Two Coupled Oscillators with Chirality

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1 The Model

1.1 Raw model

$$\dot{x}_{1,2} = v\cos\theta_{1,2}\ , \tag{1}$$

$$\dot{y}_{1,2} = v\sin\theta_{1,2}\ , \tag{2}$$

$$\dot{\theta}_{1,2} = \omega_{1,2} + \lambda f\left(r\right)\sin\left(\theta_{2,1} - \theta_{1,2}\right)\ , \tag{3}$$

where $f\left(r\right)$ is a function of $r = |\mathbf{r}_1 - \mathbf{r}_2|$, and λ is the coupling strength. The function $f\left(r\right)$ can be defined as

1. $f\left(r\right)_H = H\left(r_0 - r\right),\ r_0 > 0$;
2. $f\left(r\right)_P = \left(1 + \frac{r}{r_0}\right)^{-\frac{1}{r_0}},\ r_0 > 0$;
3. $\ldots$

1.2 Model under polar coordinates

Let

$$x_i = r_i\cos\varphi_i\ , \tag{4}$$

$$y_i = r_i\sin\varphi_i\ , \tag{5}$$

then we have

$$\dot{r}_i = \frac{1}{r_i}\left(x_i\dot{x}_i + y_i\dot{y}_i\right) = v\cos\varphi_i\cos\theta_i + v\sin\varphi_i\sin\theta_i = v\cos\left(\varphi_i - \theta_i\right)\ , \tag{6}$$

$$\dot{\varphi}_i = \frac{1}{r_i^2}\left(x\dot{y} - y\dot{x}\right) = \frac{v}{r_i}\left(\sin\varphi_i\cos\theta_i - \cos\varphi_i\sin\theta_i\right) = \frac{v}{r_i}\sin\left(\varphi_i - \theta_i\right)\ . \tag{7}$$

Introduce $\alpha_i = \varphi_i - \theta_i$, $\Delta\theta = \theta_2 - \theta_1$, $\Delta\varphi = \varphi_1 - \varphi_2$, $\Delta\omega = \omega_2 - \omega_1$, then the model becomes

$$
\begin{aligned}
\dot{r}_{1,2} &= v \cos \alpha_{1,2} \ , && (8)\\
\dot{\alpha}_{1,2} &= \frac{v}{r_{1,2}} \sin \alpha_{1,2} - \omega_{1,2} \mp \lambda f\left(r\right) \sin \Delta\theta \ , && (9)\\
\Delta\dot{\varphi} &= \frac{v}{r_1} \sin \alpha_1 - \frac{v}{r_2} \sin \alpha_2 \ , && (10)\\
\Delta\dot{\theta} &= \Delta\omega - 2\lambda f\left(r\right) \sin \Delta\theta \ , && (11)
\end{aligned}
$$

where

$$
\begin{aligned}
r &= \sqrt{\left(r_1 \cos \varphi_1 - r_2 \cos \varphi_2\right)^2 + \left(r_1 \sin \varphi_1 - r_2 \sin \varphi_2\right)^2} \\
&= \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos \Delta\varphi}
\end{aligned} \ . \tag{12}
$$

So the function $f\left(r\right)$ can be defined as $f\left(r_1, r_2, \Delta\varphi\right)$.

1.3 Single direction driving

Assuming that $\dot{\theta}_2 = \omega_2, \alpha_2 = \dfrac{\pi \mathrm{sgn}\omega_2}{2}$, which means that the second oscillator is rotating around the origin with a constant angular velocity ω_2, and the first oscillator is driven by the second one. Then the model becomes

$$
\begin{aligned}
\dot{r}_1 &= v \cos\left(\Delta\varphi + \Delta\theta + \alpha_2\right) \ , && (13)\\
\Delta\dot{\varphi} &= \omega_2 - \frac{v}{r_1} \sin\left(\Delta\varphi + \Delta\theta + \alpha_2\right) \ , && (14)\\
\Delta\dot{\theta} &= \Delta\omega - \lambda f\left(r_1, \Delta\varphi\right) \sin \Delta\theta \ . && (15)
\end{aligned}
$$

When $2\lambda f\left(r\right) \geqslant \left|\Delta\omega\right|$, the system has fixed points $\mathbf{x}$, which are

$$
\begin{aligned}
r_1 &= \frac{v}{\omega_2} \ , && (16)\\
\Delta\varphi &= C_{\Delta\varphi} \ , && (17)\\
\Delta\theta &= C_{\Delta\theta} \ , && (18)
\end{aligned}
$$

where C_φ and C_θ are constants determined by the initial conditions. Linearizing the governing equations yields

$$
M = \begin{bmatrix}
0 & -v \sin \alpha_1 & 0 & 0 \\
-\frac{v}{r_1^2} \sin \alpha_1 & \frac{v}{r_1} \cos \alpha_1 & 0 & 0 \\
\frac{v}{r_1^2} \sin \alpha_1 & -\frac{v}{r_1} \cos \alpha_1 & 0 & 0 \\
-\lambda f_{r_1} \sin \Delta\theta & 0 & -\lambda f_{\Delta\varphi} \sin \Delta\theta & -\lambda f \cos \Delta\theta
\end{bmatrix} \tag{19}
$$

where $f_{r_1} = \dfrac{\partial f}{\partial r_1}$ and $f_{\Delta\varphi} = \dfrac{\partial f}{\partial \Delta\varphi}$. Evaluating M at the fixed points results in

$$
M = \begin{bmatrix}
0 & -v \mathrm{sgn}\omega_2 & 0 & 0 \\
-\frac{\omega_2^2}{v} \mathrm{sgn}\omega_2 & 0 & 0 & 0 \\
\frac{\omega_2^2}{v} \mathrm{sgn}\omega_2 & 0 & 0 & 0 \\
-\lambda f_{r_1}\left(\mathbf{x}\right) \sin C_\theta & 0 & -\lambda f_{\Delta\varphi}\left(\mathbf{x}\right) \sin C_\theta & -\lambda f\left(\mathbf{x}\right) \cos C_\theta
\end{bmatrix} \tag{20}
$$

1.4 For $f(r) = f(r)_P$

When $f(r) = f(r)_P$, the partial derivatives of $f(r)$ are

$$\frac{\partial f}{\partial r_1} = r_1 g(r_1, \Delta\varphi) \ , \tag{21}$$

$$\frac{\partial f}{\partial \Delta\varphi} = \frac{v^2}{\omega_2^2} g(r_1, \Delta\varphi) \sin\Delta\varphi \ , \tag{22}$$

where

$$g(r_1, \Delta\varphi) = -\frac{f^{1+r_0}(r_1, \Delta\varphi)}{r_0^2\sqrt{r_1^2 - \frac{2v^2\cos\Delta\varphi}{\omega_2^2} + \frac{v^2}{\omega_2^2}}} \ , \tag{23}$$

$$f(r_1, \Delta\varphi) = \left(1 + \frac{\sqrt{r_1^2 - 2v^2\cos\Delta\varphi/\omega_2^2 + v^2/\omega_2^2}}{r_0}\right)^{-\frac{1}{r_0}} \ . \tag{24}$$

At the fixed points, the matrix M becomes

$$M = \begin{bmatrix} 0 & -v\mathrm{sgn}\omega_2 & 0 & 0 \\ -\frac{\omega_2^2}{v}\mathrm{sgn}\omega_2 & 0 & 0 & 0 \\ \frac{\omega_2^2}{v}\mathrm{sgn}\omega_2 & 0 & 0 & 0 \\ -\lambda\frac{v}{\omega_2}g(\mathbf{x})\sin C_\theta & 0 & -\lambda\frac{v^2}{\omega_2^2}g(\mathbf{x})\sin C_{\Delta\varphi}\sin C_\theta & -\lambda f(\mathbf{x})\cos C_\theta \end{bmatrix} \tag{25}$$

where

$$g(\mathbf{x}) = -\frac{|\omega_2| f^{1+r_0}\left(\frac{v}{\omega_2}, C_{\Delta\varphi}\right)}{v r_0^2\sqrt{2 - 2\cos C_{\Delta\varphi}}} \ , \tag{26}$$

$$f(\mathbf{x}) = \left(1 + \frac{v\sqrt{2 - 2\cos C_{\Delta\varphi}}}{|\omega_2| r_0}\right)^{-\frac{1}{r_0}} \ . \tag{27}$$

The eigenvalues of M are

$$\lambda_{1,2} = \pm\sqrt{\frac{g(\mathbf{x})\lambda v^2 \sin C_{\Delta\theta}}{|\omega_2|} - \frac{\omega_2^4}{|\omega_2|^2}} \ , \tag{28}$$

$$\lambda_3 = -f(\mathbf{x})\lambda\cos C_{\Delta\theta} \ , \tag{29}$$

$$\lambda_4 = 0 \ . \tag{30}$$