Two Coupled Oscillators with Chirality

April 28, 2024

1 The Model

1.1 Raw model

$$
\begin{cases}
\dot{x}_{1,2} = v\cos\theta_{1,2} \\
\dot{y}_{1,2} = v\sin\theta_{1,2} \\
\dot{\theta}_{1,2} = \omega_{1,2} + \lambda f\left(r\right)\sin\left(\theta_{2,1} - \theta_{1,2}\right)
\end{cases}
$$

where $f\left(r\right)$ is a function of $r = \left|\mathbf{r}_1 - \mathbf{r}_2\right|$, and λ is the coupling strength. The function $f\left(r\right)$ can be defined as

1. $f\left(r\right)_H = H\left(r_0 - r\right),\ r_0 > 0;$
2. $f\left(r\right)_P = \left(1 + \frac{r}{r_0}\right)^{-\frac{1}{r_0}},\ r_0 > 0;$
3. $\ldots$

1.2 Model under polar coordinates

Let

$$x_i = r_i\cos\varphi_i\ ,$$

$$y_i = r_i\sin\varphi_i\ ,$$

then we have

$$
\begin{aligned}
\dot{r}_i &= \frac{1}{r_i}\left(x_i\dot{x}_i + y_i\dot{y}_i\right) \\
&= v\cos\varphi_i\cos\theta_i + v\sin\varphi_i\sin\theta_i \\
&= v\cos\left(\varphi_i - \theta_i\right)\ , \\
\dot{\varphi}_i &= \frac{1}{r_i^2}\left(x\dot{y} - y\dot{x}\right) \\
&= \frac{v}{r_i}\left(\sin\varphi_i\cos\theta_i - \cos\varphi_i\sin\theta_i\right) \\
&= \frac{v}{r_i}\sin\left(\varphi_i - \theta_i\right)\ .
\end{aligned}
$$

Introduce $\alpha_i = \varphi_i - \theta_i$, $\Delta\theta = \theta_2 - \theta_1$, $\Delta\varphi = \varphi_2 - \varphi_1$, $\Delta\omega = \omega_2 - \omega_1$, then we have

$$\begin{cases} \dot{r}_{1,2} = v\cos\alpha_{1,2} \\ \dot{\alpha}_{1,2} = \frac{v}{r_{1,2}}\sin\alpha_{1,2} - \omega_{1,2} \mp \lambda f(r)\sin\Delta\theta \\ \Delta\dot{\varphi} = \frac{v}{r_2}\sin\alpha_2 - \frac{v}{r_1}\sin\alpha_1 \\ \Delta\dot{\theta} = \Delta\omega - 2\lambda f(r)\sin\Delta\theta \end{cases} \tag{1}$$

where

$$\begin{aligned} r &= \sqrt{(r_1\cos\varphi_1 - r_2\cos\varphi_2)^2 + (r_1\sin\varphi_1 - r_2\sin\varphi_2)^2} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\Delta\varphi} \end{aligned} .$$

So the function $f(r)$ can be defined as $f(r_1, r_2, \Delta\varphi)$.

1.3 Single direction driving

Assuming that $\dot{\theta}_2 = \omega_2, \alpha_2 = \dfrac{\pi \mathrm{sgn}\omega_2}{2}$, which means that the second oscillator is rotating around the origin with a constant angular velocity ω_2, and the first oscillator is driven by the second one. Then the model becomes

$$\begin{cases} \dot{r}_1 = v\cos\alpha_1 \\ \dot{\alpha}_1 = \frac{v}{r_1}\sin\alpha_1 - \omega_1 - \lambda f(r)\sin\Delta\theta \\ \Delta\dot{\varphi} = \omega_2 - \frac{v}{r_1}\sin\alpha_1 \\ \Delta\dot{\theta} = \Delta\omega - \lambda f(r)\sin\Delta\theta \end{cases} .$$

When $2\lambda f(r) \geqslant |\Delta\omega|$, the system has fixed points $\mathbf{x}$, which are

$$\begin{aligned} r_1 &= \frac{v}{\omega_2} , \\ \alpha_1 &= \frac{\pi \mathrm{sgn}\omega_2}{2} , \\ \Delta\varphi &= C_\varphi , \\ \Delta\theta &= C_\theta , \end{aligned}$$

where C_φ and C_θ are constants determined by the initial conditions. Linearizing the governing equations yields

$$M = \begin{bmatrix} 0 & -v\sin\alpha_1 & 0 & 0 \\ -\frac{v}{r_1^2}\sin\alpha_1 - \lambda f_{r_1}\sin\Delta\theta & \frac{v}{r_1}\cos\alpha_1 & -\lambda f_{\Delta\varphi}\sin\Delta\theta & -\lambda f\cos\Delta\theta \\ \frac{v}{r_1^2}\sin\alpha_1 & -\frac{v}{r_1}\cos\alpha_1 & 0 & 0 \\ -\lambda f_{r_1}\sin\Delta\theta & 0 & -\lambda f_{\Delta\varphi}\sin\Delta\theta & -\lambda f\cos\Delta\theta \end{bmatrix}$$

where $f_{r_1} = \dfrac{\partial f}{\partial r_1}$, $f_{\Delta\varphi} = \dfrac{\partial f}{\partial \Delta\varphi}$, and $f_{\Delta\theta} = \dfrac{\partial f}{\partial \Delta\theta}$. Evaluating M at the fixed points results in

$$M = \begin{bmatrix} 0 & -v\text{sgn}\omega_2 & 0 & 0 \\ -\frac{\omega_2^2}{v}\text{sgn}\omega_2 - \lambda f_{r_1}\left(\mathbf{x}\right)\sin C_\theta & 0 & -\lambda f_{\Delta\varphi}\left(\mathbf{x}\right)\sin C_\theta & -\lambda f\left(\mathbf{x}\right)\cos C_\theta \\ \frac{\omega_2^2}{v}\text{sgn}\omega_2 & 0 & 0 & 0 \\ -\lambda f_{r_1}\left(\mathbf{x}\right)\sin C_\theta & 0 & -\lambda f_{\Delta\varphi}\left(\mathbf{x}\right)\sin C_\theta & -\lambda f\left(\mathbf{x}\right)\cos C_\theta \end{bmatrix}$$