Sample Title:
with Forced Linebreak

A. Author,[1, a)] B. Author,[1] and C. Author[2, b)]
[1)]*Authors' institution and/or address*
[2)]*Second institution and/or address*

(*Electronic mail: Second.Author@institution.edu.)

(Dated: 16 March 2024)

An article usually includes an abstract, a concise summary of the work covered at length in the main body of the article. It is used for secondary publications and for information retrieval purposes.

The "lead paragraph" is encapsulated with the LaTeX The lead paragraph will only be found in an article being prepared for the journal *Chaos*.

I. INTRODUCTION

The color of the background represents the order parameter r of the system. The color of the snapshots represents the phase of the oscillators. The color of the arrows represents the direction of the velocity of the oscillators. The size of the arrows represents the speed of the oscillators.

II. MODEL

Oscillators have a spatial position $\mathbf{r}_i = (x_i, y_i)$ and an internal phase θ_i which evoleve according to equations:

$$\dot{x}_i = v\cos\theta_i \ , \tag{1}$$

$$\dot{y}_i = v\sin\theta_i \ , \tag{2}$$

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} A_{ij}\sin\left(\theta_j - \theta_i\right) \tag{3}$$

for $i = 1, 2, \ldots, N$, where N is the number of oscillators. As per Eq. (1) and (2), each oscillator moves with a constant speed v in the direction of its current phase θ_i. The phase θ_i evolves according to Eq. (3), where ω_i is the natural frequency of the ith oscillator, λ is the coupling strength, and A is the adjacency matrix of the network, with $A_{ij} = 1$ if there is a connection from ith to jth oscillator, and $A_{ij} = 0$ otherwise. We can consider Eq. (1)-(3) as a generalization of the Kuramoto model and the Vicsek model in the sense that it includes both the phase and the spatial position of the oscillators.

Each oscillator i is connected to all the oscillators within a action radius d_0 of its position. The adjacency matrix A is defined as:

$$A_{ij} = \begin{cases} 1, & \left|\mathbf{r}_i - \mathbf{r}_j\right| \leq d_0 \\ 0, & \left|\mathbf{r}_i - \mathbf{r}_j\right| > d_0 \end{cases} \tag{4}$$

where $\left|\mathbf{r}_i - \mathbf{r}_j\right|$ is the Euclidean distance between the ith and jth oscillators.

For simplicity, we consider oscillators are initially distributed uniformly in a two-dimensional square with side length L and periodic boundary conditions. Their positions $\mathbf{r}_i(t) = (x_i(t), y_i(t))$ at given time t are given by:

$$\begin{aligned} x_i(t+\Delta t) &= x_i(t) + v\cos\theta_i(t)\,\Delta t \bmod L, \\ x_i(t+\Delta t) &= x_i(t) + v\cos\theta_i(t)\,\Delta t \bmod L, \end{aligned} \tag{5}$$

where Δt is the discrete time step. When two oscillators are on opposite sides of the square, the absolute value of the difference between one of their coordinates is greater than $L/2$. In this case, we take the minimum distance between them, which is the distance between the two points in the periodic boundary conditions. For a given pair of points $\mathbf{r}_i$ and $\mathbf{r}_j$, the distance between them is $\left|\mathbf{r}_i - \bar{\mathbf{r}}_j\right|$, where $\bar{\mathbf{r}}_j = (\bar{x}_j, \bar{y}_j)$ is the adjusted position of the jth oscillator, given by:

$$\bar{x}_j = \begin{cases} x_j, & \left|x_i - x_j\right| \leq L/2 \\ x_j + L, & x_i - x_j > L/2 \\ x_j - L, & x_i - x_j > L/2 \end{cases} , \tag{6}$$

$$\bar{y}_j = \begin{cases} y_j, & \left|y_i - y_j\right| \leq L/2 \\ y_j + L, & y_i - y_j > L/2 \\ y_j - L, & y_i - y_j > L/2 \end{cases} . \tag{7}$$

$\left|\mathbf{r}_i - \bar{\mathbf{r}}_j\right|$ can be proved to be the minimum distance between $\mathbf{r}_i$ and $\mathbf{r}_j$ in the periodic boundary conditions (see the proof in Appendix A).

Finally, we consider that the natural frequencies ω_i are distributed in two symmetric uniform distributions. Exactly half of the oscillators have natural frequencies in the range $[\omega_{\min}, \omega_{\max}]$ $(\omega_i \sim U(\omega_{\min}, \omega_{\max}), i = 1, 2, \ldots, N/2)$ and the other half in the range $[-\omega_{\max}, -\omega_{\min}]$ $(\omega_i \sim U(-\omega_{\max}, -\omega_{\min}), i = N/2+1, N/2+2, \ldots, N)$.

III. BEHAVIOR

We performed numerical simulations of the model to probe the behavior of its solutions (see Appendix B for details on the

[a)]Also at Physics Department, XYZ University.
[b)]http://www.Second.institution.edu/~Charlie.Author.

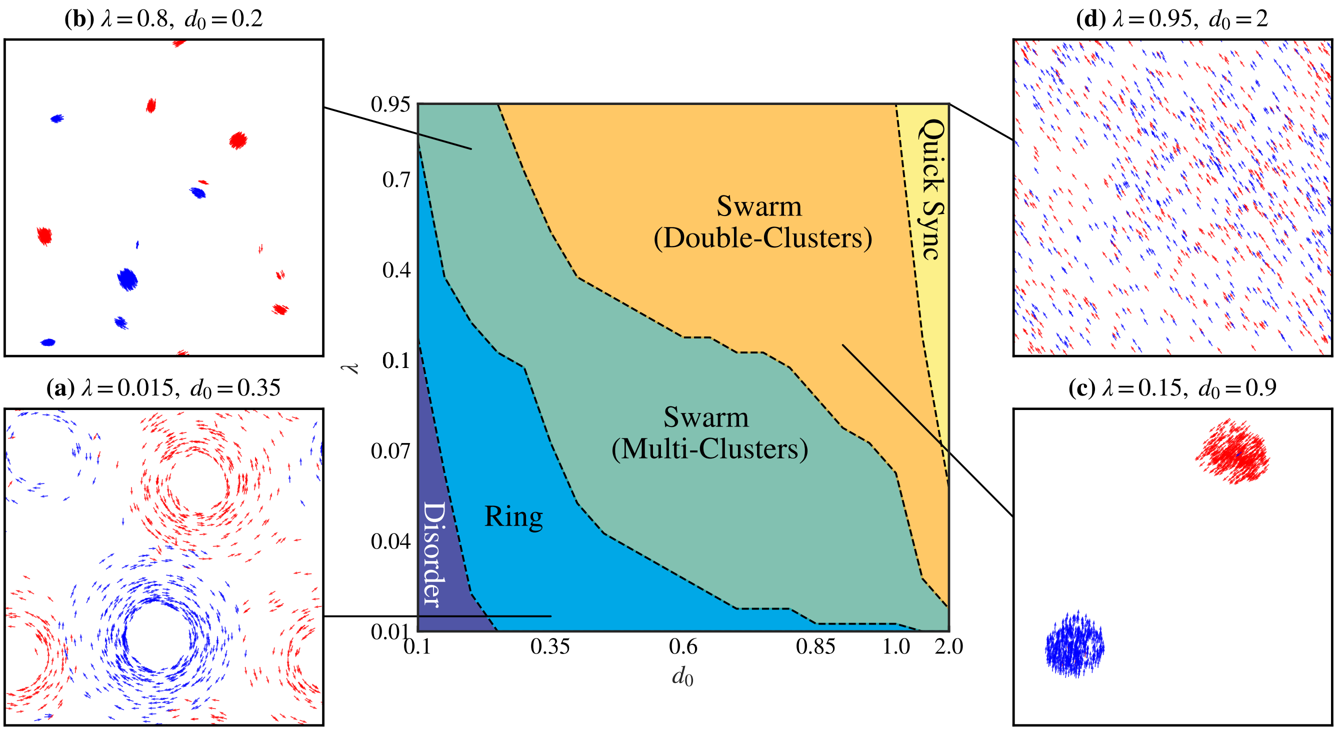

FIG. 1. Phase diagram of model Eq. (1)-(3) in the (λ-d_0) plane. The boundary between the states is analytic approximations given by XXXXXXX. For the sake of clarity, the scale of λ and d_0 is non-linear (For λ in $[0.01, 0.1]$ and $[0.1, 1]$, step length is 0.1 and 0.05, respectively. For d_0 in $[0.1, 1]$ and $[1, 2]$, step length is 0.05 and 0.5, respectively). **(a)**, The snapshots of Ring ($\lambda = 0.015,\ d_0 = 0.35$). **(b)**, Swarm (Multi-Clusters) ($\lambda = 0.8,\ d_0 = 0.2$). **(c)**, Swarm (Double-Clusters) ($\lambda = 0.15,\ d_0 = 0.9$). **(d)**, Quick Sync ($\lambda = 0.95,\ d_0 = 2$).

numerical methods). $N = 1000$ oscillators were distributed uniformly in the square of length $L = 10$ and their natural frequencies were distributed in the range $[\omega_{\min}, \omega_{\max}] = [1, 3]$ and $[-\omega_{\max}, -\omega_{\min}] = [-3, -1]$. Two-parameter of coupling strength λ and action radius d_0 are presented in the phase diagram in Fig. 1. We found the system settles into five states: **Disorder**, **Ring**, **Swarm** (which can be further divided into **Multi-Clusters** and **Double-Clusters**), and **Quick Sync**. In Fig. 1 we show the snapshots of the last four states and where these states are located in the phase diagram. We next discuss these five states.

A. Disorder State

When both λ and d_0 are small, the system is in the Disorder state, as shown in Fig. 3a. In this state, the oscillators are not synchronized and move in a way which similar to uncoupled oscillators ($\lambda = 0$). According to Eq. (1)-(3), when $\lambda = 0$, the equations of oscillators' motion can be written as:

$$\begin{aligned} x_i(t) &= x_i(0) + \tfrac{v}{\omega_i} \sin \omega_i t \ , \\ y_i(t) &= y_i(0) + \tfrac{v}{\omega_i} \cos \omega_i t \ . \end{aligned} \tag{8}$$

In such a setup, oscillators move in a circular trajectory with a radius v/ω_i and the phase θ_i increases linearly with time, as show in Fig. 3b. To calculate the real-time rotational radius,

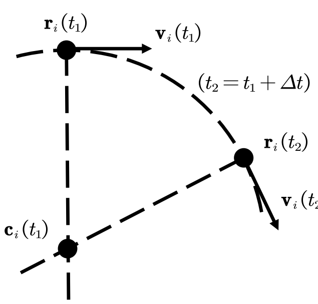

FIG. 2. Estimation for real-time centers. **c**

we first estimate real-time centers $\mathbf{c}(t)$ of the circular trajectory with method in Fig. 2 and then solve the following linear equations:

$$\begin{aligned} \mathbf{c}_i(t_1) \cdot \mathbf{v}_i(t_1) &= \mathbf{x}_i(t_1) \cdot \mathbf{v}_i(t_1) \ , \\ \mathbf{c}_i(t_2) \cdot \mathbf{v}_i(t_2) &= \mathbf{x}_i(t_2) \cdot \mathbf{v}_i(t_2) \end{aligned} \tag{9}$$

where $\mathbf{r}_i$

B. Ring State

The Ring state is characterized by the oscillators forming several ring clusters, each of which is composed of oscillators

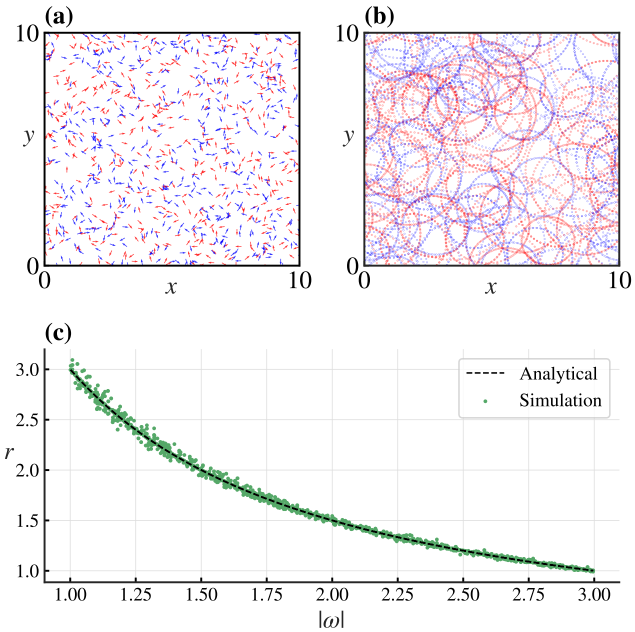

FIG. 3. A figure caption. The figure captions are automatically numbered.

with the same chirality.

Appendix A: PROOF OF THE ADJUSTED POSITION

Appendix B: NUMERICAL METHODS

All the simulations of the model Eq. (1)-(3) were run on Python using Euler integration, with a time step $\Delta t = 0.01$, and a total time of $T = 60000$.