Radii of Emergent Patterns in Swarmalator Systems

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Abstract:
For a system of swarmalators converging to different types of circular patterns, we provide expressions for the outer and inner radii of these patterns and examine their dependence on the model parameters. Derivations are made for three static patterns with an infinite number of entities and a generalized swarmalator model with parameterized attraction and repulsion kernels. Simulations of finite systems show good agreement with the asymptotic expressions.

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Model equations:

$$\dot{x}_i = \frac{1}{N} \sum_{j \neq i}^{N} e_{ij} d_{ij}^{\alpha} \left(1 + J \cos\theta_{ij}\right) - e_{ij} d_{ij}^{\beta}$$

$$\dot{\theta}_i = \frac{K}{N} \sum_{j \neq i}^{N} d_{ij}^{\gamma} \sin\theta_{ij}$$

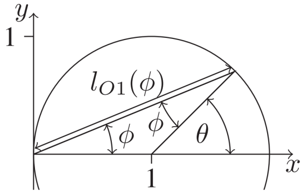
where $d_{ij} = \|x_j - x_i\|$, $\theta_{ij} = \theta_i - \theta_j$, $e_{ij} = \frac{1}{d_{ij}}(x_j - x_i)$.

Choosing $\alpha = 0$ and $\beta = \gamma = -1$ gives the original model.

In the converged state of the static patterns, the entities no longer move, so we have $\dot{x}_i = 0$, which yields:

$$\underbrace{r_{\text{out}}^{\alpha} \sum_{j \neq i}^{N} e_{ij} g_{ij}^{\alpha} \left(1 + J \cos \theta_{ij}\right)}_{\text{Attraction} A} = \underbrace{r_{\text{out}}^{\beta} \sum_{j \neq i}^{N} e_{ij} g_{ij}^{\alpha}}_{\text{Repulsion} R}$$

where $r_{\text{out}} = \max\limits_{1 \le i \le N} \|x_i - \bar{x}\|$, $g_{ij} = \frac{d_{ij}}{r_{\text{out}}}$.



(a) Calculating the length $l_{O1}(\phi)$.

- Pattern center $\bar{x}$ is positioned at $(1,0)$ and a swarmalator is present at the origin.
- using polar coordinates (r,ϕ), hecnce $e_{ij} = (\cos\phi, \sin\phi)$ and $g_{ij} = r$.
- $l_{O1}(\phi) = 2\cos\phi = \frac{2\sin\phi\cos\phi}{\sin\phi} = \frac{\sin 2\phi}{\sin\phi}$

Due to symmetry, $\sum_{j\neq i}^{N} e_{ij,y} g_{ij}^{\alpha} = 0$, $\sum_{j\neq i}^{N} e_{ij,y} g_{ij}^{\beta} = 0$, which yields

$$r_{\text{out}}^{\alpha} \sum_{j\neq i}^{N} e_{ij,x} g_{ij}^{\alpha} \left(1 + J\cos\theta_{ij}\right) = r_{\text{out}}^{\beta} \sum_{j\neq i}^{N} e_{ij,x} g_{ij}^{\alpha}$$

Next, the authors consider the limiting case $N \to \infty$ and rewrite the sums as integrals ($\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\phi$)

$$r_{\text{out}}^{\alpha-\beta} = \frac{\text{Repulsion}}{\text{Attraction}} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi \int_0^{l_{O1}(\phi)} r^{1+\beta} \mathrm{d}r\mathrm{d}\phi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi \int_0^{l_{O1}(\phi)} \left(1 + J\cos\theta_{ij}\right) r^{1+\alpha} \mathrm{d}r\mathrm{d}\phi}$$

The solution in the first integration:

$$\int_0^{l_{O1}(\phi)} r^{1+m} dr = \frac{1}{2+m} l_{O1}^{2+m}(\phi) = \frac{\left(2\cos\phi\right)^{2+m}}{2+m}$$

Supposed θ_{ij} is a constant, we have

$$\begin{aligned} r_{\text{out}}^{\alpha-\beta} &= \frac{\frac{2^{2+\beta}}{2+\beta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{3+\beta}\phi \mathrm{d}\phi}{\frac{(1+J\cos\theta_{ij})2^{2+\alpha}}{2+\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{3+\alpha}\phi \mathrm{d}\phi} \\ &= \frac{2^{\beta-\alpha}\left(2+\alpha\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{3+\beta}\phi \mathrm{d}\phi}{\left(1+J\cos\theta_{ij}\right)\left(2+\beta\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{3+\alpha}\phi \mathrm{d}\phi} \end{aligned}$$

- **Static Sync.** In Static Sync State, $J\cos\theta_{ij} = J\ \forall\,(i,j)$, For $\alpha = 0$ and $\beta = -1$, we have

$$r_{\text{out}} = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi \frac{\sin(2\phi)}{\sin\phi} \mathrm{d}\phi}{(1+J)\,\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi \frac{\sin^2(2\phi)}{\sin^2\phi} \mathrm{d}\phi} = \frac{3\pi}{8\,(1+J)}$$

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