

Two Coupled Oscillators with Chirality

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1 The Model

1.1 Raw model

$$\dot{x}_{1,2} = v \cos \theta_{1,2} , \quad (1)$$

$$\dot{y}_{1,2} = v \sin \theta_{1,2} , \quad (2)$$

$$\dot{\theta}_{1,2} = \omega_{1,2} + \lambda f(r) \sin(\theta_{2,1} - \theta_{1,2}) , \quad (3)$$

where $f(r)$ is a function of $r = |\mathbf{r}_1 - \mathbf{r}_2|$, and λ is the coupling strength. The function $f(r)$ can be defined as

1. $f(r)_H = H(r_0 - r)$, $r_0 > 0$;
2. $f(r)_P = \left(1 + \frac{r}{r_0}\right)^{-\frac{1}{r_0}}$, $r_0 > 0$;
3. ...

1.2 Model under polar coordinates

Let

$$x_i = r_i \cos \varphi_i , \quad (4)$$

$$y_i = r_i \sin \varphi_i , \quad (5)$$

then we have

$$\dot{r}_i = \frac{1}{r_i} (x_i \dot{x}_i + y_i \dot{y}_i) = v \cos \varphi_i \cos \theta_i + v \sin \varphi_i \sin \theta_i = v \cos(\varphi_i - \theta_i) , \quad (6)$$

$$\dot{\varphi}_i = \frac{1}{r_i^2} (x_i \dot{y}_i - y_i \dot{x}_i) = \frac{v}{r_i} (\sin \varphi_i \cos \theta_i - \cos \varphi_i \sin \theta_i) = \frac{v}{r_i} \sin(\varphi_i - \theta_i) . \quad (7)$$

Introduce $\alpha_i = \varphi_i - \theta_i$, $\Delta\theta = \theta_2 - \theta_1$, $\Delta\varphi = \varphi_1 - \varphi_2$, $\Delta\omega = \omega_2 - \omega_1$, then the model becomes

$$\dot{r}_{1,2} = v \cos \alpha_{1,2} , \quad (8)$$

$$\dot{\alpha}_{1,2} = \frac{v}{r_{1,2}} \sin \alpha_{1,2} - \omega_{1,2} \mp \lambda f(r) \sin \Delta \theta , \quad (9)$$

$$\Delta \dot{\varphi} = \frac{v}{r_1} \sin \alpha_1 - \frac{v}{r_2} \sin \alpha_2 , \quad (10)$$

$$\Delta \dot{\theta} = \Delta \omega - 2\lambda f(r) \sin \Delta \theta , \quad (11)$$

where

$$\begin{aligned} r &= \sqrt{(r_1 \cos \varphi_1 - r_2 \cos \varphi_2)^2 + (r_1 \sin \varphi_1 - r_2 \sin \varphi_2)^2} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos \Delta \varphi} . \end{aligned} \quad (12)$$

So the function $f(r)$ can be defined as $f(r_1, r_2, \Delta \varphi)$.

1.3 Single direction driving

Assuming that $\dot{\theta}_2 = \omega_2, \alpha_2 = -\frac{\pi \text{sgn} \omega_2}{2}$, which means that the second oscillator is rotating around the origin with a constant angular velocity ω_2 , and the first oscillator is driven by the second one. Then the model becomes

$$\dot{r}_1 = v \cos (\Delta \varphi + \Delta \theta + \alpha_2) , \quad (13)$$

$$\Delta \dot{\varphi} = \omega_2 - \frac{v}{r_1} \sin (\Delta \varphi + \Delta \theta + \alpha_2) , \quad (14)$$

$$\Delta \dot{\theta} = \Delta \omega - \lambda f(r_1, \Delta \varphi) \sin \Delta \theta . \quad (15)$$

When $\lambda f(r) \geq |\Delta \omega|$, the system has fixed points \mathbf{x} , which are

$$r_1 = \frac{v}{|\omega_2|} , \quad (16)$$

$$\Delta \varphi = -\Delta \theta = C_\Delta , \quad (17)$$

where C_Δ is a constant determined by the initial conditions. Linearizing the governing equations yields

$$M = \begin{bmatrix} 0 & -v \sin (\Delta \varphi + \Delta \theta + \alpha_2) & -v \sin (\Delta \varphi + \Delta \theta + \alpha_2) \\ \frac{v \sin (\Delta \varphi + \Delta \theta + \alpha_2)}{r_1^2} & -\frac{v \cos (\Delta \varphi + \Delta \theta + \alpha_2)}{r_1} & -\frac{v \cos (\Delta \varphi + \Delta \theta + \alpha_2)}{r_1} \\ -\lambda f_{r_1} \sin \Delta \theta & -\lambda f_{\Delta \varphi} \sin \Delta \theta & -\lambda f \cos \Delta \theta \end{bmatrix} \quad (18)$$

where $f_{r_1} = \frac{\partial f}{\partial r_1}$ and $f_{\Delta \varphi} = \frac{\partial f}{\partial \Delta \varphi}$. Evaluating M at the fixed points \mathbf{x} results in

$$M = \begin{bmatrix} 0 & v \text{sgn} \omega_2 & v \text{sgn} \omega_2 \\ -\frac{\omega_2^2 \text{sgn} \omega_2}{v} & 0 & 0 \\ -\lambda f_{r_1}(\mathbf{x}) \sin C_\Delta & -\lambda f_{\Delta \varphi}(\mathbf{x}) \sin C_\Delta & -\lambda f(\mathbf{x}) \cos C_\Delta \end{bmatrix} \quad (19)$$

The eigenvalues of M are

1.4 For $f(r) = f(r)_P$

When $f(r) = f(r)_P$, the partial derivatives of $f(r)$ are

$$\frac{\partial f}{\partial r_1} = r_1 g(r_1, \Delta\varphi) , \quad (20)$$

$$\frac{\partial f}{\partial \Delta\varphi} = \frac{v^2}{\omega_2^2} g(r_1, \Delta\varphi) \sin \Delta\varphi , \quad (21)$$

where

$$g(r_1, \Delta\varphi) = -\frac{f^{1+r_0}(r_1, \Delta\varphi)}{r_0^2 \sqrt{r_1^2 - \frac{2v^2 \cos \Delta\varphi}{\omega_2^2} + \frac{v^2}{\omega_2^2}}} , \quad (22)$$

$$f(r_1, \Delta\varphi) = \left(1 + \frac{\sqrt{r_1^2 - 2v^2 \cos \Delta\varphi / \omega_2^2 + v^2 / \omega_2^2}}{r_0} \right)^{-\frac{1}{r_0}} . \quad (23)$$

At the fixed points \mathbf{x} , the matrix M becomes

$$M = \begin{bmatrix} 0 & v \operatorname{sgn} \omega_2 & v \operatorname{sgn} \omega_2 \\ -\frac{\omega_2^2 \operatorname{sgn} \omega_2}{v} & 0 & 0 \\ -\frac{\lambda v}{|\omega_2|} g(\mathbf{x}) \sin C_\Delta & \frac{\lambda v^2}{\omega_2^2} g(\mathbf{x}) \sin^2 C_\Delta & -\lambda f(\mathbf{x}) \cos C_\Delta \end{bmatrix} \quad (24)$$

where

$$g(\mathbf{x}) = -\frac{|\omega_2| f^{1+r_0}(\mathbf{x})}{v r_0^2 \sqrt{2 - 2 \cos C_\Delta}} , \quad (25)$$

$$f(\mathbf{x}) = \left(1 + \frac{v \sqrt{2 - 2 \cos C_\Delta}}{|\omega_2| r_0} \right)^{-\frac{1}{r_0}} . \quad (26)$$