Sequences and Power Series

Leif Mejlbro

bookboon
The eBook company

Leif Mejlbro

Sequences and Power Series Guidelines for Solutions of Problems

Calculus 3b

Sequences and Power Series – Guidelines for Solutions of Problems – Calculus 3b

ISBN 978-87-7681-239-3

Contents

Preface

Here follow some guidelines for solution of problems concerning sequences and power series. It should be emphasized that my purpose has never been to write an alternative textbook on these matters. If I would have done so, I would have arranged the subject differently. Nevertheless, it is my hope that the present text can be a useful supplement to the ordinary textbooks, in which one can find all the necessary proofs which are skipped here.

The text presupposes some knowledge of *Calculus 1a, Functions in One Variable*, and it will itself be the basis for the following *Calculus 4b: Fourier Series, Differential Equations and Eigenvalue Problems*. The previous text, *Calculus 2b, Functions in Several Variables* will only be necessary occasionally.

Chapter 1 is a repetition of useful formulæ – some of them already known from high school – which will be used over and over again. The reader should read this chapter carefully together with Appendix A, which is a short collection of formulæ known previously. These will be assumed in the text without further reference, so it would be a good idea to learn these formulæ by heart, since they can be considered as the tools of Calculus which should be mastered before one can proceed.

The text itself falls into two main parts, 1) *Sequences of numbers and functions*, and 2) *Series of numbers and power series*. The more general series of functions occur only rarely in this text. I felt that the main case of Fourier series should be put into a later text, because the natural concept of convergence is not the same as the convergence dealt with here. I have seen too many students being confused by the different types of convergence to let these two main cases clash in the same volume.

Comments, remarks and examples will always be ended by the symbol $\Diamond$, so the reader can see when the main text starts again.

In general, every text in the Calculus series is given a number – here 3 – and a letter – here b – where

a means “compendium”,

b means “guidelines for solutions of standard problems”,

c means “examples”.

Since this is the first edition of this text, there may still be some errors, which the reader hopefully will forgive me.

21st June 2014
Leif Mejlbro

1 Repetition of important formulæ

1.1 Decomposition

Today this technique is less practised than earlier because it has become easier to use the command `expand` or similarly on a pocket calculator. Unfortunately this method is not always successful, and even MAPLE may sometimes give some very strange results concerning decomposition. Now, decomposition occurs in the most unexpected places in Calculus (and in the technical sciences), so in order to amend the shortcomings of the pocket calculators, the reader should at least know how in principle one can decompose a fraction of two polynomials so that one is able to modify the method when e.g. an application of `expand` fails.

The practical performance of decomposition is best illustrated by an example with a list of the standard steps needed. This will in several ways differ from the method given in *Calculus 1a, Functions in One Variable*, because the reader must be considered as been at a higher level when reading the present text than earlier.

Example 1.1 *Decompose the fractional function, i.e. the quotient between two polynomials*

$$\begin{aligned} f(x) &= \frac{x^4}{(x-1)^2(x^2+1)} = \textit{polynomial} + \textit{basic fractions} \\ &= \textit{polynomial} + \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c+d\cdot x}{x^2+1}. \end{aligned}$$

We see that the task is to find the polynomial and the constants a, b, c and d, where the theory from *Calculus 1a, Functions in One Variable* assures that this representation is unique.

1) Factorize the denominator.

This has already been done.

2) If the degree of the numerator is $\geq$ the degree of the denominator, we separate a polynomial by division. This polynomial is the first part of the result, cf. the description of the task. We shall first use this polynomial again in the last step.

In the actual case we see that

$$f(x) = \frac{x^4}{(x-1)^2(x^2+1)} = 1 + \frac{2x^3-2x^2+2x-1}{(x-1)^2(x^2+1)}.$$

The polynomial (here the constant 1) is saved for the final result.

3) (Deviation from Calculus 1a, Functions in One Variable). *A trick here is in the next step to choose the simplest of the two fractional functions describing $f(x)$, i.e. before and after the separation of the polynomial. The following method will give the same result, no matter which representation of the fractional part is chosen.*

In the actual case we shall make a choice between

$$\frac{x^4}{(x-1)^2(x^2+1)} \qquad \text{og} \qquad \frac{2x^3-2x^2+2x-1}{(x-1)^2(x^2+1)}.$$

The first fraction looks "nicest", even though the degree of the numerator is 4. We shall therefore choose this one in the following.

4) The crux of the procedure: Choose any root in the denominator. (This is the reason why we start by factorizing the denominator, so the roots can easily be found). *Hold your hand or finger over this root and insert the root in the rest of the fraction.*

In this case the denominator has the real double root $x = 1$. Remove $(x-1)^2$ from the denominator (done in practice by holding a hand or finger over it) and insert $x = 1$ in the rest of the fraction. Then we automatically get the coefficient a of $\dfrac{1}{(x-1)^2}$, i.e.

$$a = \left[\frac{x^4}{x^2+1}\right]_{x=1} = \frac{1^4}{1^2+1} = \frac{1}{2}.$$

Then reduce $f(x)$ to

$$f(x) = \frac{x^4}{(x-1)^2(x^2+1)} = \frac{1}{2} \cdot \frac{1}{(x-1)^2} + f_1(x),$$

where $f_1(x)$ denotes the rest, which should not yet be calculated.

Remark 1.1 This method can in principle also be applied for the complex roots $x = \pm\,\mathrm{i}$. One should here always think about if the complex calculations will become simple or not by applying this method. $\Diamond$

5) Continue in this way with all the different real roots in the denominator. Think it over if it would be profitable also to use it on some of the complex roots.

Returning to the example under consideration we see that 1 is the only real root. The method applied to the complex roots will give some heavy calculations, although they will lead us directly to the result. Since we here are more interested in giving some standard guidelines in the real case, I shall decline from giving the complex variant, leaving this task to the reader as an exercise.

6) Find by reduction explicitly the simpler function $f_1(x)$, which is obtained by removing all the basic fractions in 4) *and* 5)*.*

Since we have not chosen the complex variant, we have already given $f_1(x)$ as a part of the example belonging to 4). By a rearrangement and a reduction we get

$$f_1(x) = \frac{x^4}{(x-1)^2(x^2+1)} - \frac{1}{2} \cdot \frac{1}{(x-1)^2} \cdot \frac{x^2+1}{x^2+1} = \frac{1}{2} \cdot \frac{2x^4-x^2-1}{(x-1)^2(x^2+1)}.$$

If we have not introduced some error, then $x-1$ *must* necessarily be a divisor in the numerator:

$$2x^4 - x^2 - 1 = 2(x^2)^2 - x^2 - 1 = (2x^2+1)(x^2-1) = (x-1)(x+1)(2x^2+1).$$

By insertion we finally get by a reduction,

$$f_1(x) = \frac{1}{2} \cdot \frac{(x-1)(x+1)(2x^2+1)}{(x-1)^2(x^2+1)} = \frac{1}{2} \cdot \frac{(x+1)(2x^2+1)}{(x-1)(x^2+1)}.$$

7) Repeat the procedures 4), 5) and 6) on $f_1(x)$.

In the present case we shall hold our hand over $x-1$ in the denominator and insert $x=1$ in the rest. This will give us the constant b of $\dfrac{1}{x-1}$, i.e.

$$b=\left[\frac{1}{2}\cdot\frac{(x+1)(2x^2+1)}{x^2+1}\right]_{x=1}=\frac{1}{2}\cdot\frac{(1+1)(2+1)}{1+1}=\frac{3}{2}.$$

Another insertion gives

$$f_1(x)=\frac{1}{2}\cdot\frac{(x+1)(2x^2+1)}{(x-1)(x^2+1)}=\frac{3}{2}\cdot\frac{1}{x-1}+f_2(x),$$

hence by a rearrangement and a reduction,

$$\begin{aligned} f_2(x) &= \frac{1}{2}\cdot\frac{(x+1)(2x^2+1)}{(x-1)(x^2+1)}-\frac{3}{2}\cdot\frac{1}{x-1}\cdot\frac{x^2+1}{x^2+1} \\ &= \text{(some longer calculations, which are not given here)} \\ &= 1+\frac{1}{2}\cdot\frac{x}{x^2+1}. \end{aligned}$$

8) Repeat 7), as long as possible.

In the considered case we have finished the task.

9) Finally, collect all the results found previously in order to get the final decomposition.

In the chosen example we get

$$\begin{aligned} \frac{x^4}{(x-1)^2(x^2+1)} &= \frac{1}{2}\cdot\frac{1}{(x-1)^2}+f_1(x) \\ &= \frac{1}{2}\cdot\frac{1}{(x-1)^2}+\frac{3}{2}\cdot\frac{1}{x-1}+f_2(x) \\ &= 1+\frac{1}{2}\cdot\frac{x}{x^2+1}+\frac{1}{2}\cdot\frac{1}{(x-1)^2}+\frac{3}{2}\cdot\frac{1}{x-1}. \qquad \Diamond \end{aligned}$$

An important special case is

Theorem 1.1 Heaviside's expansion theorem. *Let $f(x)=\dfrac{P(x)}{Q(x)}$ be a fraction of two polynomials where the degree of the numerator is strictly smaller than the degree of the denominator.*
Assume that the denominator only has simple *roots, e.g.*

$$Q(x)=(x-a_1)(x-a_2)\cdots(x-a_n).$$

Define $Q_j(x)$, $j=1,\dots,n$, by deleting $x-a_j$ in $Q(x)$, i.e.

$$Q_j(x)=\frac{Q(x)}{x-a_j}=(x-a_1)\cdots(x-a_{j-1})\cdot(x-a_{j+1})\cdots(x-a_n).$$

Then we get the decomposition

$$f(x) = \frac{P(x)}{Q(x)} = \frac{P(a_1)}{Q_1(a_1)} \cdot \frac{1}{x-a_1} + \cdots + \frac{P(a_n)}{Q_n(a_n)} \cdot \frac{1}{x-a_n}.$$

Proof. This follows immediately by using the method of holding your hand over the simple roots. ◊

A variant is the following:

Theorem 1.2 *Let* $f(x) = \dfrac{P(x)}{Q(x)}$ *be a quotient between two polynomials where the degree of the numerator is* smaller than *the degree of the denominator.*
Assume that the denominator has only simple *roots,* $a_1, \cdots, a_n$. *Then the decomposition can be written*

$$f(x) = \frac{P(x)}{Q(x)} = \frac{P(a_1)}{Q'(a_1)} \cdot \frac{1}{x-a_1} + \cdots + \frac{P(a_n)}{Q'(a_n)} \cdot \frac{1}{x-a_n}.$$

The two theorems above can also be applied for simple complex roots in the denominator.

Example 1.2 Decompose

$$\frac{x}{x^4-1} = \frac{x}{(x-1)(x-\,\mathrm{i})(x+1)(x+\,\mathrm{i})}.$$

An application of Theorem 1.1 gives a lot of calculations,

$$Q_1(x) = (x+1)(x^2+1), \qquad Q_2(x) = (x^2-1)(x+\,\mathrm{i}),$$

$$Q_3(x) = (x-1)(x^2+1), \qquad Q_4(x) = (x^2-1)(x-\,\mathrm{i}),$$

where

$$Q_1(1) = 4, \quad Q_2(\mathrm{i}) = -4\,\mathrm{i}, \quad Q_3(-1) = -4, \quad Q_4(-\,\mathrm{i}) = 4\,\mathrm{i}.$$

By insertion we get

$$\frac{x}{x^4-1} = \frac{1}{4}\left\{\frac{1}{x-1} - \frac{1}{x-\,\mathrm{i}} + \frac{1}{x+1} - \frac{1}{x+\,\mathrm{i}}\right\}.$$

Here, Theorem 1.2 is much easier to apply, because

$$\frac{P(x)}{Q'(x)} = \frac{x}{4x^3} = \frac{1}{4}\cdot\frac{x^2}{x^4}.$$

Since all the roots satisfy the equation $a_j^4 = 1$, it follows that

$$\frac{P(a_j)}{Q'(a_j)} = \frac{1}{4}\cdot a_j^2,$$

where $a_j^2 = 1$ for the real roots and $a_j^2 = -1$ for the imaginary roots, from which we get

$$\frac{x}{x^4-1} = \frac{1}{4}\left\{\frac{1}{x-1} - \frac{1}{x-\,\mathrm{i}} + \frac{1}{x+1} - \frac{1}{x+\,\mathrm{i}}\right\}. \qquad \Diamond$$

1.2 Trigonometric formulæ

We get from e.g. *Calculus 1a, Functions of one Variable*, the *addition formulæ*

(1) $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$,

(2) $\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$,

(3) $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$,

(4) $\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$.

Mnemonic rule: $\cos x$ is *even*, and $\sin x$ is *odd*. Since $\cos(x \pm y)$ is *even*, the reduction can only contain the terms $\cos x \cdot \cos y$ *(even times even)* and $\sin x \cdot \sin y$ *(odd times odd)*. Notice the *change of sign* in front of $\sin x \cdot \sin y$. $\Diamond$

Analogously $\sin(x \pm y)$ is *odd*, hence the reduction can only contain the terms $\sin x \cdot \cos y$ *(odd times even)* and $\cos x \cdot \sin y$ *(even times odd)*. Notice that we here have no change of sign. $\Diamond$

From time to time we need to simplify products of the type

$$\sin x \cdot \sin y, \qquad \cos x \cdot \cos y, \qquad \sin x \cdot \cos y.$$

even even odd

We derive the simplifications of the *addition formulæ above*:

$$2\sin x \cdot \sin y = \cos(x-y) - \cos(x+y), \qquad (2)-(1),$$

$$2\cos x \cdot \cos y = \cos(x-y) + \cos(x+y), \qquad (2)+(1),$$

$$2\sin x \cdot \cos y = \sin(x-y) + \sin(x+y), \qquad (3)+(4).$$

The searched formulæ are then obtained by division by 2. They are called the *antilogarithmic formulæ:*

$$\sin x \cdot \sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\}, \qquad \text{even},$$

$$\cos x \cdot \cos y = \frac{1}{2}\{\cos(x-y) + \cos(x+y)\}, \qquad \text{even},$$

$$\sin x \cdot \cos y = \frac{1}{2}\{\sin(x-y) + \sin(x+y)\}, \qquad \text{odd}.$$

1.3 Notations and conventions

One of the main subjects in this text is concerned with power series. Some of these have already been given in *Calculus 1a, Functions of one Variable.*

It is of paramount importance that the student is able to recognize the *structure* of the elementary *standard series*. We shall here based on *Calculus 1a, Functions in one Variable*, once again go through them. We shall also add a couple of new concepts which only will give sense later, but which are most conveniently put here.

1) *The faculty function* $n!$

This is defined by

$$n! := 1 \cdot 2 \cdots n \quad \text{for } n \in \mathbb{N}, \qquad \text{and } 0! := 1,$$

i.e. the product of the first n natural numbers with the convention $0! = 1$ (the product of no natural number is put equal to 1).

Warning. The notation is a little treacherous. In order to warn again later misunderstandings we here calculate explicitly

$$(2n+1)! = 1 \cdot 2 \cdots (2n-1) \cdot (2n)(2n+1) = (2n-1)!(2n) \cdot (2n+1),$$

$$\begin{aligned}(2n)! &= 1 \cdot 2 \cdots (2n-2) \cdot (2n-1) \cdot (2n)\\ &= (2n-2)!(2n-1) \cdot (2n) = (2(n-1))! \cdot (2n-1) \cdot (2n).\end{aligned}$$

When one later applies the method of power series in the solution of differential equations, one often makes errors in these formulæ, where there is a factor 2 (or in general $\neq 1$) in front of n. $\Diamond$

2) *The binomial coefficients* $\left(\begin{array}{c}\alpha\\ n\end{array}\right)$.

These were introduced in e.g. *Calculus 1a, Functions of one Variable*, by

$$\left(\begin{array}{c}\alpha\\ n\end{array}\right)=\frac{\alpha\cdot(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n}\qquad\text{for } \alpha\in\mathbb{R},\quad n\in\mathbb{N},$$

i.e. we have n factors in both the numerator and the denominator. Notice that the sum of a "column" from the numerator and the denominator is a constant.

$$(\alpha-j+1)+\qquad j\qquad=\alpha+1,\qquad j\text{-th factor},$$

numerator denominator

and that one subtracts nothing in the first factor of the numerator. Due to this displacement of the indices we only subtract $n-1$ in the n-th factor of the denominator, because $0,1,\dots,n-1$ are the n consecutive numbers, starting with 0.

Note also the *recursion formula*

$$\binom{\alpha}{n+1} = \frac{\alpha-n}{n+1}\binom{\alpha}{n},$$

which is often preferred in numerical computations.

A practical convention is the extension to $n = 0$,

$$\binom{\alpha}{0} := 1.$$

If $\alpha = p \in \mathbb{N}$ and $n \in \mathbb{N}$, then

$$\binom{p}{n} = \frac{p\cdot(p-1)\cdots(p-n+1)}{1\cdot 2\cdots n} = \begin{cases} \dfrac{p!}{n!(p-n)!} & \text{for } n \le p, \\ 0 & \text{for } n > p. \end{cases}$$

In fact, if $n > p$, and they are both natural numbers, then $j = n - p \in \mathbb{N}$, and 0 must occur as a factor in the numerator.

We emphasize

$$\binom{p}{n} = \frac{p!}{n!(p-n)!} \qquad \text{for } n \le p \text{ and } p \in \mathbb{N}, \text{ and 0 otherwise.}$$

3) *The notation* 0^0.

According to *Calculus 1a, Functions of one Variable*, 0^0 *does not make sense*. However, when we restrain ourselves to power series, we have a latent limit $x \to 0$ for x^0. Therefore, when power series are considered, we shall always use the practical convention

$$0^0 := 1,$$

even if this does not make sense in general!

The introduced conventions

$$0! := 1, \qquad \binom{\alpha}{0} := 1, \qquad 0^0 := 1,$$

correctly applied, will mean a huge relaxation in the theory of power series.

1.4 Standard power series

It is of paramount importance that one can *recognize the most elementary standard power series*. These have already been given in *Calculus 1a, Functions of One Variable*.

It will here be convenient to split them into two different groups:

a) *power like series,* i.e. *the radius of convergence* is finite (for standard series usually 1),

b) *exponential like series,* i.e. *the radius of convergence* is always ∞.

The notion of *radius of convergence* will formally be defined later. Here it is just mentioned to explain why we split the standard power series into to different classes.

1.5 Power like standard series

These are again in a natural way divided into three subgroups:

$$\text{i)}\quad \begin{cases} \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n & \text{for } |x|<1, \\[2mm] \dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n & \text{for } |x|<1, \end{cases}$$

$$\text{ii)}\quad \begin{cases} (1+x)^p = \sum_{n=0}^{p} \begin{pmatrix} p \\ n \end{pmatrix} x^n & \text{for } p \in \mathbb{N} \text{ and } x \in \mathbb{R} \qquad \text{(a polynomial)}, \\[2mm] (1+x)^\alpha = \sum_{n=0}^{\infty} \begin{pmatrix} \alpha \\ n \end{pmatrix} x^n & \text{for } |x|<1 \text{ and } \alpha \in \mathbb{R}, \end{cases}$$

$$\text{iii)}\quad \begin{cases} \ln(1+x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} x^n, & \text{for } |x|<1, \\[2mm] \operatorname{Arctan} x = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}, & \text{for } |x|<1. \end{cases}$$

Remark 1.2 It is not possible here in the calculus of real functions to explain why $\ln(1+x)$ and Arctan x are naturally put into the same subgroup. This can only be seen clearly if one also has *Complex Function Theory* at hand. Therefore, the reader just has to accept that this is a convenient fact which still cannot be explained with the means we so far have at hand. $\Diamond$

In general, a power series is notated

$$\sum_{n=0}^{\infty} a_n x^n,$$

where the index n in a_n is in accordance with the exponent n in x^n. The idea is to recognize the structure of a_n in the three cases above.

Group i) is characterized by a_n is equal to either 1 or $(-1)^n$, i.e. by a *constant*, and possibly with a changing sign.

Group ii) is characterized by a_n being a *binomial coefficient*.

Group iii) is more tricky:

- $\ln(1+x)$ is characterized by

  $$a_n = \frac{(-1)^{n-1}}{n},$$

  i.e. the index occurs only in the *denominator* supplied by a changing sign.

- Arctan x is characterized by

  $$a_{2n} = 0 \qquad \text{and } a_{2n+1} = \frac{(-1)^n}{2n+1},$$

  where only *odd* exponents occur. As with $\ln(1+x)$ the index is only occurring in the denominator.

Remark 1.3 One may wonder why

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n}\, x^{2n}$$

cannot be found on the list. The reason is the following formal calculations, which later will be proved to be true for $|x| < 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n}\, x^{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(x^2\right)^n = \frac{1}{2}\ln\left(1+x^2\right),$$

where one in the last equality substitute $y = x^2$, then use the series for $\ln(1+y)$ and finally substitute back again. $\Diamond$

1.6 Recognition of power like series

Since we still do not have a sufficient pool of examples, we can only set up a general procedure.

a) Given a power like series $\sum_{n=0}^{\infty} a_n x^n$, i.e. the radius of convergence is finite (check the radius of convergence).

b) Strip the coefficient a_n of its sign, $|a_n|$. If $|a_n|$ is a fraction of polynomials in n, we decompose after n, cf. section 1.1. Then each basic fraction, named b_n in the following, is treated separately.

c) If

b_n is:	think of:	
constant,	$\dfrac{1}{1 \pm y}$,	$|y| < 1$,
binomial coefficient,	$(1 \pm y)^{\alpha}$,	$|y| < 1$,
$\dfrac{1}{n}$,	$\ln(1 \pm y)$,	$|y| < 1$,
$\dfrac{1}{2n+1}$,	$\operatorname{Arctan} y$,	$|y| < 1$.

d) In c) we get a hint of the type of the series. Substitute y in a suitable way, expressed by x, and add, if necessary suitable powers of x [remember to divide by this outside the sum, and remember to add the additional assumption that $x \neq 0$, because one is never allowed to divide by 0.] With some luck this procedure will succeed in many cases – and in courses of Calculus in almost every case.

Remark 1.4 It should be mentioned that one cannot reduce every power like series in this way. The advantage of the theory of power series is that one by using it one can define new functions which lie beyond the elementary theory of functions from e.g. *Calculus 1a, Functions of one Variable* . In practical applications in engineering problems one can in this way design one's own functions which are convenient for the solution of a given technical problem. $\Diamond$

1.7 Exponential like standard series

These are also in a natural way divided into three subgroups:

i) $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad \exp(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n, \quad x \in \mathbb{R},$

ii) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$

iii) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$

They are all characterized by having a faculty function in the denominator.

Group i) has a_n given by $\frac{1}{n!}$, possibly supplied by a change of sign.

Group ii) contains only odd exponents, and a_{2n+1} is $\frac{1}{(2n+1)!}$, possibly supplied by a change of sign $(-1)^n$.

Group iii) contains only even exponents, and a_{2n} is $\frac{1}{(2n)!}$, possibly supplied by a change of sign.

Remark 1.5 The exponential like standard series are in some sense easier to treat than the power like ones. There is, however, a small pitfall in the case of trigonometric and hyperbolic functions, where the leap in the indices is 2. When we e.g. identify a_n in

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} a_n x^n$$

one is wrongly inclined to identify a_n by $(-1)^n/(2n+1)!$. This is of course wrong, because the index *must* follow the exponent, thus

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} \qquad \text{og} \qquad a_{2n} = 0. \qquad \Diamond$$

1.8 Recognition of exponential like series

a) Given an exponential like series $\sum_{n=0}^{\infty} a_n x^n$, i.e. the radius of convergence is infinite (check), and the faculty function occurs only in the denominator.

b) Reduce a_n in a convenient way to a sum of terms, the numerators of which are constants – possibly supplied by a factor $(-1)^n$ – and the denominators are pure faculty functions. Any such term is denoted b_n in the following.

c) If the *denominator* in b_n is:

$n!$	think of	$\exp(y)$ or $\exp(-y)$,
$(2n+1)!$	think of	$\sin y$ or $\sinh y$,
$(2n)!$	think of	$\cos y$ or $\cosh y$.

d) In c) we get a hint of the type of the series. Choose a convenient substitution of y, expressed by x. In particular be very careful by writing the correct exponent for the trigonometric and hyperbolic functions. By some small pottering this procedure is usual successful – at least in courses of Calculus.

Example 1.3 In order to illustrate the technique of introducing the auxiliary variable y, we shall here show how we can find the function which is described by the series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^n.$$

a) Since $(2n)!$ occurs in the denominator, the series is of exponential-type.

b) It is seen by an identification that $a_n = \dfrac{1}{(2n)!}$, and we are apparently ended in the pitfall mentioned in the remark on page 16.

c) According to the list $\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^n$ must be written on either of the two ways

$$\cos y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} \qquad \text{or} \qquad \cosh y = \sum_{n=0}^{\infty} \frac{1}{(2n)!} y^{2n}.$$

d) It is again seen by an identification that we have the two possibilities

$$(-1)^n y^{2n} = x^n \qquad \text{i.e.} \qquad y^{2n} = (-1)^n x^n = (-x)^n \geq 0, \qquad n \in \mathbb{N},$$

or

$$y^{2n} = x^n \geq 0, \qquad n \in \mathbb{N}.$$

The former possibility can only occur, when $x \leq 0$, and the latter possibility can only be satisfied, when $x \geq 0$. We therefore have the two cases:

i) If $x \leq 0$, we can use $y = \sqrt{-x} = \sqrt{|x|}$, so

$$f(x) = \cos y = \cos(\sqrt{-x}) \qquad \text{for } x \leq 0.$$

ii) If $x \geq 0$, we use instead $y = \sqrt{x}$, so

$$f(x) = \cosh y = \cosh(\sqrt{x}) \qquad \text{for } x \geq 0.$$

Summarizing we get

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^n = \begin{cases} \cos(\sqrt{-x}) & \text{for } x \leq 0, \\ \cosh(\sqrt{x}) & \text{for } x \geq 0, \end{cases}$$

which could not be expected, if one has never seen applications of this time before. $\Diamond$

1.9 Integration of trigonometric polynomials

Problem: Find

$$\int \sin^m x \cdot \cos^n x \, dx, \qquad m, n \in \mathbb{N}_0.$$

We shall in the following only consider one single term of the type $\sin^m x \cdot \cos^n x$, of a trigonometric polynomial, where m and $n \in \mathbb{N}_0$.

We define the *degree* of $\sin^m x \cdot \cos^n x$ as the *sum* $m + n$.

There are here two main cases what integration is concerned: Is the term of *odd* or *even* degree? These two cases are then again divided into two subcases, giving us a total of four different variants by integration of a trigonometric function of the type above:

1) The degree $m + n$ is odd.

   a) $m = 2p$ even and $n = 2q + 1$ odd.

   b) $m = 2p + 1$ odd and $n = 2q$ even.

2) The degree $m + n$ is even.

   a) $m = 2p + 1$ and $n = 2q + 1$ are both odd.

   b) $m = 2p$ and $n = 2q$ are both even.

1a) $m = 2p$ even and $n = 2q + 1$ odd.

Apply the substitution $u = \sin x$ (corresponding to $m = 2p$ even), and write

$$\cos^{2q+1} x\, dx = (1 - \sin^2 x)^q \cos x\, dx = (1 - \sin^2 x)^q\, d \sin x,$$

so

$$\int \sin^{2p} x \cdot \cos^{2q+1} x\, dx = \int \sin^{2p} x (1 - \sin^2 x)^q\, d \sin x = \int_{u=\sin x} u^{2p} \cdot (1 - u^2)^q\, du,$$

i.e. the problem is reduced to integration of a polynomial, followed by a substitution.

1b) $m = 2p + 1$ odd and $n = 2q$ even.

Apply the substitution $u = \cos x$ (corresponding to $n = 2q$ even), and write

$$\sin^{2p+1} x\, dx = (1 - \cos^2 x)^p \cos x\, dx = -(1 - \cos^2)^p\, d \cos x,$$

so

$$\int \sin^{2p+1} x \cdot \cos^{2q} x\, dx = -\int (1 - \cos^2 x)^p \cdot \cos^{2q} x\, d \cos x = -\int_{u=\cos x} (1 - u^2)^p \cdot u^{2q}\, du,$$

i.e. the problem is again reduced to integration of a polynomial followed by a substitution.
2) When the degree $m + n$ is even, the trick is to use the double angle instead as integration variable.

Here we use the formulæ

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin x \cdot \cos x = \frac{1}{2}\sin 2x.$$

2a) $m = 2p+1$ and $n = 2q+1$ are both odd.

Rewrite the integrand in the following way:

$$\sin^{2p+1} x \cdot \cos^{2q+1} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p \left\{\frac{1}{2}(1+\cos 2x)\right\}^q \cdot \frac{1}{2}\sin 2x.$$

The problem is now reduced to the case 1b), so by the substitution $u = \cos 2x$ we get

$$\int \sin^{2p+1} x \cdot \cos^{2q+1} x\, dx = -\frac{1}{2^{p+q+1}} \cdot \frac{1}{2}\int_{u=\cos 2x} (1-u)^p(1+u)^q\, du.$$

We see that the problem is again reduced to integration of a polynomial followed by a substitution.

2b) $m = 2p$ and $n = 2q$ are both even.

This is the most difficult case. First rewrite the integrand in the following way:

$$\sin^{2p} x \cdot \cos^{2q} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p \left\{\frac{1}{2}(1+\cos 2x)\right\}^q.$$

We see that on the left hand side the degree is $2p+2q$ in $(\cos x, \sin x)$, while the degree is halved on the right hand side to $p+q$ in $(\cos 2x, \sin 2x)$, i.e. we now use the double angle. On the other hand we are forced to replace one single term by many terms, which now must be treated separately.

Since we halve the degree, whenever 2b) is applied and since the other cases can be calculated immediately, the problem can be solved in a finite number of steps.

Example 1.4 Let us calculate the integral

$$\int \cos^6 x\, dx.$$

The degree $0+6=6$ is even, and both $m=0$ and $n=6$ are even. Hence, we are in case 2b). When we switch to the double angle we get the following calculation of the integrand

$$\cos^6 x = \left\{\frac{1}{2}(1+\cos 2x)\right\}^3 = \frac{1}{8}(1+3\cos 2x+3\cos^2 2x+\cos^3 2x).$$

Integrations of the first two terms are straightforward:

$$\frac{1}{8}\int (1+3\cos 2x)\, dx = \frac{1}{8}x + \frac{3}{16}\sin 2x.$$

The third term is again of type 2b), so we have to double the angle again,

$$\frac{1}{8}\int 3\cos^2 2x\, dx = \frac{3}{8}\int \frac{1}{2}(1+\cos 4x)\, dx = \frac{3}{16}x + \frac{3}{64}\sin 4x.$$

The last term is of the type 1a), so

$$\frac{1}{8}\int \cos^3 2x\,dx = \frac{1}{8}\int (1-\sin^2 2x)\cdot\frac{1}{2}\,d\sin 2x = \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x.$$

Summarizing we get after a reduction

$$\int \cos^6 x\,dx = \frac{5}{16}\,x + \frac{1}{4}\,\sin 2x - \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x. \qquad \Diamond$$

1.10 Use of pocket calculators

The use of pocket calculators will usually be admitted; but they may be dangerous to use on series. The reason is that there are still lacking a lot of recognizable series in the memory of the pocket calculator (the known series are typically the standard series in the previous section, and no more). If one e.g. type in

$$\sum(\cdots, n, 1, \infty)$$

on a TI-92 or TI-89 or HP-48, one of the following three events will occur:

1) We are so lucky that the pocket calculator actually recognizes the series. Then we get the right answer, but since the pocket calculator typically only knows the standard series above we might as well have used tables instead. This is, however, a minor point.

2) The pocket calculator does not recognize the series and it chooses to stop. The pocket calculator is rescued, but we have not obtained the desired solution.

3) (Worst case!) The pocket calculator does not recognize the series, but continues its calculations! I have once myself in a test experienced this phenomenon, where the calculations did not stop, until I had removed all the batteries (including the backup battery). All my information was lost, but I rescued the pocket calculator. This test was provoked by one of my students who did not know how to stop the calculations. The guarantee had to give him a new pocket calculator, and he was an experience richer!

It is always one's own responsibility if one relies on results from pocket calculators. These also contains errors. For instance the older versions of TI-92 and TI-89 will give wrong results by calculating

$$\int_{-2}^{x} \frac{1}{\sqrt{t^2-1}}\,dt \qquad \text{for } x < -1,$$

because they were simply missing some numerical signs in their catalogues of standard functions. This has been reported back to Texas Instruments, so I guess that at least this error does not exist any more

Finally, I have also found wrong results in earlier versions of MAPLE in integrals like e.g.

$$\int_0^{\pi/2} \frac{1}{1+\tan^{\alpha} x}\, dx = \frac{\pi}{4}, \qquad \alpha \in \mathbb{R},$$

when $\alpha = 1/2$, $3/2$, $5/2$, etc., and there are continuously found new examples. In some cases I have found some other more advanced examples where even MAPLE cannot give the right answer without a very active help of the applier. Hence,

> Never trust blindly a result found by a pocket calculator or by MAPLE or Mathematica. These utilities also contain errors.
>
> On the other hand, since they exist, they should also be used, but do not forget to use your brain as well!

2 Real sequences, folklore

In this short chapter we present some "dirty tricks" which may be useful when solving problems with simple sequences.

2.1 Rules of magnitude

From *Calculus 1a, Functions of one Variable* we already know that

$$\frac{\ln x}{x^{\alpha}} \to 0 \quad \text{for } x \to \infty, \qquad \text{when } \alpha > 0;$$

$$x \cdot \ln x \to 0 \quad \text{for } x \to 0+,$$

(a power function "dominates" any logarithm).

$$\frac{x^{\alpha}}{a^{x}} \to 0 \quad \text{for } x \to \infty, \qquad \text{when } \alpha > 0 \text{ and } a > 1,$$

(an exponential "dominates" any power function).

We here add

$$\frac{a^n}{n!} \to 0 \qquad \text{for } n \to \infty, \qquad \text{when } a > 0,$$

(a faculty function "dominates" any exponential),

Proof. By choosing $N \geq 2a$, it is easily seen that for $p \in \mathbb{N}$ and $p \to \infty$ we have

$$|a_{N+p} - 0| = a_{N+p} = \frac{a \cdot a \cdots a \cdot a \cdots a}{1 \cdot 2 \cdots N \cdot (N+1) \cdots (N+p)} \leq a_N \cdot \left(\frac{1}{2}\right)^p \to 0.$$

Furthermore,

$$\frac{n!}{n^n} \to 0 \qquad \text{for } n \to \infty.$$

Just modify the proof above,

2.2 Square roots etc.

Problem: Assume that $a_n \to \infty$. How can we estimate expressions like

$$\sqrt{a_{n+1}} - \sqrt{a_n},$$

where the type of convergence is "$\infty - \infty$"? Cf. page 26.

Rewrite the difference in the following way:

$$\sqrt{a_{n+1}} - \sqrt{a_n} = \frac{\left(\sqrt{a_{n+1}} - \sqrt{a_n}\right) \cdot \left(\sqrt{a_{n+1}} + \sqrt{a_n}\right)}{\sqrt{a_{n+1}} + \sqrt{a_n}} = \frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}},$$

and proceed with the right hand side (rules of calculation etc.).

This method can be extended. We have for instance

$$\sqrt[3]{a_{n+1}} - \sqrt[3]{a_n} = \frac{a_{n+1} - a_n}{\left(\sqrt[3]{a_{n+1}}\right)^2 + \sqrt[3]{a_{n+1}} \cdot \sqrt[3]{a_n} + \left(\sqrt[3]{a_n}\right)^2},$$

which follows immediately by a multiplication by the denominator.

2.3 Taylor's formula

We let $\varepsilon(x) \to 0$ for $x \to 0$ denote some unspecified function, which tends towards 0 for x tending towards 0. From *Calculus 1a, Functions of one Variable*, we retrieve the following important expansions of first order:

$\frac{1}{1-x} = 1 + x + x \cdot \varepsilon(x)$, $\qquad \frac{1}{1+x} = 1 - x + x \cdot \varepsilon(x)$,

$(1+x)^\alpha = 1 + \alpha x + x \cdot \varepsilon(x)$, $\qquad \sqrt{1+x} = 1 + \frac{1}{2}x + x \cdot \varepsilon(x)$,

$\ln(1+x) = x + x \cdot \varepsilon(x)$, $\qquad \operatorname{Arctan} x = x + x^2\varepsilon(x)$,

$\exp(x) = 1 + x + x \cdot \varepsilon(x)$, $\qquad \exp(-x) = 1 - x + x \cdot \varepsilon(x)$,

$\sin x = x + x^2\varepsilon(x)$, $\qquad \cos x = 1 + x \cdot \varepsilon(x)$,

$\sinh x = x + x^2\varepsilon(x)$, $\qquad \cosh x = 1 + x \cdot \varepsilon(x)$.

These are the most common cases, but some expansions of higher order may occur, cf. the following examples. They are typically applied by a first approximation.

Example 2.1 a) We get from $\sin x = x + x^2\varepsilon(x)$ that

$$\frac{\sin x}{x} = \frac{x + x^2\varepsilon(x)}{x} = 1 + x \cdot \varepsilon(x) \to 1 \qquad \text{for } x \to 0.$$

A variant for sequences is obtained by replacing x by $\frac{1}{n}$ for $n \to \infty$:

$$n \cdot \sin \frac{1}{n} = 1 + \frac{1}{n}\varepsilon\left(\frac{1}{n}\right) \to 1 \qquad \text{for } n \to \infty.$$

b) When the same first order approximation is used on $(\sin x - x)/x^3$, we get

$$\frac{\sin x - x}{x^3} = \frac{x + x^2\varepsilon(x) - x}{x^3} = \frac{\varepsilon(x)}{x},$$

which is of the type "0/0" for $x \to 0$. The solution here is to expand $\sin x$ to a higher order

$$\sin x = x - \frac{1}{3!}x^3 + x^4\varepsilon(x) = x - \frac{1}{6}x^3 + x^4\varepsilon(x).$$

Let us try again,

$$\frac{\sin x - x}{x^3} = \frac{x - \frac{1}{6}x^3 + x^4\varepsilon(x) - x}{x^3} = -\frac{1}{6} + x \cdot \varepsilon(x) \to -\frac{1}{6} \quad \text{for } x \to 0,$$

and this time we succeed.

c) In order to find the order of expansion in general, always start by finding the order of the roots of the *denominator*. This is 1 in a), because $x = x^1$, and it is 3 in b). We shall now consider

$$\frac{\cos x - 1 + \frac{1}{2}x^2}{\cos(x^2) - 1} \qquad \text{for } x \to 0.$$

If we put $y = x^2$, we get the *denominator*

$$\cos(x^2) - 1 = \cos y - 1 = 1 - \frac{1}{2}y^2 + y^2\varepsilon(y) - 1 = -\frac{1}{2}y^2 + y^2\varepsilon(y) = -\frac{1}{2}x^4 + x^4\varepsilon(x),$$

because both $\varepsilon(y)$ and $\varepsilon(x)$ tend towards 0 for $x \to 0$, since $y = x^2$.

The reduction shows that the denominator has a root of order 4 in $x = 0$. Consequently, the *numerator* shall also be expanded up to the fourth order. Since

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + x^5\varepsilon(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + x^5\varepsilon(x),$$

we get for the numerator

$$\cos x - 1 + \frac{1}{2}x^2 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + x^5\varepsilon(x) - 1 + \frac{1}{2}x^2 = \frac{1}{24}x^4 + x^5\varepsilon(x),$$

hence by insertion

$$\frac{\cos x + \frac{1}{2}x^2}{\cos(x^2) - 1} = \frac{\frac{1}{24}x^4 + x^5\varepsilon(x)}{-\frac{1}{2}x^4 + x^4\varepsilon(x)} = \frac{\frac{1}{24} + x\varepsilon(x)}{-\frac{1}{2} + \varepsilon(x)} \to \frac{\frac{1}{24}}{-\frac{1}{2}} = -\frac{1}{12} \text{ for } x \to 0.$$

A variant for sequences is obtained, when x is replaced by $\frac{1}{n}$ for $n \to \infty$:

$$\frac{\cos\left(\frac{1}{n}\right) - 1 + \frac{1}{2}\frac{1}{n^2}}{\cos\left(\frac{1}{n^2}\right) - 1} \to -\frac{1}{12} \text{ for } n \to \infty. \qquad \Diamond$$

In the applications we have more typically a quotient like $\dfrac{f(n)}{g(n)}$ for $n \to \infty$. This type of problem is transformed back to Taylor's formula by the substitution $n = 1/x$, i.e. $x = 1/n \to 0$ for $n \to \infty$.

2.4 Standard sequences

It should in general be allowed to refer to the following standard sequences without any proof:

$\dfrac{1}{n} \to 0$ for $n \to \infty$, $\qquad \dfrac{1}{n^\alpha} \to 0$ for $n \to \infty$ og $\alpha > 0$,

$\sqrt[n]{n} \to 1 \qquad$ for $n \to \infty$,

$\left(1 + \dfrac{1}{n}\right)^n \to e$ for $n \to \infty$, $\qquad \left(1 + \dfrac{a}{n}\right)^n \to e^a$ for $n \to \infty, a \in \mathbb{R}$,

$(-1)^{n+1}$ is divergent, though bounded,

$1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$ is divergent, and unbounded.

3 Real sequences; description of procedures for different types of problems

In this chapter we suggest some methods for solving problems containing sequences. The list of methods is of course far from complete, if such a list does exist at all.

3.1 Sequences

1) Given a sequence (a_n) and a possible limit $a \in \mathbb{R}$. Show that $a_n \to a$ for $n \to \infty$.

   a) Show directly that $|a - a_n| \to 0$ for $n \to \infty$.

   Example 3.1 Show that $a_n = \dfrac{n}{n+1}$ converges towards $a = 1$:

$$|a - a_n| = \left|1 - \frac{n}{n+1}\right| = \left|\frac{(n+1)-n}{n+1}\right| = \frac{1}{n+1} \to 0 \quad \text{for } n \to \infty. \qquad \Diamond$$

b) Split the sequence (a_n) by using the rules of calculations into subsequences and exploit the convergence of each of these.

Example 3.2 Show that $a_n = \dfrac{n}{n+1} + (-1)^n \cdot 2^{-n}$ converges towards $a = 1$.

We write the sequence in the form $a_n = b_n + c_n$, where $b_n = \dfrac{n}{n+1} \to 1$ for $n \to \infty$ due to the example above, and $c_n = (-1)^n \cdot 2^{-n} \to 0$ for $n \to \infty$, because

$$|0 - c_n| = \frac{1}{2^n} \to 0 \qquad \text{for } n \to \infty. \qquad \Diamond$$

2) Given a sequence (a_n) with no hint of a prescribed limit. Examine whether (a_n) is convergent or divergent.

a) Check if there is some obvious candidate for a *limit*. If so, go back to 1) above.

b) Split (a_n) by means of the rules of calculations into subsequences like e.g.

$$b_n \pm c_n, \qquad b_n \cdot c_n, \qquad \frac{b_n}{c_n},$$

and combinations of these possibilities. We treat separately each of the subsequences. If they are all convergent, and no denominator is 0, then the convergence follows by the rules of calculations.

Example 3.3 For

$$a_n = \frac{\cos\left(\dfrac{1}{n}\right)}{\cosh\left(\dfrac{1}{n}\right)}$$

we put

$$b_n = \cos\left(\frac{1}{n}\right) \to \cos 0 = 1, \quad c_n = \cosh\left(\frac{1}{n}\right) \to \cosh 0 = 1 \qquad \text{for } n \to \infty,$$

hence,

$$a_n = \frac{b_n}{c_n} \to \frac{b}{c} = \frac{1}{1} = 1 \qquad \text{for } n \to \infty. \qquad \Diamond$$

By the splitting in b) there will often occur some illegal *types of convergence* like

$$\infty - \infty, \qquad 0 \cdot \infty, \qquad \frac{\infty}{\infty}, \quad \text{eller} \quad \frac{0}{0}.$$

We shall deal with these types in the following.

c) c) *Type* $\infty - \infty$, i.e. $a_n = b_n - c_n$, where $b_n \to \infty$ and $c_n \to \infty$ for $n \to \infty$.

Rewrite the sequence by e.g. putting everything on the same fraction line or by using that

$$b_n - c_n = \frac{b_n^2 - c_n^2}{b_n + c_n}, \qquad \text{or similarly.}$$

There are of course here many variants.

Example 3.4 Let $a_n = \sqrt{n^2+n+1} - n$. Put $b_n = \sqrt{n^2+n+1}$ and $c_n = n$. Then $b_n \to \infty$ and $c_n \to \infty$ for $n \to \infty$. Rewrite the expression like above,

$$\begin{aligned} a_n &= b_n - c_n = \frac{b_n^2 - c_n^2}{b_n + c_n} = \frac{(n^2+n+1)-n^2}{\sqrt{n^2+n+1}+n} = \frac{n+1}{n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1\right)} \\ &= \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}} \to \frac{1}{1+1} = \frac{1}{2} \qquad \text{for } n \to \infty, \end{aligned}$$

because

$$\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} \to \sqrt{1+0+0} = 1 \qquad \text{for } n \to \infty. \qquad \Diamond$$

d) *Type* $0 \cdot \infty$, i.e. $a_n = b_n \cdot c_n$, where $b_n \to 0$ and $c_n \to \infty$ for $n \to \infty$.

Here we have several variants:

i) Introduce some $d_n \to \infty$ for $n \to \infty$, where c_n/d_n is convergent, and where d_n is simpler than c_n. Then consider

$$a_n = (b_n \cdot d_n) \cdot \left(\frac{c_n}{d_n}\right).$$

Example 3.5 Let $a_n = \sin\left(\frac{1}{n}\right) \cdot \sqrt{n^2+1}$. Then $b_n = \sin\left(\frac{1}{n}\right) \to 0$ and $c_n = \sqrt{n^2+1} \to \infty$ for $n \to \infty$. Choose the simpler sequence $d_n = n \to \infty$. Then

$$\begin{aligned} a_n &= (d_n \cdot b_n) \cdot \left(\frac{c_n}{d_n}\right) = \left(n \cdot \sin\left(\frac{1}{n}\right)\right) \cdot \frac{\sqrt{n^2+1}}{n} \\ &= \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \cdot \sqrt{1+\frac{1}{n^2}} \to 1 \cdot 1 = 1 \end{aligned}$$

for $n \to \infty$, because $\frac{\sin x}{x} \to 1$ for $x = \frac{1}{n} \to 0$ and $\sqrt{1+\frac{1}{n^2}} \to 1$ for $n \to \infty$. $\Diamond$

ii) If $b_n \to 0+$, then $\frac{1}{b_n} \to \infty$. In this case we rewrite the expression in the following way,

$$a_n = b_n \cdot c_n = \frac{c_n}{\left(\frac{1}{b_n}\right)},$$

which is of the type $\frac{\infty}{\infty}$. Then contain like in med e).

e) *Type* $\frac{\infty}{\infty}$, i.e. $a_n = b_n/c_n$, where $b_n \to \infty$ and $c_n \to \infty$ [or $-\infty$].

Isolate the dominating terms in both the numerator and the denominator, and divide by the dominating term of the denominator.

Example 3.6 Let

$$a_n = \frac{n^3+2n^2-7n-20}{n^4+173n^3+1135}.$$

If we put $b_n = n^3+2n^2-7n-20$ and $c_n = n^4+173n^3+1135$ (> 0 for all n, so we never divide by 0), we see that $b_n \to \infty$ and $c_n \to \infty$ for $n \to \infty$. The dominating term in b_n is n^3, and the dominating term in c_n is n^4. Rewrite in the following way,

$$a_n = \frac{n^3+2n^2-7n-20}{n^4+173n^3+1135} = \frac{n^3\left\{1+\frac{2}{n}-\frac{7}{n^2}-\frac{20}{n^3}\right\}}{n^4\left\{1+\frac{173}{n}+\frac{1135}{n^4}\right\}}$$

$$= \frac{1}{n}\cdot\frac{1+\frac{2}{n}-\frac{7}{n^2}-\frac{20}{n^3}}{1+\frac{173}{n}+\frac{1135}{n^4}} \to 0\cdot\frac{1+0-0-0}{1+0+0} = 0 \qquad \text{for } n \to \infty. \qquad \Diamond$$

f) *Type* $\frac{0}{0}$, i.e. $a_n = b_n/c_n$, where $b_n \to 0$ and $c_n \to 0$ for $n \to \infty$.

In this case one will usually apply Taylor's formula.

Example 3.7 Let

$$a_n = \frac{\operatorname{Arctan}\frac{1}{n}}{\tan\frac{1}{n}} = \cos\left(\frac{1}{n}\right)\cdot\frac{\operatorname{Arctan}\frac{1}{n}}{\sin\frac{1}{n}}.$$

Here $\cos\frac{1}{n} \to 1$ for $n \to \infty$, so this factor does no harm. When we write $x = \frac{1}{n} \to 0$ for $n \to \infty$, we get by Taylor's formula, that

$$\frac{\operatorname{Arctan}\frac{1}{n}}{\sin\frac{1}{n}} = \frac{\operatorname{Arctan} x}{\sin x} = \frac{x+x^2\varepsilon(x)}{x+x^2\varepsilon(x)} = \frac{1+x\varepsilon(x)}{1+x\varepsilon(x)} \to 1 \quad \text{for } n \to \infty.$$

Using the rules of calculations we get that $a_n \to 1\cdot 1 = 1$ for $n \to \infty$. $\Diamond$

g) In some malicious cases it is not possible to find the exact value of the limit. In such cases it is *extremely important* first to prove the convergence, by e.g. monotonous convergence.

i) Show that the sequence is weakly increasing (decreasing).

ii) Show that the sequence is bounded from above (from below).

It is first after this analysis that one can trust a limit found on e.g. a pocket calculator. (Be careful here, because experience has shown that in some cases the pocket calculator cannot be stopped again!)

Example 3.8 It is easily proved that

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is increasing and unbounded i.e. divergent. If one dares to risk one's pocket calculator (check the guarantee in advance!) and type in a_n, and then let the pocket calculator perform the limit $n \to \infty$, we can expect one of the following three things to happen:

i) The pocket calculator recognizes the sequence, which nevertheless may be considered to be very unlikely.

ii) The most likely situation is that the pocket calculator just cannot be stopped. If e.g. the rounding off is 10^{-14}, then the pocket calculator must add 10^{14} numbers in even the most reasonable setup. We have had some bad experiences, where the command could not be stopped.

iii) Finally, if the pocket calculator is able to give a finite number as the result, we may also have the problem that due to the rounding off, a_n will be considered as a constant for $n \geq N$, and every element in the sequence, hence also a_N, is finite. But due to the rounding off this cannot be the true answer. $\Diamond$

h) Sequences of the structure of segments like e.g.

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

(sum of n terms following following some pattern) can in some cases be considered as *means for some integral*. The limit of a_n is then the value of this integral. However, students of today will probably not use this method by their first encounter with the theory of convergent series.

3.2 Iterative sequences

When we are concerned with practical solutions of mathematical models, sequences of this type are very important because the nature in general does not behave in such a way that calculus taught in high schools or in the first years at universities will suffice. In the applications there are especially three types of problems which are of interest:

1) Solution of a transcendent equation like

$$f_1(x) = 0 \qquad \text{or} \qquad f_2(x) = x,$$

where $f_1(x)$ and $f_2(x)$ are given *continuous* functions.

2) Solutions of linear difference equations like e.g.

$$x_{n+2} + a_{n+1}x_{n+1} + b_n x_n = 0, \qquad n \in \mathbb{N}_0,$$

where (a_n) and (b_n) are given sequences.

3) Solution of linear differential equations with variable coefficients.

The linear difference equations can in particular be found in connection with solution of linear differential equations of second order by the method of power series, a section which we refer to in this short description. They may also occur in other technical applications, like e.g. in theoretical telecommunications.

Remark 3.1 When some linear differential equation with variable coefficients cannot be solved by the method of power series it may still be solved by iteration. The idea is very simple, though no longer in the usual examination requirements at the universities: A hint is to take an outdated textbook in mathematics, in which the *proof* of the existence and uniqueness theorem for linear differential equations still can be found. Rewrite this proof for e.g. MAPLE. It is remarkable that an old proof which a long time ago sunk into oblivion now again can be used with the new advanced computer programs at hand. $\Diamond$

The proofs of the first and third case above rely on the important *fix point theorem*. This theorem will be treated more thoroughly in the next chapter. It remains the solution of a *transcendent equation* like e.g.

$$f_1(x) = 0 \qquad \text{or} \qquad f_2(x) = x,$$

where we assume that f_1 and f_2 are *continuous*. In the former equation we shall find the real zeros of $f(x)$, and in the latter equation we shall find fix points of the function f, if zeros or fix points do exist. The two problems are obviously equivalent,

$$g_1(x) := f_1(x) + x = x \qquad \text{or} \qquad g_2(x) := f_2(x) - x = 0.$$

The fix point version is rewritten as an iterative sequence by

$$a_{n+1} = f(a_n), \qquad n \in \mathbb{N}_0,$$

where a_0 is some prescribed initial value which we to some extent can choose ourselves.

Analysis. Draw the graphs of $y = f(x)$ and $y = x$, respectively. The fix points are the intersection points. Use the figure to find a suitable initial value a_0 for the iterative process.

We get a variant if the *iterative sequence* is given by the equation $a_{n+1} = f(a_n)$, where $f(x)$ is a continuous function. In that case we argue in the following way:

If (a_n) is convergent with the limit a, then we get by taking the limit in the recursion formula, that

$$\lim a_{n+1} = f(\lim a_n), \qquad \text{i.e.} \qquad a = f(a),$$

where the continuity of f assures that one can interchange the function and the limiting process. Then we solve the equation $f(a) = a$ in order to find the *possible* limits a. (We may get more possibilities, where some of them do not have to be the correct limit).

Check of convergence. When we have found the possible limits a, we must not forget also to *prove* that the sequence is convergent. We shall typically show that

$$|a - a_n| \to 0 \qquad \text{for } n \to \infty.$$

Here we can use the methods, which have been sketched in the previous section in this chapter.

3.3 Sequences of functions

The concept of a sequence of functions is derived from the concept of sequence of numbers, so it is quite reasonable to treat problems of sequences of functions right after the treatment of sequences of numbers. It will be shown later in the text why the sequences of functions are so important. The first problem follows quite naturally:

Problem. Given a sequence of functions $(f_n(x))$, $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$.

Examine whether $(f_n(x))$ is pointwise convergent or not.

Choose any fixed $x \in I$. Check whether the *sequence of numbers* $(f_n(x))$ is convergent or divergent, cf. 1) and 2) in section 3.1..

Since $x \in I$ usually is considered as a variable, it may be helpful in the beginning of the learning process temporarily to put $x = a$ and then examine the *sequence of numbers* $(f_n(a))$ instead, because a traditionally is considered as a constant. By the end of such a course in calculus this trick should no longer be necessary.

Problem. Given a sequence of functions $(f_n(x))$, $f_n : I \to \mathbb{R}$, $n \in \mathbb{N}$.

Examine whether (f_n) is uniformly convergent or not.

a) First check if all the functions $f_n(x)$ are continuous. They usually are, but in principal they do not have to be continuous. When this is not the case one has to start from the very beginning only using the definition, and one cannot proceed with b) etc. below.

b) Check if $(f_n(x))$ is pointwise convergent.

1) If 'no', then the sequence is neither pointwisely nor uniformly convergent.
2) If 'yes', find the pointwise limit function $f(x)$, and proceed with c) below.

c) If the pointwise limit function $f(x)$ is *not* continuous, it follows from a) and a theorem in the next chapter that $(f_n(x))$ is *not* uniformly convergent.

If instead $f(x)$ is continuous proceed with d) below.

d) *Remove the variable* x by an estimate like

$$|f_n(x) - f(x)| \le \sup_{x \in I} |f_n(x) - f(x)| \le a_n, \qquad n \in \mathbb{N},$$

where a_n is *not depnding of* x.

If $a_n \to 0$ for $n \to \infty$, then (f_n) is *uniformly convergent* with the limit function $f(x)$.

The method of examination of *uniform convergence* may be expressed in a flow diagram. See the next page.

Remark 3.2 In the real life one may obtain the last possibility. However, exercises in calculus courses are always constructed in such a way that if one does not make errors in one's calculations, then one will always end up in one of the three boxes to the right, and the question of uniform convergence will at the same time have been answered. $\Diamond$

Interchange of the limit process and the integration.

a) Show that $(f_n) \to f$ *uniformly*, cf. the above.

b) Refer to some suitable theorem.

c) Interchange the limit process and the integration,

$$\lim_{n\to\infty} \int_I f_n(x)\,dx = \int_I \lim_{n\to\infty} f_n(x)\,dx = \int_I f(x)\,dx.$$

Remark 3.3 Point c) must *not* be applied, if we only have got pointwise (and *not* uniform) convergence, $f_n(x) \to f(x)$ for $n \to \infty$. Instead one has to calculate $a_n = \int_I f_n(x)\,dx$ and check if (a_n) is convergent or not, cf. 2), because (a_n) is a *sequence of numbers* and not a sequence of functions. $\Diamond$

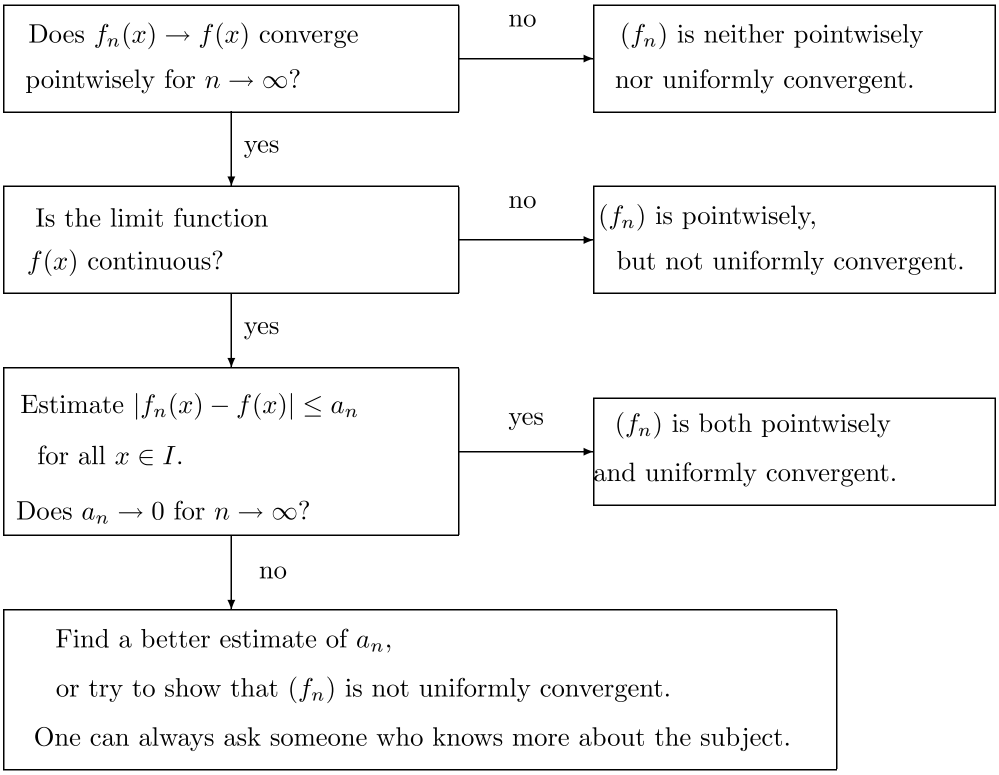
Interchange of the limit process and the differentiation.

Let (f_n), $f_n : I \to \mathbb{R}$, be a sequence of differentiable functions with *continuous* derivatives f_n'. (Check!)

a) Check if $f_n(x) \to f(x)$ *pointwisely* for $n \to \infty$, cf. above.

If 'no', we cannot interchange the two processes.

If 'yes', then $f(x)$ exists, so we can proceed with b).



Figur 1: Flow diagram for uniform convergency, assuming that all $f_n(x)$ are *continuous*.

b) Check if the *derivatives* $f_n'(x) \to g(x)$ *converge uniformly* for $n \to \infty$, cf. above.

If 'yes', it follows from some theorem in the text book (quote it!) that

$$f'(x) = g(x) = \lim_{n \to \infty} f_n'(x).$$

If 'no', we have got a problem, which needs some rethinking.

Remark 3.4 One should in elementary courses on sequences *never* get the answer 'no' in b). If one does (still in elementary courses), one must have made an error. (Find it!) In real life one can easily get 'no' in b), because the nature is here dictating the mathematics, and not the other way round. Fortunately there exist some more advanced mathematical disciplines, which can solve the problem. $\Diamond$

4 General series; tricks and methods in solutions of problems

4.1 Definition

The concept of a series is *derived* from the concept of sequences (cf. the previous chapters) and they must not be confused. One should therefore notice the difference between the definition below and the definition of a sequence.

By a *series* we shall understand a symbol like $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=N}^{\infty} a_n$ or similarly, where one imagine that all the numbers a_n have been added for n in the index set $\{1, 2, \dots\}$ or $\{N, N+1, \dots\}$, or similarly, given by the summation sign in the prescribed order.

In order to be able to use series in our calculations we need a new concept, namely *convergence*.

Definition 4.1 *We say that a series $\sum_{n=1}^{\infty} a_n$ is convergent (divergent), if and only if the corresponding sequence of segments*

$$s_n := a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is convergent (divergent).

In case of convergence the series is given the sum

$$s = \sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Remark 4.1 In words, this means that the meaningless concept of an infinite sum is traced back to finite sums where one adds more and more terms, until the index set has been exhausted. We are in this interpretation closely bound to the fact that the index set should be set up in a sequence – one says that the index set is ordered. ♢

Contrary to so many other mathematical definitions it is actually possible from time to time to use definition 4.1 in practical problems. More of this later.

4.2 Rules of calculus

We have only got rules of calculus for *convergent* series, and even these may be very treacherous. One should in particular be very careful when one applies the following one:

Theorem 4.1 *Assume that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series (check!) with the same summation index set (check!), and α and β are two real numbers. Then*

$$\alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\alpha a_n + \beta b_n).$$

Remark 4.2 (Important). This equality sign is *not* symmetric! The right hand side may be convergent, while neither of the two series on the left hand side is convergent!

If e.g. $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, and $\alpha = \beta = 1$, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent, while

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \sum_{n=1}^{\infty} 0 = 0 \qquad \text{is convergent!} \qquad \diamondsuit$$

Later in the applications we shall often use this rule *from the right towards the left*, i.e. the *illegal way!* In that case one shall always check afterwards if the two subseries $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ indeed are convergent. $\Diamond$

Remark 4.3 There is *no similar rule* for the product:

$$\sum_{n=1}^{\infty} a_n b_n \text{ is } \textit{not} \text{ equal to } \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n,$$

a statement which unfortunately is often seen at examinations. Of course one cannot just plug in summation signs at pleasure in front of factors of a product! $\Diamond$

4.3 Change of index

If the domains of summation for two convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=p}^{\infty} b_n$ are *not* the same, then we cannot directly apply the rules of calculus. We must first perform a change of indices on one of the two index sets. In the example above it will be quite natural to change the index of $\sum_{n=p}^{\infty} b_n$, such that the summation starts by $n = 0$. Notice, however, that one can easily get into that situation where it would be more convenient to change the summation domain in $\sum_{n=0}^{\infty} a_n$ to start from $n = p$. There is here no general rule.

The procedure is the following:

a) Introduce a new variable m by $n = m + p$. Then $m = 0$ for $n = p$, and $m = 1$ for $n = p + 1$, etc. Hence we get by the substitution

$$\sum_{n=p}^{\infty} b_n = \sum_{m=0}^{\infty} b_{m+p}.$$

As a *check* one examines the first term in the two series: The first term on the left hand side is b_p, and the first term on the right hand side is $b_{0+p} = b_p$, so the two series contain the same terms as previously, this time arranged in the same order.

b) Delete one of the arcs in the letter "m", so that one again writes "n",

$$\sum_{n=p}^{\infty} b_n = \sum_{m=0}^{\infty} b_{m+p} = \sum_{n=0}^{\infty} b_{n+p}.$$

In the example in the beginning we therefore get

$$\underbrace{\sum_{n=0}^{\infty} a_n + \sum_{n=p}^{\infty} b_n}_{\text{different summation sets}} = \underbrace{\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_{n+p}}_{\text{same summation set}} = \sum_{n=0}^{\infty} \{a_n + b_{n+p}\},$$

assuming that both series are convergent.

Warnings of the transformation: If an index is multiplied by a constant, which often occurs in connection with a *power series*, then one must be very careful with step a) above. We have for instance

$$\sum_{n=1}^{\infty} a_{\underbrace{2n}_{\text{even}}} = \sum_{m=0}^{\infty} a_{2(m+1)} = \sum_{m=0}^{\infty} a_{2m+2} = \sum_{n=0}^{\infty} a_{2n+2} \neq \sum_{n=0}^{\infty} a_{\underbrace{2n+1}_{\text{odd}}},$$

where the following wrong variant is often seen at examinations

$$\sum_{n=1}^{\infty} \frac{1}{(2n)!}\, a_{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n+2)!}\, a_{2n+2} \neq \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, a_{2n+1}. \qquad \Diamond$$

4.4 A general advice

Never use the General Principle of Convergency in practical problems! The *General Principle of Convergency* for series is very important in *theoretical considerations*, but not at all in practice. And even if one should be so lucky to use it correctly, there will later be shown some other criteria which will give the same result in a much easier way with much more information.

4.5 Elementary standard series

We collect here some standard series for later reference.

1) *A quotient series*

If $-1 < k < 1$, then

$$\sum_{n=0}^{\infty} k^n = 1 + k + k^2 + \cdots + k^n + \cdots = \frac{1}{1-k}$$

where we have the variant

$$\sum_{n=1}^{\infty} k^n = k \sum_{n=0}^{\infty} k^n = \frac{k}{1-k}.$$

The quotient series is *coarsely divergent* for $|k| \geq 1$.

Notice that the *sequence of segments* s_n for every $k \in \mathbb{C}$ is given by

$$s_n = \sum_{j=0}^{n} k^j = 1 + k + \cdots + k^n = \begin{cases} \dfrac{k^{n+1}-1}{k-1} & \text{for } k \neq 1, \\[2mm] n+1 & \text{for } k = 1. \end{cases}$$

This sequence of segments occur fairly often in problems. When $k \neq 1$, we get the result by multiplying by $k-1$.

2) *The harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \textit{is divergent!}$$

Remark 4.4 The traditional wrong argument here is the following: "Since $a_n = 1/n \to 0$ for $n \to \infty$, the series must be convergent, because it satisfies the necessary condition of convergence." The necessary condition of convergence is unfortunately *not* sufficient in this case. $\Diamond$

3) *The alternating harmonic series* is convergent with the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = \ln 2.$$

4) *slowly convergent/divergent series.*

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}, \quad \sum_{n=2}^{\infty} \frac{1}{n} \cdot \frac{1}{(\ln n)^{\alpha}}, \quad \sum_{n=3}^{\infty} \frac{1}{n \cdot \ln n} \cdot \frac{1}{(\ln \ln n)^{\alpha}}, \quad \text{etc.}$$

are (slowly) convergent for $\alpha > 1$ and (slowly) divergent for $\alpha \leq 1$.

We here say that the convergence/divergence is *slow*, in order to express that we have absolutely no chance in calculating the value of the series by using pocket calculators or MAPLE. For instance, if we let a pocket calculator add the first 10^6 terms of the (divergent) harmonic series above, it is easy by applying the integral test that the sum has barely passed 21, so one would be tempted wrongly to conclude that the series is convergent. (Intuitively 21 lies "very far away" from infinity). The importance of these slowly convergent series does not lie in their explicit sums, but in the fact that they can be used in the *comparison test* in order to decide if another series is convergent or divergent.

5) *Important special cases.*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

4.6 Types of Convergence

1) *Coarse divergence.*

   We say that the series is *coarsely divergent* if the *necessary* condition for convergence is *not* satisfied, i.e. if

   $$a_n \nrightarrow 0 \qquad \text{for } n \to \infty.$$

   Although this terminology is very practical, it is not commonly used, so one may not find the term "coarse divergence" in the textbooks. The reason for its introduction here will be made clear in the following.

2) *Absolute convergence.*

   The series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent*, if

   $$\sum_{n=1}^{\infty} |a_n| \qquad \text{is convergent.}$$

   Notice that we have

   $$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

3) *Conditional convergence.*

   The series $\sum_{n=1}^{\infty} a_n$ is said to be *conditionally convergent*, if it is convergent, but *not* absolutely convergent.

4) If a series is not of any of the types above, it is called *divergent* (though not coarsely divergent).

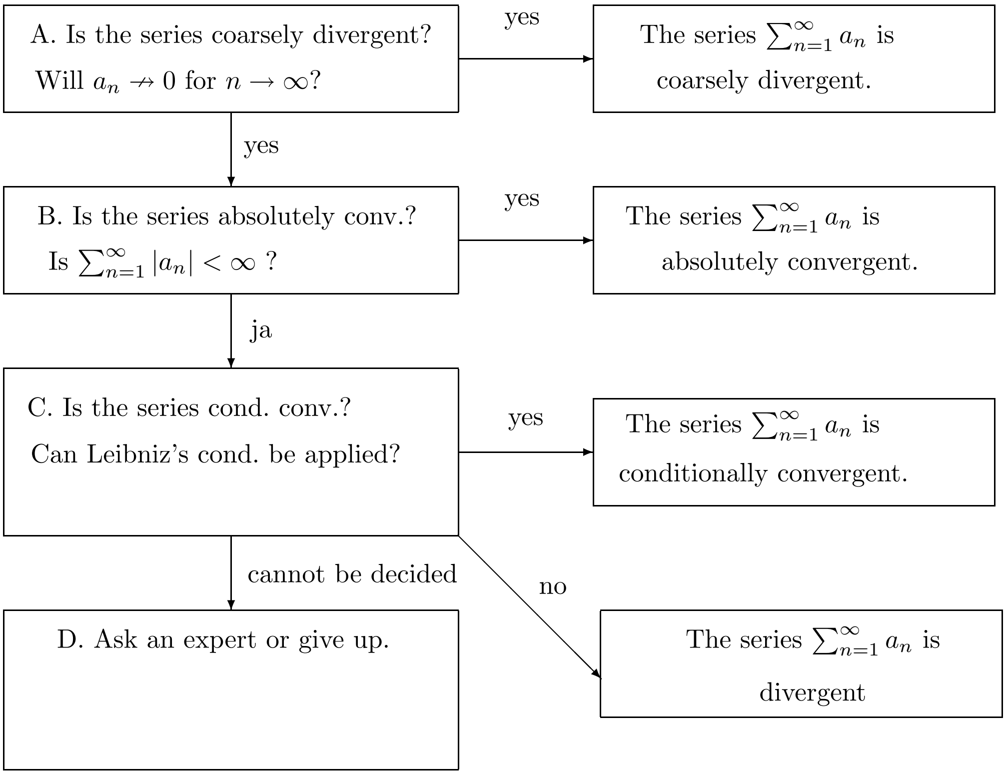
The *type of convergence* for a series can be found by going through the following flow diagram.

In any course of Calculus the problems will always be of a type such that all five results can *only* occur in connection with *Fourier series*, where one often can make use of other results. The Fourier series are treated in the following *Calculus 4b, Fourier Series, Systems of Differential Equations and Eigenvalue Problems.*

Whenever one is only considering power series and one has made no error in the calculations one will usually only end in one of the three upper boxes on the right hand side in the flow diagram. We therefore have a weak test saying that if we do not end here in this particular case, then we have probably made an error in our calculations. In that case one should start again from the very beginning.

In the real life where the nature is governing we can of course get any of the five boxes of results.

No matter how a problem is formulated, experience shows that this flow diagram is optimal and the closest one can get to some standard procedure.



Figur 2: Flow diagram for types of convergence.

4.7 An elaboration on the flow diagram.

A) Always start by checking the condition $a_n \nrightarrow 0$ for $n \to \infty$.

B) If one has reached this box, then the *necessary condition* $a_n \to 0$ for $n \to \infty$ for convergence is fulfilled, so we shall no longer bother with that condition in the following.

Write $\sum_{n=1}^{\infty} |a_n|$ (reduce it if possible) and apply one of the *criteria of convergence* (see the following) in order to check whether $\sum_{n=1}^{\infty} |a_n|$ is convergent or divergent.

C) If one has come to this box, then the only method known within the elementary Calculus is *Leibniz's criterion*. Since Leibniz's criterion is considered as difficult by the students, it is recommended *not to use it, unless it is absolutely necessary!*

D) If one in an elementary course of Calculus ends in this box, one has most probably made an error in the calculations. When we end here in real life, we shall later in the section on Leibniz's criterion give a more general test.

4.8 Convergence tests

The common convergence tests are in a natural way divided into five groups. Notice that no convergence test is universal in the sense that it can be applied in all cases, so it is recommended that one always starts by searching that particular group of tests which is best suited for a given situation.

Furthermore, there exist some overlaps, so *more* tests can often be applied on the same problem.

If one has the access to the literature from about 1900, one sees that the tests in each group can be supplied by more of the same kind.

I *Divergence test.*

1) Test for coarse divergence, $a_n \nrightarrow 0$ for $n \to \infty$.

II *General criteria of convergence.*

2) Comparison test.

5) Equivalence test.

III *Comparison with a quotient series.*

3) Quotient test (remember to check $|a_n| \neq 0$ for $n \geq N$).

4) Root test (remember the numerical sign).

IV *More advanced tests* (check the assumptions!)

6) Integral test.

7) Leibniz's criterion.

V *Other methods.*

8) Telescopic series by decomposition and convergence of a sequence of segments of a series.

Elaborating comments to the convergence tests:

Warning. We introduce in some of these tests an *auxiliary sequence*. The limit of this auxiliary sequence must *not* be confused with the sum of the series (which is often postulated by students at their examination). They are only introduced in order to decide whether we have convergence or divergence.

1) **Test of coarse divergence.**

If $a_n \nrightarrow 0$ for $n \to \infty$, then the *necessary condition* for convergence is *not* fulfilled, and the series $\sum_{n=1}^{\infty} a_n$ is *coarsely divergent.*

If some series is coarsely divergent, then we have obtained our answer, and we can save ourselves for any other investigation of convergence. It is therefore highly recommended that one always start with this test.

2) **Test of comparison** (two variants).

Let $0 \leq a_n \leq b_n$ for every $n \in \mathbb{N}$, i.e. every terme is ≥ 0. (Check!) Then we have formally,

$$0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n \leq \infty.$$

A) If the larger series $\sum_{n=1}^{\infty} b_n$ is convergent, then the smaller series $\sum_{n=1}^{\infty} a_n$ is also convergent,

$$0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n < \infty.$$

B) If the smaller series $\sum_{n=1}^{\infty} a_n$ is divergent, then the larger series $\sum_{n=1}^{\infty} b_n$ is also divergent,

$$\sum_{n=1}^{\infty} b_n \geq \sum_{n=1}^{\infty} a_n = \infty.$$

3) **The ratio test.**

An important assumption: It is (still) *not* allowed to divide by 0, so one should *always check* that $a_n \neq 0$ for $n \geq N$ for some constant N.

Assume that the *numerical sequence of quotients* is convergent,

$$\left|\frac{a_{n+1}}{a_n}\right| \to c \qquad \text{for } n \to \infty \quad (\text{og } n \geq N).$$

Check that $c \geq 0$. If this is not true one has forgotten the numerical sign!

A) If $c < 1$, Then the series $\sum_{n=1}^{\infty} a_n$ is convergent (but its sum is *not* c).

B) If $c = 1$, then neither the ratio test nor the root test can be applied. Try some other test outside group III, since comparison with a series of quotients cannot be applied.

C) If $c > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is *coarsely divergent.*
The latter case can also be found by test 1).

Remark 4.5 The ratio test is in particular applied, when a_n contains *faculty functions* like $n!$, $(2n)!$ or $(2n+1)!$ or similarly. $\Diamond$

Remark 4.6 The ratio test can *never be applied*, when a_n is a rational function in n, i.e. a quotient of two polynomials in n. If one tries, one will *always* end up in case B, where nothing can be decided). $\Diamond$

Remark 4.7 *Never forget the numerical sign!* For instance, the series $\sum_{n=1}^{\infty}(-1)^{n-1}n$ is *coarsely divergent*, because $a_n = (-1)^{n-1}n \nrightarrow 0$ for $n \to \infty$. Nevertheless one may occasionally come across the following "proof" of "convergence" made by students:

$$\frac{a_{n+1}}{a_n} = -\frac{n+1}{n} = -\left(1+\frac{1}{n}\right) \to -1 < 1 \qquad \text{for } n \to \infty,$$

from which one erroneously conclude that the series is "convergent". $\Diamond$

Remark 4.8 We notice that one can find *convergent* series, where $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ is *not* convergent. In this case one will often – though not always – be able to solve the problem by using the *extension* of the *root test*. In this respect the *root test* is slightly stronger than the *ratio test*. $\Diamond$

4) **The root test.**

Assume that the sequence of n-th roots is convergent,

$$\sqrt[n]{|a_n|} = |a_n|^{1/n} \to c \qquad \text{for } m \to \infty.$$

A) If $c < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent (but its sum is *not* c).

B) If $c = 1$, then neither the root test nor the ratio test can be applied. Try some of the other tests outside group III.

C) If $c > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is coarsely divergent.
The latter case can also be obtained by test 1).

Remark 4.9 Contrary to the quotient test we here do not have to assume that $a_n \neq 0$; remember the numerical sign. $\Diamond$

Remark 4.10 Neither the ratio test nor the root test can be applied, when a_n is a rational function in n. One should instead try some other test. (One does not give up, or does one?) $\Diamond$

Remark 4.11 It is difficult to apply the root test, when the faculty function $n!$ occurs, because the student cannot be expected to be able to calculate $\sqrt[n]{n!}$.

In order to give some help we here add *Stirling's formula*

$$n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n \quad \text{for large } n, \qquad \text{i.e. } \sqrt[n]{n!} \sim \sqrt[2n]{2\pi n}\cdot\frac{n}{e}.$$

A more accurate version is

$$n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n}\right) \qquad \text{for large } n,$$

where the den relative error decreases when n increases.

To demonstrate the accuracy of e.g. the latter approximation formula we calculate for $n = 10$,

$$10! = 3\,628\,800$$

and

$$\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n}\right) \approx 3\,628\,810,05 \qquad \text{for } n = 10,$$

so this (classical) approximation is very accurate, indeed.

Stirling's formula is also applied in Probability and Statistics. $\Diamond$

Remark 4.12 Since both the ratio test and the root test are proved by a comparison with a series of quotients $k\sum_{n=1}^{\infty} c^n$ with the same $c > 0$, we get the following very strange result:

If both $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ and $\left(\sqrt[n]{|a_n|}\right)$ are convergent, then they have the same limit,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = c = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

This observation has been used to find the limit of some "exotic" sequences, but it will not occur here further. $\Diamond$

Remark 4.13 Contrary to the ratio test the root test can be generalized to

$$\limsup_{n\to\infty}\sqrt[n]{|a_n|} = c$$

with the same classification A), B) and C). Since $\limsup_{n\to\infty}\sqrt[n]{|a_n|}$ [which always exists] is not commonly defined in most textbooks on Calculus of today, we shall not elaborate on this aspect. $\Diamond$

5) **The equivalence test.**
Consider to series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n_1}^{\infty} b_n$, where all $a_n > 0$ and all $b_n > 0$ for $n \geq N$ (check!). Assume that

$$\frac{a_n}{b_n} \to 1 \quad \text{for } n \to \infty \qquad \left(\text{possibly } \frac{a_n}{b_n} \to c > 0 \quad \text{for } n \to \infty\right).$$

Then $\sum_{n=1}^{\infty} a_n$ is convergent, if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.

Remark 4.14 Any person has his own favorites. I personally only apply the equivalence test very rarely, because $a_n/b_n \to 1$ for $n \to \infty$ implies that

$$\frac{1}{2}\, b_n < a_n < 2b_n \qquad \text{for } n \geq N,$$

and then the *comparison test* will give the same resultat. $\Diamond$

The two tests in the next group IV Advanced tests, are somewhat more sophisticated, and they demand great care in their applications. (Never forget to check the assumptions etc.)

6) **The integral test.**

Let $f : [1, \infty[\to \mathbb{R}_+$ be a *decreasing* function. (Check!) Then

$$\sum_{n=1}^{\infty} f(n) \text{ is convergent, if and only if } \int_1^{\infty} f(t)\,dt \text{ is convergent.}$$

When we have convergence we have the following useful estimates

$$\int_1^{\infty} f(t)\,dt < \sum_{n=1}^{\infty} f(n) < f(1) + \int_1^{\infty} f(t)\,dt,$$

or more generally for $p \in \mathbb{N}$,

$$(5)\quad \sum_{n=1}^{p} f(n) + \int_{p+1}^{\infty} f(t)\,dt < \sum_{n=1}^{\infty} f(n) < \sum_{n=1}^{p+1} f(n) + \int_{p+1}^{\infty} f(t)\,dt.$$

Remark 4.15 If one is only interested in the question of *convergence/divergence*, then it suffices that f is decreasing for $t \geq t_0 \geq 1$ for some fixed t_0 (the same conclusion). It is still possible to apply the general estimate (5), if only $p + 1 \geq t_0$. $\diamond$

Remark 4.16 The estimate (5) shows that if $f(t)$ is decreasing for $t \geq p+1$, then we can find an $\alpha \in]0, 1[$, such that

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{p} f(n) + \int_{p+1}^{\infty} f(t)\,dt + \alpha \cdot f(p+1).$$

If we delete $\alpha \cdot f(p+1)$, we get an approximation of the series by a finite sum and an integral, where the error is smaller than $f(p+1)$. $\diamond$

Remark 4.17 The integral test gives a long list if incredibly slowly convergent series. The standard examples are for $a > 1$,

1) $\sum_{n=1}^{\infty} \dfrac{1}{n^a}$, $\qquad f(t) = \dfrac{1}{t^a}$ $\qquad$ og $\quad \int_1^{\infty} \dfrac{1}{t^a}\,dt = \dfrac{1}{a-1}$,

2) $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^a}$, $\qquad f(t) = \dfrac{1}{t(\ln t)^a}$ $\qquad$ og $\quad \int_2^{\infty} \dfrac{dt}{t(\ln t)^a} = \int_{\ln 2}^{\infty} \dfrac{du}{u^a} = \dfrac{(\ln 2)^{1-a}}{a-1}$,

3) $\sum_{n=3}^{\infty} \dfrac{1}{n \ln n (\ln \ln n)^a}$, $\quad$ analogt til 2),

etc., where the reader should check that the chosen functions $f(t)$ are decreasing for $t > t_0$. $\diamond$

The same series are ("slowly") *divergent*, if $a \leq 1$ (with a special exception for $a = 1$, and where we for $a \leq 0$ obtain coarse divergence). It is absolutely not a good idea to type any of these series in a pocket calculator or in MAPLE.

The series above can conveniently be applied in connection with the *comparison test*, in case of the ratio test or the root test give $c = 1$, i.e. case B, where these tests cannot be applied.

7) **Leibniz's criterion** ("the problem child").

If

a) $b_n \to 0$ for $n \to \infty$,

b) (b_n) is decreasing, for n bigger than some N,

$$0 \leq b_{n+1} < b_n \qquad \text{for all } n \geq N,$$

then the *alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = s$$

is convergent, and we have the error estimate

$$|s - s_n| = \left| s - \sum_{k=1}^{n} (-1)^{k-1} b_k \right| < b_{n+1} \qquad \text{for all } n \geq N,$$

i.e. the error is always smaller than the numerical value $|(-1)^n b_{n+1}| = b_{n+1}$ of the first deleted term (interval narrowing).

Remark 4.18 Whenever Leiniz's criterion is applied, *always* remember to check the assumptions a) and b). At exams, this is the first thing which the teacher will check. $\Diamond$

Remark 4.19 Error estimates are extremely important by numerical calculations – even in technical disciplins. $\Diamond$

Remark 4.20 (Outside the usual syllabus.) An important extension of Leibniz's criterion is the following complicated monstrosity:

Let (a_n) and (b_n) be two sequences which satisfy

a) $b_n \to 0$ for $n \to \infty$.

b) (b_n) is decreasing for $n \geq N$ for some N.

c) The sequence $\sum_{n=1}^{\infty} a_n$ does *not* have to be convergent, but there must *exist* a constant $c > 0$, such that

$$|s_n| = \left|\sum_{k=1}^{n} a_k\right| \leq c \qquad \text{for all } n \in \mathbb{N},$$

i.e. *the sequence of segments* $s_n = \sum_{k=1}^{n} a_k$ is *bounded* (though not necessarily convergent).

Then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

For completion we show two applications of this extended criterion (only for special interested readers).

1) *Leibniz's criterion* is obtained when $a_n = (-1)^{n-1}$. In fact, in this case we get from condition c) that

$$|s_n| = \left|\sum_{k=1}^{n} a_k\right| = \left|\sum_{k=1}^{n} (-1)^{k-1}\right| = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even,} \end{cases}$$

hence the sequence of segments is bounded, which is sufficient. The sequence of segments is of course not convergent.

2) *The Fourier series* $\sum_{n=1}^{\infty} \dfrac{1}{n} \sin nx$ is pointwise convergent.

For $x = p\pi$, $p \in \mathbb{Z}$, there is nothing to prove, because the zero series is convergent. Choose a fixed $x_0 \neq p\pi$, $p \in \mathbb{Z}$. Then $e^{ix} = \cos x + i \sin x \neq 1$, hence we get by complex calculations on $a_k = \sin kx_0 = \text{ Im } e^{ikx_0}$,

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \sin kx_0 = \text{ Im } \sum_{k=1}^{n} \left(e^{ix_0}\right)^k = \text{ Im } \left\{ \frac{\left(e^{ix_0}\right)^{n+1} - 1}{e^{ix_0} - 1} \right\}.$$

Since $\sin x_0/2 = \frac{1}{2i}\{\exp(ix_0/2) - \exp(-ix_0/2)\}$, we get

$$\begin{aligned} s_n &= \operatorname{Im}\left\{\frac{(e^{ix_0})^{n+1}-1}{e^{ix_0}-1}\cdot\frac{\exp(-x_0/2)}{\exp(-x_0/2)}\right\} = \operatorname{Im}\left\{\frac{(e^{ix_0})^{n+1/2}}{e^{ix_0/2}-e^{-ix_0/2}}\right\} \\ &= \operatorname{Im}\left\{\frac{2i}{\sin x_0/2}\left\{\exp\left(i\left(n+\frac{1}{2}\right)x_0\right)\right\}\right\} \\ &= \frac{1}{\sin x_0/2}\operatorname{Re}\left\{\exp\left(i\left(n+\frac{1}{2}\right)x_0\right)\right\} \\ &= \frac{2}{\sin x_0/2}\left\{\cos\left(n+\frac{1}{2}\right)x_0 - \cos\frac{x_0}{2}\right\}. \end{aligned}$$

Then we remove n by an estimation (notice that $\sin x_0/2 \neq 0$ for $x_0 \neq p\pi$, $p \in \mathbb{Z}$)

$$|s_n| \le \frac{2}{|\sin x_0/2|}\{1+1\} = \frac{4}{|\sin x_0/2|} \qquad \text{for all } n \in \mathbb{N},$$

and we have proved c).

Since $b_n = 1/n$ trivially fulfils a) and b), it follows from the extended criterion that the series is pointwise convergent for every $x \neq p\pi$, $p \in \mathbb{Z}$, i.e. for every $x \in \mathbb{R}$, because we have already proved the pointwise convergence for $x \in p\pi$, $p \in \mathbb{Z}$. This result is of course at this stage far from obvious. $\Diamond$

8) **Telescopy.**

If each term a_n is a quotient of two polynomials in n, and

$$a_n = \frac{P(n)}{Q(n)} \to 0 \qquad \text{for } n \to \infty$$

(i.e. the numerator is of lower degree than the denominator), then it is possible in some (but not in all) cases to find the sum $s = \sum_{n=1}^{\infty} a_n$ directly by means of telescopy.

The procedure is the following:

a) Decompose $a_n = \dfrac{P(n)}{Q(n)}$, cf. *Calculus 1a, Functions in one Variable.*

b) Calculate and reduce the sequence of segments $s_n = a_1 + \cdots + a_n$.

c) Finally, take the limit $n \to \infty$. If

$$\lim_{n\to\infty} s_n = s,$$

then the sum s of the series is according to definition 3.1,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = s.$$

We illustrate the method by an example. Considering the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty} a_n,$$

it is seen that $a_n = \dfrac{1}{n(n+1)}$.

a) We get by a *decomposition*

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

b) The *sequence of segments* s_n is written down and reduced,

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

It is seen that the series is reduced like an old-fashioned telescope, therefore the name.

c) Finally, we get by taking the *limit*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

The principle of *telescopy* can also be applied in other cases where a_n is not a quotient between two polynomials. The main case is, however, what has been described above.

4.9 Series of functions

Just as for sequences we have here several types of convergence.

Let $\sum_{n=1}^{\infty} f_n(x)$ be a series with functions as terms. We say that the series is *pointwise convergent*, whenever the corresponding *sequence of segments*

$$s_n(x_0) = \sum_{k=1}^{n} f_k(x_0), \qquad n \in \mathbb{N},$$

is pointwise convergent as a sequence of numbers for any fixed $x = x_0 \in I$.

If a series is pointwise convergent, then its sum function $s(x)$ is given by

$$s(x) = \sum_{n=1}^{\infty} f_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) = \lim_{n\to\infty} s_n(x) \qquad \text{for } x \in I.$$

If we only remember to keep $x \in I$ *fixed*, there is no real difference between pointwise convergence of a series of functions and convergence of series of numbers.

Since the pointwise convergence of series is derived from the pointwise convergence of sequences, we have analogously a lack of good mathematical properties.

Let $s(x) = \sum_{n=1}^{\infty} f_n(x)$ be pointwise convergent.

1) Even if every $f_n(x)$ is continuous, we *cannot* conclude that $s(x)$ is continuous.

2) We cannot expect integration and summation to be interchangeable. Both

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx \quad \text{and} \quad \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

may be convergent without being equal.

3) Differentiation and summation cannot be interchanged either. Both

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) \quad \text{and} \quad \sum_{n=1}^{\infty} f_n'(x)$$

may be convergent without being equal.

In order to in some cases to obtain these desirable properties we introduce the concept *uniform convergence*. Formally we have:

The series $\sum_{n=1}^{\infty} f_n(x)$, $x \in I$, is called *uniformly convergent* with the sum function $s(x)$, if

$\sum_{n=1}^{\infty} f_n(x) = s(x)$ pointwisely for $x \in I$.

To any $\varepsilon > 0$ we can find a N, such that for any $n \geq N$ and any $x \in I$ we have

$$|s_n(x) - s(x)| = \left|\sum_{k=1}^{n} f_k(x) - s(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| < \varepsilon.$$

The definition of *uniform convergence* above is fairly complicated and it is *rarely used* in practice. Instead we apply the *important*

Test by a majoring series. Let $\sum_{n=1}^{\infty} f_n(x)$, $x \in I$, be a series of functions. If

a) one can estimate

$$|f_n(x)| \le a_n \qquad \text{for every } x \in I \text{ and every } n \in \mathbb{N},$$

where the a_n *do not* depend on $x \in I$, and

b) the auxiliary series $\sum_{n=1}^{\infty} a_n$ is convergent (as a series of numbers),

then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.

The series of numbers $\sum_{n=1}^{\infty} a_n$ is here called a *majoring series* for $\sum_{n=1}^{\infty} f_n(x)$.

Notice that the test by a majoring series *does not* give us the true sum function (here we shall instead use the usual methods from pointwise convergence). We only obtain the *estimate*

$$|s(x)| = \left|\sum_{n=1}^{\infty} f_n(x)\right| \le \sum_{n=1}^{\infty} |f_n(x)| \le \sum_{n=1}^{\infty} a_n, \qquad \text{for } x \in I,$$

where it is obvious that it in general is not a good idea to approximate $s(x)$ by a constant.

Strategy by investigation of uniform convergence of series.

The two most commonly used possibilities are the following.

1) If every term $f_n(x)$ in the series is *continuous*, while the pointwise sum function $s(x) = \sum_{n=1}^{\infty} f_n(x)$ is *not* continuous (or does not exist), then the convergence *cannot* be uniform.

   This principle is in particular used in the theory of *Fourier series*, where one typically in problems in calculus starts with e.g. a discontinuous function (the pointwise sum function) and then use it to *construct* the corresponding Fourier series, the terms of which all are continuous.

2) Assuming instead that "the series is probably uniformly convergent", we may use the following procedure:

   a) Remove $x \in I$ by an estimation in the numerical value of $|f_n(x)|$, e.g.

   $$|f_n(x)| \le \sup_{x \in I} |f_n(x)| \le a_n \qquad \text{for } n \in \mathbb{N}.$$

   One may often be able to choose $a_n = \sup_{x\in I} |f_n(x)|$, but in some cases this expression may be rather complicated. If so, then give an *upper estimate* by a simpler expression for a_n (convenient for the following calculations in the next point b)).

   b) Prove that $\sum_{n=1}^{\infty} a_n$ is *convergent*. If this is the case, we have solved our problem. If not, we have got a problem. Something may be rescued by giving a better estimate in a), which means that one in one's first try has given a too crude estimate. This is illustrated by

   Example 4.1 The series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$ has the two majoring series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, because

   $$\left|\frac{1}{n^2} \sin nx\right| < \frac{1}{n}, \quad \text{and even } \left|\frac{1}{n^2} \sin nx\right| \le \frac{1}{n^2}.$$

The first majoring series is *divergent* (hence no conclusion, because the estimate is too crude), and the second one is *convergent*, hence the trigonometric series is uniformly convergent. (The first majoring series is actually taken from an examination, where the conclusion unfortunately became wrong). $\Diamond$

The importance of *uniform konvergence* is illustrated by the following results:

Theorem 4.2 *Let every function $f_n(x)$, $x \in I$, be continuous.* (Check!) *If $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent, then the sum function $f(x)$ is continuous.*

It happens that the theorem is used in precisely this form, but one more commonly applies it in the following equivalent version, cf. the previous mentioned strategy, pkt. 1.

Theorem 4.3 *Let every function $f_n(x)$, $x \in I$, be continuous. if*

a) *The series is pointwise convergent with the sum function $f(x) = \sum_{n=1}^{\infty} f_n(x)$,*

b) *The sum function $f(x)$ is* not *continuous in the whole of I,*

Then $\sum_{n=1}^{\infty} f_n(x)$ is not *uniformly convergent in the whole of I.*

The next result is concerned with the interchanging of summation and integration. We hereby obtain some very important relaxations of the calculations.

Theorem 4.4 *If $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent, then we can* interchange *summation and integration (over a finite interval):*

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

The consequence of this theorem is of paramount importance, in particular for the theory of power series.

Finally we have the usual complications concerning *differentiation.*

Theorem 4.5 *Let (f_n), $f_n : I \to \mathbb{R}$, be a sequence of functions for which the derivatives $f_n' : I \to \mathbb{R}$ all exist and are continuous. If*

a) *The series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is* pointwise convergent *for $x \in I$,*

b) *The series $g(x) = \sum_{n=1}^{\infty} f_n'(x)$ is* uniformly convergent *over I,*

then f is differentiable in I with $f'(x) = g(x)$, i.e.

$$f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{df_n}{dx}(x) = g(x).$$

By the general applications in this chapter one should therefore always check that every $f_n'(x)$ is continuous and check a) and b).

Notice, however, that when one is considering *power series*, then everything is simpler and more streamlined, if only one stays inside the *open* interval of convergence. We shall deal with this in the next chapter.

Finally, it should be mentioned that there also exist other meaningful concepts of convergence. The most important of these is the *convergence in squared mean*, also called convergence in L^2. This concept is closely connected with *energy* in Physics, and in this sense it is more important for an engineer than pointwise convergence. In fact, it is never possible to *measure exactly* a physical function at *every* point, while considerations concerning energy are far more robust.

At this stage I admit that the formal mathematical definition is impossible to explain, so I shall only give a sketch. A series $\sum_{n=1}^{\infty} f_n(x)$, $x \in I$, *converges in* $L^2(I)$ towards a function $f(x)$ [which does not have to be equal to e.g. some existing pointwise limit function], if

$$\int_I \left| f(x) - \sum_{k=1}^{n} f_k(x) \right|^2 dx \to 0 \qquad \text{for } n \to \infty.$$

This squaring does not look nice at this early step of the student's education! It is nevertheless the *right* concept of convergence in e.g. the theory of *Fourier series*, where one in some presentations for some obscure reason denotes *convergence in* $L^2(I)$ by the symbol $\sim$, i.e.

$$f \sim \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}, \qquad I = [-\pi, \pi[.$$

In many important cases, which will be explained in *Calculus 4b*, the series on the right hand side is actually *pointwise convergent* with the adjusted function f^* as its sum function. We can then write

$$f^*(x) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}, \qquad x \in [-\pi, \pi[,$$

and we have reduced the task to investigating where $f^*(x) = f(x)$.

Here we have only used the example in order to explain that the difference in the notations $\sim$ and $=$ in the theory of Fourier series relies on the difference in the applied concept of convergence.

5 Power series; methods in solution of problems

This chapter of survey assumes that the reader can handle the concepts from chapter 4. It contains

- Standard power series.
- Recognition of the *structure* of the standard power series.
- Convergence of power series.
- Review of important theorems.
- Solution of differential equations by means of power series.
- Solution of recursion formulæ *(simple difference equations)*.
- Second order differential equations *(straight tips)*.
- Second order differential equations *(solution formula)*.

5.1 Standard power series

These are the building stones in the theory of power series and *the reader is therefore encouraged to learn these by heart!* They are all known from the usual textbooks in Calculus. We shall here subdivide them into two groups:

a) *Quotient like series.*

All these series are variations of the power series $\sum_{n=0}^{\infty} x^n$. With only one obvious exception, they have all the *radius of convergence* $\varrho = 1$.

b) *Exponential like series.*

These are all variations of the exponential series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. They are all convergent in the whole of $\mathbb{R}$, hence the radius of convergence is $\varrho = \infty$.

a) *Quotient like series,* $\varrho = 1$. (Learn both columns by heart!)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}, \qquad |x| < 1,$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^{\alpha}, \qquad |x| < 1, \qquad \alpha \in \mathbb{R}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \ln(1+x), \qquad |x| < 1,$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \arctan x, \qquad |x| < 1.$$

If in the third line $\alpha = n \in \mathbb{N}_0$, then $(1+x)^n$ is a *polynomial.* The series only contains a *finite* number of terms, hence the *radius of convergence* is $\varrho = \infty$.
In some languages one may use the notation "Arctan" instead of "arctan".

b) *Exponential like series*, $\varrho = \infty$. (Learn both columns by heart!)

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n, \qquad \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n = \exp(x), \qquad x \in \mathbb{R},$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} = \sin x, \qquad x \in \mathbb{R},$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n} = \cos x, \qquad x \in \mathbb{R},$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1}, \qquad \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1} = \sinh x, \qquad x \in \mathbb{R},$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n}, \qquad \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n} = \cosh x, \qquad x \in \mathbb{R}.$$

5.2 Recognition of the structure of standard power series

Problem: Given a power series $\sum_{n=0}^{\infty} a_n x^n$. Is it possible to make a shortcut and directly identify the structure of one of the standard power series above and thereby obtain a) the sum function, and b) the radius of convergence and the interval of convergence?

Answer: This is actually possible in many cases (whereby one saves a lot of time). We here sketch one procedure:

First rewrite the *coefficient* a_n, in such a way that it consists of a finite sum (or difference) of terms of the structure

$$(6)\quad \left\{\begin{array}{ccccccc} 1 & \dfrac{1}{n} & \dfrac{1}{2n+1} & \dfrac{1}{n!} & \dfrac{1}{(2n+1)!} & \dfrac{1}{(2n)!} & \dbinom{\alpha}{n} \\ & & & & & & \\ \text{const.} & \text{all} & \text{odd} & \text{all} & \text{odd} & \text{even} & \text{rare} \end{array}\right.$$

possibly supplied by an *alternating sign* $(-1)^n$.

The series is then formally split into the corresponding subseries, i.e. here disregarding any discussion of the convergence (we shall take care of this later). Here we can encounter variants like

$$\frac{1}{n+1}, \quad \frac{1}{2n-1}, \quad \frac{1}{(n+1)!}, \quad \frac{1}{(2n-1)!}, \quad \frac{1}{(2n+2)!}, \quad \text{etc.}$$

By some change of the summation variable,

$$m = n+1, \quad 2m+1 = 2n-1, \quad m = n+1, \quad 2m+1 = 2n-1, \quad 2m = 2n+2, \quad \text{etc.}$$

where the summation index set is changed correspondingly (always check the first term before and after the transform) we arrive to (6).

Constants in the power n, like e.g. a^n, where $a > 0$, are built into x^n, because $a^n x^n = (ax)^n$.

The rest is only some pottering: Let b_n be one of the possibilities.

a) $b_n = 1$ or $b_n = (-1)^n$.

Set up

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \quad \text{or} \quad \frac{1}{1+y} = \sum_{n=0}^{\infty} (-1)^n y^n, \quad |y| < 1.$$

Compare with the actual subseries and find y expressed by x. Adjust the lower bound of summation by

1) either removing additional terms,

2) or by adding missing terms and subtract them again.

Hereby we the the *sum function* and the *interval of convergence*, where the condition $|y| < 1$ is translated into a condition on x, when one solves the equation $y = y(x)$ in x.

b) $b_n = \dfrac{1}{n}$ or $b_n = \dfrac{(-1)^{n-1}}{n}$.

We rewrite the first case to

$$b_n = -\frac{(-1)^{n-1}}{n} \cdot (-1)^n.$$

The last alternating sign $(-1)^n$ is then combined with x^n, to $(-1)^n x^n = (-x)^n$. Then set up

$$\ln(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, y^n, \qquad |y| < 1.$$

Repeat the procedure from a).

c) $b_n = \dfrac{1}{2n+1}$ or $b_n = \dfrac{(-1)^n}{2n+1}$.

In the first case rewrite to

$$b_n = \frac{(-1)^n}{2n+1} \cdot (-1)^n,$$

where the last alternating sign $(-1)^n$ is built into the x-term. Set up

$$\arctan y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, y^{2n+1}, \qquad |y| < 1.$$

Adjust the x-series such that only *odd* exponents occur, e.g.

$$x^{2n+2} = x \cdot x^{2n+1}, \qquad \text{put } x \text{ outside the sum.}$$

$$x^{2n} = \frac{1}{x} \cdot x^{2n+1} \quad \text{for } x \neq 0.$$

Check especially $x = 0$ afterwards,

$$x^n = (\sqrt{x})^{2n} = \frac{1}{\sqrt{x}} \left(\sqrt{x}\right)^{2n+1} \qquad \text{for } x > 0,$$

etc. Repeat the procedure from a).

d) $b_n = \dfrac{1}{n!}$.

Set up the exponential series $\exp(y) = \sum_{n=0}^{\infty} \dfrac{y^n}{n!}$ and proceed as in a).

e) $b_n = \dfrac{(-1)^n}{(2n+1)!}$ or $b_n = \dfrac{1}{(2n+1)!}$ (odd $2n+1$)

Write the series for sin(h) x. Adjust the exponents like in c), and proceed like in a).

f) $b_n = \dfrac{(-1)^n}{(2n)!}$ or $b_n = \dfrac{1}{(2n)!}$ (even $2n$).

Write the series for cos(h) x. Adjust the exponents like in c) in a way, such they are all *even!* Then proceed like in a).

There calculations are correct in the *intersection set* of all the the intervals of convergence, from which one also *directly can find the radius of convergence.*

5.3 Convergence of power series

The convergence of power series $\sum_{n=0}^{\infty} a_n x^n$ is very simple to describe, because there are only three possibilities.

A) The series is *only* convergent for $x = 0$, the *radius of convergence is* $\varrho = 0$.
This case is of no interest, because the series cannot be used in practical calculations.

B) The series is *absolutely convergent* for every $x \in \mathbb{R}$, and *uniformly convergent* in every *bounded* interval; the *radius of convergence* is $\varrho = \infty$.

C) The *radius of convergence* $\varrho \in \mathbb{R}_+$ is positive and finite.

The series is *absolutely convergent* for every $x \in]-\varrho, \varrho[$, and *uniformly convergent* in every *closed* subinterval $[a, b] \subset]-\varrho, \varrho[$.

It should be noted that one can find examples where the series is both absolutely and uniformly convergent in the whole of $[-\varrho, \varrho]$. A simple example is er $\sum_{n=0}^{\infty} \frac{1}{n^2} x^n$, $x \in [-1, 1]$, where $\varrho = 1$, However, this is not the general rule.

The series is *divergent* for $|x| > \varrho$.

For $|x| = \varrho$ a special investigation is needed, in which one may possibly apply Abel's theorem (see below). In general, convergence at the end points is only of limited interest, even though this problem can often be met with at examinations.

The *radius of convergence* ϱ is formally defined by

$$\varrho = \sum \left\{ |x| \;\middle|\; \sum_{n=0}^{\infty} a_n x^n \text{ is convergent} \right\}.$$

However, one rarely applies this definition. Instead one typically derive the radius of convergence from the *root test* or the *ratio test:*

1) *Preparations* (in order to avoid errors). One will often see the so-called *lacunar series*, i.e. series with "gaps" in the index set, like e.g. $\sum_{n=0}^{\infty} a_n x^{2n}$ or $\sum_{n=0}^{\infty} a_n x^{2n+1}$, where one has made an abuse on the notation, because the index n should follow the exponent n, which is not the case here. In principle this can always be amended.

   In order to avoid *fallacies* one may introduce b_n as the numerical value of the n-th term different from zero.

   Example 5.1 a) For $\sum_{n=0}^{\infty} a_n x^n$ we put $b_n = |a_n| \cdot |x|^n$. (Nothing new).
   b) For $\sum_{n=0}^{\infty} a_n x^{2n}$ we put $b_n = |a_n| \cdot x^{2n}$. (Lacunar series; the terms x^{2n+1} are missing), etc. $\Diamond$

2) We choose the *ratio test*, if the faculty function enters b_n.

   Assuming that $x \neq 0$, (never divide by zero, however, we have a trivial convergence for $x = 0$), then $b_n > 0$ and

$$\frac{b_{n+1}}{b_n} = \frac{|a_{n+1}|}{|a_n|} \cdot \frac{|x|^p}{|x|^q} = |x|^{p-q} \cdot \frac{|a_{n+1}|}{|a_n|},$$

where we in each case must find p and q.

Main case: If $b_n = |a_n| \cdot |x|^n$, then $p = n+1$ and $q = n$, hence $p - q = 1$.

The *lacunar case:* If the series is $\sum_{n=0}^{\infty} a_n x^{2n}$, then $p = 2n+2$ and $q = 2n$, hence $p - q = 2$.

When $\dfrac{|a_{n+1}|}{|a_n|} \to c$, then the condition for convergence is

$$|x|^{p-q} \cdot c < 1, \qquad \text{i.e. } |x| < \left(\frac{1}{c}\right)^{1/(p-q)} = \varrho \qquad \text{for } c > 0,$$

and $\varrho = \infty$, for $c = 0$.

3) The *root test* cannot be applied if the faculty function enters b_n, unless one will use *Stirling's formula*,

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n .$$

If

$$\sqrt[n]{b_n} = \sqrt[n]{|a_n|} \cdot |x|^{p(n)/n} \to |x|^{\alpha} \cdot c \qquad \text{for } n \to \infty$$

is convergent, i.e. $\sqrt[n]{|a_n|} \to c$ for $n \to \infty$, we obtain the radius of convergence by

$$|x| < \left(\frac{1}{c}\right)^{1/\alpha} = \varrho \qquad \text{for } c > 0,$$

where $\varrho = \infty$, if $c = 0$.

Main case: If $b_n = |a_n| \cdot |x|^n$, then $p(n) = n$, hence $\dfrac{p(n)}{n} = 1 = \alpha$.

Lacunar case: If e.g. (with odd exponents) $b_n = |a_n| \cdot |x|^{2n+1}$, then $p(n) = 2n+1$, hence $\dfrac{p(n)}{n} = \dfrac{2n+1}{n} = 2 + \dfrac{1}{n} \to 2 = \alpha$ for $n \to \infty$.

We see that we have many variants. However, in the *main case*, where we do *not* have "gaps" in the series, we have

$$\varrho = \frac{1}{c}, \quad \text{where either } \left|\frac{a_{n+1}}{a_n}\right| \to c \text{ or } \sqrt[n]{|a_n|} \to c \qquad \text{for } n \to \infty.$$

5.4 Review of some important theorems

The following results are the most common ones used in the theory of power series.

If $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence $\varrho > 0$ (incl. $\varrho = \infty$), then

a) the series is *absolutely convergent* for every $|x| < \varrho$,

b) the series is *uniformly convergent* in every *closed and bounded* interval $[a, b] \subset]-\varrho, \varrho[$ (though it is not necessarily equal to $]-\varrho, \varrho[$.

The consequences of these results are immense:

1) The sum function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous for $|x| < \varrho$.

2) Summation and integration can be interchanged (termwise integration),

$$\int_0^x \left\{\sum_{n=0}^{\infty} a_n t^n\right\} dt = \sum_{n=0}^{\infty} a_n \int_0^x t^n \, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \qquad \text{for } |x| < \varrho,$$

   and the termwise integrated series has the *same* radius of convergence ϱ.

3) Summation and differentiation are also interchangeable (termwise differentiation), and the differentiated series has the *same* radius of convergence ϱ.

On grounds of the method of solution of differential equations by using *power series* we here mention the important formulæ

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < \varrho,$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \qquad |x| < \varrho,$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n, \quad |x| < \varrho,$$

etc. We note that since differentiation of a *constant* is 0, *the lower bound of summation increases by each differentiation.*

However, be aware of the *lacunar series*, where we have to apply a different rule.

For $f(x) = \sum_{n=0}^{\infty} a_n x^{2n}$ we get

$$f'(x) = \sum_{n=1}^{\infty} 2n \cdot a_n x^{2n-1} = \sum_{n=0}^{\infty} (2n+2)\, a_{n+1} x^{2n+1},$$

because the constant vanishes by differentiation. Now, the first term in the series of $f'(x)$ is no longer a constant, $2a_1 x$, so the lower bound of summation does not change by the next differentiation,

$$f''(x) = \sum_{n=1}^{\infty} 2n(2n-1) a_n x^{2n-2} = \sum_{n=0}^{\infty} (2n+2)(2n+1) a_{n+1} x^{2n}.$$

Abel's theorem can be difficult to handle (there often appear fallacies here). Check *always* the following assumptions:

a) The series $\sum_{n=0}^{\infty} a_n x^n$ is *convergent* for $|x| < \varrho$ with the *sum function* $f(x)$.

b) The series $\sum_{n=0}^{\infty} a_n \varrho^n$ (or $\sum_{n=0}^{\infty} a_n (-\varrho)^n$, if we instead consider the end point $x = \varrho$) is convergent!

The student is often inclined to forget the last condition.

It is only when both these conditions are fulfilled that we can conclude that

$$\sum_{n=0}^{\infty} a_n \varrho^n = \lim_{x \to \varrho-} f(x), \qquad \text{resp.} \qquad \sum_{n=0}^{\infty} a_n (-\varrho)^n = \lim_{x \to -\varrho+} f(x).$$

The problem is that $\lim_{x \to \varrho-} f(x) = f(\varrho)$ may exist by a continuous extension, while the series $\sum_{n=0}^{\infty} a_n \varrho^n$ is not convergent.

For $x = -\varrho$ we apply a similar method.

Splitting of series. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < \varrho$, and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ for $|x| < \lambda$, and α and β are constants, then

$$\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) x^n = \alpha f(x) + \beta g(x)$$

at least for $|x| < \min\{\varrho, \lambda\}$.

When $\lambda = \varrho$ we may in some cases obtain that the left hand side has a *larger* interval of convergence. Usually it is the indicated one above.

Cauchy-multiplication is not presented here! *Avoid it like the plague!* I have seen too many errors here in the students' calculations to recommend it. Notice also that one does *not* multiply two series by removing one of the sum signs (a frequent error by examinations). In general we have

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n \neq \sum_{n=0}^{\infty} a_n b_n x^n \qquad \left(\text{and } \neq \sum_{n=0}^{\infty} a_n b_n x^{2n} \right).$$

Theorem 5.1 (The identity theorem for power series). (Important!)
If two power series $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *have the same* sum function $f(x)$ *in a neighbourhood of 0, then they are* identical *and*

$$a_n = b_n = \frac{1}{n!}\, f^{(n)}(0) \qquad \text{for every } n \in \mathbb{N}_0.$$

A typical and frequent application in the following is:

If $\sum_{n=0}^{\infty} a_n x^n = 0 = \sum_{n=0}^{\infty} 0 \cdot x^n$,then $a_n = 0$ for every $n \in \mathbb{N}_0$.

5.5 Determination of the sum function by termwise differentiation or integration

It is possible in some cases (though one shall not always do it) determine the sum function of a series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < \varrho,$$

by first *differentiating* (integrating) it termwise, then find the sum function of the *differentiated* (integrated) series, and then finally find $f(x)$ by the inverse operation, i.e. *integration* (differentiation).

Unfortunately the process is rarely so easy as described above. There are usually some "noise" which confuses the student. We therefore here sketch a possible procedure.

1) *Choice of method*. Where is the unwanted factor n, $n+1$, etc., situated?
   a) If the factor is found in the *denominator*, we choose *differentiation*.
   b) If the factor is found in the *numerator*, we choose *integration*.
2) *Adjustment of the series*. Calculate $f(0)$ separately and then assume that $x \neq 0$.
   a) If the factor is in the *denominator*, then the *exponent* must be *equal to (a factor in) the denominator*. Multiply or divide by a convenient factor, and put the new series equal to $g(x)$, which then is found by a *differentiation*.

Example 5.2 Let $f(x) = \sum_{n=0}^{\infty} \dfrac{1}{n+1}\, x^n$, $|x| < 1$. Then $f(0) = 1$, and for $x \neq 0$ we have

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\, x^n = \frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{n+1}\, x^{n+1} = \frac{1}{x}\, g(x), \quad \text{hvor } g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

It follows that $g(0) = 0$ and

$$g'(x) = \frac{d}{dx}\left\{ \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right\} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

so by an integration,

$$g(x) = g(0) + \int_0^x g'(t)\, dt = 0 + \int_0^x \frac{dt}{1-t} = -\ln(1-x) \quad \text{for } |x| < 1.$$

By insertion we get

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\, x^n = \frac{1}{x}\, g(x) = -\frac{\ln(1-x)}{x} \quad \text{for } 0 < |x| < 1,$$

(never *divide* by 0), supplied with

$$f(0) = 1 \qquad \text{for } x = 0. \qquad \Diamond$$

b) If instead the factor is in the *numerator* (the case of integration), then the *exponent* must be the *numerator minus 1*. Multiply or divide by a convenient factor and put the new series equal to $g(x)$, which then is determined by *integration*.

Example 5.3 Let $f(x) = \sum_{n=1}^{\infty} nx^n$, $|x| < 1$. Then $f(0) = 0$, and we get for $x \neq 0$

$$f(x) = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \cdot g(x), \quad \text{where } g(x) = \sum_{n=1}^{\infty} nx^{n-1}.$$

It follows that $g(0) = 1$, and by an *integration*,

$$\begin{aligned}G(x) &= g(0)+\int_0^x g(t)\,dt = 1+\int_0^x\left\{\sum_{n=1}^{\infty} nt^{n-1}\right\}dt \\ &= 1+\sum_{n=1}^{\infty}\int_0^x nt^{n-1}\,dt = 1+\sum_{n=1}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x|<1.\end{aligned}$$

From this we get by a *differentiation* that

$$g(x) = G'(x) = \frac{1}{(1-x)^2},$$

hence by insertion

$$f(x) = \sum_{n=1}^{\infty} nx^n = x\cdot g(x) = \frac{x}{(1-x)^2} \qquad \text{for } |x|<1.$$

Notice that in this case we do not need the assumption $x \neq 0$, because the function $f(x)$ also satisfies $f(0) = 0$. Nevertheless it is a good idea always *to calculate $f(0)$ separately and then calculate $f(x)$, because one then avoids inadvertently a division by 0.* $\diamond$

Before the method is applied one should always first try other method, e.g. *decomposition* of a_n after n. The reason is that one often makes errors in the calculations described above. Note also that we shall perform an *integration* in both cases, where the result cannot always be expressed by *elementary functions*, known by the students.

Outside the usual syllabus we mention the important

Theorem 5.2 Weierstraß's approximation theorem.
Every continuous function $f(x)$ defined on a closed and bounded interval I can be uniformly approximated by a sequence of polynomials.

In the literature one can find some explicit constructions, e.g. by means of the so-called *Bernstein-polynomials.* It will go beyond the purpose of this text to go further into them here.

5.6 The method of power series by solution of linear differential equations with polynomial coefficients

This is a very important technique, and at the same time it looks fairly complicated when the student first encounters it. For that reason we here sketch a possible standard procedure, where we have lots of possible variants. The method is illustrated by an attendant example.

1) *Notice that the coefficients of the equation are polynomials in x.*

This is the case for the attendant example defined by

$$x\,\frac{d^2y}{dx^2}+2\,\frac{dy}{dx}+xy=0, \quad x\in\mathbb{R}, \qquad y(0)=1, \quad y'(0)=0.$$

2) *Assume that the equation has a solution which can be described by a power series* $y = \sum_{n=0}^{\infty} a_n x^n$ *with the interval of convergence* $]-\varrho, \varrho[$.

 Notice that this is an *assumption*, so we shall later *test* it when we have found a *candidate*. In the following the unknowns are the infinitely many coefficients a_n, $n \in \mathbb{N}_0$, and last but not least the *radius of convergence* ϱ. The latter is *often forgotten*, which means that the calculations become without any value. This is also the case when $\varrho = 0$, because then the series only "lives" in $]-\varrho, \varrho[$, which is the empty set when $\varrho = 0$.

3) *Write the equation from the right to the left and insert mechanically the series*

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

 into the equation.

 Notice the changes in the *lower summation limits* by the differentiations.

 In the example under consideration we get

$$\begin{aligned} 0 &= x\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + xy \\ &= x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

 In this setup we see that we are able to continue the calculations in the next step without repeating "$= 0$".

4) *All factors outside the sums are put inside.*

 When the factors contain more terms (like e.g. in$1 + x$) we introduce a new summation for each of the terms.

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

5) *Is it possible to add convenient zero terms?*

 In the given example we see that $n(n-1)a_n x^{n-1}$ is defined when $n - 1 \geq 0$, i.e. $n \geq 1$. The corresponding sum starts by $n = 2$, but if we insert $n = 1$, we see that $n(n-1)a_n x^{n-1} = 0$ for $n = 1$. Therefore, we can change $n = 2$ in the lower summation bound to $n = 1$ by adding 0,

$$0 = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

6) *The various series are collected in groups according to their type*, i.e. according to $a_n x^{n+j}$, or similarly.

 By "type" we here mean the *difference* between the index of a_n and the corresponding exponent.

In the attendant example we get

$$0=\left\{\sum_{n=1}^{\infty} n(n-1)a_n x^{n-1}+\sum_{n=1}^{\infty} 2na_n x^{n-1}\right\}+\sum_{n=0}^{\infty} a_n x^{n+1}.$$

The series in the brackets are all of the type

$$\text{index} \; - \; \text{exponent} = n-(n-1)=1,$$

and analogously the type of the last series is $n-(n+1)=-1$, i.e. of a different type.

7) *Inside each group we adjust the lower summation bound*, either by removing terms or by adding terms, which then must be subtracted outside the sums.

Here the lower bound is $n = 1$ for every sum in the group of type 1, and the lower bound is $n = 0$ for the group of type -1.

If we had neglected point 5) for the attendant example, then we would have got different lower limits in the group of type 1

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 2na_n x^{n-1} = \left\{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=2}^{\infty} 2na_n x^{n-1}\right\} + 2a_1.$$

We see that we here get some technical problems because the series cannot be added as long they do not have the same lower bound. I therefore highly *recommend* the student to the troubles of adding zero terms in 5).

8) *Inside each group the series are now of the same kind, i.e. with the same summation set and the same cluster $a_n x^{n+j}$ under the summation sign (supplied by polynomial coefficients in n). We can therefore put them under the same sum.*

Reduce the expression in the example and factorize the n-polynomial

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty}\{n(n-1)+2n\}a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=1}^{\infty} n(n+1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

The factorization is here $n(n-1) + 2n = n^2 - n + 2n = n^2 + n = n(n+1)$.

9) Every type is now represented by exactly one sum. They still cannot be added because they have different *exponents. Choose a common* exponent*, and transform the series, such that they all have the same* exponent.

In the example we choose the exponent $n+1$. The first series is not of this type, so put $m+1 = n-1$, i.e. $n = m+2$. Then the lower bound $n = 1$ corresponds to the *new lower bound* $m = 1-2 = -1$, so we have the calculation

$$\sum_{n=1}^{\infty} n(n+1)a_n x^{n-1} = \sum_{m=-1}^{\infty}(m+2)(m+3)a_{m+2}x^{m+1} = \sum_{n=-1}^{\infty}(n+2)(n+3)a_{n+2}x^{n+1},$$

where we in the latter equality just write "n" instead of "m". Then we get by insertion

$$0 = \sum_{n=-1}^{\infty}(n+2)(n+3)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1}.$$

10) Every series has now the same exponent, though not necessarily the same lower bound. *Adjust the lower bound by removing the unwanted terms.* The removed terms may now be reduced to a polynomial with increasing exponents.

By a continuation of the example we here get

$$\begin{aligned} 0 &= 1\cdot 2\cdot a_1 + \sum_{n=0}^{\infty}(n+2)(n+3)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= 2a_1 + \sum_{n=0}^{\infty}(n+2)(n+3)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

11) *The series have now the same* exponent *and the same* lower bound. *They can therefore be collected in one single sum.*

In the specific example we get

$$0 = 2a_1 + \sum_{n=0}^{\infty}\{(n+2)(n+3)a_{n+2} + a_n\}x^{n+1}.$$

We note that x^{n+1} now stands alone as a factor and that the coefficients do *not* contain x.

12) *Check that the polynomial outside the sum* does not *contain terms with an exponent which also occurs in the series.*

If the polynomial contains some terms with exponents which are interfering with exponents in the series, this is an indication of a miscalculation during the process. Since this error can have been made anywhere, the only advice is to *start from the very beginning!*

In the given example, $2a_1$ is a constant, i.e. of degree 0. The smallest degree in the series is obtained for the *lower bound*, i.e. for $n = 0$, where the corresponding *exponent* is $n + 1 = 1 > 0$. Therefore, we have no interference here, and we may proceed.

13) *Set up the* recursion formula *by means of the* identity theorem. *The recursion formula is* valid *in precisely the* summation index set *of the series. Therefore, indicate this summation index set explicitly!*

Since

$$0 = \sum_{n=0}^{\infty} 0\cdot x^n = 2a_1 + \sum_{n=0}^{\infty}\{(n+2)(n+3)a_{n+2} + a_n\}x^{n+1},$$

and since there is no interference between the series and the polynomial outside the sum, we conclude from the *identity theorem* (quote it!) that

$2a_1 = 0,$ from the polynomisl

and

(7) $(n+2)(n+3)a_{n+2} + a_n = 0$ for $n = 0, 1, 2, \cdots$ from the series.

We conclude that

$a_1 = 0,$

and that we get the *recursion formula* (note that $(n+2)(n+3) \neq 0$ for $n \in \mathbb{N}_0$)

$$a_{n+2} = -\frac{1}{(n+2)(n+3)}\, a_n \qquad \text{for } n \in \mathbb{N}_0.$$

Here we have specified where the recursion formula holds, namely for $n \in \mathbb{N}_0$. This detail is often forgotten by the student.

In general, do not forget to check that we *never* unintendedly divide by 0 in the set where the recursion formula holds. If the denominator becomes 0 for some n_0, we have to consider this value separately by directly checking (7).

Always remember that we get an extra minus sign when we rewrite (7). This is also a very frequent error in practice.

14) *Check if there are "leaps" in the recursion formula.*

When the leap is 1, i.e. when a_{n+1} is expressed by a_n, proceed with point 15).

If the leap is 2, then divide into the two cases of $n = 2p$ *even* and $n = 2p+1$ *odd* and set up new recursion formulæ in each of the two cases, before one proceeds with 15).

Analogously when the leap is > 2.

In our chosen example we express a_{n+2} by a_n, hence the leap is 2. Therefore we must consider the two cases of $n = 2p$ even and $n = 2n+1$ odd separately.

The original recursion formula is

$$a_{n+2} = -\frac{1}{(n+2)(n+3)}\, a_n \qquad \text{for } n \in \mathbb{N}_0.$$

a) For $n = 2p$ even it is seen that we stay inside $n \in \mathbb{N}_0$, when $p \in \mathbb{N}_0$. The we get by insertion

$$a_{2p+2} = -\frac{1}{(2p+2)(2p+3)}\, a_{2p}, \qquad p \in \mathbb{N}_0,$$

i.e. when we put $b_p = a_{2p}$ (the calculations become easier in the following)

$$b_{p+1} = -\frac{1}{(2p+2)(2p+3)}\, b_p, \qquad p \in \mathbb{N}_0.$$

b) For $n = 2p+1$ odd we see that we stay inside $n \in \mathbb{N}_0$, when r $p \in \mathbb{N}_0$. We get by insertion

$$a_{2p+3} = -\frac{1}{(2p+3)(2p+4)}\, a_{2p+1}, \qquad p \in \mathbb{N}_0.$$

The following calculations also become easier to perform, if we introduce $c_p = a_{2p+1}$. If we do this, we get

$$c_{p+1} = -\frac{1}{(2p+3)(2p+4)}\, c_p, \qquad p \in \mathbb{N}_0.$$

In the real life we may see that a_{n+2} is expressed by a combination of both a_{n+1} and a_n. These difference equations are in general too difficult to solve in a first course of calculus, so one will avoid them. If the student therefore at this stage has derived such an equation, this will most likely be an indication of some earlier miscalculation. (Check!)

15) *Solve the recursion formula.*

There are here so many variants that we instead refer to the next section concerned with *solution of difference equations.*

We can proceed with the example. It is given in 1) that

$$y(0) = a_0 = 1 \qquad \text{and} \qquad y'(0) = 1 \cdot a_1 = 0.$$

We also proved the latter in 13), which means that we formally has an over-determined problem, i.e. too much information: If we demand a power series solution, then we are *forced* to put $a_1 = 0$. Fortunately the given data is also giving $a_1 = 0$.

a) For $n = 2p$ even we have found the recursion formula

$$b_{p+1} = -\frac{1}{(2p+2)(2p+3)}\, b_p, \quad p \in \mathbb{N}_0, \qquad b_p = a_{2p}.$$

Thus $a_0 = b_0 = 1$. If we replace p by $p-1$, we instead get

$$b_p = -\frac{1}{(2p+1)(2p)}\, b_{p-1}, \qquad p \in \mathbb{N}.$$

By *recursion* (the indices decreases, i.e. b_{p-1} is expressed by b_{p-2}, etc.) we get

$$b_p = \frac{-1}{(2p+1)(2p)} \cdot \frac{-1}{(2p-1)(2p-2)} \cdots \frac{-1}{(2\cdot 1+1)\cdot(2\cdot 1)}\, b_{1-1},$$

by successively calculating the factor in front of b_{p-1}, in front of b_{p-2}, etc. until we have found the factor in front of b_0, corresponding to $p = 1$.

A counting gives p factors, hence the numerator must be $(-1)^p$. The denominator becomes

$$(2p+1)(2p)(2p-1)\cdots 3\cdot 2\cdot 1 = (2p+1)!,$$

Hence

$$a_{2p} = b_p = \frac{(-1)^p}{(2p+1)!}\, b_0 = \frac{(-1)^p}{(2p+1)!}, \qquad \text{because } b_0 = a_0 = 1.$$

b) For $n = 2p+1$ odd we found the recursion formula

$$c_{p+1} = -\frac{1}{(2p+3)(2p+4)}\, c_p, \qquad p \in \mathbb{N}_0, \qquad c_p = a_{2p+1}.$$

Here we already have $c_0 = a_1 = 0$. Therefore, we conclude by *induction* that $c_p = 0$ for every $p \in \mathbb{N}_0$. In fact,

i) $c_0 = 0$.

ii) If some $c_p = 0$, then also the successor $c_{p+1} = 0$, i.e. $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, etc.

16) *Set up the* formal *series (i.e. the candidate).*

Remember here that we have $b_p = a_{2p}$ and $c_p = a_{2p+1}$, and that a_n is the coefficient of x^n.

We found in the illustrating example that

$$a_{2p} = \frac{(-1)^p}{(2p+1)!}, \quad p \in \mathbb{N}_0, \quad \text{and} \quad a_{2p+1} = c_p = 0, \quad p \in \mathbb{N}_0.$$

We therefore have the *formal* power series (write n instead of p)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n} \qquad \left(= \sum_{n=0}^{\infty} a_{2n} x^{2n} \right).$$

Note that the exponent $2n$ belongs to a_{2n} and *not* to x^n.

17) *Find the radius of convergence* ϱ.

This is a very important part of the solution. In fact, if $\varrho = 0$, then the series is *divergent* for $x \neq 0$, and the equation has *no* power series as a solution.

There are three methods.

a) We recognize the series as a variant of one of the standard series.
b) The radius of convergence is determined from the recursion formula in 13).
c) The radius of convergence is determined directly from the series in 16).

Let us consider the obtained formal power series $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}\, x^{2n}$.

a) When we insert $x = 0$ and use the convention $x^0 = 0$, we get the sum

$$f(0) = \frac{(-1)^0}{(2 \cdot 0 + 1)!} \cdot 1 = 1.$$

When $x \neq 0$, the coefficient $\dfrac{(-1)^n}{(2n+1)!}$ resembles "something like $\sin y$", cf. the list of standard series. When we write down the series for $\sin y$, we get

$$\sin y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, y^{2n+1} \qquad \text{for } y \in \mathbb{R}.$$

This expression is equal to the formal x-series when we choose $y = x$, apart from a missing factor x. Since $\dfrac{x}{x} = 1$ for $x \neq 0$, this is easily repaired for $x \neq 0$,

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} = \frac{\sin x}{x} \quad \text{for } x \neq 0.$$

By this recognition method we found i) the interval of convergence $\mathbb{R}$, i.e. $\varrho = \infty$, and also ii) the sum function $\dfrac{\sin x}{x}$ for $x \neq 0$, and $f(0) = 1$ for $x = 0$.

b) From 13) we have the recursion formula

$$a_{n+2} = -\frac{1}{(n+2)(n+3)}\, a_n \qquad \text{for } n \in \mathbb{N}_0.$$

Since $a_n = 0$ for n odd, we are only interested in

$$a_{2n+2} = -\frac{1}{(2n+2)(2n+3)}\, a_{2n},$$

so the series is written $\sum_{n=0}^{\infty} a_{2n} x^{2n}$, Where all $a_{2n} \neq 0$. Then put $d_n = |a_{2n} x^{2n}| > 0$ for $x \neq 0$. We get by the *ratio test* that

$$\left|\frac{d_{n+1}}{d_n}\right| = \left|\frac{a_{2n+2}}{a_{2n}}\right| \cdot \frac{x^{2n+2}}{x^{2n}} = \frac{x^2}{(2n+2)(2n+3)} \to 0 < 1 \quad \text{for } n \to \infty$$

for every fixed $x \neq 0$. Therefore, the series is convergent in $\mathbb{R}$ and $\varrho = \infty$, where we combine the two cases $x = 0$ and $x \neq 0$.

c) If instead we apply the *ratio test* directly on $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}\, x^{2n}$, and use that

$$k_n := \left| \frac{(-1)^n}{(2n+1)!}\, x^{2n} \right| = \frac{x^{2n}}{(2n+1)!} > 0 \qquad \text{for } x \neq 0,$$

then we get

$$\left| \frac{k_{n+1}}{k_n} \right| = \frac{x^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n}} = \frac{x^2}{(2n+3)(2n+2)} \to 0 < 1 \quad \text{for } n \to \infty$$

for every fixed $x \neq 0$. Then proceed as above.

18) *A typical demand is an explicit expression of the sum function.* This can be found by means of the earlier described methods concerning the recognition of the structure as well as use of tables of standard power series.

We have already shown the method in the example in 17). We shall only collect all the results,

$$y = f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n} = \begin{cases} \dfrac{\sin x}{x} & \text{for } x \neq 0, \\[2mm] 1 & \text{for } x = 0. \end{cases}$$

Although it is not necessary it is always a good idea to test the result, i.e. we insert $y = \dfrac{\sin x}{x}$, $x \neq 0$, into the original differential equation, and we check the initial conditions. Notice that

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{1}{x}\{x + x\,\varepsilon(x)\} = \lim_{x \to 0}\{1 + \varepsilon(x)\} = 1,$$

hence the function $y = f(x)$ is *continuous*, what it should be for theoretical reasons.

It "only" remains to elaborate on 15), i.e. describe how one solves recursion formulæ of the given type. This will be done in the next section.

5.7 Solution of recursion formulæ (difference equations)

The simple recursion formulæ occurring in courses in Calculus will for pedagogical reasons all be of the following two types:

I Of first order, i.e. the leap is 1 in the indices,

$$p(n)a_{n+1} = q(n)a_n.$$

II Of second order, i.e. the leap is 2 in the indices

$$p(n)a_{n+2} = q(n)a_n.$$

A weak test is that if one in a problem from a course i Calculus does not get one of these structures, then one has probably made a miscalculation.

Here, $p(n)$ and $q(n)$ are *polynomials in n*, when the corresponding differential equation has polynomial coefficients in x.

1) If we have not already factorized $p(n)$ and $q(n)$ this should be done now!

2) Determine whether $p(n)$ and $q(n)$ have common factors.

   If $p(n) = (n - \alpha)p_1(n)$ and $q(n) = (n - \alpha)q_1(n)$, it is tempting to reduce the expressions by removing $n - \alpha$ in e.g. (we shall only treat case I here, since case II is treated analogously)

   $$(n - \alpha)p_1(n)a_{n+1} = (n - \alpha)q_1(n)a_n, \qquad n = n_0, n_0 + 1, n_0 + 2, \dots .$$

   However, one must not do this without first checking whether it can be done or not.

   a) If α does *not* belong to the domain of the recursion formula, $\{n_0, n_0 + 1, \dots\}$, this can be done immediately, i.e. the recursion formula is reduced to the simpler

      $$p_1(n)a_{n+1} = q_1(n)a_n, \qquad n = n_0, n_0 + 1, \dots,$$

      because $p_1(n)$ and $q_1(n)$ have a lower degree.

   b) If on the other hand there is some $j \in \mathbb{N}_0$, such that $\alpha = n_0 + j$, we reduce the recursion formula for $n = n_0 + j = \alpha$ to the triviality

      $$0 \cdot a_{n_0+j+1} = 0 \cdot a_{n_0+j}.$$

We see that there is *no connection* between a_{n_0+j} and a_{n_0+j+1}, so concerning *formal* calculations they can be chosen independently of each other.
(We may get some explicit values later by using other conditions, but it is not possible to see this at this stage).
In this case we must divide the domain of the recursion formula into to subdomains,

$$\{n_0, n_0+1, \dots, n_0+j-1\} \quad \text{and} \quad \{n_0+j+1, n_0+j+2, \dots\},$$

much each must be treated separately! Notice that we here now miss the index $n = n_0 + j$!.
(This phenomenon occurs actually from time to time).
We can remove $n - n_0 - j$ by reduction of the recursion formula in each of the two subdomains.

3) When we have got rid of common factors (and possibly divided the domain into subdomains and exceptional points) we continue by investigating whether $p_1(n) = 0$ or $q_1(n) = 0$ for some n in a subdomain.

   a) If $p_1(n_1) = 0$, then $q_1(n_1) \neq 0$, because we have removed common factors in point 2). Therefore we conclude from

   $$0 = p_1(n_1) \cdot a_{n_1+1} = q_1(n_1) \cdot a_{n_1},$$

   that $a_{n_1} = 0$. This value is then applied *recursively and decreasingly* in order to get a_{n_1-1} $(= 0$, if we also have $q_1(n_1-1) \neq 0$, etc.; if $q_1(n_1-1) = 0$, or possibly $q_1(n_1-j) = 0$, we here proceed to the next point).

   b) If $q_1(n_2) = 0$, then $p_1(n_2) \neq 0$, because we have already removed common factors in 2). It then follows from

   $$0 = p_1(n_2) a_{n_2+1} = q_1(n_2) a_{n_2},$$

   that $a_{n_2+1} = 0$. We use this value *inductively and increasingly* in order to determine $a_{n_2+2} = 0$, if also $p_1(n_2+1) \neq 0$, etc.; if $p_1(n_2+1) = 0$, or possibly $p_1(n_2+j) = 0$, we return to the first point.nkt.

After these general remarks which can be applied in both cases I and II, we turn to the solution itself. Here we have three possibilities,

- induction (we let the indices increase),
- recursion (we let the indices decrease),
- the "Deus ex machina" solution, or the so-called "divine inspiration". The first time one sees this method it looks like magic, but with some training it often becomes very easy to apply and it leads straight to the solution.

We shall demonstrate all three methods on two examples

I $a_{n+1} = \dfrac{1}{n+1}\, a_n$, $n \in \mathbb{N}_0$, and $a_0 = 1$, (first order)

II $a_{n+2} = \dfrac{-1}{(n+2)(n+3)}\, a_n$, $n \in \mathbb{N}_0$, and $a_0 = 1$, $a_1 = 0$, (second order)

where we have treated II *recursively* previously in 15).

4) *Inductive solution.*
Calculate the first 3–4 elements of the sequence. Try to find some pattern which can be derived from these elements, i.e. set up an hypothesis. *Assume* that this pattern holds for some n. Then show that this implies that the pattern is also valid for the successive step $n+1$. Conclude by induction that this pattern is true for *every* n.

I When $a_0 = 1$ and $a_{n+1} = \dfrac{1}{n+1}\, a_n$, $n \in \mathbb{N}_0$, we get successively

$$a_1 = \frac{1}{0+1}\, a_0 = 1, \quad a_2 = \frac{1}{1+1}\, a_1 = \frac{1}{2},$$

$$a_3 = \frac{1}{2+1}\, a_2 = \frac{1}{6} = \frac{1}{3 \cdot 3 \cdot 1} = \frac{1}{3!}.$$

This gives us the hint that the general pattern is $a_n = \dfrac{1}{n!}$. At least this holds for $n = 0$, 1, 2, 3.

Then assume that we have $a_n = \dfrac{1}{n!}$ for some n. Then we get for its successor that

$$a_{n+1} = \frac{1}{n+1} \cdot a_n = \frac{1}{n+1} \cdot \frac{1}{n!} = \frac{1}{(n+1)!}.$$

This is of the same form as the assumption, only with n replaced by $n+1$, hence the pattern is *the same* for $n+1$. Since the hypothesis holds for $n = 0$, 1, 2, 3, it follows from above that it holds for $n = 4$, $n = 5$, $n = 6$, etc., i.e. for every $n \in \mathbb{N}_0$.

II When $a_0 = 1$ and $a_1 = 0$, and $a_{n+2} = \dfrac{-1}{(n+2)(n+3)}\, a_n$, $n \in \mathbb{N}_0$, the leap of 2 in the indices indicates that we must divide into the cases of n even or odd. We see immediately that since $a_1 = 0$, we must necessarily have $a_3 = 0$, and then $a_5 = 0$, etc., hence it follows by induction that $a_{2p+1} = 0$. (The reader should try to prove this based on the argument in I).

Now $a_0 = 1$, hence we get for first terms with *even* index that

$$a_2 = \frac{-1}{(0+2)(0+3)}\, a_0 = -\frac{1}{3 \cdot 2}, \qquad a_4 = \frac{-1}{(2+2)(2+3)}\, a_3 = \frac{+1}{5 \cdot 4 \cdot 3 \cdot 2},$$

$$a_6 = \frac{-1}{6 \cdot 7}\, a_4 = \frac{-1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 1} = \frac{(-1)^3}{7!}.$$

These all fit in the pattern

$$a_{2p} = \frac{(-1)^p}{(2p+1)!}$$

for $p = 0$, 1, 2, 3.

Assume that this holds for some $n = 2p$. For the successor $n+2 = 2(p+1)$ (notice the leap of 2) we then get

$$\begin{aligned} a_{2p+2} &= \frac{-1}{(2p+2)(2p+3)}\, a_{2p} = \frac{-1}{(2p+3)(2p+2)} \cdot \frac{(-1)^{p+1}}{(2p+1)!} \\ &= \frac{(-1)^{p+1}}{(2p+3)!} = \frac{(-1)^{p+1}}{((2p+2)+1)!}, \end{aligned}$$

hence the same pattern as in the assumption, only with $2p$ replaced by $2p+2$. Then the assumption follows in general by *induction*.

By *induction* we let the indices *increase*, i.e. we derive the *successor*.

5) *Recursive solution.*
Write the recursion formula, such that n (in case I) or $2p$ $(2p+1)$ (in case II) is the largest index. Repeat this iteration on a_{n-1} (case I) or on a_{2p-2} (a_{2p-1}) (case II) etc., as far as we can inside the *domain* and reduce. (Here we can get a counting problem, i.e. how many factors do we have?)

I The recursion formula $a_{n+1} = \dfrac{1}{n+1}\, a_n$, $n \in \mathbb{N}_0$, is equivalent to

$$a_n = \frac{1}{n}\, a_{n-1} \qquad \text{for } n \in \mathbb{N}.$$

Check the first term: For $n=0$ we have $a_{0+1} = a_1 = \dfrac{1}{0+1}\, a_0$ in the first formula, and for $n=1$ we have $a_1 = \dfrac{1}{1}\, a_{1-1} = a_0$ in the second formula; hence they are equivalent. When we repeat the formula for $n-1$, etc., we get

$$\begin{aligned} a_n &= \frac{1}{n}\, a_{n-1} = \frac{1}{n}\left\{\frac{1}{n-1}\, a_{n-2}\right\} = \cdots \\ &= \frac{1}{n}\cdot\frac{1}{n-1}\cdot\frac{1}{n-2}\cdots\frac{1}{1}\cdot a_{1-1} = \frac{1}{n!}\, a_0 = \frac{1}{n!}, \end{aligned}$$

because $a_0 = 1$ was given. This holds for every $n \in \mathbb{N}$, and trivially for $n=0$, thus $a_n = \dfrac{1}{n!}$ for $n \in \mathbb{N}_0$.

II When $a_1 = 0$, we always apply the *inductive method* to conclude that $a_{2p+1} = 0$ for every *odd* indices $2p+1 \in \mathbb{N}_0$, i.e. for $p \in \mathbb{N}_0$.

For the *even* indices $n = 2p$, we write the recursion formula

$$a_{2p+2} = \frac{-1}{(2p+2)(2p+3)}\, a_2 p \qquad \text{for } p \in \mathbb{N}_0.$$

When p is replaced by $p-1$ (be very careful with the factor 2 in the formula; here one often sees miscalculations), we get

$$a_{2p} = \frac{-1}{(2p+1)(2p)}\, a_{2p-2} \qquad \text{for } p \in \mathbb{N} \quad \text{(check the first term).}$$

This gives by *recursion* (i.e. a repetition of the pattern on a_{2p-2} etc.)

$$\begin{aligned} a_{2p} &= \frac{-1}{(2p+1)(2p)}\, a_{2p-2} = \frac{-1}{(2p+1)(2p)}\cdot\frac{-1}{(2p-1)(2p-2)}\cdot a_{2p-4} = \cdots \\ &= \frac{-1}{(2p+1)(2p)}\cdot\frac{-1}{(2p-1)(2p-2)}\cdots\frac{-1}{3\cdot 2\, a_2 2} \\ &\qquad (p \text{ factors, e.g. } 2p,\ 2(p-1),\ \ldots,\ 2\cdot 1) \\ &= \frac{(-1)^p}{(2p+1)!}\, a_0 = \frac{(-1)^p}{(2p+1)!}, \end{aligned}$$

because $a_0 = 1$. Therefore, $a_{2p} = \dfrac{(-1)^p}{(2p+1)!}$ for $p \in \mathbb{N}$, and since this is trivially true for $p=0$, we have proved the formula for every $p \in \mathbb{N}_0$.

6) *The "divine inspiration".*
Multiply the recursion formula by some factor $\neq 0$, such that the two sides of the equation obtain the same *structure*, only for different indices. Introduce if necessary an auxiliary sequence b_n, and reduce.

I The formula $a_{n+1} = \dfrac{1}{n+1}\, a_n$, $n \in \mathbb{N}_0$, is multiplied by $(n+1)! \neq 0$. We get

$$(n+1)!\, a_{n+1} = n!\, a_n, \qquad n \in \mathbb{N}_0,$$

where the left hand side can be derived from the right hand side by writing $n+1$ instead of n. Writing $b_n = n!\, a_n$, we obtain the very simple recursion formula

$$b_{n+1} = b_n = b_{n-1} = \cdots = b_0 = 0!\, a_0 = 1.$$

Thus

$$b_n = n! a_n = 1, \qquad \text{i.e. } a_n = \frac{1}{n!} \quad \text{for } n \in \mathbb{N}_0.$$

II First we inspect the formula

$$a_{2p+2} = \frac{-1}{(2p+3)(2p+2)}\, a_{2p}, \qquad p \in \mathbb{N}_0.$$

We see that there is a leap of 2 in the indices, as well as two successive factors in the denominator, so we may expect a multiplication by $(2p+3)! \neq 0$. There is furthermore a change of sign. We compensate for this by a multiplication by $(-1)^{p+1}$. All things considered we multiply by $(-1)^{p+1}(2p+3)! \neq 0$. Then we get the equivalent recursion formula

$$\begin{aligned}(-1)^{p+1}((2p+3)!)a_{2p+2} &= (-1)^{p+1}(2(p+1)+1)!a_{2(p+1)}\\ &= (-1)^p(2p+1)!a_{2p}, \quad p \in \mathbb{N}_0.\end{aligned}$$

We see that the left hand side can be derived from the right hand side by writing $p+1$ instead of p. If we put $b_p = (-1)^p(2p+1)!\, a_{2p}$, we get the trivial recursion formula

$$b_{p+1} = b_p = b_{p-1} = \cdots = b_0 = (-1)^0 \cdot (2 \cdot 0 + 1)!\, a_0 = 1 \cdot 1 \cdot 1 = 1.$$

But then

$$b_p = (-1)^p(2p+1)!\, a_{2p} = 1, \quad \text{i.e. } a_{2p} = \frac{(-1)^p}{(2p+1)!}, \quad \text{for } p \in \mathbb{N}_0.$$

In general we investigate the recursion formula for *odd* indices in the same way, but since we already know that $a_1 = 0$, it is here easier to conclude *inductively* that $a_{2p+1} = 0$ for every $p \in \mathbb{N}_0$, cf. point 4).

The argument is not at all "divine" although it may look like magic the first time one sees it. In fact, we have a true argument in the beginning of II. But the method demands a lot of training, and one should not in general choose this as one's favorite standard method. I have here ordered the three methods according to what I believe that the student will consider the easiest one [point 4)], the middle one [point 5)] and the most difficult one [point 6)]. There may, however, be students whose priorities are different from this order.

5.8 Second order differential equations (straight tips)

There exist some non-authorized straight tips which are only rarely described in usual textbooks in Calculus. I shall here present some of them.

Given a differential equation

$$(8)\quad p_0(x)\,\frac{d^2y}{dx^2} + p_1(x)\,\frac{dx}{dx} + p_2(x)\,y = g(x), \qquad x \in \mathbb{R},$$

where $p_0(x)$, $p_1(x)$, $p_2(x)$ are polynomials in x without any common zero, and where $g(x)$ has a convergent power series expansion valid in all of $\mathbb{R}$.

Find all *complex* zeros to $p_0(x)$ (the coefficient of the term of highest order $\dfrac{d^2y}{dx^2}$). If $p_0(x)$ is a polynomial of degree m, then we have at most m different zeros $\{x_1, x_2, \dots, x_q\}$, $q \leq m$, and $x_j \in \mathbb{C}$.

The radius of convergence ϱ for any power series solution of (8) is then one of the numbers $|x_1|$, $|x_2|$, $\dots$, $|x_q|$ or ∞.

We note that if $p_0(0) \neq 0$, then there will always exist a power series solution.

If $p_0(0) = 0$, there does not necessarily exist a power series solution.

We call the zeros of $p_0(x)$ for the *singular points* of the equation. It is seen that they determine the radius of convergence ϱ of a power series solution.

The numerical values of the singular points indicate the *possible* value of the radius of convergence. One shall still find ϱ by one of the earlier methods, e.g. by an application of the *ratio test.* The straight tip above, however, gives us a possibility of a weak control. If one by e.g. the ration test finds a ϱ, which is *not* either ∞ or the numerical value of some singular point, then we can conclude that we have made a miscalculation of ϱ.

Example 5.4 In the differential equation

$$x\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + xy = 0, \qquad x \in \mathbb{R},$$

we see that $p_0(x) = x$. The only singular point is $x = 0$. The *possible* values of ϱ are either 0 or ∞. It can be shown that the power series solution in fact has $\varrho = \infty$. $\Diamond$

Example 5.5 *(A Bessel equation; not usually in the syllabus, although it is extremely relevant in some engineering courses).* We see that the differential equation

$$x^2\,\frac{d^2y}{dx^2} + \left(x^2 + \frac{1}{4}\right) y = 0, \qquad x \in \mathbb{R},$$

has $p_0(x) = x^2$. The only zero is $x = 0$ (of multiplicity 2), hence we must have $\varrho \in \{0, \infty\}$ for a power series solution. It is here possible to find a power series solution where $\varrho = \infty$. $\Diamond$

Example 5.6 *(Outside the usual syllabus).* For the differential equation

$$\frac{d^4y}{dx^4} + (\lambda - x)\,\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

we have $p_0(x) = 1$. This is a constant $\neq 0$, hence it has no zeros. Therefore, i) there exist power series solutions, and ii) they all have radius of convergence $\varrho = \infty$. $\Diamond$

Example 5.7 If $p_0(x) = 1 + x^2$, then $p_0(x) \neq 0$ for every $x \in \mathbb{R}$. One could then be misled to the wrong conclusion that ϱ can only be ∞. This is *not* true. The equation $p_0(x) = 1 + x^2 = 0$ has the two *complex* roots $\pm i$. Since $|\pm i| = 1 \neq 0$, we conclude that i) such an equation has power series solutions, and ii) every one of these has a radius of convergence $\varrho \in \{1, \infty\}$, i.e. either $\varrho = 1$ or $\varrho = \infty$. $\Diamond$

The next trick demands a lot of training. In principle *integration* is the inverse operation of *differentiation.* Therefore, if we can rewrite the equation

$$p_0(x)\,\frac{d^2y}{dx^2} + p_1(x)\,\frac{dy}{dx} + p_2(x)\,y = g(x)$$

in the form

$$\frac{d}{dx}\left\{F_2(x)\,\frac{d}{dx}\,\{F_1(x)\,y\}\right\} = G(x)$$

by possibly adding a factor $a(x)$, such that $G(x) = a(x)g(x)$, then we get by integration that

$$F_2(x)\,\frac{d}{dx}\{F_1(x)y\} = c_2 + \int_{x_0}^{x} G(t)\,dt, \qquad c_2 \text{ arbitrær konstant,}$$

i.e.

$$\frac{d}{dx}\,\{F_1(x)y\} = \frac{c_2}{F_2(x)} + \frac{1}{F_2(x)}\int_{x_0}^{x} G(t)\,dt.$$

Another integration gives

$$F_1(x)y = c_1 + c_2\int_{x_0}^{x}\frac{dt}{F_2(t)} + \int_{x_0}^{x}\frac{1}{F_2(t)}\left\{\int_{x_0}^{t} G(u)\,du\right\}dt,$$

where c_1 and c_2 are arbitrary constants. Thus, if we neglect possible zeros of $F_1(t)$ and $F_2(t)$ etc., the complete solution becomes

$$y = \frac{c_1}{F_1(x)} + \frac{c_2}{F_1(x)}\int_{x_0}^{x}\frac{dt}{F_2(t)} + \frac{1}{F_1(x)}\int_{x_0}^{x}\frac{1}{F_2(t)}\left\{\int_{x_0}^{t} G(u)\,du\right\}dt,$$

where c_1 and c_2 are arbitrary constants.

The solution of the corresponding *homogeneous equation* is obtained by putting $G(x) = 0$, i.e.

$$y = c_1\varphi_1(x) + c_2\varphi_2(x) = \frac{c_1}{F_1(x)} + \frac{c_2}{F_1(x)}\int_{x_0}^{x}\frac{dt}{F_2(t)},$$

from which it is seen that the linearly independent solutions of the homogeneous equation are given by

$$\varphi_1(x) = \frac{1}{F_1(x)} \quad \text{and} \quad \varphi_2(x) = \frac{1}{F_1(x)}\int_{x_0}^{x}\frac{dt}{F_2(t)}.$$

By recalling the well-known *rules of differentiation*

$$f\,\frac{dg}{dx} + g\,\frac{df}{dx} = \frac{d}{dx}\,(f\cdot g),$$

and

$$f\,\frac{dg}{dx} - g\,\frac{df}{dx} = f^2\,\frac{d}{dx}\left(\frac{g}{f}\right), \qquad \text{provided that } f(x) \neq 0,$$

we shall now demonstrate how we often can *guess* the structure

$$\frac{d}{dx}\left\{F_2(x)\,\frac{d}{dx}\,\{F_1(x)y\}\right\} = G(x),$$

or possibly

$$(9)\quad \frac{d}{dx}\left\{F_2(x)\,\frac{d}{dx}\left\{\frac{y}{\varphi_1(x)}\right\}\right\} = G(x),$$

if only we have given a solution $\varphi_(x)$ of the homogeneous equation. In the case of (9) we can even calculate the expression and compare it with the given equation in order to find explicitly the functions $F_2(x)$ and $G(x) = a(x)g(x)$, i.e. find $a(x)$.

All this is theoretically very easy; but the procedure is actually difficult in practice, even though most of the equations in simple courses of Calculus can be solved in this way by inspection. Of the known equations so far it is only the *Bessel equation* in the example above which *cannot* be solved in this way.

On the other hand, the student should *not* only rely on this method, but only consider it as a valuable alternative.

Example 5.8 The example above can now be solved in the following way:

$$\begin{aligned}
0 &= x\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + xy && \text{The trick is to write } 2 = 1+1 \\
&= \left\{x\,\frac{d^2y}{dx^2} + 1\cdot\frac{dy}{dx}\right\} + \frac{dy}{dx} + xy && \text{Write } 1 = \frac{dx}{dx} \text{ and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \left\{x\,\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dx}{dx}\cdot\frac{dy}{dx}\right\} + \frac{dy}{dx} + xy && \text{Differentiation of a product} \\
&= \frac{d}{dx}\left\{x\,\frac{dy}{dx}\right\} + \frac{dy}{dx} + xy && \text{Differentiation is linear} \\
&= \frac{d}{dx}\left\{x\cdot\frac{dy}{dx} + 1\cdot y\right\} + xy && \text{Differentiation of a product} \\
&= \frac{d}{dx}\left\{\frac{d}{dx}(xy)\right\} + xy.
\end{aligned}$$

Put $z = xy$. The equation is then reduced to

$$\frac{d^2z}{dx^2} + z = 0,$$

with the complete solution

$$z = xy = c_1 \sin x + c_2 \cos x, \qquad c_1, c_2 \text{ arbitrar constants.}$$

For $x \neq 0$ we get the complete solution

$$y = c_1\,\frac{\sin x}{x} + c_2\,\frac{\cos x}{x}, \qquad c_1, c_2 \text{ arbitrar constants,}$$

and we see that the calculations are somewhat easier to perform than by inserting a power series. Unfortunately, the method relies heavily on that on immediately sees the right way of putting the equation by writing

$$2\,\frac{dy}{dx} = \frac{dy}{dx} + \frac{dy}{dx},$$

and then treat the to terms on the right hand side differently. $\Diamond$

Example 5.9 *(Outside the usual syllabus).* We shall here in a more difficult example add the arguments which are indicated at the end of the example above. We rewrite the equation in the following way:

$$\begin{aligned}
0 &= \frac{d^4y}{dx^4} + (\lambda - x)\,\frac{d^2y}{dx^2} - \frac{dy}{dx} && \text{Write } -1 = \frac{d}{dx}(\lambda - x) \\
&= \frac{d^4x}{dx^4} + \left\{(\lambda - x)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d}{dx}(\lambda - x)\,\frac{dy}{dx}\right\} && \text{Differentiation of a product} \\
&= \frac{d^4y}{dx^4} + \frac{d}{dx}\left\{(\lambda - x)\,\frac{dy}{dx}\right\} && \text{Differentiation is linear} \\
&= \frac{d}{dx}\left\{\frac{d^3y}{dx^3} + (\lambda - x)\,\frac{dy}{dx}\right\},
\end{aligned}$$

hence by an integration

$$\frac{d^3y}{dx^3} + (\lambda - x)\,\frac{dy}{dx} = c.$$

Putting $x = \lambda$ we get $c = 0$ from the given boundary conditions. Hence

$$\frac{d^3y}{dx^3} + (\lambda - x)\frac{dy}{dx} = 0.$$

Putting $z = \dfrac{dy}{dx}$ we see that this equation with the corresponding boundary values is equivalent to the system

$$\begin{cases} \dfrac{d^2z}{dx^2} + (\lambda - x)z = 0, & z(0) = z'(\lambda) = z''(\lambda) = 0, \\ \dfrac{dy}{dx} = z, & y(0) = 0. \end{cases}$$

The next trick is to put $z(x) = \varphi(\lambda - x)$ and then perform the change of variable $t = \lambda - x$. Then we get by the *chain rule* that

$$\frac{dz}{dx} = \frac{dt}{dx}\cdot\frac{d\varphi}{dt} = -\frac{d\varphi}{dt},$$

$$\frac{d^2z}{dx^2} = \frac{dt}{dx}\cdot\frac{d}{dx}\left(\frac{dz}{dx}\right) = -\frac{d}{dt}\left(-\frac{d\varphi}{dt}\right) = \frac{d^2\varphi}{dt^2},$$

and we have transformed the first equation into

$$\frac{d^2\varphi}{dt^2} + t\cdot\varphi(t) = 0, \qquad \varphi(\lambda) = \varphi'(0) = \varphi''(0) = 0.$$

Now, it can be proved that this equation *does not have an elementary function as a solution* $\varphi(t)$, so at this stage we have somehow to use the method of inserting power series.

Replace the boundary problem with the following *initial value problem*:

$$\frac{d^2\varphi}{dt^2} + t\cdot\varphi(t) = 0, \qquad \varphi(0) = 1 \text{ and } \varphi'(0) = 0.$$

When $t = 0$ we see that we must have $\varphi''(0) = 0$.
Since $p_0(t) = 1$ there are no singular points, hence i) the equation has a (unique) power series solution, and ii) the radius of convergence is $\varrho = \infty$.

By insertion of the power series solution

$$\varphi(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad \varphi''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2},$$

we get $a_0 = \varphi(0) = 1$ and $a_1 = 1\cdot\varphi'(0) = 0$, and

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+1} \\ &= \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+1} \qquad (n-2 = m+1 \text{ etc.}) \\ &= 2\cdot 1\cdot a_2 + \sum_{n=0}^{\infty} \left\{(n+3)(n+2)a_{n+3} + a_n\right\} t^{n+1}. \end{aligned}$$

It follows from the *identity theorem* that $a_2 = 0$ and that we have the recursion formula

$$(n+3)(n+2)a_{n+3} + a_n = 0 \quad \text{for } n \in \mathbb{N}_0 \quad \text{(the summation domain).}$$

Since $a_0 = 1$ and $a_1 = 0$ and $a_2 = 0$, and since there is a leap of 3 in the indices, we get by *induction* (because $(n+3)(n+2) \neq 0$ for $n \in \mathbb{N}_0$) that

$$a_{3p+1} = 0 \quad \text{and} \quad a_{3p+2} = 0, \qquad \text{for every } p \in \mathbb{N}_0,$$

hence the power series solution is of the form

$$\varphi(t) = \sum_{p=0}^{\infty} a_{3p} t^{3p},$$

and $b_p = a_{3p}$ satisfies the recursion formula $(n = 3p)$

$$(3p+3)(3p+2)b_{p+1} = -b_p \qquad \text{for } p \in \mathbb{N}_0, \text{ and } b_0 = 1.$$

This is not a "nice" formula. We find the first coefficients

$$b_0 = 1, \quad b_1 = -\frac{1}{2 \cdot 3}, \quad b_2 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}, \quad b_3 = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \quad \dots,$$

so

$$\varphi(t) = \sum_{p=0}^{\infty} b_p t^{3p} = 1 - \frac{1}{6}\,t^3 + \frac{1}{180}\,t^6 - \frac{1}{12960}\,t^9 + \cdots .$$

The ratio test applied on the *recursion formula* confirms that we indeed have $\varrho = \infty$.

When we have found $\varphi(t)$ we shall still find λ such that $\varphi(\lambda) = 0$, $\lambda > 0$. The primitive method – which works – consists of successively to determine possible positive zeros in

$$\begin{aligned}
&\varphi_1(\lambda) = 1 - \frac{1}{6}\,\lambda^3, && \lambda_{1,1} = 1,81712,\\
&\varphi_2(\lambda) = 1 - \frac{1}{6}\,\lambda^3 + \frac{1}{180}\,\lambda^6, && \lambda_{2,1} = 2,02403,\\
&\varphi_3(\lambda) = 1 - \frac{1}{6}\,\lambda^3 + \frac{1}{180}\,\lambda^6 - \frac{1}{12960}\,\lambda^9, && \lambda_{3,1} = 1,98444,\\
&\text{etc.} && \text{etc.,}
\end{aligned}$$

and then arrange these according to size. In the example I have calculated the smallest positive zero on a pocket calculator. It is seen that the convergence is *very fast*. It can be shown that if the j-th zero $\lambda_{n,j}$ exists as a real positive number for $n \geq N$, then the true j-th eigenvalue will always lie between two successive terms in the sequence for $n \geq N$ (and closest to the term of highest index). In the given example we get

$$1,98444 < \lambda_1 < 2,02403,$$

where the first eigenvalue λ_1 lies closest to the lower bound.

The convergence of the second eigenvalue λ_2 is somewhat slower, For instance we see that $\lambda_{3,2}$ is complex because $\varphi_4(\lambda)$ has only one real root. However, after a couple of extra iterations we obtain a reasonable approximation (and at the same time we see that the approximations of the previous eigenvalue also is enhanced).

The interested reader may as an exercise on the pocket calculator of this method find the first positive zero π of

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n},$$

by solving successively

$$\varphi_1(x) = 1 - \frac{1}{6}\, x^2, \qquad \varphi_2(x) = 1 - \frac{1}{6}\, x^2 + \frac{1}{120}\, x^4, \qquad \text{etc..}$$

It is seen that when the complex roots disappear, then the convergence towards the true value $\lambda_1 = \pi$ is in fact fairly fast. $\Diamond$

5.9 Differential equations of second order, a general solution formula

It is convenient here to mention a solution formula for a general differential equation of second order

$$(10) \quad f_2(t)\, \frac{d^2y}{dt^2} + f_1(t)\, \frac{dy}{dt} + f_0(t)\, y = u(t), \qquad t \in I,$$

because it is closely connected to the method of power series.

There does not exist a general solution formula of (10), but if one knows just one solution $y_1(t) \neq 0$ of the corresponding homogeneous equation

$$f_2(t)\, \frac{d^2y_1}{dt^2} + f_1(t)\, \frac{dy_1}{dt} + f_0(t)\, y_1 = 0,$$

then equation (10) can be *completely solved!*

The simplest method is to apply a theorem which should be included in every textbook on this matter, and which I shall recommend as the *standard method*. The procedure is the following:

1) First find a nontrivial solution $y_1(t)$ of the homogeneous equation.

   This may be i) either given, ii) or found by a lucky guess (possibly by using a given hint), iii) or it can be found by the method of power series (when $f_2(t)$ and $f_1(t)$ and $f_0(t)$ are polynomials).

2) Normalize the equation (10), i.e. divide by $f_2(t)$,

$$\frac{d^2y}{dt^2} + \frac{f_1(t)}{f_2(t)}\, \frac{dy}{dt} + \frac{f_0(t)}{f_2(t)}\, y = \frac{u(t)}{f_2(t)}.$$

   (If one forgets to perform this part of the process, we shall later obtain *very wrong* formulæ). The zeros of the denominator $f_2(t)$ divide I into subintervals. The method of solution can only be applied on these *open* subintervals, so one should always afterwards also *test* whether the found solution also is valid in these points. The same can be said of the zeros for $y_1(t)$.

3) Calculate the function

$$\Omega(t) = \exp\left(\frac{f_1(t)}{f_2(t)}dt\right).$$

Any antiderivative of $f_1(t)/f_2(t)$ may be used. Choose that one which after a reduction gives the simplest $\Omega(t)$.

4) Calculate

$$\int y_1(t)\,\Omega(t)\cdot\frac{u(t)}{f_2(t)}\,dt = F(t).$$

Again we may use any antiderivative. Do not forget the normalizing factor $f_2(t)$ in the denominator (a frequent error).

5) Calculate

$$\int \frac{F(t)}{\Omega(t)}\cdot\frac{1}{y_1(t)^2}\,dt = G_1(t) \quad \text{and} \quad \int \frac{1}{\Omega(t)}\cdot\frac{1}{y_1(t)^2}\,dt = G_2(t).$$

Here we can also apply any antiderivative.

6) The solution of (10) in then in every of the open subintervals of I, where $f_2(t) \neq 0$, given by

$$y(t) = y_1(t)\cdot G_1(t) + c_1 y_1(t) + c_2 y_1(t) G_2(t),$$

where c_1 and c_2 are arbitrary constants.

Notice that $y_2(t) = y_1(t)G_2(t)$ is a solution of the homogeneous equation, and that $y_1(t)$ and $y_2(t)$ are linearly independent.

7) Test the result, i.e. check if the complete solution can be extended to the zeros of $f_2(t)$ and $y_1(t)$.

A compact form of 4)–6) is

$$y(t) = y_1(t)\int \frac{1}{\Omega(t)y_1(t)^2}\left\{\int \Omega(t)y_1(t)\cdot\frac{u(t)}{f_2(t)}\,dt\right\}dt,$$

where $f_2(t)$ is the coefficient of $\dfrac{d^2y}{dt^2}$ in (10) and $y_1(t)$ is introduced in 1), and $\Omega(t)$ is introduced in 3). By each inside integration *we add an arbitrary constant* (c_2 by the innermost integral, and c_1 by the outermost integral). Notice that $\Omega(t)$ occurs once in the "numerator" and once in the "denominator", and that $y_1(t)$ occurs twice in the "numerator" and also twice in the "denominator".

We see that it is only the normalizing factor $f_2(t)$ in the denominator which is not put in this "symmetric way".

Remark 5.1 The method of Wrońskians, which is described in every textbook of Calculus, is of course correct. However, it is also unnecessarily complicated. It is an unhappy historical relict, originally aiming at quite another object. The theory of determinants were introduced in Linear Algebra in the beginning of the 1800s. Like any other new inventions there was a large opposition among scientists in those days against this theory. In order to show that determinants actually were useful Liouville developed in 1829 this theory for differential equations of second order. For that reason the formulæ became really “contaminated” by determinants.

This example convinced the established mathematical world, and Liouville’s solution was from then on called *the method*, and it was further developed by many mathematicians in the rest of the 19th century, and then the theory fossilized. It looks like that only Belmann (in the 20th century) had realized that the determinants were not at all necessary, but no one listened to him, because he was notorious for his hot temper and his very egocentric appearance.

It is also a mystery why Wroński got his name attached to this formula. He has apparently had nothing to do with the development on the determinant, which now is named after him. The point is of course that it is very difficult to find a formula years before one is even born! It is possibly a misquote from some one in the 19th century, and then his name got stuck to it.

Finally it should be mentioned that the Wrońskian $W(t)$ is connected with the theory here by the formula

$$W(t)\Omega(t) = c,$$

where the constant $c \neq 0$ depends on the chosen system $y_1(t)$, $y_2(t)$ of linear independent solution of the homogeneous equation.

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}, \qquad \text{(often very difficult to calculate).}$$

Note that the calculation of $\Omega(t)$ *only* applies two of the coefficient functions of the equation, so $\Omega(t)$ should be easy to find. $\Diamond$

A Formulæ

Some of the following formulæ can be assumed to be known from high school. Others are introduced in Calculus 1a. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

A.1 Squares etc.

The following simple formulæ occurs very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

A.2 Powers etc.

Logarithm:

$$
\begin{array}{rcll}
\ln|xy| &=& \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| &=& \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| &=& r\ln|x|, & x\neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\ y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\ y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark A.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

A.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

A.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \ \ \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ og } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1,1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]1,+\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark A.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\diamondsuit$

A.5 Integration

The most obvious rules are about linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and about that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark A.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. ◊

Remark A.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. See also Chapter 4. ◊

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark A.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ og $\varphi'(y) = 2y$. $\Diamond$

A.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1,} \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R},\ \text{og } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

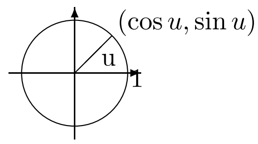
$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

A.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a piont P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.



Figur 3: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\, u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\, u)$ and $\exp(-\mathrm{i}\, u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\, u) + \exp(-\mathrm{i}\, u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\, u) - \exp(-\mathrm{i}\, u)),$$

.

Moivre's formula: By expressing $\exp(inu)$ in two different ways we get:

$$\exp(inu) = \cos nu + \text{ i } \sin nu = (\cos u + \text{i } \sin u)^n.$$

Example A.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical appliction,

$$\begin{aligned}
\cos(3u) + \text{ i } \sin(3u) &= (\cos u + \text{ i } \sin u)^3 \\
&= \cos^3 u + 3\text{i } \cos^2 u \cdot \sin u + 3\text{i}^2 \cos u \cdot \sin^2 u + \text{ i}^3 \sin^3 u \\
&= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \text{ i}\{3\cos^2 u \cdot \sin u - \sin^3 u\} \\
&= \{4\cos^3 u - 3\cos u\} + \text{ i}\{3\sin u - 4\sin^3 u\}
\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \lozenge$$

Addition formulæ:

$$\sin(u+v) = \sin u \, \cos v + \cos u \, \sin v,$$

$$\sin(u-v) = \sin u \, \cos v - \cos u \, \sin v,$$

$$\cos(u+v) = \cos u \, \cos v - \sin u \, \sin v,$$

$$\cos(u-v) = \cos u \, \cos v + \sin u \, \sin v.$$

Products of trigonometric functions to a sum:

$$\sin u \, \cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u \, \sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u \, \sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u \, \cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u \, \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

A.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

A.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x.$$

A.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and*exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

A.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$