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Ring states in swarmalator systems

Kevin P. O'Keeffe, Joep H.M. Evers, Theodore Kolokolnikov

Synchronization is a universal phenomenon, seen in systems as diverse as superconducting Josephson junctions and discharging pacemaker cells. Here the elements have rhythmic state variables whose mutual influence promotes temporal order. A parallel form of order is seen in swarming systems, such as schools of fish or flocks of birds. Now the degrees of freedom are the individuals' positions, which get redistributed through interactions to form spatial structures. Systems capable of both swarming and synchronizing, dubbed swarmalators, have recently been proposed [O'Keeffe, Kevin P., and Steven H. Strogatz. "Swarmalators: Oscillators that sync and swarm." arXiv preprint arXiv:1701.05670 (2017)] and analyzed in the continuum limit. Here we extend this work by studying finite populations of swarmalators, whose phase similarity affects both their spatial attraction and repulsion. We find ring states, and compute criteria for their existence and stability. Larger populations can form annular distributions, whose density and inner and outer radii we calculate explicitly. These states may be observable in groups of Japanese tree frogs, magnetic colloids, and other systems with an interplay between swarming and synchronization.

Subjects: **Adaptation and Self-Organizing Systems (nlin.AO)**; Pattern Formation and Solitons (nlin.PS)

Contents

Model equations:

$$\dot{\mathbf{x}}_k = \frac{1}{N}\sum_{j\neq i}^{N}\left[\left(\mathbf{x}_j - \mathbf{x}_k\right)\left(A + J_1\cos\left(\theta_j - \theta_k\right)\right) - \left(B - J_2\cos\left(\theta_j - \theta_k\right)\right)\frac{\mathbf{x}_j - \mathbf{x}_k}{\left|\mathbf{x}_j - \mathbf{x}_k\right|^2}\right]$$

$$\dot{\theta}_k = \frac{K}{N}\sum_{j\neq i}^{N}\frac{\sin\left(\theta_j - \theta_k\right)}{\left|\mathbf{x}_j - \mathbf{x}_k\right|^2}$$

$A = B = 1$

Ring phase waves

Spatial angle $\phi_k = \tan^{-1}\left(\frac{\mathbf{x}_{k,2}}{\mathbf{x}_{k,1}}\right)$

In this state, the spatial angle of each swarmalator is correlated with its phase (i.e. $\phi_k = \theta_k + C$)

Existence: In the ring phase wave state, the position and phase of the kth swarmalator are

$$\mathbf{x}_k = R\cos\left(2\pi k/N\right)\hat{x} + R\sin\left(2\pi k/N\right)\hat{y}$$

$$\theta_k = 2\pi k/N + C$$

$$R = \sqrt{\frac{N - 1 + J_2}{N\left(2 - J_1\right)}}$$

Stability of Ring phase waves

When $K = 0$.

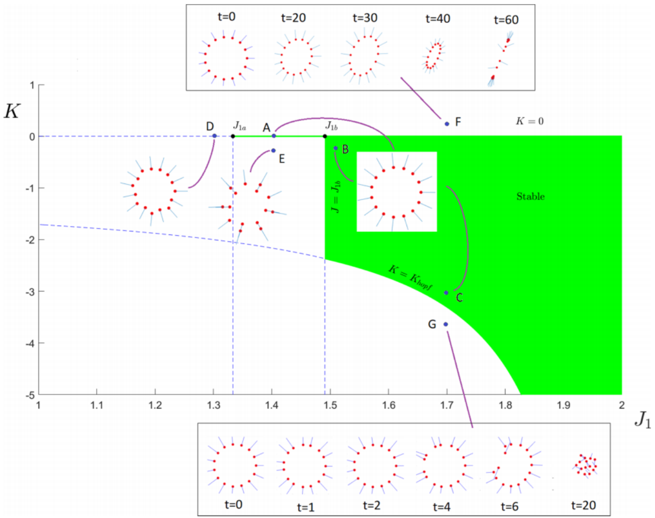
$$J_{1a} := \begin{cases} 2 - 8\frac{(N-1+J_2)}{(N-2)^2(1-J_2)}, & N\ even, N > 4 \\ 2 - 8\frac{(N-1+J_2)}{(N-1)(N-3)(1-J_2)}, & N\ odd, N > 4 \end{cases}$$

When $K > 0$ the swarmalators' phases are no longer frozen.

When $K < 0$

$$J_{1b} := \begin{cases} 2\left(\frac{1}{1-4/N^2}\right) - \frac{1}{1-J_2}\frac{8}{(N-4/N)}, & N\ even, N > 4 \\ 2\left(\frac{1}{1-4/(N^2-1)}\right) - \frac{1}{1-J_2}\frac{8}{(N-5/N)}, & N\ odd, N > 4 \end{cases}$$

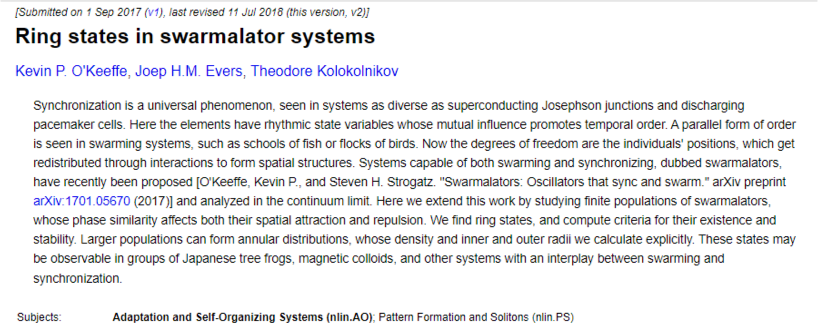
$$K_{\mathrm{Hopf}} = \begin{cases} -\frac{(J_2-1)(-2+J_1)N^2+[(-4J_2+4)J_1+8J_2]N+4J_1(J_2-1)}{N(N-4)(2-J_1)} \\ -\frac{(J_2-1)(-2+J_1)N^2+[(-4J_2+4)J_1+8J_2]N+(3J_2-3)J_1+2J_2-2}{(N^2-4N-1)(2-J_1)} \end{cases}$$


t=0
t=20
t=30
t=40
t=60
K
J1
1
0
-1
-2
-3
-4
-5
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
D
J1a
A
J1b
E
F
K = 0
B
J = J1b
Stable
K = Khopf
C
G
t=0
t=1
t=2
t=4
t=6
t=20

$$N \to \infty, J_{1a} \sim J_{1b} \sim 2 - \frac{8}{1 - J_2}$$

$$N_{\max} \sim \frac{8}{(2 - J_1)(1 - J_2)}$$

Contents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$A = B = 1$

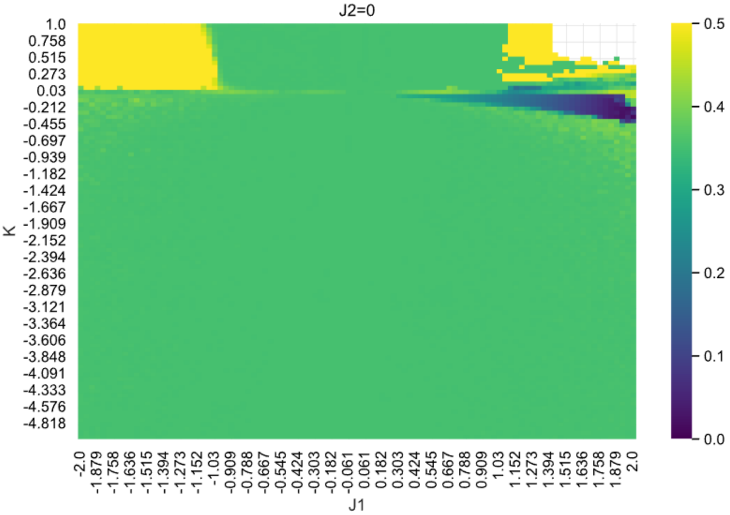
$$R_j = \sqrt{\hat{x}_j^2 + \hat{y}_j^2}$$

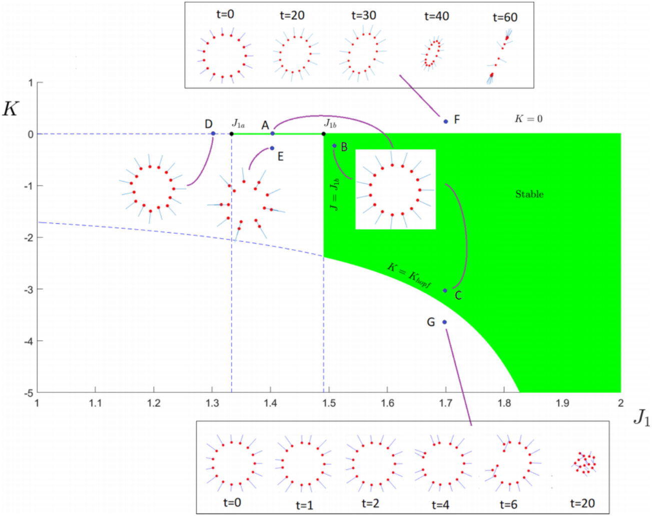
$$\bar{R} = \frac{1}{N}\sum_{j=1}^{N} R_j, \text{(Mean radius)}$$

Order parameters:

$$V = \frac{\sqrt{\frac{\sum_{j=1}^{N}\left(R_j - \bar{R}\right)^2}{n-1}}}{\bar{R}}, \text{(Coefficient of Variation)}$$

$$K \in [-5,-1], J_1 \in [-2,2], J_2 \in \{-2,-1,0,1,2,3\}$$

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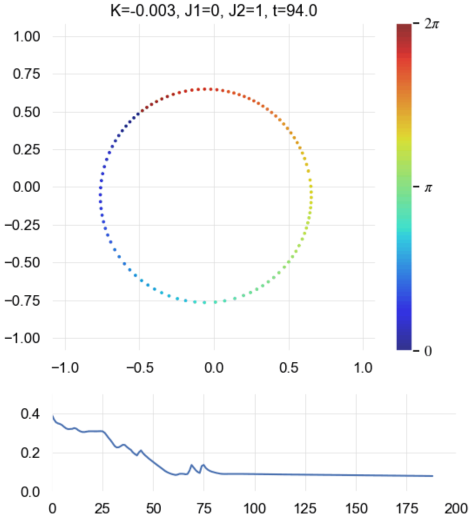
t=20
t=30
t=40
t=60
K
1
0
-1
-2
-3
-4
-5
D
J_{1a}
A
J_{1b}
E
F
$K = 0$
B
$J = J_{1b}$
Stable
$K = K_{hopf}$
C
G
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
J_1
t=0
t=1
t=2
t=4
t=6
t=20


K=-0.003, J1=0, J2=1, t=94.0
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.0
−0.5
0.0
0.5
1.0
2π
π
0
0.4
0.2
0.0
0
25
50
75
100
125
150
175
200

$$\rho\left(r,\phi,\theta\right) = \frac{1}{2\pi} g\left(r\right)\delta\left(\phi-\theta\right), \quad R_1 \le r \le R_2$$
$$= 0, \qquad \textit{elsewhere}$$

$$\underline{v} = \left(\mathbf{v_x}, v_\theta\right)$$
$$\nabla \cdot \underline{v} \equiv 0$$
$$v_r = \int \left(s\cos\left(\phi'-\phi\right) - r\right)\left(1 + J_1\cos\left(\phi'-\phi\right) - \frac{1 - J_2\cos\left(\theta'-\theta\right)}{s^2 - 2rs\cos\left(\theta'-\theta\right) + r^2}\right)$$
$$s\rho\left(s,\phi',\theta'\right) ds d\phi' d\theta'$$
$$v_r = \int s\cos\left(\phi'-\phi\right)\left(1 + J_1\cos\left(\phi'-\phi\right) - \frac{1 - J_2\cos\left(\theta'-\theta\right)}{s^2 - 2rs\cos\left(\theta'-\theta\right) + r^2}\right)$$
$$s\rho\left(s,\phi',\theta'\right) ds d\phi' d\theta'$$
$$v_\theta = K\int \frac{\sin\left(\theta'-\theta\right)}{s^2 - 2rs\cos\left(\theta'-\theta\right) + r^2} s\rho\left(s,\phi',\theta'\right) ds d\phi' d\theta'$$