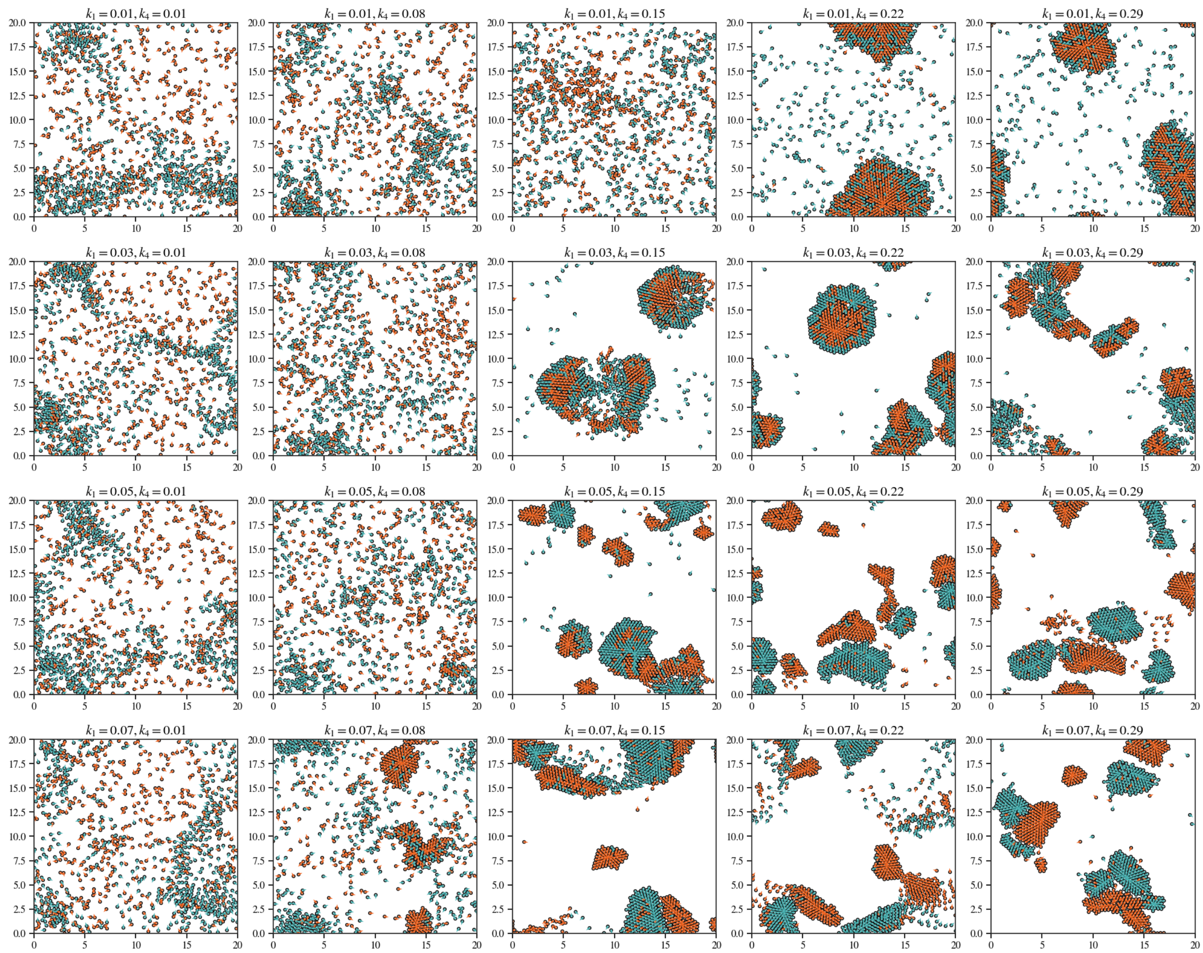
$$\dot{\mathbf{r}}_i^{1,2} = v\mathbf{p}\left(\theta_i^{1,2}\right) - \sum_{j\in A_i^{1,2}} \mathbf{I}_{ij}^{1,2} \ , \tag{9a}$$

$$\dot{\theta}_i^{1,2} = |\nabla c_{1,2}| \sin\left(\varphi_{c_{1,2}} - \theta_i^{1,2}\right) + F\left(\theta, \mathbf{r}\right) \ , \tag{9b}$$

$$\dot{c}_1 = D_1 \nabla^2 c_1 + c_1\left(k_1 - k_2 c_2\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^1\right) \ , \tag{9c}$$

$$\dot{c}_2 = D_2 \nabla^2 c_2 + c_2\left(k_3 c_1 - k_4\right) \sum_{j=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_j^2\right) \ , \tag{9d}$$

2 Behaviors


$k_1 = 0.01, k_4 = 0.01$
$k_1 = 0.01, k_4 = 0.08$
$k_1 = 0.01, k_4 = 0.15$
$k_1 = 0.01, k_4 = 0.22$
$k_1 = 0.01, k_4 = 0.29$
$k_1 = 0.03, k_4 = 0.01$
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$k_1 = 0.07, k_4 = 0.01$
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$k_1 = 0.07, k_4 = 0.22$
$k_1 = 0.07, k_4 = 0.29$

3 Continuum model

The continuum model can be derived from a Boltzmann-like approach for the probability density $P(\mathbf{r}, \theta, t)$ of finding a particle at position $\mathbf{r}$ with orientation θ at time t.