Enhancement of Phase Separation in Swarmalators with Chirality-induced Frustration

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1 The Model

Swarmalators have a spatial position $\mathbf{r}_i = (x_i, y_i)$ and an internal phase θ_i which evolve according to equations:

$$\dot{\mathbf{r}}_i = v\mathbf{p}(\theta_i) \ , \tag{1a}$$

$$\dot{\theta}_i = \omega_i + K \sum_{j \in A_i} \left[\sin(\theta_j - \theta_i + \alpha_{ij}) - \sin\alpha_{ij} \right] , \tag{1b}$$

Mean-field definition:

$$\dot{\mathbf{r}}_i = v\mathbf{p}(\theta_i) \ , \tag{2a}$$

$$\dot{\theta}_i = \omega_i + \frac{K}{|A_i|} \sum_{j \in A_i} \left[\sin(\theta_j - \theta_i + \alpha_{ij}) - \sin\alpha_{ij} \right] , \tag{2b}$$

for $i = 1, 2, \ldots, N$. Here in Eq. (2a), $\mathbf{p}(\theta) = (\cos\theta, \sin\theta)$, which means each swarmalator rotates with a constant speed v in the direction of its instantaneous phase $\theta_i(t)$. As per Eq. (2b), the sum runs over neighbors within a coupling radius d_0 around swarmalator i:

$$A_i(t) = \{ j \mid |\mathbf{r}_i(t) - \mathbf{r}_j(t)| \leqslant d_0 \} \ , \tag{3}$$

K is the coupling strength, and ω_i is the natural frequency of the i-th swarmalator. This means that a swarmalator will rotate with the angular velocity $|\omega_i|$ in the absence of mutual coupling ($K = 0$), and the sign of ω_i represents the direction of rotation, namely, the tribute of the chirality of the i-th swarmalator. A positive (negative) chirality (ω) describes the counterclockwise (clockwise) rotations of the swarmalator in space. Here, we consider parallels with two types of chiralities with both positive and negative natural frequencies uniformly distributed in two symmetric regimes, namely, half of the swarmalators possess positive natural frequencies $\omega_i \sim U(\omega_{\min}, \omega_{\max})$ and the other half have negative natural frequencies $\omega_i \sim U(-\omega_{\max}, -\omega_{\min})$, where $\omega_{\min,\max} > 0$.

Additionally, α_{ij} is the phase frustration between two neighboring swarmalators, which is defined as:

$$\alpha_{ij} = \begin{cases} \alpha_0, & \omega_i\omega_j < 0 \\ 0, & \omega_i\omega_j \geqslant 0 \end{cases} \tag{4}$$

When $\alpha_0 = 0$, the dynamics reduces to the normal chiral model.

For simplicity, we assume that swarmalators are initially distributed uniformly in a two-dimensional $L \times L$ square with periodic boundary conditions. When two swarmalators are on opposite sides of the square, the absolute value of the difference between at least one of their coordinates is larger than $L/2$. In this case, we take the minimum distance between them, which is the relative distance between the two points in the periodic boundary conditions.

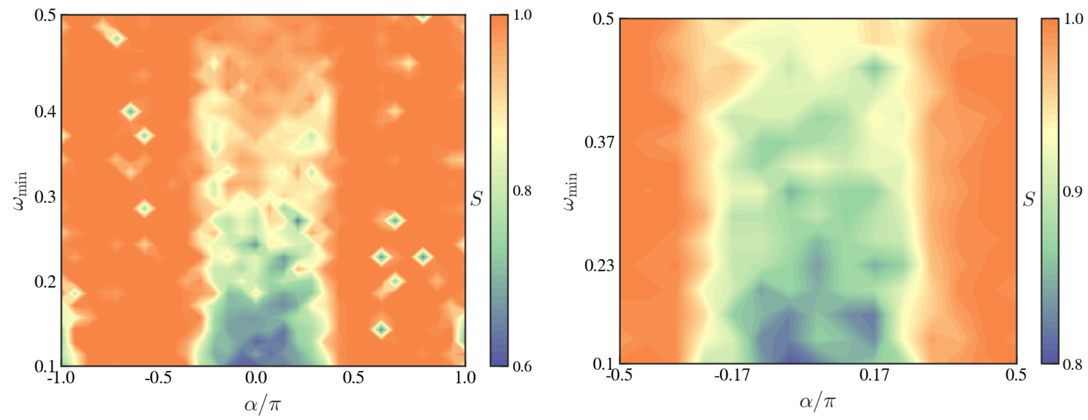
Some order parameters can be introduced to measure the level of spatiotemporal coordination among swarmalators and distinguish the different collective states of the system. Firstly, the usual order parameter to measure the chiral phase separation among swarmalators can be defined as:

$$S(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{\sum_{j \in A_i} H(\omega_i\omega_j)}{|A_i(t)|} \ , \tag{5}$$

where $H(x) = (x > 0)$ is the Heaviside step function. $S(t)$ is the fraction of the pairs of neighboring swarmalators with the same chirality. When $S(t) = 1$, all the neighboring pairs of swarmalators have the same chirality, and the system is in a completely phase-separated state. When $S(t) < 1$, the system is in a mixed state.

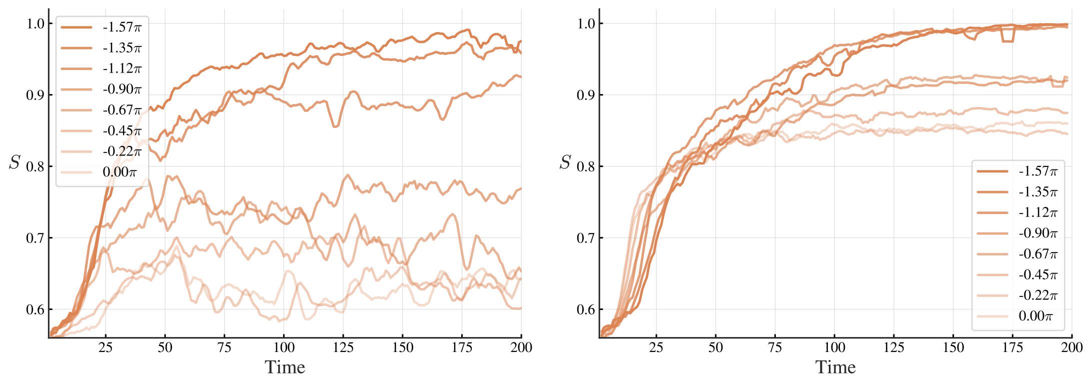
We conducted numerical simulations to investigate the performance and characteristics of our system under various conditions. All the numerical simulations of the model Eq. (1) were run on Python using Euler integration with box size of $L = 10$, population sizes of $N = 500, 1000$ for single/double-chiral swarmalators, respectively, maximum and minimum absolute values of natural frequencies $\omega_{\max} = 3$, $\omega_{\min} = 1$, coupling strength λ in $[0.01, 0.1] \cup [0.1, 1]$ with step length 0.1, 0.05, respectively, action radius of d_0 in $[0.1, 1] \cup [1, 2]$ with step length 0.05, 0.5, respectively, a time step $\Delta t = 0.01$, and total number of iterations of $T = 60000$. Unless otherwise stated, each data point of order parameters R, R_c and N_r was collected by averaging last 1000 time steps of the simulation to discard the transients.

2 Frustration-enhanced phase separation



(a) $\Delta\omega = 1$ (b) $\Delta\omega = 2$

Figure 1: Phase diagrams of the system with different natural frequency differences $\Delta\omega$. The color represents the order parameter S of the system.



(a) $\Delta\omega = 1$ (b) $\Delta\omega = 2$

Figure 2: The order parameter S v.s. time of the system with different natural frequency differences $\Delta\omega$ and $\omega_{\min} = 0.1$

3 Abstracted to a synchronization problem

The dynamics of the system (2) can be abstracted as a synchronization problem in multilayer networks. The system can be described by the following equations:

$$\dot{\theta}_i^+ = \omega_i^+ + \frac{K}{N}\sum_{j=1}^{N+} \sin\left(\theta_j^+ - \theta_i^+\right) + \frac{K}{N}\sum_{j=1}^{N-} \left[\sin\left(\theta_j^- - \theta_i^+ + \alpha_0\right) - \sin\alpha_0\right], \tag{6a}$$

$$\dot{\theta}_i^- = \omega_i^- + \frac{K}{N}\sum_{j=1}^{N-} \sin\left(\theta_j^- - \theta_i^-\right) + \frac{K}{N}\sum_{j=1}^{N+} \left[\sin\left(\theta_j^+ - \theta_i^- + \alpha_0\right) - \sin\alpha_0\right]. \tag{6b}$$

$$R_{\max} = \max_{t\in[T,T+h]} \{R(t)\} \tag{7a}$$

$$R_{\min} = \min_{t\in[T,T+h]} \{R(t)\} \tag{7b}$$

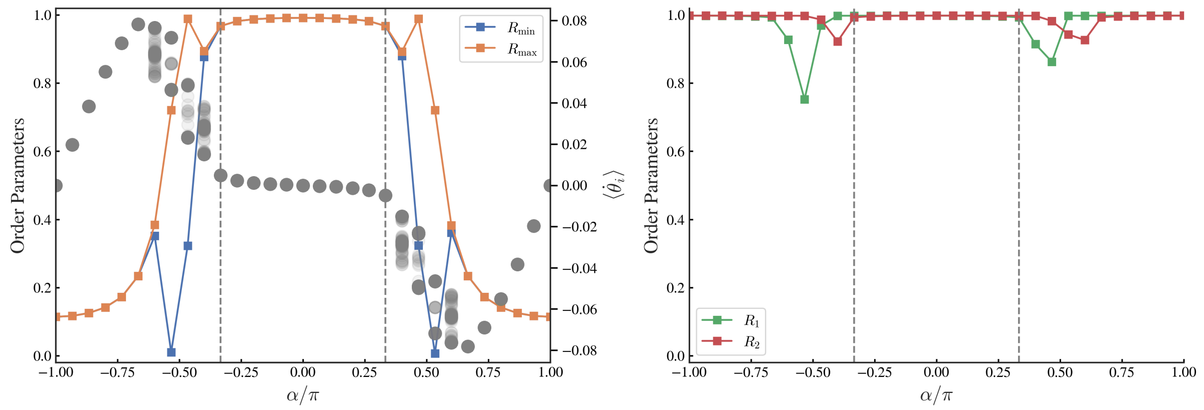


Figure 3: The order parameter R v.s. α_0 and effective frequency $\left\langle\dot{\theta}_i\right\rangle$ of the oscillator system with $\Delta\omega = 2$ and $\omega_{\min} = 0.1$

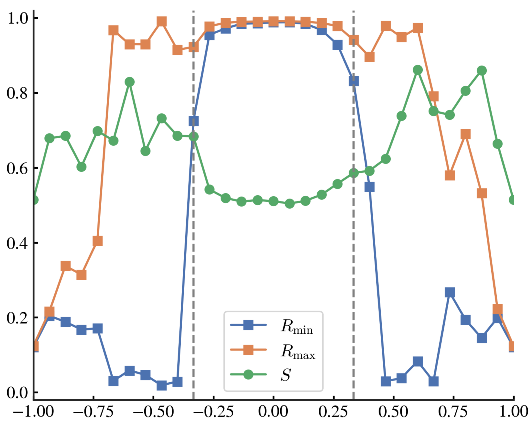


Figure 4: The order parameter R v.s. α_0 and effective frequency $\left\langle\dot{\theta}_i\right\rangle$ of the swarmalator system with $\Delta\omega = 2$ and $\omega_{\min} = 0.1$

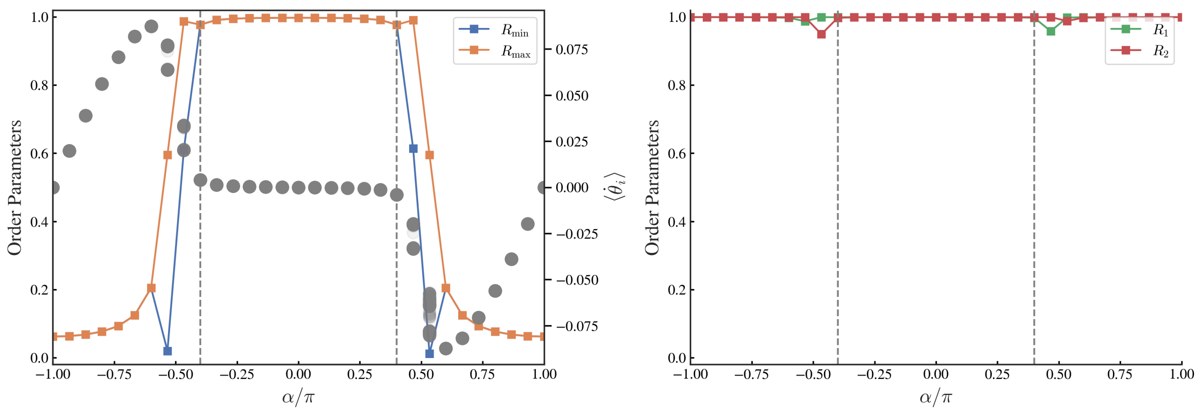


Figure 5: The order parameter R v.s. α_0 and effective frequency $\left\langle \dot{\theta}_i \right\rangle$ of the oscillator system with $\Delta\omega = 1$ and $\omega_{\min} = 0.1$

4 Coarse grained equations and phase diagrams

We begin with Eq. (1), replacing the finite coupling distance alignment interaction with a pseudopotential (the 'δ'-interaction). This substitution is justified when the interaction is sufficiently short-ranged, making the specific shape of the associated interaction potential irrelevant to the dynamics of many swarmalators. The pseudopotential is defined as:

$$\dot{\mathbf{r}}_i^{\pm} = v\mathbf{p}\left(\theta_i^c\right) \ , \tag{8a}$$

$$\dot{\theta}_i^{\pm} = \omega_i^{\pm} + K \sum_{j=1} \left\{ \delta\left(\mathbf{r}_j^{\pm} - \mathbf{r}_i^{\pm}\right) \sin\left(\theta_j^{\pm} - \theta_i^{\pm}\right) \right.$$
$$\left. + \delta\left(\mathbf{r}_j^{\mp} - \mathbf{r}_i^{\pm}\right) \left[\sin\left(\theta_j^{\mp} - \theta_i^{\pm} + \alpha_0\right) - \sin\alpha_0\right] \right\} \ , \tag{8b}$$

where $c \in \{+,-\}$ is the chirality of the swarmalator i and $b = +$ if $c = -$ and vice versa. Then following [?] we derive a continuum equation of motion for the combined N-swarmalator probability density

$$\rho^{\pm}\left(\mathbf{r}, \theta, t\right) = \sum_{i=1} \rho_i^{\pm}\left(\mathbf{r}, \theta, t\right) \ , \tag{9}$$

where $\rho_i^{\pm}\left(\mathbf{r}, \theta, t\right) = \delta\left(\mathbf{r}_i^{\pm}(t) - \mathbf{r}\right) \delta\left(\theta_i^{\pm}(t) - \theta\right)$ is the probability density of finding i-th swarmalator at position $\mathbf{r}$ with phase θ and chirality $+$ or $-$ at time t. Since the deterministic dynamical equation Eq. (1) conserves the number of oscillators with a given natural frequency over time, the distribution function evolves according to a continuity equation of the following form:

$$\frac{\partial \rho_i^{\pm}}{\partial t} = -\nabla \cdot \left(\rho_i^{\pm} v_{\mathbf{r}}\right) - \frac{\partial}{\partial \theta}\left(\rho_i^{\pm} v_{\theta}^{\pm,i}\right) \ . \tag{10}$$

Here, the velocity fields read

$$v_{\mathbf{r}}\left(\mathbf{r}, \theta, t\right) = v\mathbf{p}\left(\theta\right) \ , \tag{11a}$$

$$v_{\theta}^{\pm,i}\left(\mathbf{r}, \theta, t\right) = \omega_i^{\pm} + K \int_{-\pi}^{\pi} \mathrm{d}\phi \left\{ \rho^{\pm}\left(\mathbf{r}, \phi, t\right) \sin\left(\phi - \theta\right) \right.$$
$$\left. + \rho^{\mp}\left(\mathbf{r}, \phi, t\right) \left[\sin\left(\phi - \theta + \alpha_0\right) - \sin\alpha_0\right] \right\} \ . \tag{11b}$$

Summing Eq. (10) over the i and $\pm$ indices, and using the definition of the density $\rho^{\pm}$ in Eq. (9), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\rho^{\pm}\left(\mathbf{r}, \theta, t\right) &= -v\mathbf{p}\left(\theta\right) \cdot \nabla \rho^{\pm}\left(\mathbf{r}, \theta, t\right) - \frac{\partial}{\partial \theta}\Omega^{\pm} \\ &- K \frac{\partial}{\partial \theta}\rho^{\pm}\left(\mathbf{r}, \theta, t\right) \int_{-\pi}^{\pi} \mathrm{d}\phi \left\{ \rho^{\pm}\left(\mathbf{r}, \phi, t\right) \sin\left(\phi - \theta\right) \right. \\ &\left. + \rho^{\mp}\left(\mathbf{r}, \phi, t\right) \left[\sin\left(\phi - \theta + \alpha_0\right) - \sin\alpha_0\right] \right\} \ , \end{aligned} \tag{12}$$

where $\Omega^{\pm}(\mathbf{r},\theta,t) = \sum_{i=1} \rho_i^{\pm}(\mathbf{r},\theta,t)\,\omega_i^{\pm}$. Spatial homogeneity and phase synchronization of the ISS indicates $\forall i$, $\rho_i^{\pm}(\mathbf{r},\theta,t) \equiv \rho_{\text{ISS}}(\mathbf{r},\theta,t)$, which yields

$$\Omega^{\pm}(\mathbf{r},\theta,t) = \rho^{\pm}(\mathbf{r},\theta,t)\,\bar{\omega}^{\pm} \ , \tag{13}$$

where $\bar{\omega}^{\pm} = \left\langle \omega_i^{\pm} \right\rangle$.

Transforming $\rho^{\pm}$ in a Fourier series in θ, we have

$$\rho^{\pm}(\mathbf{r},\theta,t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} \varrho_k^{\pm}(\mathbf{r},t)\mathrm{e}^{-\mathrm{i}k\theta} \ , \tag{14}$$

where $\varrho_k^{\pm}$ is the k-th Fourier coefficient of $\rho^{\pm}$, defined as

$$\varrho_k^{\pm}(\mathbf{r},t) = \int \rho^{\pm}(\mathbf{r},\theta,t)\,\mathrm{e}^{\mathrm{i}k\theta}\mathrm{d}\theta \ . \tag{15}$$

To streamline the analysis, we express Eq. (12) in its complex form by multiplying both sides by $\mathrm{e}^{\mathrm{i}k\theta}$ and then integrating with respect to θ:

$$\begin{aligned}
\dot{\varrho}_k^{\pm} &= -\frac{v}{2}\left[\partial_x\left(\varrho_{k+1}^{\pm} + \varrho_{k-1}^{\pm}\right) - \mathrm{i}\partial_y\left(\varrho_{k+1}^{\pm} - \varrho_{k-1}^{\pm}\right)\right] \\
&+ \mathrm{i}k\bar{\omega}^{\pm}\varrho_k^{\pm} \\
&+ \frac{\mathrm{i}kK}{2\pi}\sum_{m=-\infty}^{\infty} \varrho_{k-m}^{\pm}F_{-m}\varrho_m^{\pm} \\
&+ \frac{\mathrm{i}kK}{2\pi}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\alpha_0}\varrho_{k-m}^{\pm}F_{-m}\varrho_m^{\mp} \\
&- \mathrm{i}kK\varrho_0^{\mp}\varrho_k^{\pm}\sin\alpha_0 \ ,
\end{aligned} \tag{16}$$

where F_m is the m-th Fourier coefficient of $\sin\theta$. Evaluating Eq. (16) for $k = 0, 1, \ldots$ leads to a hierarchy of equations for $\left\{\varrho_k^{\pm}\right\}$ with

$$\rho^{\pm}(\mathbf{r},t) = \varrho_0^{\pm}(\mathbf{r},t) = \int_{-\pi}^{\pi} \rho^{\pm}(\mathbf{r},\theta,t)\,\mathrm{d}\theta \tag{17}$$

being the probability density of finding a swarmalator at position $\mathbf{r}$ at time t and

$$\mathbf{w}^{\pm}(\mathbf{r},t) = \left(\mathrm{Re}\varrho_1^{\pm}, \mathrm{Im}\varrho_1^{\pm}\right) = \int_{-\pi}^{\pi} \mathbf{p}(\theta)\,\rho^{\pm}(\mathbf{r},\theta,t)\,\mathrm{d}\theta \tag{18}$$

being the momentum field at position $\mathbf{r}$ at time t. The degree modulus $R^{\pm}(\mathbf{r},t) = |\mathbf{w}^{\pm}(\mathbf{r},t)|$ is the absolute value of the mean normalized velocity of local swarmalators, which can be used as a measure of the local degree of synchronization.

To close the hierarchy of equations Eq. (16), we truncate the series at a finite order K and neglect the higher-order terms. This truncation is justified when the interaction is sufficiently short-ranged, making the specific shape of the associated interaction potential irrelevant to the dynamics of many swarmalators. Specifically, we assume that $\varrho_2^{\pm}(\mathbf{r},t)$ follows changes in $\varrho_0^{\pm}(\mathbf{r},t)$ and $\varrho_1^{\pm}(\mathbf{r},t)$ adiabatically ($\dot{\varrho}_2^{\pm}(\mathbf{r},t) \approx 0$) and neglect the higher-order terms ($\varrho_{k\geqslant 3}^{\pm}(\mathbf{r},t) \approx 0$). Truncating at order 3, we are left with

$$\dot{\varrho}_0^{\pm} = -\mathrm{Re}\bar{\nabla}\varrho_1^{\pm} \tag{19a}$$

$$\begin{aligned}
\dot{\varrho}_1^{\pm} &= -\frac{v}{2}\left(\nabla\varrho_0^{\pm} + \bar{\nabla}\varrho_2^{\pm}\right) + \mathrm{i}\varrho_1^{\pm}\left(\bar{\omega}^{\pm} - K\varrho_0^{\mp}\sin\alpha_0\right) \\
&+ \frac{K}{2}\varrho_0^{\pm}\left(\varrho_1^{\pm} + \mathrm{e}^{\mathrm{i}\alpha_0}\varrho_1^{\mp}\right) \ ,
\end{aligned} \tag{19b}$$

$$\begin{aligned}
\dot{\varrho}_2^{\pm} &= -\frac{v}{2}\nabla\varrho_1^{\pm} + \mathrm{i}2\varrho_k^{\pm}\left(\bar{\omega}^{\pm} - K\varrho_0^{\mp}\sin\alpha_0\right) \\
&+ K\varrho_1^{\pm}\left(\varrho_1^{\pm} + \mathrm{e}^{\mathrm{i}\alpha_0}\varrho_1^{\mp}\right) \ ,
\end{aligned} \tag{19c}$$

where $\nabla = \partial_x + \mathrm{i}\partial_y$ denotes the gradient operator in the complex plane, and $\bar{\nabla} = \partial_x - \mathrm{i}\partial_y$ is the complex conjugate of ∇. Let $\dot{\varrho}_2^{\pm} = 0$, then $\varrho_2^{\pm}$ can be solved as

$$\varrho_2^{\pm} = -\frac{\mathrm{i}K\varrho_1^{\pm}}{a^{\pm}}\left(\varrho_1^{\pm} + \mathrm{e}^{-\mathrm{i}\alpha_0}\varrho_1^{\mp}\right) + \frac{\mathrm{i}\nabla\varrho_1^{\pm}v}{2a^{\pm}} \tag{20}$$

where $a^{\pm} = 2\left(K\rho^{\mp}\sin\alpha_0 - \bar{\omega}^{\pm}\right)$. Substituting $\varrho_2^{\pm}$ into Eq. (19b), we have

$$\begin{aligned}\dot{\varrho}_1^{\pm} = &-\frac{v}{2}\nabla\varrho_0^{\pm} + \frac{\mathrm{i}vK\bar{\nabla}\varrho_1^{\pm}}{2a^{\pm}}\left(\varrho_1^{\pm} + \mathrm{e}^{-\mathrm{i}\alpha_0}\varrho_1^{\mp}\right) \\ &+ \frac{\mathrm{i}vK\varrho_1^{\pm}}{2a^{\pm}}\left(\bar{\nabla}\varrho_1^{\pm} + \mathrm{e}^{-\mathrm{i}\alpha_0}\bar{\nabla}\varrho_1^{\mp}\right) - \frac{\mathrm{i}v^2\Delta\varrho_1^{\pm}}{4a^{\pm}} \\ &+ \mathrm{i}\varrho_1^{\pm}\left(\bar{\omega}^{\pm} - K\varrho_0^{\mp}\sin\alpha_0\right) + \\ &\frac{K}{2}\varrho_0^{\pm}\left(\varrho_1^{\pm} + \mathrm{e}^{\mathrm{i}\alpha_0}\varrho_1^{\mp}\right)\end{aligned} \tag{21}$$