Two Coupled Swarmalators with Chirality

July 3, 2024

1 Reference

1.1 Farrell F D C, Marchetti M C, Marenduzzo D, et al. Pattern formation in self-propelled particles with density-dependent motility[J]. Physical review letters, 2012, 108(24): 248101.

Microscopic dynamics:

$$\begin{aligned}
&\dot{\mathbf{r}}_i = v\mathbf{e}_{\theta_i} \ , \\
&\dot{\theta}_i = \gamma \sum_{j=1}^{N} F\left(\theta_j - \theta_i, \mathbf{r}_j - \mathbf{r}_i\right) + \sqrt{2\epsilon}\tilde{\eta}_i\left(t\right) \ .
\end{aligned} \tag{1}$$

The microscopic density of particles at position $\mathbf{r}$ with angle θ is given by

$$f\left(\mathbf{r}, \theta\right) = \sum_{i=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_i\right) \delta\left(\theta - \theta_i\right) . \tag{2}$$

Using Itô's formula, a stochastic dynamical equation for the density Eq. (2) can be derived:

$$\begin{aligned}
&\partial_t f\left(\mathbf{r}, \theta\right) + \mathbf{e}_\theta \cdot \nabla\left[vf\right] \\
&= \epsilon\tfrac{\partial^2 f}{\partial\theta^2} - \tfrac{\partial}{\partial\theta}\sqrt{2\epsilon f}\eta - \gamma\tfrac{\partial}{\partial\theta}\int \mathrm{d}\theta'\mathrm{d}\mathbf{r}' f\left(\mathbf{r}', \theta'\right) \times f\left(\mathbf{r}, \theta\right) F\left(\theta' - \theta, \mathbf{r} - \mathbf{r}'\right) \ .
\end{aligned} \tag{3}$$

Drop the noise term, and Fourier transform Eq. (3) to get equations of motion for

$$f_k \equiv \int f\left(\mathbf{r}, \theta\right) e^{ik\theta} \mathrm{d}\theta \ . \tag{4}$$

2 Our Work

We replace the finite range alignment interaction by a pseudopotential (δ-interaction) in the model:

$$\begin{aligned}
&\dot{\mathbf{r}}_i = v\mathbf{p}_i \\
&\dot{\theta}_i = \omega_i + \lambda \sum_{j\neq i} \delta\left(\mathbf{r}_j - \mathbf{r}_i\right) \sin\left(\theta_j - \theta_i\right)
\end{aligned} \tag{5}$$

where $\mathbf{p}_i = (\cos\theta_i, \sin\theta_i)$. The combined probability density of finding a particle at position $\mathbf{r}_i$ with angle θ_i is given by

$$f_i\left(\mathbf{r}, \theta\right) = \delta\left(\mathbf{r} - \mathbf{r}_i\right) \delta\left(\theta - \theta_i\right) \ . \tag{6}$$

The basic idea is to replace $f_i\omega_i$ by its mean plus a typical fluctuation

$$\omega_i f_i \longrightarrow \bar{\omega} f + \sqrt{\Delta_\omega f}\eta \ , \tag{7}$$

where η describes Gaussian random numbers with zero mean and unit variance

$$\langle \eta(\mathbf{r}, \theta, t)\eta\left(\mathbf{r}'\theta', t\right)\rangle = \delta\left(\mathbf{r} - \mathbf{r}'\right)\delta\left(\theta - \theta'\right) \ , \tag{8}$$

and $\bar{\omega}, \Delta\omega$ are defined by

$$\langle \omega_i \omega_j \rangle = \bar{\omega}^2 + \Delta_\omega \delta_{ij} \tag{9}$$

Physically, we should understand Eq.(7) as a local replacement and interpret the above averages as **mesoscopic** ones over all particles within the interaction range around a given point x rather than a global average over the whole rotor ensemble. Accordingly, we allow $\bar{\omega}$ and η to fluctuate in space and time. We now define a field $\omega(\mathbf{x}, t)$ representing the deviation from the global (time-independent) average rotation frequency $\bar{\omega}_0 = \langle \omega_i \rangle$.

$$\bar{\omega} = \bar{\omega}_0 + \omega\left(\mathbf{x}, t\right) \tag{10}$$

We now expand $\sqrt{f}$ for modest deviations from isotropy as follows

$$\sqrt{f} = \sqrt{\sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{-\mathrm{i}k\theta}} \approx \frac{\sqrt{f_0}}{2} + \frac{1}{2\sqrt{f_0}} \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{-\mathrm{i}k\theta} \ . \tag{11}$$

After a long calculation, we find the following equations

$$\begin{aligned}
\dot{\rho} &= -v\nabla\cdot\mathbf{w} \\
\dot{\mathbf{w}} &= (\lambda\rho - 2)\frac{\mathbf{w}}{2} - \frac{v}{2}\nabla\rho + \frac{v^2}{2b}\nabla^2\mathbf{w} - \frac{\lambda^2}{b}|\mathbf{w}|^2\mathbf{w} \\
&\quad + \frac{\lambda v}{4b}\left[5\nabla\mathbf{w}^2 - 10\mathbf{w}(\nabla\cdot\mathbf{w}) - 6(\mathbf{w}\cdot\nabla)\mathbf{w}\right] \\
&\quad + \omega\mathbf{w}_\perp + \frac{v^2\omega}{4b}\nabla^2\mathbf{w}_\perp - \frac{\lambda^2\omega}{2b}|\mathbf{w}|^2\mathbf{w}_\perp \\
&\quad - \frac{\lambda v\omega}{8b}\left[3\nabla_\perp\mathbf{w}^2 - 6\mathbf{w}\left(\nabla_\perp\cdot\mathbf{w}\right) - 10\left(\mathbf{w}\cdot\nabla_\perp\right)\mathbf{w}\right]
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\omega &= \langle \omega_i \rangle + \omega\left(\mathbf{x}, t\right) + \sqrt{\frac{\Delta_\omega}{f}}\eta \\
b &= 2\left(4 + \omega^2\right) \\
\mathbf{w}_\perp &= \left(-w_y, w_x\right) \\
\nabla_\perp &= \left(-\partial_y, \partial_x\right)
\end{aligned} \tag{13}$$