Phase Frustration-Induced Spatial Lattice Symmetry in the Vicsek-Kuramoto-Sakaguchi Model

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October 30, 2025

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1 The Model

Particles are characterized by their spatial position $\mathbf{r}_i = (x_i, y_i)$ and a phase angle θ_i, whose dynamics are governed by the following equations:

$$\dot{\mathbf{r}}_i = v\mathbf{p}(\theta_i) \ , \tag{1a}$$

$$\dot{\theta}_i = \frac{K}{|A_i|} \sum_{j \in A_i} [\sin(\theta_j - \theta_i + \alpha) - \sin\alpha] \ , \tag{1b}$$

for $i = 1, 2, \ldots, N$. Here in Eq. (1a), $\mathbf{p}(\theta) = (\cos\theta, \sin\theta)$denotes the direction vector, implying that each particle moves at a constant speed v along the direction of its instantaneous phase $\theta_i(t)$. According to Eq. (1b), the phase evolution involves a local average over neighbors within a coupling radius d_0 of particle i:

$$A_i(t) = \{j \mid |\mathbf{r}_i(t) - \mathbf{r}_j(t)| \leqslant d_0\} \ , \tag{2}$$

where $K\,(\geqslant 0)$ represents the coupling strength and α is the phase frustration between two neighboring particles. The introduction of counter term $-\sin\alpha$ ensures that the frustration vanishes exactly when phase differences vanish ($\theta_j - \theta_i = 0$), thereby guaranteeing that perfect synchronization remains an equilibrium state. Without this term, synchronized oscillators would experience a net force $K\sin\alpha$, artificially shifting their frequencies [?]. This model generalizes both aligning [? ? ? ? ?] and anti-aligning [? ?] interaction models. When $\alpha_0 = 0$, the dynamics reduces to the Vicsek-Kuramoto model. In the case where $\alpha = \pi$, the system exhibits anti-aligning interactions, causing particles to adopt phases opposite to those of their neighbors.