Two Coupled Swarmalators with Chirality

July 12, 2024

1 Reference

1.1 Farrell F D C, Marchetti M C, Marenduzzo D, et al. Pattern formation in self-propelled particles with density-dependent motility[J]. Physical review letters, 2012, 108(24): 248101.

Microscopic dynamics:

$$\begin{aligned} \dot{\mathbf{r}}_i &= v\mathbf{e}_{\theta_i} \ , \\ \dot{\theta}_i &= \gamma \sum_{j=1}^{N} F\left(\theta_j - \theta_i, \mathbf{r}_j - \mathbf{r}_i\right) + \sqrt{2\epsilon}\tilde{\eta}_i\left(t\right) \ . \end{aligned} \tag{1}$$

The microscopic density of particles at position $\mathbf{r}$ with angle θ is given by

$$f\left(\mathbf{r}, \theta\right) = \sum_{i=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_i\right) \delta\left(\theta - \theta_i\right) \ . \tag{2}$$

Using Itô's formula, a stochastic dynamical equation for the density Eq. (2) can be derived:

$$\begin{aligned} &\partial_t f\left(\mathbf{r}, \theta\right) + \mathbf{e}_\theta \cdot \nabla\left[vf\right] \\ &= \epsilon \tfrac{\partial^2 f}{\partial \theta^2} - \tfrac{\partial}{\partial \theta}\sqrt{2\epsilon f}\eta - \gamma \tfrac{\partial}{\partial \theta} \int \mathrm{d}\theta' \mathrm{d}\mathbf{r}' f\left(\mathbf{r}', \theta'\right) \times f\left(\mathbf{r}, \theta\right) F\left(\theta' - \theta, \mathbf{r} - \mathbf{r}'\right) \ . \end{aligned} \tag{3}$$

Drop the noise term, and Fourier transform Eq. (3) to get equations of motion for

$$f_k \equiv \int f\left(\mathbf{r}, \theta\right) e^{ik\theta} \mathrm{d}\theta \ . \tag{4}$$

2 Our Work

We replace the finite range alignment interaction by a pseudopotential (δ-interaction) in the model:

$$\begin{aligned} \dot{\mathbf{r}}_i &= v\mathbf{p}_i \\ \dot{\theta}_i &= \omega_i + \lambda \sum_{j \neq i} \delta\left(\mathbf{r}_j - \mathbf{r}_i\right) \sin\left(\theta_j - \theta_i\right) \end{aligned} \tag{5}$$

where $\mathbf{p}_i = (\cos\theta_i, \sin\theta_i)$. Assuming that we have M species, consisting of $N_1, \ldots, N_M$ particles with identical frequencies $\tilde{\Omega}_1, \ldots, \tilde{\Omega}_M$ respectively, and that $N_1, \ldots, N_M$ are all macroscopic in an area element over which macroscopic quantities (density, polarization) vary significantly, allows us to derive a continuum theory for the particle dynamics. The combined probability density to find a particle of given species j at position $\mathbf{r}$ with angle θ at time t is given by

$$f^{(j)}\left(\mathbf{r},\theta,t\right)=\sum_{i=1}^{N}\delta\left(\mathbf{r}-\mathbf{r}_i\left(t\right)\right)\delta\left(\theta-\theta_i\left(t\right)\right)\delta_{\Omega_i,\tilde{\Omega}_j}\ . \tag{6}$$

A Boltzmann-like equation for the combined density $f^{(j)}$ can be derived:

$$\begin{aligned}\dot{f}^{(j)}(\mathbf{r},\theta,t)=&-\mathrm{Pe}\mathbf{p}\cdot\left[\nabla_{\mathbf{r}}f^{(j)}(\mathbf{r},\theta,t)\right]-\Omega_j\partial_\theta f^{(j)}(\mathbf{r},\theta,t)+\partial_\theta^2 f^{(j)}(\mathbf{r},\theta,t)\\&-g\partial_\theta\left[f^{(j)}(\mathbf{r},\theta,t)\int\mathrm{d}\theta'\sin(\theta'-\theta)\sum_{i=1}^{M}f^{(i)}(\mathbf{r},\theta',t)\right]-\partial_\theta\sqrt{2f^{(j)}(\mathbf{r},\theta,t)}\eta_j(\mathbf{r},\theta,t)\end{aligned} \tag{7}$$

where η_j represents spatiotemporal white noise with zero mean and unit-variance (the subscript j denotes that the noise-realization of a given ensemble is individual for each species).

In the following, we focus on a mean-field description and neglect the (multiplicative) noise term in Eq. (7). Now transforming Eq. (7) to Fourier space yields a hierarchy of dynamical equations for the Fourier modes $f_k^{(j)}(\mathbf{r},t)=\int\mathrm{d}\theta f^{(j)}(\mathbf{r},\theta,t)\mathrm{e}^{\mathrm{i}k\theta}$ and $2\pi f^{(j)}(\mathbf{r},\theta,t)=\sum_{k=-\infty}^{\infty}f_k^{(j)}(\mathbf{r},t)\mathrm{e}^{-\mathrm{i}k\theta}$, reading

$$\begin{aligned}\dot{f}_k^{(j)}(\mathbf{r},t)=&-\frac{\mathrm{Pe}}{2}\left[\partial_x\left(f_{k+1}^{(j)}+f_{k-1}^{(j)}\right)-\mathrm{i}\partial_y\left(f_{k+1}^{(j)}-f_{k-1}^{(j)}\right)\right]\\&+\left(\mathrm{i}k\Omega_j-k^2\right)f_k^{(j)}+\frac{\mathrm{i}gk}{2\pi}\sum_{m=-\infty}^{\infty}f_{k-m}^{(j)}F_{-m}\sum_{i=1}^{M}f_m^{(i)}\end{aligned} \tag{8}$$

After a long calculation, we find the following equations

$$\begin{aligned}\dot{\rho}&=-v\nabla\cdot\mathbf{w}\\\dot{\mathbf{w}}&=(\lambda\rho-2)\frac{\mathbf{w}}{2}-\frac{v}{2}\nabla\rho+\frac{v^2}{2b}\nabla^2\mathbf{w}-\frac{\lambda^2}{b}|\mathbf{w}|^2\mathbf{w}\\&+\frac{\lambda v}{4b}\left[5\nabla\mathbf{w}^2-10\mathbf{w}(\nabla\cdot\mathbf{w})-6(\mathbf{w}\cdot\nabla)\mathbf{w}\right]\\&+\omega\mathbf{w}_\perp+\frac{v^2\omega}{4b}\nabla^2\mathbf{w}_\perp-\frac{\lambda^2\omega}{2b}|\mathbf{w}|^2\mathbf{w}_\perp\\&-\frac{\lambda v\omega}{8b}\left[3\nabla_\perp\mathbf{w}^2-6\mathbf{w}\left(\nabla_\perp\cdot\mathbf{w}\right)-10\left(\mathbf{w}\cdot\nabla_\perp\right)\mathbf{w}\right]\end{aligned} \tag{9}$$

where

$$\begin{aligned}\omega&=\langle\omega_i\rangle+\omega\left(\mathbf{x},t\right)+\sqrt{\frac{\Delta_\omega}{f}}\eta\\b&=2\left(4+\omega^2\right)\\\mathbf{w}_\perp&=\left(-w_y,w_x\right)\\\nabla_\perp&=\left(-\partial_y,\partial_x\right)\end{aligned} \tag{10}$$