

# Enhancement of Demixing in Swarmalators with Chirality-induced Frustration

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\* All green parts are the parts that were tired but did not work.

## 1 The Model

Swarmalators have a spatial position  $\mathbf{r}_i = (x_i, y_i)$  and an internal phase  $\theta_i$  which evolve according to equations:

$$\dot{\mathbf{r}}_i = v\mathbf{p}(\theta_i) , \quad (1a)$$

$$\dot{\theta}_i = \omega_i + \frac{K}{|A_i|} \sum_{j \in A_i} [\sin(\theta_j - \theta_i + \alpha_{ij}) - \sin \alpha_{ij}] , \quad (1b)$$

for  $i = 1, 2, \dots, N$ . Here in Eq. (1a),  $\mathbf{p}(\theta) = (\cos \theta, \sin \theta)$ , which means each swarmalator rotates with a constant speed  $v$  in the direction of its instantaneous phase  $\theta_i(t)$ . As per Eq. (1b), the sum runs over neighbors within a coupling radius  $d_0$  around swarmalator  $i$ :

$$A_i(t) = \{j \mid |\mathbf{r}_i(t) - \mathbf{r}_j(t)| \leq d_0\} , \quad (2)$$

$K$  is the coupling strength, and  $\omega_i$  is the natural frequency of the  $i$ -th swarmalator. This means that a swarmalator will rotate with the angular velocity  $|\omega_i|$  in the absence of mutual coupling ( $K = 0$ ), and the sign of  $\omega_i$  represents the direction of rotation, namely, the tribute of the chirality of the  $i$ -th swarmalator. A positive (negative) chirality ( $\omega$ ) describes the counterclockwise (clockwise) rotations of the swarmalator in space. We will study swarmalators including two types of chiralities with both positive and negative natural frequencies uniformly distributed in two symmetric regimes as

$$g_U^D(\omega; \mu, \sigma) = \frac{1}{2} [g_U(\omega; -\mu, \sigma) + g_U(\omega; \mu, \sigma)] . \quad (3)$$

where  $g_U(\omega; \mu, \sigma)$  is the uniform distribution:

$$g_U(\omega; \mu, \sigma) = \begin{cases} \frac{1}{\sigma}, & \omega_{\min} \leq \omega \leq \omega_{\max} \\ 0, & \text{otherwise} \end{cases} . \quad (4)$$

Here  $\sigma = \omega_{\max} - \omega_{\min}$  is the natural-frequency span, and  $\mu = (\omega_{\max} + \omega_{\min})/2$  is the average.

Additionally,  $\alpha_{ij}$  is the phase frustration between two neighboring swarmalators, which is defined as:

$$\alpha_{ij} = \begin{cases} \alpha_0, & \omega_i \omega_j < 0 \\ 0, & \omega_i \omega_j \geq 0 \end{cases} \quad (5)$$

When  $\alpha_0 = 0$ , the dynamics reduces to the normal chiral model.

Some order parameters can be introduced to measure the level of spatiotemporal coordination among swarmalators and distinguish the different collective states of the system. Firstly, the usual order parameter to measure the chiral phase separation among swarmalators can be defined as:

$$S(t) = \frac{1}{N} \sum_{i=1}^N \frac{\sum_{j \in A_i} H(\omega_i \omega_j)}{|A_i(t)|}, \quad (6)$$

where  $H(x) = (x > 0)$  is the Heaviside step function.  $S(t)$  is the fraction of the pairs of neighboring swarmalators with the same chirality. When  $S(t) = 1$ , all the neighboring pairs of swarmalators have the same chirality, and the system is in a completely phase-separated state. When  $S(t) < 1$ , the system is in a mixed state.

We conducted numerical simulations to investigate the performance and characteristics of our system under various conditions. For simplicity, we assume that swarmalators are initially distributed uniformly in a two-dimensional  $L \times L$  square with periodic boundary conditions. All the numerical simulations of the model Eq. (1) were run on Python using Euler integration with box size of  $L = 10$ , population sizes of  $N = 1000$ , natural-frequency span  $\sigma = 1, 2$ , minimal absolute value of frequencies  $\omega_{\min}$  in  $[0.1, 5]$ , action radius of  $d_0 = 1$ , a time step  $\Delta t = 0.01$ , and total number of iterations of  $T = 30000$ . Unless otherwise stated, each data point of order parameters  $R$  and  $S$  was collected by averaging last 1000 time steps of the simulation to discard the transients.

## 2 Frustration-enhanced demixing

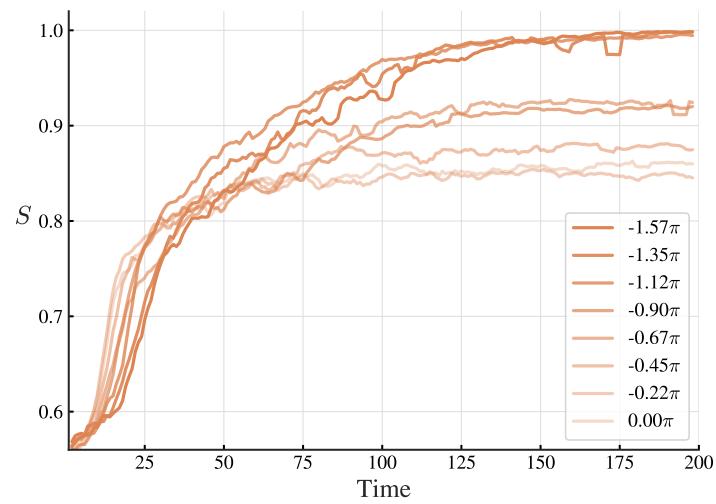


Figure 1: The order parameter  $S$  v.s. time of the system with different natural frequency differences  $\Delta\omega$  and  $\omega_{\min} = 0.1$

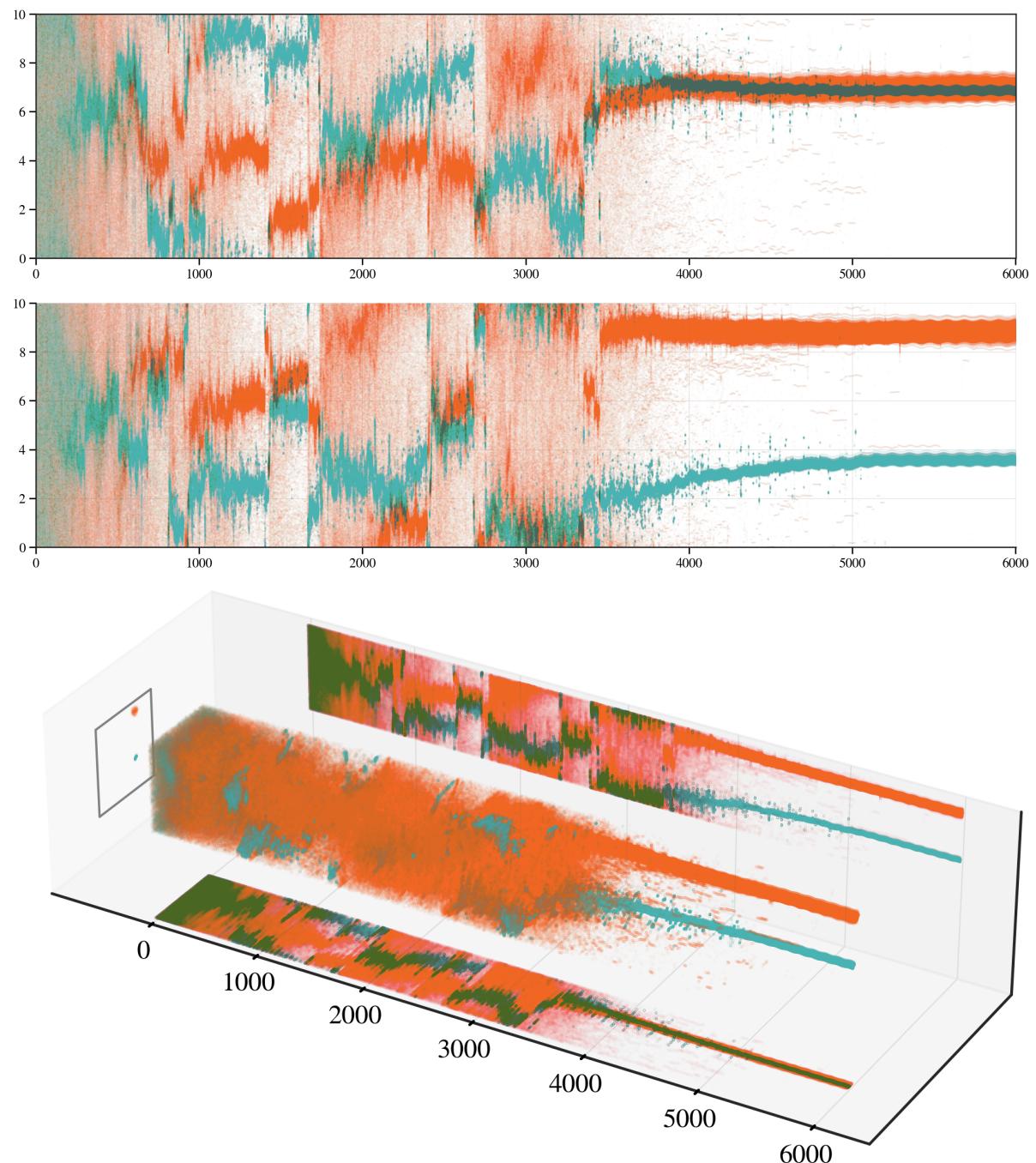


Figure 2: The spatial distribution of real rotation centers

### 3 Phase diagrams and critical frustration

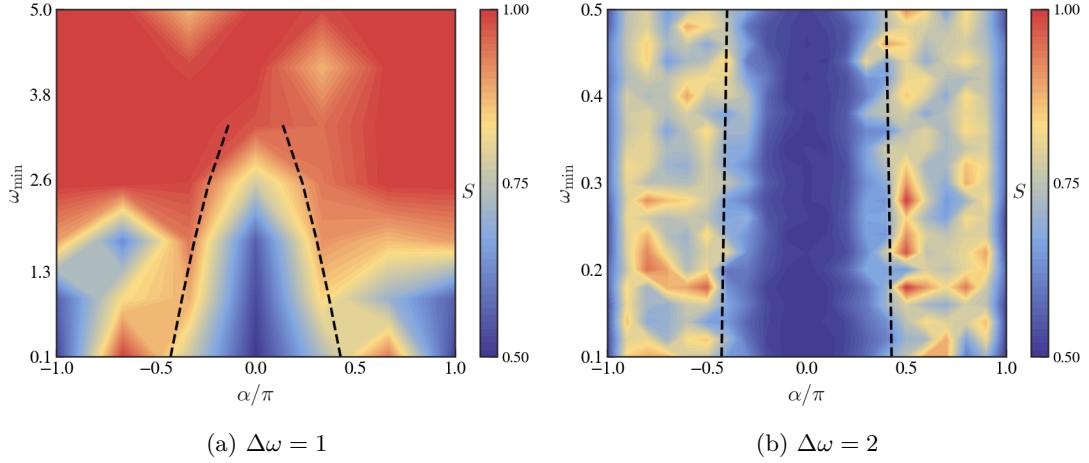


Figure 3: Phase diagrams of the system. The color represents the order parameter  $S$  of the system.

The dynamics of the system (1) in the ISS can be abstracted as a synchronization problem in multilayer networks. This problem can be described by the following equations:

$$\begin{aligned} \dot{\theta}_i^{\pm} = & \omega_i^{\pm} + \frac{K}{N} \sum_{j=1}^{N^{\pm}} \sin(\theta_j^{\pm} - \theta_i^{\pm}) \\ & + \frac{K}{N} \sum_{j=1}^{N^{\mp}} [\sin(\theta_j^{\mp} - \theta_i^{\pm} + \alpha_0) - \sin \alpha_0], \end{aligned} \quad (7)$$

where  $N^{\pm} = N/2$  is the numbers of swarmalators with positive and negative chirality, respectively. Each layer  $\pm$  represents the swarmalators with the same chirality, and the second, third terms in Eq. (7) describe the intralayer and interlayer interactions, respectively.

To quantify the inter-layer coherence of the system, it is convenient to define the generalized complex-valued order parameters, i.e.,

$$Z^{\pm}(t) = R^{\pm}(t) e^{i\psi^{\pm}(t)} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j^{\pm}(t)}, \quad (8)$$

where  $n = 1, 2, \dots, N$ , and  $r^{\pm}(t)$ ,  $\psi^{\pm}(t)$  are the amplitudes and arguments of the  $n$ -order parameter.

$$R_{\max} = \max_{t \in [T, T+h]} \{R(t)\} \quad (9a)$$

$$R_{\min} = \min_{t \in [T, T+h]} \{R(t)\} \quad (9b)$$

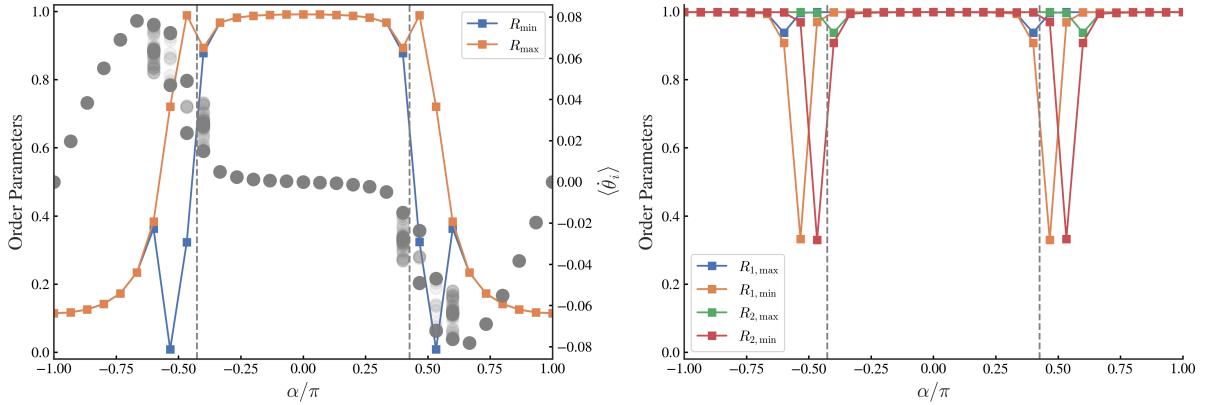


Figure 4: The order parameter  $R$  v.s.  $\alpha_0$  and effective frequency  $\langle \dot{\theta}_i \rangle$  of the oscillator system with  $\sigma = 2$

Let  $\theta_i^\pm(t) = \theta^\pm$  and introduce  $\Delta\theta = \theta^+ - \theta^-$ ,  $\Delta\omega = \mu^+ - \mu^-$ , the phase locking dynamics of two layers can be described by the following equations:

$$\Delta\dot{\theta} = \Delta\omega - K \sin(\theta^+ - \theta^-) \cos \alpha_0 . \quad (10)$$

When  $K^{-1}\Delta\omega \leq 1$ , Eq. (10) has fixed points solutions  $\Delta\theta = \arcsin(K \sin \alpha_0 \Delta\omega)$ , which means the two layers are phase-locked. When  $K^{-1}\Delta\omega > 1$ , the two layers are not phase-locked, and the phase difference  $\Delta\theta$  will drift with time. Therefore, the critical frustration  $\alpha_0$  can be driven by the natural frequency difference  $\Delta\omega$  and coupling strength  $K$  as

$$\alpha_0 = \arccos \frac{|\Delta\omega|}{K} . \quad (11)$$

The critical line of the phase separation is shown in Fig. 3. The critical frustration  $\alpha_0$  descends with the minimal natural frequency  $\Delta\omega$  increasing until the fixed point solution of Eq. (10) is absent.

### 3.1 OA Reduction

Our starting point is to analyze the critical point for the pulsation of the order parameter. In the thermodynamic limit  $N \rightarrow \infty$ , where  $N^\pm \rightarrow \infty$ . Then Eq. (7) gives rise to the continuity equations

$$\frac{\partial \rho^\pm}{\partial t} + \frac{\partial}{\partial \theta} (\rho^\pm v^\pm) = 0 \quad (12)$$

where  $\rho^\pm(\theta, \omega, t)$  is the probability density of oscillators in layer  $\pm$  with phase  $\theta$  at fixed frequency  $\omega$  and  $t$ , it satisfies the normalization condition

$$\int_0^{2\pi} \rho^\pm(\theta, \omega, t) d\theta = 1 . \quad (13)$$

Here,  $v^\pm(\theta, \omega, t)$  is the velocity field, given by

$$\begin{aligned} v^\pm(\theta, \omega, t) &= \omega - \frac{K}{2} \sin \alpha_0 \\ &+ \frac{K}{2} \int \sin(\theta' - \theta) \rho^\pm(\theta', \omega', t) d\theta' d\omega' \\ &+ \frac{K}{2} \int \sin(\theta' - \theta + \alpha_0) \rho^\mp(\theta', \omega', t) d\theta' d\omega' . \end{aligned} \quad (14)$$

Correspondingly, the 1-order parameter defined in Eq. (8) in the thermodynamic limit reads

$$z^\pm(t) = \int_{-\infty}^{+\infty} g^\pm(\omega) \int_0^{2\pi} e^{i\theta} \rho^\pm(\theta, \omega, t) d\theta d\omega , \quad (15)$$

then  $v^\pm(\theta, \omega, t)$  simplifies to

$$v^\pm = \omega - \frac{K}{2} \sin \alpha_0 + \frac{K}{2} \operatorname{Im} [Z^\pm(t) e^{-i\theta}] \quad (16)$$

with the mean-field being

$$Z^\pm(t) = z^\pm(t) + z^\mp(t) e^{i\alpha_0} \quad (17)$$

To proceed, the probability  $\rho^\pm(\theta, \omega, t)$  can be expressed as a Fourier series of the following form

$$\rho^\pm(\theta, \omega, t) = \frac{1}{2\pi} \sum_{-\infty}^{+\infty} a_n^\pm(\omega, t) e^{in\theta}, \quad (18)$$

where  $a_n^\pm(\omega, t)$  is the  $n$ -th Fourier coefficient.

According to Ott and Antonsen ansatz [2, 3], the dynamics of the system can be reduced to a set of ordinary differential equations for the 1st Fourier coefficients  $a^\pm(\omega, t)$ :

$$\dot{\rho}^\pm = \frac{1}{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} [a^\pm(\omega, t) e^{-i\theta}]^n + \text{c.c.} \right\}. \quad (19)$$

To streamline the analysis, we express Eq. (12) in its complex form by multiplying both sides by  $e^{i\theta}$  and then integrating with respect to  $\theta$ :

$$\begin{aligned} \dot{a}^\pm(\omega, t) = & i a^\pm(\omega, t) \left( \omega - \frac{K}{2} \sin \alpha_0 \right) \\ & + \frac{K}{4} [Z^\pm(t) - \bar{Z}^\pm(a^\pm(\omega, t))^2(t)]. \end{aligned} \quad (20)$$

Herein, “ $\bar{\cdot}$ ” denotes the complex conjugate. It is closed by the order parameter,

$$z^\pm(t) = \int_{-\infty}^{+\infty} a^\pm(\omega, t) g^\pm(\omega) d\omega = \hat{g} a^\pm(\omega, t), \quad (21)$$

where the integral operator  $\hat{g}$  is introduced to ease notation.

We aim to obtain the steady solutions to Eq. (20), specifically where  $\dot{a}^\pm(\omega, t) = 0$  and  $z^\pm(t) = r^\pm$ . Some calculations give

$$a_0^\pm(\omega) = \begin{cases} \frac{i\Omega + \sqrt{(R^\pm)^2 - \Omega^2}}{R^\pm}, & |\Omega| \leq R^\pm, \\ i \frac{\Omega - \operatorname{sgn}(\Omega) \sqrt{\Omega^2 - (R^\pm)^2}}{R^\pm}, & |\Omega| > R^\pm, \end{cases} \quad (22)$$

where  $R^\pm = K(r^\pm + r^\mp e^{i\alpha_0})$  and  $\Omega = \omega - \frac{1}{2}K \sin \alpha_0$ . In fact, the first part of the resulting expression corresponds to the locked oscillators, and the second one accounts for drafting populations.

We next explore the stability of  $a_0^\pm(\omega)$  by linearizing Eq. (20). To this end, we introduce the perturbation  $a^\pm(\omega, t) = a_0^\pm(\omega) + \varepsilon \eta^\pm(\omega, t)$ , where  $0 < \varepsilon \ll 1$  denotes the magnitude of the weak perturbation, and  $\eta^\pm(\omega, t)$  is the perturbed function. As a result, the order parameter under the perturbation now becomes

$$z^\pm(t) = r^\pm + \varepsilon \hat{g} \eta^\pm(\omega, t). \quad (23)$$

Substituting all these terms into Eq. (20) and neglecting higher-order terms of  $\varepsilon$ , we obtain

$$\frac{\partial \eta^\pm}{\partial t} = \nu^\pm(\omega) \eta^\pm + \frac{K}{4} [\hat{g} \eta^\pm - (a_0^\pm(\omega))^2 \hat{g} \bar{\eta}^\pm], \quad (24)$$

where the coefficient  $\nu(\omega)$  is

$$\nu^\pm(\omega) = i \left( \omega + \frac{K}{2} \sin \alpha_0 \right) - \frac{1}{2} K R^\pm a_0^\pm(\omega). \quad (25)$$

To ensure clarity and thoroughness, we define the vector  $\mathbf{v} = (\eta^+, \bar{\eta}^+, \eta^-, \bar{\eta}^-)^\top$ . We can then express Eq. (24) in a compact matrix form.

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{M}\mathbf{v} + \frac{K}{4}\mathbf{P}\hat{g}\mathbf{v}, \quad (26)$$

where the multiplication matrix is

$$\mathbf{M} = \begin{bmatrix} \nu^+(\omega) & 0 & 0 & 0 \\ 0 & \bar{\nu}^+(\omega) & 0 & 0 \\ 0 & 0 & \nu^-(\omega) & 0 \\ 0 & 0 & 0 & \bar{\nu}^-(\omega) \end{bmatrix}, \quad (27)$$

and the coefficient matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & -(a_0^+)^2 & 0 & 0 \\ (a_0^+)^2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -(a_0^-)^2 \\ 0 & 0 & (a_0^-)^2 & 1 \end{bmatrix}. \quad (28)$$

To determine the stability of the fixed point, we let  $\frac{\partial \mathbf{v}}{\partial t} = \lambda \mathbf{v}$  and analyze the eigenvectors of the matrix, which are given by

$$\mathbf{v} = \frac{K}{4}(\lambda \mathbf{I} - \mathbf{M})^{-1}\mathbf{P}\hat{g}\mathbf{v}. \quad (29)$$

Applying the operator  $\hat{g}$  to both sides of the equation, we have

$$(\mathbf{I} - K\mathbf{J})\hat{g}\mathbf{v} = 0, \quad (30)$$

where the matrix  $\mathbf{J}$  is defined as

$$\mathbf{J} = \frac{1}{4}\hat{g}\left[(\lambda \mathbf{I} - \mathbf{M})^{-1}\mathbf{P}\right]. \quad (31)$$

A nontrivial solution to Eq. (30) is equivalent to the condition  $\det(\mathbf{I} - K\mathbf{J}) = 0$ . Straightforward calculations yield a group of eigenvalue equations, which are

$$\frac{1}{K} = h_c^\pm(\lambda) = \frac{1}{2}\hat{g}\frac{1 - (\alpha_0^\pm)^2}{\lambda - \eta(\omega)}, \quad (32a)$$

$$\frac{1}{K} = h_s^\pm(\lambda) = \frac{1}{2}\hat{g}\frac{1 + (\alpha_0^\pm)^2}{\lambda - \eta(\omega)}. \quad (32b)$$

The sign of eigenvalue  $(\lambda + in\mathbb{R})$  symbols the stability property of partial locking. In other words, it is stable for  $\lambda < 0$ ; otherwise, it is unstable. Determining the exact solution of the eigenvalue equation is challenging, nevertheless, some qualitative features of  $\lambda$  with respect to  $R^\pm$  can be inferred from Eqs. (32). The first observation is that  $h_c^\pm(\lambda) > 0$  for  $\lambda > 0$  and  $\lim_{\lambda \rightarrow +\infty} h_c^\pm(\lambda) = 0$ . Second, it follows from Eq. (22) that

$$\frac{da_0^\pm}{dR^\pm} = \begin{cases} \frac{1}{\sqrt{(R^\pm)^2 - \omega^2}} - \frac{i\Omega + \sqrt{(R^\pm)^2 - \Omega^2}}{(R^\pm)^2}, & |\Omega| < R^\pm, \\ \frac{i\text{sgn}(\Omega)}{\sqrt{\Omega^2 - (R^\pm)^2}} - i\frac{(\Omega + \text{sgn}(\Omega)\sqrt{\Omega^2 - (R^\pm)^2})}{(R^\pm)^2}, & |\Omega| > R^\pm. \end{cases} \quad (33)$$

It can be verified that

$$\lim_{\lambda \rightarrow 0^+} h_c^\pm(\lambda) = \hat{g}\left(\frac{da_0^\pm}{dR^\pm}\right) = \frac{d}{dR^\pm}(\hat{g}a_0^\pm). \quad (34)$$

Therefore, we have the identity

$$h_c^\pm(0^+) - \frac{1}{K} = R^\pm \frac{dQ(R^\pm)}{dR^\pm}, \quad (35)$$

where  $Q(R^\pm)$  given by the self-consistent equation

$$\frac{1}{K} = Q(R^\pm) = \frac{1}{R^\pm} \int_{|\Omega| < R^\pm} \sqrt{1 - \frac{\Omega^2}{(R^\pm)^2}} g(\omega) d\omega. \quad (36)$$

To close the system, we express the complex order parameter  $z^\pm(t)$  in Eq. (15) by using Cauchy's residue theorem with analytical continuation of  $a^\pm(\omega, t)$  into the right/left half complex plane for  $g^+(\omega)$  and  $g^-(\omega)$ , respectively. Then we have  $z^\pm = \bar{a}^\pm(\pm\omega_0 + i\Delta, t)$ .

As a result, the low-dimensional evolution of the order parameter can be obtained by letting  $\omega = i\Delta$  and substituting Eq. (21) into Eq. (20), which yields

$$\begin{aligned}\dot{z}^\pm &= z^\pm \left( \pm\omega_0 - \frac{iK}{2} \sin \alpha_0 - \Delta \right) + \frac{K}{4} (z^\pm + z^\mp e^{-i\alpha_0}) \\ &\quad - \frac{K}{4} (z^\pm)^2 (\bar{z}^\pm + \bar{z}^\mp e^{i\alpha_0}).\end{aligned}\tag{37}$$

Then we rewrite the above equation using polar coordinates  $z^\pm = r^\pm e^{-i\phi^\pm}$ . (The negative sign in the exponent is chosen to ensure that  $\rho^\pm$  converges to  $\delta(\theta - \phi^\pm)$ , not  $\delta(\theta + \phi^\pm)$  as  $t \rightarrow \infty$ .) Thus,  $\phi^\pm$  can be interpreted as the phase of the chirality  $\pm$ , and  $r^\pm$  measures the degree of synchronization of the chirality  $\pm$ . Then Eq. (37) becomes

$$\dot{r}^\pm = -r^\pm \Delta + K \frac{1 - (r^\pm)^2}{4} [r^\pm + r^\mp \cos(\phi^\pm - \phi^\mp - \alpha_0)],\tag{38a}$$

$$\dot{\phi}^\pm = \pm\omega_0 + \frac{K}{2} \sin \alpha_0 - Kr^\mp \frac{1 + (r^\pm)^2}{4r^\pm} \sin(\phi^\pm - \phi^\mp - \alpha_0).\tag{38b}$$

Introducing  $\varphi = \phi^+ - \phi^-$ , the dynamics of the system can be further simplified to

$$\dot{r}^+ = -r^+ \Delta + K \frac{1 - (r^+)^2}{4} [r^+ + r^- \cos(\varphi - \alpha_0)],\tag{39a}$$

$$\dot{r}^- = -r^- \Delta + K \frac{1 - (r^-)^2}{4} [r^- + r^+ \cos(\varphi + \alpha_0)],\tag{39b}$$

$$\begin{aligned}\dot{\varphi} &= 2\omega_0 - Kr^- \frac{1 + (r^+)^2}{4r^+} \sin(\varphi - \alpha_0) \\ &\quad - Kr^+ \frac{1 + (r^-)^2}{4r^-} \sin(\varphi + \alpha_0)\end{aligned}\tag{39c}$$

## 4 Coarse grained equations and phase diagrams

We begin with Eq. (1), replacing the finite coupling distance alignment interaction with a pseudopotential (the ' $\delta'$ -interaction). This substitution is justified when the interaction is sufficiently short-ranged, making the specific shape of the associated interaction potential irrelevant to the dynamics of many swarmalators. The pseudopotential is defined as:

$$\dot{\mathbf{r}}_i^\pm = v \mathbf{p}(\theta_i^c),\tag{40a}$$

$$\begin{aligned}\dot{\theta}_i^\pm &= \omega_i^\pm + K \sum_{j=1} \{ \delta(\mathbf{r}_j^\pm - \mathbf{r}_i^\pm) \sin(\theta_j^\pm - \theta_i^\pm) \\ &\quad + \delta(\mathbf{r}_j^\mp - \mathbf{r}_i^\pm) [\sin(\theta_j^\mp - \theta_i^\pm + \alpha_0) - \sin \alpha_0] \},\end{aligned}\tag{40b}$$

where  $c \in \{+, -\}$  is the chirality of the swarmalator  $i$  and  $b = +$  if  $c = -$  and vice versa. Then following [1] we derive a continuum equation of motion for the combined  $N$ -swarmalator probability density

$$\rho^\pm(\mathbf{r}, \theta, t) = \sum_{i=1} \rho_i^\pm(\mathbf{r}, \theta, t),\tag{41}$$

where  $\rho_i^\pm(\mathbf{r}, \theta, t) = \delta(\mathbf{r}_i^\pm(t) - \mathbf{r}) \delta(\theta_i^\pm(t) - \theta)$  is the probability density of finding  $i$ -th swarmalator at position  $\mathbf{r}$  with phase  $\theta$  and chirality  $+$  or  $-$  at time  $t$ . Since the deterministic dynamical equation Eq. (1) conserves the number of oscillators with a given natural frequency over time, the distribution function evolves according to a continuity equation of the following form:

$$\frac{\partial \rho_i^\pm}{\partial t} = -\nabla \cdot (\rho_i^\pm v_\mathbf{r}) - \frac{\partial}{\partial \theta} (\rho_i^\pm v_\theta^{\pm, i}).\tag{42}$$

Here, the velocity fields read

$$v_{\mathbf{r}}(\mathbf{r}, \theta, t) = v \mathbf{p}(\theta) , \quad (43a)$$

$$\begin{aligned} v_{\theta}^{\pm, i}(\mathbf{r}, \theta, t) &= \omega_i^{\pm} + K \int_{-\pi}^{\pi} d\phi \left\{ \rho^{\pm}(\mathbf{r}, \phi, t) \sin(\phi - \theta) \right. \\ &\quad \left. + \rho^{\mp}(\mathbf{r}, \phi, t) [\sin(\phi - \theta + \alpha_0) - \sin \alpha_0] \right\} . \end{aligned} \quad (43b)$$

Summing Eq. (42) over the  $i$  and  $\pm$  indices, and using the definition of the density  $\rho^{\pm}$  in Eq. (41), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \rho^{\pm}(\mathbf{r}, \theta, t) &= -v \mathbf{p}(\theta) \cdot \nabla \rho^{\pm}(\mathbf{r}, \theta, t) - \frac{\partial}{\partial \theta} \Omega^{\pm} \\ &\quad - K \frac{\partial}{\partial \theta} \rho^{\pm}(\mathbf{r}, \theta, t) \int_{-\pi}^{\pi} d\phi \left\{ \rho^{\pm}(\mathbf{r}, \phi, t) \sin(\phi - \theta) \right. \\ &\quad \left. + \rho^{\mp}(\mathbf{r}, \phi, t) [\sin(\phi - \theta + \alpha_0) - \sin \alpha_0] \right\} , \end{aligned} \quad (44)$$

where  $\Omega^{\pm}(\mathbf{r}, \theta, t) = \sum_{i=1} \rho_i^{\pm}(\mathbf{r}, \theta, t) \omega_i^{\pm}$ . Spatial homogeneity and phase synchronization of the ISS indicates  $\forall i, \rho_i^{\pm}(\mathbf{r}, \theta, t) \equiv \rho_{\text{ISS}}(\mathbf{r}, \theta, t)$ , which yields

$$\Omega^{\pm}(\mathbf{r}, \theta, t) = \rho^{\pm}(\mathbf{r}, \theta, t) \bar{\omega}^{\pm} , \quad (45)$$

where  $\bar{\omega}^{\pm} = \langle \omega_i^{\pm} \rangle$ .

Transforming  $\rho^{\pm}$  in a Fourier series in  $\theta$ , we have

$$\rho^{\pm}(\mathbf{r}, \theta, t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \varrho_k^{\pm}(\mathbf{r}, t) e^{-ik\theta} , \quad (46)$$

where  $\varrho_k^{\pm}$  is the  $k$ -th Fourier coefficient of  $\rho^{\pm}$ , defined as

$$\varrho_k^{\pm}(\mathbf{r}, t) = \int_{-\pi}^{\pi} \rho^{\pm}(\mathbf{r}, \theta, t) e^{ik\theta} d\theta . \quad (47)$$

To streamline the analysis, we express Eq. (44) in its complex form by multiplying both sides by  $e^{ik\theta}$  and then integrating with respect to  $\theta$ :

$$\begin{aligned} \dot{\varrho}_k^{\pm} &= -\frac{v}{2} [\partial_x (\varrho_{k+1}^{\pm} + \varrho_{k-1}^{\pm}) - i\partial_y (\varrho_{k+1}^{\pm} - \varrho_{k-1}^{\pm})] \\ &\quad + ik\bar{\omega}^{\pm} \varrho_k^{\pm} \\ &\quad + \frac{ikK}{2\pi} \sum_{m=-\infty}^{\infty} \varrho_{k-m}^{\pm} F_{-m} \varrho_m^{\pm} \\ &\quad + \frac{ikK}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\alpha_0} \varrho_{k-m}^{\pm} F_{-m} \varrho_m^{\mp} \\ &\quad - ikK \varrho_0^{\mp} \varrho_k^{\pm} \sin \alpha_0 , \end{aligned} \quad (48)$$

where  $F_m$  is the  $m$ -th Fourier coefficient of  $\sin \theta$ . Evaluating Eq. (48) for  $k = 0, 1, \dots$  leads to a hierarchy of equations for  $\{\varrho_k^{\pm}\}$  with

$$\rho^{\pm}(\mathbf{r}, t) = \varrho_0^{\pm}(\mathbf{r}, t) = \int_{-\pi}^{\pi} \rho^{\pm}(\mathbf{r}, \theta, t) d\theta \quad (49)$$

being the probability density of finding a swarmalator at position  $\mathbf{r}$  at time  $t$  and

$$\mathbf{w}^{\pm}(\mathbf{r}, t) = (\text{Re} \varrho_1^{\pm}, \text{Im} \varrho_1^{\pm}) = \int_{-\pi}^{\pi} \mathbf{p}(\theta) \rho^{\pm}(\mathbf{r}, \theta, t) d\theta \quad (50)$$

being the momentum field at position  $\mathbf{r}$  at time  $t$ . The degree modulus  $R^{\pm}(\mathbf{r}, t) = |\mathbf{w}^{\pm}(\mathbf{r}, t)|$  is the absolute value of the mean normalized velocity of local swarmalators, which can be used as a measure of the local degree of synchronization.

To close the hierarchy of equations Eq. (48), we truncate the series at a finite order  $K$  and neglect the higher-order terms. This truncation is justified when the interaction is sufficiently short-ranged, making the specific shape of the associated interaction potential irrelevant to the dynamics of many swarmalators. Specifically, we assume that  $\dot{\varrho}_2^\pm(\mathbf{r}, t)$  follows changes in  $\varrho_0^\pm(\mathbf{r}, t)$  and  $\varrho_1^\pm(\mathbf{r}, t)$  adiabatically ( $\dot{\varrho}_2^\pm(\mathbf{r}, t) \approx 0$ ) and neglect the higher-order terms ( $\varrho_{k \geq 3}^\pm(\mathbf{r}, t) \approx 0$ ). Truncating at order 3, we are left with

$$\dot{\varrho}_0^\pm = -\text{Re}\bar{\nabla}\varrho_1^\pm \quad (51a)$$

$$\begin{aligned} \dot{\varrho}_1^\pm &= -\frac{v}{2}(\nabla\varrho_0^\pm + \bar{\nabla}\varrho_2^\pm) + i\varrho_1^\pm(\bar{\omega}^\pm - K\varrho_0^\mp \sin\alpha_0) \\ &\quad + \frac{K}{2}\varrho_0^\pm(\varrho_1^\pm + e^{i\alpha_0}\varrho_1^\mp), \end{aligned} \quad (51b)$$

$$\begin{aligned} \dot{\varrho}_2^\pm &= -\frac{v}{2}\nabla\varrho_1^\pm + i2\varrho_k^\pm(\bar{\omega}^\pm - K\varrho_0^\mp \sin\alpha_0) \\ &\quad + K\varrho_1^\pm(\varrho_1^\pm + e^{i\alpha_0}\varrho_1^\mp), \end{aligned} \quad (51c)$$

where  $\nabla = \partial_x + i\partial_y$  denotes the gradient operator in the complex plane, and  $\bar{\nabla} = \partial_x - i\partial_y$  is the complex conjugate of  $\nabla$ . Let  $\dot{\varrho}_2^\pm = 0$ , then  $\varrho_2^\pm$  can be solved as

$$\varrho_2^\pm = -\frac{iK\varrho_1^\pm}{a^\pm}(\varrho_1^\pm + e^{-i\alpha_0}\varrho_1^\mp) + \frac{i\nabla\varrho_1^\pm v}{2a^\pm} \quad (52)$$

where  $a^\pm = 2(K\rho^\mp \sin\alpha_0 - \bar{\omega}^\pm)$ . Substituting  $\varrho_2^\pm$  into Eq. (51b), we have

$$\begin{aligned} \dot{\varrho}_1^\pm &= -\frac{v}{2}\nabla\varrho_0^\pm + \frac{ivK\bar{\nabla}\varrho_1^\pm}{2a^\pm}(\varrho_1^\pm + e^{-i\alpha_0}\varrho_1^\mp) \\ &\quad + \frac{ivK\varrho_1^\pm}{2a^\pm}(\bar{\nabla}\varrho_1^\pm + e^{-i\alpha_0}\bar{\nabla}\varrho_1^\mp) - \frac{iv^2\Delta\varrho_1^\pm}{4a^\pm} \\ &\quad + i\varrho_1^\pm(\bar{\omega}^\pm - K\varrho_0^\mp \sin\alpha_0) + \\ &\quad \frac{K}{2}\varrho_0^\pm(\varrho_1^\pm + e^{i\alpha_0}\varrho_1^\mp) \end{aligned} \quad (53)$$

## References

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