Fourier Series and Systems of Differential...

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Fourier Series and Systems of Differential Equations and Eigenvalue Problems

Guidelines for Solutions of Problems

Calculus 4b

Fourier Series and Systems of Differential Equations and
Eigenvalue Problems – Guidelines for Solutions of Problems – Calculus 4b

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Contents

Preface

In this volume I shall give some guidelines for solving problems in the theories of Fourier series and Systems of Differential Equations and eigenvalue problems. The reader should be aware of that it has never been my intention to write an alternative textbook, since then I would have disposed of the subject in another way. It is, however, my hope that this text can be used as a supplement to the normal textbooks in which one can find all the necessary proofs.

This text is a successor of *Calculus 1a, Functions of one Variable* and *Calculus 3b, Sequences and Power Series*, which will be assumed in the following.

Chapter 1 in this book is a short review of some important trigonometric formulæ, which will be used over and over again in connection with Fourier series. This is a part of the larger Chapter 1 in *Calculus 3b, Sequences and Power Series.* Here we shall we concentrate on the trigonometric functions. This introducing chapter should be studied carefully together with Appendix A, which is a collection of the important formulæ already known from high school and previous courses in Calculus. Since we shall assume this, we urge the student to learn most of the formulæ of Appendix A by heart.

The text in the following chapters is more difficult than the previous mentions texts on Calculus. The Fourier series have always been included in the syllabus, but they have also been considered by the student as very difficult. I have here added a chapter with a catalogue over standard examples and standard problems with their results, though without their corresponding calculations.

Then follows a little about linear systems of differential equations, where some results from Linear Algebra are applied. I have tried always to find the simplest methods of solution, because the traditional textbooks follow the usual tendency of using a style which is more common in advanced books on mathematics without giving the innocent reader any hint of how one may use this theory in practice. In one of the variants we use the *Caley Hamilton's theorem* known from Linear Algebra. This theorem may, however, not be known to all readers. The theory is illustrated by (2×2)-matrices.

At last we give a short review of *eigenvalue problems.* This is really a difficult subject, and it is only possible to benefit from it, when one at least knows the theory of Fourier series. On the other hand, the eigenvalue problem is extremely relevant for the engineering sciences – here demonstrated by the theory for bending of beams and columns. I also know of applications in the theory of chloride ingress into concrete, let alone the periodic signals in emission theory. These applications have convinced me that the eigenvalue problems are very important for the applications. On the other hand, the theory is also difficult, so it is usually played down in the teaching which to my opinion is a pity. I shall not dare here to claim that I have found the right way to present these matters, but I shall at least give it a try.

All notes from in this series of Calculus are denoted by a number – here 4 – and a letter – here b – where

a stands for "compendium",

b stands for "solution procedures of standard problems",

c stands for "examples".

Since this is the first edition of this text in English, I cannot avoid some errors. I hope that the reader will see mildly on such errors, as long as they are not really misleading.

26th June 2014
Leif Mejlbro

1 Review of some important trigonometric formulæ

1.1 Trigonometric formulæ

We quote from *Calculus 1a, Functions in one Variable*, the *addition formulæ*

(1) $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$,

(2) $\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$,

(3) $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$,

(4) $\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$.

Mnemonic rule: $\cos x$ is *even*, and $\sin x$ is *odd*. Since $\cos(x \pm y)$ is *even*, the reduction can only contain the terms $\cos x \cdot \cos y$ *(even times even)* and $\sin x \cdot \sin y$ *(odd times odd)*. We have furthermore a *change of sign* on the term $\sin x \cdot \sin y$.

Analogously, $\sin(x \pm y)$ is *odd*, thus the reduction can only contain the terms $\sin x \cdot \cos y$ *(odd times even)* and $\cos x \cdot \sin y$ *(even times odd)*. Here there is no change of sign on sinus. $\Diamond$

We shall also sometimes need to simplify products like

$$\sin x \cdot \sin y, \qquad \cos x \cdot \cos y, \qquad \sin x \cdot \cos y.$$

even even odd

We obtain these simplifications from the *addition formulæ above*:

$$2\sin x \cdot \sin y = \cos(x-y) - \cos(x+y), \qquad (2)-(1),$$

$$2\cos x \cdot \cos y = \cos(x-y) + \cos(x+y), \qquad (2)+(1),$$

$$2\sin x \cdot \cos y = \sin(x-y) + \sin(x+y), \qquad (3)+(4).$$

We get our formulæ by a division by 2. They are called the *antilogarithmic formulæ*:

$$\sin x \cdot \sin y = \frac{1}{2}\{\cos(x-y) - \cos(x+y)\}, \qquad \text{even},$$

$$\cos x \cdot \cos y = \frac{1}{2}\{\cos(x-y) + \cos(x+y)\}, \qquad \text{even},$$

$$\sin x \cdot \cos y = \frac{1}{2}\{\sin(x-y) + \sin(x+y)\}, \qquad \text{odd}.$$

1.2 Integration of trigonometric polynomials

Problem: Find

$$\int \sin^m x \cdot \cos^n x \, dx, \qquad m, n \in \mathbb{N}_0.$$

We shall in the following only consider one term of the form $\sin^m x \cdot \cos^n x$ of a trigonometric polynomial, where m and $n \in \mathbb{N}_0$.

We define the *degree* of $\sin^m x \cdot \cos^n x$ as the *sum* $m+n$ of the exponents.

Concerning integration of such a term we have two main cases: Is the term of *odd* or *even* degree? These two cases are again each divided into two subcases, so all things considered we are left with four different variants of integration of a trigonometric function of the type described above:

1) The degree $m+n$ is odd.

   a) $m = 2p$ is even and $n = 2q+1$ is odd.

   b) $m = 2p+1$ is odd and $n = 2q$ is even.

2) The degree $m+n$ is even.

   a) $m = 2p+1$ and $n = 2q+1$ are both odd.

   b) $m = 2p$ and $n = 2q$ are both even.

1a) $m = 2p$ is even and $n = 2q+1$ is odd.

Use the substitution $u = \sin x$ (corresponding to $m = 2p$ even), and write

$$\cos^{2q+1} x\, dx = (1-\sin^2 x)^q \cos x\, dx = (1-\sin^2 x)^q\, d\sin x.$$

Then we get

$$\int \sin^{2p} x \cdot \cos^{2q+1} x \, dx = \int \sin^{2p} x (1-\sin^2 x)^q \, d\sin x = \int_{u=\sin x} u^{2p} \cdot (1-u^2)^q \, du,$$

and the problem has been reduced to an integration of a polynomial followed by a substitution.

1b) $m = 2p+1$ is odd and $n = 2q$ is even.

Use the substitution $u = \cos x$ (corresponding to $n = 2q$ even), and write

$$\sin^{2p+1} x \, dx = (1-\cos^2 x)^p \cos x \, dx = -(1-\cos^2)^p \, d\cos x.$$

Then

$$\int \sin^{2p+1} x \cdot \cos^{2q} x \, dx = -\int (1-\cos^2 x)^p \cdot \cos^{2q} x \, d\cos x = -\int_{u=\cos x} (1-u^2)^p \cdot u^{2q} \, du,$$

and the problem has again been reduced to an integration of a polynomial followed by a substitution.
2) Then consider the case where the degree $m+n$ is even. Here the trick is to pass to the double angle by the formulæ

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin x \cdot \cos x = \frac{1}{2}\sin 2x.$$

2a) $m = 2p+1$ and $n = 2q+1$ are both odd.

Rewrite the integrand in the following way:

$$\sin^{2p+1} x \cdot \cos^{2q+1} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p \left\{\frac{1}{2}(1+\cos 2x)\right\}^q \cdot \frac{1}{2}\sin 2x.$$

Then the problem is reduced to case 1b), and we get by the substitution $u = \cos 2x$ that

$$\int \sin^{2p+1} x \cdot \cos^{2q+1} x \, dx = -\frac{1}{2^{p+q+1}} \cdot \frac{1}{2} \int_{u=\cos 2x} (1-u)^p (1+u)^q \, du.$$

The problem is again reduced to an integration of a polynomial followed by a substitution.

2b) $m = 2p$ and $n = 2q$ are both odd.

This is actually the worst case. First we rewrite the integrand in the following way:

$$\sin^{2p} x \cdot \cos^{2q} x = \left\{\frac{1}{2}(1-\cos 2x)\right\}^p \left\{\frac{1}{2}(1+\cos 2x)\right\}^q.$$

We see that the left hand side has degree $2p+2q$ in $(\cos x, \sin x)$, while the right hand side has got its degree halved $p+q$ with respect to $(\cos 2x, \sin 2x)$, i.e. described by the double angle. On the other hand we have also got one term written as a sum of more terms which now should be handled one by one.

Since the degree is halved whenever 2b) is applied, and since the other cases can be calculated straight away, we see that the problem can be solved in a finite number of steps.

Example 1.1 Consider the integral

$$\int \cos^6 x\, dx.$$

The degree $0+6=6$ is even and both $m=0$ and $n=6$ are even, so we are in case 2b). By using the double angle we get the calculation

$$\cos^6 x = \left\{\frac{1}{2}(1+\cos 2x)\right\}^3 = \frac{1}{8}(1+3\cos 2x + 3\cos^2 2x + \cos^3 2x).$$

Integration of the first two terms is easy:

$$\frac{1}{8}\int (1+3\cos 2x)\, dx = \frac{1}{8}\,x + \frac{3}{16}\,\sin 2x.$$

The third term is again of type 2b), hence we shall here consider the quadruple angle,

$$\frac{1}{8}\int 3\cos^2 2x\, dx = \frac{3}{8}\int \frac{1}{2}(1+\cos 4x)\, dx = \frac{3}{16}\,x + \frac{3}{64}\sin 4x.$$

Finally, the last term is of type 1a), hence

$$\frac{1}{8}\int \cos^3 2x\, dx = \frac{1}{8}\int (1-\sin^2 2x)\cdot\frac{1}{2}\, d\sin 2x = \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x.$$

Collecting all results we finally get after a reduction

$$\int \cos^6 x\, dx = \frac{5}{16}\,x + \frac{1}{4}\,\sin 2x - \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x. \qquad \Diamond$$

2 Fourier series; methods of calculation

2.1 General

The *Theory of Fourier Series* is a special case of the theory of series. Its purpose is to break up a periodic function

$$f(t+T) = f(t) \qquad \text{for every } t \in \mathbb{R}$$

into *basic oscillations*. The main case is when $T = 2\pi$, where one gets

$$f \sim \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} \{a_n \cos nt + b_n \sin nt\}$$

with the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt,$$

and where the integration over $[-\pi,\pi[$ can be replaced by any other interval of length 2π, e.g. $[0,2\pi[$.

In the general case we get by the change of variable

$$\tau = \frac{T}{2\pi}\,t \in [0,T[\qquad \text{for } t \in [0,2\pi[.$$

The symbol $\sim$ requires a comment. It means that the *sequence of segments*

$$s_n(t) = \frac{1}{2}\,a_0 + \sum_{k=1}^{n} \{a_k \cos kt + b_k \sin kt\}\,, \qquad n \in \mathbb{N},$$

converges in the sense of *"energy"* (also called in *square mean*, or *in* L^2) towards f. This means more explicitly,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx = 0.$$

In particular, one can within the error of any $\varepsilon > 0$ approximate the *energy* of a signal $f(t)$ by a *finite* sum $s_n(x)$, which will make the engineering considerations somewhat easier.

This terminology sounds like a course in transmission of e.g. radio waves, but actually Fourier Analysis can be found in many other disciplines, like e.g. *diffusion* over some *finite* interval. Furthermore, it is closely connected with the *eigenvalue problems*, cf. Chapter 5. The generalized Fourier series occur very frequently in technical literature, where properties of materials may be built into the eigenfunctions.

Seen from this point of view

Theorem 2.1 Parseval's equation.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = \frac{1}{2}\,a_0^2 + \sum_{n=1}^{\infty} \left\{a_n^2 + b_n^2\right\}.$$

becomes the most important theorem in the engineering applications. On the other hand, most students will in their beginning of their studies consider this theorem as "quite odd".

The explanation is that the space

$$L^2(([-\pi,\pi[) = \left\{ f : [-\pi,\pi[\to \mathbb{R} \;\middle|\; \int_{-\pi}^{\pi} f(t)^2 dt < \infty \right\}$$

can be considered as a generalization of the *Euclidean space* to an infinite dimensional space, when we supply it with the *inner product* (the chosen notation here comes from Quantum Mechanics, where one also can meet Fourier series)

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t) \, dt.$$

With this inner product we obtain that the system

$$\left\{ \frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt, \dots \right\}$$

becomes an infinite orthonormal basis, and the Fourier series of f is exactly the *description of f with respect to this orthonormal basis* (cf. Linear Algebra)

$$
\begin{aligned}
f &\sim \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \{\langle f, \cos nt\rangle \cos nt + \langle f, \sin nt\rangle \sin nt\} \\
&= \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} \{a_n \cos nt + b_n \sin nt\},
\end{aligned}
$$

because

$$\left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} = \frac{1}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos 0t \, dt = \frac{1}{2}\,a_0,$$

and

$$\langle f, \cos nt\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = a_n, \qquad n \in \mathbb{N},$$

$$\langle f, \sin nt\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = n_n, \qquad n \in \mathbb{N}.$$

Then *Parseval's equation* becomes nothing but *Pythagoras's theorem* in infinite dimensions. The strange factor $\frac{1}{2}$ is due to the fact that $\frac{1}{\sqrt{2}}$ is the normed basic element, while the constant 1 is not normed. On the other hand, 1 is easier to work with than $\frac{1}{\sqrt{2}}$, so usually one chooses to stay with the factor $\frac{1}{2}$ in the formulæ.

The students' precariousness of the Theory of Fourier series is caused by the "unusual" convergence in *energy* instead of in a *pointwise* sense. At this stage of the education the teaching is still mostly focussing on *pointwisely* defined functions. Fortunately the *main theorem for Fourier series* gives a *very useful pointwise* result.

Let $f : [-\pi, \pi[\to \mathbb{R}$ be a given function. Whenever we consider a *point of discontinuity* t_0 of the type where the limits exist from the left and from the right without being equal, we redefine f to the so-called *normalized* function f^* by

$$f^*(t_0) = \begin{cases} \dfrac{1}{2}\{f(t_0-) + f(t_0+)\} & \text{for } t_0 \in]-\pi, \pi[, \\[2ex] \dfrac{1}{2}\{f(\pi-) + f(-\pi+)\} & \text{for } t_0 = -\pi, \end{cases}$$

where $f(t_0-) = \lim_{t \to t_0-} f(t)$ and $f(t_0+) = \lim_{t \to t_0+} f(t)$, i.e. the limits from the left and from the right respectively.

The function f (or f^*) is called *piecewise differentiable*, if one can remove a *finite number* of points $x_1, \dots, x_n$ from $[-\pi, \pi[$, such that the restriction of f to each of the open subintervals is *continuously differentiable*, i.e. f' is continuous.

If the periodical function f is piecewise differentiable *without vertical half tangents*, i.e. $f'(t)$ does not tend to $\pm\infty$ anywhere, we say that f belongs to the class $K_{2\pi}^*$ (or in general K_T^*), and we write $f \in K_{2\pi}^*$.

Remark 2.1 We shall never see in the elementary courses of Calculus that f has a vertical half tangent in a Fourier problem. The obvious reason is that the corresponding integrals cannot be calculated explicitly. But there are also other reasons. It is true that Volterra once constructed an example with a vertical half tangent and where the Fourier series does not converge in the corresponding point, but this example lies far beyond what can be expected in an elementary course in Calculus. Furthermore, it can be proved that e.g. $f(t) = \sqrt{\pi^2 - t^2}$ for $t \in [-\pi, \pi[$ has vertical half tangents for $t = \pm\pi$, *and* its Fourier series is actually pointwise convergent everywhere with the right sum $f(t)$, so this condition which occurs in some textbooks is somewhat restrictive. Finally, one may refer to Carleson's theorem which states that if f only is of finite energy, (more precisely, f is "measurable" and squared integrable over $[-\pi, \pi[$; every function occurring in engineering sciences is "measurable" in the mathematical sense), then its Fourier series is pointwise convergent "almost everywhere" (i.e. outside some "meagre set") with the right sum $f(t)$. Therefore, it is no point in demanding of the student that he should check that the function $f(t)$ does not have vertical half tangents, when this never will have an impact on the technical applications. $\Diamond$

We have

Theorem 2.2 Main theorem. (Riemann's theorem).
Assume that $f \in K_{2\pi}^*$. then the Fourier series of f is pointwise convergent everywhere *with $f^*(t)$ as its sum. Hence*

$$f \sim \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n \cos nt + b_n + \sin nt\}$$

and

$$f^*(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n \cos nt + b_n \sin nt\} \qquad \text{for all } t \in [-\pi, \pi[.$$

2.2 Standard procedure for solution of problems in Fourier series, $T = 2\pi$

1) *Sketch the graph of f over a periodic interval and also into the two neighbouring intervals.*

   One should always do this *no matter how the problem is formulated.* By this extension into the neighbouring intervals we can immediately see if the end points of the interval are continuity points or not.

2) *Normalize the function $f(t)$ to $f^*(t)$ by the formula*

   $$f^*(t) = \frac{1}{2}\{f(t-) + f(t+)\}.$$

   In a point of continuity we get $f^*(t) = f(t)$, and in every point of discontinuity the function $f^*(t)$ is the mean of the limits from the left and from the right. (It is possible to construct examples, where one cannot apply this definition, but such examples are far too difficult for an introductory course in Fourier Analysis). Since we already have sketched the graph of f in 1), it is now easy also to sketch the graph of f^* on the same figure by e.g. using another colour for the points in which $f^*(t) \neq f(t)$.

3) *Explain why f^* (or f) belongs to the class $K_{2\pi}^*$.*

   Check that

a) f^* is piecewise C^1, i.e. the periodic interval can be divided by a *finite* number of points into a finite number of *open* subintervals, such that f is differentiable with a continuous derivative in each of these subintervals.

   It is of no importance if f^* is *discontinuous* in a finite number of points. We choose the division points as the points of discontinuity and the points of continuity in which f' either does not exist or is discontinuous.

b) f^* does not have a vertical half tangent in any end point for any of the open subintervals mentioned above (if one is so unlucky to have a textbook, in which this unnecessary condition should be checked as well).

Any function occurring in the introductory courses in Calculus or in engineering applications will always belong to the class $K_{2\pi}^*$. Even if there exist many periodic functions which do *not* lie in $K_{2\pi}^*$, none of these has any importance when a physical or technical situation is modelled.

4) *Refer to the main theorem and conclude that the Fourier series is pointwise convergent everywhere with the normalized function $f^*(t)$ as its sum function.*

From now on we can replace $\sim$ by $=$, and we have answered a question which always will occur in a problem on Fourier series. Notice that one should *always* give an argument for writing $=$ instead of $\sim$.

These first preparatory four steps can always be made without calculating one single integral.

5) *Calculate the Fourier coefficients*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad og \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt.$$

(Possibly an integration over some other periodic interval, though it usually is the *symmetric* interval $[-\pi,\pi]$ because we get some extra information when $f(t)$ is either an even or an odd function).

During these calculations we may get into some integration problems:

a) If there is given a *hint*, set up the integrals and use the hint.

b) If $f(t)$ is composed of *polynomials, exponentials* or *hyperbolic functions*, apply *partial integration.*

c) If $f(t)$ contains *trigonometric functions*, we first reduce the products of these by the *antilogarithmic formulæ*

$$\cos A\cos B = \frac{1}{2}\{\cos(A-B)+\cos(A+B)\},$$

$$\sin A\cos B = \frac{1}{2}\{\sin(A-B)+\sin(A+B)\},$$

$$\sin A\sin B = \frac{1}{2}\{\cos(A-B)-\cos(A+B)\}.$$

   After this reduction we continue with a *partial integration.*

Remark 2.2 Be very careful when a partial integration is applied because one may *unawares* divide by 0. This is actually the most common error made by students in problems like this. Therefore, *always check* whether some value of n gives 0 in the denominator. Such values of n require a separate treatment.$\diamondsuit$

d) If $f^*(t)$ is *even*, i.e. $f^*(-t) = f^*(t)$ for every t, then

$$a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt \quad \text{and} \quad b_n = 0.$$

e) If $f^*(t)$ is *odd*, i.e. $f^*(-t) = -f^*(t)$ for every t, then

$$a_n = 0 \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^\pi f(t)\sin nt\,dt.$$

Point 5) is usually the hard work in problems in Fourier series.

6) *Set up the Fourier series*, i.e. insert the values of a_n and b_n found in 5) into the *pointwise equation* [cf. point 4)]

$$f^*(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n\cos nt + b_n \sin nt\}.$$

Do not forget the factor $\frac{1}{2}$ in front of a_0. Refer if necessary again to the main theorem.

7) *Concerning uniform convergence of the Fourier series* we have two (not exhaustive) possibilities:

a) If $f^*(t)$ has at least one point of discontinuity (which e.g. can be seen from the graph sketched in point 1)), then it follows from a theorem in any textbook on Fourier Analysis (refer to this theorem!) that the convergence *cannot* be uniform. In fact, all the functions $\cos nt$ and $\sin nt$ are continuous, and the sum function is not.

b) If $f^*(t) = f(t)$ is *continuous*, we consider the *majoring series*

$$\frac{1}{2}|a_0| + \sum_{n=1}^{\infty}\{|a_n| + |b_n|\}.$$

If this majoring series is convergent (check!), then it follows from another theorem in any textbook (refer to it!) that the Fourier series is *uniformly convergent*.

8) If one in connection with a Fourier series is asked about *the sum of a series of numbers*, then we have two possibilities:

a) (The most frequent case). If the terms in the series of numbers *look like* the Fourier coefficients, then apply the *main theorem*, i.e. point 6) with equality and the adjusted function $f^*(t)$. Insert some suitable t-values, typically $t = 0$, π or $\frac{\pi}{2}$, and more rarely $t = \frac{\pi}{4}$.

Then reduce.

b) (This does not occur so often.) If the terms of the series look like the *sum of squares* $a_n^2 + b_n^2$, and the series does *not* contain negative terms, then use *Parseval's equation*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = \frac{1}{2}\,a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

In this case one should do the following

i) Calculate $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$ explicitly and then $\frac{1}{2}\,a_0^2$.

It is due to the lack of the factor 1/2 of normalization that the errors usually occur here. The student is wrongly inclined to square $\frac{1}{2}\,a_0$ from the Fourier series itself, but then we only get $\frac{1}{4}\,a_0^2$. Therefore, be careful here.

ii) Since $f(t)$ is given from the very beginning we can always calculate $\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt$.

iii) Insert and reduce.

9) It is possible here to set up the following task: Given a Fourier series. Find its pointwise sum function $f^*(t)$.

One will, however, *never* get this task in an elementary course of Calculus, because the answer requires some knowledge of *Complex Function Theory*, and even with such a knowledge this task may be very difficult. The problem is of course relevant in engineering applications because one by using various measuring devices implicitly determines the coefficients a_n and b_n without knowing the sum function. Here the mathematics becomes too difficult for Calculus.

2.3 Standard Fourier series with pointwise results and Parseval's equation

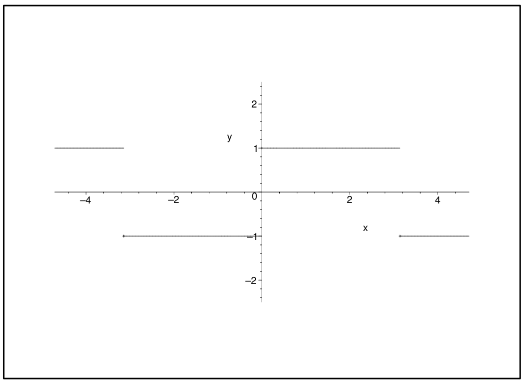
Some functions from $K_{2\pi}^*$ appear very frequently in the engineering sciences. In this section we set up a catalogue including the main results from these examples. The catalogue is a poor replacement of the missing treatment in point 9) above. In the next section we give a similar list for a different purpose sorted out according to "type".

We suppose in the following that all functions are periodic, so they will only be specified in a periodic interval.

2.4 The square function

The *normalized function* is given by

$$f^*(t) = \begin{cases} 1 & \text{for } t \in]0,\pi[, \\ 0 & \text{for } t = 0 \text{ or } t = \pi, \\ -1 & \text{for } t \in]-\pi,0[. \end{cases}$$



The function is *odd* and

$$f^*(t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\sin(2n+1)t \qquad \text{for every } t \in \mathbb{R}.$$

Special pointwise results. For $t = \dfrac{\pi}{2}$ we get $f^*(1) = 1$ and $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$, thus

$$1 = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4},$$

cf. the Taylor series of arctan.

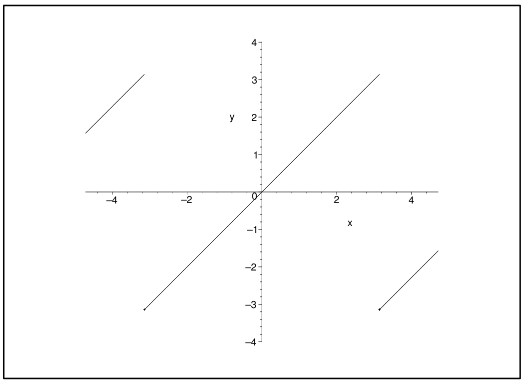
Parseval. Since $a_n = 0$ and $b_{2n} = 0$ and $b_{2n+1} = \dfrac{4}{\pi}\cdot\dfrac{1}{2n+1}$, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = 2 = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

2.5 The identical function in $]-\pi,\pi[$

The *normalized function* is given by

$$f^*(t) = \begin{cases} t & \text{for } t \in]-\pi,\pi[, \\ \\ 0 & \text{for } t = -\pi \text{ or } t = \pi. \end{cases}$$



The function is *odd*, and

$$f^*(t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt \qquad \text{for every } t \in \mathbb{R}.$$

Special pointwise results. If we put $t = \dfrac{\pi}{2}$ we have $\sin n\,\dfrac{\pi}{2} = 0$ for n *even.* For *odd* indices we replace n by $2n+1$, and then $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$. Thus

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4},$$

i.e. the same result as for the square function.

Parseval. Since $a_n = 0$ and $b_n = 2\,\dfrac{(-1)^{n+1}}{n}$, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} b_n^2 = 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

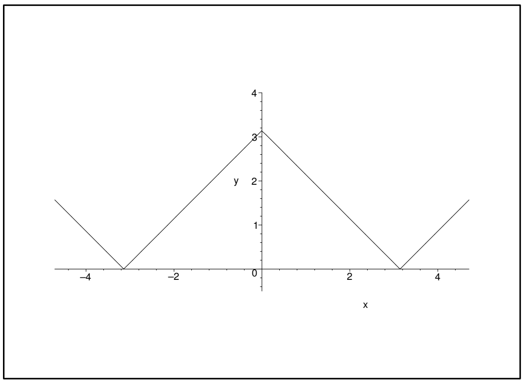
From this we derive the *important result*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2.6 The first sawtooth function

The *normalized function* is given by

$$f^*(t) = \pi - |t| \qquad \text{for } t \in [-\pi,\pi].$$



The function is *even* and *continuous*

$$f^*(t) = f(t) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\,\cos(2n+1)t.$$

Special pointwise results. When $t = 0$ we have $\cos(2n+1)0 = 1$, hence

$$f(0) = \pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \quad \text{from which} \quad \sum_{n=0}^{\infty} \frac{1}{2n+1)^2} = \frac{\pi^2}{8}.$$

Parseval. Since $a_0 = \pi$ (NB) and $a_{2n+1} = \dfrac{4}{\pi}\,\dfrac{1}{(2n+1)^2}$, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = \frac{2}{\pi}\int_{-\pi}^{\pi} (\pi - t)^2 dt = \frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4},$$

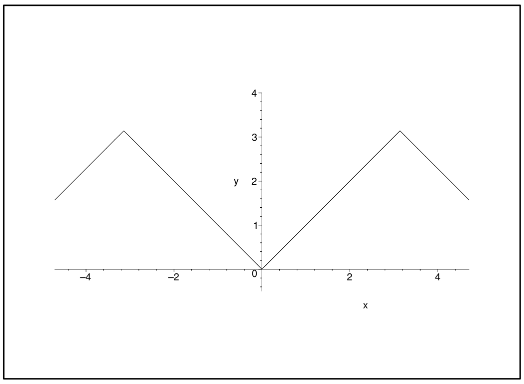
so by a rearrangement

$$\sum_{n=0}^{\infty} \frac{1}{(2b+1)^4} = \frac{\pi^4}{96}.$$

2.7 The second sawtooth function

The *normalized function* is given by

$$f^*(t) = |t| \qquad \text{for } t \in [-\pi, \pi].$$



Notice the periodic continuation. The function is *even* and *continuous.*

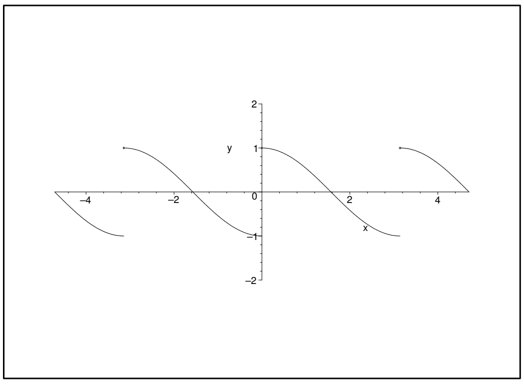
$$f^*(t) = f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos(2n+1)t.$$

Notice also that if the two sawtooth functions are added we get the constant π, which agrees with the fact that the sum of the two series is π. For the same reason neither the *pointwise results* nor *Parseval* will give anything new.

2.8 Expansion of cosine in a sinus series over a half period

The *normalized function* is given by

$$f^*(t) = \begin{cases} \cos t & \text{for } t \in]0, \pi[, \\ 0 & \text{for } t = -\pi, 0, \pi, \\ -\cos t & \text{for } t \in]-\pi, 0[. \end{cases}$$



The function is *odd*,

$$f^*(t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2nt \qquad \text{for every } t \in \mathbb{R}.$$

Special pointwise results. These are of no interest for $t = 0$, $\frac{\pi}{2}$, π (both sides of the equality are zero).
For $t = \frac{\pi}{4}$ we get after some calculation that

$$f^*\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin n\,\frac{\pi}{2} = \frac{1}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} + \frac{1}{4n+3}\right) \cdot (-1)^n,$$

from which

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{4n+1} + \frac{1}{4n+3}\right) = \frac{\pi\sqrt{2}}{4}.$$

Parseval. Since $a_n = 0$ and $b_{2n+1} = 0$ and $b_{2n} = \frac{8}{\pi} \cdot \frac{n}{4n^2-1}$, we get

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 t \, dt = 1 = \sum_{n=1}^{\infty} = \frac{64}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{n}{4n^2-1}\right)^2,$$

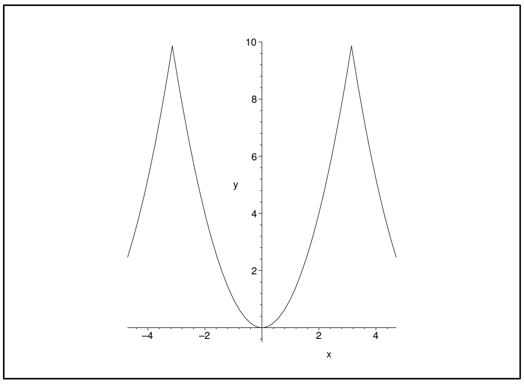
hence

$$\sum_{n=1}^{\infty} \left(\frac{n}{4n^2-1}\right)^2 = \frac{\pi^2}{64}.$$

2.9 Symmetric parabolic arc in $[-\pi, \pi]$

The *normalized function* is given by

$$f^*(t) = f(t) = t^2 \qquad \text{for } t \in [-\pi, \pi].$$



The function is *continuous* and *even*,

$$f^*(t) = f(t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt, \qquad t \in \mathbb{R}.$$

Special pointwise results. For $t = 0$ we get

$$f(0) = 0 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

For $t = \pi$ we get

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{from which} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Parseval. Since $a_0 = 2 \cdot \dfrac{\pi^2}{3}$ (NB), and $a_n = 4 \cdot \dfrac{(-1)^n}{n^2}$, $b_n = 0$ for $n \in \mathbb{N}$, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = \frac{2}{\pi}\int_0^{\pi} t^4 dt = \frac{2\pi^4}{5} = \frac{1}{2}\, a_0^2 + \sum_{n=1}^{\infty} a_n^2 = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty} \frac{1}{n^4},$$

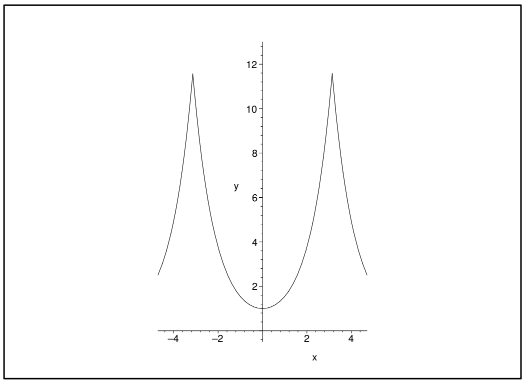
hence by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2.10 Hyperbolic cosine in $[-\pi, \pi]$

The *normalized function* is given by

$$f^*(t) = f(t) = \cosh t \qquad \text{for } t \in [-\pi, \pi].$$



The function is *continuous* and *even*,

$$f^*(t) = f(t) = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \cos nt, \quad t \in \mathbb{R}.$$

Special pointwise results. For $t = 0$ we get

$$f(0) = 1 = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}, \text{ from which } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1} = \frac{1}{2} - \frac{\pi}{2\,\sinh \pi}.$$

For $t = \pi$ we get

$$f(\pi) = \cosh \pi = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2+1}, \text{ from which } \sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2} \coth \pi - \frac{1}{2}.$$

We see that $n = 0$ corresponds to -1 or 1 in both series. Therefore, if we add 1, then (note the change of sign in the first series)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{1}{2} + \frac{\pi}{2\sinh \pi} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi \cosh \pi}{2\sinh \pi}.$$

Parseval. Since $a_0 = \dfrac{2\sinh \pi}{\pi}$ and $a_n = \dfrac{2\sinh \pi}{\pi} \cdot \dfrac{(-1)^n}{n^2+1}$ and $b_n = 0$ for $n \in \mathbb{N}$, we get

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cosh^2 t \, dt = \frac{\sinh 2\pi}{2\pi} + 1 = \frac{\sinh \pi \cosh \pi}{\pi} + 1 = \frac{2\sinh^2 \pi}{\pi^2} + \frac{4\sinh^2 \pi}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2},$$

hence by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4} \coth \pi + \frac{\pi^2}{4\sinh^2 \pi} - \frac{1}{2}.$$

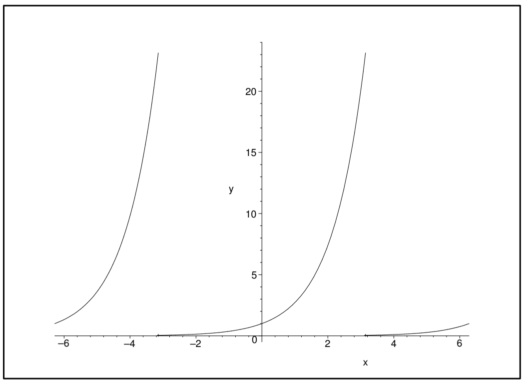
Since $n = 0$ corresponds to the constant 1, we also get that

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4} \coth \pi + \frac{\pi^2}{4\,\sinh^2 \pi} + \frac{1}{2}.$$

2.11 Exponential function over $]-\pi, \pi]$

The *normalized function* is given by

$$f^*(t) = \begin{cases} e^{\alpha t} & \text{for } t \in]-\pi, \pi[, \\ \cosh(\alpha\pi) & \text{for } t = -\pi, \pi. \end{cases}$$



The function is neither continuous nor even or odd.

$$f^*(t) = \frac{\sinh \alpha\pi}{\alpha\pi} + 2\,\frac{\sinh \alpha\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2+n^2} \left\{\alpha \cos nt - n \sin nt\right\}.$$

Special pointwise results. For $t = 0$ we get

$$f^*(0) = 1 = \frac{\sinh \alpha\pi}{\pi} \left\{ \frac{1}{\alpha} + 2\alpha \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2+n^2} \right\},$$

hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha^2+n^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \sinh \alpha\pi}.$$

For $t = \pi$ we get

$$f^*(\pi) = \cosh \alpha\pi = \frac{\sinh \alpha\pi}{\alpha\pi} + \frac{2\alpha \sinh \alpha\pi}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha^2+n^2},$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2+n^2} = \frac{\pi \coth \alpha\pi}{2\alpha} - \frac{1}{2\alpha^2}.$$

When we start the the summation form $n = 0$ (and change one single sign), we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^2+n^2} = \frac{1}{2\alpha^2} + \frac{\pi}{2\alpha \sinh \alpha\pi}, \qquad \sum_{n=0}^{\infty} \frac{1}{\alpha^2+n^2} = \frac{1}{2\alpha^2} + \frac{\pi \cosh \alpha\pi}{2\alpha \sinh \alpha\pi}.$$

Since $a_n^2 + b_n^2 = \dfrac{\alpha^2+n^2}{(\alpha^2+n^2)^2} = \dfrac{1}{\alpha^2+n^2}$ for $n \in \mathbb{N}$, we do not get anything new by an application of *Parseval's equation.*

3 A list of problems in the Theory of Fourier series

It should be of some interest also to have a larger list of Fourier series. When the calculations are not too difficult the usual pattern will be the following:

a) Indication of a periodic interval.

b) Specification of f in a periodic interval.

We shall here tacitly assume that there will be no trouble with the "vertical half tangents", i.e. we shall always assume that the half tangents do exist *everywhere*. It will be too cumbersome to mention it every time.

c) The Fourier series with an indication of $=$ whenever this is possible.

d) Possible pointwise results.

e) Parseval's equation.

The subsections follow the type of the function. These are shortly listed here:

- A piecewise constant function.
- A piecewise linear function.
- A piecewise polynomial of second degree.
- A piecewise polynomial of third degree.
- A piecewise polynomial of fourth degree.
- A piecewise sinus.
- A piecewise cosine.
- Mixed sinus and cosine.
- A piecewise function composed by a polynomial times a trigonometric function.
- A function, in which the exponential function enters.
- The problem of the condition of vertical half tangent.

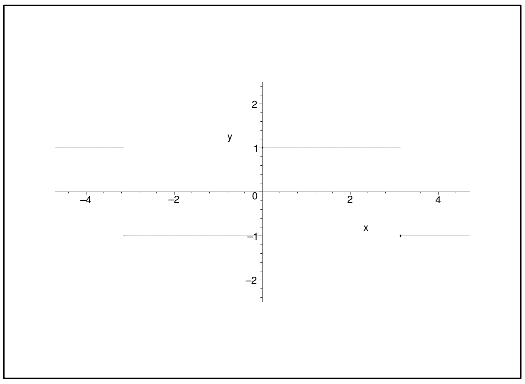
In the last mentioned subsection we discuss what is known and where this condition of avoiding vertical half tangents stems from. The correct condition is that the function should be of *bounded variation* (not defined in these notes). Since every *monotonous function* in particular is of bounded variation, we must necessarily have pointwise results for functions which are piecewise C^1 and which are monotonous in the neighbourbood of every exception point.

We note that this list will contain more information than what is usually asked for.

3.1 A piecewise constant function

Example 3.1 *The period is T.*

$$f(t)=\begin{cases} +A & \text{for } t\in\left]0,\dfrac{T}{2}\right[, \\[2ex] -A & \text{for } t\in\left]-\dfrac{T}{2},0\right[. \end{cases} \qquad \text{supplied by } f^*(nT)=0 \text{ for } n\in\mathbb{Z}.$$



Fourier series:

$$f \sim \frac{4A}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin n\omega t, \qquad \text{where } \omega = \frac{2\pi}{T}.$$

Pointwise:

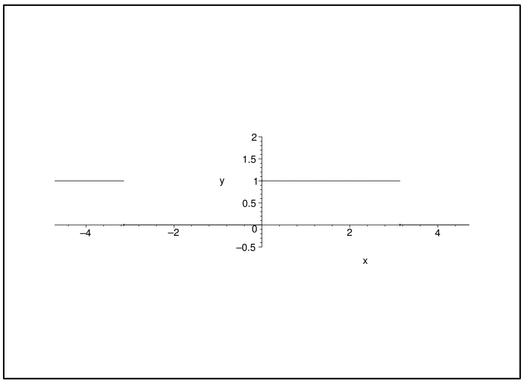
$$A = \frac{4A}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin n\omega t, \text{ i.e. } \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin n\omega t = \frac{\pi}{4} \text{ for } t \in \left]0, \frac{T}{2}\right[.$$

Parseval:

$$\frac{A^2}{2} = \frac{4A^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \qquad \Diamond$$

Example 3.2 *The period is 2π.*

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi, \\ 0 & \text{for } -\pi \le t \le 0, \end{cases} \quad \text{Adjusted by } f^*(n\pi) = \frac{1}{2},\ n \in \mathbb{Z}.$$



Fourier series:

$$f \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\,\sin(2n+1)t.$$

Pointwise: For $t \in]0,\pi[$ we get

$$1 = \frac{1}{2} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\,\sin(2n+1)t, \text{ i.e. } \sum_{n=0}^{\infty}\frac{1}{2n+1}\,\sin(2n+1)t = \frac{\pi}{4}.$$

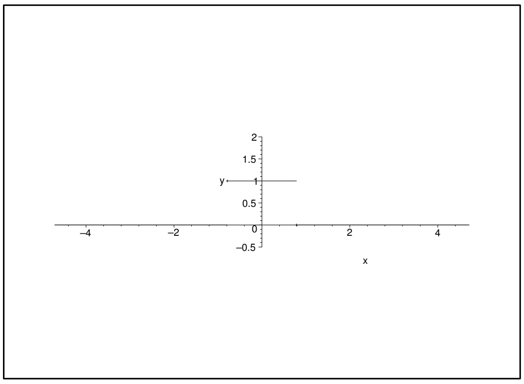
For $t = \frac{\pi}{2}$ we obtain the series of $\arctan 1$.

Parseval:

$$1 = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}, \text{ i.e. } \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \qquad \Diamond$$

Example 3.3 *The period is* 2π.

$$f(t)=\begin{cases} 1 & \text{for } |t|\le \dfrac{\pi}{4}, \\ 0 & \text{for } \dfrac{\pi}{4}<|t|\le \pi. \end{cases} \qquad \text{Adjusted by } f^*\left(\pm\frac{\pi}{4}+2n\pi\right)=\frac{1}{2}.$$



Fourier series:

$$f \sim \frac{1}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\,\frac{\pi}{4}\right)}{n}\,\cos nt.$$

Pointwise: For $t=\dfrac{\pi}{4}$ we get by applying $2\sin\left(n\,\dfrac{\pi}{4}\right)\cos\left(n\,\dfrac{\pi}{4}\right)=\sin\left(n\,\dfrac{\pi}{2}\right)$ the same series as before.

For $t=0$ we get

$$1=\frac{1}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\,\frac{\pi}{4}\right), \text{ i.e. } \sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\,\frac{\pi}{4}\right)=\frac{3\pi}{8}.$$

Parseval:

$$\frac{1}{2}=\frac{1}{8}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(n\,\frac{\pi}{4}\right), \text{ i.e. } \sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(n\,\frac{\pi}{4}\right)=\frac{3\pi^2}{32}. \qquad \Diamond$$

Example 3.4 *The period is* 2π.

$$f(t)=\begin{cases} 0 & \text{for } |t|< \dfrac{\pi}{2}, \\ 1 & \text{for } \dfrac{\pi}{2}<|t|\le \pi. \end{cases} \qquad \text{Supplied with } f^*\left(\pm\frac{\pi}{2}+2n\pi\right)=\frac{1}{2}.$$

Fourier series:

$$f \sim \frac{1}{2}+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\,\cos(2n+1)t.$$

Pointwise: We get $\arctan 1$ for both $t=0$ and $t=\pi$.

The case $t=\dfrac{\pi}{2}$ is of no interest $\left(\dfrac{1}{2}=\dfrac{1}{2}+\sum_{n=0}^{\infty}0\right)$.

Parseval:

$$1=\frac{1}{2}+\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}. \qquad \Diamond$$

Example 3.5 *The period is 2ℓ.*

$$f(t)=\begin{cases} -q_0 & \text{for } -\dfrac{\ell}{2}<t<0,\\ q_0 & \text{for } 0<t<\dfrac{\ell}{2},\\ 0 & \text{for } \dfrac{\ell}{2}<|t|<\ell.\end{cases}$$

Adjustments in the discontinuity points.

Fourier series:

$$f\sim\frac{2q_0}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{2n-1}\,\sin(2n-1)\,\frac{\pi}{\ell}\,t+\frac{1-(-1)^n}{2n}\,\sin 2n\,\frac{\pi}{\ell}\,t\right\}.$$

By investing some effort one can further reduce this result since $1-(-1)^n$ is either 0 or 2.

Pointwise: Here we get some extremely ugly results which are not worth mentioning.

Parseval: This formula is here reduced to the well-known $\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}=\dfrac{\pi^2}{8}$. $\Diamond$

Example 3.6 *The period is 4ℓ.*
Even function with

$$f(t)=\begin{cases} 1 & \text{for } 0\le t\le\dfrac{\ell}{2},\\ \dfrac{1}{2} & \text{for } \dfrac{\ell}{2}<t<\dfrac{3\ell}{2},\\ 0 & \text{for } \dfrac{3\ell}{2}<t<2\ell,\end{cases}$$

with adjustments in the discontinuity points.

Fourier series:

$$f\sim\frac{1}{2}+\frac{\sqrt{2}}{\pi}\left\{\cos\left(\frac{\pi t}{2\ell}\right)+\frac{1}{3}\cos\left(\frac{3\pi t}{2\ell}\right)-\frac{1}{5}\cos\left(\frac{5\pi t}{2\ell}\right)-\frac{1}{7}\cos\left(\frac{7\pi t}{2\ell}\right)++--\cdots\right\}.$$

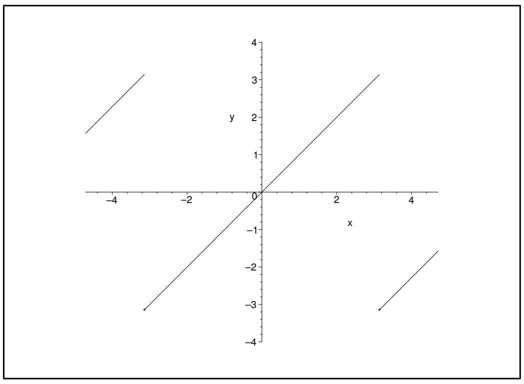
Pointwise: It is not worth mentioning any of these ugly results.

Parseval: This formula is here reduced to the well-known $\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}=\dfrac{\pi^2}{8}$. $\Diamond$

3.2 Piecewise linear functions

Example 3.7 *The period is* 2π.

$$f(t) = t \quad \text{for } t \in]-\pi, \pi]. \quad \text{Adjustment } f^*(\pi + 2n\pi) = 0.$$



Fourier series:

$$f \sim \sum_{n=1}^{\infty} \frac{2}{n}\,(-1)^{n+1}\,\sin nt.$$

Pointwise:

$$t = \sum_{n=1}^{\infty} \frac{2}{n}\,(-1)^{n+1}\sin nt \quad \text{for } |t| < \pi,$$

and especially $\arctan 1$ for $t = \dfrac{\pi}{2}$.

Parseval:

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \Diamond$$

Example 3.8 *The period is 4.*

$f(t) = t \quad \text{for } t \in]-2,2[$. Supplied by $f(2+4n) = 0$.

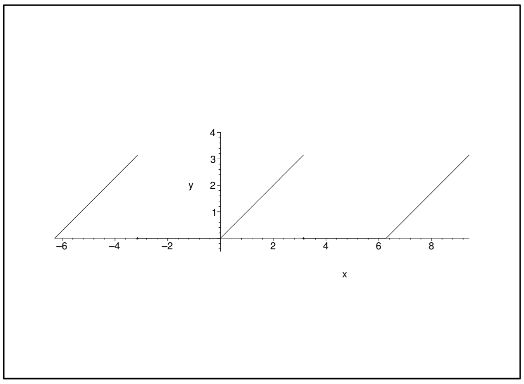
Fourier series:

$$f \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,\sin\left(\frac{n\pi t}{2}\right).$$

The rest: Cf. a previous example. $\Diamond$

Example 3.9 *The period is 2π.*

$$f(t) = \begin{cases} t & \text{for } 0 < t < \pi, \\ 0 & \text{for } \pi < t \le 2\pi, \end{cases} \quad \text{supplied by } f^*(\pi + 2n\pi) = \frac{\pi}{2}.$$



Fourier series:

$$\begin{aligned} f \quad &\sim \quad \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{\pi n^2}\,\cos nt + \frac{(-1)^{n-1}}{n}\,\sin nt\right\} \\ &= \quad \frac{\pi}{4} - \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\,\cos(2n+1)t + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,\sin nt. \end{aligned}$$

Pointwise: We derive from $t = 0$ that $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^2} = \dfrac{\pi^2}{8}$. (The same for $t = \pi$).

When $t = \dfrac{\pi}{2}$ we get the series for $\arctan 1$.
Parseval:

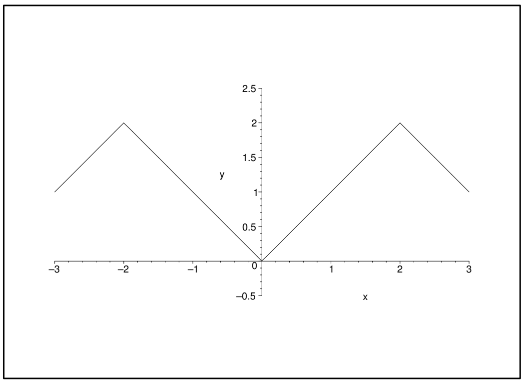
$$\frac{\pi^2}{3} = \frac{\pi^2}{8} + \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, we get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}. \qquad \Diamond$$

Example 3.10 *The period is 4.*

$$f(t) = |t| \text{ for } t \in [-2, 2],$$



The function is continuous and piecewise C^1 without vertical half tangents, thus the Fourier series can be written pointwisely with "=" instead of "$\sim$":

$$f(t) = 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left(\left(n + \frac{1}{2}\right)\pi t\right).$$

Pointwise: For $t = 0$ we derive that $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^2} = \dfrac{\pi^2}{8}$.

Parseval:

$$\frac{8}{3} = 2w + \frac{64}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}, \quad \text{i.e.} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}. \qquad \Diamond$$

Example 3.11 *The period is 8.*

$$f(t) = |t| \qquad \text{for } t \in [-4, 4].$$

Apart from the scaling the figure is the same as in a previous example.

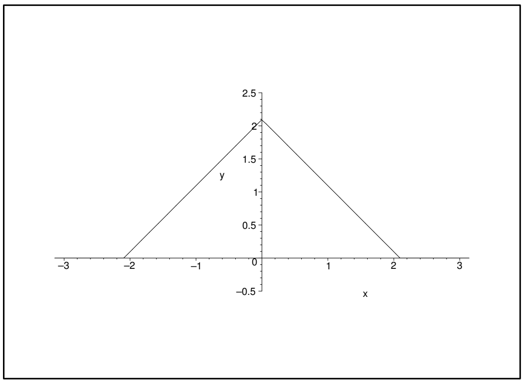
The function is continuous and piecewise C^1 and without any vertical half tangents. We can therefore write the Fourier series with "=" instead of "$\sim$":

$$f(t) = 2 - \frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)\,\frac{\pi}{4}\,t.$$

By a substitution one is led back to a previous example. $\Diamond$

Example 3.12 *The period is* 2π.

$$f(t) = \begin{cases} \dfrac{2\pi}{3} - |t|, & \text{for } |t| \leq \dfrac{2\pi}{3}, \\ \quad 0 & \text{for } \dfrac{2\pi}{3} < |t| \leq \pi. \end{cases}$$



The function is continuous and piecewise C^1 and without any vertical half tangents. We can therefore according to the main theorem write the **Fourier series** with a pointwise equality sign instead of with "$\sim$":

$$f(t) = \frac{2\pi}{9} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \left\{ 1 - \cos\left(n\,\frac{2\pi}{3}\right)\right\} \cos nt.$$

Pointwise for $t = 0$: By applying the well-known formula $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, we get

$$\frac{2\pi}{3} = \frac{2\pi}{9} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(n\,\frac{2\pi}{3}\right),$$

from which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(n\,\frac{2\pi}{3}\right) = -\frac{\pi^2}{18}.$$

Parseval:

$$\frac{16\pi^2}{81} = \frac{8\pi^2}{81} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^4}\left\{1-\cos\left(n\,\frac{2\pi}{3}\right)\right\}^2,$$

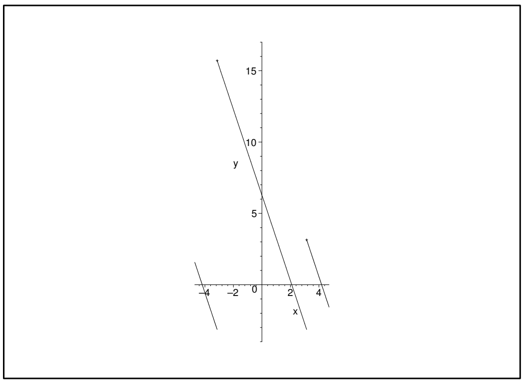
i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n^4}\left\{1-\cos\left(n\,\frac{2\pi}{3}\right)\right\}^2 = \frac{2\pi^4}{81}.$$

It is here possible to reduce $\left\{1-\cos\left(n\,\frac{2\pi}{3}\right)\right\}^2$ further. $\Diamond$

Example 3.13 *The period is 2π.*

$f(t) = 2\pi - 3t, \quad t \in]-\pi,\pi[, \quad$ supplied by $f(\pi + 2n\pi) = 2\pi$.



Fourier series:

$$f \sim 2\pi + \sum_{n=1}^{\infty} (-1)^n \frac{6}{n} \sin nt.$$

Pointwise: For $t = \dfrac{\pi}{2}$ we obtain the series for $\arctan 1$.

Parseval:

$$14\pi^2 = 8\pi^2 + 36 \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \Diamond$$

Example 3.14 *The period is 2π.*

$$f(t) = \begin{cases} k - k^2|t| & \text{for } |t| \in \left[0, \dfrac{1}{k}\right], \\[2ex] 0 & \text{for } |t| \in \left]\dfrac{1}{k}, \pi\right]. \end{cases}$$

The function is continuous and piecewise C^1 and without any vertical half tangents. We can therefore write the **Fourier series** with a pointwise equality sign instead of with "$\sim$":

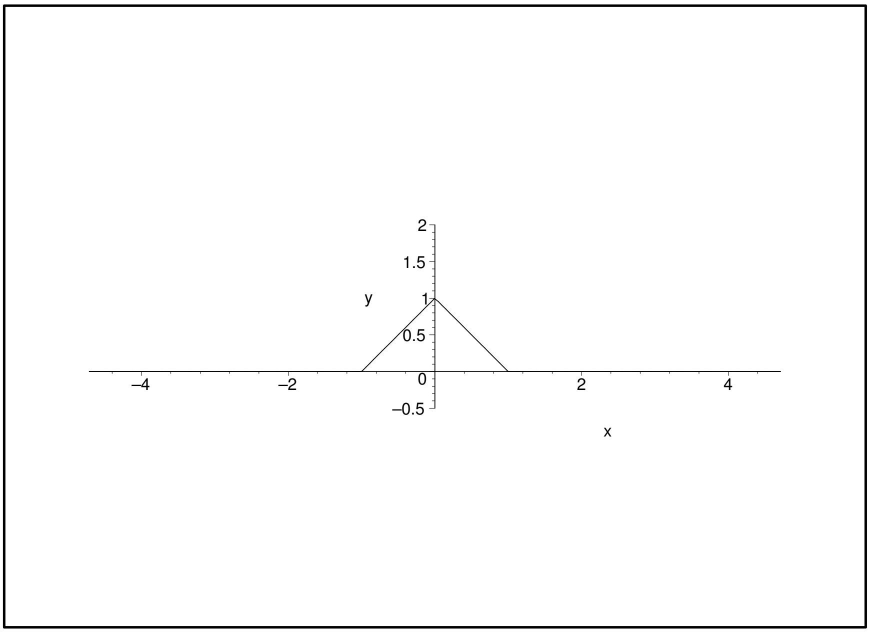
$$f(t) = \frac{1}{2\pi} + \frac{2k^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ 1 - \cos\left(\frac{n}{k}\right) \right\} \cos nt.$$

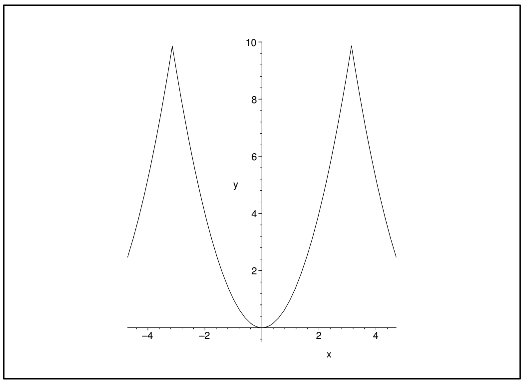
The rest is then only variations of a previous example. $\Diamond$

3.3 A piecewise polynomial of second degree

Example 3.15 *The period is 2π.*

$f(t) = t^2, \qquad t \in [-\pi,\pi]$.





The function is continuous and piecewise C^1 and without vertical half tangents. According to the main theorem we can then write the **Fourier series** with a pointwise equality sign instead of only with "$\sim$":

$$f(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\,(-1)^n \cos nt.$$

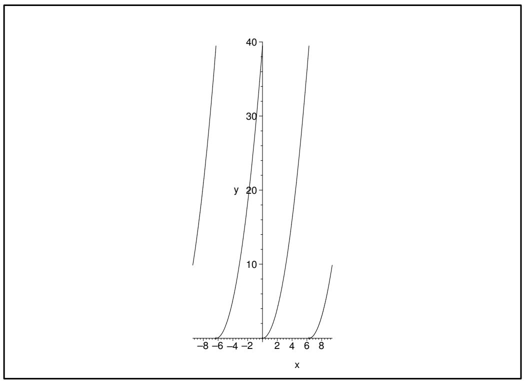
Pointwise: For $t = 0$ fås $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$.

Parseval:

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}, \quad \text{from which } \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \qquad \Diamond$$

Example 3.16 *The period is* 2π.

$$f(t) = t^2, \qquad t \in\,]0, 2\pi[. \quad \text{Adjustment } f^*(2\pi) = 2\pi^2.$$



Fourier series:

$$f \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left\{ \frac{4}{n^2}\,\cos nt - \frac{4\pi}{n}\,\sin nt \right\}.$$

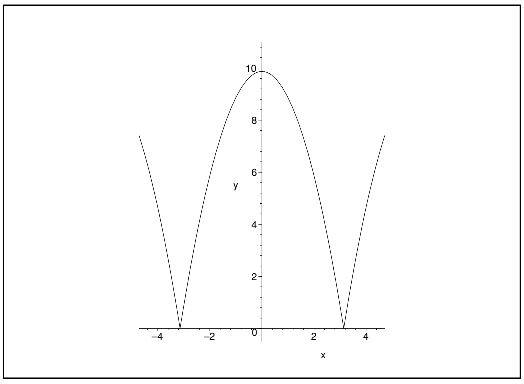
Pointwise: When $t = \pi$ we get $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$.

Parseval:

$$\frac{32\pi^4}{5} = \frac{32\pi^4}{9} + 16\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \qquad \Diamond$$

Example 3.17 *The period is 2π.*

$f(t) = \pi^2 - t^2, t \in [-\pi, \pi]$.



The function is continuous and piecewise C^1 without vertical half tangents. According to the main theorem we can then use a pointwise equality sign instead of only "$\sim$" in the **Fourier series:**

$$f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n^2} \cos nt.$$

One can also get the Fourier series by subtracting the result of a previous example from π^2, thus the remaining questions will only be variants of this previous example. $\Diamond$

Example 3.18 *The period is 2π.*

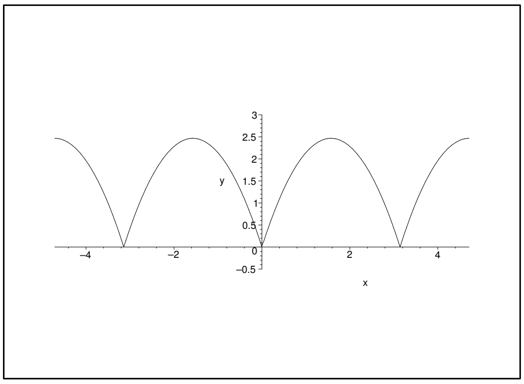
$f(t) = t(\pi - t), \quad t \in [0, \pi]$ and odd.

The function is continuous and piecewise C^1 without vertical half tangents. According to the main theorem the **Fourier series** can then be written with a pointwise equality sign instead of with "$\sim$":

$$f(t) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin(2n+1)t.$$

Pointwise: For $t = \dfrac{\pi}{2}$ we get

$$\frac{\pi^2}{4} = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}, \quad \text{from which} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$



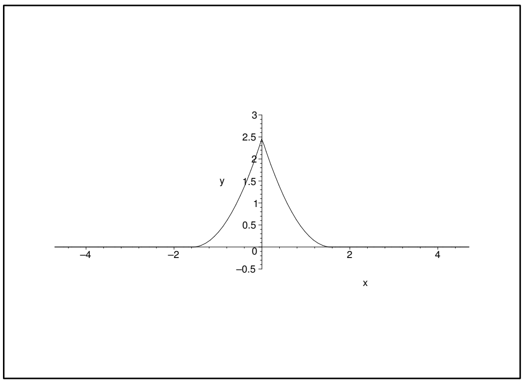
Remark 3.1 One still does not know the exact values of $\sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^3}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$. $\Diamond$

Parseval:

$$\frac{\pi^4}{15} = \frac{64}{\pi^4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}, \quad \text{from which} \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}. \qquad \Diamond$$

Example 3.19 *The period is* 2π.

$$f(t) = \begin{cases} \left(t - \dfrac{\pi}{2}\right)^2, & t \in \left[0, \dfrac{\pi}{2}\right], \\[2ex] 0, & t \in \left]\dfrac{\pi}{2}, \pi\right], \end{cases} \qquad \text{og lige.}$$



The function is continuous and piecewise C^1 and without vertical half tangents. According to the main theorem the **Fourier series** can then be written with a pointwise equality sign instead of with "$\sim$":

$$f(t) = \frac{\pi^2}{24} + 2 \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{2}{\pi n^3 e} \sin n \, \frac{\pi}{2} \right) \cos nt.$$

Pointwise: For $t = 0$ we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{24} + 2\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin n \frac{\pi}{2},$$

hence by a rearrangement

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

It is possible to obtain the same result, though it is more difficult, for $t = \pi$.

Parseval:

$$\frac{\pi^4}{80} = \frac{\pi^4}{288} + 4\sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{16}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} - \frac{16}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ and $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^5} = \frac{5\pi^5}{1536}.$$

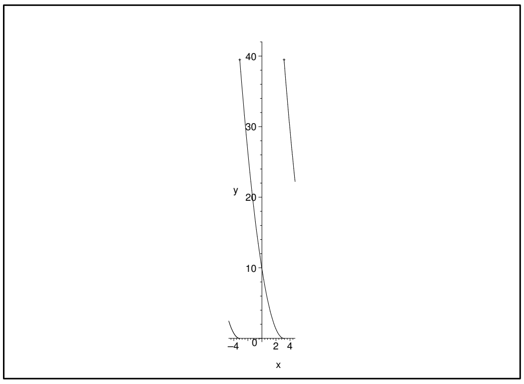
Remark 3.2 One does not know the exact value of any series

$$\sum_{n=0}^{\infty} \frac{1}{n^{2k+1}} \qquad \text{or} \qquad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k+1}}$$

without the change of sign for any odd exponent $2k+1$, $k \in \mathbb{N}$. On the other hand one can in principle always find the exact value for $\sum_{n=0}^{\infty} \frac{1}{n^{2k}}$, $k \in \mathbb{N}$ for any even exponent $2k$. $\Diamond$

Example 3.20 *The period is* 2π.

$$f(t) = (t-\pi)^2 \quad \text{for } t \in]-\pi, \pi[. \quad \text{Adjustment } f^*(\pi + 2n\pi) = 2\pi^2.$$



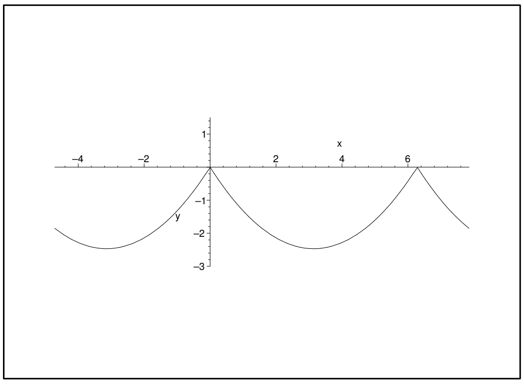
Fourier series:

$$f \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{n^2} \cos nt + \frac{(-1)^n \pi}{n} \sin nt \right\}.$$

There is nothing new concerning **pointwise** results.

Parseval:

$$\frac{32\pi^4}{5} = \frac{35\pi^4}{9} + 16\sum_{n=1}^{\infty} \frac{1}{n^4} + 16\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{from which } \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \qquad \Diamond$$



Example 3.21 *The period is* 2π.

$$f(t) = \frac{t^2}{4} - \frac{\pi t}{2}, \qquad t \in]0, 2\pi[.$$

The function is continuous and piecewise C^1 and without vertical half tangents. According to the main theorem we can use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = -\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}.$$

By some calculations and reductions we get the **pointwise** result

$$\sum_{n=1}^{\infty} \frac{\cos nt}{n^2} = \frac{1}{4}\left\{(\pi - t)^2 - \frac{\pi^2}{3}\right\} \qquad \text{for } t \in]0, 2\pi[.$$

Parseval:

$$\frac{\pi^4}{15} = \frac{\pi^4}{18} + \sum_{n=1}^{\infty} \frac{1}{n^4}, \qquad \text{from which } \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \qquad \Diamond$$

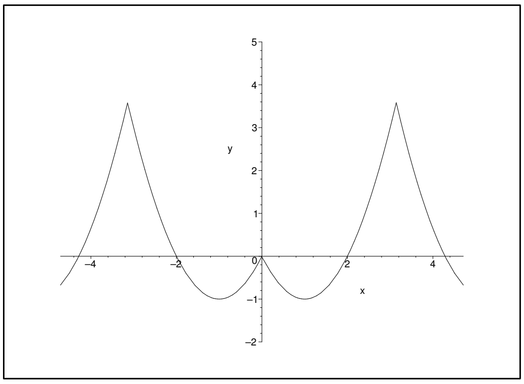
Example 3.22 *The period is* 2π

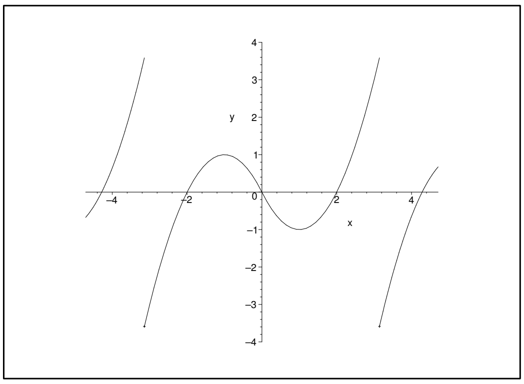
$$f(t) = t^2 - 2t, \qquad t \in [0, \pi] \text{ even function.}$$

The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem write pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{\pi^2}{3} - \pi + \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\{1 + (-1)^n(\pi - 1)\}\cos nt.$$

In this case both the **pointwise results** and **Parseval's formula** become very messy, so we shall not give any of these. $\Diamond$





Example 3.23 *The period is* 2π.

$$f(t) = t^2 - 2t, \qquad t \in [0,\pi], \text{ odd function.}$$

The adjustment is $f^*(\pi + 2n\pi) = 0$.
Fourier series:

$$f \sim \sum_{n=1}^{\infty} \left\{ \frac{2(\pi-2)}{n} (-1)^n - \frac{4}{\pi n^3}(1 - 1(-1)^n) \right\} \sin nt.$$

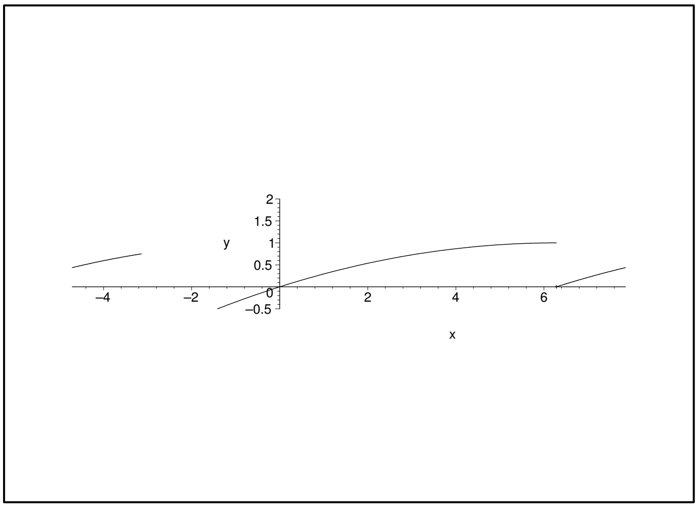
Both the **pointwise results** and **Parseval's formula** are rather messy, so we shall not give any of them. $\Diamond$

Example 3.24 *The period is* 2π.

$$f(t) = \frac{1}{4\pi^2}\, t(4\pi - t), \qquad t \in [0, 2\pi].$$

The adjustment is $f^*(2n\pi) = \dfrac{1}{2}$.
Fourier series:

$$f \sim \frac{2}{3} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2}\, \cos nt + \frac{\pi}{n}\, \sin nt \right\}.$$



Pointwise: Nothing new.

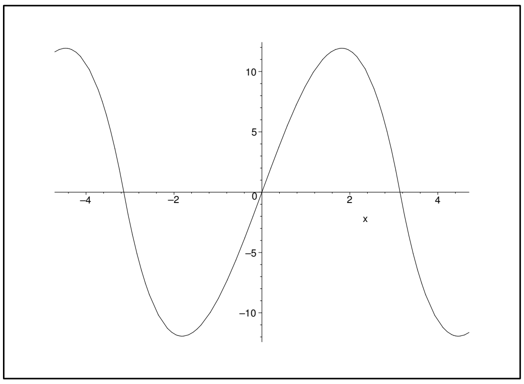
Parseval:

$$\frac{16}{15} = \frac{8}{9} + \frac{1}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}. \qquad \Diamond$$

3.4 A piecewise polynomial of third degree

Example 3.25 *The period is 2π.*

$$f(t) = t(\pi^2 - t^2), \qquad t \in [-\pi, \pi],$$



The function is continuous and piecewise C^1 without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nt.$$

Pointwise: For $t = \frac{\pi}{2}$ we get

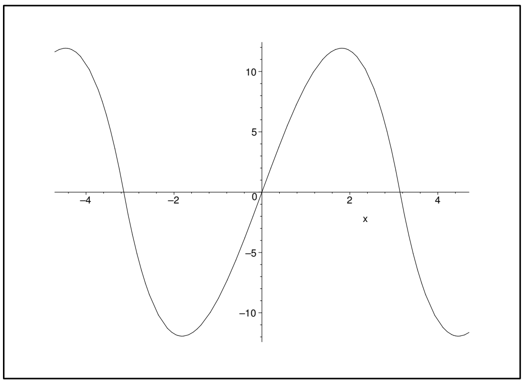
$$\frac{3\pi^2}{8} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}, \qquad \text{from which } \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

Parseval:

$$\frac{16\pi^6}{105} = 144 \sum_{n=1}^{\infty} \frac{1}{n^6}, \qquad \text{from which } \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}. \qquad \Diamond$$

Example 3.26 *The period is* 2π.

$$f(t) = t\left(t - \frac{\pi}{2}\right)(t-\pi), \quad t \in [0,\pi] \text{ odd function.}$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

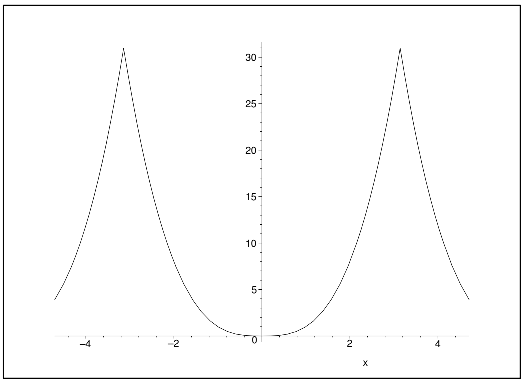
$$f(t) = \frac{3}{2} \sum_{n=1}^{\infty} \frac{\sin 2nt}{n^3}.$$

Pointwise: For $n = \frac{\pi}{4}$ we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$.
Parseval:

$$\frac{\pi^6}{420} = \frac{9}{4} \sum_{n=1}^{\infty} \frac{1}{n^6}, \qquad \text{from which } \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}. \qquad \Diamond$$

Example 3.27 *The period is* 2π.

$$f(t) = |t|^3, \qquad t \in [-\pi,\pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{\pi^3}{4} + 6\pi \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi^2 n^4} \right\} \cos nt.$$

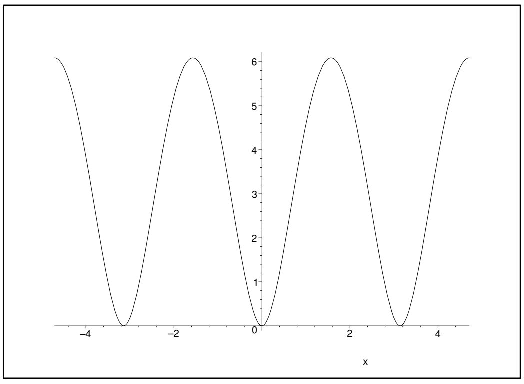
Pointwise: Nothing new.

Parseval gives a mess. $\Diamond$

3.5 A piecewise polynomial of fourth degree

Example 3.28 *The period is* 2π.

$$f(t) = \{\pi|t| - t^2\}^2, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{\pi^4}{30} - \sum_{n=1}^{\infty} \frac{3}{n^4} \cos 2nt.$$

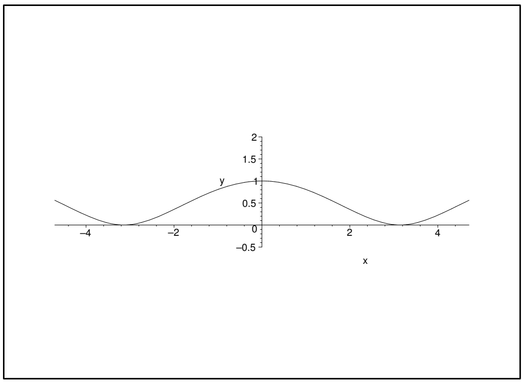
Pointwise: For $t=0$ we get immediately that $\sum_{n=1}^{\infty} \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$.

Parseval:

$$\frac{\pi^8}{315} = \frac{\pi^8}{450} + 9\sum_{n=1}^{\infty} \frac{1}{n^8}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^4}{9450}. \qquad \Diamond$$

Example 3.29 *The period is* 2π.

$$f(t) = \frac{1}{\pi^4}\,(t^2-\pi^2)^2, \qquad t \in [-\pi,\pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{8}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}\cos nt.$$

Pointwise: For $t=0$ we get

$$1 = \frac{8}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}.$$

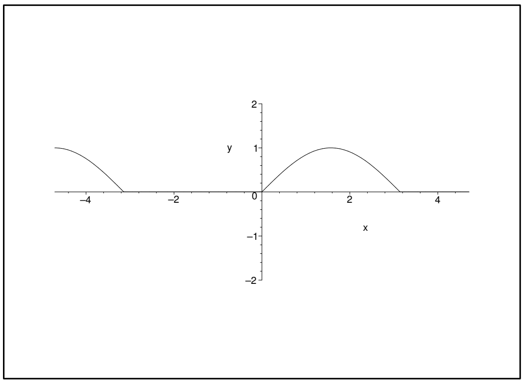
Parseval:

$$\frac{256}{315} = \frac{128}{225} + \frac{2304}{\pi^8}\sum_{n=1}^{\infty} \frac{1}{n^8}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}. \qquad \Diamond$$

3.6 Piecewise sinus

Example 3.30 *The period is* 2π.

$$f(t) = \begin{cases} \sin t, & t \in \,]0,\pi[, \\ 0 & t \in [-\pi,0], \end{cases}$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nt.$$

Pointwise: For $t = 0$ we get

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

For $t = \dfrac{\pi}{4}$ we get

$$\frac{\sqrt{2}}{2} = \frac{1}{\pi} + \frac{\sqrt{2}}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{16n^2-1}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{16n^2-1} = \frac{\pi\sqrt{2}}{8} - \frac{1}{2}.$$

For $t = \dfrac{\pi}{2}$ we get

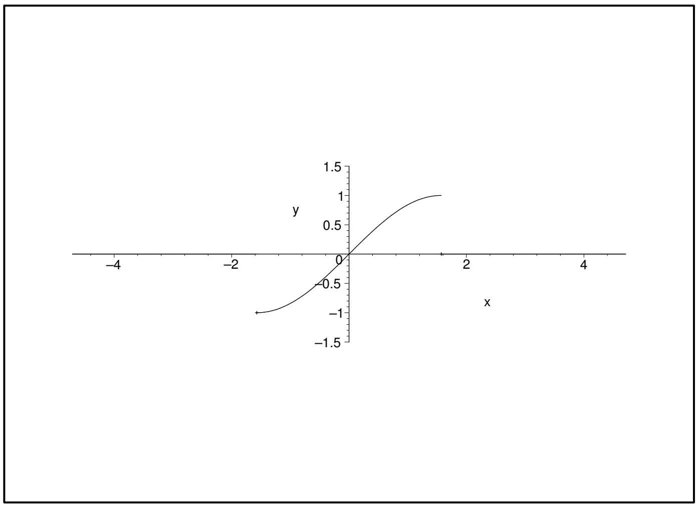
$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi}{4} - \frac{1}{2}.$$

Parseval:

$$\frac{1}{2} = \frac{2}{\pi^2} + \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}. \qquad \Diamond$$

Example 3.31 *The period is* 2π.

$$f(t) = \begin{cases} \sin t, & |t| \le \dfrac{\pi}{2}, \\[2ex] 0, & \dfrac{\pi}{2} < |t| \le \pi. \end{cases} \qquad \text{Adjustment } f^*\left(\pm\frac{\pi}{2} + 2n\pi\right) = \pm\frac{1}{2}.$$



Fourier series:

$$f \sim \frac{1}{2}\,\sin t + \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{4n}{4n^2-1}\,\sin 2nt.$$

Pointwise: For $t = \dfrac{\pi}{4}$ we get

$$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{n}{4n^2-1}\,\sin n\,\frac{\pi}{2} = \frac{\sqrt{2}}{4} + \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{2n+1}{4(2n+1)^2-1},$$

from which

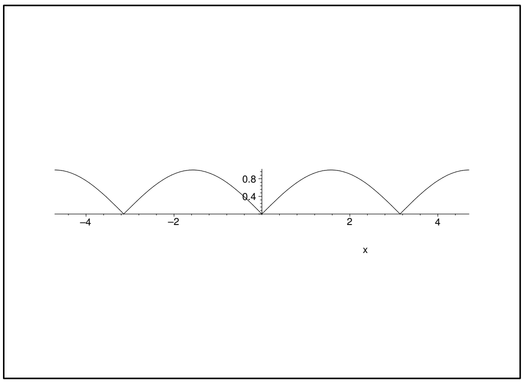
$$\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{4(2n+1)^2-1} = \frac{\sqrt{2}\pi}{16}.$$

Parseval:

$$\frac{1}{2} = \frac{1}{4} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2}, \quad \text{from which} \quad \sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}. \qquad \Diamond$$

Example 3.32 *The period is* 2π*.*

$$f(t) = |\sin t|, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nt.$$

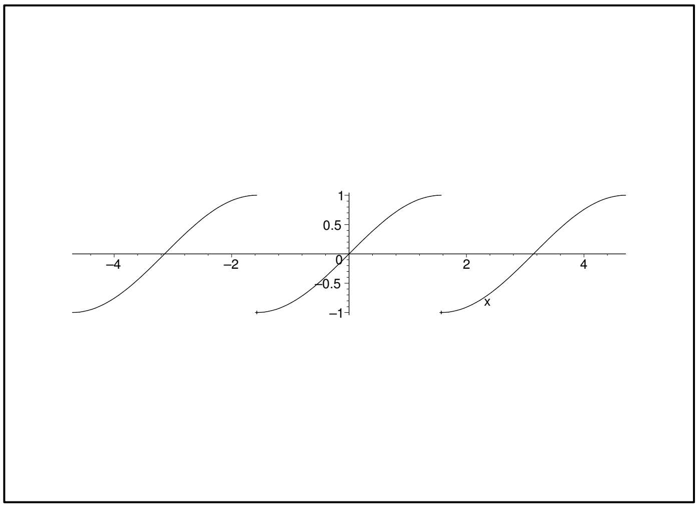
Pointwise: Same results as in a previous example.

Parseval:

$$1 = \frac{8}{\pi^2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} = \frac{\pi^2}{16} - \frac{1}{2}. \qquad \Diamond$$

Example 3.33 *The period is* 2π*.*

$$f(t) = \begin{cases} \sin t, & t \in \left[0, \frac{\pi}{2}\right], \\ -\sin t, & t \in \left]\frac{\pi}{2}, \pi\right], \end{cases} \qquad \text{odd.}$$



Adjustment: $f^*\left(\dfrac{\pi}{2}+2n\pi\right)=0$.
Fourier series:

$$f \sim \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n}{(2n-1)(2n+1)}\,\sin 2nt = \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n}{4n^2-1}\,\sin 2nt.$$

Pointwise: For $t=\dfrac{\pi}{4}$ we get

$$\frac{\sqrt{2}}{2}=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{4n^2-1}\,\sin n\,\frac{\pi}{2}=\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{4(2n+1)^2-1},$$

i.e.

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{4(2n+1)^2-1}=\frac{\sqrt{2}\pi}{16}.$$

Parseval:

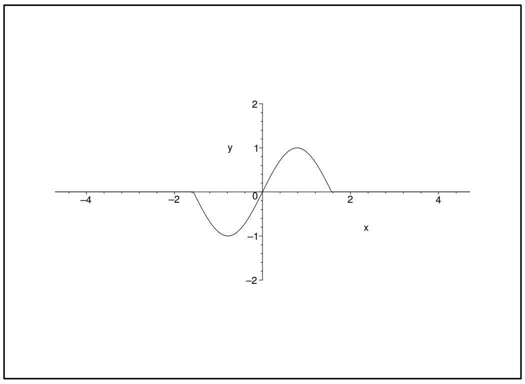
$$1=\frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2},\quad \text{i.e.}\quad \sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2}=\frac{\pi^2}{64}. \qquad \Diamond$$

Example 3.34 *The period is* 2π*.*

$$f(t)=\begin{cases}\sin 2t, & |t|<\dfrac{\pi}{2},\\[2ex] 0, & \dfrac{\pi}{2}\le|t|\le\pi.\end{cases}$$

The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t)=\frac{1}{2}\,\sin 2t+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+3)}\,\sin(2n+1)t.$$



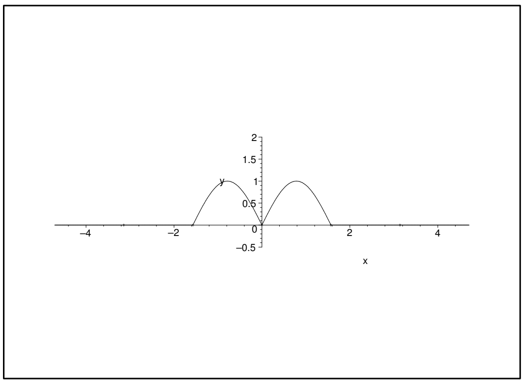
Pointwise: Nothing of interest.

Parseval:

$$\frac{1}{2}=\frac{1}{4}+\frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n-1)^2(2n+3)^2},\quad \text{i.e.}\quad \sum_{n=0}^{\infty}\frac{1}{(2n-1)^2(2n+3)^2}=\frac{\pi^2}{64}.\qquad\Diamond$$

Example 3.35 *The period is* 2π.

$$f(t) = \begin{cases} |\sin 2t|, & |t| \le \dfrac{\pi}{2}, \\[2ex] 0, & \dfrac{\pi}{2} < |t| \le \pi. \end{cases}$$



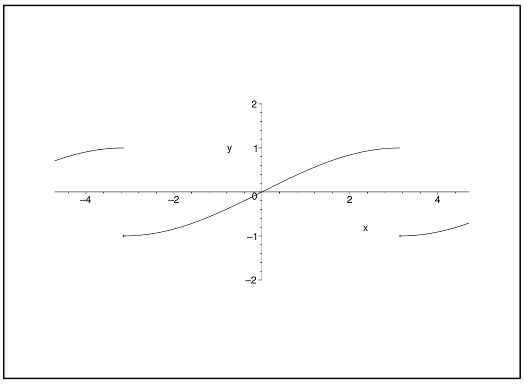
The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{1}{\pi} + \frac{4}{3\pi}\,\cos t - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{2\cos(4n-1)t}{(4n-3)(4n+1)} + \frac{\cos 4nt}{(2n-1)(2n+1)} + \frac{2\cos(4n+1)t}{(4n-1)(4n+3)}\right\}.$$

The **pointwise results** and **Parseval's equation** become again quite messy. $\Diamond$

Example 3.36 *The period is* 2π.

$$f(t) = \sin\frac{t}{2}, \quad t \in \left]-\pi, \pi\right]. \qquad \text{Adjustment: } f^*(\pi + 2n\pi) = 0.$$



Fourier series:

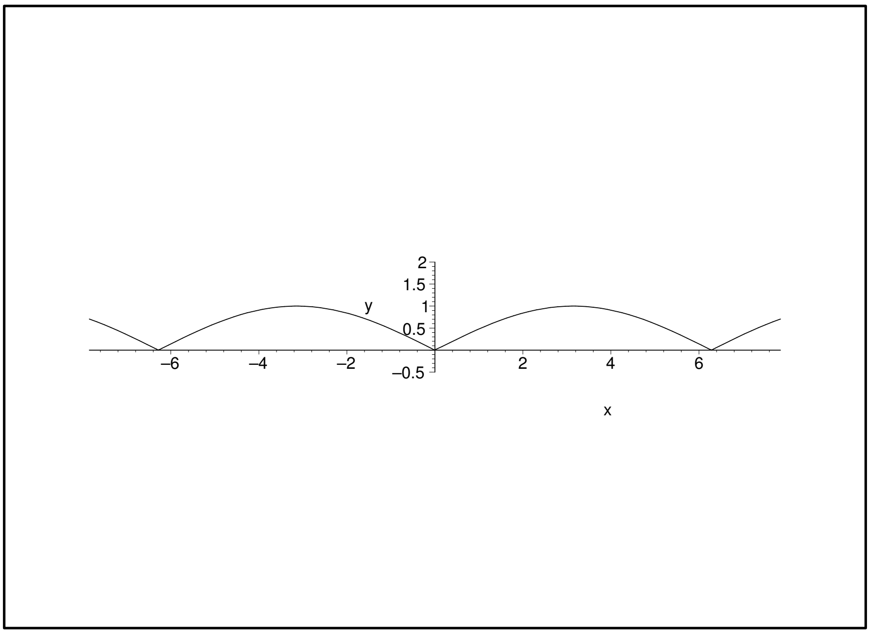
$$f \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{1-4n^2}\,\sin nt = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{4n^2-1}\,\sin nt.$$

The **pointwise results** and **Parseval's formula** are almost like in a previous example. $\Diamond$

Example 3.37 *The period is* 2π.

$$f(t) = \left|\sin\frac{t}{2}\right|, \qquad t \in [-\pi, \pi].$$

The function is continuous and piecewise C^1 and without vertical half tangents. We can according



to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

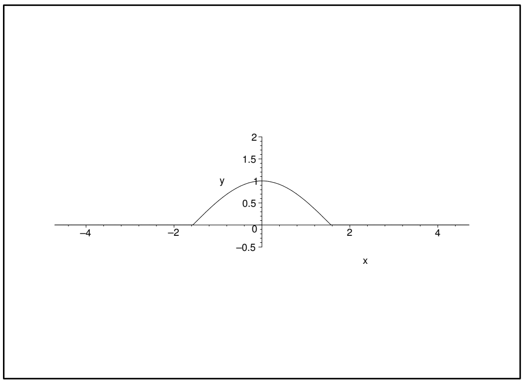
$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\,\cos nt.$$

The **pointwise results** and **Parseval's equation** are approximately as in a previous example. $\Diamond$

3.7 Piecewise cosine

Example 3.38 *The period is* 2π.

$$f(t) = \begin{cases} \cos t & \text{for } |t| < \dfrac{\pi}{2}, \\[2ex] 0 & \text{for } \dfrac{\pi}{2} \le |t| \le \pi. \end{cases}$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos t + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}\cos 2nt.$$

Pointwise: For $t = 0$ we get

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi}{4} - \frac{1}{2}.$$

Parseval:

$$\frac{1}{2} = \frac{2}{\pi^2} + \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}. \qquad \Diamond$$

Example 3.39 *The period is* 2π.

$$f(t) = \begin{cases} \cos t & \text{for } t \in \,]0,\pi[, \\ -\cos t & \text{for } t \in \,]-\pi,0[. \end{cases} \qquad \text{Adjustment: } f^*(n\pi) = 0.$$

Fourier series:

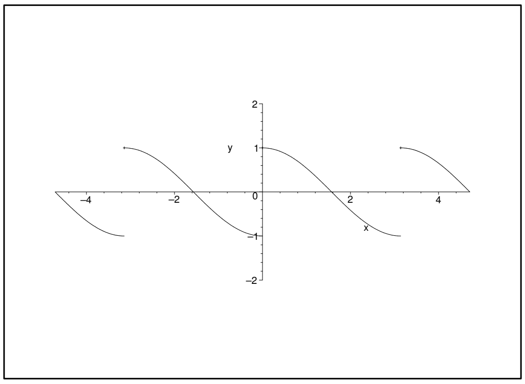
$$f \sim \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1}\sin 2nt.$$

Pointwise: For $t = \dfrac{\pi}{4}$ we get

$$\frac{\sqrt{2}}{2} = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1}\sin n\frac{\pi}{2} = \frac{8}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{4(2n+1)^2-1},$$

i.e.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{4(2n+1)^2-1} = \frac{\pi\sqrt{2}}{16}.$$



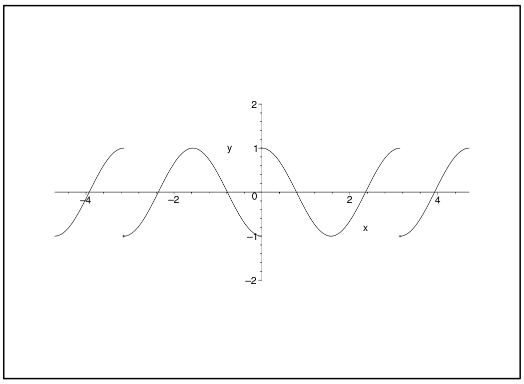
Parseval:

$$1 = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2}, \quad \text{from which } \sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}. \qquad \Diamond$$

Example 3.40 *The period is* 2π.

$$f(t)=\begin{cases}\cos 2t & \text{for } t\in\,]0,\pi[,\\ -\cos 2t & \text{for } t\in\,]-\pi,0[.\end{cases}\qquad \text{Adjustment: } f^*(n\pi)=0.$$

Fourier series:



$$f\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{2n+1}{(2n-1)(2n+3)}\,\sin(2n+1)t.$$

Pointwise: For $t=\dfrac{\pi}{2}$ we get

$$-1=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n-1)(2n+3)},$$

from which

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{(2n-1)(2n+3)}=\frac{\pi}{4}.$$

Parseval:

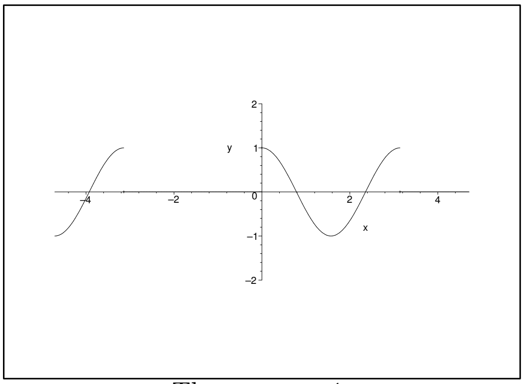
$$1=\frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{(2n+1)^2}{(2n-1)^2(2n+3)^2},\qquad\text{i.e.}\qquad\sum_{n=0}^{\infty}\frac{(2n+1)^2}{(2n-1)^2(2n+3)^2}=\frac{\pi^2}{16}.\qquad\Diamond$$

Example 3.41 *The period is* 2π.

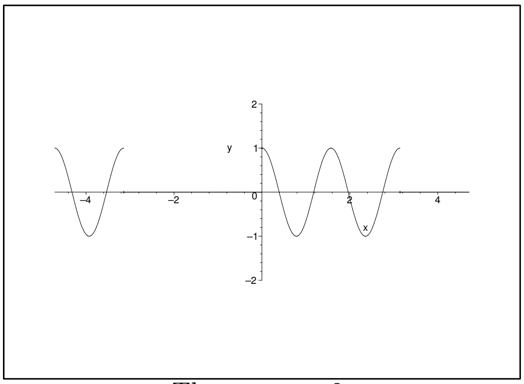
$$f(t)=\begin{cases}\cos 2qt, & t\in[0,\pi],\,q\in\mathbb{N},\\ 0, & t\in\,]-\pi,0[.\end{cases}\qquad \text{Adjustment: } f^*(n\pi)=\frac{1}{2}.$$

Fourier series:

$$f\sim\frac{1}{2}\,\cos 2qt+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{2n+1}{(2n+1)^2-4q^2}\,\sin(2n+1)t.$$



The case $q = 1$.



The case $q = 2$.

Pointwise: For $t = \frac{\pi}{2}$ we get

$$(-1)^q = \frac{1}{2}\,(-1)^q + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2-4q^2},$$

i.e.

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2-4q^2} = (-1)^q\,\frac{\pi}{4}.$$

Parseval:

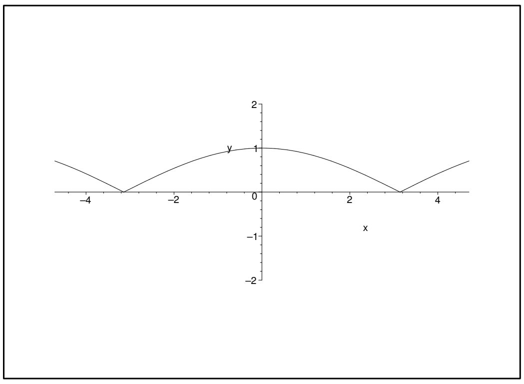
$$\frac{1}{2} = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=0}^{\infty} \frac{(2n+1)^2}{\left\{(2n+1)^2-4q^2\right\}^2},$$

from which

$$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{\left\{(2n+1)^2-4q^2\right\}^2} = \frac{\pi^2}{16}. \qquad \Diamond$$

Example 3.42 *The period is* 2π.

$$f(t) = \cos\frac{t}{2}, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{2}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}} \cos nt.$$

Pointwise: For $t = 0$ we get

$$1 = \frac{2}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}} = \pi - 2.$$

For $t = \pi$ we get

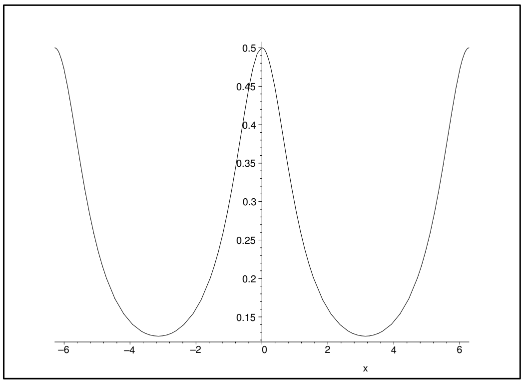
$$0 = \frac{2}{\pi} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}} = 2.$$

Parseval:

$$1 = \frac{2}{\pi^2} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{\left(n^2 - \frac{1}{4}\right)^2}, \quad \text{from which} \quad \sum_{n=1}^{\infty} \frac{1}{\left(n^2 - \frac{1}{4}\right)^2} = \pi^2 - 2. \qquad \Diamond$$

Example 3.43 *The period is* 2π.

$$f(t) = \frac{1}{5 - 3\cos t}, \qquad t \in \mathbb{R}.$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{3^n}\,\cos nt.$$

Pointwise: Nothing of interest.

Parseval:

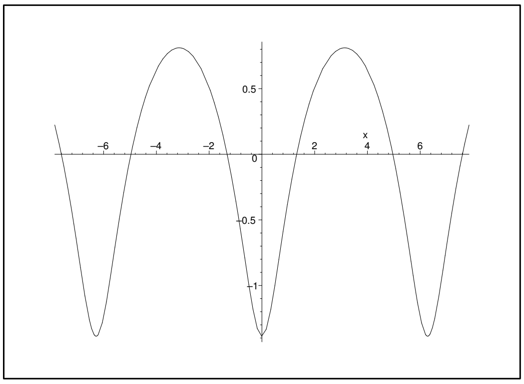
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{dt}{(5-3\cos t)^2}=\frac{1}{8}+\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{3^n}\right)^2=\frac{5}{32},$$

i.e.

$$\int_{-\pi}^{\pi}\frac{dt}{(5-3\cos t)^2}=\frac{5\pi}{32}. \qquad \Diamond$$

Example 3.44 *The period is 2π.*

$$f_r(t)=\ln(1+r^2-2r\cos t), \quad t\in\mathbb{R} \text{ and } r\in\,]0,1[.$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f_r(t)=-2\sum_{n=1}^{\infty}\frac{r^2}{n}\cos nt, \qquad r\in\,]0,1[.$$

Pointwise: For $t=0$ and $t=\dfrac{\pi}{2}$ and $t=\pi$ we get some logarithmic series.

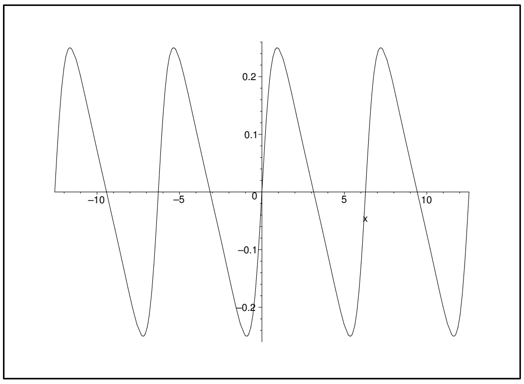
Parseval:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\ln(1+r^2-2r\cos t)\right\}^2 dt=4\sum_{n=1}^{\infty}\frac{r^{2n}}{n^2}, \qquad r\in\,]0,1[. \qquad \Diamond$$

3.8 Mixed sinus and cosine

Example 3.45 *The period is 2π.*

$$f(t)=\frac{\sin t}{5-3\cos t}, \qquad t\in\mathbb{R}.$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{2}{3}\sum_{n=1}^{\infty} \frac{1}{3^n}\, \sin nt.$$

Pointwise: Nothing of interest.

Parseval:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin^2 t}{(5-3\cos t)^2}\, dt = \frac{4}{9}\sum_{n=1} \frac{1}{9^2} = \frac{1}{18},$$

i.e.

$$\int_{-\pi}^{\pi} \frac{\sin^2 t}{(5-3\cos t)^2}\, dt = \frac{\pi}{18}. \qquad \Diamond$$

Example 3.46 *The period is* 2π.

$$f(t) = \begin{cases} \sin t & \text{for } t \in \,]-\pi,0], \\ \cos t & \text{for } t \in \,]0,\pi], \end{cases} \qquad \text{Adjustment: } f^*(n\pi) = (-1)^n \cdot \frac{1}{2}.$$

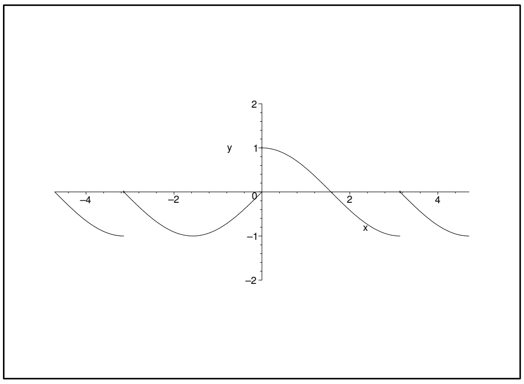
Fourier series:

$$\begin{aligned} f \quad \sim \quad & -\frac{1}{\pi} + \frac{1}{2}\{\cos t + \sin t\} \\ & \qquad + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{ \frac{1}{4n^2-1}\, \cos 2nt + \frac{2n}{4n^2-1}\, \sin 2nt \right\}. \end{aligned}$$

Pointwise: Nothing of interest.

Parseval:

$$\begin{aligned} 1 \quad &= \quad \frac{1}{\pi^2} + \frac{1}{4} + \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{4n^2}{(4n^2-1)^2} \\ &= \quad \frac{1}{2} + \frac{2}{\pi^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{4n^2+1}{(4n^2-1)^2}, \end{aligned}$$



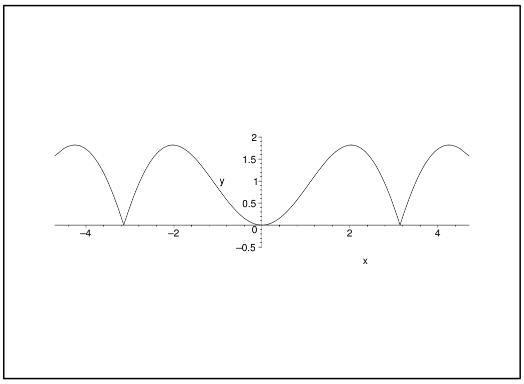
from which

$$\sum_{n=1}^{\infty} \frac{4n^2+1}{(4n^2-1)^2} = \frac{\pi^2}{8} - \frac{1}{2}. \qquad \Diamond$$

3.9 A piecewise polynomial times a trigonometric function

Example 3.47 *The period is* 2π

$$f(t) = t\sin t, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = 1 - \frac{1}{2}\cos t + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1}\cos nt.$$

Pointwise: Nothing of interest.

Parseval:

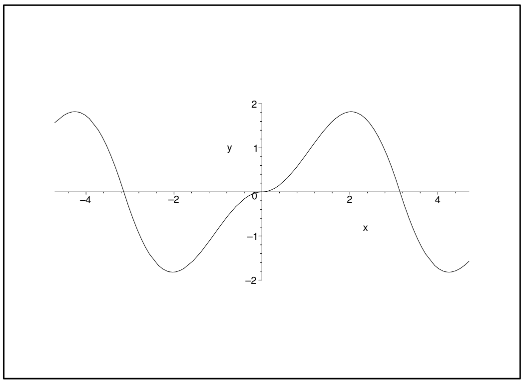
$$\frac{\pi^2}{3} - \frac{1}{2} = 2 + \frac{1}{4} + 4\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2},$$

from which

$$\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2} = \frac{\pi^2}{12} - \frac{11}{16}. \qquad \Diamond$$

Example 3.48 *The period is* 2π.

$$f(t) = \begin{cases} t\sin t, & t \in [0,\pi], \\ -t\sin t, & t \in\,]-\pi,0[, \end{cases}$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{\pi}{2}\,\sin t - \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)^2}\,\sin 2nt.$$

Pointwise: Nothing of interest.

Parseval:

$$\frac{\pi^2}{3} - \frac{1}{2} = \frac{\pi^2}{4} + \frac{256}{\pi^2}\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^4},$$

i.e.

$$\sum_{n=1}^{\infty}\frac{n^2}{(4n^2-1)^4} = \frac{\pi^4}{3072} - \frac{\pi^2}{512}. \qquad \Diamond$$

Example 3.49 *The period is* 2π.

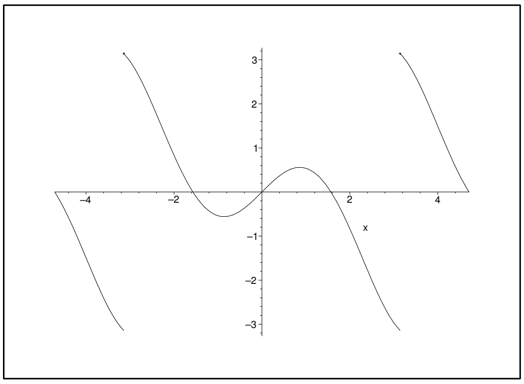
$$f(t) = t\cos t, \qquad t \in\,]-\pi,\pi[, \quad \text{Adjustment: } f^*(n\pi) = 0.$$

Fourier series:

$$f \sim -\frac{1}{2}\,\sin t + \sum_{n=2}^{\infty}(-1)^n \cdot \frac{2n}{n^2-1}\,\sin nt.$$

Pointwise: For $t = \dfrac{\pi}{2}$ we get

$$0 = -\frac{1}{2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n+1)}{(2n+1)^2-1}, \quad \text{i.e.} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n+1)}{(2n+1)^2-1} = \frac{1}{4}.$$

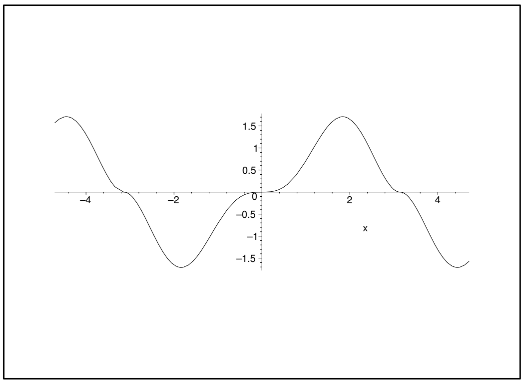


Parseval:

$$\frac{\pi^2}{3}+\frac{1}{2}=\frac{1}{4}+4\sum_{n=2}^{\infty}\frac{n^2}{(n^2-1)^2},\quad \text{from which}\ \sum_{n=2}^{\infty}\frac{n^2}{(n^2-1)^2}=\frac{\pi^2}{12}+\frac{1}{16}. \qquad \Diamond$$

Example 3.50 *The period is 2π.*

$$f(t)=t\cdot\sin^2 t,\qquad t\in[-\pi,\pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t)=\frac{4}{3}\sin t-\frac{3}{8}\sin 2t+\sum_{n=3}^{\infty}(-1)^n\cdot\frac{1}{n(n^2-4)}\sin nt.$$

Pointwise: For $t=\dfrac{\pi}{2}$ we get

$$\frac{\pi}{2}=\frac{4}{3}+\sum_{n=3}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)},$$

i.e.

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)(2n+3)} = \frac{\pi}{2} - \frac{4}{3}.$$

Parseval:

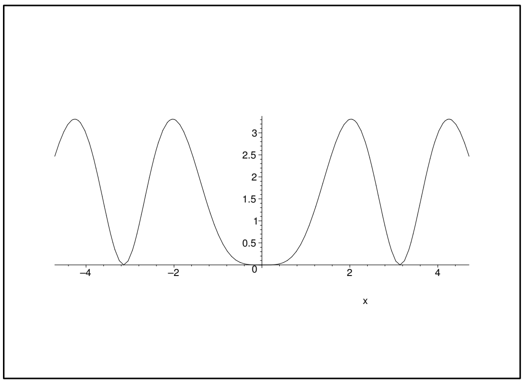
$$\frac{\pi^2}{4} - \frac{15}{32} = \frac{16}{9} + \frac{9}{64} + \sum_{n=3}^{\infty} \frac{1}{n^2-4)^2},$$

i.e.

$$\sum_{n=3}^{\infty} \frac{1}{n^2(n^2-4)^2} = \frac{\pi^2}{4} - \frac{1375}{576}. \qquad \Diamond$$

Example 3.51 *The period is* 2π.

$$f(t) = t^2 \sin t, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \left(\frac{\pi^2}{3} - \frac{1}{2}\right) \sin t + \sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{8n}{(n^2-1)^2} \sin nt.$$

Pointwise: For $t = \dfrac{\pi}{2}$ we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{3} - \frac{1}{2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)}{n(n+1)},$$

from which

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+1)}{n(n+1)} = \frac{\pi^2}{24} - \frac{1}{4}.$$

Parseval:

$$\frac{\pi^4}{5} - \pi^2 + \frac{3}{2} = \left(\frac{\pi^2}{3} - \frac{1}{2}\right)^2 + 64 \sum_{n=2}^{\infty} \frac{n^2}{(n^2-1)^4},$$

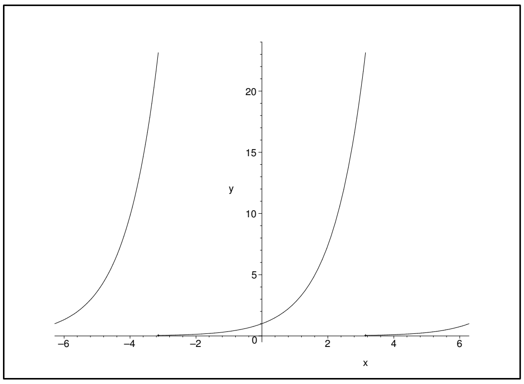
from which

$$\sum_{n=2}^{\infty} \frac{n^2}{(n^2-1)^4} = \frac{\pi^4}{720} - \frac{\pi^2}{96} + \frac{5}{256}. \qquad \Diamond$$

3.10 The exponential function occurs

Example 3.52 *The period is 2π.*

$$f(t) = e^{\alpha t}, \qquad t \in]-\pi, \pi[, \quad \alpha \neq 0. \quad \text{Adjustment: } f^*(\pi + 2n\pi) = \cosh(\alpha\pi).$$



Fourier series:

$$f \sim \frac{\sinh \alpha\pi}{\alpha\pi} + \frac{2\sinh\alpha\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2+n^2} \{\alpha \cos nt - n \sin nt\}.$$

Pointwise: For $t = 0$ we get

$$1 = \frac{\sinh\alpha\pi}{\alpha\pi} + \frac{2\alpha\sinh(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^2+n^2},$$

from which

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha^2+n^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha\sinh(\alpha\pi)}.$$

Parseval:

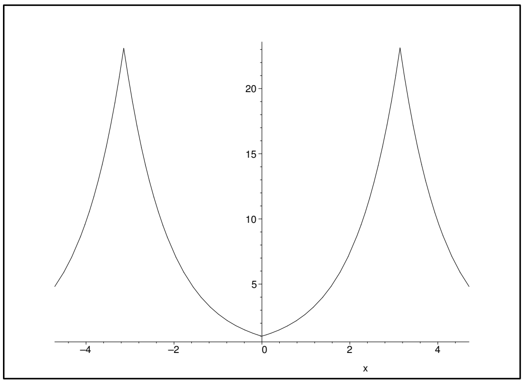
$$\frac{\sinh 2\alpha\pi}{\alpha\pi} = \frac{4\sinh^2\alpha\pi}{\pi^2} \left\{ \frac{1}{2\alpha^2} + \sum_{n=1}^{\infty} \frac{1}{\alpha^2+n^2} \right\},$$

from which

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2+n^2} = \frac{\pi\coth\alpha\pi}{2\alpha} - \frac{1}{2\alpha^2}. \qquad \Diamond$$

Example 3.53 *The period is 2π.*

$$f(t) = e^{|t|}, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{e^\pi - 1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-e^\pi(-1)^n}{n^2+1}\cos nt.$$

Pointwise: For $t = \dfrac{\pi}{2}$ we get

$$\begin{aligned}\exp\left(\frac{\pi}{2}\right) &= \frac{1}{\pi}\{e^\pi - 1\} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-e^\pi}{4n^2+1}(-1)^2 \\ &= \frac{1}{\pi}\{e^\pi - 1\} + \frac{2}{\pi}\{e^\pi - 1\}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2+1},\end{aligned}$$

from which

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1} = \frac{1}{2} - \frac{\pi}{4\sinh\left(\frac{\pi}{2}\right)}$$

and

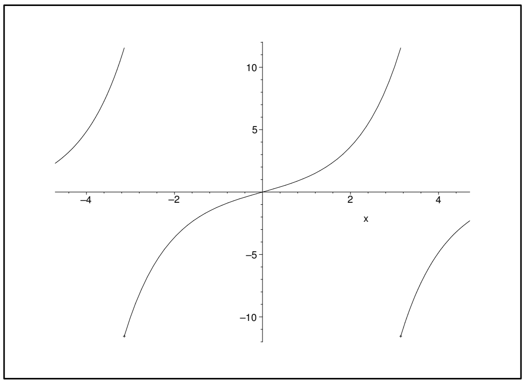
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4n^2+1} = \frac{1}{2} + \frac{\pi}{4\sinh\left(\frac{\pi}{2}\right)}.$$

Parseval's equation does not look nice in this case:

$$\frac{1}{\pi}(e^{2\pi}-1) = \frac{2}{\pi^2}(e^\pi - 1)^2 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\{1-e^\pi(-1)^n\}^2}{(n^2+1)^2}. \qquad \Diamond$$

Example 3.54 *The period is 2π.*

$$f(t) = \sinh t, \qquad t \in \,]-\pi, \pi]. \quad \text{Adjustment: } f^*(\pi + 2n\pi) = 0.$$



Fourier series:

$$f \sim \frac{2\sinh\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1} \, \sin nt.$$

Pointwise: For $t = \frac{\pi}{2}$ fås

$$\sinh \frac{\pi}{2} = \frac{2 \sinh \pi}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 + 1},$$

from which

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 + 1} = \frac{\pi}{4 \cosh \left(\frac{\pi}{2}\right)}.$$

Parseval:

$$\frac{\sinh 2\pi}{2\pi} - 1 = \frac{4 \sinh^2 \pi}{\pi^2} \sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2},$$

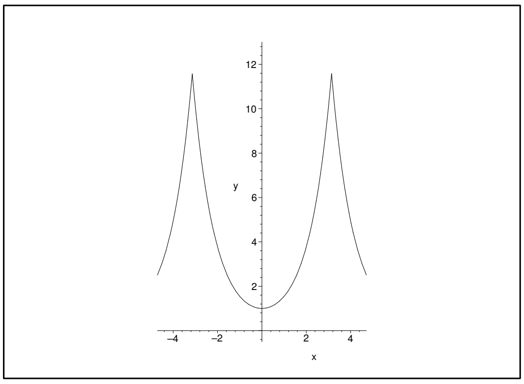
from which

$$\sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2} = \frac{\pi}{4} \coth \pi - \frac{\pi^2}{4 \sinh^2 \pi}. \qquad \Diamond$$

Example 3.55 *The period is* 2π.

$$f(t) = \cosh t, \qquad t \in [-\pi, \pi].$$

The function is continuous and piecewise C^1 and without vertical half tangents. We can according



to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \cos nt.$$

Pointwise: For $t = 0$ we get

$$1 = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1},$$

hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1} = \frac{1}{2} - \frac{\pi}{2\sinh\pi}.$$

For $t = \pi$ we get

$$\cosh\pi = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1},$$

hence

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{1}{2} - \frac{\pi\coth\pi}{2}.$$

Parseval:

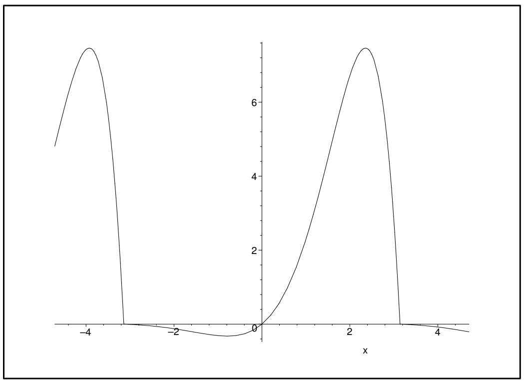
$$\frac{\sinh 2\pi}{2\pi} + 1 = \frac{2\sinh^2\pi}{\pi^2} + \frac{4\sinh^2\pi}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2},$$

from which

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+1)^2} = \frac{\pi}{4}\,\coth\pi + \frac{\pi^2}{4\sinh^2\pi} - \frac{1}{2}. \qquad \Diamond$$

Example 3.56 *The period is 2π.*

$$f(t) = e^t \sin t, \qquad t \in [-\pi, \pi].$$



The function is continuous and piecewise C^1 and without vertical half tangents. We can according to the main theorem use pointwise equality instead of "$\sim$" in the **Fourier series:**

$$f(t) = \frac{\sinh\pi}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n(4-2n^2)}{n^4+4}\,\cos nt + \frac{(-1)^{n+1}4n}{n^4+4}\,\sin nt\right\}\right\}.$$

Pointwise: For $t = 0$ we get

$$0 = \frac{\sinh \pi}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n (4 - 2n^2)}{n^4 + 4} \right\}, \quad \text{hence} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2 - n^2)}{n^4 + 4} = \frac{1}{4}.$$

Parseval:

$$\begin{aligned} \frac{\sinh 2\pi}{4\pi} &= \frac{\sinh^2 \pi}{\pi^2} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(4 - 2n^2)^2}{(n^4 + 4)^2} + \sum_{n=1}^{\infty} \frac{16n^2}{(n^4 + 4)^2} \right\} \\ &= \frac{\sinh^2 \pi}{\pi^2} \left\{ \frac{1}{2} + 4 \sum_{n=1}^{\infty} \frac{1}{n^4 + 4} \right\}, \end{aligned}$$

from which

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 4} = \frac{\pi}{2} \coth \pi - \frac{1}{2}. \qquad \Diamond$$

3.11 The problem of the assumption of no vertical half tangent

This condition, together with the assumption that the one-sided limits always exist are sufficient for the pointwise convergence, though very far from necessary. It is rather awkward at exams, because the student for a good reason do not understand why this condition is added. It is also awkward for experts because it is not at all necessary. At this stage of Calculus some more convenient conditions are the following:

1) The function is piecewise C^1 as defined in the textbook.

2) We demand in each of the open intervals in which the function is of class C^1 that the derivative $f'(t)$ has a limit (possibly $\pm\infty$), when t tends to any of the end points of the subinterval.

3) The function is bounded. (Less can do it. It suffices e.g. that the function is squared integrable,

$$\int_{-\pi}^{\pi} f(t)^2 dt < \infty,$$

provided the period is 2π).

We mention as an example that the Fourier series for the periodic function $f(t)$, which is given in the periodic interval $[-\pi, \pi]$ by

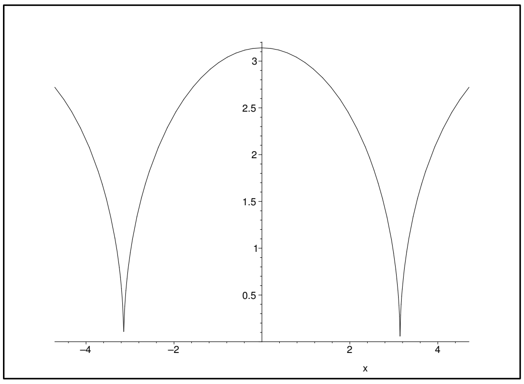
$$f(t) = \sqrt{\pi^2 - t^2}, \qquad t \in [-\pi, \pi],$$

converges pointwise everywhere towards $f(t)$, in spite of the fact that $f(t)$ has vertical half tangents. Notice that the Fourier coefficients of $f(t)$ in this case can only be found by numerical integration.

I shall below sketch what one knows about pointwise convergence of Fourier series:

1) *Carleson, 1965.* If $f \in L^2([-\pi, \pi])$, then the Fourier series of f is pointwise convergent *almost everywhere* with sum $f(t)$. This result has later been extended by *Hunt* to $f \in L^p([-\pi, \pi])$, $p > 1$, i.e. $\int_{-\pi}^{\pi} |f(t)|^p dt < \infty$, cf. e.g. *Jørsboe and Mejlbro, Carleson-Hunt's theorem, Springer.*

2) *Kolmogorov* constructed once in the beginning of the 1920s an example of a function $f \in L^1([-\pi, \pi])$, the Fourier series of which is pointwise divergent everywhere.



3) *Dirichlet, 1837*, proved that if f is *piecewise continuous and monotonous* in a periodic interval, then the Fourier series is pointwise convergent everywhere with the sum $\frac{1}{2}\{f(t-)+f(t+)\}$.

4) *Jessen, Mat 2, 1962–63, MI 124*, quotes *Jordan* without any further reference that if the function f of period 2π if of *bounded variation* (i.e. it can be written as a difference between two to monotonous functions) on every finite interval, then the Fourier series is pointwise convergent for every t with the sum

$$\frac{1}{2}\{f(t+)+f(t-)\}.$$

5) *du Bois-Reymond, 1876*, and *Fejér, 1911*, both constructed continuous functions of period 2π, the Fourier series of which are discontinuous in some points. Fejér's example can e.g. be found in Jessen, Mat2, 1962–63, MI 126.

The functions under consideration are all bounded, piecewise continuous and piecewise differentiable. I practice they will also be of bounded variation.

The justification of the condition is that there exist continuous functions which are even continuously differentiable with the exception of one single point and which are *not* of bounded variation. With the additional assumptions given here this means that the limit of $f'(t)$ does not exist at all, when t tends to one of the points, in which f is not differentiable. One example is

$$f(t)=\begin{cases} t\cos\left(\dfrac{\pi}{2t}\right) & \text{for } x\neq 0,\\[2mm] 0 & \text{for } x=0.\end{cases}$$

In fact, if we choose the subdivision $\left\{0,\frac{1}{2n},\frac{1}{2n-1},\dots,\frac{1}{3},\frac{1}{2},1\right\}$ of $[0,1]$, then

$$\sum_{k=1}^{n}|\Delta f_k| = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\to\infty \qquad \text{for } n\to\infty.$$

Notice, however, that

$$f(t)=\begin{cases} t^2\cos\left(\dfrac{1}{t}\right) & \text{for } x\neq 0,\\ 0 & \text{for } x=0,\end{cases} \qquad x\in[0,1],$$

is of bounded variation, because one by some pottering can show that $|f'(t)| \leq 3$ for $t \in [0,1]$.

These considerations indicate that one must really think in a strange way if one wants to create a pointwise divergent Fourier series.

4 Systems of linear differential equations; methods

4.1 The Existence and Uniqueness Theorem

Before we show the practical solution procedures we must mention the theoretical background, namely the *Existence and the Uniqueness Theorem.* This is given in two versions of which the second one is derived from the first one.

Theorem 4.1 Existence and Uniqueness Theorem, 1st version. *Let* $\mathbf{A}(t)$ *be a continuous* $(n \times n)$*-matrix for* $t \in]a,b[$*, and let* $t_0 \in]a,b[$*. Let* $\mathbf{v} \in \mathbb{R}^n$ *be a given vector, and* $\mathbf{u}(t)$ *a given continuous vector function. Then the initial value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{v},$$

has precisely one solution.

Theorem 4.2 Existence and Uniqueness Theorem, 2nd version. *Let* $a_0(t)$*,* $a_1(t)$*, ...,* $a_{n-1}(t)$ *and* $u(t)$ *be continuous functions in* $]a,b[$*. Let* $t_0 \in]a,b[$*, and let* $\mathbf{v} \in \mathbb{R}^n$ *be a given vector. Then the initial value problem*

$$\begin{cases} \dfrac{d^n x}{dt^n} + a_{n-1}(t)\,\dfrac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t)\,\dfrac{dx}{dt} + a_0(t)x(t) = u(t), \\[2ex] x(t_0) = v_1, \quad x'(t_0) = v_2, \quad \dots, \quad x^{(n-1)}(t_0) = v_n, \end{cases}$$

has precisely one solution.

It is very **important** in the 2nd version that the coefficient of the term of highest order $\dfrac{d^n x}{dt^n}$ is 1 (or is a function, which is never zero in $]a,b[$). This is frequently neglected by the students. One can make a similar remark to the 1st version of the Existence and Uniqueness Theorem, but the notation here invites less to making an error.

There exists in general in the 2nd version only a solution formula, if one already knows $n-1$ linearly independent solutions of the corresponding homogeneous equation. Analogously for the 1st version (because the 2nd version is derived from the 1st version). However, if the matrix of the system $\mathbf{A}$ is *constant*, then we can in principle always find the complete solution.

4.2 Solution of a linear homogeneous differential equation system with a constant system matrix A

The most common problem is to find the complete solution of the matrix equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}, \qquad \mathbf{x} = \mathbf{x}(t),$$

where $\mathbf{A}$ is a constant $(n \times n)$-matrix.

If we put $\mathbf{u} \equiv \mathbf{0}$ in the 1st version of the Existence and Uniqueness Theorem we conclude that there to any vector $\mathbf{v} = \mathbf{c} \in \mathbb{R}^n$ exists precisely one solution. Considering $\mathbf{c} = (c_1, \dots, c_n)$ as a vector consisting of n arbitrary constants we see that the complete solution *must* contain n arbitrary constants.

We shall here present five solution methods of this problem:

1) The eigenvalue method, real eigenvalues.

2) The eigenvalue method, complex eigenvalues, complex calculations.

3) The eigenvalue method, complex eigenvalues, real calculations.

4) Direct determination of the exponential matrix.

5) The fumbling method.

Of these. at least the 1st and the 2nd must be mastered by the student. Method number 4 is elegant, but it may be considered as difficult the first time it is seen. The methods of the 3rd and 5th case are some kind of emergency brakes. They should only be applied when everything else fails.

One should *always* (cf. the name "eigenvalue method") start by finding the *eigenvalues* of the matrix $\mathbf{A}$ *with constant coefficients* (check!), i.e. the roots of the *characteristic polynomial*

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}).$$

We shall only go through these methods for (2×2)-matrices. The generalization to $(n \times n)$-matrices is left to the reader.

4.3 Eigenvalue method, real eigenvalues, A a constant (2×2)-matrix

We have here two possibilities: 1) two different eigenvalues, and 2) two identical (multiple) eigenvalues.

1) Two different real eigenvalues $\lambda_1 \neq \lambda_2$.

   The corresponding *eigenvectorer* are (e.g.)

$$\mathbf{W}_1 = \begin{pmatrix} a_{12} \\ \lambda_1 - a_{11} \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} a_{12} \\ \lambda_2 - a_{11} \end{pmatrix}, \quad \text{where } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

   Proof. For $i = 1\ 2$ we get

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{w}_i = \begin{pmatrix} a_{11} - \lambda_i & a_{12} \\ a_{21} & a_{22} - \lambda_i \end{pmatrix} \begin{pmatrix} a_{12} \\ \lambda_i - a_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ \star \end{pmatrix},$$

   where $\star$ is also 0, because the two rows in $\mathbf{A} - \lambda_i \mathbf{I}$ are linearly dependent when λ_i is an eigenvalue and $\lambda_1 \neq \lambda_2$. An eigenvector is therefore perpendicular to any of the *rows* in $\mathbf{A} - \lambda_i \mathbf{I}$. (This means that one interchanges the coordinates and changes the sign at only one place). If the first row should be the $\mathbf{0}$-vector, then choose the second row instead. □

   The *complete solution* is

$$\mathbf{x}(t) = c_1 \exp(\lambda_1 t)\, \mathbf{w}_1 + c_2 \exp(\lambda_2 t)\, \mathbf{w}_2,$$

   where c_1 and c_2 are arbitrary constants.

   A *fundamental matrix* is

$$\mathbf{\Phi}(t) = \left(e^{\lambda_1 t} \mathbf{w}_1 \ \vdots \ e^{\lambda_2 t} \mathbf{w}_2 \right) = \begin{pmatrix} w_{1,1} e^{\lambda_1 t} & w_{2,1} e^{\lambda_1 t} \\ w_{1,2} e^{\lambda_1 t} & w_{2,2} e^{\lambda_2 t} \end{pmatrix}.$$

   The *exponential matrix* is

$$\exp(\mathbf{A}t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}.$$

   This can be found in a more elegant way by one of the methods in 4); cf. later.

   The reader is here reminded of the method of *inversion of a regular* (2×2)*-matrix*, cf. Linear Algebra:

$$\mathbf{B}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \mathbf{B}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det \mathbf{B} = ad - bc \neq 0.$$

   (Interchange the elements of the diagonal; change sing on the two remaining elements; divide by the determinant).

2) Multiple eigenvalues, $\lambda_1 = \lambda_2 = \lambda$.

   Here we have two cases, a) the eigenspace is of rang 2, or b) the eigenspace is of rang 1.

   a) If the eigenspace corresponding to the eigenvalue λ of multiplicity 2 has *rang 2*, then choose two linearly independent eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$, and continue as in 1) above.

b) If the eigenspace corresponding to λ of multiplicity only has *rang 1*, then find the eigenvector $\mathbf{w}_1$ as in 1) above, e.g.

$$\mathbf{w}_1 = \begin{pmatrix} a_{12} \\ \lambda - a_{11} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \lambda_1 = \lambda_2 = \lambda.$$

Then determine $\mathbf{w}_2$ as a solution of the **singular equation**

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 = \mathbf{w}_1.$$

The *complete solution* is then

$$\mathbf{x}(t) = c_1\, e^{\lambda t}\,\mathbf{w}_1 + c_2\left\{t\, e^{\lambda t}, \mathbf{w}_1 + e^{\lambda t}\,\mathbf{w}_2\right\},$$

where c_1 and c_2 are arbitrary constants.

Proof. It suffices to prove that $\mathbf{x}(t) = te^{\lambda t}\mathbf{w}_1 + e^{\lambda t}\mathbf{w}_2$ is a solution. By a calculation we get

$$\frac{d\mathbf{x}}{dt} = te^{\lambda t} \cdot \lambda\mathbf{w}_1 + e^{\lambda t}(\mathbf{w}_1 + \lambda\mathbf{w}_2)$$

and

$$\mathbf{A}\,\mathbf{x} = te^{\lambda t} \cdot \mathbf{A}\,\mathbf{w}_1 + e^{\lambda t} \cdot \mathbf{A}\,\mathbf{w}_2.$$

Since $e^{\lambda t}$ and $te^{\lambda t}$ are *linearly independent*, we have $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}$, if and only if the vectors of the coefficients agree,

$$\mathbf{A}\,\mathbf{w}_1 = \lambda\,\mathbf{w}_1 \quad \text{og} \quad (\mathbf{A} - \lambda\,\mathbf{I})\mathbf{w}_2 = \mathbf{w}_1.$$

Now, $\mathbf{w}_1$ and $\mathbf{w}_2$ were precisely constructed in such a way that these two matrix equations are satisfied. □

A *fundamental matrix* is

$$\mathbf{\Phi}(t) = \left(e^{\lambda t}\,\mathbf{w}_1 \;\vdots\; te^{\lambda t}\,\mathbf{w}_1 + e^{\lambda t}\,\mathbf{w}_2\right).$$

Since $\mathbf{\Phi}(0) = (\mathbf{w}_1 \;\vdots\; \mathbf{w}_2)$, we easily get the *exponential matrix* by the formula

$$\exp(\mathbf{A}\,t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}.$$

4.4 The eigenvalue method, complex conjugated eigenvalues, comples calculations

When a *real* (2×2)-matrix has *complex* eigenvalues, then they are *complex conjugated*,

$$\lambda = a + i\omega \qquad (\text{og } \tilde{\lambda} = a - i\omega), \quad a, \omega \in \mathbb{R}, \quad \omega \neq 0.$$

Choose any one of these, e.g. $\lambda = a + i\omega$, where $\omega > 0$. A corresponding *complex* eigenvector is then e.g.

$$\alpha + i\beta = \begin{pmatrix} a_{12} \\ a - a_{11} \end{pmatrix} + i\begin{pmatrix} 0 \\ \omega \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

(Interchange the first row in $\mathbf{A} - (a + i\omega)\mathbf{I}$ and change one sign).

If we instead use the second row in $\mathbf{A} - (a + i\omega)\mathbf{I}$ for this construction, we get

$$\tilde{\alpha} + i\tilde{\beta} = \begin{pmatrix} a - a_{22} \\ a_{21} \end{pmatrix} + i \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Coinciding *complex* eigenvalues can for real matrices only occur when $n \geq 4$. Such systems are too big for problems in an elementary course of Calculus, so they will not be met here.

Expressed by the *real* vectors α and β the corresponding *fundamental matrix* becomes

$$\begin{aligned} \mathbf{\Phi}(t) &= \left(\mathrm{Re}\left\{ e^{(a+i\omega)t}(\alpha + i\beta) \right\} \ \vdots \ \mathrm{Im}\ \left\{ e^{(a+i\omega)t}(\alpha + i\beta) \right\} \right) \\ &= e^{at}\cos(\omega t)(\alpha \ \vdots \ \beta) + e^{at}\sin(\omega t)(-\beta \ \vdots \ \alpha). \end{aligned}$$

Since $\mathbf{\Phi}(0) = (\alpha \ \vdots \ \beta)$, it is easy to calculate the *exponential matrix*:

$$\exp(\mathbf{A}\, t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}.$$

Note here the *alternative method* in section 4.6.

The *complete solution løsning* is

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{\Phi}(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \left\{ e^{at}\cos(\omega t) \cdot \alpha - e^{at}\sin(\omega t) \cdot \beta \right\} + c_2 \left\{ e^{at}\cos(\omega t) \cdot \beta + e^{at}\sin(\omega t) \cdot \alpha \right\}, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

4.5 The eigenvalue method, complex conjugated eigenvalues, real calculations

This variant is an *emergency brake*, which should *only* be applied if one does not like complex calculations at all. The *punishment* is that even if the calculations can be performed they are always *extremely big*.

1) If the eigenvalues are $\lambda = a \pm i\omega$, $\omega \neq 0$, then the structure of the solution is necessarily

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_1 e^{at}\cos\omega t + a_2 e^{at}\sin\omega t \\ b_1 e^{at}\cos\omega t + b_2 e^{at}\sin\omega t \end{pmatrix}.$$

2) Calculate

$$\frac{d\mathbf{x}}{dt} \quad \text{og} \quad \mathbf{A}\,\mathbf{x},$$

and identify the coefficients of $e^{at}\cos\omega t$ and $e^{at}\sin\omega t$.

3) Then we get *two* equations in the *four* unknown a_1, a_2, b_1, b_2. The two "missing" equations correspond to the two arbitrary constants c_1 and c_2. Once this has been realized it becomes easy to solve the system of equations.

One should be aware of that there in practice always will occur some very tricky calculations by this method.

4.6 Direct determination of the exponential matrix $\exp(\mathbf{A}\,t)$

Let $P(\lambda) = \det(\mathbf{A} - \lambda\,\mathbf{I})$ be the characteristic polynomial. Then by *Caley-Hamilton's theorem*, possibly known from Linear Algebra,

$$P(\mathbf{A}) = \mathbf{0}.$$

For (2×2)-matrices we exploit this result in the following way:

First set up the characteristic polynomial in the following way,

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = (\lambda - a)^2 + b,$$

i.e.

$$a = -\frac{a_1}{2} \quad \text{og} \quad b = a_2 - \left(\frac{a_1}{2}\right)^2 = \frac{4a_2 - a_1^2}{4}.$$

When we put $\mathbf{B} = \mathbf{A} - a\,\mathbf{I}$, it follows from Caley-Hamilton's theorem that

$$\mathbf{B}^2 = -b\,\mathbf{I}.$$

This result has the following nice consequence,

(5) $\mathbf{B}^{2n} = (-b)^n \mathbf{I}$ and $\mathbf{B}^{2n+1} = (-b)^n \mathbf{B}$.

Since $\mathbf{I}$ commutes with everything we get by an insertion into the exponential series that

$$\begin{aligned}
\exp(\mathbf{A}\,) &= \exp(\mathbf{B}\,t + at\,\mathbf{I}) = e^{at}\exp(\mathbf{B}\,t) \qquad (\mathbf{I}\text{ commutes with }\mathbf{B})\\
&= e^{at}\sum_{n=0}^{\infty}\frac{1}{n!}\,\mathbf{B}^n t^n \qquad \text{divide into even/odd indices}\\
&= e^{at}\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\mathbf{B}^{2n}t^{2n} + e^{at}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\mathbf{B}^{2n+1}t^{2n+1} \qquad [\text{apply (5)}]\\
&= e^{at}\left\{\sum_{n=0}^{\infty}\frac{(-b)^n}{(2n)!}\,t^{2n}\right\}\mathbf{I} + e^{at}\left\{\sum_{n=0}^{\infty}\frac{(-b)^n}{(2n+1)!}\,t^{2n+1}\right\}\mathbf{B}.
\end{aligned}$$

We have now three possibilities which each should be considered more closely:

1) $b=0$, 2) $b>0$, 3) $b<0$.

1) When $b=0$, i.e. $\lambda = a$ is a root of multiplicity 2 in the characteristic polynomial, then apart from the first term every term in the series are zero, hence

$$\begin{aligned}
\exp(\mathbf{A}\,t) &= e^{at}\mathbf{I} + te^{at}\mathbf{B} \qquad (\mathbf{B} = \mathbf{A} - a\mathbf{I})\\
&= e^{at}(1-at)\mathbf{I} + te^{at}\mathbf{A}.
\end{aligned}$$

2) When $b>0$ (i.e. the characteristic polynomial has the two complex conjugated roots $\lambda = a \pm i\sqrt{b}$), then we perform the following rearrangements of the series

$$\sum_{n=0}^{\infty}\frac{(-b)^n}{(2n)!}\,t^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,(\sqrt{b}\cdot t)^{2n} = \cos(\sqrt{b}\cdot t)$$

and

$$\sum_{n=0}^{\infty}\frac{(-b)^n}{(2n+1)!}\,t^{2n+1} = \frac{1}{\sqrt{b}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(\sqrt{b}\cdot t)^{2n+1} = \frac{1}{\sqrt{b}}\,\sin(\sqrt{b}\cdot t).$$

Since $\mathbf{B} = \mathbf{A} - a\mathbf{I}$, we get by insertion

$$\begin{aligned}
\exp(\mathbf{A}\,t) &= e^{at}\cos(\sqrt{b}\cdot t)\mathbf{I} + \frac{1}{\sqrt{b}}\,e^{at}\sin(\sqrt{b}\cdot t)\mathbf{B}\\
&= e^{at}\left\{\cos(\sqrt{b}\cdot t) - \frac{a}{\sqrt{b}}\,\sin(\sqrt{b}\cdot t)\right\}\mathbf{I} + \frac{1}{\sqrt{b}}\,e^{at}\sin(\sqrt{b}\cdot t)\mathbf{A}.
\end{aligned}$$

In this case the roots in the characteristic polynomial are traditionally written $\lambda = a \pm i\omega$, so $\omega = \sqrt{b}$. Therefore also

$$\exp(\mathbf{A}\,t) = e^{at}\left\{\cos\omega t - \frac{a}{\omega}\,\sin\omega t\right\}\mathbf{I} + \frac{1}{\omega}\,e^{at}\sin\omega t\cdot\mathbf{A}.$$

3) When $b < 0$ (i.e. the characteristic polynomial has the two *real* roots $\lambda a = a \pm \omega$, where $\omega^2 = -b > 0$), we get by analogous calculations (trigonometric functions are replaced by hyperbolic functions)

$$\exp(\mathbf{A}\,t) = e^{at}\left\{\cosh\omega t - \frac{a}{\omega}\,\sinh\omega t\right\}\mathbf{I} + \frac{1}{\omega}\,e^{at}\,\sinh\omega t \cdot \mathbf{A}.$$

This can be further reduced by inserting

$$\cosh\omega t = \frac{1}{2}\left\{e^{\omega t} + e^{-\omega t}\right\}, \quad \sinh\omega t = \frac{1}{2}\left\{e^{\omega t} - e^{-\omega t}\right\}.$$

A solution formula is in this case not so easy to comprehend as the expression above in $\cosh\omega t$ and $\sinh\omega t$:

If we write the two roots of the characteristic polynomial

$$\lambda_1 = a + \omega, \qquad \lambda_2 = a - \omega,$$

it follows that $2\omega = \lambda_1 - \lambda_2$. Then the reduction above gives after some calculation that

$$\exp(\mathbf{A}\,t) = \frac{1}{\lambda_1 - \lambda_2}\left\{-\lambda_2 e^{\lambda_1 t} + \lambda_1 e^{\lambda_2 t}\right\}\mathbf{I} + \frac{1}{\lambda_1 - \lambda_2}\left\{e^{\lambda_1 t} - e^{\lambda_2 t}\right\}\mathbf{A}.$$

4.7 The fumbling method

I have coined this name because the method typically is applied by intelligent students who did *not* read the theory in the textbook and who at the examination nevertheless by chance fumble themselves to the right idea. This method should *only* be applied when everything else fails, because the calculations often are fairly complicated. The method is, however, far *better* than this name. when the coefficients are *variable*, because in that case all the methods mentioned above fail, since they all assume that the matrix of the system $\mathbf{A}$ is *constant*.

1) Write the system of equations $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x}$ in full:

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1(t) + a_{12}x_2(t), \\ \dfrac{dx_2}{dt} = a_{21}x_1(t) + a_{22}x_2(t). \end{cases}$$

   a) If e.g. $a_{12} = 0$, the first equation is reduced to

   $$\frac{dx_1}{dt} = a_{11}x_1(t),$$

   the solution of which is $x_1(t) = c_1\exp(a_{11}t)$. This solution is then inserted into the second equation which then is solved by methods already mentioned in *Calculus 1a, Functions in one Variable*. Analogously if $a_{21} = 0$.

   b) If both $a_{12} \neq 0$ and $a_{21} \neq 0$, we get from the first equation

   $$(6)\quad x_2(t) = \frac{1}{a_{12}}\,\frac{dx_1}{dt} - \frac{a_{11}}{a_{12}}\,x_1(t).$$

   Insertion of this expression into the second equation gives

   $$\frac{1}{a_{12}}\,\frac{d^2x_1}{dt^2} - \frac{a_{11}}{a_{12}}\,\frac{dx_1}{dt} = a_{21}x_1(t) + \frac{a_{22}}{a_{12}}\,\frac{dx_1}{dt} - \frac{a_{11}a_{22}}{a_{12}}\,x_1(t),$$

i.e. after a small rearrangement,

$$(7)\quad \frac{d^2x_1}{dt^2} - (a_{11}+a_{22})\frac{dx_1}{dt} + \{a_{11}a_{22} - a_{12}a_{21}\}x_1(t) = 0,$$

which is a linear differential equation of second order and constant coefficients. This can therefore be solved completely. The result is then put into (6), and one gets the corresponding function $x_2(t)$.

Equation (7) is also written

$$\frac{d^2x_1}{dt^2} - \text{ trace } \mathbf{A}\,\frac{dx_1}{dt} + \det \mathbf{A}\, x_1(t) = 0,$$

where trace $\mathbf{A}$ is called the trace of the matrix, i.e. the sum of the elements in the diagonal, and where $\det \mathbf{A}$ is the determinant. Incidentally this shows that the problems are best sorted according to the trace of the matrix of the system.

4.8 Solution of an inhomogeneous linear system of differential equations

An inhomogeneous system of differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x} + \mathbf{u}(t)$$

is solved by *superposition*, i.e. we first determine all solutions of the corresponding *homogeneous* system, and then find a *particular* solution. Finally, all solutions of the homogeneous equation are added to the particular solution.

Here I shall give *two* methods for determining the *particular* solution:

1) *Systematic guessing.* Examine the structure of the functions in $\mathbf{u}(t)$ and make a guess of a solution. Test your guess by insertion. This method is *recommended*, because the alternative method below can produce some very complicated calculations. (Problems in elementary Calculus can *always* be solved by a clever guessing).

2) *The general solution formula*

$$\mathbf{x}(t) = \varphi(t)\mathbf{c} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}(\tau)^{-1}\mathbf{u}(\tau)\,d\tau, \qquad t \in\,]a,b[.$$

This should be avoided as much as possible! If one nevertheless is forced to use it, I shall here ease matters a little:

a) First calculate $\mathbf{\Phi}(t)^{-1}\mathbf{u}(t) = \mathbf{v}(t)$.

b) Find the antiderivatives

$$\int v_1(t)\,dt \quad \text{and} \quad \int v_2(t)\,dt$$

separately! (i.e. *not* in some form of a matrix).

c) Insert finally into the solution formula.

The task will become big, but it should be possible to finish one's calculations.

5 The eigenvalue problem of differential equations

This chapter is an alternative exposition of the usual one seen in textbooks, based on my experience that the students have difficulties in seeing why this complex of problem is of interest. One must at least have the Fourier series at hand in order to profit from this fairly abstract set up, and this assumption is not always followed in the textbooks. Let us begin by a review of the basis, already given in *Calculus 1a, Functions in one Variable*.

5.1 Linear differential equations with constant coefficients

Let a_0, $a_1, \dots$, a_{n-1} be real constant. (The extension to complex constants should not make any trouble). Let $q(x)$ be a continuous function, defined in an interval I. We shall find the complete solution in I of the following equation of order n:

$$(8)\quad L(t) := \frac{d^n y}{dx^n} + a_{n-1}\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\,\frac{dy}{dx} + a_0\,y = q(x), \qquad x \in I.$$

We have in most cases either $n = 1$ or $n = 2$. In some courses in Mechanics one may see $n = 4$ as well. For instance the differential equation of the bending of a column is given by

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) - \frac{d}{dx}\left(N\,\frac{dw}{dx}\right) - p(x) = 0.$$

Assuming that E, I and N are all constants, this equation is reduced to

$$(9)\quad \frac{d^4w}{dx^4} - \frac{N}{EI}\,\frac{d^2w}{dx^2} = \frac{p(x)}{EI},$$

i.e. a special equation of the type (8).

A sound principle for linear differential equations like e.g. (8) is that a differential equation of order n should be solved by n successive integrations, each giving a new arbitrary constant. We shall now demonstrate that this is in fact true for (8).

The starting point is that the simplest possible differential equation

$$(10)\quad \frac{dy}{dx} = q(x), \qquad x \in I,$$

is solved by just an *integration* and the complete solution is

$$(11)\quad y = c + \int^x g(t)\,dt, \qquad x \in I, \quad c \text{ arbitrar constant.}$$

Every exact solution formula will at some point go through the step from (10) to (11).

We shall demonstrate this by

$$(12)\quad \frac{dy}{dx} - a\,y = q(x), \qquad x \in I, \quad a \text{ some complex constant.}$$

Since we always have $e^{-ax} \neq 0$, we see that (12) is equivalent to

$$(13)\quad \frac{d}{dx}\left(e^{-ax}y\right) = e^{-ax}\left\{\frac{dy}{dx} - a\,y\right\} = e^{-ax}q(x),$$

i.e. precisely a formula of the type (10). We therefore get by an integration that

$$e^{-ax}y = c + \int^x e^{-at}q(t)\,dt,$$

and we have derived the well-known formula, also presented in *Calculus 1a, Functions in one Variable*,

$$(14)\quad y = c\,e^{ax} + e^{ax}\int^x e^{-at}q(t)\,dt, \quad x \in I, \quad c \text{ arbitrary.}$$

Remark 5.1 Recall that if $a = \alpha + i\beta$ is complex then

$$e^{ax} = exp((\alpha + i\beta)x) := e^{\alpha x}\{\cos\beta x + i\sin\beta x\}. \quad \Diamond$$

Theorem 5.1 *Consider the linear equation of second order*

$$(15)\quad \frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=q(x), \qquad x\in I.$$

If R_1 and R_2 are the roots of the corresponding characteristic polynomial

$$R^2+a_1R+a_0,$$

which is obtained from the differential equation by replacing j differentiations by R^j, and no differentiation by the constant 1, then the complete solution of (15) is given by

$$(16)\quad y=e^{R_1x}\int^x e^{(R_2-R_1)t}\left\{\int^t e^{-R_2u}q(u)\,du\right\}dt+c_1e^{R_1x}+c_2e^{R_1x}\int^x e^{(R_2-R_1)t}dt,$$

where c_1 and c_2 are arbitrary constants.

Proof. Since the roots are R_1 and R_2, we have

$$R^2+a_1R+a_0=(R-R_1)(R-R_2)=R^2-(R_1+R_2)R+R_1R_2,$$

hence

$$a_1=-(R_1+R_2)\quad \text{og}\quad a_0=R_1R_2.$$

If we put

$$z=\frac{dy}{dx}-R_1y,$$

we get

$$\frac{dz}{dx}-R_2z=\frac{d^2y}{dx^2}-R_1\frac{dy}{dx}-R_2\frac{dy}{dx}+R_1R_2\,y=\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0\,y=q(x).$$

Therefore, equation (15) is equivalent to the two equations of first order

$$\frac{dz}{dx}-R_2\,z=q(x)\quad\text{and}\quad\frac{dy}{dx}-R_1\,y=z(x).$$

According to (14) the first equation has the complete solution

$$z(x)=c_2e^{R_2x}+e^{R_2x}\int^x e^{-R_2t}q(t)\,dt.$$

By insertion into the second equation we again obtain an equation of the type (12), so by the trick in (13) we get precisely (16). $\Diamond$

Remark 5.2 Notice that we do not make any particular assumptions on R_1, R_2, if only we mention both of them. By this we mean that if R_1 is a root of multiplicity 2, then we shall take R_1 twice. Therefore, R_1 and R_2 are allowed to be equal or different, real or complex, and the succession of them is also of no importance. This means that (16) is a general result.

We can only see the difference when we *compare* the case of $R_1 \neq R_2$ with the case of $R_1 = R_2$. In the first case we shall always get a factor of the type $e^{\alpha t}$, $\alpha \neq 0$, on every integral, while we for $R_1 = R_2 = R$ get the following:

$$\begin{aligned} y &= e^{Rx}\int^x \left\{\int^t e^{-Ru}q(u)\,du\right\}dt + c_1 e^{Rx} + c_2 e^{Rx}\int^x dt \\ &= e^{Rx}\int^x (x-t)e^{-Rt}q(t)\,dt + \tilde{c}_1 e^{Rx} + \tilde{c}_2 x\, e^{Rx}. \end{aligned}$$

This shows that when we have a root of multiplicity 2, then we shall get a factor x [and $(x-t)$] in the formulæ. $\Diamond$

The corresponding *homogeneous equation*

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0\, y = 0$$

is solved by putting $q(x) \equiv 0$ in (15) and (16),

$$(17)\;\; y = c_1 e^{R_1 x} + c_2 e^{R_1 x}\int^x e^{(r_2 - R_1)t} dt.$$

It is again seen that the solution of the inhomogeneous equation (15) is obtained by adding all solutions (17) of the homogeneous equation to any *particular solution*, which is either found by guessing or is calculated by the formula

$$y_0 = e^{R_1 x}\int^x e^{(R_2-R_1)t}\left\{\int^t e^{-R_2 u}q(u)\,du\right\}dt.$$

This principle is typically used by the *method of guessing*, where one can save oneself of some very long and tedious integrations.

5.2 Special case; the guessing method

For linear equations of second order with constant coefficients like in (15) it is always easy to find the complete solution of the corresponding *homogeneous equation*, cf. (17) above. Then one can in principle use the solution formula (16) to determine the complete solution. More general one may use other solution formulæ occurring in the traditional textbooks.

The disadvantage of these solution formulæ is that one often is forced to go through some heavy calculations of integrals, where even the pocket calculator or MAPLE do not like them. (It is therefore recommended that one *alternatively* also try the *guessing method* for the determination of the solution of homogeneous equation.

The idea is simple: If one e.g. by some humbug (one can also test the equation afterwards) has found a solution of the *inhomogeneous* equation, then we get the *complete* solution by adding every solution of the corresponding *homogeneous* equation.

The words *"guessing method"* sends actually a wrong signal to the reader. The method contains a lot of systematism in spite of the name.

In lots of engineering applications (and to some extent also in mathematical applications) the inhomogeneous term $q(x)$ in e.g. (15) is usually of one of the following types:

1) a polynomial
   Ex.: $q(x) = x^2$;
2) an exponential function
   Ex.: $q(x) = e^{-x}$;
3) a trigonometric function
   Ex.: $q(x) = \sin 3x$;
4) a product of functions of the types above
   Ex.: $x \sin 2x$;
5) a sum of functions of the types above
   Ex.: $e^{-x} + xe^{-x}$.

Notice that by differentiation each of the classes above is either mapped into itself or into a simpler class.

In the first step we split functions from class 5) into single functions each from one of the classes 1)–4), and we solve each of the new equations separately. (For polynomials it is even possible directly to proceed further).

The *main rule* is: Always guess a solution of the same structure and test it by insertion into the equation. (We here assume that the coefficients are constant).

1) If e.g. $q(x) = x^2$, guess $y_0(x) = k_2x^2 + k_1x + k_0$, where k_2, k_1, k_0 are determined by testing the original equation. Since the degree of a polynomial is lowered by each differentiation, we can guess the full polynomial of the same degree as $q(x)$.

2) If e.g. $q(x) = 6e^{-x}$, guess $y_0(x) = k \cdot e^{-x}$. The test in the equation will give us the explicit value of k.

3) If e.g. $q(x) = \sin 3x$, guess

   $$y_0(x) = k_1 \cos 3x + k_2 \sin 3x,$$

   and then k_1, k_2 is determined by testing this solution. Note that one always guess the *full* trigonometric polynomial because e.g. $\dfrac{d}{dx} \sin 3x = 3 \cos 3x$.

4) If e.g. $q(x) = x^2e^{-x}$, guess $y_0(x) = (k_2x^2+k_1x+k_0)e^{-x}$, just like in 1) and 2) above. The constants are found by insertion into the differential equation-

5) Add all the solutions we have found already. Here we exploit that the differential equation is linear.

Exceptional case. If $q(x)$ is a solution of the homogeneous equation (and of one of the types 1)–4) above), then the method fails, because a guess of this type will always give 0.

The *trick* is here first to guess the structure as above *and then add a factor* x. If this guess also is a solution of the homogeneous equation we add another factor x, etc.. Since the equation is of finite order, we shall only modify in this way a finite number of times.

Example 5.1 Find the complete solution of

$$\frac{d^3y}{dx^3} - 4\,\frac{d^2y}{dx^2} + 5\,\frac{dy}{dx} - 2y = 2e^x - 2x.$$

The characteristic polynomial

$$R^3 - 4R^2 + 5R - 2 = (R^2 - 2R + 1)(R - 2) = (R - 1)^2(R - 2)$$

has the roots $R_1 = R_2 = 1$ and $R_3 = 2$.
the complete solution of the *homogeneous* equation is

$$c_1e^x + c_2x\,e^x + c_3e^{2x}.$$

The right hand side $q(x) = 2e^x - 2x$ is of type 5), so we split into

a) $q_1(x) = -2x$, and b) $q_2(x) = 2e^x$.

a) Since $q_1(x) = -2x$ is *not* a solution of the homogeneous equation we guess $y_1(x) = ax + b$. By insertion into the equation we get

$$\frac{d^3y_1}{dx^3} - 4\,\frac{d^2y_1}{dx^2} + 5\,\frac{dy_1}{dx} - 2y_1 = 5a - (2ax + b) = -2ax + 5a - b.$$

This expression is equal to $q_1(x) = -2x$ when $a = 1$ and $b = 5$, thus

$$y_1(x) = x + 5.$$

b) Since $q_2(x) = 2e^x$ *is* a solution of the homogeneous equation, we first guess $c\,x\,e^x$. However, this is also a solution of the homogeneous equation, so in our *second guess* we choose $y_2(x) = c\,x^2 e^x$. It follows from

$$y_2(x) = x\,x^2 e^x, \qquad \frac{dy_2}{dx} = c(x^2+2x)e^x,$$

$$\frac{d^2y_2}{dx^2} = c(x^2+4x+2)e^x, \quad \frac{d^3y_2}{dx^3} = c(x^2+6x+6)e^x,$$

that we get by insertion

$$\frac{d^3y_2}{dx^3} - 4\,\frac{d^2y_2}{dx^2} + 5\,\frac{dy_2}{dx} - 2\,y_2$$

$$= ce^x\left\{(x^2+6x+6) - 4(x^2+4x+2) + 5(x^2+2x) - 2x^2\right\} = -2c\,e^x.$$

This expression is equal to $q_2(x) = 2e^x$ for $c = -1$, thus $y_2(x) = -x^2 e^x$.

c) We have now found a particular solution given by

$$y_0(x) = y_1(x) + y_2(x) = x + 5 - x^2 e^x.$$

d) The complete solution is then

$$y = x + 5 - x^2 e^x + c_1 e^x + c_2 x e^x + c_3 e^{2x}, \qquad x \in \mathbb{R},$$

where c_1 and c_2 and c_3 are arbitrary constants. $\Diamond$

5.3 The initial value problem

Given the linear differential equation of order n

$$\frac{d^n y}{dx^n} + a_{n-1}\,\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\,\frac{dy}{dx} + a_0\,y = q(x), \qquad x \in I.$$

Find the particular solution $y = \varphi(x)$, $x \in I$, which contains the line element $(x_0; y_0, y_1, \ldots, y_{n-1})$. This means that we demand for $x = x_0$ that

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, \quad \ldots, \quad \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}.$$

Since we specify y, $\dfrac{dy}{dx}$, $\ldots$, $\dfrac{d^{n-1}y}{dx^{n-1}}$ to the *initial "time"* x_0 we call this problem an *initial value problem.* It is solved by first finding the *complete solution* as described in the first two sections above.

Then we calculate $y(x)$, $\dfrac{dy}{dx}$, $\ldots$, $\dfrac{d^{n-1}y}{dx^{n-1}}$ and insert $x = x_0$. This gives us n linear equations in the n unknown constants c_1, c_2, $\ldots$, c_n from the complete solution. It can be proved that if the equation has constant coefficients, then the solution is unique.

The constants c_1, c_2, $\ldots$, c_n are found by methods from Linear Algebra.

Therefore, initial value problems will hardly give one difficulties.

5.4 The boundary value problem

Contrary to the initial value problem it is much more difficult to understand the meaning by the boundary value problem, and why it is so interesting. Fortunately, *Krenk: Mechanics and Analysis of Beams, Columns and Cables*, p. 160 sqq., gives an (ideal) example of applications, which I shall briefly sketch here.

Consider an ideal column exposed to a variable load $P = -N > 0$. The column is then compressed a little, which usually is of no importance. For small loads P the column does not deviate much from the vertical. However, for some critical load P_E, called the *Euler load*, the column loses its stability and it will bend from the vertical.

The task is not to describe and solve the problem of determining the critical load P_E.

Assume that the column has the length ℓ and that its bending module EI is *constant*. If we put

$$(18)\quad \lambda = k^2 = \frac{P}{EI},$$

where the load P is allowed to vary, the differential equation of the *bending* is in the normalized form given by

$$(19)\quad \frac{d^4 w}{dx^4} + k^2\,\frac{d^2 w}{dx^2} = 0 \qquad \left(\text{evt. } \frac{d^4 w}{dx^4} + \lambda\,\frac{d^2 w}{dx^2} = 0\right)$$

where

$$(20)\quad \frac{M(x)}{EI} = -\frac{d^2 w}{dx^2}.$$

The reason why $\dfrac{P}{EI}$ in (18) is put equal to both λ and k^2 is that one has different traditions in the two disciplines. In mathematics one will here typically write λ.

From a mathematical, though not an engineering point of view, it is "new" that one has given conditions in *both* end points of the interval $[0,\ell]$, namely

$$w(0) = w(\ell) = 0 \quad \text{og} \quad M(0) = M(\ell) = 0.$$

The model can now be formulated equivalently in the two "languages":

$$(21)\quad \begin{cases} \text{Mechanics} \\ \dfrac{d^4 w}{dx^4} + k^2\,\dfrac{d^2 w}{dx^2} = 0, \\ w(0) = w(\ell) = 0, \\ M(0) = M(\ell) = 0, \end{cases} \qquad \begin{cases} \text{Mathematics} \\ \dfrac{d^4 w}{dx^4} + \lambda\,\dfrac{d^2 w}{dx^2} = 0, \\ w(0) = w(\ell) = 0, \\ w''(0) = w''(\ell) = 0. \end{cases}$$

the task is to find the critical load P_E, i.e. in reality k^2 and λ in (21), such that this equation has *non-trivial* solutions.

In Mechanics these non-trivial solutions to (21) are called the *buckling modes* $w_n(x)$ with the corresponding *buckling loads* $P_n = k_n^2\,EI$.

In Mathematics these non-trivial solutions of (21) are called the *eigenfunctions* $w_n(x)$ with the corresponding *eigenvalues* λ_n.

Since the conditions on the solutions are not specified in a single point as it is by the initial value problem, but instead in the *boundary points* of the interval $[0, \ell]$, we call this problem for a *boundary value problem*.

Remark 5.3 *(Important!)* Apart from the fact that an *initial value problem* is specified in *1 point*, while a *boundary value problem* is specified in *2 points*, there is also a fundamental difference in what we want by the two problems:

1) Considering an *initial value problem* we want to specify an *unique solution*. Here $x = t$ is typically the time, and y is the coordinate of some particle to time t. This coordinate ought to be unique for a given line element, if the model should be useful.

2) Considering a *boundary value problem* we want on the contrary to find out when the solution is *not* unique ("when something breaks down"), i.e. the opposite of the task in the initial value problem. When this is the purpose, we also call the boundary value problem an *eigenvalue problem*. $\Diamond$

It should be obvious that it is important for a civil engineer to be able to estimate the load where some given structure breaks down. When the maximal load is known we know something about the dimensioning of the structure. As shown above this problem can be formulated as a boundary value problem, and this should be sufficient for considering such problems. There is, however, much more in the eigenvalue problem than one would guess at this stage.

In more advanced problems, e.g. when the stiffness EI is *not* a constant, it is roughly speaking possible to construct the solutions by means of the *eigenfunctions* of

$$(22)\quad \begin{cases} \dfrac{d^4w}{dx^4} + \dfrac{\mu}{E(x)I(x)}\,\dfrac{d^2w}{dx^2} = 0, \\[2ex] w(0) = w(\ell) = 0, \\[2ex] w''(0) = w''(\ell) = 0. \end{cases}$$

One determines (e.g. by means of computer programs like MAPLE, since the task is too big for the old fashioned use of tables, and even advanced pocket calculators will meet their shortcomings here) a sequence of *eigenfunctions* $w_n(x) = \varphi_n(x)$ with corresponding *eigenvalues* μ_n. These can be used to describe the full technical model.

Here we need a small comment. It can be proved that the eigenfunctions uniquely determine a so-called *weight function* $k(x)$, such that the eigenfunctions $\{\varphi_n(x) \mid n \in \mathbb{N}\}$ become *orthonormal* with respect to the "strange" *inner product*,

$$(23)\quad \langle f, g\rangle = \int_0^\ell f(x)g(x)\cdot k(x)\,dx, \qquad \langle \varphi_n, \varphi_n\rangle = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases}$$

Usually $k(x)$ contains an ugly constant, which one chooses to omit. The price is that $\{\varphi_n(x) \mid n \in \mathbb{N}\}$ then only become *orthogonal*; apart from this the calculations become easier to perform.

We have actually already seen all this before, namely by the expansion of a function in a *sinus series* over $[0, \ell]$, in which case $k(x) \equiv 1$. The expansion of a function after $\{\varphi_n(x) \mid n \in \mathbb{N}\}$ by the inner product (23) can therefore be regarded as a *theory of generalized Fourier series.* Many linear models can in fact with advantage alternatively be described by its system of corresponding eigenfunctions, even in the case of differential equations with variable coefficients.

One has known this theory during the last century, but so far only the constant case (where we get the Fourier series) can be calculated explicitly. First by the appearance of programs like Mathematica and MAPLE it has become possible to calculate also more general cases. This means that these computer programs have opened up for a lot of new possibilities of applications.

Since these notes are only meant for *elementary* Calculus the student will at this stage for the time being never meet eigenvalue problems where the eigenfunctions can *only* be determined by programs like MAPLE.

The formal definition of a linear boundary value problem is:

Definition 5.1 Boundary value problem. *Given a linear differential equation of order n,*

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = q(x), \qquad x \in [a,b] \subset \mathbb{R}.$$

Find a solution $y = \varphi(x)$, $x \in [a,b]$, which satisfies the m linear boundary conditions

$$r_{1,1}\varphi(a) + \cdots + r_{1,n}\varphi^{(n-1)}(a) + r_{1,n+1}\varphi(b) + \cdots + r_{1,2n}\varphi^{(n-1)}(b) = s_1,$$

$$\dots$$

$$r_{m,1}\varphi(a) + \cdots + r_{m,n}\varphi^{(n-1)}(a) + r_{m,n+1} + \cdots + r_{m.,2n}\varphi^{(n-1)}(b) = s_m.$$

Remark 5.4 This of course a formal and complicated definition, which should be reformulated as soon as possible to matrices, if we shall keep our general view. Unfortunately one is then easily diverted from the real purpose of the problem, namely the applications in the engineering disciplines. $\Diamond$

Remark 5.5 In the practical applications we have $m = n$. the most commonly met applications are given for $m = n = 2$. This is the reason why one can find a lot of literature of this case, where the boundary value problem has been reduced to

$$(24)\qquad \begin{cases} \dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_0\,y = q(x), \qquad x \in [a,b], \\[2mm] r_{1,1}Y(a) + r_{1,2}y'(a) + r_{1,3}y(b) + r_{1,4}y'(b) = s_1, \\[2mm] r_{2,1}y(a) + r_{2,2}y'(a) + r_{2,3}y(b) + r_{2,4}y'(b) = s_2. \end{cases}$$

Even in this reduced case there will usually only be two of the $r_{i,j}$s in each row of the boundary conditions which are $\neq 0$.

It has been noticed that concerning beams it is more natural to consider $m = n = 4$, cf. *Krenk*. There is no need explicitly to write down the general equations in this case. They are of course analogous to (24). $\Diamond$

Remark 5.6 If we put

$$r_{1,2} = r_{1,3} = r_{1,4} = 0 \quad \text{and} \quad r_{2,1} = r_{2,3} = r_{2,4} = 0$$

and $r_{1,1} = 1$ and $r_{2,2} = 1$ in (24), we get

$$\begin{cases} \dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_0\,y = q(x), \qquad x \in [a,b], \\[2mm] y(a) = s_1 \quad \text{og} \quad y'(a) = s_2, \end{cases}$$

which from a formal point of view is also an *initial value problem.* This is quite confusing! Formally the definition is OK; but in practice one should always demand that both boundary points a and b should enter the boundary conditions at least once, when we consider *boundary value problem.* $\Diamond$

Remark 5.7 By practical applications we are not at all interested in boundary value problems; rather in the closely connected *eigenvalue problems.* For that reason we continue with the eigenvalue problems in the next section without giving further examples of the boundary value problem. $\Diamond$

5.5 The eigenvalue problem

The most common form of the eigenvalue problem is derived from the boundary value problem.

Definition 5.2 Eigenvalue problem I. *Consider the following boundary value problem in* $[a,b]$,

$$\left\{\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0\,y\right\} + \lambda\left\{\frac{d^p y}{dx^p} + b_{p-1}\frac{d^{p-1}y}{dx^{p-1}} + \cdots + b_0\,y\right\} = 0$$

with m *linear homogeneous* boundary conditions, *i.e.*

$$r_{i,1}y(a) + \cdots + \tilde{r}_{i,1}y(b) + \cdots = 0 \qquad \textit{for } i = 1,\ldots,m.$$

Find the values of the parameter λ, *for which this problem has* non-trivial *solutions*

$$y_\lambda(x).$$

The given solutions λ *are called* eigenvalues *with the corresponding* eigenfunctions $y_\lambda(x)$.

Remark 5.8 The differences from the general boundary value problem are

1) We put $q(x) \equiv 0$ and $s_1 = s_2 = \cdots = s_m = 0$.

2) It is the *parameter* λ, which first is the unknown. In the solution we get additional the corresponding *eigenfunctions* $u_\lambda(x) \neq 0$, which often in the applications can be given a physical interpretation.

3) The *eigenspace* $\{c\,u_\lambda(x) \mid c \in \mathbb{R}\}$ is usually of dimension 1 with a conveniently chosen eigenfunction $u_\lambda(x)$ as its basis vector. One is often careless about this and write instead that $u_\lambda(x)$ is the eigenfunction, where we tacitly mean that $c \cdot u_\lambda(x)$ for $c \neq 0$ is also a eigenfunction.

   Occasionally we can have eigenspaces of a higher dimension. I have in another application seen eigenspaces of dimension 2, i.e. all eigenfunctions belonging to some eigenvalue λ are of the form

   $$c_1 u_{1,\lambda}(x) + c_2 u_{2,\lambda}(x), \qquad c_1, c_2 \in \mathbb{R},$$

   where $u_{1,\lambda}(x)$ and $u_{2,\lambda}(x)$ are linear independent. Such eigenspaces are difficult to handle, so they are also avoided in the elementary courses. One cannot exclude eigenspaces of dimension ≥ 3, but I have never seen any in the applications. $\Diamond$

Remark 5.9 The formulation of the *eigenvalue problem I* assumes that the interval $[a,b]$ is fixed. This definition is too narrow, and it is in fact easy to find an eigenvalue problem, which is *not* of this type, e.g. when the interval $[0,\lambda]$ *depends on the eigenvalue* λ. I call such problems rather vaguely *eigenvalue problems of type II.* They are difficult to define mathematically though they intuitively are obvious eigenvalue problems if we stick to that the *task is to determine the values of the parameter* λ, *for which the boundary value problem has non-trivial solutions.* $\Diamond$

5.6 Examples

In this section we go through some examples of *eigenvalue problems*, mainly based on examples from textbooks. The general procedure is

1) First determine the complete solution of the differential equation, typically based on the first sections of this chapter, as well as more advanced theorems, when the coefficients are variable.

2) Insert the boundary conditions and reduce.

3) Determine the values of the parameter λ, for which at least one of the arbitrary constants in the complete solution can be chosen freely. Then these λ are the *eigenvalues*.

4) Insert the eigenvalues λ from above in order to identify the *eigenfunctions*.

Example 5.2 (The original eigenvalue problem). Consider the eigenvalue problem

$$(25)\quad \begin{cases} \dfrac{d^2y}{dx^2} + \lambda\, y = 0, \qquad x \in [0,\ell], \\ \\ y(0) = 0 \quad \text{og} \quad y(\ell) = 0. \end{cases}$$

For every fixed λ this is a *boundary value problem* of order $n = 2$ and with $m = 2$ boundary conditions.

Notice that one often in textbooks writes $y(0) = y(\ell) = 0$ for short. Thus we have latently *two* boundary conditions, namely $y(0) = 0$ and $y(\ell) = 0$. We shall here avoid the short version because one may lose some information by using it.

1) First find the complete solution of the differential equation

$$\frac{d^2y}{dx^2} + \lambda\, y = 0, \qquad x \in [0,\ell],$$

without the boundary conditions.

the equation has constant coefficients. We therefore set up the characteristic equation

$$R^2 + \lambda = 0, \qquad \text{i.e. } R^2 = -\lambda.$$

Then we get the usual mess of (here) three different cases:

a) If $\lambda < 0$, then we can write $\lambda = -k^2$ for some $k > 0$. The characteristic equation $R^2 = -\lambda = +k^2$ has the two *real* roots $\pm k$, so the complete solution is

$$y = \tilde{c}_1 e^{kx} + \tilde{c}_2 e^{-kx}, \qquad \tilde{c}_1, \tilde{c}_2 \in \mathbb{R} \text{ arbitrary.}$$

Since

$$e^{kx} = \cosh(kx) + \sinh(kx) \text{ og } e^{-kx} = \cosh(kx) - \sinh(kx),$$

the complete solution is more conveniently written in the form

$$y = c_1 \cosh(kx) + c_2 \sinh(kx), \qquad c_1, c_2 \text{ arbitrary.}$$

Be aware of this trick, which always should be applied when one of the boundary points is 0. Notice also the analogy with the trigonometric case below.

b) If $\lambda = 0$, then the characteristic equation $R^2 = 0$ has the root $R = 0$ of *multiplicity 2*. The complete solution is

$$y = c_1 + c_2 x, \qquad c_1, c_2 \text{ arbitrary.}$$

c) If $\lambda > 0$, we write $\lambda = k^2$ for some $k > 0$. The characteristic equation $R^2 = -\lambda = -k^2$ has the two *complex conjugated* roots $\pm i\,k$. Thus the complete solution is

$$y = c_1 \cos(kx) + c_2 \sin(kx), \qquad c_1, c_2 \text{ arbitrary.}$$

Notice the similarity between the structure in c) and in a). In the former case we use trigonometric functions while in the latter case the hyperbolic functions. Notice also that b) (the "parabolic case") always is messy.

We have now found the complete solution in all three cases. We now proceed to the next step.

2) We shall use the boundary conditions $y(0) = 0$ and $y(\ell) = 0$ on every of the three cases of the complete solutions.

a) When $\lambda = -k^2 < 0$, we have the complete solution

(26) $y(x) = c_1 \cosh(kx) + c_2 \sinh(kx).$

By insertion of $x = 0$ we get

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1 = 0.$$

Hereby (26) is reduced to the possibility

$$y(x) = c_2 \sinh(kx).$$

Here we put $x = \ell$ and we get

$$(27)\quad y(\ell) = c_2 \sinh(k\ell) = 0.$$

Since $\sinh(k\ell) > 0$, we must have $c_2 = 0$. Therefore the only solution is obtained for $c_1 = 0$ and $c_2 = 0$, i.e. according to (26),

$$y(x) \equiv 0, \qquad \text{when } \lambda = -k^2 < 0.$$

Since we only get the zero solution, we conclude that *no* $\lambda < 0$ is an eigenvalue.

b) When $\lambda = 0$ we get the complete solution

$$y(x) = c_1 + c_2 x.$$

By putting $x = 0$ we get

$$y(0) = c_1 + c_2 \cdot 0 = c_1 = 0,$$

and the set of candidates is reduced to

$$y(x) = c_2 x.$$

Inserting $x = \ell$ we get

$$y(\ell) = c_2 \cdot \ell = 0, \qquad \text{i.e. } c_2 = 0, \text{ because } \ell > 0.$$

Since both $c_1 = 0$ and $c_2 = 0$, we only obtain the *zero solution*, proving that $\lambda = 0$ is *not an eigenvalue.*

c) When $\lambda = k^2 > 0$ the complete solution is

$$y(x) = c_1 \cos kx + c_2 \sin kx.$$

Inserting $x = 0$ we get

$$y(0) = c_1 + c_2 \cdot 0 = c_1 = 0.$$

The candidates are then reduced to

$$(28) \quad y(x) = c_2 \sin(kx).$$

Inserting $x = \ell$ we get

$$(29) \quad y(\ell) = c_2 \sin(k\ell) = 0.$$

3) We have only reached the present step in case c) above. We want non-trivial solutions of the type (28), i.e. we want that $c_2 \neq 0$. This can according to (29) only be achieved when

$$\sin(k\ell) = 0, \qquad \text{i.e. when } k_n \ell = n\pi > 0, \quad n \in \mathbb{N}.$$

This gives us the possibilities:

$$k_n = \frac{n\pi}{\ell}, \qquad n \in \mathbb{N},$$

and it is seen that each of these in fact produces a solution.

The *eigenvalues* are then

$$\lambda_n = k_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \qquad n \in \mathbb{N}.$$

4) According to (28) [with $c_2 = 1$] an *eigenfunction* $y_n(x)$ corresponding to the eigenvalue $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n \in \mathbb{N}$, is given by

$$y_n(x) = \sin(k_n x) = \sin\left(\frac{n\pi x}{\ell}\right), \qquad n \in \mathbb{N}.$$

The complete set of *eigenfunctions* corresponding to $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $n \in \mathbb{N}$, is given by

$$c \cdot \sin\left(\frac{n\pi x}{\ell}\right), \qquad x \in [0, \ell], \quad c \text{ arbitrary}.$$

Note that all the eigenfunctions $\left\{ \sin\left(\frac{n\pi x}{\ell}\right) \;\middle|\; n \in \mathbb{N} \right\}$ form a basis for a *sinus expansion* of a function over $[0, \ell]$. $\Diamond$

Example 1.5 is the model of all eigenvalue problems. For that reason it has been described fairly thoroughly.

Example 5.3 Consider the eigenvalue problem (21), which was used here as a motivation, se also *Krenk, p. 160 sqq.*. In the mathematical formulation we have according to (21) the eigenvalue problem

$$\begin{cases} \dfrac{d^4w}{dx^4} + \lambda \dfrac{d^2w}{dx^2} = 0, \qquad x \in [0, \ell], \\ \\ w(0) = 0, \quad w(\ell) = 0, \quad w''(0) = 0, \qquad w''(\ell) = 0, \end{cases}$$

i.e. the order is $n = 4$ and we have $m = 4$ boundary conditions.

1) Since the equation has constant coefficients, we consider the characteristic polynomial

$$R^4 + \lambda R^2 = R^2(R^2 + \lambda).$$

Formally we again must consider three cases:

a) When $\lambda = -k^2 < 0$, the complete solution is

$$y(x) = c_1 + c_2 x + c_3 \cosh(kx) + c_4 \sinh(kx).$$

b) When $\lambda = 0$, we see that $R = 0$ is a root of multiplicity 4, hence the complete solution is

$$y(x) = c_1 + c_2 x + c_3 x 2 + c_4 x^3.$$

c) When $\lambda = k^2 > 0$, the complete solution is

$$y(x) = c_1 + c_2 x + c_3 \cos(kx) + c_4 \sin(kx).$$

Here we can reuse a lot from the previous example.

2) By a long and tedious analysis it is shown like in the example above that no $\lambda \leq 0$ can ever be an eigenvalue of the problem. The proof is left to the reader.

We are again left with c), i.e. $\lambda = k^2 > 0$, and the *candidates* of the eigenfunctions are

(30) $y(x) = c_1 + c_2 x + c_3 \cos(kx) + c_4 \sin(kx)$

with

(31) $y''(x) = -k^2\{c_3 \cos(kx) + c_4 \sin(kx)\}$.

By insertion of $x = 0$ into (30) we get from the boundary condition that

$$y(0) = c_1 + c_3 = 0.$$

By insertion of $x = 0$ in (31) we get from the boundary condition

$$y''(0) = -k^2 c_3 = 0.$$

Since $k > 0$, the last equation gives $c_3 = 0$, which put into the first one immediately gives $c_1 = 0$.

We have hereby narrowed the set of candidates to

(32) $y(x) = c_2 x + c_4 \sin(kx)$

with

(33) $y''(x) = -k^2 c_4 \sin(kx)$.

When we put $x = \ell$ in (32), it follows from the boundary condition that

(34) $y(\ell) = c_2 \ell + c_4 \sin(k\ell) = 0$.

When we put $x = \ell$ in (33), it follows from the boundary condition that

$$y''(\ell) = -k^2 c_4 \sin(k\ell) = 0,$$

i.e.

(35) $c_4 \sin(k\ell) = 0$.

Finally, when (35) is inserted into (34), we get $c_2 \ell = 0$, from which $c_2 = 0$.

We have now reduced the set of candidates to

(36) $y(x) = c_4 \sin(kx)$.

Since we want *non-trivial solutions*, we must have $c_4 \neq 0$. According to (35) this gives us the condition

$$\sin(k\ell) = 0, \qquad \text{i.e. } k_n \ell = n\pi > 0, \quad n \in \mathbb{N},$$

from which $k_n = \dfrac{n\pi}{\ell}$, $n \in \mathbb{N}$ (the same as in the modelling example).

3) The *eigenvalues* are

$$\lambda_n = k_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \qquad n \in \mathbb{N}.$$

4) The corresponding *eigenfunctions* are according to (36)

$$c \cdot y_n(x) = c \cdot \sin\left(\frac{n\pi x}{\ell}\right), \qquad n \in \mathbb{N}.$$

The physical interpretation of the eigenvalues is that

$$P_n = \lambda_n EI = k_n^2 EI = n^2 \left(\frac{\pi}{\ell}\right)^2 EI, \qquad n \in \mathbb{N},$$

are the critical loads or *(buckling loads)*. The smallest of these

$$P_1 = P_E = \left(\frac{\pi}{\ell}\right)^2 EI, \qquad n = 1,$$

is called the *Euler load*, for which the ideal column breaks down. $\Diamond$

That the eigenvalue problem in practice far from always gives "nice" high school solutions is seen in the following example.

Example 5.4 *Cf. Krenk, Example 3.3.* Consider an ideal column with fixed and simply supported end points. We shall find the Euler load. Let P be the variable load. We derive by the usual reductions (cf. *Krenk, p. 160, p. 164 and p. 165*) that the problem can be described by the following eigenvalue problem

$$(37)\quad \begin{cases} \dfrac{d^4 w}{dx^4} + k^2 \dfrac{d^2 w}{dx^2} = 0 \qquad \text{where } k^2 = \dfrac{P}{EI} \\[2mm] w(0) = 0, \qquad w'(0) = 0, \qquad w(\ell) = 0, \qquad w''(\ell) = 0, \end{cases}$$

i.e. of order $n = 4$ and with $m = 4$ boundary conditions.

We are considering a problem of compression, so we can for physical reasons immediately exclude $\lambda \le 0$ and only consider the case $\lambda = k^2 > 0$.

1) The equation is the same as before, so we reuse the complete solution of the differential equation from (37),

$$(38)\quad w(x) = c_1 + c_2 x + c_3 \cos kx + c_4 \sin kx.$$

2) By the insertion into the boundary conditions we shall also need

(39) $w'(x) = c_2 - kc_3 \sin kx + kc_4 \cos kx$

and

(40) $w''(x) = -k^2\{c_3 \cos kx + c_4 \sin kx\}$.

Putting $x = 0$ in (38) we get by the boundary condition

$$w(0) = c_1 + c_3 = 0, \qquad \text{i.e. } c_3 = -c_1.$$

Putting $x = 0$ in (39)we get by the boundary condition

$$w'(0) = c_2 + kc_4 = 0, \qquad \text{dvs. } c_2 = -kc_4.$$

These conditions narrow the set of candidates to

(41) $w(x) = c_1(1 - \cos kx) + c_4(-kx + \sin kx)$

with [cf. (40)]

(42) $w''(x) = -k^2\{-c_1 \cos kx + c_4 \sin kx\}$.

Putting $x = \ell$ in (41) we get by the boundary condition

(43) $w(\ell) = c_1(1 - \cos k\ell) + c_4(-k\ell + \sin k\ell) = 0$.

Putting $x = \ell$ in (42) we get by the boundary condition

$$w''(\ell) = -k^2\{-c_1 \cos k\ell + c_4 \sin k\ell\} = 0,$$

i.e. since $k^2 \neq 0$,

(44) $-c_1 \cos k\ell + c_4 \sin k\ell = 0$.

And then another small *trick:* When we subtract (44) from (43) we get the simpler equation

(45) $c_1 - c_4 k\ell = 0, \qquad \text{dvs. } c_1 = c_4 k\ell,$

and the set of candidates has now been restricted to

(46) $w(x) = c_4\{k\ell(1 - \cos kx) + (-kx + \sin kx)\}$.

We only get non-trivial solutions by choosing $c_4 \neq 0$ in (46), and it is actually possible by using this information to fiddle with the solutions by repeating the investigation of (46), when e.g. $c_4 = 1$. We shall not, however, do this here, but instead use the *standard procedure.*

The problem has now been reduced to finding solutions $(c_1, c_2) \neq (0, 0)$ to (44) and (45). Let us set up in the usual manner of Linear Algebra *where c_1 and c_4 are the unknowns:*

$$\begin{cases} -\cos(k\ell) \cdot c_1 & + & \sin(k\ell) \cdot c_4 & = & 0, \\ 1 \cdot c_1 & - & k\ell \cdot c_4 & = & 0. \end{cases}$$

This system does only have solutions $\neq (0,0)$ *when the corresponding determinant is 0*, i.e.dvs.

$$0 = \begin{vmatrix} -\cos(k\ell) & \sin(k\ell) \\ 1 & -k\ell \end{vmatrix} = k\ell \cdot \cos(k\ell) - \sin(k\ell).$$

If $\cos(k\ell) = 0$, then $\sin(k\ell) = \pm 1$, and the equation is not fulfilled. Therefore, any solution must satisfy the *transcendent equation*

(47) $\tan(k\ell) = k\ell.$

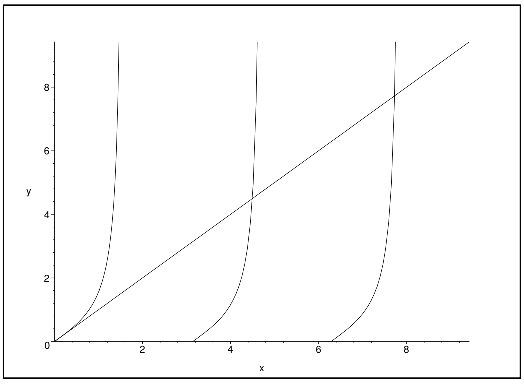


Figure 1: Graphical solution of the equation $\tan(k\ell) = k\ell$.

By considering the graph we see that this equation has infinitely many positive solutions

$$k_n\ell \in \left] n\pi, \left(n+\frac{1}{2}\right)\pi \right[, \qquad n \in \mathbb{N},$$

and that they tend to $\left(n+\frac{1}{2}\right)\pi$ from below when $n \to \infty$, i.e.

$$0 < \left(n+\frac{1}{2}\right)\pi - k_n\ell \to 0 \qquad \text{for } n \to \infty.$$

The values shall be found by *iteration*, cf. e.g. *Krenk, p. 166*, or try yourself to set up a *Newton-Raphson procedure* for the problem.

The eigenfunctions become very complicated, cf. (46). $\Diamond$

We have now shown the most important modelling examples of eigenvalue problems. Whenever one comes across another eigenvalue problem, these should be the first ones to copy. For completeness we shall here continue with a couple of more advanced eigenvalue problems.

Example 5.5 Considering eigenvalue problems of type II one can also allow the eigenvalue to enter the boundary conditions, like in e.g.

$$\begin{cases} \dfrac{d^2y}{dx^2} + \lambda\, y = 0, \qquad x \in [0,1], \\ \\ y(0) = 0, \\ \\ y(1) - f(\lambda) y'(1) = 0, \end{cases}$$

where $f(\lambda)$ is a given function in the eigenvalue λ. Since the purpose here is only to show the principles, we shall restrain ourselves to the determination of the *positive* eigenvalues $\lambda > 0$.

1) Since $\lambda > 0$, the complete solution is

$$y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

where

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\, x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\, x).$$

2) Putting $x = 0$ we get from the boundary condition that

$$y(0) = c_1 = 0,$$

thus the candidates can only be of the form

$$y(x) = c_2 \sin(\sqrt{\lambda}\, x).$$

Since we are going for *non-trivial* solutions, we must have $c \neq 0$. Due to the linearity it is sufficient (for $c_2 = 1$) to consider

(48) $y(x) = \sin(\sqrt{\lambda}\, x) \qquad \text{where } y'(x) = \sqrt{\lambda} \cdot \cos(\sqrt{\lambda}\, x)$

and find the λ, for which (48) is a solution.

When we put $x = 1$ we get from the latter boundary condition that

$$y(1) = f(\lambda) y'(1) = \sin(\sqrt{\lambda}) - f(\lambda)\sqrt{\lambda}\,\cos(\sqrt{\lambda}) = 0.$$

Since $\cos(\sqrt{\lambda}) = 0$ implies that $\sin(\sqrt{\lambda}) \neq 0$, we must have $\cos(\sqrt{\lambda}) \neq 0$ for any solution of this equation. Hence we get the equivalent transcendental equation

(49) $f(\lambda) = \dfrac{\tan(\sqrt{\lambda})}{\sqrt{\lambda}}, \qquad \lambda > 0, \quad [\text{possibly } \sqrt{\lambda}\, f(\lambda) = \tan(\sqrt{\lambda})]$

3) In order to solve (49) we first draw the graphs of the two functions $\sqrt{\lambda} f(\lambda)$ and $\tan(\sqrt{\lambda})$ to see where we can expect the eigenvalues. Once we know approximately where the eigenvalues are situated we proceed with *iteration* in order to improve the approximate values. First try *Newton-Raphson iteration*, and if it fails, apply the *fix point theorem.*

From an application point of view the example is somewhat laboured, so there is absolutely no reason to insert specific functions $f(x)$. The method has already been demonstrated in *Krenk, Example 3.3.* ◊

Example 5.6 A more difficult example is concerned with the modelling of the bending of a vertical one-sided constrained thin column of the length λ under the impact of its own weight. This is actually an old and misunderstood problem from an examination which never should have been given at this stage. The problem is relevant, but it is far too difficult. The corresponding eigenvalue problem is of *type II*,

$$\begin{cases} \dfrac{d^4y}{dx^4} + (\lambda - x)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0, \qquad x \in [0\lambda], \\ \\ y(0) = 0, \quad y'(0) = 0, \quad y''(\lambda) = 0, \quad y'''(\lambda) = 0, \end{cases}$$

because the eigenvalue λ also enters the definition of the interval. Hence, the problem is of the same structure as an eigenvalue problem, and yet it is not of the type, which is described by the usual definition.

The poor students did not have the right mathematical tools for solving this problem, so they could only – even with a lot of help – get the estimate

$$\lambda \approx \sqrt[3]{2} = 1,25992$$

of the first (and critical) eigenvalue. I have later calculated this value and found

$$\lambda \approx 1,98635,$$

which is far from the first estimate stated at the examination. $\Diamond$

Example 5.7 The third example is also from textbook which retrieved this example from a scientific paper. This is actually an example of *"Murphy's Law: What can go wrong, must go very wrong!"* First the textbook does not quote the equation correctly. Then the physical dimensions of the equations do not fit at all. And finally, the problem was not solved. (It was not either in the paper.) Such an example should never be included in a textbook of elementary Calculus.

In order to help possible readers of the textbook mentioned above I shall here briefly add the following: The correct eigenvalue problem is (after some normalization such that all the physical dimensions disappear)

$$\frac{d^4y}{dx^4} + \lambda\left[\left\{\alpha(1-x) - \frac{1}{2}(1-x^2)\right\}\frac{d^2y}{dx^2} - (\alpha - x)\frac{dy}{dx} - y\right] = 0$$

with the boundary conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

and $\alpha = \dfrac{R}{\ell}$, where R is the radius of the circle and ℓ is the length of the rod.

For an unspecified α one can only set up an iteration process, which converges all right, though is not the easiest one to perform, not even for a specified α. I used 14 pages in my draft for the iteration! The conclusion was that the iteration cannot be performed, unless α is given as a fixed number, i.e. we cannot derive a general result, but we have to start from the very beginning for each new value of α! Such an example cannot be used as a motivation in an elementary textbook. It will only confuse the students. $\Diamond$

A Formulæ

Some of the following formulæ can be assumed to be known from high school. Others are introduced in Calculus 1a. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

A.1 Squares etc.

The following simple formulæ occurs very frequently in the most different situations.

$$
\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}
$$

A.2 Powers etc.

Logarithm:

$$
\begin{array}{rll}
\ln|xy| = & \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| = & \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| = & r\ln|x|, & x\neq 0.
\end{array}
$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \quad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\ y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \quad \text{(extensions for some } x,\ y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \quad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark A.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

A.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

A.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \quad \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}(a^x) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ og } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\text{ Arcsin } x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1, 1\,[,$$

$$\frac{d}{dx}\text{ Arccos } x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \,]-1, 1\,[,$$

$$\frac{d}{dx}\text{ Arctan } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arccot } x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\text{ Arsinh } x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\text{ Arcosh } x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \,]\,1, +\infty\,[,$$

$$\frac{d}{dx}\text{ Artanh } x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\text{ Arcoth } x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark A.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

A.5 Integration

The most obvious rules are about linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and about that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark A.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark A.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. See also Chapter 4. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x))\cdot\varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \text{“}\int f(\varphi(x))\,d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\,dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark A.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ og $\varphi'(y) = 2y$. $\Diamond$

A.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1,} \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x\,dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x\,dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ og } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x\,dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x\,dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x\,dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x\,dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x}\,dx = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x\,dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x\,dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x\,dx = \ln\cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x\,dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x}\,dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x}\,dx = 2\operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x}\,dx = \frac{1}{2}\ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

A.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a piont P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

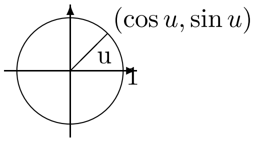


Figure 2: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\, u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\, u)$ and $\exp(-\mathrm{i}\, u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\, u) + \exp(-\mathrm{i}\, u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\, u) - \exp(-\mathrm{i}\, u)),$$

.

Moivre's formula: By expressing $\exp(inu)$ in two different ways we get:

$$\exp(inu) = \cos nu + \text{ i } \sin nu = (\cos u + \text{i } \sin u)^n.$$

Example A.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical appliction,

$$\begin{aligned}
&\cos(3u) + \text{ i } \sin(3u) = (\cos u + \text{ i } \sin u)^3 \\
&\qquad = \cos^3 u + 3\text{i } \cos^2 u \cdot \sin u + 3\text{i}^2 \cos u \cdot \sin^2 u + \text{ i}^3 \sin^3 u \\
&\qquad = \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \text{ i}\{3\cos^2 u \cdot \sin u - \sin^3 u\} \\
&\qquad = \{4\cos^3 u - 3\cos u\} + \text{ i}\{3\sin u - 4\sin^3 u\}
\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \diamondsuit$$

Addition formulæ:

$$\sin(u + v) = \sin u \, \cos v + \cos u \, \sin v,$$

$$\sin(u - v) = \sin u \, \cos v - \cos u \, \sin v,$$

$$\cos(u + v) = \cos u \, \cos v - \sin u \, \sin v,$$

$$\cos(u - v) = \cos u \, \cos v + \sin u \, \sin v.$$

Products of trigonometric functions to a sum:

$$\sin u \, \cos v = \frac{1}{2}\sin(u + v) + \frac{1}{2}\sin(u - v),$$

$$\cos u \, \sin v = \frac{1}{2}\sin(u + v) - \frac{1}{2}\sin(u - v),$$

$$\sin u \, \sin v = \frac{1}{2}\cos(u - v) - \frac{1}{2}\cos(u + v),$$

$$\cos u \, \cos v = \frac{1}{2}\cos(u - v) + \frac{1}{2}\cos(u + v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u \, \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \, \cos^2 u - 1 = 1 - 2 \, \sin^2 u,$$

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

A.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

A.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x.$$

A.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $=1$ for the standard series) and*exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

A.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,
2) power functions,
3) exponential functions,
4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \to 0 \qquad \text{for } x \to \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^\alpha}{a^x} \to 0 \qquad \text{for } x \to \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^\alpha \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$