Nonlinear Sciences > Adaptation and Self-Organizing Systems

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Swarmalators on a ring with distributed couplings

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We study a simple model of identical swarmalators, generalizations of phases oscillators that swarm through space. We confine the movements to a one-dimensional (1D) ring and consider distributed (non-identical) couplings; the combination of these two effects captures an aspect of the more realistic 2D swarmalator model \cite{o2017oscillators}. We find new collective states as well as generalizations of previously reported ones which we describe analytically. These states imitate the behavior of vinegar eels, catalytic microswimmers, and other swarmalators which move on quasi-1D rings.

Subjects: **Adaptation and Self-Organizing Systems (nlin.AO)**; Disordered Systems and Neural Networks (cond-mat.dis-nn); Soft Condensed Matter (cond-mat.soft)

Model equations:

$$\dot{x}_i = \omega + \frac{1}{N}\sum_{j}^{N} J_j \sin\left(x_j - x_i\right)\cos\left(\theta_j - \theta_i\right)$$

$$\dot{\theta}_i = \nu + \frac{1}{N}\sum_{j}^{N} K_j \sin\left(\theta_j - \theta_i\right)\cos\left(x_j - x_i\right)$$

Order parameters:

$$W_{\pm} = S_{\pm}e^{i\Phi_{\pm}} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\left(x_j + \theta_i\right)}$$

$$V = \frac{1}{N}\sum_{j=1}^{N}\left\langle\sqrt{\dot{x}_j^2 + \dot{\theta}_j^2}\right\rangle_t$$

'Double delta' distribution

$$g\left(J\right) = \delta\left(J - 1\right)$$
$$h\left(K\right) = p\delta\left(K - K_p\right) + \left(1 - p\right)\delta\left(K - K_n\right)$$

- **Static sync** for $p > p_c$, Order parameters are $S_{\pm} = 1$ and $V = 0$
- **Static phase wave** for $p_0 < p < p_c$: $x_i = \theta_i \pm C$, $(S_+, S_-) = (1, 0)$ or $(0, 1)$
- **Buckled phase wave** near p_c: A static phase wave with a 'buckle', so $S_+ \approx 1, S_- = 0, V = 0$, which means $x_i \to \theta_i + C$, x_i and $-\theta_i$ are uncorrelated.
- **Noisy phase wave** for $p < p_0$
- **Async** for $p \approx 0$

Static sync

$$\begin{bmatrix} \dot{x}_i \\ \dot{\theta}_i \end{bmatrix} = M \begin{bmatrix} x_i \\ \theta_i \end{bmatrix}$$

where the Jacobian M for the static sync at this fixed point has a block structure:

$$M = \frac{1}{N}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where

$$A := \begin{bmatrix} -\sum_{j\neq 1} J_j & J_2 & \cdots & J_N \\ J_1 & -\sum_{j\neq 2} J_j & \cdots & J_N \\ \vdots & \vdots & & \vdots \\ J_1 & J_2 & \cdots & -\sum_{j\neq N} J_j \end{bmatrix}$$

$$A := \begin{bmatrix} -\sum_{j\neq 1} J_j & J_2 & \cdots & J_N \\ J_1 & -\sum_{j\neq 2} J_j & \cdots & J_N \\ \vdots & \vdots & & \vdots \\ J_1 & J_2 & \cdots & -\sum_{j\neq N} J_j \end{bmatrix}$$

Sync:

$$\begin{aligned} y_i &= \omega + \frac{1}{N}\sum_{j}^{N} J_j \sin\left(x_j - x_i\right)\cos\left(\theta_j - \theta_i\right) \\ &= \frac{1}{N}\sum_{j}^{N} J_j \sin\left(x_j - x_i\right) \quad \left(sync \Longrightarrow \theta_j - \theta_i = 0\right) \end{aligned}$$

Static:

$$\begin{aligned} Static \Longrightarrow y_i = 0 \Longrightarrow \sin\left(x_j - x_i\right) &= 0 \\ \Longrightarrow \cos\left(x_j - x_i\right) &= 1 \end{aligned}$$

$$\frac{\partial y_i}{\partial x_i} = -\frac{1}{N}\sum_{j}^{N} J_j \cos\left(x_j - x_i\right) = -\frac{1}{N}\sum_{j}^{N} J_j$$

$$
A := \begin{bmatrix} J_1 & J_2 & \cdots & J_N \\ J_1 & J_2 & \cdots & J_N \\ \vdots & \vdots & & \vdots \\ J_1 & J_2 & \cdots & J_N \end{bmatrix} + \begin{bmatrix} -\langle J \rangle & 0 & \cdots & 0 \\ 0 & -\langle J \rangle & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\langle J \rangle \end{bmatrix} = A_1 + A_2
$$

$$
\lambda_{A_1} = 0, \langle J \rangle, \lambda_{A_2} = -\langle J \rangle
$$

Two matrices have the same eigenvector:

$$
\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T
$$

So $\lambda_A = \lambda_{A_1} + \lambda_{A_2}$

$$
\begin{aligned} \lambda_{A,1} &= 0 & &\text{(w.m. 1)} \\ \lambda_{A,2} &= -\langle J \rangle & &\text{(w.m. } N-1\text{)} \end{aligned}
$$

Putting this together gives

$$
\begin{aligned}
\lambda_0 &= 0 && (\text{w.m.}2) \\
\lambda_1 &= -\langle J \rangle && (\text{w.m.}N-1) \\
\lambda_2 &= -\langle K \rangle && (\text{w.m.}N-1)
\end{aligned}
$$

For the double delta distribution working example

$$
\begin{aligned}
\langle K \rangle &= pK_p + (1-p) K_n \\
&= K_p \left[p(1+Q) - Q \right] \quad \left(Q \equiv -K_n/K_p \right) \\
&= 0
\end{aligned}
$$

Setting this to zero gives the critical fraction of positively coupled swarmlators

$$
p_c = \frac{Q}{1+Q}
$$

Buckled phase wave

First the authors move to (ξ, η) coordinates

$$\xi_i = x_i + \theta_i, \eta_i = x_i + \theta_i$$

The governing equations become

$$\begin{aligned}
\dot{\xi}_i = & \frac{U_+}{2} \sin\left(\Psi_+ - \xi_i\right) + \frac{V_+}{2} \sin\left(\Phi_+ - \xi_i\right) \\
& + \frac{U_+}{2} \sin\left(\Psi_+ - \eta_i\right) + \frac{V_-}{2} \sin\left(\Phi_- - \eta_i\right) \\
\dot{\eta}_i = & \frac{U_+}{2} \sin\left(\Psi_+ - \xi_i\right) - \frac{V_+}{2} \sin\left(\Phi_+ - \xi_i\right) \\
& + \frac{U_-}{2} \sin\left(\Psi_+ - \eta_i\right) + \frac{V_-}{2} \sin\left(\Phi_- - \eta_i\right)
\end{aligned}$$

where

$$U_\pm e^{i\Psi_\pm} = \frac{1}{N} \sum_j J_j e^{i\left(x_j \pm \theta_j\right)}, V_\pm e^{i\Phi_\pm} = \frac{1}{N} \sum_j K_j e^{i\left(x_j \pm \theta_j\right)}$$

are 'glassy' order parameters [1].

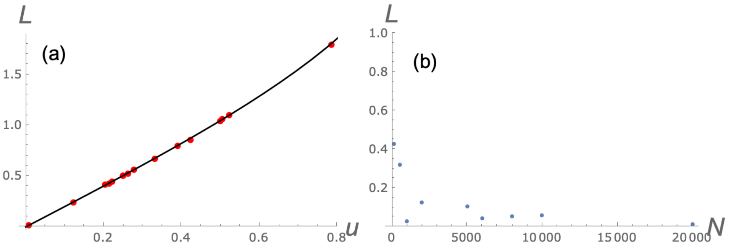
[1] I. M. Kloumann, I. M. Lizarraga, and S. H. Strogatz, *Phase diagram for the Kuramoto model with van Hemmen interactions*, Physical Review E 89, 012904 (2014).

Next the authors set $\dot{\xi}_i, \dot{\eta}_i$ to zero. then the authors add and subtract the equations to produce

$$\begin{aligned} 0 &= U_+ \sin \xi_i + U_- \sin \eta_i \\ 0 &= V_+ \sin\left(\Phi_+ - \xi_i\right) - V_- \sin\left(\Phi_- - \xi_i\right) \end{aligned} \tag{1}$$

where the authors set $\Psi_\pm = 0$ wlog. Eq 1 are nullclines, curves in (ξ, η) space, derive the the 1D manifold $\Gamma_1(x, \theta) = 0$, $\Gamma_2(x, \theta) = 0$ which defines the state.
The authors observe that (i) the nullclines must be identical and (ii) describe buckled phase wave $\Gamma_1 = \Gamma_2 = \Gamma$. (1) implies

$$\frac{U_-}{U_+} = \frac{V_-}{V_+}$$
$$\Phi_+ - \Phi_+ = \pi$$



$$\Gamma(\xi, \eta) = \sin \xi + u \sin \eta = 0$$

where $u := U_- / U_+$