Random variables II

Probability Examples c-3

Leif Mejlbro

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Contents

	Introduction	**5**
1	**Some theoretical results**	**6**
2	**Law of total probability**	**20**
3	**Correlation coefficient and skewness**	**23**
4	**Examples concerning the Poisson distribution**	**60**
5	**Miscellaneous examples**	**70**
	Index	**116**

Introduction

This is the third book of examples from the *Theory of Probability.* This topic is not my favourite, however, thanks to my former colleague, Ole Jørsboe, I somehow managed to get an idea of what it is all about. The way I have treated the topic will often diverge from the more professional treatment. On the other hand, it will probably also be closer to the way of thinking which is more common among many readers, because I also had to start from scratch.

The topic itself, *Random Variables*, is so big that I have felt it necessary to divide it into three books, of which this is the second one. We shall here continue the study of frequencies and distribution functions in 1 and 2 dimensions, and consider the correlation coefficient. We consider in particular the Poisson distribution.

The prerequisites for the topics can e.g. be found in the *Ventus: Calculus 2* series, so I shall refer the reader to these books, concerning e.g. plane integrals.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
26th October 2009

1 Some theoretical results

The abstract (and precise) definition of a *random variable* X is that X is a real function on Ω, where the triple $(\Omega, \mathcal{F}, P)$ is a probability field, such that

$$\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F} \qquad \text{for every } x \in \mathbb{R}.$$

This definition leads to the concept of a *distribution function* for the random variable X, which is the function $F : \mathbb{R} \to \mathbb{R}$, which is defined by

$$F(x) = P\{X \leq x\} \qquad (= P\{\omega \in \Omega \mid X(\omega) \leq x\}),$$

where the latter expression is the mathematically precise definition which, however, for obvious reasons everywhere in the following will be replaced by the former expression.

A distribution function for a random variable X has the following properties:

$0 \leq F(x) \leq 1 \qquad$ for every $x \in \mathbb{R}$.

The function F is weakly increasing, i.e. $F(x) \leq F(y)$ for $x \leq y$.

$\lim_{x \to -\infty} F(x) = 0 \qquad$ and $\qquad \lim_{x \to +\infty} F(x) = 1$.

The function F is continuous from the right, i.e. $\lim_{h \to 0+} F(x+h) = F(x)$ for every $x \in \mathbb{R}$.

One may in some cases be interested in giving a crude description of the behaviour of the distribution function. We define a *median* of a random variable X with the distribution function $F(x)$ as a real number $a = (X) \in \mathbb{R}$, for which

$$P\{X \leq a\} \geq \frac{1}{2} \qquad \text{and} \qquad P\{X \geq a\} \geq \frac{1}{2}.$$

Expressed by means of the distribution function it follows that $a \in \mathbb{R}$ is a median, if

$$F(a) \geq \frac{1}{2} \qquad \text{and} \qquad F(a-) = \lim_{h \to 0-} F(x+h) \leq \frac{1}{2}.$$

In general we define a *p-quantile*, $p \in]0,1[$, of the random variable as a number $a_p \in \mathbb{R}$, for which

$$P\{X \leq a_p\} \geq p \qquad \text{and} \qquad P\{X \geq a_p\} \geq 1-p,$$

which can also be expressed by

$$F(a_p) \geq p \qquad \text{and} \qquad F(a_p-) \leq p.$$

If the random variable X only has a finite or a countable number of values, x_1, x_2, ..., we call it *discrete*, and we say that X has a *discrete distribution*.

A very special case occurs when X only has one value. In this case we say that X is *causally distributed*, or that X is constant.

The random variable X is called *continuous*, if its distribution function $F(x)$ can be written as an integral of the form

$$F(x) = \int_{-\infty}^{x} f(u)\,du, \qquad x \in \mathbb{R},$$

where f is a nonnegative integrable *function*. In this case we also say that X has a continuous *distribution*, and the integrand $f : \mathbb{R} \to \mathbb{R}$ is called a *frequency* of the random variable X.

Let again $(\Omega, \mathcal{F}, P)$ be a given probability field. Let us consider *two* random variables X and Y, which are both defined on Ω. We may consider the pair (X, Y) as a *2-dimensional random variable*, which implies that we then shall make precise the extensions of the previous concepts for a single random variable.

We say that the *simultaneous distribution*, or just the *distribution*, of (X, Y) is known, if we know

$$P\{(X, Y) \in A\} \qquad \text{for every Borel set } A \subseteq \mathbb{R}^2.$$

When the simultaneous distribution of (X, Y) is known, we define the *marginal distributions* of X and Y by

$$P_X(B) = P\{X \in B\} := P\{(X, Y) \in B \times \mathbb{R}\}, \qquad \text{where } B \subseteq \mathbb{R} \text{ is a Borel set,}$$

$$P_Y(B) = P\{Y \in B\} := P\{(X, Y) \in \mathbb{R} \times B\}, \qquad \text{where } B \subseteq \mathbb{R} \text{ is a Borel set.}$$

Notice that we can always find the *marginal distributions* from the *simultaneous distribution*, while it is far from always possible to find the *simultaneous distribution* from the *marginal distributions*. We now introduce

The *simultaneous distribution function* of the 2-dimensional random variable (X, Y) is defined as the function $F : \mathbb{R}^2 \to \mathbb{R}$, given by

$$F(x, y) := P\{X \leq x \wedge Y \leq y\}.$$

We have

- If $(x, y) \in \mathbb{R}^2$, then $0 \leq F(x, y) \leq 1$.
- If $x \in \mathbb{R}$ is kept fixed, then $F(x, y)$ is a weakly increasing function in y, which is continuous from the right and which satisfies the condition $\lim_{y\to-\infty} F(x, y) = 0$.
- If $y \in \mathbb{R}$ is kept fixed, then $F(x, y)$ is a weakly increasing function in x, which is continuous from the right and which satisfies the condition $\lim_{x\to-\infty} F(x, y) = 0$.
- When both x and y tend towards infinity, then
$$\lim_{x,\, y\to+\infty} F(x, y) = 1.$$
- If x_1, x_2, y_1, $y_2 \in \mathbb{R}$ satisfy $x_1 \leq x_2$ and $y_1 \leq y_2$, then
$$F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_2) \geq 0.$$

Given the simultaneous distribution function $F(x, y)$ of (X, Y) we can find the distribution functions of X and Y by the formulæ

$$F_X(x) = F(x, +\infty) = \lim_{y\to+\infty} F(x, y), \qquad \text{for } x \in \mathbb{R},$$

$$F_y(x) = F(+\infty, y) = \lim_{x\to+\infty} F(x, y), \qquad \text{for } y \in \mathbb{R}.$$

The 2-dimensional random variable (X, Y) is called *discrete*, or that it has a *discrete distribution*, if both X and Y are discrete.

The 2-dimensional random variable (X, Y) is called *continuous*, or we say that it has a *continuous distribution*, if there exists a nonnegative integrable function (a *frequency*) $f : \mathbb{R}^2 \to \mathbb{R}$, such that the distribution function $F(x, y)$ can be written in the form

$$F(x, y) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{y} f(t, u)\, du \right\} dt, \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

In this case we can find the function $f(x, y)$ at the differentiability points of $F(x, y)$ by the formula

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}.$$

It should now be obvious why one should know something about the theory of integration in more variables, cf. e.g. the *Ventus: Calculus 2* series.

We note that if $f(x, y)$ is a frequency of the continuous 2-dimensional random variable (X, Y), then X and Y are both continuous 1-dimensional random variables, and we get their (marginal) frequencies by

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy, \qquad \text{for } x \in \mathbb{R},$$

and

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx, \qquad \text{for } y \in \mathbb{R}.$$

It was mentioned above that one far from always can find the simultaneous distribution function from the marginal distribution function. It is, however, possible in the case when the two random variables X and Y are *independent*.

Let the two random variables X and Y be defined on the same probability field $(\Omega, \mathcal{F}, P)$. We say that X and Y are *independent*, if for all pairs of Borel sets A, $B \subseteq \mathbb{R}$,

$$P\{X \in A \wedge Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\},$$

which can also be put in the simpler form

$$F(x,y) = F_X(x) \cdot F_Y(y) \qquad \text{for every } (x,y) \in \mathbb{R}^2.$$

If X and Y are *not* independent, then we of course say that they are *dependent*.

In two special cases we can obtain more information of independent random variables:

If the 2-dimensional random variable (X,Y) is *discrete*, then X and Y are independent, if

$$h_{ij} = f_i \cdot g_j \qquad \text{for every } i \text{ and } j.$$

Here, f_i denotes the probabilities of X, and g_j the probabilities of Y.

If the 2-dimensional random variable (X,Y) is *continuous*, then X and Y are independent, if their frequencies satisfy

$$f(x,y) = f_X(x) \cdot f_Y(y) \qquad \text{almost everywhere.}$$

The concept *"almost everywhere"* is rarely given a precise definition in books on applied mathematics. Roughly speaking it means that the relation above holds outside a set in $\mathbb{R}^2$ of area zero, a so-called *null set*. The common examples of null sets are either finite or countable sets. There exists, however, also non-countable null sets. Simple examples are graphs of any (piecewise) C^1-curve.

Concerning maps of random variables we have the following very important results,

Theorem 1.1 *Let X and Y be independent random variables. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be given functions. Then $\varphi(X)$ and $\psi(Y)$ are again independent random variables.*

If X is a continuous random variable of the frequency I, then we have the following important theorem, where it should be pointed out that one always shall check *all* assumptions in order to be able to conclude that the result holds:

Theorem 1.2 *Given a continuous random variable X of frequency f.*

1) Let I be an open interval, such that $P\{X \in I\} = 1$.

2) Let $\tau : I \to J$ be a bijective *map of I onto an* open *interval J.*

3) Furthermore, assume that τ is differentiable *with a* continuous *derivative τ', which satisfies*

$$\tau'(x) \neq 0 \qquad \text{for alle } x \in I.$$

Under the assumptions above $Y := \tau(X)$ is also a continuous random variable, and its frequency $g(y)$ is given by

$$g(y) = \begin{cases} f\left(\tau^{-1}(y)\right) \cdot \left|\left(\tau^{-1}\right)'(y)\right|, & \text{for } y \in J, \\ 0, & \text{otherwise.} \end{cases}$$

We note that if just one of the assumptions above is *not* fulfilled, then we shall instead find the distribution function $G(y)$ of $Y := \tau(X)$ by the general formula

$$G(y) = P\{\tau(X) \in]-\infty\,,\,y]\} = P\left\{X \in \tau^{\circ -1}(]-\infty\,,\,y])\right\},$$

where $\tau^{\circ -1} = \tau^{-1}$ denotes the inverse set map.

Note also that if the assumptions of the theorem are all satisfied, then τ is necessarily monotone.

At a first glance it may be strange that we at this early stage introduce 2-dimensional random variables. The reason is that by applying the simultaneous distribution for (X,Y) it is fairly easy to define the elementary operations of calculus between X and Y. Thus we have the following general result for a *continuous* 2-dimensional random variable.

Theorem 1.3 *Let (X,Y) be a* continuous *random variable of the frequency $h(x,y)$.*

The frequency of the sum $X+Y$ is $\qquad k_1(z) = \int_{-\infty}^{+\infty} h(x, z-x)\,dx.$

The frequency of the difference $X-Y$ is $\qquad k_2(z) = \int_{-\infty}^{+\infty} h(x, x-z)\,dx.$

The frequency of the product $X \cdot Y$ is $\qquad k_3(z) = \int_{-\infty}^{+\infty} h\left(x\,,\,\frac{z}{x}\right) \cdot \frac{1}{|x|}\,dx.$

The frequency of the quotient X/Y is $\qquad k_4(z) = \int_{-\infty}^{+\infty} h(zx\,,\,x) \cdot |x|\,dx.$

Notice that one must be very careful by computing the product and the quotient, because the corresponding integrals are improper.

If we furthermore assume that X and Y are *independent*, and $f(x)$ is a frequency of X, and $g(y)$ is a frequency of Y, then we get an even better result:

Theorem 1.4 *Let X and Y be* continuous *and* independent *random variables with the frequencies $f(x)$ and $g(y)$, resp..*

The frequency of the sum $X+Y$ is $k_1(z) = \int_{-\infty}^{+\infty} f(x)g(z-x)\,dx.$

The frequency of the difference $X-Y$ is $k_2(z) = \int_{-\infty}^{+\infty} f(x)g(x-z)\,dx.$

The frequency of the product $X \cdot Y$ is $k_3(z) = \int_{-\infty}^{+\infty} f(x)\,g\left(\frac{z}{x}\right) \cdot \frac{1}{|x|}\,dx.$

The frequency of the quotient X/Y is $k_4 = \int_{-\infty}^{+\infty} f(zx)g(x) \cdot |x|\,dx.$

Let X and Y be *independent* random variables with the distribution functions F_X and F_Y, resp.. We introduce two random variables by

$$U := \max\{X, Y\} \qquad \text{and} \qquad V := \min\{X, Y\},$$

the distribution functions of which are denoted by F_U and F_V, resp.. Then these are given by

$$F_U(u) = F_X(u) \cdot F_Y(u) \qquad \text{for } u \in \mathbb{R},$$

and

$$F_V(v) = 1 - (1 - F_X(v)) \cdot (1 - F_Y(v)) \qquad \text{for } v \in \mathbb{R}.$$

These formulæ are general, provided only that X and Y are independent.

If X and Y are *continuous and independent*, then the frequencies of U and V are given by

$$f_U(u) = F_X(u) \cdot f_Y(u) + f_X(u) \cdot F_Y(u), \qquad \text{for } u \in \mathbb{R},$$

and

$$f_V(v) = (1 - F_X(v)) \cdot f_Y(v) + f_X(v) \cdot (1 - F_y(v)), \qquad \text{for } v \in \mathbb{R},$$

where we note that we shall apply both the frequencies and the distribution functions of X and Y.

The results above can also be extended to bijective maps $\underline{\varphi} = (\varphi_1, \varphi_2) : \mathbb{R}^2 \to \mathbb{R}^2$, or subsets of $\mathbb{R}^2$. We shall need the *Jacobian* of $\underline{\varphi}$, introduced in e.g. the *Ventus: Calculus 2* series.

It is important here to define the notation and the variables in the most convenient way. We start by assuming that D is an open domain in the $(x_1\, x_2)$ plane, and that $\tilde{D}$ is an open domain in the (y_1, y_2) plane. Then let $\underline{\varphi} = (\varphi_1, \varphi_2)$ be a bijective map *of* $\tilde{D}$ *onto* D with the inverse $\underline{\tau} = \underline{\varphi}^{-1}$, i.e. the opposite of what one probably would expect:

$$\underline{\varphi} = (\varphi_1, \varphi_2) : \tilde{D} \to D, \qquad \text{with } (x_1, x_2) = \underline{\varphi}(y_1, y_2).$$

The corresponding *Jacobian* is defined by

$$J_{\underline{\varphi}} = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \dfrac{\partial \varphi_1}{\partial y_1} & \dfrac{\partial \varphi_2}{\partial y_1} \\ \dfrac{\partial \varphi_1}{\partial y_1} & \dfrac{\partial \varphi_2}{\partial y_2} \end{vmatrix},$$

where the independent variables (y_1, y_2) are in the "denominators". Then recall the *Theorem of transform of plane integrals*, cf. e.g. the *Ventus: Calculus 2* series: If $h : D \to \mathbb{R}$ is an integrable function, where $D \subseteq \mathbb{R}^2$ is given as above, then for every (measurable) subset $A \subseteq D$,

$$\int_A h(x_1, x_2)\, dx_1 dx_2 = \int_{\underline{\varphi}^{-1}(A)} h(x_1, x_2) \cdot \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| dy_1 dy_2.$$

Of course, this formula is not mathematically correct; but it shows intuitively what is going on: Roughly speaking we "delete the y-s". The correct mathematical formula is of course the well-known

$$\int_A h(x_1, x_2)\, dx_1 dx_2 = \int_{\underline{\varphi}^{-1}(A)} (\varphi_1(y_1, y_2), \varphi_2(y_1, y_2)) \cdot \left| J_{\underline{\varphi}}(y_1, y_2) \right| dy_1 dy_2,$$

although experience shows that it in practice is more confusing then helping the reader.

Theorem 1.5 *Let* (X_1, X_2) *be a continuous 2-dimensional random variable with the frequency* $h(x_1\,,\,x_2)$. *Let* $D \subseteq \mathbb{R}^2$ *be an open domain, such that*

$$P\{(X_1\,,\,X_2) \in D\} = 1.$$

Let $\underline{\tau} : D \to \tilde{D}$ *be a bijective map of* D *onto another open domain* $\tilde{D}$, *and let* $\underline{\varphi} = (\varphi_1\,,\,\varphi_2) = \underline{\tau}^{-1}$, *where we assume that* φ_1 *and* φ_2 *have continuous partial derivatives and that the corresponding Jacobian is different from 0 in all of* $\tilde{D}$.
Then the 2-dimensional random variable

$$(Y_1\,,\,Y_2) = \underline{\tau}(X_1\,,\,X_2) = (\tau_1(X_1\,,\,X_2)\,,\,\tau_2(X_1\,,\,X_2))$$

has the frequency $k(y_1\,,\,y_2)$, *given by*

$$k(y_1\,,\,y_2) = \begin{cases} h(\varphi_1(y_1\,,\,y_2)\,,\,\varphi_2(y_1\,,\,y_2)) \cdot \left|\dfrac{\partial(x_1\,,\,x_2)}{\partial(y_1\,,\,y_2)}\right|, & \text{for } (y_1\,,\,y_2) \in \tilde{D}, \\[2ex] 0, & \text{otherwise} \end{cases}$$

We have previously introduced the concept *conditional probability.* We shall now introduce a similar concept, namely the *conditional distribution.*

If X and Y are discrete, we define the *conditional distribution of* X *for given* $Y = y_j$ by

$$P\{X = x_i \mid Y = y_j\} = \frac{P\{X = x_i \wedge Y = y_j\}}{P\{Y = y_j\}} = \frac{h_{ij}}{g_j}.$$

It follows that for fixed j we have that $P\{X = x_i \mid Y = y_j\}$ indeed is a distribution. We note in particular that we have the *law of the total probability*

$$P\{X = x_i\} = \sum_j P\{X = x_i \mid Y = y_j\} \cdot P\{Y = y_j\}.$$

Analogously we define for two continuous random variables X and Y the *conditional distribution function of* X *for given* $Y = y$ by

$$P\{X \le x \mid Y = y\} = \frac{\int_{-\infty}^{x} f(u,y)\,du}{f_Y(y)}, \qquad \text{forudsat, at } f_Y(y) > 0.$$

Note that the conditional distribution function is not defined at points in which $f_Y(y) = 0$.

The corresponding frequency is

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}, \qquad \text{provided that } f_Y(y) = 0.$$

We shall use the convention that "0 times undefined = 0". Then we get the *Law of total probability*,

$$\int_{-\infty}^{+\infty} f(x \mid y) \cdot f_Y(y)\,dy = \int_{-\infty}^{+\infty} f(x,y)\,dy = f_X(x).$$

We now introduce the *mean*, or *expectation* of a random variable, provided that it exists.

1) Let X be a *discrete* random variable with the possible values $\{x_i\}$ and the corresponding probabilities $p_i = P\{X = x_i\}$. The *mean*, or *expectation*, of X is defined by

$$E\{X\} := \sum_i x_i \, p_i,$$

provided that the series is absolutely convergent. If this is not the case, the mean does not exists.

2) Let X be a *continuous* random variable with the frequency $f(x)$. We define the *mean*, or *expectation* of X by

$$E\{X\} = \int_{-\infty}^{+\infty} x \, f(x) \, dx,$$

provided that the integral is absolutely convergent. If this is not the case, the mean does not exist.

If the random variable X only has nonnegative values, i.e. the image of X is contained in $[0, +\infty[$, and the mean exists, then the mean is given by

$$E\{X\} = \int_0^{+\infty} P\{X \geq x\} \, dx.$$

Concerning maps of random variables, means are transformed according to the theorem below, provided that the given expressions are absolutely convergent.

Theorem 1.6 *Let the random variable $Y = \varphi(X)$ be a function of X.*

1) If X is a discrete *random variable with the possible values $\{x_i\}$ of corresponding probabilities $p_i = P\{X = x_i\}$, then the* mean *of $Y = \varphi(X)$ is given by*

$$E\{\varphi(X)\} = \sum_i \varphi(x_i) \, p_i,$$

provided that the series is absolutely convergent.

2) If X is a continuous *random variable with the frequency $f(x)$, then the* mean *of $Y = \varphi(X)$ is given by*

$$E\{\varphi(X)\} = \int_{-\infty}^{+\infty} \varphi(x) \, g(x) \, dx,$$

provided that the integral is absolutely convergent.

Assume that X is a random variable of mean μ. We add the following concepts, where $k \in \mathbb{N}$:

The k-th moment,	$E\left\{X^k\right\}$.
The k-th absolute moment,	$E\left\{\lvert X\rvert^k\right\}$.
The k-th central moment,	$E\left\{(X-\mu)^k\right\}$.
The k-th absolute central moment,	$E\left\{\lvert X-\mu\rvert^k\right\}$.
The variance, i.e. the second central moment,	$V\{X\} = E\left\{(X-\mu)^2\right\}$,

provided that the defining series or integrals are absolutely convergent. In particular, the *variance* is very important. We mention

Theorem 1.7 *Let X be a random variable of mean $E\{X\} = \mu$ and variance $V\{X\}$. Then*

$$
\begin{aligned}
&E\left\{(X-c)^2\right\} = V\{X\} + (\mu - c)^2 && \textit{for every } c \in \mathbb{R},\\
&V\{X\} = E\left\{X^2\right\} - (E\{X\})^2 && \textit{for } c = 0,\\
&E\{aX + b\} = a\,E\{X\} + b && \textit{for every } a,\, b \in \mathbb{R},\\
&V\{aX + b\} = a^2 V\{X\} && \textit{for every } a,\, b \in \mathbb{R}.
\end{aligned}
$$

It is not always an easy task to compute the distribution function of a random variable. We have the following result which gives an estimate of the probability that a random variable X differs more than some given $a > 0$ from the mean $E\{X\}$.

Theorem 1.8 (Čebyšev's inequality). *If the random variable X has the mean μ and the variance σ^2, then we have for every $a > 0$,*

$$P\{|X - \mu| \geq a\} \leq \frac{\sigma^2}{a^2}.$$

If we here put $a = k\sigma$, we get the equivalent statement

$$P\{\mu - k\sigma < X < \mu + k\sigma\} \geq 1 - \frac{1}{k^2}.$$

These concepts are then generalized to 2-dimensional random variables. Thus,

Theorem 1.9 *Let $Z = \varphi(X,Y)$ be a function of the 2-dimensional random variable (X,Y).*

1) If (X,Y) is discrete*, then the* mean *of $Z = \varphi(X,Y)$ is given by*

$$E\{\varphi(X,Y)\} = \sum_{i,j} \varphi\left(x_i\,,\,y_j\right) \cdot P\left\{X = x_i \,\wedge\, Y = y_j\right\},$$

provided that the series is absolutely convergent.

2) If (X,Y) is continuous*, then the* mean *of $Z = \varphi(X,Y)$ is given by*

$$E\{\varphi(X,Y)\} = \int_{\mathbb{R}^2} \varphi(x,y)\, f(x,y)\, dxdy,$$

provided that the integral is absolutely convergent.

It is easily proved that if (X,Y) is a 2-dimensional random variable, and $\varphi(x,y) = \varphi_1(x) + \varphi_2(y)$, then

$$E\left\{\varphi_1(X) + \varphi_2(Y)\right\} = E\left\{\varphi_1(X)\right\} + E\left\{\varphi_2(Y)\right\},$$

provided that $E\left\{\varphi_1(X)\right\}$ and $E\left\{\varphi_2(Y)\right\}$ exists. In particular,

$$E\{X+Y\} = E\{X\} + E\{Y\}.$$

If we furthermore assume that X and Y are independent and choose $\varphi(x,y) = \varphi_1(x) \cdot \varphi_2(y)$, then also

$$E\left\{\varphi_1(X) \cdot \varphi_2(Y)\right\} = E\left\{\varphi_1(X)\right\} \cdot E\left\{\varphi_2(Y)\right\},$$

provided that $E\left\{\varphi_1(X)\right\}$ and $E\left\{\varphi_2(Y)\right\}$ exists. In particular we get under the assumptions above that

$$E\{X \cdot Y\} = E\{X\} \cdot E\{Y\},$$

and

$$E\{(X - E\{X\}) \cdot (Y - E\{Y\})\} = 0.$$

These formulæ are easily generalized to n random variables. We have e.g.

$$E\left\{\sum_{i=1}^{n} X_i\right\} = \sum_{i=1}^{n} E\left\{X_i\right\},$$

provided that all means $E\left\{X_i\right\}$ exist.

If two random variables X and Y are not independent, we shall find a measure of how much they "depend" on each other. This measure is described by the *correlation*, which we now introduce.

Consider a 2-dimensional random variable (X,Y), where

$$E\{X\} = \mu_X, \qquad E\{Y\} = \mu_Y, \qquad V\{X\} = \sigma_X^2 > 0, \qquad V\{Y\} = \sigma_Y^2 > 0,$$

all exist. We define the *covariance* between X and Y, denoted by $\mathrm{Cov}(X,Y)$, as

$$\mathrm{Cov}(X,Y) := E\left\{(X-\mu_X)\cdot(Y-\mu_Y)\right\}.$$

We define the *correlation* between X and Y, denoted by $\varrho(X,Y)$, as

$$\varrho(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}.$$

Theorem 1.10 *Let X and Y be two random variables, where*

$$E\{X\} = \mu_X, \qquad E\{Y\} = \mu_Y, \qquad V\{X\} = \sigma_X^2 > 0, \qquad V\{Y\} = \sigma_Y^2 > 0,$$

all exist. Then

$$\mathrm{Cov}(X,Y) = 0, \qquad \textit{if } X \textit{ and } Y \textit{ are independent},$$

$$\mathrm{Cov}(X,Y) = E\{X\cdot Y\} - E\{X\}\cdot E\{Y\},$$

$$|\mathrm{Cov}(X,Y)| \le \sigma_X \cdot \sigma_y,$$

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X),$$

$$V\{X+Y\} = V\{X\} + V\{Y\} + 2\mathrm{Cov}(X,Y),$$

$$V\{X+Y\} = V\{X\} + V\{Y\}, \qquad \textit{if } X \textit{ and } Y \textit{ are independent},$$

$$\varrho(X,Y) = 0, \qquad \textit{if } X \textit{ and } Y \textit{ are independent},$$

$$\varrho(X,X) = 1, \qquad \varrho(X,-X) = -1, \qquad |\varrho(X,Y)| \le 1.$$

Let Z be another random variable, for which the mean and the variance both exist- Then

$$\mathrm{Cov}(aX+bY,Z) = a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z), \qquad \textit{for every } a,\, b \in \mathbb{R},$$

and if $U = aX+b$ and $V = cY+d$, where $a>0$ and $c>0$, then

$$\varrho(U,V) = \varrho(aX+b, cY+d) = \varrho(X,Y).$$

Two independent random variables are always non-correlated, while two non-correlated random variables are not necessarily independent.

By the obvious generalization,

$$V\left\{\sum_{i=1}^n X_i\right\} = \sum_{i=1}^n V\left\{X_i\right\} + 2\sum_{j=2}^n \sum_{i=1}^{j-1} \mathrm{Cov}\left(X_i, X_j\right).$$

If all X_1, X_2, $\dots$, X_n are independent of each other, this is of course reduced to

$$V\left\{\sum_{i=1}^n X_i\right\} = \sum_{i=1}^n V\left\{X_i\right\}.$$

Finally we mention the various types of convergence which are natural in connection with sequences of random variables. We consider a sequence X_n of random variables, defined on the same probability field $(\Omega, \mathcal{F}, P)$.

1) We say that X_n *converges in probability* towards a random variable X on the probability field $(\Omega, \mathcal{F}, P)$, if

$$P\{|X_n - X| \geq \varepsilon\} \to 0 \qquad \text{for } n \to +\infty,$$

for every fixed $\varepsilon > 0$.

2) We say that X_n *converges in probability* towards a constant c, if every fixed $\varepsilon > 0$,

$$P\{|X_n - c| \geq \varepsilon\} \to 0 \qquad \text{for } n \to +\infty.$$

3) If each X_n has the distribution function F_n, and X has the distribution function F, we say that the sequence X_n of random variables *converges in distribution* towards X, if at every point of continuity x of $F(x)$,

$$\lim_{n \to +\infty} F_n(x) = F(x).$$

Finally, we mention the following theorems which are connected with these concepts of convergence. The first one resembles *Čebyšev's inequality.*

Theorem 1.11 (The weak law of large numbers). *Let X_n be a sequence of* independent *random variables, all defined on $(\Omega, \mathcal{F}, P)$, and assume that they all have the same mean and variance,*

$$E\{X_i\} = \mu \qquad \textit{and} \qquad V\{X_i\} = \sigma^2.$$

Then for every fixed $\varepsilon > 0$,

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right\} \to 0 \qquad \textit{for } n \to +\infty.$$

A slightly different version of the weak law of large numbers is the following

Theorem 1.12 *If X_n is a sequence of* independent identical distributed *random variables, defined on $(\Omega, \mathcal{F}, P)$ where $E\{X_i\} = \mu$, (notice that we do not assume the existence of the variance), then for every fixed $\varepsilon > 0$,*

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right\} \to 0 \qquad \textit{for } n \to +\infty.$$

We have concerning convergence in distribution,

Theorem 1.13 (Helly-Bray's lemma). *Assume that the sequence X_n of random variables converges in distribution towards the random variable X, and assume that there are real constants a and b, such that*

$$P\{a \leq X_n \leq b\} = 1 \qquad \textit{for every } n \in \mathbb{N}.$$

If φ is a continuous function on the interval $[a, b]$, then

$$\lim_{n \to +\infty} E\{\varphi(X_n)\} = E\{\varphi(X)\}.$$

In particular,

$$\lim_{n \to +\infty} E\{X_n\} \qquad \textit{and} \qquad \lim_{n \to +\infty} V\{X_n\} = V\{X\}.$$

Finally, the following theorem gives us the relationship between the two concepts of convergence:

Theorem 1.14 *1) If X_n converges in probability towards X, then X_n also converges in distribution towards X.*

2) If X_n converges in distribution towards a constant c, then X_n also converges in probability towards the constant c.

2 Law of total probability

Example 2.1 *Given a countable number of boxes: U_1, U_2, ..., U_n, ... Let box number n contain n slips of paper with the numbers 1, 2, ..., n. We choose at random with probability p_n the box U_n, and from this box we choose randomly one of the slips of paper. Let X denote the random variable, which indicates the number of the chosen box, and let Y denote the random variable, which gives the number on the chosen slip of paper.*

1) Find the distribution of the random variable Y.

2) Prove that the mean $E\{Y\}$ exists if and only if the mean $E\{X\}$ exists. When both these means exist one shall express $E\{Y\}$ by means of $E\{X\}$.

3) Assume that $p_n = pq^{n-1}$, where $p > 0$, $q > 0$ and $p + q = 1$. Find

$$P\{Y = 1\}.$$

1) It is given that

$$\sum_{n=1}^{\infty} p_n = 1, \qquad p_n \geq 0,$$

and

$$P\{X = b\} = p_n, \qquad n \in \mathbb{N},$$

and

$$P\{Y = k \mid X = n\} = \begin{cases} \dfrac{1}{n}, & k = 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

When we apply the law of total probability, it follows for any $k \in \mathbb{N}$ that

$$\begin{aligned} P\{Y = k\} &= \sum_{n=1}^{\infty} P\{Y = k \mid X = n\} \cdot P\{X = n\} = \sum_{n=k}^{\infty} P\{Y = k \mid X = n\} \cdot P\{X = n\} \\ &= \sum_{n=k}^{\infty} \frac{1}{n}\, p_n. \end{aligned}$$

2) Assume that $E\{Y\}$ exists. Since all terms are ≥ 0, we can interchange the summations,

$$\begin{aligned} E\{Y\} &= \sum_{k=1}^{\infty} k\, P\{Y = k\} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{k}{n} \cdot p_n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} k \cdot \frac{1}{n}\, p_n = \sum_{n=1}^{\infty} \frac{1}{2}\, n(n+1)\, \frac{1}{n}\, p_n \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (n+1) p_n = \frac{1}{2} \sum_{n=1}^{\infty} n p_n + \frac{1}{2} \sum_{n=1}^{\infty} p_n = \frac{1}{2} + \frac{1}{2}\, E\{X\}. \end{aligned}$$

If on the other hand $E\{X\}$ exists, then we can reverse all computations above and conclude that $E\{Y\}$ exists. In fact, every term is ≥ 0, so the summations can be interchanged, which gives

$$E\{Y\} = \frac{1}{2}\,(1 + E\{X\}).$$

3) If $p_n = pq^{n-1}$, it follows from (1) that

$$P\{Y=1\} = \sum_{n=1}^{\infty} \frac{1}{n}\, p\, q^{n-1} = \frac{p}{q}\sum_{n=1}^{\infty}\frac{1}{n}\, q^n = \frac{p}{q}\{-\ln(1-q)\} = \frac{p}{1-p}\,\ln\left(\frac{1}{p}\right).$$

Example 2.2 *Throw once an (honest) dice and let the random variable N denote the number given by the dice.*
Then flip a coin N times, where N is the random variable above, and let X denote the number of heads in these throws.

1) Find $P\{X=0 \wedge N=i\}$ for $i = 1, 2, 3, 4, 5, 6$.

2) Find $P\{X=0\}$.

3) Find the mean $E\{X\}$.

1) If $N=i$, then $X=0$ means that we get tails i times, thus

$$P\{X=0 \wedge N=i\} = \left(\frac{1}{2}\right), \qquad i = 1,2,3,4,5,6.$$

2) By the law of total probability,

$$P\{X=0\} = \sum_{i=1}^{6} P\{X=0 \wedge N=i\}\cdot P\{N=i\} = \sum_{i=1}^{6}\left(\frac{1}{2}\right)\cdot\frac{1}{6} = \frac{1}{6}\left(1-\frac{1}{2^6}\right) = \frac{21}{128}.$$

3) We get for $j \in \{1,\dots,i\}$, $i \in \{1,\dots,6\}$,

$$P\{X=j \wedge N=i\} = \binom{i}{j}\cdot\left(\frac{1}{2}\right)^j\cdot\left(\frac{1}{2}\right)^{i-j} = \binom{i}{j}\left(\frac{1}{2}\right)^i,$$

hence

$$P\{X=j\} = \sum_{i=j}^{6} P\{X=j \wedge N=i\}\cdot P\{N=i\} = \frac{1}{6}\sum_{i=j}^{6}\binom{i}{j}\left(\frac{1}{2}\right)^i.$$

Then by interchanging the order of summation,

$$\begin{aligned} E\{X\} &= \sum_{j=1}^{6} j\, P\{X=j\} = \sum_{j=1}^{6}\frac{j}{6}\sum_{i=j}^{6}\binom{i}{j}\left(\frac{1}{2}\right)^i = \frac{1}{6}\sum_{i=1}^{6}\left(\frac{1}{2}\right)^i\sum_{j=1}^{i} j\binom{i}{j} \\ &= \frac{1}{6}\sum_{i=1}^{6}\left(\frac{1}{2}\right)^i\cdot i\sum_{j=1}^{i}\binom{i-1}{j-1} = \frac{1}{6}\sum_{i=1}^{6} i\left(\frac{1}{2}\right)^i\sum_{k=0}^{i-1}\binom{i-1}{k} = \frac{1}{6}\sum_{i=1}^{6} i\left(\frac{1}{2}\right)^i 2^{i-1} \\ &= \frac{1}{12}\sum_{i=1}^{6} i = \frac{1}{12}\cdot\frac{1}{2}\cdot 6\cdot 7 = \frac{7}{4}. \end{aligned}$$

Example 2.3 *A box contains N balls with the numbers 1, 2, ..., N. Choose at random a ball from the box and note its number X, without returning it to the box. Then select another ball and note its number Y.*

1) Find the distribution of the 2-dimensional random variable (X, Y).

2) Find the distribution of the random variable $Z = |X - Y|$.

1) It is obvious that

$$P\{(X,Y) = (k,n)\} = \begin{cases} \dfrac{1}{N(N-1)} & \text{for } k, n \in \{1, \dots, N\} \text{ and } k \neq n, \\ \\ 0 & \text{otherwise.} \end{cases}$$

2) Since $X \neq Y$, the random variable $Z = |X - Y|$ can only attain the values 1, 2, ..., $N - 1$. If $n \in \{1, 2, \dots, N-1\}$, then

$$\begin{aligned} P\{Z = n\} &= P\{|X - Y| = n\} = P\{X - Y = n\} + P\{Y - X = n\} \\ &= \sum_{k=1}^{B} P\{(X,Y) = (n+k, k)\} + \sum_{k=1}^{N} P\{(X,Y) = (k, n+k)\} \\ &= 2\sum_{k=1}^{N} P\{(X,Y) = (k, n+k)\} = 2\sum_{k=1}^{N-n} P\{(X,Y) = (k, n+k)\} = 2\,\frac{N-n}{N(N-1)}. \end{aligned}$$

CONTROL. It follows that

$$\sum_{n=1}^{N-1} P\{Z = n\} = \sum_{n=1}^{N-1} 2 \cdot \frac{N-n}{N(N-1)} = \frac{2}{N(N-1)} \sum_{n=1}^{N-1} n = \frac{2}{N(N-1)} \cdot \frac{1}{2}\,(N-1)N = 1.$$

3 Correlation coefficient and skewness

Example 3.1 *A random variable X has its distribution given by*

$$P\{X = i\} = \frac{1}{100}, \qquad i = 1, 2, 3, \ldots, 98, 99, 100.$$

Two random variables Y and Z depend on X, such that

$$Y = \begin{cases} 1, & \text{if } X \text{ can be divided by at least one of the numbers 2 or 3,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Z = \begin{cases} 1, & \text{if } X \text{ can be divided by 3,} \\ 0, & \text{otherwise.} \end{cases}$$

Compute the correlation coefficient $\varrho(Y, Z)$.

We shall find

$$\varrho(Y,Z) = \frac{\operatorname{Cov}(Y,Z)}{\sigma_1\sigma_2},$$

where

$$\operatorname{Cov}(Y,Z) = E\{YZ\} - E\{Y\}E\{Z\},$$

and

$$\sigma_1^2 = V\{Y\} \qquad \text{and} \qquad \sigma_2^2 = V\{Z\}.$$

The distribution functions of Y and Z are found by simply counting,

$$\begin{aligned} P\{Y=1\} &= P\{X \text{ even}\} + P\{X \text{ odd, and } X \text{ is divisible by } 3\} \\ &= \sum_{n=1}^{50} P\{X=2n\} + \sum_{n=1}^{17} P\{X=6n-3\} = \frac{50}{100} + \frac{17}{100} = \frac{67}{100}, \end{aligned}$$

and

$$P\{Z=1\} = P\{X \text{ is divisible by } 3\} = \sum_{n=1}^{33} P\{X=3n\} = \frac{33}{100}.$$

Since Y and Z can only have the values 0 and 1 (where $0^2 = 0$ and $1^2 = 1$), we get

$$E\left\{Y^2\right\} = E\{Y\} = \sum_{i=0}^{1} i^{(2)} P\{Y=i\} = P\{Y=1\} = \frac{67}{100},$$

and

$$E\left\{Z^2\right\} = E\{Z\} = \sum_{i=0}^{1} i^{(2)} P\{Z=i\} = P\{Z=1\} = \frac{33}{100},$$

hence

$$\sigma_1^2 = V\{Y\} = E\left\{Y^2\right\} - (E\{Y\})^2 = \frac{67}{100} - \left(\frac{67}{100}\right)^2 = \frac{67}{100}\cdot\frac{33}{100},$$

and

$$\sigma_2^2 = V\{Z\} = E\left\{Z^2\right\} - (E\{Z\})^2 = \frac{33}{100} - \left(\frac{33}{100}\right)^2 = \frac{33}{100}\cdot\frac{67}{100},$$

whence

$$\sigma_1\sigma_2 = \sqrt{\frac{67}{100}\cdot\frac{33}{100}\cdot\frac{33}{100}\cdot\frac{67}{100}} = \frac{33}{100}\cdot\frac{67}{100}.$$

Finally,

$$\begin{aligned} E\{YZ\} &= \sum_{i=0}^{1}\sum_{j=0}^{1} ij\, P\{Y=i \wedge Z=j\} = P\{Y=1 \wedge Z=1\} \\ &= P\{X \text{ is divisible by } 3\} = P\{Z=1\} = \frac{33}{100} = E\{Z\}, \end{aligned}$$

so

$$\mathrm{Cov}(Y,Z) = E\{YZ\} - E\{Y\}E\{Z\} = \frac{33}{100}\left(1 - \frac{67}{100}\right) = \frac{33^2}{100^2}.$$

We derive that the correlation coefficient is

$$\varrho(Y,Z) = \frac{\mathrm{Cov}(Y,Z)}{\sigma_1\sigma_2} = \frac{\frac{33^2}{100^2}}{\frac{67}{100}\cdot\frac{33}{100}} = \frac{33}{67}.$$

Example 3.2 *Let X denote a random variable, for which $E\{X\} = \mu$, $V\{X\} = \sigma^2$ and $E\left\{X^3\right\}$ all exist.*

1. *Prove the formula*

$$E\left\{(X-\mu)^3\right\} = E\left\{X^3\right\} - \mu\left(3\sigma^2 + \mu^2\right).$$

When $V\{X\}$ is bigger than 0, we define the skewness *(asymmetry) of the distribution by the number $\gamma(X)$, given by*

$$\gamma(X) = \frac{E\left\{(X-\mu)^3\right\}}{\sigma^3}.$$

A random variable X has the possible values 0, 1, 2, of the corresponding probabilities p, $\frac{1}{2}$, $\frac{1}{2} - p$, where $0 \le p \le \frac{1}{2}$.

2. *Find the number $\gamma(X)$ of this distribution.*

3. *Find the values of p, for which $\gamma(X) = 0$.*

4. *Find $\gamma(X)$ for $p = \frac{1}{8}$.*

1) The claim is proved in the continuous case. The proof in the discrete case is analogous. A straightforward computation gives

$$\begin{aligned}
E\left\{(X-\mu)^3\right\} &= \int_{-\infty}^{\infty} (x-\mu)^3 f(x)\,dx = \int_{-\infty}^{\infty} \left\{x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3\right\} f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^3 f(x)\,dx - \mu \int_{-\infty}^{\infty} \left\{3x^2 - 3\mu x + \mu^2\right\} f(x)\,dx \\
&= E\left\{X^3\right\} - \mu \int_{-\infty}^{\infty} \left\{3x^2 - 6\mu x + 3\mu^2 + 3\mu x - 2\mu^2\right\} f(x)\,dx \\
&= E\left\{X^3\right\} - 3\mu \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx - 3\mu^2 \int_{-\infty}^{\infty} x\, f(x)\,dx + 2\mu^3 \int_{-\infty}^{\infty} f(x)\,dx \\
&= E\left\{X^3\right\} - 3\mu\sigma^2 - 2\mu^2\mu + 2\mu^3 = E\left\{X^3\right\} - \mu\left(3\sigma^2 + \mu^2\right).
\end{aligned}$$

Alternatively, apply the following direct proof (all cases),

$$\begin{aligned}
E\left\{(X-\mu)^3\right\} &= E\left\{X^3-3\mu X^2+3\mu^2X-\mu^3\right\} = E\left\{X^3\right\}-3\mu\,E\left\{X^2\right\}+3\mu^2\,E\{X\}-\mu^3 \\
&= E\left\{X^3\right\}-3\mu\left[E\left\{X^2\right\}-(E\{X\})^2\right]-3\mu\,(E\{X\})^2+3\mu^3-\mu^3 \\
&= E\left\{X^3\right\}-3\mu\sigma^2-3\mu^3+3\mu^3-\mu^3 \\
&= E\left\{X^3\right\}-\mu\left(3\sigma^2+\mu^2\right).
\end{aligned}$$

2) If

$$P\{X=0\}=p, \qquad P\{X=1\}=\frac{1}{2} \quad \text{og} \quad P\{X=2\}=\frac{1}{2}-p,$$

where $0\le p\le \frac{1}{2}$, then

$$\mu=E\{X\}=\sum_{i=0}^{2} i\,P\{X=i\}=0\cdot p+1\cdot\frac{1}{2}+2\left(\frac{1}{2}-p\right)=\frac{3}{2}-2p,$$

and

$$E\left\{X^2\right\}=\sum_{i=0}^{2} i^2P\{X=i\}=0\cdot p+1\cdot\frac{1}{2}+4\left(\frac{1}{2}-p\right)=\frac{5}{2}-4p,$$

hence

$$\begin{aligned}
\sigma^2 &= E\left\{X^2\right\}-(E\{X\})^2=\frac{5}{2}-4p-\left(\frac{3}{2}-2p\right)^2=\frac{5}{2}-4p-\left(\frac{9}{4}-6p+4p^2\right) \\
&= \frac{1}{4}+2p-4p^2=\frac{1}{4}\left\{1+8p-16p^2\right\} \qquad \left(\ge\frac{1}{4}\right).
\end{aligned}$$

Finally,

$$E\left\{X^3\right\}=\sum_{i=0}^{2} i^3P\{X=i\}=0\cdot p+1\cdot\frac{1}{2}+8\left(\frac{1}{2}-p\right)=\frac{9}{2}-8p,$$

thus

$$\begin{aligned}
E\left\{(X-\mu)^3\right\} &= E\left\{X^3\right\}-\mu\left(3\sigma^2+\mu^2\right)=\frac{9}{2}-8p-\left(\frac{3}{2}-2p\right)\left(\frac{3}{4}+6p-12p^2+\left[\frac{3}{2}-2p\right]^2\right) \\
&= \frac{9}{2}-8p-\left(\frac{3}{2}-2p\right)\left(\frac{3}{4}+6p-12p^2+\frac{9}{4}-6p+4p^2\right) \\
&= \frac{9}{2}-8p-\left(\frac{3}{2}-2p\right)\left(3-8p^2\right)=\frac{9}{2}-8p-\left\{\frac{9}{2}-12p^2-6p+16p^3\right\} \\
&= \frac{9}{2}-8p-\frac{9}{2}+12p^2+6p-16p^3=-2p+12p^2-16p^3=-p\left\{16p^2-12p+2\right\} \\
&= -16p\left(p-\frac{1}{4}\right)\left(p-\frac{1}{2}\right).
\end{aligned}$$

This implies that

$$\gamma(X)=\frac{E\left\{(X-\mu)^3\right\}}{\sigma^3}=\frac{-16p\left(p-\frac{1}{4}\right)\left(p-\frac{1}{2}\right)}{\frac{1}{8}\left(1+8p-16p^2\right)^{3/2}}=-\frac{128p\left(p-\frac{1}{2}\right)\left(p-\frac{1}{4}\right)}{\left\{2-(4p-1)^2\right\}^{3/2}}.$$

3) It follows immediately that $\gamma(X)=0$ for $p=0,\ \frac{1}{4},\ \frac{1}{2}$.

4) If $p=\frac{1}{8}$, then

$$\gamma(X)=-\frac{128\cdot\frac{1}{8}\left(\frac{1}{8}-\frac{1}{2}\right)\left(\frac{1}{8}-\frac{1}{4}\right)}{\left\{2-\left(\frac{1}{2}-1\right)^2\right\}^{3/2}}=-\frac{16\cdot\frac{3}{8}\cdot\frac{1}{8}}{\left\{2-\frac{1}{4}\right\}^{3/2}}=-\frac{\frac{3}{4}}{\frac{7}{4}\sqrt{\frac{7}{4}}}=-\frac{6}{7\sqrt{7}}\approx-0.324.$$

Example 3.3 *Given for any $n \in \mathbb{N}$ a random variable X_n of the frequency*

$$f_n(x) = \begin{cases} \dfrac{1}{(n-1)!}\, a^n x^{n-1} e^{-ax}, & x > 0, \\ \\ 0, & \textit{otherwise}, \end{cases}$$

where a is a positive constant.
Compute the skewness $\gamma(X_n)$, and show that $\gamma(X_n) \to 0$ for $n \to \infty$.

According to Example 3.2 the skewness $\gamma(X_n)$ is defined by

$$\gamma(X_n) = \frac{E\left\{(X_n - \mu_n)^3\right\}}{\sigma_n^3},$$

where

$$E\left\{(X_n - \mu_n)^3\right\} = E\left\{X_n^3\right\} - \mu_n\left(3\sigma_n^2 + \mu_n^2\right).$$

By some small computations,

$$\mu_n = E\{X_n\} = \frac{a^n}{(n-1)!}\int_0^\infty x^n e^{-ax}\, dx = \frac{1}{a(n-1)!}\int_0^\infty t^n e^{-1}\, dt = \frac{n!}{a(n-1)!} = \frac{n}{a},$$

and

$$E\left\{X_n^2\right\} = \frac{a^n}{(n-1)!}\int_0^\infty x^{n+1} e^{-ax}\, dx = \frac{(n+1)!}{a^2(n-1)!} = \frac{n(n+1)}{a^2},$$

hence

$$\sigma_n^2 = E\left\{X_n^2\right\} - (E\{X_n\})^2 = \frac{n(n+1)}{a^n} - \frac{n^2}{a^2} = \frac{n}{a^2},$$

and

$$E\left\{X_n^3\right\} = \frac{a^n}{(n-1)!}\int_0^\infty x^{n+2} e^{-ax}\, dx = \frac{(n+2)!}{a^3(n-1)!} = \frac{n(n+1)(n+2)}{a^3},$$

whence

$$\begin{aligned} E\left\{(X_n - \mu_n)^3\right\} &= E\left\{X_n^3\right\} - \mu_n\left(3\sigma_n^2 + \mu_n^2\right) = \frac{n(n+1)(n+2)}{a^3} - \frac{n}{a}\cdot\left\{\frac{3n}{a^2} + \frac{n^2}{a^2}\right\} \\ &= \frac{n}{a^3}\left\{n^2 + 3n + 2 - 3n - n^2\right\} = \frac{2n}{a^3}. \end{aligned}$$

The skewness is

$$\gamma(X_n) = \frac{E\left\{(X_n - \mu_n)^3\right\}}{\sigma_n^3} = \frac{2n}{a^3}\cdot\frac{a^3}{n^{3/2}} = \frac{2}{\sqrt{n}} \to 0 \qquad \text{for } n \to \infty.$$

Example 3.4 *Assume that the 2-dimensional random variable (X,Y) has the frequency*

$$f(x,y)=\begin{cases}\dfrac{2}{A^2}, & 0<y<x<A,\\[2ex] 0, & \textit{otherwise},\end{cases}$$

where A is a positive constant.

1) Find the frequencies of X and Y.

2) Find the means of X and Y.

3) Find the variances of X and Y.

4) Compute the correlation coefficient ϱ between X and Y, and prove that it does not depend on A.

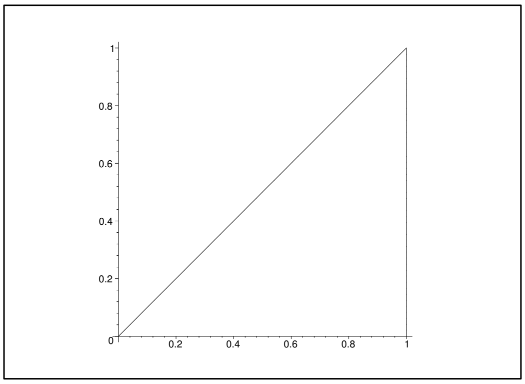


Figure 1: The domain where $f(x,y)>0$ for $A=1$.

1) If $x\in\,]0,A[$, then

$$f_X(x)=\int_0^x\frac{2}{A^2}\,dy=\frac{2x}{A^2},\quad\text{and}\quad f_X(x)=0\text{ otherwise.}$$

If $y\in\,]0,A[$, then

$$f_Y(y)=\int_y^A\frac{2}{A^2}\,dx=\frac{2(A-y)}{A^2}=\frac{2}{A}-\frac{2y}{A^2},\quad\text{og}\quad f_Y(y)=0\text{ otherwise.}$$

2) The means are

$$E\{X\}=\int_0^A\frac{2x^2}{A^2}\,dx=\frac{2}{3}\,A,$$

and

$$E\{Y\}=\int_0^A\left\{\frac{2y}{A}-\frac{2y^2}{A^2}\right\}dy=\left[\frac{y^2}{A}-\frac{2}{3}\,\frac{y^3}{A^2}\right]_0^A=\frac{1}{3}\,A.$$

3) It follows from

$$E\left\{X^2\right\} = \int_0^A \frac{2x^3}{A^2}\,dx = \left[\frac{x^4}{2A^2}\right]_0^A = \frac{A^2}{2}$$

that

$$V\{X\} = E\left\{X^2\right\} - (E\{X\})^2 = \frac{A^2}{2} - \frac{4}{9}\,A^2 = \frac{A^2}{18}.$$

It follows from

$$E\left\{Y^2\right\} = \int_0^A \left\{\frac{2y^2}{A} - \frac{2y^3}{A^2}\right\} dy = \left[\frac{2y^3}{3A} - \frac{y^4}{2A^2}\right]_0^A = \left(\frac{2}{3} - \frac{1}{2}\right) A^2 = \frac{A^2}{6}$$

that

$$V\{Y\} = E\left\{Y^2\right\} - (E\{Y\})^2 = \frac{A^2}{6} - \frac{A^2}{9} = \frac{A^2}{18}.$$

4) First compute

$$\begin{aligned} E\{XY\} &= \int\!\!\int_{\mathbb{R}^2} xy\, f(x,y)\,dx dy = \frac{2}{A^2}\int_0^A \left\{\int_0^x yx\,dy\right\} dx = \frac{2}{A^2}\int_0^A \left[\frac{xy^2}{2}\right]_{y=0}^x dx \\ &= \frac{1}{A^2}\int_0^A x^3\,dx = \frac{A^2}{4}. \end{aligned}$$

Then by insertion,

$$\operatorname{Cov}(X,Y) = E\{XY\} - E\{X\}\cdot E\{Y\} = \frac{A^2}{4} - \frac{2}{3}\,A\cdot\frac{1}{3}\,A = \left(\frac{1}{4}-\frac{2}{9}\right)A^2 = \frac{A^2}{36}.$$

Finally, we obtain

$$\varrho(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sigma_X\sigma_y} = \frac{\frac{1}{36}\,A^2}{\frac{1}{18}\,A^2} = \frac{1}{2},$$

which is independent of A.

Example 3.5 *Consider a 2-dimensional random variable (X,Y), which in the parallelogram given by the inequalities*

$$0 \le x \le 1 \qquad \text{and} \qquad x \le y \le x+1$$

has the frequency

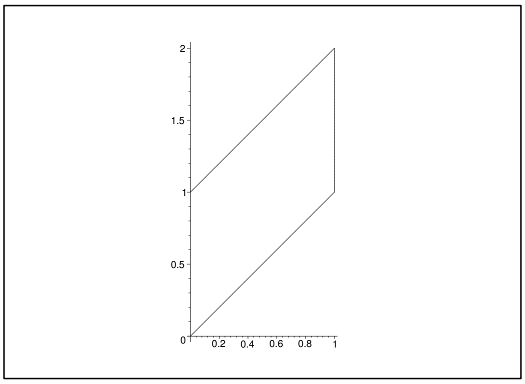
$$f(x,y) = \frac{2}{3}\,(x+y),$$

while the frequency is equal to 0 anywhere else in the (x,y) plane.

1) Find the frequencies of the de random variables X and Y.

2) Find the means of each of the random variables X and Y.

3) Find the covariance $\operatorname{Cov}(X,Y)$.



1) When $x \in\,]0,1[$, it follows by a vertical integration that

$$f_X(x) = \frac{2}{3}\int_x^{x+1}(x+y)\,dy = \frac{1}{3}\left[(x+y)^2\right]_{y=x}^{x+1} = \frac{1}{3}\left\{(2x+1)^2-(2x)^2\right\} = \frac{4}{3}\,x+\frac{1}{3},$$

thus

$$f_X(x) = \begin{cases} \dfrac{4}{3}\,x + \dfrac{1}{3}, & x \in \,]0,1[, \\ \\ \quad 0, & \text{otherwise}. \end{cases}$$

If $y \notin \,]0,2[$, then $f_Y(y) = 0$.
If $y \in \,]0,1[$, then by a horizontal integration,

$$f_Y(y) = \frac{2}{3}\int_0^y (x+y)dx = \frac{1}{3}\,\left[(x+y)^2\right]_{x=0}^{y} = \frac{1}{3}\left\{(2y)^2 - y^2\right\} = y^2.$$

If $y \in \,]1,2[$, it follows again by a horizontal integration that

$$\begin{aligned} f_Y(y) &= \frac{2}{3}\int_{y-1}^1 (x+y)\,dy = \frac{1}{3}\,\left[(x+y)^2\right]_{x=y-1}^{1} = \frac{1}{3}\left\{(y+1)^2 - (2y-1)^2\right\} \\ &= \frac{1}{3}\left\{y^2 + 2y + 1 - 4y^2 + 4y - 1\right\} = 2y - y^2, \end{aligned}$$

hence

$$f_Y(y) = \begin{cases} y^2, & y \in \,]0,1[, \\ \\ 2y - y^2 = 1 - (y-1)^2, & y \in \,]1,2[, \\ \\ 0, & \text{otherwise}. \end{cases}$$

2) The means are

$$E\{X\} = \int_0^1 x\left\{\frac{4}{3}\,x + \frac{1}{3}\right\}dx = \int_0^1 \left\{\frac{4}{3}\,x^2 + \frac{1}{3}\,x\right\}dx = \frac{4}{9} + \frac{1}{6} = \frac{11}{18},$$

and

$$\begin{aligned} E\{Y\} &= \int_0^1 y^3 dy = \int_1^2 \left\{2y^2 - y^3\right\}dy = \frac{1}{4} + \left[\frac{2}{3}\,y^3 - \frac{1}{4}\,y^4\right]_1^2 \\ &= \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4} = \frac{14}{3} + \frac{1}{2} - 4 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}. \end{aligned}$$

3) We first compute

$$\begin{aligned}
E\{XY\} &= \frac{2}{3}\int_0^1 \left\{\int_x^{x+1} xy(x+y)\,dy\right\}dx = \frac{2}{3}\int_0^1 \left\{\int_x^{x+1} \left(x^2y+xy^2\right)\,dy\right\}dx \\
&= \frac{2}{3}\int_0^1 \left[\frac{1}{2}\,x^2y^2+\frac{1}{3}\,xy^3\right]_{y=x}^{x+1} dx \\
&= \frac{2}{3}\int_0^1 \left[\frac{1}{2}\,x^2\left\{(x+1)^2-x^2\right\}+\frac{1}{3}\left\{(x+1)^3-x^3\right\}\right] dx \\
&= \frac{2}{3}\int_0^1 \left\{\frac{1}{2}\,x^2(2x+1)+\frac{1}{3}\,x\left(3x^2+3x+1\right)\right\} dx \\
&= \frac{2}{3}\int_0^1 \left\{x^3+\frac{1}{2}\,x^2+x^3+x^2+\frac{1}{3}\,x\right\} dx \\
&= \frac{2}{3}\int_0^1 \left\{2x^3+\frac{3}{2}\,x^2+\frac{1}{3}\,x\right\} dx = \frac{2}{3}\left\{\frac{1}{2}+\frac{1}{2}+\frac{1}{6}\right\} = \frac{2}{3}\cdot\frac{7}{6} = \frac{7}{9}.
\end{aligned}$$

Then by insertion,

$$\mathrm{Cov}(X,Y) = E\{XY\}-E\{X\}E\{Y\} = \frac{7}{9}-\frac{11}{18}\cdot\frac{7}{6} = \frac{7}{9}\cdot\left(1-\frac{11}{12}\right) = \frac{7}{108}.$$

Example 3.6 *Consider a 2-dimensional random variable (X,Y), which in the first quadrant has the frequency*

$$h(x,y) = \frac{a}{(1+x+y)^5},$$

while the frequency is equal to 0 anywhere else in the (x,y) plane.

1) Find the constant a.

2) Find the distribution function and the frequency of random variable $Z = X+Y$.

3) Find the mean $E\{Z\}$ and the variance $V\{Z\}$.

1) When we integrate over the first quadrant we obtain

$$\begin{aligned}
1 &= \int_0^\infty\int_0^\infty h(x,y)\,dx\,dy = a\int_0^\infty\int_0^\infty (1+x+y)^5\,dx\,dy \\
&= a\int_0^\infty \left[-\frac{1}{4}\,(1+x+y)^{-4}\right]_{x=0}^\infty dy = \frac{a}{4}\int_0^\infty (1+y)^{-4}\,dy = \frac{a}{12},
\end{aligned}$$

from which we conclude that $a = 12$. Hence the frequency is

$$h(x,y) = \begin{cases} \dfrac{12}{(1+x+y)^5} & \text{for } x>0 \text{ and } y>0, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

2) The frequency of $Z = X + Y$ for $z > 0$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} h(x, z-x)\, dx = \int_0^z h(x, z-x)\, dx = \int_0^z \frac{12}{(1+x+z-x)^5}\, dx = \frac{12z}{(1+z)^5},$$

i.e.

$$f_Z(z) = \begin{cases} \dfrac{12z}{(1+z)^5} & \text{for } z > 0, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

The distribution function is $F_Z(z) = 0$ for $z \le 0$.
If $z > 0$, then

$$\begin{aligned} F_Z(z) &= \int_0^z f_Z(t)\, dt = 12 \int_0^z \frac{t+1-1}{(t+1)^5}\, dt = \int_0^z \left\{12(t+1)^{-4} - 12(t+1)^{-5}\right\} dt \\ &= \left[-4(t+1)^{-3} + 3(t+1)^{-4}\right]_0^z = 1 - \frac{4}{(z+1)^3} + \frac{3}{(z+1)^4} = 1 - \frac{4z+1}{(z+1)^4}. \end{aligned}$$

Summing up we get

$$F_Z(z) = \begin{cases} 1 - \dfrac{4z+1}{(z+1)^4} & \text{for } z > 0, \\[2mm] 0 & \text{for } z \le 0. \end{cases}$$

3) The mean is

$$\begin{aligned}
E\{Z\} &= \int_0^\infty \frac{12z^2}{(z+1)^5}\,dz = 12\int_0^\infty \frac{z^2+2z+1-2z-2+1}{(z+1)^5}\,dz \\
&= \int_0^\infty \left\{12(z+1)^{-3} - 24(z+1)^{-4} + 12(z+1)^{-5}\right\}\,dz \\
&= \left[-6(z+1)^{-2} + 8(z+1)^{-3} - 3(z+1)^{-4}\right]_0^\infty = 6-8+3 = 1.
\end{aligned}$$

We get in the same way,

$$\begin{aligned}
E\left\{Z^2\right\} &= \int_0^\infty \frac{12z^3}{(z+1)^5} = 12\int_0^\infty \frac{(z^3+3z^2+3z+1)-(3z^2+6z+3)+(3+3z)-1}{(z+1)^5}\,dz \\
&= \int_0^\infty \left\{12(z+1)^{-2} - 36(z+1)^{-3} + 36(z+1)^{-4} - 12(z+1)^{-5}\right\}\,dz \\
&= \left[-12(z+1)^{-1} + 18(z+1)^{-2} - 12(z+1)^{-3} + 3(z+1)^{-4}\right]_0^\infty \\
&= 12-18+12-3 = 3.
\end{aligned}$$

Then finally,

$$V\{Z\} = E\left\{Z^2\right\} - (E\{Z\})^2 = 3-1 = 2.$$

Example 3.7 *A 2-dimensional random variable (X,Y) has the frequency*

$$h(x,y) = \begin{cases} \dfrac{1}{2}\,x^3\,e^{-x(y+1)} & \textit{for } x>0 \textit{ and } y>0, \\[2mm] 0 & \textit{otherwise.} \end{cases}$$

1) Find the frequencies of X and Y.

2) Find $\varrho(X,Y)$.

1) If $x>0$, then

$$f_X(x) = \frac{1}{2}\,x^3\int_0^\infty e^{-x(y+1)}\,dy = \frac{1}{2}\,x^2\,e^{-x},$$

and if $y>0$, then

$$f_Y(y) = \frac{1}{2}\int_0^\infty x^3\,e^{-x(y+1)}\,dx = \frac{1}{2}\cdot\frac{1}{(y+1)^4}\int_0^\infty t^3\,e^{-t}\,dt = \frac{3}{(y+1)^4},$$

hence, by summing up,

$$f_X(x) = \begin{cases} \dfrac{1}{2}\,x^2\,e^{-x} & \text{for } x>0, \\[2mm] 0 & \text{for } x\le 0, \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{3}{(y+1)^4} & \text{for } y > 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

2) Then we get

$$E\{X\} = \frac{1}{2}\int_0^{\infty} x^3\, e^{-x}\, dx = \frac{3!}{2} = 3,$$

and

$$E\left\{X^2\right\} = \frac{1}{2}\int_0^{\infty} x^4\, e^{-x}\, dx = \frac{4!}{2} = 12,$$

hence

$$V\{X\} = E\left\{X^2\right\} - (E\{X\})^2 = 12 - 3^2 = 3.$$

Analogously we obtain

$$E\{Y\} = 3\int_0^{\infty} \frac{y+1-1}{(y+1)^4}\, dy = 3\int_0^{\infty} \left\{\frac{1}{(y+1)^3} - \frac{1}{(y+1)^4}\right\} dy = 3\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{2},$$

and

$$\begin{aligned} E\left\{Y^2\right\} &= 3\int_0^{\infty} \frac{y^2+2y+1-2y-2+1}{(y+1)^4}\, dy \\ &= 3\int_0^{\infty} \left\{\frac{1}{(y+1)^2} - \frac{2}{(y+1)^3} + \frac{1}{(y+1)^4}\right\} dy = 3\left\{1 - 1 + \frac{1}{3}\right\} = 1, \end{aligned}$$

so the variance of Y is

$$V\{Y\} = E\left\{Y^2\right\} - (E\{Y\})^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

Finally,

$$\begin{aligned} E\{XY\} &= \int_0^{\infty} \left(\int_0^{\infty} \frac{1}{2}\, x^4\, y\, e^{-x(y+1)}\, dy\right) dx = \int_0^{\infty} \frac{1}{2}\, x^4 e^{-x} \left\{\int_0^{\infty} y\, e^{-xy}\, dy\right\} dx \\ &= \int_0^{\infty} \frac{1}{2}\, x^2 e^{-x}\, dx = 1. \end{aligned}$$

hence

$$\mathrm{Cov}(X,Y) = E\{XY\} - E\{X\}\cdot E\{Y\} = 1 - 3\cdot\frac{1}{2} = -\frac{1}{2},$$

and the correlation coefficient is

$$\varrho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V\{X\}\cdot V\{Y\}}} = \frac{-\frac{1}{2}}{\sqrt{3\cdot\frac{3}{4}}} = -\frac{1}{3}.$$

Example 3.8 *Let X_1 and X_2 be independent, identically distributed random variables of the frequency*

$$f(x)=\begin{cases}\dfrac{1}{\sqrt{2\pi x}}\,\exp\left(-\dfrac{x}{2}\right), & x>0,\\[2ex] 0, & x\le 0.\end{cases}$$

1) Find the frequency of $Y=\dfrac{X_1}{X_2}$.

2) Check if $E\{Y\}$ exists, and if so, find $E\{Y\}$.

1) Let $f_Y(y)$ be the frequency of $Y=\dfrac{X_1}{X_2}$. Then

$$f_Y(y)=\int_{-\infty}^{\infty} f(yx)\,f(x)\,|x|\,dx.$$

Clearly, $f_Y(y)=0$ for $y\le 0$.

If $y>0$, then

$$\begin{aligned}f_Y(y) &= \int_0^{\infty}\frac{1}{\sqrt{2\pi yx}}\,\exp\left(-\frac{yx}{2}\right)\cdot\frac{1}{\sqrt{2\pi x}}\,\exp\left(-\frac{x}{2}\right)|x|\,dx\\ &= \frac{1}{2\pi\sqrt{y}}\int_0^{\infty}\exp\left(-\frac{y+1}{2}\,x\right)dx=\frac{1}{2\pi\sqrt{y}}\cdot\frac{2}{y+1}=\frac{1}{\pi}\cdot\frac{1}{y+1}\cdot\frac{1}{\sqrt{y}},\end{aligned}$$

hence

$$f_Y(y)=\begin{cases}\dfrac{1}{\pi}\cdot\dfrac{1}{y+1}\cdot\dfrac{1}{\sqrt{y}} & \text{for } y>0,\\[2ex] 0 & \text{for } y\le 0.\end{cases}$$

2) Since $f_Y(y)\ne 0$ is equivalent to $y>0$ and $f_Y(y)>0$, the integrand satisfies $y\,f_Y(y)\ge 0$, hence the check of the existence is reduced to check the convergence for $A\to\infty$ of

$$\begin{aligned}\int_0^A y\,f_Y/y)\,dy &= \frac{1}{\pi}\int_0^A\frac{y}{y+1}\cdot\frac{1}{\sqrt{y}}\,dy=\frac{1}{\pi}\int_0^A\frac{y+1-1}{y+1}\cdot\frac{1}{\sqrt{y}}\,dy\\ &= \frac{1}{\pi}\int_0^A\frac{1}{\sqrt{y}}\,dy-\frac{1}{\pi}\int_0^A\frac{1}{y+1}\cdot\frac{1}{\sqrt{y}}\,dy\\ &= \frac{1}{\pi}\,[2\sqrt{y}]_0^A-\frac{2}{\pi}\,[\text{Arctan}\,\sqrt{y}]_0^A\\ &= \frac{2}{\pi}\,\sqrt{A}-\frac{2}{\pi}\,\text{Arctan}\,\sqrt{A}.\end{aligned}$$

Since $-\dfrac{2}{\pi}\,\text{Arctan}\,\sqrt{A}\to-\dfrac{2}{\pi}\cdot\dfrac{\pi}{2}=-1$ and $\dfrac{2}{\pi}\,\sqrt{A}\to\infty$ for $A\to\infty$, we conclude that $E\{Y\}$ does *not* exist.

Alternatively, it follows that the integrand $\dfrac{y}{y+1}\cdot\dfrac{1}{\sqrt{y}}\sim\dfrac{1}{\sqrt{y}}$, and since $\int_0^{\infty}\dfrac{1}{\sqrt{y}}\,dy$ is divergent, $\int_0^{\infty}\dfrac{y}{y+1}\,\dfrac{1}{\sqrt{y}}\,dy$ is also divergent, and the mean $E\{Y\}$ does not exist. $\Diamond$

Example 3.9 *A 2-dimensional random variable (X, Y) has in the first quadrant the frequency*

$$h(x, y) = \frac{1}{2}\,(x+y)\,e^{-(x+y)},$$

while the frequency is 0 everywhere else in the (x, y) plane.

1) Find the frequencies of X and Y.

2) Find frequency of $Z = X + Y$.

3) Find the mean and the variance of the random variable Z.

4) Find the correlation coefficient $\varrho(X, Y)$.

1) If $x > 0$, then

$$\begin{aligned} f_X(x) &= \frac{1}{2}\int_0^{\infty}(x+y)\,e^{-(x+y)}\,dy = \frac{1}{2}\,x\,e^{-x}\int_0^{\infty} e^{-y}\,dy + \frac{1}{2}\int_0^{\infty} y\,e^{-y}\,dy \\ &= \frac{1}{2}\,x\,e^{-x} + \frac{1}{2}\,e^{-x} = \frac{1}{2}\,(x+1)e^{-x}. \end{aligned}$$

By the symmetry,

$$f_X(x) = \begin{cases} \dfrac{1}{2}\,(x+1)\,e^{-x}, & x > 0, \\ \\ 0, & x \le 0, \end{cases}$$

and

$$f_Y(x) = \begin{cases} \dfrac{1}{2}\,(y+1)\,e^{-y}, & y > 0, \\ \\ 0, & y \le 0, \end{cases}$$

2) If $z > 0$, then $Z = X + Y$ has the frequency

$$f_Z(z) = \int_0^z h(x, z-x)\,dx = \int_0^z \frac{1}{2}\,z\,e^{-z}\,dx = \frac{1}{2}\,z^2\,e^{-z},$$

and if $z \le 0$, the frequency is 0, thus

$$f_Z(z) = \begin{cases} \frac{1}{2}\,z^2 e^{-z} & \text{for } z > 0, \\ 0 & \text{for } z \le 0. \end{cases}$$

3) We get

$$E\{Z\} = \int_0^\infty \frac{1}{2}\,z^3\,e^{-z}\,dz = 3,$$

$$E\left\{Z^2\right\} = \int_0^\infty \frac{1}{2}\,z^4\,e^{-z}\,dz = 12,$$

and

$$V\{Z\} = 12 - 3^2 = 3.$$

4) First notice that

$$E\{X\} = E\{Y\} = \frac{1}{2}\,(E\{X\} + E\{Y\}) = \frac{1}{2}\,E\{Z\} = \frac{3}{2}.$$

Then

$$E\left\{X^2\right\} = E\left\{Y^2\right\} = \frac{1}{2}\int_0^\infty \left(t^3 e^{-t} + t^2 e^{-t}\right)\,dt = \frac{1}{2}\,(3! + 2!) = 4,$$

hence

$$V\{X\} = V\{Y\} = E\left\{X^2\right\} - (E\{X\})^2 = 4 - \frac{9}{4} = \frac{7}{4}.$$

We finally compute

$$\begin{aligned} E\{XY\} &= \frac{1}{2}\int_0^\infty \int_0^\infty xy(x+y)\,e^{-(x+y)}\,dx\,dy \\ &= \frac{1}{2}\int_0^\infty \left\{ ye^{-y} \int_0^\infty x^2\,e^{-x}\,dx + y^2 e^{-y} \int_0^\infty x\,e^{-x}\,dx \right\} dy \\ &= \frac{1}{2}\int_0^\infty \left\{2!\,y\,e^{-y} + 1!\,y^2 e^{-y}\right\} dy = \frac{1}{2}\,(2 \cdot 1! + 1 \cdot 2!) = 2, \end{aligned}$$

thus

$$\operatorname{Cov}(X,Y) = E\{XY\} - E\{X\}E\{Y\} = 2 - \frac{3}{2}\cdot\frac{3}{2} = 2 - \frac{9}{4} = -\frac{1}{4},$$

and

$$\varrho(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{V\{X\}V\{Y\}}} = \frac{-\frac{1}{4}}{\frac{7}{4}} = -\frac{1}{7}.$$

Alternatively, it follows from

$$V\{Z\} = V\{X\} + V\{Y\} + 2\operatorname{Cov}(X,Y),$$

that

$$\operatorname{Cov}(X,Y) = -\frac{1}{4},$$

and hence

$$\varrho(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{V\{X\}V\{Y\}}} = \frac{-1/4}{7/4} = -\frac{1}{7}.$$

Example 3.10 *A compound experiment can be described by first choosing at random a real number X in the interval $]0,1[$, and then at random to choose a real number Y in the interval $]X,1[$. The frequency of the 2-dimensional random variable (X,Y) is denoted by $h(x,y)$.*

1) *Prove that $h(x,y)$ is 0 outside the triangle in the (x,y) plane of the vertices $(0,0)$, $(0,1)$ and $(1,1)$, and that $h(x,y)$ inside the mentioned triangle above is given by*

$$h(x,y) = \frac{1}{1-x}.$$

2) *Find the frequencies $f(x)$ and $g(y)$ of the random variables X and Y.*

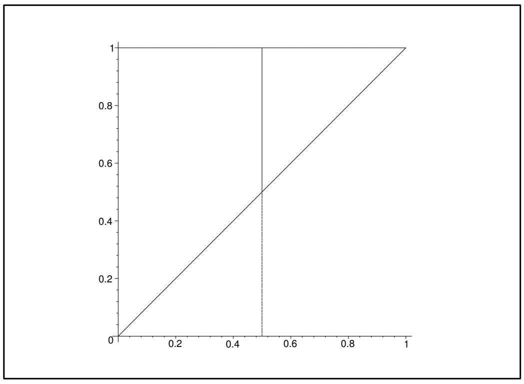
3) *Find the mean and variance of the random variables X and Y.*

1) We see that

$$f_X(x) = \begin{cases} 1 & \text{for } x \in\,]0,1[, \\ 0 & \text{otherwise.} \end{cases}$$

If we keep $x \in\,]0,1[$ fixed, then

$$f(y \mid x) = \begin{cases} \dfrac{1}{1-x} & \text{for } y \in\,]x,1[, \\ 0 & \text{otherwise.} \end{cases}$$

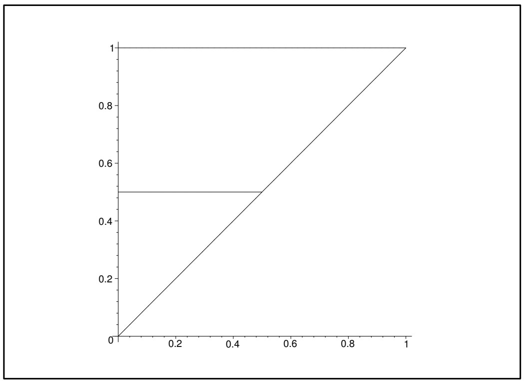


Hence, if $x \in]0,1[$, then

$$f(y \mid x) = \frac{h(x,y)}{f_X(x)} = h(x,y),$$

and we have proved that

$$h(x,y) = \begin{cases} \dfrac{1}{1-x} & \text{for } 0 < x < y < 1, \\[2mm] 0 & \text{otherwise.} \end{cases}$$



2) Obviously,

$$f(x) = f_X(x) = \begin{cases} 1 & \text{for } x \in]0,1[, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

If $y \in]0,1[$, then

$$g(y) = \int_0^y h(x,y)\,dx = \int_0^y \frac{dx}{1-x} = [-\ln|1-x|]_0^y = \ln\frac{1}{1-y},$$

hence

$$g(y)=\begin{cases}\ln\dfrac{1}{1-y}=-\ln(1-y) & \text{for } y\in\,]0,1[,\\[2mm] 0 & \text{otherwise.}\end{cases}$$

3) Clearly,

$$E\{X\}=\frac{1}{2}\qquad\text{and}\qquad E\left\{X^2\right\}=\int_0^1 x^2\,dx=\frac{1}{3},$$

so

$$V\{X\}=E\left\{X^2\right\}-(E\{X\})^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}.$$

One may of course instead notice that X is rectangularly distributed, so

$$E\{X\}=\frac{1}{2}\qquad\text{and}\qquad V\{X\}=\frac{1}{12}.$$

Then turn to Y. We get by the change of variable $t = 1 - y$,

$$\begin{aligned}
E\{Y\} &= \int_0^1 y\,\{-\ln(1-y)\}\,dy = -\int_0^1 (1-t)\ln t\,dt = -\int_0^1 \ln t\,dt + \int_0^1 t\,\ln t\,dt \\
&= -[t\,\ln t - t]_0^1 + \left\{\frac{t^2}{2}\,\ln t\right]_0^1 - \frac{1}{2}\int_0^1 t\,dt = -\{0-1\} + 0 - \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4},
\end{aligned}$$

and

$$\begin{aligned}
E\left\{Y^2\right\} &= \int_0^1 y^2\{-\ln(1-y)\}\,dy = -\int_0^1 (1-t)^2\ln t\,dt \\
&= -\int_0^1 \ln t\,dt + 2\int_0^1 t\,\ln t\,dt - \int_0^1 t^2\ln t\,dt \\
&= -[t\,\ln t - t]_0^1 + 2\left[\frac{t^2}{2}\,\ln t\right]_0^1 - \int_0^1 t\,dt - \left[\frac{t^3}{3}\,\ln t\right]_0^1 + \frac{1}{3}\int_0^1 t^2\,dt \\
&= 1 + 2\cdot 0 - \frac{1}{2} - 0 + \frac{1}{9} = \frac{1}{2} + \frac{1}{9} = \frac{11}{18}.
\end{aligned}$$

Alternatively, perform the computations

$$E\{Y\} = \int_{x=0}^1 \left\{\int_{y=x}^1 y\cdot\frac{1}{1-x}\,dy\right\}dx = \frac{1}{2}\int_0^1 \frac{1-x^2}{1-x}\,dx = \int_0^1 \frac{1}{2}\,(1+x)\,dx = \frac{3}{4},$$

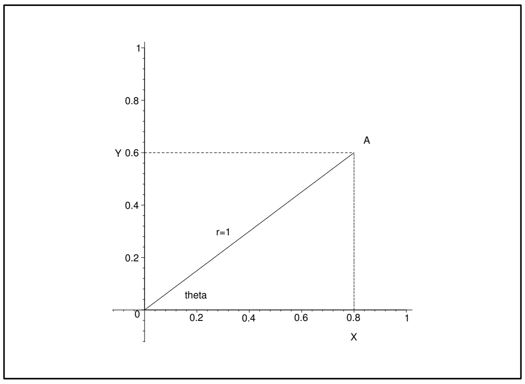
and

$$\begin{aligned}
E\left\{Y^2\right\} &= \int_{x=0}^1 \left\{\int_{y=x}^1 y^2\cdot\frac{1}{1-x}\,dy\right\}dx = \frac{1}{3}\int_0^1 \frac{1-x^3}{1-x}\,dx \\
&= \frac{1}{2}\int_0^1 \left\{1+x+x^2\right\}\,dx = \frac{11}{18}.
\end{aligned}$$

This gives us the variance,

$$\begin{aligned}
V\{Y\} &= E\left\{Y^2\right\} - (E\{Y\})^2 = \frac{11}{18} - \frac{9}{16} \\
&= \frac{1}{2}\left(\frac{11}{9} - \frac{9}{8}\right) = \frac{1}{2}\left(\frac{2}{9} - \frac{1}{8}\right) = \frac{16-9}{144} = \frac{7}{144}.
\end{aligned}$$

Example 3.11 *The point A is in the (x,y) plane given by its polar coordinates $r = OA = 1$ and $\angle(x, OA) = \Theta$. The projections of A onto the two coordinate axes are called X and Y.*



We assume that Θ is a rectangularly distributed random variable over the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

1) Find the distribution functions and the frequencies of the two random variables X and Y.

2) Find the means $E\{X\}$ and $E\{Y\}$.

3) Find the variances $V\{X\}$ and $V\{Y\}$.

4) Explain that the random variables X and Y are non-correlated, though not independent of each other.

The frequency of Θ is

$$f(\theta) = \begin{cases} \dfrac{1}{\pi} & \text{for } x \in \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $X = \cos\Theta$ and $Y = \sin\Theta$.

1) Since $\cos\theta > 0$ for $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, where $\cos\theta$ is *not* monotonous, we get for $x \in]0,1[$,

$$\begin{aligned} F_X(x) &= P\{X \le x\} = P\{\cos\theta \le x\} = P\{\text{Arccos } x \le \theta \le \pi - \text{ Arccos } x\} \\ &= \frac{1}{\pi}\int_{\text{Arccos } x}^{\pi - \text{ Arccos } x} d\theta = 1 - \frac{2}{\pi}\text{Arccos } x, \end{aligned}$$

hence

$$F_X(x) = \begin{cases} 1, & x \ge 1, \\ 1 - \dfrac{2}{\pi}\text{Arccos } x, & 0 < x < 1, \\ 0, & x \le 0, \end{cases}$$

and

$$f_X(x) = \begin{cases} \dfrac{2}{\pi} \dfrac{1}{\sqrt{1-x^2}}, & x \in]0,1[, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Analogously, we get for $y \in]-1,1[$,

$$\begin{aligned} F_Y(y) &= P\{Y \le y\} = P\{\sin\theta \le y\} = P\{\theta \le \text{ Arcsin } y\} \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\text{Arcsin } y} d\theta = \frac{1}{2} + \frac{1}{\pi} \text{Arcsin } y, \end{aligned}$$

hence

$$F_Y(y) = \begin{cases} 1, & y \ge 1, \\ \\ \dfrac{1}{2} + \dfrac{1}{\pi} \text{Arcsin } y, & -1 < y < 1, \\ \\ 0, & y \le -1, \end{cases}$$

and

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi} \dfrac{1}{\sqrt{1-y^2}}, & y \in]-1,1[, \\ \\ 0, & \text{otherwise.} \end{cases}$$

2) The means are

$$E\{X\} = \frac{2}{\pi} \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx = \frac{2}{\pi} \left[-\sqrt{1-x^2} \right]_0^1 = \frac{2}{\pi},$$

and

$$E\{Y\} = \frac{1}{\pi} = \frac{1}{\pi} \int_{-1}^1 \frac{y}{\sqrt{1-y^2}}\, dy = 0.$$

3) We get by the substitution $x = \sin t$,

$$\begin{aligned} E\left\{X^2\right\} &= \frac{2}{\pi} \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}} \cdot \cos t\, dt \\ &= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left\{\sin^2 t + \cos^2 t\right\} dt = \frac{1}{2}. \end{aligned}$$

Furthermore,

$$E\left\{Y^2\right\} = \frac{1}{\pi} \int_{-1}^1 \frac{y^2}{\sqrt{1-y^2}}\, dy = \frac{2}{\pi} \int_0^1 \frac{y^2}{\sqrt{1-y^2}}\, dy = E\left\{X^2\right\} = \frac{1}{2}.$$

The variances are

$$V\{X\} = E\left\{X^2\right\} - (E\{X\})^2 = \frac{1}{2} - \left(\frac{2}{\pi}\right)^2 = \frac{1}{2} - \frac{4}{\pi^2} \quad (\approx 0{,}095),$$

and

$$V\{Y\} = E\left\{Y^2\right\} - (E\{Y\})^2 = \frac{1}{2}.$$

4) Since $X^2 + Y^2 = 1$, it is obvious that X and Y are not independent.

Let $f(x, y)$ be the frequency of $Z = (X, Y)$. Then

$$f(x, y) = f(x \mid y) \cdot f_Y(y) = f(x \mid y) \cdot \frac{1}{\pi} \cdot \frac{1}{\sqrt{1-y^2}} \qquad \text{for } y \in]-1, 1[,$$

where

$$f(x \mid y) = \begin{cases} 1 & \text{for } x = \sqrt{1-y^2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E\{XY\} = \int_{-1}^{1} \sqrt{1-y^2} \cdot y \cdot \frac{1}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}\, dy = \frac{1}{\pi} \int_{-1}^{1} y\, dy = 0,$$

Thus

$$\text{Cov}(X,Y) = E\{XY\} - E\{X\} \cdot E\{Y\} = 0 - \frac{2}{\pi} \cdot 0 = 0,$$

so X and Y are non-correlated.

Example 3.12 *A 2-dimensional random variable (X,Y) has the frequency*

$$h(x,y) = \begin{cases} 2a^2\, e^{-a(x+y)}, & 0 < x < y, \\ 0, & \textit{otherwise}, \end{cases}$$

where a is a positive constant.

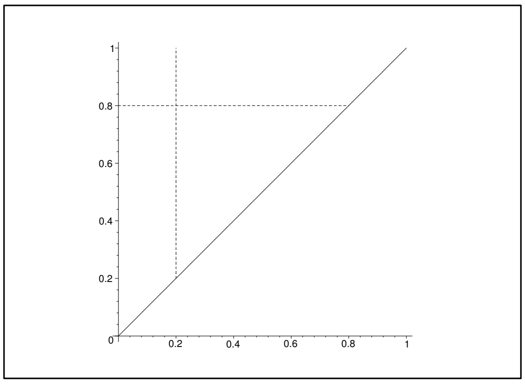
1) Find the frequencies of the random variables X and Y.

2) Find the means $E\{X\}$ and $E\{Y\}$.

3) Find $\text{Cov}(X,Y)$.

4) Find the frequency of $Z = X + Y$.

5) Find the mean $E\{Z\}$ and the variance $V\{Z\}$.



1) When $x > 0$, we get by a vertical integration,

$$f_X(x) = \int_x^{\infty} 2a^2\, e^{-a(x+y)}\, dy = 2a\, e^{-ax} \left[-e^{-ay}\right]_x^{\infty} = 2a\, e^{-2ax},$$

hence

$$f_X(x) = \begin{cases} 2a\, e^{-2ax} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

When $y>0$, we get by a horizontal integration,

$$f_Y(y)=\int_0^y 2a^2\,e^{-a(x+y)}\,dx=2a\,e^{-ay}\,\left[-e^{-ax}\right]_0^y=2a\,e^{-ay}-2a\,e^{-2ay},$$

hence

$$f_Y(y)=\begin{cases}2a\,e^{-ay}-2a\,e^{-2ay} & \text{for } y>0,\\[2ex] 0 & \text{otherwise.}\end{cases}$$

2) The means are given by

$$E\{X\}=\int_0^\infty 2a\,x\,e^{-2ax}\,dx=\frac{1}{2a}\int_0^\infty t\,e^{-t}\,dt=\frac{1}{2a},$$

and

$$\begin{aligned}E\{Y\} &= \int_0^\infty 2a\,y\,e^{-ay}\,dy-\int_0^\infty 2a\,y\,e^{-2ay}\,dy=\frac{2}{a}\int_0^\infty t\,e^{-t}\,dt-\frac{1}{2a}\int_0^\infty t\,e^{-t}\,dt\\ &= \frac{2}{a}-\frac{1}{2a}=\frac{3}{2a}.\end{aligned}$$

3) Then we compute

$$\begin{aligned}E\{XY\} &= \int_0^\infty\left\{\int_0^y \mathbf{x}y\cdot 2a^2e^{-a(x+y)}\,\mathbf{dx}\right\}dy=\int_0^\infty 2y\,e^{-ay}\left\{\int_0^y a\,\mathbf{x}\,e^{-a\mathbf{x}}\,a\,\mathbf{dx}\right\}dy\\ &= \int_0^\infty 2y\,e^{-ay}\left\{\int_0^{ay} t\,e^{-t}\,dt\right\}dt=\int_0^\infty 2y\,e^{-ay}\,\left[-t\,e^{-t}-e^{-t}\right]_0^{ay}dy\\ &= \int_0^\infty 2y\,e^{-ay}\left\{1-ay\,e^{-ay}-e^{-ay}\right\}dy\\ &= \int_0^\infty 2y\,e^{-ay}\,dy-\int_0^\infty 2a\,y^2\,e^{-2ay}\,dy-\int_0^\infty 2y\,e^{-2ay}\,dy\\ &= \frac{2}{a^2}\int_0^\infty t\,e^{-t}\,dt-\frac{1}{4a^2}\int_0^\infty t^2e^{-t}\,dt-\frac{1}{2a^2}\int_0^\infty t\,e^{-t}\,dt\\ &= \frac{2}{a^2}-\frac{1}{2a^2}-\frac{1}{2a^2}=\frac{1}{a^2}.\end{aligned}$$

It follows that

$$\operatorname{Cov}(X,Y)=E\{XY\}-E\{X\}\cdot E\{Y\}=\frac{1}{a^2}-\frac{1}{2a}\cdot\frac{3}{2a}=\frac{1}{4a^2}.$$

4) Clearly, $f_Z(z)=0$ for $z\le 0$. Når $z>0$, so

$$f_Z(z)=\int_{-\infty}^\infty h(x,z-x)\,dx=\int_0^\infty h(x,z-x)\,dx.$$

The integrand is only $\ne 0$, if $x<y=z-x$, i.e. when $x<\dfrac{1}{2}\,z$, hence

$$f_Z(z)=\int_0^{\frac{z}{2}} g(x,z-x)\,dx=2a^2e^{-az}\int_0^{\frac{z}{2}}dz=a^2z\,e^{-az},$$

and thus

$$f_Z(z) = \begin{cases} a^2 z\, e^{-az} & \text{for } z > 0, \\ 0 & \text{otherwise.} \end{cases}$$

5) The mean is

$$E\{Z\})E\{X\} + E\{Y\} = \frac{1}{2a} + \frac{3}{2a} = \frac{2}{a},$$

or ALTERNATIVELY and more elaborated,

$$E\{Z\} = \int_0^\infty a^2 z^2 e^{-az}\, dz = \frac{1}{a}\int_0^\infty t^2 e^{-t}\, dt = \frac{2}{a}.$$

Furthermore,

$$E\left\{Z^2\right\} = \int_0^\infty a^2 z^3 e^{-az}\, dz = \frac{1}{a^2}\int_0^\infty t^3 e^{-t}\, dt = \frac{6}{a^2},$$

hence

$$V\{Z\} = E\left\{Z^2\right\} - (E\{Z\})^2 = \frac{6}{a^2} - 4a^2 = \frac{2}{a^2}.$$

Example 3.13 *A 2-dimensional random variable* (X, Y) *has the frequency* $h(x, y) = 1$ *inside the triangle in the* (x, y) *plane of vertices at the points* $(0, 0)$, $(0, 2)$ *and* $(1, 1)$, *while the frequency is 0 anywhere else outside this triangle.*

1) Find the frequencies of the random variables X *and* Y.

2) Prove that X *and* Y *are non-correlated, though not independent.*

3) Find the distribution function and the frequency for each of the random variables $Z = X + Y$ *and* $V = X - Y$.

1) If $x \in\,]0, 1[$, then

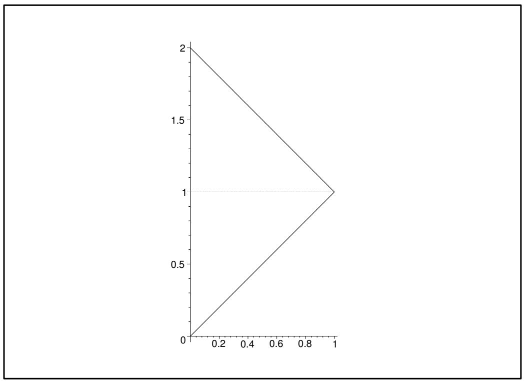
$$f_X(x) = \int_x^{2-x} dy = 2 - 2x,$$

hence

$$f_X(x) = \begin{cases} 2 - 2x & \text{for } x \in\,]0, 1[, \\ 0 & \text{otherwise.} \end{cases}$$

If $y \in\,]0, 1]$, then

$$f_Y(y) = \int_0^y dx = y.$$



If $y \in]1,2[$, then

$$f_Y(y) = \int_0^{2-y} dx = 2 - y.$$

Summing up,

$$f_Y(y) = \begin{cases} y & \text{for } y \in]0,1], \\ 2-y & \text{for } y \in]1,2[, \\ 0 & \text{otherwise.} \end{cases}$$

2) It follows by considering a figure that $E\{Y\}=1$. Furthermore,

$$E\{X\}=\int_0^1\left(2x-2x^2\right)\,dx=1-\frac{2}{3}=\frac{1}{3}.$$

Then by a double integration, where we start in the inner integral to integrate vertically after y),

$$\begin{aligned}E\{XY\} &= \int_0^1\left\{\int_x^{2-x} xy\,dy\right\}dx=\int_0^1 x\left[\frac{y^2}{2}\right]_x^{2-x}dx\\ &= \frac{1}{2}\int_0^1 x(4-4x)\,dx=\int_0^1\left(2x-2x^2\right)\,dx=\frac{1}{3}.\end{aligned}$$

Since

$$\operatorname{Cov}(X,Y)=E\{XY\}-E\{X\}E\{Y\}=\frac{1}{3}-\frac{1}{3}\cdot 1=0,$$

it follows that X and Y are non-correlated.
Since $f_X(x)\cdot f_Y(y)\neq 0$ in the square $]0,1[\times]0,2[$, we see that $f_X(x)\cdot f_Y(y)$ cannot be equal to $h(x,y)$. [This can of course also be seen directly.] Hence, X and Y are not independent.

3) The frequency of $Z=X+Y$ is

$$f_Z(z)=\int_0^1 h(x,z-x)\,dx.$$

The integrand is $\neq 0$, when $y=z-x\in]x,2-x[$, e.g. $2x<z<2$, hence

$$f_Z(z)=\int_0^{\frac{z}{2}} h(x,z-x)\,dx=\int_0^{\frac{z}{2}}dx=\frac{z}{2},$$

and we find the frequency

$$f_Z(z)=\begin{cases}\dfrac{z}{2} & \text{for } z\in]0,2[,\\[2mm] 0 & \text{otherwise,}\end{cases}$$

and the distribution function

$$F_Z(z)=\begin{cases}0 & \text{for } z\leq 0,\\[2mm] \dfrac{z^2}{4} & \text{for } z\in]0,2[,\\[2mm] 1 & \text{for } z\geq 2.\end{cases}$$

Then we note that $X=X-Y$ has values in $]-2,0[$. If $v\in]-2,0[$, then

$$F_V(v)=P\{X-Y\leq v\}=\int_{\{x-y\leq v\}} h(x,y)\,dx\,dy=\int_0^2\left\{\int_0^{v+y} h(x,y)\,dx\right\}dy.$$

We get by a differentiation,

$$f_V(v)=F_V'(v)=\int_0^2 h(v+y,y)\,dy.$$

The integrand is $\neq 0$ for

$$0 < v + y < 1 \qquad \text{and} \qquad v + y < y < 2 - v - y,$$

hence

$$0 < -v < y < 1 - \frac{v}{2} < 2.$$

If $v \in]-2,0[$, then

$$f_V(v) = \int_{-v}^{1-\frac{v}{2}} dy = 1 - \frac{v}{2} + v = 1 + \frac{v}{2},$$

thus the frequency of V is

$$f_V(v) = \begin{cases} 1 + \dfrac{v}{2} & \text{for } v \in]-2,0[, \\[2mm] 0 & \text{otherwise,} \end{cases}$$

and the corresponding distribution function is

$$F_V(v) = \begin{cases} 0, & \text{for } v \le -2, \\ \left(1 + \dfrac{v}{2}\right)^2, & \text{for } v \in]-2,0[, \\[2mm] 1, & \text{for } v \ge 0. \end{cases}$$

Example 3.14 *Given the functions*

$$f(x)=\begin{cases}12x^2(1-x), & 0<x<1,\\[2mm] 0, & \textit{otherwise,}\end{cases}\qquad g(y)=\begin{cases}12y(1-y)^2, & 0<y<1,\\[2mm] 0, & \textit{otherwise.}\end{cases}$$

1. *Prove that f and g are frequencies.*

In the remaining part of the example we let X and Y denote random variables, where X has the frequency $f(x)$, and Y has the frequency $g(y)$.

2. *Find the mean and variance of X.*

3. *Prove that Y has the same distribution as $1-X$.*

4. *Find the mean and the variance of Y.*

5. *Prove that $X+Y$ and $X-Y$ are non-correlated.*

6. *We now assume that X and Y are independent. Explain why the two probabilities*

$$P\left\{X+Y>\frac{1}{2}\right\}\qquad \textit{and}\qquad P\left\{X-Y>\frac{1}{2}\right\}$$

are positive (one shall not compute the probabilities). Check, e.g. by applying this result, if $X+Y$ and $X-Y$ are independent.

7. *Here we assume that $\mathrm{Cov}(X,Y)=-\dfrac{1}{25}$. Prove that Y is then a function of X, and find this function.*
HINT: *Compute e.g. the variance of $X+Y$.*

1) It is obvious that $f(x)\geq 0$ for every $x\in\mathbb{R}$. Since furthermore

$$\int_0^1 12x^2(1-x)\,dx=12\int_0^1\left(x^2-x^3\right)\,dx=12\left(\frac{1}{3}-\frac{1}{4}\right)=1,$$

it follows that $f(x)$ is a frequency.
Since $g(y)=f(1-y)$ and $\left|\dfrac{dx}{dy}\right|=|-1|=1$, it follows that $g(y)$ is also a frequency.

2) The mean of X is

$$E\{X\}=12\int_0^1\left(x^3-x^4\right)\,dx=12\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{12}{20}=\frac{3}{5}.$$

Since furthermore,

$$E\left\{X^2\right\}=12\int_0^1\left(x^4-x^5\right)\,dx=12\left(\frac{1}{5}-\frac{1}{6}\right)=\frac{12}{30}=\frac{2}{5},$$

the variance is

$$V\{X\}=\frac{2}{5}-\left(\frac{3}{5}\right)^2=\frac{10-9}{25}=\frac{1}{25}.$$

3) The frequency of $\varphi(X) = 1 - X$ is

$$f(1-x) \cdot \left|\frac{d(1-x)}{dx}\right| = f(1-x) = \begin{cases} 12x(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

This is precisely the structure of the frequency of Y, with x instead of y, thus Y and $1-X$ have the same distribution.

4) It follows from **(3)** that

$$E\{Y\} = E\{1-X\} = 1 - E\{X\} = 1 - \frac{3}{5} = \frac{2}{5}$$

and

$$V\{Y\} = V\{1-X\} = V\{1\} + V\{X\} = 0 + \frac{1}{25} = \frac{1}{25} = V\{X\}.$$

5) It follows from the definition,

$$\text{Cov}(X+Y, X-Y) = V\{X\} - V\{Y\} + \text{ Cov}(Y,X) - \text{ Cov}(X,Y) = \frac{1}{25} - \frac{1}{25} = 0,$$

hence $X+Y$ and $X-Y$ are non-correlated.

6) It is obvious that X and Y both have their values in $]0,1[$ with a positive probability for every open, non-empty subinterval of $]0,1[$. Then both

$$\left\{X + Y > \frac{3}{2}\right\} \qquad \text{and} \qquad \left\{X - Y > \frac{1}{2}\right\}$$

have a positive probability. Since $2X = (X+Y) + (X-Y)$, we get

$$\{X > 1\} = \{2X > 2\} \supseteqq \left\{X + Y > \frac{3}{2}\right\} \cap \left\{X - Y > \frac{1}{2}\right\}.$$

Since

$$P\{X > 1\} = 0, \qquad P\left\{X + Y > \frac{3}{2}\right\} > 0, \qquad P\left\{X - Y > \frac{1}{2}\right\} > 0,$$

we get

$$\begin{aligned} 0 &= P\{X > 1\} = P\left(\left\{X + Y > \frac{3}{2}\right\} \cap \left\{X - Y > \frac{1}{2}\right\}\right) \\ &\neq P\left\{X + Y > \frac{3}{2}\right\} \cdot P\left\{X - Y > \frac{1}{2}\right\}, \end{aligned}$$

proving that $X+Y$ and $X-Y$ are not independent.

7) Since

$$V\{X+Y\} = V\{X\} + V\{Y\} - 2\,\text{Cov}(X,Y) = \frac{1}{25} + \frac{1}{25} - \frac{2}{25} = 0,$$

it follows that $X+Y$ is causal, so $X + Y = X + (1-X) = 1 = a$ with the only possibility

$Y = 1 - X$.

Example 3.15 *A rectangular triangle has the two smaller sides X_1 and X_2, where X_1 and X_2 are independent random variables of the frequencies*

$$f_{X_1}(x_1) = \begin{cases} 1, & 0 < x_1 < 1, \\ 0, & \textit{otherwise}, \end{cases} \qquad f_{X_2}(x_2) = \begin{cases} \frac{1}{2}, & 0 < x_2 < 2, \\ 0, & \textit{otherwise}. \end{cases}$$

Let $Y_1 = X_1 + X_2$ denote the sum of the lengths of the two smaller sides and let $Y_2 = \dfrac{1}{2} X_1 X_2$ denote the area of the triangle.

1) Compute the mean and the variance of Y_1.

2) Compute the mean and variance of Y_2.

3) Prove that

$$\operatorname{Cov}(X_1 + X_2, X_1 X_2) = E\{X_1\} V\{X_2\} + E\{X_2\} C\{X_1\},$$

and then compute $\operatorname{Cov}(Y_1, Y_2)$.

4) Find the frequency of Y_1.

1) The mean of $Y_1 = X_1 + X_2$ is

$$E\{Y_1\} = E\{X_1\} + E\{X_2\} \frac{1}{2} + 1 = \frac{3}{2}.$$

Since X_1 and X_2 are independent, the variance is

$$V\{Y_1\} = V\{X_1\} + V\{X_2\} = \frac{1}{12}\left\{1^2 + 2^2\right\} = \frac{5}{12}.$$

2) Since X_1 and X_2 are independent, we find that

$$E\{Y_2\} = \frac{1}{2} E\{X_1\} \cdot E\{X_2\} = \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4},$$

and

$$\begin{aligned} V\{Y_2\} &= \frac{1}{4} V\{X_1 X_2\} = \frac{1}{4}\left(E\left\{X_1^2 X_2^2\right\} - \left(E\{X_1\} E\{X_2\}\right)^2\right) \\ &= \frac{1}{4}\left(E\left\{X_1^2\right\} E\left\{X_2^2\right\} - \left(E\{X_1\} E\{X_2\}\right)^2\right) = \frac{1}{4}\left\{\frac{1}{3} \cdot \frac{4}{3} - \left(\frac{1}{2}\right)^2\right\} = \frac{1}{4}\left\{\frac{4}{9} - \frac{1}{4}\right\} = \frac{7}{144}. \end{aligned}$$

3) By a direct computation,

$$\begin{aligned} &\operatorname{Cov}(X_1 + X_2, X_1 X_2) = E\{(X_1 + X_2 - E\{X_1\} - E\{X_2\})(X_1 X_2 - E\{X_1\} \cdot E\{X_2\})\} \\ &\quad = E\{(X_1 - E\{X_1\})(X_1 - E\{X_1\}) X_2\} + E\{(X_1 - E\{X_1\}) X_2\} \cdot E\{X_1\} \\ &\qquad\quad + E\{(X_2 - E\{X_2\})(X_2 - E\{X_2\}) X_1\} + E\{X_1 (X_2 - E\{X_2\})\} \cdot E\{X_2\} \\ &\qquad\quad - E\{X_1\} E\{X_2\} \cdot E\{(X_1 - E\{X_1\} + X_2 - E\{X_2\})\} \\ &\quad = V\{X_1\} E\{X_2\} + 0 + V\{X_2\} E\{X_1\} + 0 + 0 \\ &\quad = E\{X_1\} V\{X_2\} + E\{X_2\} V\{X_1\}. \end{aligned}$$

Then

$$\begin{aligned}
\operatorname{Cov}(Y_1, Y_2) &= \frac{1}{2}\operatorname{Cov}(X_1+X_2, X_1X_2) = \frac{1}{2}\left(E\{X_1\}V\{X_2\}+E\{X_2\}V\{X_1\}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{3}+1\cdot\frac{1}{12}\right) = \frac{1}{2}\left(\frac{1}{6}+\frac{1}{12}\right) = \frac{1}{8}.
\end{aligned}$$

4) Since X_1 takes its values in $]0,1[$, and X_2 takes its values in $]0,2[$, the sum $Y_1 = X_1 + X_2$ will take its values in $]0,3[$. If $y \in]0,3[$, then the frequency of Y is given by

$$f_Y(y) = \int_0^y f_{X_1}(x) f_{X_2}(y-x)\,dx.$$

Then we must split the investigation according to the different subintervals.

a) If $y \in]0,1]$, then

$$f_Y(y) = \int_0^y 1\cdot\frac{1}{2}\,dx = \frac{y}{2}.$$

b) If $y \in]1,2]$, then

$$f_Y(y) = \int_0^1 1\cdot\frac{1}{2}\,dx = \frac{1}{2}.$$

c) If $y \in]2,3]$, then
$$f_Y(y) = \int_0^1 1 \cdot f_{X_2}(y-x)\,dx = \int_{y-2}^1 1 \cdot \frac{1}{2}\,dx = \frac{1}{2}\,(3-y).$$
Summing up,
$$f_Y(y) = \begin{cases} \dfrac{y}{2}, & \text{for } y \in]0,1], \\[2mm] \dfrac{1}{2}, & \text{for } y \in]1,2], \\[2mm] \dfrac{1}{2}\,(3-y), & \text{for } y \in]2,3[, \\[2mm] 0, & \text{otherwise.} \end{cases}$$

Example 3.16 *A 2-dimensional random variable (X,Y) has the frequency*
$$h(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1, \\[2mm] 0, & \textit{otherwise.} \end{cases}$$
1. *Find the marginal frequencies of X and Y.*

2. *Find the means of X and Y.*

3. *Find the variances of X and Y.*

4. *Compute the covariance between X and Y, and the correlation coefficient between X and Y.*

Let the random variables U and V be given by
$$U = \max\{X,Y\} \qquad \textit{and} \qquad V = \min\{X,Y\}.$$
5. *Compute the probability $P\left\{U \leq \frac{1}{2}\right\}$ and the probability $P\left\{V \leq \frac{1}{2}\right\}$.*

1) Due to the symmetry, X and Y have the same marginal frequency. If $x \in [0,1]$, then
$$f(x) = \int_0^1 (x+y)\,dy = \left[\frac{(x+y)^2}{2}\right]_{y=0}^1 = \frac{1}{2}\,\left\{(x+1)^2 - x^2\right\} = x + \frac{1}{2},$$
hence
$$f(x) = \begin{cases} x + \dfrac{1}{2} & \text{for } x \in [0,1], \\[2mm] 0 & \text{otherwise,} \end{cases}$$
and
$$g(y) = \begin{cases} y + \dfrac{1}{2} & \text{for } y \in [0,1], \\[2mm] 0 & \text{otherwise.} \end{cases}$$

2) It also follows from the symmetry that

$$E\{X\}=E\{Y\}=\int_0^1 x\left(x+\frac{1}{2}\right)dx=\int_0^1\left(x^2+\frac{x}{2}\right)dx=\left[\frac{x^3}{3}+\frac{x^2}{4}\right]_0^1=\frac{7}{12}.$$

3) For the same reason,

$$E\left\{X^2\right\}=E\left\{Y^2\right\}=\int_0^1 x^2\left(x+\frac{1}{2}\right)dx=\int_0^1\left(x^3+\frac{x^2}{2}\right)dx=\left[\frac{x^4}{4}+\frac{x^3}{6}\right]_0^1=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}.$$

Hence

$$V\{X\}=V\{Y\}=E\left\{X^2\right\}-(E\{X\})^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{60-49}{144}=\frac{11}{144}.$$

4) According to a formula, the covariance is

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= E\{XY\}-E\{X\}\cdot E\{Y\}=\int_0^1\left\{\int_0^1 xy(x+y)\,dy\right\}dx-\frac{7}{12}\cdot\frac{7}{12}\\
&= \int_0^1 x\left\{\int_0^1\left(yx+y^2\right)dy\right\}dx-\frac{49}{144}=\int_0^1 x\left\{\frac{1}{2}x+\frac{1}{3}\right\}dx-\frac{49}{144}\\
&= \int_0^1\left\{\frac{x^2}{2}+\frac{x}{3}\right\}dx-\frac{49}{144}=\frac{1}{6}+\frac{1}{6}-\frac{49}{144}=\frac{1}{3}-\frac{49}{144}=-\frac{1}{144}.
\end{aligned}$$

Then we get the correlation coefficient

$$\varrho(X,Y)=\frac{\operatorname{Cov}(X,Y)}{\sqrt{V\{X\}\cdot V\{Y\}}}=\frac{-\frac{1}{144}}{\frac{11}{144}}=-\frac{1}{11}.$$

5) If $U=\max\{X,Y\}$, then

$$\begin{aligned}
P\left\{U\le\frac{1}{2}\right\} &= P\left\{X\le\frac{1}{2}\wedge Y\le\frac{1}{2}\right\}=\int_0^{\frac{1}{2}}\left\{\int_0^{\frac{1}{2}}(x+y)\,dy\right\}dx\\
&= \int_0^{\frac{1}{2}}\frac{1}{2}\left[(x+y)^2\right]_{y=0}^{\frac{1}{2}}dx=\frac{1}{2}\int_0^{\frac{1}{2}}\left\{\left(x+\frac{1}{2}\right)^2-x^2\right\}dx\\
&= \frac{1}{6}\left[\left(x+\frac{1}{2}\right)^3-x^3\right]_0^{\frac{1}{2}}=\frac{1}{6}\left\{1^3-\left(\frac{1}{2}\right)^3-\left(\frac{1}{2}\right)^3+0^3\right\}\\
&= \frac{1}{6}\left\{1-\frac{1}{4}\right\}=\frac{1}{6}\cdot\frac{3}{4}=\frac{1}{8}.
\end{aligned}$$

If $V = \min\{X, Y\}$, we get by using the complementary probability that

$$\begin{aligned}
P\left\{V \le \frac{1}{2}\right\} &= 1 - P\left\{V > \frac{1}{2}\right\} = 1 - \int_{\frac{1}{2}}^{1} \left\{ \int_{\frac{1}{2}}^{1} (x+y)\,dy \right\} dx = 1 - \frac{1}{2} \int_{\frac{1}{2}}^{1} \left[(x+y)^2\right]_{y=\frac{1}{2}}^{1} dx \\
&= 1 - \frac{1}{2} \int_{\frac{1}{2}}^{1} \left\{ (x+1)^2 - \left(x + \frac{1}{2}\right)^2 \right\} dx = 1 - \frac{1}{6} \left[(x+1)^3 - \left(x + \frac{1}{2}\right)^3 \right]_{\frac{1}{2}}^{1} \\
&= 1 - \frac{1}{6} \left\{ 2^3 - \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^3 + 1^3 \right\} = 1 - \frac{1}{6} \left\{ 8 - \frac{27}{4} + 1 \right\} \\
&= 1 - \frac{1}{6} \left\{ \frac{36 - 27}{4} \right\} = 1 - \frac{1}{6} \cdot \frac{9}{4} = 1 - \frac{3}{8} = \frac{5}{8}.
\end{aligned}$$

4 Examples concerning the Poisson distribution

Example 4.1 *Let X and Y be independent random variables, and let X have the frequency $f(x)$, and let Y have the frequency $g(y)$.*

1. *Prove that the frequency of the random variable $U = X - Y$ is given by*

$$k(u) = \int_{-\infty}^{\infty} f(x)g(x-u)dx, \qquad u \in \mathbb{R}.$$

In the remaining of the example we assume that

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x > 0, \\ 0, & x \le 0, \end{cases} \qquad g(y) = \begin{cases} \mu\, e^{-\mu y}, & y > 0, \\ 0, & y \le 0, \end{cases}$$

where λ and μ are positive constants.

2. *Find the frequency of the random variable U.*

3. *Find the mean $E\{U\}$ and the variance $V\{U\}$.*

4. *Compute the correlation coefficient $\varrho(U,X)$.*

1) Let $K(u)$ be the distribution function of U. Then

$$K(u) = P\{X - Y \le u\} = \int_{\{x-y\le u\}} f(x)g(y)\,dx\,dy = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{u+y} f(x)g(y)\,dx \right\} dy.$$

By differentiation, followed by the change of variable $x = u + y$,

$$k(u) = \int_{-\infty}^{\infty} f(u+y)g(y)\,dy = \int_{-\infty}^{\infty} g(x)g(x-u)\,dx, \qquad u \in \mathbb{R}.$$

2) It follows from

$$k(u) = \int_{-\infty}^{\infty} g(x)g(x-u)\,dx = \int_{0}^{\infty} f(x)g(x-u)\,dx$$

that if $u > 0$ then the integrand is only $\neq 0$ for $x > u$, thus

$$\begin{aligned} k(u) &= \int_u^{\infty} \lambda\, e^{-\lambda x} \cdot \mu \cdot e^{-\mu(x-u)}\,dx = \lambda\,\mu\, e^{\mu u} \int_u^{\infty} e^{-(\lambda+\mu)x}\,dx \\ &= \frac{\lambda\,\mu}{\lambda+\mu}\, e^{\mu u} \cdot e^{-(\lambda+\mu)u} = \frac{\lambda\,\mu}{\lambda+\mu}\, e^{-\lambda\, u}. \end{aligned}$$

If instead $u \le 0$, then

$$k(u) = \int_0^{\infty} \lambda\, e^{-\lambda x} \cdot \mu\, e^{-\mu(x-u)}\,dx = \lambda\,\mu\, e^{\mu u} \int_0^{\infty} e^{-(\lambda+\mu)x}\,dx = \frac{\lambda\,\mu}{\lambda+\mu}\, e^{\mu\, u}.$$

Summing up,

$$k(u)=\begin{cases}\dfrac{\lambda\,\mu}{\lambda+\mu}\,e^{-\lambda\,u} & \text{for } u>0,\\[2ex] \dfrac{\lambda\,\mu}{\lambda+\mu}\,e^{\mu\,u} & \text{for } u\le 0.\end{cases}$$

3) The mean is

$$E\{U\}=E\{X\}-E\{Y\}=\frac{1}{\lambda}-\frac{1}{\mu}=\frac{\mu-\lambda}{\lambda\,\mu}.$$

Furthermore,

$$\begin{aligned}E\left\{U^2\right\} &= \frac{\mu}{\lambda+\mu}\int_0^{\infty}\lambda\,u^2\,e^{-\lambda\,u}\,du+\frac{\lambda}{\lambda+\mu}\int_{-\infty}^{0}\mu\,u^2\,e^{\mu\,u}\,du\\ &= \frac{1}{\lambda+\mu}\cdot\frac{\mu}{\lambda^2}\int_0^{\infty}t^2\,e^{-t}\,dt+\frac{1}{\lambda+\mu}\cdot\frac{\lambda}{\mu^2}\int_0^{\infty}t^2\,e^{-t}\,dt\\ &= \frac{2}{\lambda+\mu}\left\{\frac{\lambda}{\mu^2}+\frac{\mu}{\lambda^2}\right\}=\frac{2}{\lambda+\mu}\cdot\frac{\lambda^3+\mu^3}{\lambda^2\mu^2}=2\cdot\frac{\lambda^2-\lambda\,\mu+\mu^2}{\lambda^2\mu^2}.\end{aligned}$$

The variance is

$$\begin{aligned}V\{U\} &= E\left\{U^2\right\}-(E\{U\})^2=2\cdot\frac{\lambda^2-\lambda\,\mu+\mu^2}{\lambda^2\mu^2}-\frac{\lambda^2-2\lambda\,\mu+\mu^2}{\lambda^2\mu^2}\\ &= \frac{\lambda^2+\mu^2}{\lambda^2\mu^2}=\frac{1}{\lambda^2}+\frac{1}{\mu^2}.\end{aligned}$$

4) It is well-known that

$$E\{X\}=\frac{1}{\lambda}\qquad\text{and}\qquad V\{X\}=\frac{1}{\lambda^2}.$$

Since X and Y are stochastically independent, we have

$$E\{XY\}=E\{X\}E\{Y\}.$$

By the rules of computation,

$$\operatorname{Cov}(U,X)=\operatorname{Cov}(X-Y,X)=\operatorname{Cov}(X,X)-\operatorname{Cov}(Y,X)=V\{X\}=\frac{1}{\lambda^2},$$

hence

$$\varrho(U,X)=\frac{\operatorname{Cov}(U,X)}{\sqrt{V\{U\}V\{X\}}}=\frac{1}{\lambda^2}\cdot\frac{1}{\sqrt{\dfrac{\lambda^2+\mu^2}{\lambda^2\mu^2}\cdot\dfrac{1}{\lambda^2}}}=\frac{\mu}{\sqrt{\lambda^2+\mu^2}}.$$

Example 4.2 *A radioactive material emits both α and β particles, where these two types of particles are emitted independently of each other. We shall study this emission from (and included) the time $t = 0$.*
Let X_1, $X_1 + X_2$, $X_1 + X_2 + X_3$, $\dots$, indicate the times of the emission of the first, second, third, $\dots$, α particle.
We assume that the random variables X_i, $i = 1, 2, \dots$, are mutually independent of the frequency

$$f(x) = \begin{cases} \lambda\, e^{-\lambda\, x}, & x \geq 0, \\ \\ 0, & x < 0, \end{cases} \qquad \lambda > 0.$$

Analogously, Y_1, $Y_1 + Y_2$, $Y_1 + Y_2 + Y_3$, $\dots$, indicates the times of the emission of the first, second, third, $\dots$, β particle.
We assume that the random variables Y_i, $i = 1, 2, \dots$, also are mutually independent, and then by the assumption independent of the X_i of the frequency

$$g(y) = \begin{cases} \mu\, e^{-\mu\, y}, & Y \geq 0, \\ \\ 0, & y < 0, \end{cases} \qquad \mu > 0.$$

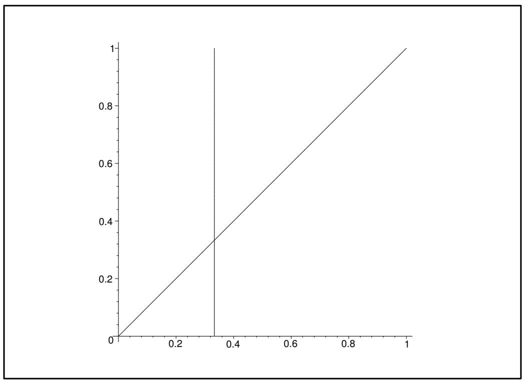
1) *Find the frequency of $X_1 + X_2$.*

2) *Find the probability that there is emitted at least two α particles before one β particle is emitted. For which value of $\dfrac{\lambda}{\mu}$ is this probability equal to $\dfrac{1}{2}$?*

1) When $x > 0$, then the frequency of $X_1 + X_2$ is given by the convolution integral

$$f_2(x) = \int_0^x f(x) f(x-t)\, dt = \int_0^x \lambda\, e^{-t\,\lambda} \cdot \lambda\, e^{-(x-t)\lambda}\, dt = \lambda^2\, x\, e^{-\lambda\, x},$$

and $f_2(x) = 0$ otherwise.



2) We shall find $P\{X_1 + X_2 < Y_1\}$.

First method. The simultaneous frequency of $(X_1 + X_2, Y_1)$ is $f_2(x)g(y)$, hence

$$\begin{aligned}
P\{X_1 + X_2 < Y_1\} &= \int_{\{x<y\}} f_2(x)g(y)\,dx\,dy \\
&= \int_{x=0}^{\infty} f_2(x)\left\{\int_{y=x}^{\infty} g(y)\,dy\right\}dx = \int_0^{\infty} \lambda^2\,x\,e^{-\lambda\,x}\left\{\int_{y=x}^{\infty} \mu\,e^{-\mu\,y}\,dy\right\}dx \\
&= \int_0^{\infty} \lambda^2\,x\,e^{-(\lambda+\mu)x}\,dx = \frac{\lambda^2}{(\lambda+\mu)^2}\int_0^{\infty} t\,e^{-t}\,dt = \left\{\frac{\frac{\lambda}{2}}{\frac{\lambda}{\mu}+1}\right\},
\end{aligned}$$

where we have applied the substitution $t = (\lambda + \mu)x$.

Remark 4.1 Here it is difficult to compute the double integral in the order $\int_{y=0}^{\infty}\left\{\int_{x=0}^{y}\cdots\right.$, so we omit this variant. $\Diamond$

Second method. (More difficult.) The frequency of $Z = \dfrac{Y_1}{X_1 + X_2}$ is computed according to some formula. If $z > 0$, then

$$\begin{aligned}
k(z) &= \int_0^{\infty} g(zx)f_2(x)\,x\,dx = \int_0^{\infty} \mu\,e^{-\mu\,z\,x}\lambda^2\,x\,e^{-\lambda\,x}x\,dx \\
&= \mu\lambda^2\int_0^{\infty} x^2 e^{-(\lambda+\mu\,z)x}\,dx = \frac{2\lambda^2\mu}{(\lambda+\mu\,z)^3},
\end{aligned}$$

hence

$$\begin{aligned}
P\{X_1 + X_2 < Y_1\} &= P\{Z > 1\} = \int_1^{\infty} k(z)\,dz \\
&= 2\lambda^2\mu\int 1^{\infty}(\lambda+\mu\,z)^{-3}\,dz = \frac{\lambda^2}{(\lambda+\mu)^2}.
\end{aligned}$$

Third method. (SKETCH). Find the frequency of $\dfrac{X_1 + X_2}{Y_1}$, cf. the second method.

Fourth method. (Even more difficult; only a SKETCH). Find the frequency of

$$U = (X_1 + X_2) - Y_1.$$

Then

$$P\{X_1 + X_2 < Y_1\} = P\{U < 0\} = \int_{-\infty}^{0} f_U(u)\,du.$$

The probability is $\frac{1}{2}$, when

$$\frac{\frac{\lambda}{\mu}}{\frac{\lambda}{\mu}+1} = \frac{1}{\sqrt{2}},$$

and we get

$$\frac{\lambda}{\mu} = \sqrt{2}+1.$$

Example 4.3 . (Continuation of Example 4.2).

1) *Find the probability that there is emitted at least three α particles, before the first β particle is emitted.*

2) *Find the probability that there is emitted precisely two α particles, before the first β particle is emitted.*

3) *Find the probability $P_n(t)$ that there in the time interval $]0,1[$ is emitted a total of n particles.*

1) It follows from Example 4.2 that $X_1 + X_2$ has the frequency

$$f_2(x) = \begin{cases} \lambda^2\, x\, e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Then X_3 has the frequency

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

so the frequency $f_3(s)$ of $X_1 + X_2 + X_3$ is zero for $s \leq 0$. If $s > 0$, then

$$f_3(s) = \int_0^s \lambda^2 x\, e^{-\lambda x} \cdot \lambda\, e^{-\lambda(s-x)}\,dx = \lambda^3 e^{-\lambda s} \int_0^s x\,dx = \frac{1}{2}\,\lambda^3 s^2 e^{-\lambda s}.$$

Then (cf. Example 4.2)

$$
\begin{aligned}
P\left\{X_1+X_2+X_3<Y_1\right\} &= \int_{-\infty}^{0}\left\{\int_0^{\infty} f_3(x)g(x-s)\,dx\right\}ds \\
&= \int_{-\infty}^{0}\left\{\int_0^{\infty}\frac{1}{2}\,\lambda^3x^2e^{-\lambda x}\cdot\mu\,e^{-\mu(x-s)}dx\right\}ds \\
&= \frac{1}{2}\,\lambda^3\mu\int_{-\infty}^{0}e^{-\mu s}ds\cdot\int_0^{\infty}x^2e^{-(\lambda+\mu)x}\,dx \\
&= \frac{1}{2}\,\lambda^3\cdot 1\cdot\frac{1}{(\lambda+\mu)^3}\int_0^{\infty}t^2e^{-t}dt=\left(\frac{\lambda}{\lambda+\mu}\right)^3.
\end{aligned}
$$

2) The probability that there is emitted precisely two α particles before one β particle is emitted is

$$
\begin{aligned}
&P\left\{X_1+X_2<Y_1\right\}-P\left\{X_1+X_2+X_3<Y_1\right\} \\
&\qquad=\left(\frac{\lambda}{\lambda+\mu}\right)^2-\left(\frac{\lambda}{\lambda+\mu}\right)^3=\left(\frac{\lambda}{\lambda+\mu}\right)^3\left(\frac{\lambda+\mu}{\lambda}-1\right)=\frac{\lambda^2\mu}{(\lambda+\mu)^3}.
\end{aligned}
$$

3) Assume that $Z_n=X_1+\cdots+X_n$ has the frequency $f_k(s)$. Then $f_k(s)=0$ for $s\le 0$, and we have for $s>0$,

$$
f_n(s)=\int_0^s f_{m-1}(x)f(s-x)\,dx=\int_0^s f_{n-1}(x)\,\lambda\,e^{-\lambda(s-x)}dx=\lambda\,e^{-\lambda s}\int_0^s e^{\lambda x}f_{n-1}(x)\,dx,
$$

i.e.

$$
f_2(s)=\lambda\,e^{-\lambda s}\textstyle\int_0^s e^{\lambda x}\lambda\,e^{-\lambda x}\,dx=\lambda^2 s\,e^{-\lambda s},\qquad s>0,
$$

$$
f_3(s)=\lambda\,e^{-\lambda s}\textstyle\int_0^s e^{\lambda x}\cdot\lambda^2x\,e^{-\lambda x}\,dx=\lambda^3\cdot\dfrac{s^2}{2!}\,e^{-\lambda s},\qquad s>0,
$$

and then by induction

$$
f_n(s)=\begin{cases}\dfrac{\lambda^n}{(n-1)!}\,s^{n-1}e^{-\lambda s}, & s>0,\\[2mm] 0, & s\le 0.\end{cases}
$$

It follows that

$$
P\left\{Z_k<t\right\}=\int_0^t f_k(s)\,ds=\frac{\lambda^k}{(k-1)!}\int_0^t s^{k-1}e^{\lambda s}\,ds,\qquad 0\le k\le n,
$$

which is the probability that there is emitted at least k of the α particles before time t.
The probability that there is emitted *precisely* k particles of α type before time t, is

$$
\begin{aligned}
P\left\{Z_k<t\right\}-P\left\{Z_{k+1}<t\right\} &= \frac{\lambda^k}{(k-1)!}\int_0^t s^{k-1}e^{-\lambda s}ds-\frac{\lambda^{k+1}}{k!}\int_0^t s^ke^{-\lambda s}ds \\
&= \frac{\lambda^k}{(k-1)!}\int_0^t s^{k-1}e^{-\lambda s}\,dx+\left[\frac{\lambda^k}{k!}\,s^ke^{-\lambda s}\right]_0^t-\frac{\lambda^k}{(k-1)!}\int_0^t s^{k-1}e^{-\lambda s}\,ds \\
&= \left[\frac{\lambda^k}{k!}\,s^ke^{-\lambda s}\right]_0^t=\frac{\lambda^k}{k!}\,t^ke^{-\lambda t}=\frac{(\lambda t)^k}{k!}\,e^{-\lambda t}.
\end{aligned}
$$

Analogously, the probability that there is emitted precisely $n-k$ particles of type β in $]0,1[$ is given by

$$\frac{(\mu t)^{n-k}}{(n-k)!}\, e^{-\mu t}.$$

Finally, the probability that there is emitted precisely n particle (of either type α or type β) in the time interval $]0,1[$ is

$$\sum_{k=0}^{n} \frac{(\lambda t)^k}{k!}\, e^{-\lambda t} \cdot \frac{(\mu t)^{n-k}}{(n-k)!}\, e^{-\mu t} = \frac{1}{n!}\, e^{-(\lambda+\mu)t} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, (\lambda t)^k (\mu t)^{n-k}$$

$$= \frac{1}{n!}\, t^n e^{-(\lambda+\mu)t} \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mu^{n-k} = \frac{1}{n!}\, (\lambda+\mu)^n t^n e^{-(\lambda+\mu)t}.$$

Example 4.4 *An instrument A contains two components, which can fail independently of each other. The instrument does not work, if just one of the components does not work.*
The lifetime for each of the two components has a distribution given by the frequency

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x > 0, \\[2mm] 0, & x \le 0, \end{cases}$$

where λ is a positive constant.
The task is to find the distribution of the lifetime of the instrument A.

There is in another instrument B only one component, the lifetime of which has the same frequency $f(x)$ as above.
We shall find the probability that the lifetime of instrument B is at least the double of the lifetime of A.
Let us imagine that we first apply instrument A, and when it is ruined, then we apply instrument B. Find the distribution of the total lifetime and find the mean of this lifetime.

Let Y_1 and Y_2 denote the lifetimes of the two components of A, and Y the lifetime of A, and X the lifetime of B.

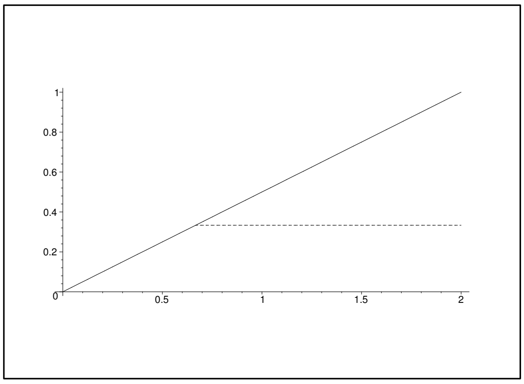
Clearly, $Y = \min\{Y_1, Y_2\}$.

Then Y is exponentially distributed of frequency

$$g(y) = \begin{cases} 2\lambda\, e^{-2\lambda y}, & y > 0, \\[2mm] 0, & y \le 0, \end{cases}$$

In the next subtask we shall find $P\{X \ge 2Y\}$.
A reasonable assumption is that A and B function independently of each other. This means that (X,Y) has the simultaneous frequency $f(x)g(y)$, thus

$$\begin{aligned} P\{X \ge 2Y\} &= \int_{\{x \ge 2y\}} f(x)g(y)\, dx\, dy = \int_{y=0}^{\infty} 2\lambda\, e^{-2\lambda y} \left\{ \int_{x=2y}^{\infty} \lambda\, e^{-\lambda x}\, dx \right\} dy \\ &= \int_{y=0}^{\infty} 2\lambda\, e^{-4\lambda y}\, dy = \frac{1}{2}. \end{aligned}$$



The frequency of $X+Y$ for $z>0$ is given by the convolution integral,

$$k(z)=\int_0^z \lambda\, e^{-\lambda x}\cdot 2\lambda\, e^{-2\lambda(z-x)}\,dx = 2\lambda^2 e^{-2\lambda z}\int_0^z e^{\lambda x}\,dx = 2\lambda\left(e^{-\lambda x}-e^{-2\lambda z}\right),$$

and $k(z)=0$ for $z\le 0$.

Finally,

$$E\{X+Y\}=E\{X\}+E\{Y\}=\frac{1}{\lambda}+\frac{1}{2\lambda}=\frac{3}{2\lambda}.$$

ALTERNATIVELY,

$$E\{X+Y\} = \int_0^{\infty} z\,k(z)\,dz = 2\lambda \int_0^{\infty} \left(z\,e^{-\lambda z} - z\,e^{-2\lambda z}\right) dz = \frac{2}{\lambda}\cdot 1! - \frac{1}{2\lambda}\cdot 1! = \frac{3}{2\lambda}.$$

Example 4.5 **1.** *Let X be a non-negative random variable of frequency $f(x)$ and mean $E\{X\}$. Prove that*

$$(1)\ \ E\{X\} = \int_0^{\infty} P\{X \geq x\}\,dx.$$

Hint: Express e.g. $P\{X \geq x\}$ by means of the frequency $f(x)$.

We shall allow in the following without proof to apply the result that the mean of every non-negative random variable is given by (1).

Two patients A_1 and A_2 arrive to a doctor's waiting room at the times X_1 and X_1+X_2, where X_1 and X_2 are independent random variables, both of the frequency

$$f(x) = \begin{cases} \lambda\,e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where λ is a positive constant.

The times of treatment of A_1 and A_2 are assumed to be the random variables Y_1 and Y_2, which are mutually independent (and also independent of X_1 and X_2), and we assume that they have the frequency

$$g(y) = \begin{cases} \mu\,e^{-\mu y}, & y > 0, \\ 0, & y \leq 0, \end{cases}$$

where μ is a positive constant.

The patient A_1 is treated immediately after his arrival, while A_2 possibly may wait to after the treatment of A_1.

2. *Describe, expressed by Y_1 and Y_2, the event that A_2 does not wait for his treatment, and find the probability of this event.*

3. *Find for every $z > 0$ the probability that the waiting time Z of A_2 is $\geq z$.*

4. *Find the mean of the random variable Z.*

1) Since $f(t) \geq 0$, and $f(t) = 0$ for $t < 0$, we get

$$\int_0^{\infty} P\{X \geq x\}\,dx = \int_0^{\infty} \left\{\int_x^{\infty} f(t)\,dt\right\} dx = \int_0^{\infty} \left\{\int_0^{t} f(t)\,dx\right\} dt = \int_0^{\infty} t\,f(t)\,dt = E\{X\}.$$

2) The condition that A_2 does not have to wait is

$$X_1 + Y_1 \leq X_1 + X_2, \qquad \text{thus} \qquad Y_1 \leq X_2,$$

hence

$$\begin{aligned} P\{X_2 \ge Y_1\} &= \int_{y=0}^{\infty} g(y) \left\{ \int_{x=y}^{\infty} f(x)\,dx \right\} dy = \int_{y=0}^{\infty} \mu\, e^{-\mu y} \left\{ \int_{x=y}^{\infty} \lambda\, e^{-\lambda x}\,dx \right\} dy \\ &= \int_{y=0}^{\infty} \mu\, e^{-(\lambda+\mu)y}\,dy = \frac{\mu}{\lambda+\mu}. \end{aligned}$$

3) When the waiting time is positive, it is described by $Z = Y_1 - X_2$. Then for $z > 0$,

$$\begin{aligned} P\{Z \ge z\} &= P\{Y_1 \ge X_2 + z\} = \int_{x=0}^{\infty} f(x) \left\{ \int_{y=x+z}^{\infty} g(y)\,dy \right\} dx \\ &= \int_{x=0}^{\infty} \lambda\, e^{-(\lambda+\mu)x}\,dx \cdot e^{-\mu z} = \frac{\lambda}{\lambda+\mu} \cdot e^{-\mu z}. \end{aligned}$$

4) It follows from (1) that

$$E\{Z\} = \int_0^{\infty} P\{Z \ge z\}\,dz = \frac{\lambda}{\lambda+\mu} \int_0^{\infty} e^{-\mu z} dz = \frac{\lambda}{\mu} \cdot \frac{1}{\lambda+\mu}.$$

Remark 4.2 The distribution of Z is of mixed type, i.e. neither discrete nor continuous. $\Diamond$

5 Miscellaneous examples

Example 5.1 *A 2-dimensional random variable (X, Y) has in the domain given by the inequalities*

$$1 \le x^2 + y^2 \le 4$$

the frequency

$$h(x, y) = \frac{1}{3\pi},$$

while the frequency is 0 everywhere else in the (x, y) plane.

1) Find the frequency of the random variable X, and sketch the graph of this function.

2) Find the variance of the random variable X.

3) Explain why the random variable X and Y are non-correlated, though not independent.

4) Find the probability that $|X| + |Y| \ge 2$.

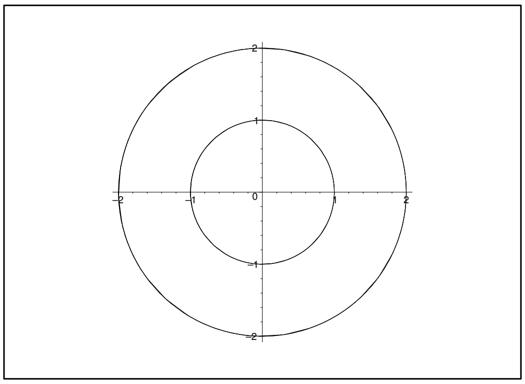


Figure 2: The frequency has its support in the annulus.

1) If $|x| \ge 2$, then $f_X(x) = 0$.
By the symmetry, $f_X(-x) = f_X(x)$. If $|x| \in [1, 2]$, then it follows by a vertical integration (a consideration of a graph) that

$$f_X(x) = \frac{1}{3\pi} \cdot 2\sqrt{4 - x^2} = \frac{2}{3\pi}\sqrt{4 - x^2}.$$

If $|x| \in [0, 1]$, then we get instead

$$f_X(x) = \frac{2}{3\pi}\left\{\sqrt{4 - x^2} - \sqrt{1 - x^2}\right\}.$$

Summing up,

$$f_X(x) \begin{cases} \dfrac{2}{3\pi}\left\{\sqrt{4 - x^2} - \sqrt{1 - x^2}\right\}, & x \in [-1, 1], \\[2ex] \dfrac{2}{3\pi}\sqrt{4 - x^2}, & 1 \le |x| \le 2, \\[2ex] 0, & \text{otherwise}. \end{cases}$$

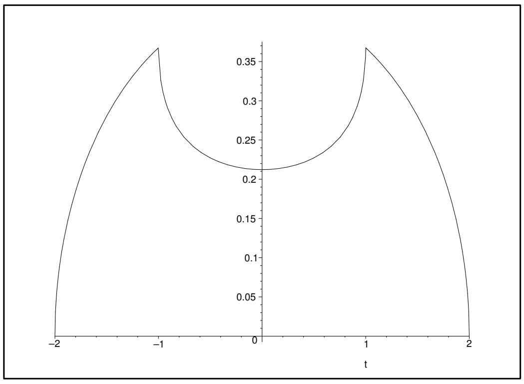


Figure 3: The graph of $f_X(x)$.

2) The mean is trivially $E\{X\}=0$, so the variance is

$$\begin{aligned}
V\{X\} &= E\left\{X^2\right\}-(E\{X\})^2=E\left\{X^2\right\} \\
&= \frac{4}{3\pi}\int_0^1 x^2\left\{\sqrt{4-x^2}-\sqrt{1-x^2}\right\}+\frac{4}{3\pi}\int_1^2 x^2\sqrt{4-x^2}\,dx \\
&= \frac{4}{3\pi}\int_0^2 x^2\sqrt{4-x^2}\,dx-\frac{4}{3\pi}\int_0^1 x^2\sqrt{1-x^2}\,dx \\
&= \frac{4}{3\pi}\int_0^{\frac{\pi}{2}} 4\sin^2 t\cdot 2\cos t\cdot 2\cos t\,dt-\frac{4}{3\pi}\int_0^{\frac{\pi}{2}}\sin^2 t\,\cos t\,\cos t\,dt \\
&= \frac{16-1}{3\pi}\int_0^{\frac{\pi}{2}} 4\sin^2 t\,\cos^2 t\,dt=\frac{5}{\pi}\int_0^{\frac{\pi}{2}}\sin^2 2t\,dt=\frac{5}{\pi}\cdot\frac{\pi}{4}=\frac{5}{4}.
\end{aligned}$$

3) The support of h (i.e. the closure of the set, where $h(x,y)\neq 0$) is not a rectangle. Hence, X and Y cannot be independent.
The annulus is denoted by Ω. By using that $E\{X\}=0$, it follows by the symmetry that

$$\operatorname{Cov}(X,Y)=E\{XY\}-E\{X\}E\{Y\}=\int\int_\Omega xy\cdot\frac{1}{3\pi}\,dx\,dy=0,$$

hence X and Y are non-correlated.

4) It follows by considering the figure that $P\{|X|+|Y|\geq 2\}$ is equal to the integral of $h(x,y)$ over the four circular segments, thus equal to $\dfrac{1}{3\pi}$ times the area of these four circular segments, hence

$$P\{|X|+|Y|\geq 2\}=\frac{1}{3\pi}\left\{\pi\cdot 2^2-(2\sqrt{2})^2\right\}=\frac{4}{3\pi}(\pi-2)=\frac{4}{3}-\frac{8}{3\pi}\approx 0.485.$$

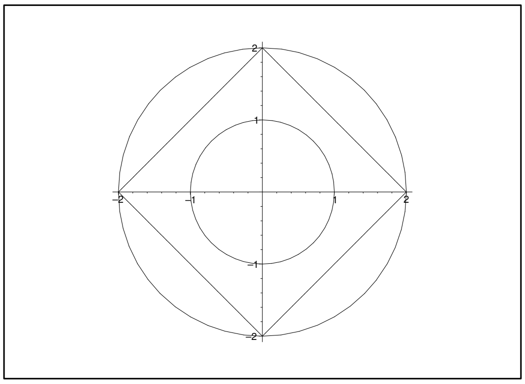


Figure 4: The domain where $|X| + |Y| \geq 2$, is the union of the four circular segments on the figure.

Example 5.2 *1) Find the pairs of numbers (a, b), for which*

$$g(x,y) = \begin{cases} ax + by & \text{for } 0 \leq x \leq 2 \text{ og } 0 \leq y \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

is the frequency of a 2-dimensional random variable (X, Y).

2) Find, expressed by a, the means $E\{X\}$ and $E\{Y\}$.

3) Find the pairs of numbers (a, b), for which the product $E\{X\}E\{Y\}$ is largest, and compute the maximum.

4) Compute for $(a, b) = \left(\frac{1}{4}, \frac{1}{2}\right)$ the covariance $\mathrm{Cov}(X, Y)$.

1) Since $g(x, y) \geq 0$ everywhere, we must have $a \geq 0$ and $b \geq 0$. Furthermore, we derive the condition

$$1 = \int_0^2 ax \left\{ \int_0^1 dy \right\} dx + \int_0^2 b \left\{ \int_0^1 y\, dy \right\} dx = 2a + 2b \cdot \frac{1}{2} = 2a + b,$$

thus $b = 1 - 2a$, where $a \in \left[0, \frac{1}{2}\right]$, hence

$$g(x,y) = \begin{cases} ax + (1-2a)y & \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad a \in \left[0, \frac{1}{2}\right].$$

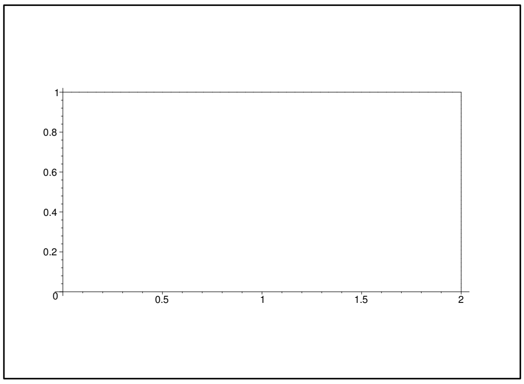


Figure 5: The support of $g(x,y)$.

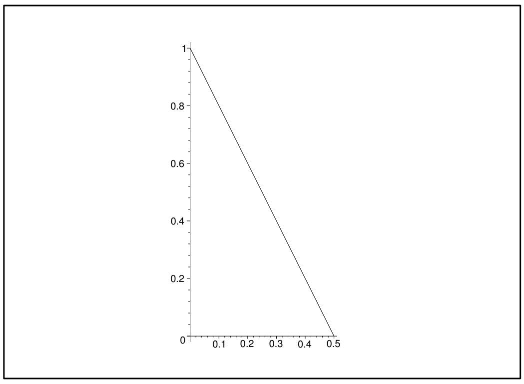


Figure 6: The possible values of (a,b) lie on the oblique line.

2) If $a \in \left[0,\dfrac{1}{2}\right]$ we get the mean

$$\begin{aligned} E\{X\} &= \int_0^2 x\, f_X(x)\,dx = \int_0^2 \left\{\int_0^1 x\{ax+(1-2a)y\}dy\right\} dx \\ &= \int_0^2 ax^2 dx + \int_0^2 x\,dx \cdot (1-2a)\int_0^1 y\,dy \\ &= \frac{8a}{3} + (1-2a)\cdot\frac{1}{2}\cdot\frac{2^2}{2} = 1+\frac{2}{3}\,a, \end{aligned}$$

and analogously

$$\begin{aligned} E\{Y\} &= \int_0^1 y\, f_Y(y)\,dy = \int_0^2 \left(\int_0^1 y\{ax+(1-2a)y\}\,dy\right) dx \\ &= a\int_0^2 x\,dx \cdot \int_0^1 y\,dy ? (1-2a)\cdot 2\int_0^1 y^2 dy \\ &= 2a\cdot\frac{1}{2}+\frac{2}{3}\,(1-2a) = a+\frac{2}{3}-\frac{4}{3}\,a = \frac{2}{3}-\frac{1}{3}\,a. \end{aligned}$$

3) If we put

$$\begin{aligned}\varphi(a) &= E\{X\}E\{Y\} = \left(1+\frac{2}{3}\,a\right)\left(\frac{2}{3}-\frac{1}{3}\,a\right)\\ &= \frac{1}{9}\,(3+2a)(2-a) = \frac{1}{9}\,\left(6+a-2a^2\right),\end{aligned}$$

then

$$\varphi'(a) = 1-4a = 0 \qquad \text{for } a = \frac{1}{4}.$$

Since $\varphi'(a) > 0$ for $a < \frac{1}{4}$, and $\varphi'(a) < 0$ for $a > \frac{1}{4}$, it follows that $a = \frac{1}{4}$ corresponds to the maximum

$$\varphi\left(\frac{1}{4}\right) = \frac{1}{9}\left(6+\frac{1}{4}-\frac{1}{8}\right) = \frac{48+2-1}{72} = \frac{49}{72}.$$

4) If $(a,b) = \left(\frac{1}{4}, \frac{1}{2}\right)$, then

$$\begin{aligned} E\{XY\} &= \int_0^2 \left\{ \int_0^1 xy\left(\frac{1}{4}x + \frac{1}{2}y\right) dy \right\} dx = \frac{1}{4}\int_0^2 x^2 dx \cdot \int_0^1 y\,dy + \frac{1}{2}\int_0^2 x\,dx \cdot \int_0^1 y^2 dy \\ &= \frac{1}{4}\cdot\frac{8}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{4}{2}\cdot\frac{1}{3})\frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \end{aligned}$$

hence

$$\operatorname{Cov}(X,Y) = E\{XY\} - E\{X\}E\{Y\} = \frac{2}{3} - \frac{49}{72} = \frac{2\cdot 24 - 49}{72} = -\frac{1}{72}.$$

Example 5.3 *A 2-dimensional random variable (X,Y) has in the square defined by $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$ the frequency*

$$h(x,y) = k(\sin x + \cos y),$$

while the frequency is 0 outside this square.

1) Prove that the constant k is equal to $\frac{1}{\pi}$.

2) Find the frequencies $f_X(x)$ and $f_Y(y)$ of the random variables X and Y.

3) Find the means $E\{X\}$ and $E\{Y\}$ of the random variables X and Y.

4) Find the frequency $f_Z(z)$ of the random variable $Z = X + Y$, and sketch the graph of the function.

1) Clearly, $h(x,y) \geq 0$, if and only if $k \geq 0$. If $h(x,y)$ is a frequency, then necessarily

$$\begin{aligned} 1 &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} h(x,y)\,dx\,dy = k\left(\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}} \sin x\,dx\right\}dy + \int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}} \cos y\,dy\right\}dx\right) \\ &= k\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = k\cdot\pi, \end{aligned}$$

and we conclude that $k = \frac{1}{\pi}$ as claimed.

2) When $x \notin \left]0, \frac{\pi}{2}\right[$, then $f_X(x) = 0$. When $x \in \left]0, \frac{\pi}{2}\right[$, it follows by a vertical integration that

$$f_X(x) = \int_0^{\frac{\pi}{2}} h(x,y)\,dy = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{\sin x + \cos y\}\,dy = \frac{1}{2}\,\sin x + \frac{1}{\pi}.$$

When $x \notin \left]0, \frac{\pi}{2}\right[$, then $f_Y(y) = 0$. When $y \in \left]0, \frac{\pi}{2}\right[$, it follows by a horizontal integration that

$$f_Y(y) = \int_0^{\frac{\pi}{2}} h(x,y)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{\sin x + \cos y\}\,dx = \frac{1}{2}\,\cos y + \frac{1}{\pi}.$$

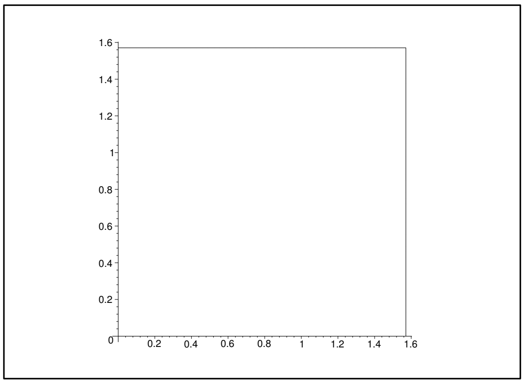


Figure 7: The square $\left]0,\frac{\pi}{2}\right[\times\left]0,\frac{\pi}{2}\right[$.

3) The means are

$$\begin{aligned} E\{X\} &= \int_0^{\frac{\pi}{2}} x\, f_X(x)\,dx = \int_0^{\frac{\pi}{2}} \left\{\frac{1}{2}\,x\,\sin x + \frac{x}{\pi}\right\} dx = \left[-\frac{x}{2}\,\cos x + \frac{x^2}{2\pi}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x\,dx \\ &= \frac{\pi}{8} + \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} E\{Y\} &= \int_0^{\frac{\pi}{2}} y\, f_Y(y)\,dy = \int_0^{\frac{\pi}{2}} \left\{\frac{1}{2}\,y\,\cos y + \frac{y}{\pi}\right\} dy = \left[\frac{1}{2}\,y\,\sin y + \frac{y^2}{2\pi}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin y\,dy \\ &= \frac{\pi}{4} + \frac{\pi}{8} - \frac{1}{2} = \frac{3\pi}{8} - \frac{1}{2}. \end{aligned}$$

4) Clearly, $X+Y$ has values in $]0,\pi[$. Since X and Y are not independent, the frequency of $Z = X+Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} h(x,z-x)\,dx = \int_0^{\frac{\pi}{2}} h(x,z-x)\,dx.$$

Now let $0<z<\pi$. The the integrand is $\neq 0$, if $0<x<\dfrac{\pi}{2}$ and $0<z-x<\dfrac{\pi}{2}$, i.e. if $z-\dfrac{\pi}{2}<x<z$. Then we must split into two cases:

a) If $0<z\le\dfrac{\pi}{2}$, then the domain of integration is $0<x<z$, so

$$\begin{aligned} f_Z &= \int_0^z h(x,z-x)\,dx = \frac{1}{\pi}\int_0^z \{\sin x + \cos(z-x)\}\,dx \\ &= \frac{1}{\pi}\left[-\cos x + \sin(x-z)\right]_0^z = \frac{1}{\pi}\{1+\sin z - \cos z\}. \end{aligned}$$

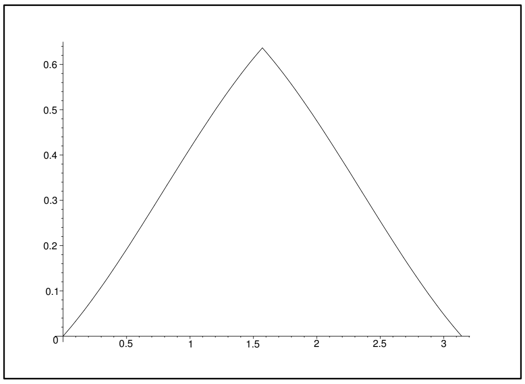


Figure 8: The graph of $f_Z(z)$.

b) If $\dfrac{\pi}{2} < z < \pi$, then the domain of integration is $z - \dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, hence

$$
\begin{aligned}
f_Z(z) &= \frac{1}{\pi}\left[-\cos x + \sin(x-z)\right]_{z-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{\pi}\left\{-0 + \sin\left(\frac{\pi}{2} - z\right) + \cos\left(z - \frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right\} = \frac{1}{\pi}\{1 + \sin z + \cos z\}.
\end{aligned}
$$

Summing up,

$$f_Z(z) = \begin{cases} \dfrac{1}{\pi}\{1+\sin z - \cos z\} & \text{for } 0 < z \le \dfrac{\pi}{2}, \\[2mm] \dfrac{1}{\pi}\{1+\sin z + \cos z\} & \text{for } \dfrac{\pi}{2} < z < \pi, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

Example 5.4 *Let X and Y be independent random variables, which both are rectangularly distributed over the interval* $]1,2[$.

1. *Find the frequency of the random variable* $Z = \dfrac{X}{Y}$.

Compute the mean of Z.

Find the median of Z.

A random variable U is given by $U = \dfrac{X}{Y} + \dfrac{Y}{Z}$.

4. *Which values can U have?*

5. *Find the probability that* $U < \dfrac{25}{12}$.

1) Clearly, Z has its values in $\left]\dfrac{1}{2}, 2\right[$.

The frequency of $Z = \dfrac{X}{Y}$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(zx)\cdot f_Y(x)\cdot |x|\,dx = \int_1^2 f_X(zx)\,x\,dx.$$

When $z \in \left]\dfrac{1}{2}, 2\right[$, then the conditions become $1 < x < 2$ and $1 < zx < 2$, hence $\dfrac{1}{z} < x < \dfrac{2}{z}$.

a) When $z \in \left]\dfrac{1}{2}, 1\right]$, then the interval of integration is $\left]\dfrac{1}{z}, 2\right[$, hence

$$f_Z(z) = \int_{\frac{1}{z}}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{\frac{1}{z}}^{2} = 2 - \frac{1}{2z^2} \qquad \left(= \frac{4z^2-1}{2z^2}\right).$$

b) When $z \in]1,2[$, then the interval of integration is $\left]1, \dfrac{2}{z}\right[$, hence

$$f_Z(z) = \int_1^{\frac{2}{z}} x\,dx = \left[\frac{x^2}{2}\right]_1^{\frac{2}{z}} = \frac{2}{z^2} - \frac{1}{2} \qquad \left(= \frac{4-z^2}{2z^2}\right).$$

Summing up,

$$f_Z) = \begin{cases} 2 - \dfrac{1}{2z^2} & \text{for } z \in \left]\dfrac{1}{2}, 1\right], \\[2mm] \dfrac{2}{z^2} - \dfrac{1}{2} & \text{for } z \in]1,2[, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

2) The mean is

$$\begin{aligned} E\{Z\} &= \int_{\frac{1}{2}}^{1} \left\{2z - \frac{1}{2z}\right\} dz + \int_{1}^{2} \left\{\frac{2}{z} - \frac{z}{2}\right\} dz = \left[z^2 - \frac{1}{2}\ln z\right]_{\frac{1}{2}}^{1} + \left[2\ln z - \frac{z^2}{4}\right]_{1}^{2} \\ &= 1 - \frac{1}{4} + \frac{1}{2}\ln\frac{1}{2} + 2\ln 2 - 1 + \frac{1}{4} = \frac{3}{2}\ln 2. \end{aligned}$$

3) For $\frac{1}{2} \le z \le 1$ the distribution function is given by

$$F_Z(z) = \int_{\frac{1}{2}}^{z} \left\{2 - \frac{1}{2t^2}\right\} dt = \left[2t + \frac{1}{2t}\right]_{\frac{1}{2}}^{z} = 2z + \frac{1}{2z} - 1 - 1 = 2z + \frac{1}{2z} - 2 = \frac{(2z-1)^2}{2z}.$$

When $z = 1$, we get $F_Z(1) = \frac{1}{2}$, so the median is $(Z) = 1$, and there is in this question no need to find the expression of the distribution function.

4) If we put $z = \frac{x}{y} \in \left]\frac{1}{2}, 2\right[$, then $u = z + \frac{1}{z}$, which has a minimum for $z = 1$ and is increasing for $z \in]1,2[$. It follows that U has its values in $\left[2, \frac{5}{2}\right[$.

The inequality $U = Z + \frac{1}{Z} < \frac{25}{12}$ is equivalent to $Z^2 - \frac{25}{12}Z + 1 < 0$, thus Z lies between the roots of the equation

$$z^2 - \frac{25}{12}z + 1 = 0.$$

These roots are

$$z = \frac{25}{24} \pm \sqrt{\left(\frac{25}{24}\right)^2 - 1} = \frac{25}{24} \pm \sqrt{\frac{49}{24} \cdot \frac{1}{24}} = \frac{25}{24} \pm \frac{7}{4} = \begin{cases} \dfrac{4}{3}, \\[2mm] \dfrac{3}{4}. \end{cases}$$

Then

$$P\left\{U < \frac{25}{12}\right\} = P\left\{\frac{3}{4} < Z < \frac{4}{3}\right\} = F_Z\left(\frac{4}{3}\right) - F_Z\left(\frac{3}{4}\right).$$

We shall now need the explicit expression of the distribution function $F_Z(z)$ when $z \in]1,2[$. We find

$$\begin{aligned} F_Z\left(\frac{4}{3}\right) &= \frac{1}{2}+\int_1^{\frac{4}{3}}\left\{\frac{2}{t^2}-\frac{1}{2}\right\}dt = \frac{1}{2}+\left[-\frac{2}{t}-\frac{t}{2}\right]_1^{\frac{4}{3}} = \frac{1}{2}-\frac{2}{\frac{4}{3}}-\frac{\frac{4}{3}}{2}+2+\frac{1}{2} \\ &= 3-\frac{3}{2}-\frac{2}{3}=\frac{3}{2}-\frac{2}{3}=\frac{5}{6}, \end{aligned}$$

hence by insertion

$$P\left\{U<\frac{25}{12}\right\}=\frac{5}{6}-\frac{\left(\frac{3}{2}-1\right)^2}{\frac{3}{2}}=\frac{5}{6}-\frac{1}{6}=\frac{4}{6}=\frac{2}{3}.$$

Example 5.5 *A 2-dimensional random variable (X,Y) has in the domain given by $0 \le x \le a$, $x \le y \le x+1$ (where $a>0$) the frequency*

$$h(x,y)=\frac{1}{a},$$

while the frequency is 0 everywhere else in the (x,y) plane.

1) *Find, possibly without first finding the marginal frequencies, the means $E\{X\}$ and $E\{Y\}$, the variances $V\{X\}$ and $V\{Y\}$, and the mean $E\{XY\}$.*

2) *Indicate, expressed by a, the correlation coefficient $\varrho(X,Y)$.*

3) *Find $\lim_{a\to\infty} \varrho(X,Y)$ and $\lim_{a\to 0} \varrho(X,Y)$.*

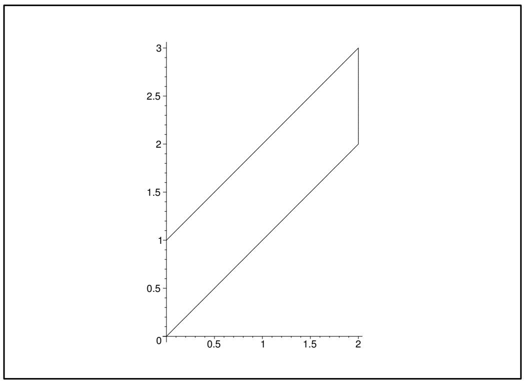


Figure 9: The domain for $a=2$.

1) It follows immediately that

$$f_X(x)=\begin{cases}\dfrac{1}{a} & \text{for } x\in]0,a[,\\[2mm] 0 & \text{otherwise.}\end{cases}$$

thus X is rectangularly distributed, hence

$$E\{X\} = \frac{a}{2} \qquad \text{and} \qquad V\{X\} = \frac{a^2}{12}.$$

Then

$$\begin{aligned} E\{Y\} &= \frac{1}{a}\int_0^a \left\{\int_x^{x+1} y\,dy\right\} dx = \frac{1}{2a}\int_0^a \left\{(x+1)^2 - x^2\right\} dx \\ &= \frac{1}{2a}\int_0^a (2x+1)\,dx = \frac{1}{2a}\left[x^2+x\right]_0^a = \frac{a^2+a}{2a} = \frac{a+1}{2}, \end{aligned}$$

and

$$\begin{aligned} E\left\{Y^2\right\} &= \frac{1}{a}\int_0^a \left\{\int_x^{x+1} y^2 dy\right\} dx = \frac{1}{3a}\int_0^a \left\{(x+1)^3 - x^3\right\} dx \\ &= \frac{1}{3a}\int_0^a \left\{3x^2+3x+1\right\} dx = \frac{1}{3a}\left[x^3 + \frac{3}{2}x^2 + x\right]_0^a \\ &= \frac{1}{3a}\left\{a^3 + \frac{3}{2}a^2 + a\right\} = \frac{1}{6}\left\{2a^2+3a+2\right\}, \end{aligned}$$

hence

$$\begin{aligned} V\{Y\} &= \frac{1}{6}\left\{2a^2+3a+2\right\} - \frac{1}{4}\,(a+1)^2 \\ &= \frac{1}{12}\left\{4a^2+6a+4-3a^2-6a-3\right\} = \frac{a^2+1}{12}. \end{aligned}$$

Finally,

$$\begin{aligned} E\{XY\} &= \frac{1}{a}\int_0^a x\left\{\int_x^{x+1} y\,dy\right\}dx = \frac{1}{2a}\int_0^a x\{2x+1\}\,dx = \frac{1}{2a}\int_0^a \left\{2x^2+x\right\}dx \\ &= \frac{1}{2a}\left\{\frac{2}{3}\,a^3+\frac{1}{2}\,a^2\right\} = \frac{1}{3}\,a^2+\frac{1}{4}\,a. \end{aligned}$$

2) It follows by insertion,

$$\begin{aligned} \mathrm{Cov}(X,Y) &= E\{XY\}-E\{X\}E\{Y\} = \frac{1}{3}\,a^2+\frac{1}{4}\,a-\frac{a}{2}\left(\frac{a+1}{2}\right) \\ &= \frac{a^2}{3}+\frac{a}{4}-\frac{a^2}{4}-\frac{a}{4} = \frac{a^2}{12}. \end{aligned}$$

This implies that

$$\varrho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V\{X\}V\{Y\}}} = \frac{a^2}{12}\cdot\frac{1}{\sqrt{\dfrac{a^2}{12}\cdot\dfrac{a^2+1}{12}}} = \frac{a}{\sqrt{a^2+1}}.$$

3) The limits are trivial,

$$\lim_{a\to\infty}\varrho(X,Y) = \lim_{\to\infty}\frac{a}{\sqrt{a^2+1}} = 1,$$

and

$$\lim_{a\to 0}\varrho(X,Y) = 0.$$

Example 5.6 *A 2-dimensional random variable* (X,Y) *has in the domain* D *given by* $0<x<1$, $0<y<1$, *the frequency*

$$f(x,y)=\frac{6}{5}\left(x+y^2\right),$$

while the frequency is 0 everywhere else in the (x,y) *plane.*

1) Find the frequencies and the distribution function of the random variables X *and* Y.

2) Find the means $E\{X\}$ *and* $E\{Y\}$, *the variances* $V\{X\}$ *and* $V\{Y\}$, *and the covariance* $\operatorname{Cov}(X,Y)$.

3) Find the distribution function $F(x,y)$ *of the 2-dimensional random variable* (X,Y) *in the domain* D.

4) Find the set M *of* all *points in the* (x,y) *plane, for which*

$$F(x,y)=\frac{7}{20},$$

and sketch the graph of the point set M.

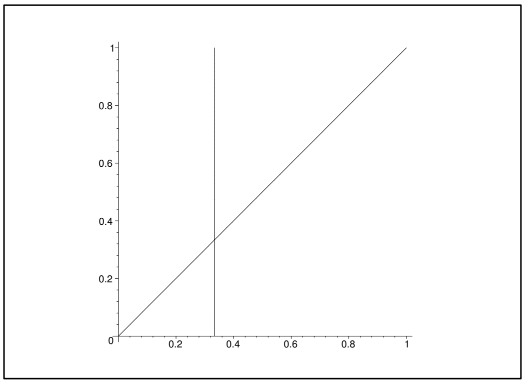


Figure 10: The domain D.

1) When $0<x<1$, then

$$f_X(x)=\frac{6}{5}\int_0^1\left(x+y^2\right)\,dy=\frac{6}{5}\left\{x+\frac{1}{3}\right\}=\frac{6}{5}\,x+\frac{2}{5},$$

and $f_X(x)=0$ otherwise.

When $0<y<1$, then

$$f_Y(y)=\frac{6}{5}\int_0^1\left(x+y^2\right)\,dx=\frac{6}{5}\left\{\frac{1}{2}+y^2\right\}=\frac{3}{5}+\frac{6}{5}\,y^2,$$

and $f_y(y)=0$ otherwise.

Summing up, the frequency of X is given by

$$f_X(x) = \begin{cases} \frac{6}{5}\,x + \frac{2}{5} & \text{for } 0 < x < 1, \\ \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding distribution function is

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ \\ \frac{3}{5}\,x^2 + \frac{2}{5}\,x, & 0 < x < 1, \\ \\ 1, & x \ge 1. \end{cases}$$

Analogously, the frequency of Y is given by

$$f_Y(y) = \begin{cases} \frac{3}{5} + \frac{6}{5}\,y^2 & \text{for } 0 < y < 1, \\ \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding distribution function is

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ \\ \frac{2}{5}\,y^3 + \frac{3}{5}\,y, & 0 < y < 1, \\ \\ 1, & y \ge 1. \end{cases}$$

2) The means are

$$E\{X\} = \int_0^1 \left\{ \frac{6}{5}\,x^2 + \frac{2}{5}\,x \right\} dx = \frac{2}{5} + \frac{1}{5} = \frac{3}{5},$$

and

$$E\{Y\} = \int_0^1 \left\{ \frac{3}{5}\,y + \frac{6}{5}\,y^3 \right\} dy = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}.$$

Furthermore,

$$E\left\{X^2\right\} = \int_0^1 \left\{ \frac{6}{5}\,x^3 + \frac{2}{5}\,x^2 \right\} dx = \frac{1}{5}\left\{ \frac{3}{2} + \frac{2}{3} \right\} = \frac{13}{30},$$

and

$$E\left\{Y^2\right\} = \frac{1}{5}\int_0^1 \left\{3y^2 + 6y^4\right\} dy = \frac{1}{5}\left\{ 1 + \frac{6}{5} \right\} = \frac{11}{25},$$

thus the variances are

$$V\{X\} = \frac{13}{30} - \frac{9}{25} = \frac{65-54}{150} = \frac{11}{150},$$

and

$$V\{Y\} = \frac{11}{25} - \frac{9}{25} = \frac{2}{25}.$$

Finally,

$$\begin{aligned} E\{XY\} &= \frac{6}{5}\int_0^1 x\left\{\int_0^1 \left(xy+y^3\right)\,dy\right\}dx = \frac{6}{5}\int_0^1 x\left\{\frac{x}{2}+\frac{1}{4}\right\}dx \\ &= \frac{3}{10}\int_0^1 \left(2x^2+x\right)\,dx = \frac{3}{10}\left\{\frac{2}{3}+\frac{1}{2}\right\} = \frac{3\cdot 7}{60} = \frac{7}{20}, \end{aligned}$$

hence the covariance is

$$\operatorname{Cov}(X,Y) = E\{XY\} - E\{X\}\cdot E\{Y\} = \frac{7}{20} - \frac{9}{25} = \frac{35-36}{100} = -\frac{1}{100}.$$

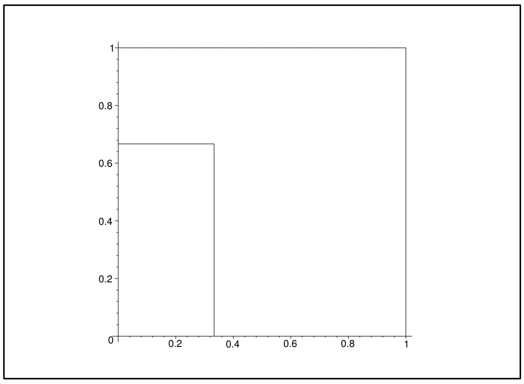


Figure 11: The domain of integration for the determination of $F(x,y)$.

3) If $(x,y) \in D$, i.e. $0 < x < 1$ and $0 < y < 1$, then the distribution function is given by

$$\begin{aligned} F(x,y) &= \frac{6}{5}\int_0^x \left\{\int_0^y \left(t+u^2\right)\,du\right\} dt = \frac{6}{5}\int_0^x \left\{ty + \frac{1}{3}\,y^3\right\} dt \\ &= \frac{6}{5}\left\{\frac{1}{2}\,x^2 y + \frac{1}{3}\,xy^3\right\} = \frac{1}{5}\,\left(3x^2y + 2xy^3\right) = \frac{1}{5}\,xy\left(3x+2y^2\right). \end{aligned}$$

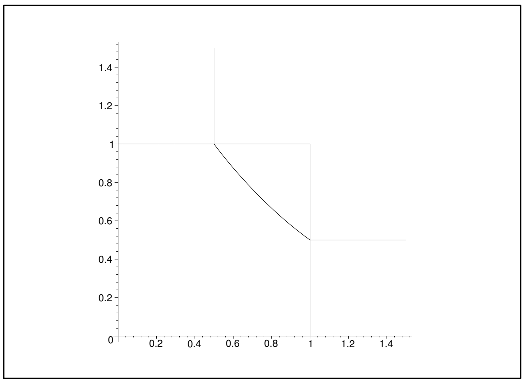


Figure 12: The curve M, where $F(x,y) = \dfrac{7}{20}$.

4) We have in D,

$$F(x,y) = \frac{3}{5}\,y\cdot x^2 + \frac{2}{5}\,y^3\cdot x = \frac{7}{20},$$

when

$$(12y)\cdot x^2 + \left(8y^3\right)x - 7 = 0.$$

Since $y \neq 0$ for every solution, we find by solving with respect to x that

$$x = \frac{-8y^3 + \sqrt{64y^6 + 4\cdot 7\cdot 12y}}{24y} = \frac{\sqrt{4y^6+21y} - 2y^3}{6y}.$$

If we in particular choose $y = 1$, then $x = \frac{1}{6}\{\sqrt{4+21} - 2\} = \frac{1}{2}$- Then $F\left(\frac{1}{2}, y\right) = \frac{7}{20}$ for every $y \geq 1$.
Choosing $x = 1$, the equation is reduced to $8y^3 + 12y - 7 = 0$, the only solution of which in $[0,1]$ is $y = \frac{1}{2}$. Then $F\left(x, \frac{1}{2}\right) = \frac{7}{20}$ for every $x \geq 1$.

Example 5.7 *A point set D in the (x,y) plane is the union of the following two sets*

$$D_1 = \left\{(x,y) \;\middle|\; 0 \leq x \leq 1,\, 0 \leq y \leq \frac{x}{2}\right\},$$

$$D_2 = \left\{(x,y) \;\middle|\; 0 \leq x \leq 1,\, \frac{1+x}{2} \leq y \leq 1\right\}.$$

A 2-dimensional random variable (X,Y) has in D the frequency $f(x,y) = 2$, while the frequency is 0 everywhere else in the plane.

1) Find the frequencies $f_X(x)$ and $f_Y(y)$ of the random variable X and Y.

2) Find the means $E\{X\}$ and $E\{Y\}$ and the variances $V\{X\}$ and $V\{Y\}$.

3) Find the covariance $\operatorname{Cov}(X,Y)$.

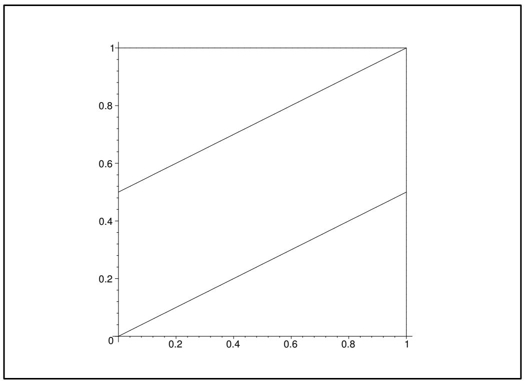


Figure 13: The subdomain D_1 is the lower triangle and the subdomain D_2 is the upper triangle.

1) By mental arithmetic (i.e. it is strictly speaking a vertical integration) it follows that

$$f_X(x) = \begin{cases} 1 & \text{for } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

so X is rectangularly distributed over $[0,1]$.

When $y \in \left[0, \frac{1}{2}\right]$, we get by a horizontal integration (also mental arithmetic) that

$$f_Y(y) = 2 \cdot (1 - 2y) = 2 - 4y.$$

If on the other hand, $y \in \left[\frac{1}{2}, 1\right]$, then analogously,

$$f_Y(y) = 2 \cdot (2y - 1) = 4Y - 2.$$

Summing up,

$$f_Y(y) = \begin{cases} 2 - 4y & \text{for } y \in \left[0, \frac{1}{2}\right], \\ 4y - 2 & \text{for } y \in \left]\frac{1}{2}, 1\right], \\ 0 & \text{otherwise,} \end{cases}$$

which is reduced to

$$f_Y(y) = \begin{cases} 2\,|2y - 1| & \text{for } y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

2) Since X is rectangularly distributed, we have

$$E\{X\} = \frac{1}{2} \qquad \text{and} \qquad V\{X\} = \frac{1}{12}.$$

It follows by the symmetry that $E\{Y\} = \frac{1}{2}$.

Alternatively, this follows by the computation

$$\begin{aligned} E\{Y\} &= \int_0^{\frac{1}{2}} \left(2y - 4y^2\right) dy + \int_{\frac{1}{2}}^1 \left(4y^2 - 2y\right) dy = \left[y^2 - \frac{4}{3}\,y^3\right]_0^{\frac{1}{2}} + \left[\frac{4}{3}\,y^3 - y^2\right]_{\frac{1}{2}}^1 \\ &= \frac{1}{4} - \frac{1}{6} + \frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

Furthermore,

$$\begin{aligned} E\left\{Y^2\right\} &= \int_0^{\frac{1}{2}} \left(2y^2 - 4y^3\right) dy + \int_{\frac{1}{2}}^1 \left(4y^3 - 2y^2\right) dy = \left[\frac{2}{3}\,y^3 - y^4\right]_0^{\frac{1}{2}} + \left[y^4 - \frac{2}{3}\,y^3\right]_{\frac{1}{2}}^1 \\ &= \frac{1}{6} - \frac{1}{16} + 1 - \frac{2}{3} - \frac{1}{16} + \frac{1}{6} = \frac{1}{3} - \frac{1}{8} + \frac{1}{3} = \frac{2}{3} - \frac{1}{8} = \frac{16 - 3}{24} = \frac{13}{24}, \end{aligned}$$

hence

$$V\{Y\} = \frac{13}{24} - \frac{1}{4} = \frac{7}{24}.$$

3) First compute

$$
\begin{aligned}
E\{XY\} &= 2\int\!\!\int_{D_1} xy\,dx\,dy + 2\int\!\!\int_{D_2} xy\,dx\,dy \\
&= 2\int_0^1 x\left\{\int_0^{\frac{x}{2}} y\,dy\right\} dx + 2\int_0^1 \left\{\int_{\frac{1+x}{2}}^1 y\,dy\right\} dx = \int_0^1 x\left\{\left(\frac{x}{2}\right)^2 + 1 - \left(\frac{1+x}{2}\right)^2\right\} dx \\
&= \int_0^1 x\left\{\frac{1+2x}{2}\cdot\left(-\frac{1}{2}\right)+1\right\} dx = \frac{1}{4}\int_0^1 x(4-1-2x)\,dx \\
&= \frac{1}{4}\int_0^1 \left(3x-2x^2\right)\,dx = \frac{1}{4}\left(\frac{3}{2}-\frac{2}{3}\right) = \frac{1}{4}\cdot\frac{0-4}{6} = \frac{5}{24}.
\end{aligned}
$$

We finally get

$$
\operatorname{Cov}(X,Y) = E\{XY\} - E\{X\}E\{Y\} = \frac{5}{24} - \frac{1}{2}\cdot 12 = -\frac{1}{24}.
$$

Example 5.8 *A 2-dimensional random variable (X,Y) has in the domain $D =]0,1[\times]0,1[$ the frequency*

$$f(x,y) = 3\left(xy^2 + yx^2\right),$$

while the frequency is 0 everywhere else in the (x,y) plane.

1. *Find the frequency $f_X(x)$ and the distribution function $F_X(x)$ of the random variable X.*

2. *Compute the mean $E\{X\}$ and the variance $V\{X\}$.*

3. *Find for every real number k the simultaneous distribution function $F(x,y)$ of (X,Y) at the point (k,k).*

4. *Find the probability that both X and Y are smaller than $\frac{1}{2}$.*

5. *Find the probability that both X and Y are bigger than $\frac{1}{2}$.*

The parabolic arcs $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$, divide D into three domains D_1, D_2, D_3.

6. *Find the probabilities*

$$P\{(X,Y)\in D_1\}, \quad P\{(X,y)\in D_2\} \quad \text{and} \quad P\{(X,Y)\in D_3\}$$

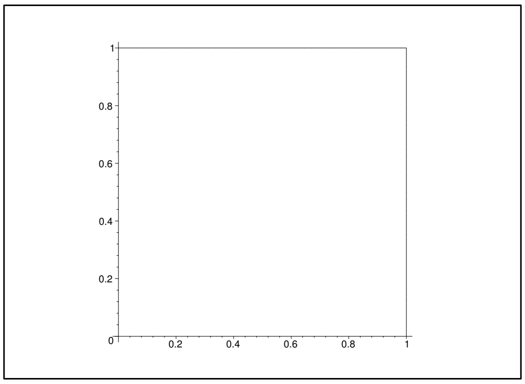


Figure 14: The domain D.

1) When $x \in]0,1[$, we get by a vertical integration,

$$f_X(x) = 3\int_0^1 \left(xy^2 + y^2x\right)\,dy = x + \frac{3}{2}\,x^2,$$

thus the frequency is

$$f_X(x) = \begin{cases} x + \dfrac{3}{2}\,x^2, & x \in]0,1[, \\[2mm] 0, & \text{otherwise,} \end{cases}$$

and the distribution function is

$$F(x) = \begin{cases} 0, & x \le 0, \\[2mm] \dfrac{1}{2}\left(x^2 + x^3\right), & 0 < x < 1, \\[2mm] 1, & x \ge 1. \end{cases}$$

2) The mean is

$$E\{X\} = \int_0^1 \left\{ x^2 + \frac{3}{2}, x^3 \right\} dx = \frac{1}{3} + \frac{3}{8} = \frac{8+9}{24} = \frac{17}{24}.$$

Since

$$E\left\{X^2\right\} = \int_0^1 \left\{ x^2 + \frac{3}{2}\, x^4 \right\} dx = \frac{1}{4} + \frac{3}{10} = \frac{5+6}{20} = \frac{11}{20},$$

the variance becomes

$$V\{X\} = \frac{11}{20} - \left(\frac{17}{24}\right)^2 = \frac{139}{2880}.$$

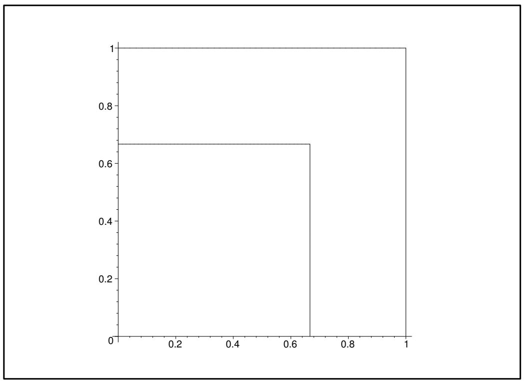


Figure 15: The domain of integration for the determination of $F(k,k)$; here $k = \dfrac{2}{3}$.

3) When $k \le 0$, then $F(k,k) = 0$, and when $k \ge 1$, then $F(k,k) = 1$.
When $0 < k < 1$, then

$$\begin{aligned} F(k,k) &= \int_0^k \left\{ \int_0^k \left(3xy^2 + 3yx^2\right)\, dy \right\} dx = \int_0^k \left[xy^3 + \frac{3}{2}\, y^2x^2 \right]_{y=0}^k dx \\ &= \int_0^k \left\{ k^3x + \frac{3}{2}\, k^2x^2 \right\} dx = \left[\frac{1}{2}\, k^3x^2 + \frac{1}{2}\, k^2x^3 \right]_0^k = k^5. \end{aligned}$$

4) The probability that both X and Y are $\leq \dfrac{1}{2}$, is

$$P\left\{X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right\} = F\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

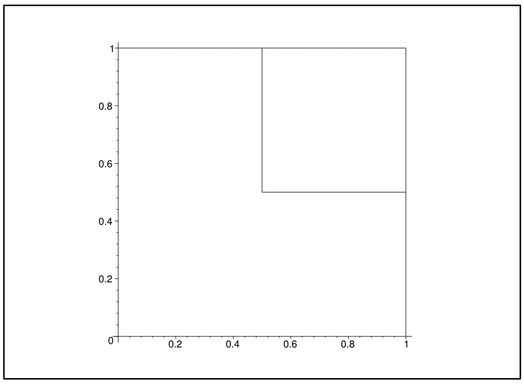


Figure 16: The domain of integration of question 5 is the upper square.

5) The probability that both X and Y are $\geq \dfrac{1}{2}$, is due to the symmetry,

$$\begin{aligned} P\left\{X \geq \frac{1}{2}, Y \geq \frac{1}{2}\right\} &= 1 - P\left\{P \geq \frac{1}{2}\right\} - P\left\{Y \geq \frac{1}{2}\right\} + P\left\{X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right\} \\ &= 1 - 2\,P\left\{X \leq \frac{1}{2}\right\} + F\left(\frac{1}{2}, \frac{1}{2}\right) = 1 - 2\,F_X\left(\frac{1}{2}\right) + F\left(\frac{1}{2}, \frac{1}{2}\right) \\ &= 1 - \left(\frac{1}{4} + \frac{1}{8}\right) + \frac{1}{32} = \frac{32 - 8 - 4 + 1}{32} = \frac{21}{32}. \end{aligned}$$

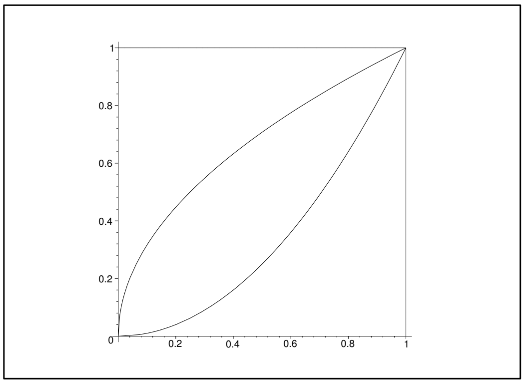


Figure 17: The domains D_1 (down most), D_2 (in the middle) and D_3 (uppermost).

6) It follows by the symmetry that

$$P\{(X,Y)\in D_1\} = P\{(X,Y)\in D_3\},$$

hence

$$P\{(X,Y)\in D_2\} = 1 - 2\,P\{(X,Y)\in D_1\}.$$

Then by a planar integral,

$$\begin{aligned} P\{(X,Y)\in D_1\} &= \int_0^1 \left\{\int_0^{x^2} \left(3xy^2+3yx^2\right)dy\right\}dx = \int_0^1 \left[xy^3+\frac{3}{2}\,y^2x^2\right]_0^{x^2} dx \\ &= \int_0^1 \left\{x^7+\frac{3}{2}\,x^6\right\}dx = \frac{1}{8}+\frac{3}{14} = \frac{7+12}{56} = \frac{19}{56}, \end{aligned}$$

hence

$$P\{(X,y)\in D_1\} = P\{(X,Y)\in D_3\} = \frac{19}{56},$$

and

$$P\{(X,Y)\in D_2\} = 1-\frac{19}{28} = \frac{9}{28}.$$

Example 5.9 *Given for every $k \in]0,1[$ a function f_k by*

$$f_k(x) = \begin{cases} k\,e^{-x} + 2(1-k)e^{-2x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

1) Prove that $f_k(x)$ is a frequency of a random variable, which is denoted by X_k.

2) Find the distribution function $F_k(x)$ of X_k.

3) Find the mean and variance of X_k.

4) Find the median of $X_{\frac{1}{2}}$.

5) The random variable Y_k is given by

$$Y_k = \exp\left(\frac{X_k}{2}\right).$$

Find the distribution function and the frequency of Y_k, and compute the mean $E\{Y_k\}$.

6) Then assume that the random variable $X_{\frac{1}{2}}$ is observed twice by independent observations. Find the probability that the second observation is bigger than the half of the first one.

1) When $k \in]0,1[$, then $f_k(x) \ge 0$. Then by an integration

$$\int_{-\infty}^{\infty} f_k(x)\,dx = \int_0^{\infty} \left\{k\,e^{-x} + 2(1-k)e^{-2k}\right\} dx = k + 2 \cdot \frac{1}{2}\,(1-k) = 1,$$

thus $f_k(x)$ is a frequency of a random variable X_k.

2) When $x \le 0$, then $F_k(x) = 0$. When $x > 0$, then

$$\begin{aligned} F_k(x) &= \int_0^x \left\{k\,e^{-t} + 2(1-k)e^{-2t}\right\} dt = \left[-k\,e^{-t} - (1-k)e^{-2t}\right]_0^x \\ &= 1 - k\,e^{-x} - (1-k)e^{-2k}, \end{aligned}$$

hence summing up,

$$F_k(x) = \begin{cases} 1 - k\,e^{-x} - (1-k)e^{-2x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

3) The mean is

$$E\{X\} = \int_0^{\infty} \left\{k \cdot x\,e^{-x} + (1-k) \cdot 2x\,e^{-2x}\right\} dx = k \cdot 1! + \frac{1}{2}\,(1-k) \cdot 1! = \frac{1+k}{2}.$$

Furthermore,

$$\begin{aligned} E\left\{X^2\right\} &= \int_0^{\infty} \left\{k \cdot x^2 e^{-x} + 2(1-k) \cdot x^2 e^{-2x}\right\} dx \\ &= k \cdot 2! + \frac{1}{4}\,(1-k)2! = 2k + \frac{1-k}{2} = \frac{1+3k}{2}, \end{aligned}$$

so the variance becomes

$$V\{X\} = \frac{1+3k}{2} - \left(\frac{1+k}{2}\right)^2 = \frac{1}{4}\left\{2+6k-1-2k-k^2\right\} = \frac{1+4k-k^2}{4}.$$

4) The median $\left(X_{\frac{1}{2}}\right)$ is the solution of $F_{\frac{1}{2}}(x) = \dfrac{1}{2}$, i.e. of the equation

$$1 - \frac{1}{2}\,e^{-x} - \frac{1}{2}\left(e^{-x}\right)^2 = \frac{1}{2}.$$

This is rewritten as the equation of second degree in e^{-x},

$$\left(e^{-x}\right)^2 + \left(e^{-x}\right) - 1 = 0,$$

hence

$$e^{-x} = -\frac{1}{2}\,(\pm)\,\frac{\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2} = \frac{2}{\sqrt{5}+1},$$

and whence

$$\left(X_{\frac{1}{2}}\right) = \ln\left(\frac{\sqrt{5}+1}{2}\right).$$

5) The image of Y_k is $]1,\infty[$. When $y \in]1,\infty[$, then

$$\begin{aligned} F_{Y_k}(y) &= P\left\{Y_k = \exp\left(\frac{X_k}{2}\right) \le y\right\} = P\left\{X_k \le 2\,\ln y\right\} = F_k(2\,\ln y) \\ &= 1 - k\,e^{-2\ln y} - (1-k)e^{-2\cdot 2\ln y} = 1 - \frac{k}{y^2} - \frac{1-k}{y^4}, \end{aligned}$$

hence the distribution function is

$$F_{Y_k}(y) = \begin{cases} 1 - \dfrac{k}{y^2} - \dfrac{1-k}{y^4} & \text{for } y > 1, \\[2ex] 0 & \text{for } y \le 1. \end{cases}$$

The corresponding frequency is obtained by a differentiation,

$$f_{Y_k}(y) = \begin{cases} \dfrac{2k}{y^3} + \dfrac{4(1-k)}{y^5} & \text{for } y > 1, \\[2ex] 0 & \text{for } y \le 1. \end{cases}$$

The mean is

$$E\left\{Y_k\right\} = \int_1^\infty y\,f_{Y_k}(y)\,dy = \int_1^\infty \left\{\frac{2k}{y^2} + \frac{4(1-k)}{y^4}\right\}dy = 2k + \frac{4}{3}\,(1-k) = \frac{2}{3}\,(k+2).$$

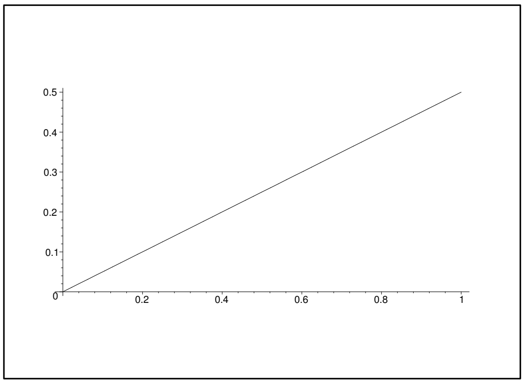


Figure 18: The domain of integration of question 6 lies in the first quadrant above the oblique line.

6) Let $X'_{1/2}$ and $X''_{1/2}$ be two independent random variables, both of the frequency $f_{1/2}$. Then

$$
\begin{aligned}
P\left\{X''_{1/2} > \frac{1}{2}\,X'_{1/2}\right\} &= \int_{x=0}^{\infty}\left\{\int_{y=\frac{1}{2}\,x}^{\infty} f_{1/2}(x)\,f_{1/2}(y)\,dy\right\}dx \\
&= \int_{x=0}^{\infty}\frac{1}{2}\left(e^{-x}+2\,e^{-2x}\right)\cdot\left\{\int_{y=\frac{1}{2}\,x}^{\infty}\frac{1}{2}\left(e^{-y}+2\,e^{-2y}\right)dy\right\}dx \\
&= \int_{x=0}^{\infty}\frac{1}{2}\left(e^{-x}+2\,e^{-2x}\right)\cdot\left(\frac{1}{2}\,e^{-x/2}+\frac{1}{2}\,e^{-x}\right)dx \\
&= \int_{0}^{\infty}\left\{\frac{1}{4}\,e^{-3x/2}+\frac{1}{4}\,e^{-2x}+\frac{1}{2}\,e^{-5x/2}+\frac{1}{2}\,e^{-3x}\right\}dx \\
&= \frac{1}{4}\cdot\frac{2}{3}+\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{2}{5}+\frac{1}{2}\cdot\frac{1}{3} = \frac{79}{120}.
\end{aligned}
$$

Example 5.10 *A rectangle has the side lengths X_1 and X_2, where X_1 and X_2 are independent random variables, and where X_1 and X_1 are both rectangularly distributed over $]1,2[$.*
Let $Y_1 = 2X_1 + 2X_2$ denote the circumference of the rectangle, and let $Y_2 = X_1X_2$ denote the area of the rectangle.

1) *Compute the mean and the variance of Y_1.*
2) *Compute the mean and the variance of Y_2.*
3) *Compute the covariance $\operatorname{Cov}(Y_1, Y_2)$.*
4) *Compute the correlation coefficient $\varrho(Y_1, Y_2)$.*
5) *Compute the frequency of Y_1.*
6) *Compute the frequency of Y_2.*

1) Since X_1 and X_2 are independent, and e.g.

$$E\{X_i\} = \int_1^2 t\,dt = \frac{3}{2},$$

which of course also can be seen directly, we get

$$E\{Y_1\} = 2\,E\{X_1\} + 2\,E\{X_2\} = 4\int_1^2 t\,dt = 4 \cdot \frac{3}{2} = 6,$$

and

$$V\{Y_1\} = 2^2 V\{X_1\} + 2^2 V\{X_2\} = 8\int_1^2 \left(t - \frac{3}{2}\right)^2 dt = \frac{8}{3}\left[\left(t-\frac{3}{2}\right)^3\right]_1^2 = \frac{2}{3},$$

just to demonstrate a couple of the possible variants. (There are of course more direct method by e.g. applying that the mean and variance are known for the rectangular distribution).

2) For the same reason,

$$E\{Y_2\} = E\{X_1\} \cdot E\{X_2\} = \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}.$$

Furthermore,

$$E\left\{Y_2^2\right\} = E\left\{X_1^2\right\} \cdot E\left\{X_2^2\right\} = \left\{\int_1^2 x^2 dx\right\}^2 = \left\{\left[\frac{1}{3}\,x^3\right]_1^2\right\}^2 = \left(\frac{7}{3}\right)^2 = \frac{49}{9},$$

hence

$$V\{Y_2\} = E\left\{Y_2^2\right\} - (E\{Y_2\})^2 = \frac{49}{9} - \left(\frac{9}{4}\right)^2 = \frac{49}{9} - \frac{81}{16} = \frac{55}{144}.$$

3) Since the covariance is bilinear, we get by insertion of $Y_1 = 2X_1 + 2X_2$ and $Y_2 = X_1 X_2$ that

$$\begin{aligned}
\operatorname{Cov}(Y_1, Y_2) &= \operatorname{Cov}(2X_1 + 2X_2, X_1 X_2) \\
&= 2\operatorname{Cov}(X_1, X_1 X_2) + 2\operatorname{Cov}(X_2, X_1 X_2) = 4\operatorname{Cov}(X_1, X_1 X_2) \\
&= 4\left(E\{X_1 \cdot X_1 X_2\} - E\{X_1\} \cdot E\{X_1 X_2\}\right) \\
&= 4\left(E\{X_1^2\} \cdot E\{X_2\} - (E\{X_1\})^2 \cdot E\{X_2\}\right) \\
&= 4\left(E\{X_1^2\} - (E\{X_1\})^2\right) \cdot E\{X_2\} = 4\,V\{X_1\} \cdot E\{X_2\} \\
&= 4 \cdot \frac{1}{12} \cdot \frac{3}{2} = \frac{1}{2},
\end{aligned}$$

because it follows by question 1 that

$$V\{X_1\} = \frac{1}{8} V\{Y_1\} = \frac{1}{8} \cdot \frac{2}{3} = \frac{1}{12},$$

which we also can obtain directly by using that X_1 is rectangularly distributed.

4) We have now

$$\operatorname{Cov}(Y_1, Y_2) = \frac{1}{2}, \qquad V\{Y_1\} = \frac{2}{3}, \qquad V\{Y_2\} = \frac{55}{144},$$

so the correlation coefficient is

$$\varrho\left(Y_1, Y_2\right)=\frac{\operatorname{Cov}\left(Y_1, Y_2\right)}{\sqrt{X\left\{Y_1\right\} \cdot V\left\{Y_2\right\}}}=\frac{\frac{1}{2}}{\sqrt{\frac{2}{3} \cdot \frac{55}{144}}}=\frac{\sqrt{3} \cdot 12}{2 \sqrt{110}}=\frac{6 \sqrt{3}}{\sqrt{110}}=\sqrt{\frac{54}{55}}=\frac{3 \sqrt{330}}{55},$$

where there are more possibilities of the indication of the result.

5) First compute the frequency of X_1+X_2:

$$g(s)=\int_{-\infty}^{\infty} f(x)\, f(s-x)\, dx,$$

where

$$f(x)= \begin{cases}1 & \text { for } x \in]1,2[, \\ 0 & \text { otherwise. }\end{cases}$$

If $g(s) \neq 0$, then we must have the restrictions

$$1<x<2 \qquad \text{og} \qquad 1<s-x<2,$$

i.e. after a rearrangement

$$1<x<2 \qquad \text{and} \qquad s-2<x<s-1.$$

Then we have two possibilities,

a) When $s \in]2,3[$, then $g(s)=\int_1^{s-1} 1\, dx=s-2$.
b) When $s \in]3,4[$, then $g(s)=\int_{s-2}^2 1\, dx=4-s$.

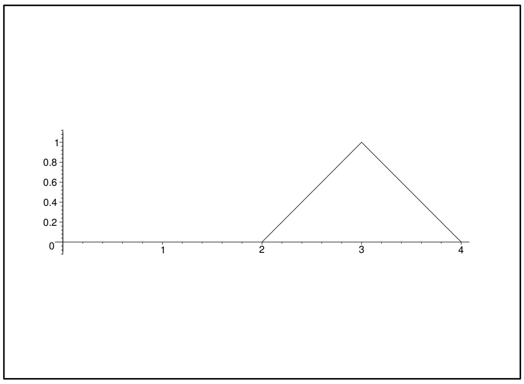


Figure 19: The graph of $g(s)$.

Summing up we get

$$g(s)= \begin{cases}s-2, & \text { for } 2 \leq s \leq 3, \\ 4-s, & \text { for } 3<s \leq 4, \\ 0, & \text { otherwise. }\end{cases}$$

The random variable $Y_1 = 2(X_1 + X_2)$ has the frequency

$$h(s) = \frac{1}{2}\, g\left(\frac{s}{2}\right),$$

where $\frac{s}{2} \in]2,4[$ for $s \in]4,8[$, i.e.

$$(2)\;\; h(s) = \begin{cases} \frac{1}{2}\left(\frac{s}{2} - 2\right) = \frac{s}{4} - 1, & \text{for } 4 \le s \le 6, \\[2mm] \frac{1}{2}\left(4 - \frac{s}{2}\right) = 2 - \frac{s}{4}, & \text{for } 6 < s \le 8, \\[2mm] 0, & \text{otherwise.} \end{cases}$$

Alternatively, it is seen that $2X_1$ and $2X_2$ are both rectangularly distributed over $]2,4[$.

Alternatively we consider a figure in order to determine the *the distribution function* of $2X_1 + 2X_2$. We have two cases:

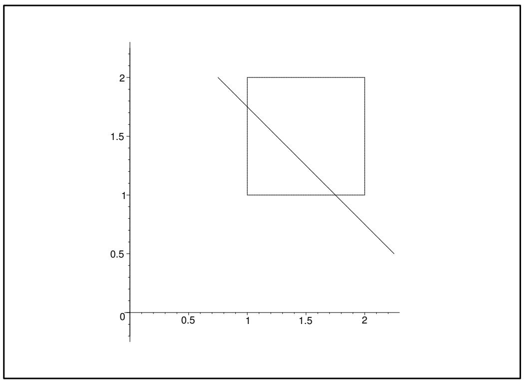


Figure 20: When $4 \le s \le 6$, then the distribution function $H(s)$ is the area of the triangle of the figure.

a) When $4 \le s \le 6$, then the distribution function is equal to the area of the triangle on figure 5, the smaller sides of which both have the length $\frac{s}{2} - 1$, thus

$$H(s) = \frac{1}{2}\left(\frac{s}{2} - 2\right)^2.$$

We get the frequency by a differentiation,

$$h(s) = \left(\frac{s}{2} - 2\right) \cdot \frac{1}{2} = \frac{s}{4} - 1.$$

b) When $6 \le s \le 8$, then the distribution function is equal to the area of the square minus the area of the triangle on figure 5a, hence

$$H(s) = 1 - \frac{1}{2}\left(4 - \frac{s}{2}\right)^2.$$

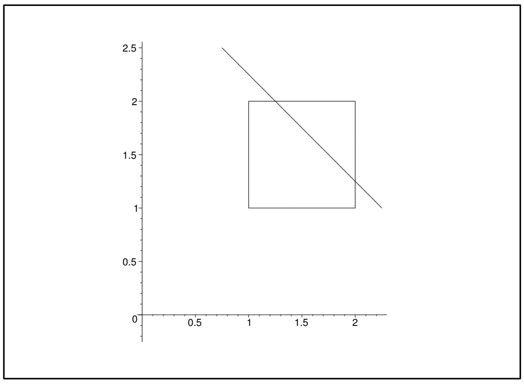


Figure 21: When $4 \le s \le 6$, then the distribution function $H(s)$ is the area of the square minus the area of the triangle on the figure.

We get the frequency by a differentiation,

$$h(s) = \left(4 - \frac{s}{2}\right) \cdot \frac{1}{2} = 2 - \frac{s}{2}.$$

Summing up that we again obtain (2).

6) The frequency of $Y_2 = X_1 X_2$ is

$$k(s) = \int_{-\infty}^{\infty} f(x)\, f\left(\frac{s}{x}\right) \frac{1}{|x|}\, dx.$$

If the integrand is $\neq 0$, then we must have $1 < x < 2$ and $1 < \dfrac{s}{x} < 2$, thus

$$1 < x < 2 \qquad \text{and} \qquad \frac{s}{2} < x < s.$$

Again we have two cases.

a) If $s \in]1,2[$, then

$$k(s) = \int_1^s \frac{1}{x}\, dx = \ln s.$$

b) If $s \in]2,4[$, then

$$k(s) = \int_{\frac{s}{2}}^{2} \frac{1}{x}\, dx = \ln 2 - \ln \frac{s}{2} = \ln 4 - \ln s.$$

Hence we get

$$(3)\quad k(s) = \begin{cases} \ln s, & \text{for } 1 < s \le 2, \\ \ln 4 - \ln s = \ln \dfrac{4}{s}, & \text{for } 2 < s < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively one may again apply a consideration of a figure in the determination of *the distribution function* of X_1X_2, where we again must consider two cases:

a) When $1 < s < 2$, then the distribution function $H(s)$ is equal to the area of the curvilinear triangle on the figure on the next page, hence

$$K(s) = \int_1^s \left(\frac{s}{x} - 1\right) dx = s \ln s - s + 1.$$

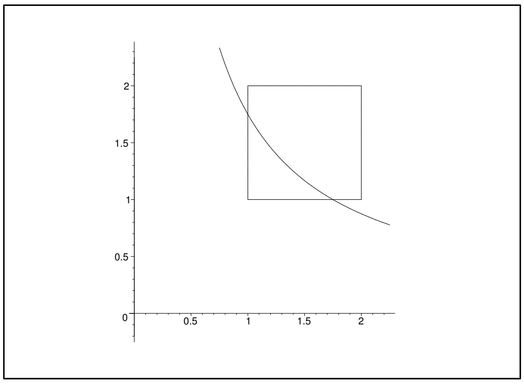


Figure 22: The distribution function $H(s)$ is the area of the curvilinear triangle.

We obtain the frequency by a differentiation,

$$k(s) = \ln 2, \qquad \text{for } 1 < s < 2.$$

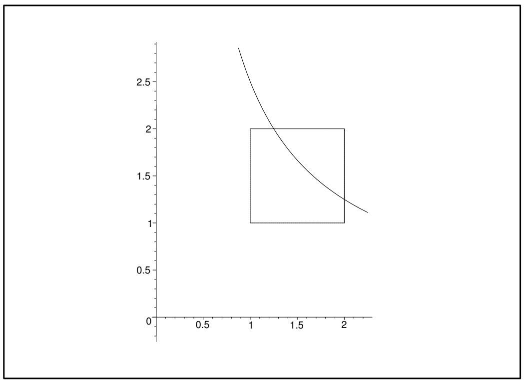


Figure 23: The distribution function is the area of the square minus the area of the curvilinear triangle.

b) When $2 < s < 4$, then $H(s)$ is the area of the square minus the area of the curvilinear triangle, hence

$$K(s) = 1 - \int_{\frac{s}{2}}^{2} \left\{1 - \left(\frac{s}{x} - 1\right)\right\} dx = 1 - \int_{\frac{s}{2}}^{2} \left(2 - \frac{s}{x}\right) dx = -3 + s\,\ln 4 + s - s\,\ln s.$$

We obtain the frequency by a differentiation,

$$k(s) = \ln 4 - \ln s, \qquad \text{for } 2 < s < 4.$$

Summing up we again obtain (3).

Example 5.11 *A 2-dimensional random variable* (X,Y) *has the frequency*

$$f(x,y)=\begin{cases} 4x(1-y), & 0<x<1,\, 0<y<1,\\[2mm] 0, & \textit{otherwise}.\end{cases}$$

1) Prove that the random variables X *and* Y *are independent.*

2) Find the means $E\{X\}$ *and* $E\{Y\}$.

3) Find the variances $V\{X\}$ *and* $V\{Y\}$.

4) Find the frequency of the random variable $X-Y$.

5) Let C *denote the disc* $x^2+y^2\le 1$. *Compute* $P\{(X,Y)\in C\}$.

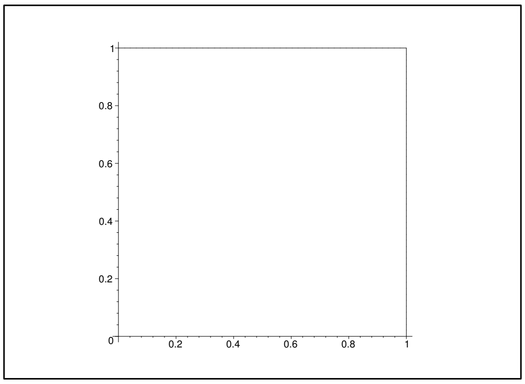


Figure 24: The domain D, where $f(x,y)\neq 0$.

1) It follows immediately that

$$f_X(x)=\begin{cases} 2x, & 0<x<1,\\[2mm] 0, & \text{otherwise},\end{cases}$$

and

$$f_Y(y)=\begin{cases} 2(1-y), & 0<y<1,\\[2mm] 0, & \text{otherwise},\end{cases}$$

Furthermore,

$$f(x,y)=f_X(x)\cdot f_Y(y),$$

so X and Y are stochastically independent.

2) The means are given by

$$E\{X\} = \int_0^1 2x^2\,dx = \frac{2}{3}$$

and

$$E\{Y\} = \int_0^1 \left(2y - 2y^2\right)\,dy = 1 - \frac{2}{3} = \frac{1}{3}.$$

3) It follows from

$$E\left\{X^2\right\} = \int_0^1 2x^3\,dx = \frac{1}{2},$$

that

$$V\{X\} = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Similarly

$$E\left\{Y^2\right\} = \int_0^1 \left(2y^2 - 2y^3\right)\,dy = \frac{2}{3} - \frac{1}{2} = \frac{1}{6},$$

implies that

$$V\{Y\} = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

4) The random variable $Z = X - Y$ has its values in $]-1,1[$. The frequency is for $-1 < z < 1$ given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\,f_Y(x-z)\,dx.$$

The integrand is $\neq 0$, when $0 < x < 1$ and $0 < x - z < 1$, i.e. when

$$0 < x < 1 \qquad \text{and} \qquad z < x < z+1.$$

We shall then split into two cases:

a) If $z \in]-1,0]$, then the domain of integration is $]0, z+1[$, thus

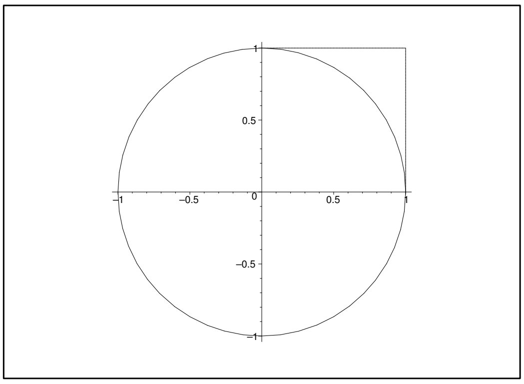
$$\begin{aligned} f_Z(z) &= \int_0^{z+1} f_X(x)\,f_Y(x-z)\,dx = 4\int_0^{z+1} x(1+z-x)\,dx \\ &= \left[2(1+z)x^2 - \frac{4}{3}\,x^3\right]_0^{z+1} = 2(1+z)^3 - \frac{4}{3}\,(1+z)^3 = \frac{2}{3}\,(1+z)^3. \end{aligned}$$

b) If $z \in]0,1[$, then the domain of integration is $]z,1[$, thus

$$\begin{aligned} f_Z(z) &= \left[2(1+z)x^2 - \frac{4}{3}\,x^3\right]_z^1 = 2(1+z) - \frac{4}{3} - 2(1+z)z^2 + \frac{4}{3}\,z^3 \\ &= \frac{2}{3} + 2z - \frac{2}{3}\,z^3 - 2z^2 = \frac{2}{3}\left\{1 - z^3 + 3z(1-z)\right\} \\ &= \frac{2}{3}\left\{1 + 3z - 3z^2 - z^3\right\} \qquad \left(= \frac{2}{3}\,(1-z)\left(1+4z+z^2\right)\right). \end{aligned}$$

Summing up,

$$f_Z(z) = \begin{cases} \dfrac{2}{3}\left(1+3z+3z^2+z^3\right) & \text{for } z \in]-1,0], \\ \dfrac{2}{3}\left(1+3z-3z^2-z^3\right) & \text{for } z \in]0,1[, \\ 0 & \text{otherwise.} \end{cases}$$



c) By considering a figure it follows by using polar coordinates.

$$\begin{aligned}
P\{(X,Y)\in C\} &= \int\!\!\int_C f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{2}}\left\{\int_0^1 4r\,\cos\varphi\,(1-r\,\sin\varphi)r\,dr\right\}d\varphi \\
&= \int_0^{\frac{\pi}{2}}\left\{\int_0^1\left(4r^2\cos\varphi-4r^3\cos\varphi\,\sin\varphi\right)dr\right\}d\varphi \\
&= \int_0^{\frac{\pi}{2}}\left(\frac{4}{3}\cos\varphi-\cos\varphi\cdot\sin\varphi\right)d\varphi = \left[\frac{4}{3}\sin\varphi-\frac{1}{2}\sin^2\varphi\right]_0^{\frac{\pi}{2}} \\
&= \frac{4}{3}-\frac{1}{2}=\frac{5}{6}.
\end{aligned}$$

Example 5.12 *A 2-dimensional random variable (X,Y) has the frequency*

$$f(x,y)=\begin{cases}\dfrac{1}{2}\,xy & 0<y<x<2,\\[2mm] 0 & \textit{otherwise.}\end{cases}$$

1) Find the frequencies $f_X(x)$ and $f_Y(y)$ of the random variables X and Y.

2) Find the means $E\{X\}$ and $E\{Y\}$ of the random variables X and Y.

3) Find the medians of the random variable X and Y.

4) Find the frequency $f_Z(z)$ of the random variable $Z=X+Y$.

5) Find the means $E\{Z\}$ and $E\left\{\dfrac{1}{Z}\right\}$ of the random variables Z and $\dfrac{1}{Z}$.

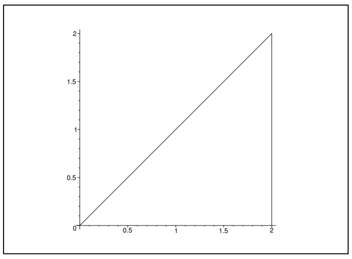


Figure 25: The domain D, where $f(x,y)\neq 0$.

1) If $0<x<2$, then we get by a vertical integration,

$$f_X(x)=\int_0^x\frac{1}{2}\,xy\,dy=\frac{1}{4}\,x^3.$$

If $0<y<2$, then we get by a horizontal integration,

$$f_Y(y)=\int_y^2\frac{1}{2}\,xy\,dx=\frac{1}{4}\,y\left(4-y^2\right)=y-\frac{1}{4}\,y^3.$$

Summing up,

$$f_X(x) = \begin{cases} \dfrac{1}{4}\,x^3 & 0 < x < 2, \\ \\ 0 & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} y - \dfrac{1}{4}\,y^3 & 0 < y < 2, \\ \\ 0 & \text{otherwise.} \end{cases}$$

2) The means are given by

$$E\{X\} = \int_0^2 \frac{1}{4}\,x^4\,dx = \left[\frac{1}{20}\,x^5\right]_0^2 = \frac{32}{20} = \frac{8}{5},$$

and

$$E\{Y\} = \int_0^2 \left\{y^2 - \frac{1}{4}\,y^4\right\}\,dy = \left[\frac{y^3}{3} - \frac{y^5}{20}\right]_0^2 = \frac{8}{3} - \frac{8}{5} = \frac{16}{15}.$$

3) The distribution function $F_X(x)$, when $0 < x < 2$, is given by

$$F_X(x) = \int_0^x \frac{1}{4}\,t^3\,dt = \frac{1}{16}\,x^4 \qquad \left[= \frac{1}{2} \quad \text{for } x = \sqrt[4]{8}\right],$$

hence the median is $(X) = \sqrt[4]{8}$.

The distribution function $F_Y(y)$, when $0 < y < 2$, is given by

$$F_Y(y) = \frac{1}{2}\,y^2 - \frac{1}{16}\,y^4.$$

The median is given by

$$\frac{1}{2}\,y^2 - \frac{1}{16}\,y^4 = \frac{1}{2}, \qquad \text{hence} \qquad 8y^2 - y^4 = 8,$$

and whence by a rearrangement,

$$\left(y^2\right)^2 - 8y^2 + 8 = 0, \qquad \text{i.e.} \qquad \left(y^2 - 4\right)^2 = 8.$$

Therefore, we get $y^2 = 4 \pm \sqrt{8}$. However, since also $0 < y < 2$, we cannot apply $+$, so we conclude that $y^2 = 4 - \sqrt{8}$, which implies that the median is

$$(Y) = \sqrt{4 - 2\sqrt{2}}.$$

4) Clearly, $Z = X + Y$ has its values in $]0, 4[$. The frequency is

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z - x)\,dx,$$

where the integrand is $\neq 0$, when $0 < z - x < x < 2$. The conditions are

$$(4)\ \ 0 < x < 2 \qquad \text{and} \qquad \frac{z}{2} < x < z,$$

which both should be fulfilled.

When $f(x, z - x) \neq 0$, then an integral is given by

$$\int f(x, z - x)\,dx = \int \frac{1}{2}\,x(z - x)\,dx = \frac{1}{4}\,zx^2 - \frac{1}{6}\,x^3 = \frac{1}{12}\,x^2(3z - 2x).$$

a) When $z \in]0,2[$, then the domain of integration is $\left]\frac{z}{2}, z\right[$, according to (4). Hence

$$f_Z(z) = \int_{\frac{z}{2}}^{z} f(x, z-x)\, dx = \left[\frac{1}{12}\, x^2(3z-2x)\right]_{\frac{z}{2}}^{z} = \frac{1}{12}\, z^3 - \frac{1}{24}\, z^3 = \frac{z^3}{24}.$$

b) When $z \in]2,4[$, then the domain of integration is $\left]\frac{z}{2}, 2\right[$, according to (4). Hence

$$f_Z(z) = \int_{\frac{z}{2}}^{2} f(x, z-x)\, dx = \left[\frac{1}{12}\, x^2(3z-2x)\right]_{\frac{z}{2}}^{2} = \frac{1}{3}\,(3z-4) - \frac{z^3}{24} = z - \frac{4}{3} - \frac{z^3}{24}.$$

Summing up,

$$f_Z(z) = \begin{cases} \dfrac{z^3}{24} & \text{for } 0 < z \le 2, \\[2mm] z - \dfrac{4}{3} - \dfrac{z^3}{24} & \text{for } 2 < z < 4, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

5) The means are

$$E\{Z\} = E\{X\} + E\{Y\})\frac{8}{5} + \frac{16}{15} = \frac{40}{15} = \frac{8}{3},$$

and

$$E\left\{\frac{1}{Z}\right\} = \int_0^2 \frac{z^2}{24}\,dz + \int_2^4 \left\{1 - \frac{4}{3}\cdot\frac{1}{z} - \frac{z^2}{24}\right\} dz = \frac{1}{9} + \frac{11}{9} - \frac{4}{3}\ln 2 = \frac{4}{3}(1 - \ln 2).$$

Example 5.13 *A 2-dimensional random variable (X,Y) has the frequency*

$$f(x,y) = \begin{cases} e^{-|x|}\cdot e^{-y}, & y > |x|, \\ 0, & \textit{otherwise.} \end{cases}$$

1) Find the frequencies $f_X(x)$ and $f_Y(y)$ of the random variables X and Y.

2) Find the means $E\{X\}$ and $E\{Y\}$ of the random variables X and Y.

3) Prove that the random variables X and Y are non-correlated.

4) Check if the random variables X are Y independent.

5) Find the frequency $f_Z(z)$ of the random variable $Z = X + Y$.

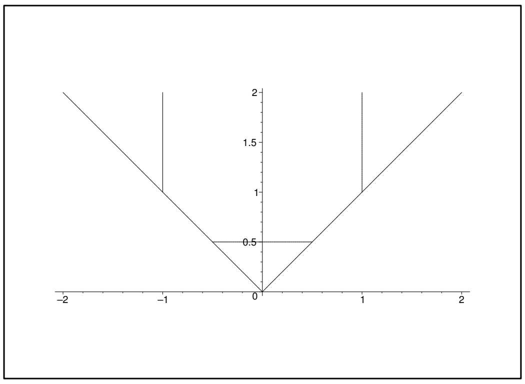


Figure 26: The support of $f(x,y)$ with a couple of paths of integration.

1) Clearly, we must split into the two cases $x \ge 0$ and $x < 0$.

a) If $x \ge 0$, then

$$f_X(x) = e^{-x}\int_{y=x}^{\infty} e^{-y}\,dy = e^{-2x}.$$

b) If $x<0$, then

$$f_X(x) = e^x \int_{y=-x}^{\infty} e^{-y}\,dy = e^{+2x} = e^{-2|x|}.$$

Summing up,

$$f_X(x) = e^{-2|x|}, \qquad x \in \mathbb{R}.$$

If $y \le 0$, then $f_Y(y) = 0$. If $y>0$, then

$$f_Y(y) = e^{-y}\int_{-y}^{y} e^{-|x|}\,dx = 2e^{-y}\int_0^y e^{-x}\,dx = 2e^{-y}\left(1-e^{-y}\right).$$

Summing up,

$$f_Y(y) = \begin{cases} 2e^{-y}\left(1-e^{-y}\right) & \text{for } y>0, \\ 0 & \text{for } y \le 0. \end{cases}$$

2) Due to the exponential factors, the integrals of the means are clearly convergent. We conclude by the symmetry that

$$E\{X\} = \int_{-\infty}^{\infty} x\,e^{-2|x|}\,dx = 0.$$

Furthermore,

$$E\{Y\} = \int_0^{\infty} \left\{2y\,e^{-y} - 2y\,e^{-2y}\right\}\,dy = 2 - \frac{1}{2} = \frac{3}{2}.$$

3) It follows from

$$E\{XY\} = \int_{y=0}^{\infty} y\,e^{-y}\left\{\int_{x=-y}^{y} x\,e^{-|x|}\,dx\right\}dy = 0 = E\{X\}\cdot E\{Y\},$$

that X and Y are non-correlated.

4) Since $f(x,y) \ne f_X(x)\,f_Y(y)$, we conclude that X and Y are not independent.

5) Since $f(x,y)$ is only $\ne 0$ for $y>|x|$, it follows that $Z=X+Y$ can only have values >0. If $z>0$, then

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx.$$

Since $z>0$, the integrand is $\ne 0$ for $x-z<x<z-x$, hence for $x<\frac{z}{2}$. Then

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\frac{z}{2}} e^{-|x|}\,e^{x-z}\,dx = \int_{-\infty}^{0} e^{2x}\,dx\cdot e^{-z} + \int_0^{\frac{z}{2}} 1\,dx\cdot e^{-z} \\ &= \frac{1}{2}\,e^{-z} + \frac{z}{2}\,e^{-z} = \frac{1}{2}\,(1+z)\,e^{-z}. \end{aligned}$$

Summing up,

$$f_Z(z) = \begin{cases} \dfrac{1}{2}(1+z)\,e^{-z} & \text{for } z > 0, \\ \\ 0 & \text{for } z \le 0. \end{cases}$$

Example 5.14 *A rectangular triangle has its shorter sides X_1 and X_2, where X_1 and X_2 are independent random variables of the frequencies*

$$f_{X_1}(x_1) = \begin{cases} \dfrac{1}{2}\,x_1, & 0 < x_1 < 2, \\ \\ 0, & \textit{otherwise.} \end{cases}$$

$$f_{X_2}(x_2) = \begin{cases} \dfrac{1}{2} & 0 < x_2 < 2, \\ \\ 0, & \textit{otherwise.} \end{cases}$$

Let $Y_1 = X_1 + X_2$ denote the sum of the lengths of the shorter sides, and let $Y_2 = \dfrac{1}{2}\,X_1X_2$ denote the area of the triangle

1) Compute the mean and the variance of Y_1.

2) Compute the mean and the variance of Y_2.

3) Compute the frequency of Y_1.

4) Compute the frequency of Y_2.

5) Check if the random variable $Z = X_2/X_1$ has a mean, and if so, find it.

We start by the following computations,

$$E\{X_1\} = \int_0^2 \frac{1}{2}\,x_1^2\,dx_1 = \left[\frac{1}{6}\,x_1^3\right]_0^2 = \frac{4}{3},$$

and

$$E\left\{X_1^2\right\} = \int_0^2 \frac{1}{2}\,x_1^3\,dx_1 = \left[\frac{1}{8}\,x_1^4\right]_0^2 = 2,$$

thus the variance of X_1 is

$$V\{X_1\} = 2 - \frac{16}{9} = \frac{2}{9}.$$

Analogously,

$$E\{X_2\} = \int_0^2 \frac{1}{2}\,x_2\,dx_2 = \left[\frac{1}{4}\,x_2^2\right]_0^2 = 1,$$

and

$$E\left\{X_2^2\right\} = \int_0^2 \frac{1}{2}\, x_2^2\, dx_2 = \left[\frac{1}{6}\, x_2^3\right]_0^2 = \frac{4}{3},$$

hence

$$V\left\{X_2\right\} = \frac{4}{3} - 1^2 = \frac{1}{3},$$

which also follows directly from the fact that X_2 is rectangularly distributed over $]0,2[$.
Since X_1 and X_2 are stochastically independent, the following computations become much easier.

1) The mean and variance of Y_1 are

$$E\left\{Y_1\right\} = E\left\{X_1 + X_2\right\} = E\left\{X_1\right\} + E\left\{X_2\right\} = \frac{4}{3} + 1 = \frac{7}{3},$$

and

$$V\left\{Y_1\right\} = V\left\{X_1 + X_2\right\} = V\left\{X_1\right\} + V\left\{X_2\right\} = \frac{2}{9} + \frac{1}{3} = \frac{5}{9}.$$

2) The mean and variance of Y_2 are

$$E\left\{Y_2\right\} = E\left\{\frac{1}{2}\,X_1X_2\right\} = \frac{1}{2}\,E\left\{X_1\right\}\cdot E\left\{X_2\right\} = \frac{1}{2}\cdot\frac{4}{3}\cdot 1 = \frac{2}{3},$$

and

$$E\left\{Y_2^2\right\} = \frac{1}{4}\,E\left\{X_1^2\right\}\cdot E\left\{X_2^2\right\} = \frac{1}{4}\cdot 2\cdot\frac{4}{3} = \frac{2}{3},$$

hence

$$V\left\{Y_2\right\} = E\left\{Y_2^2\right\} - \left(E\left\{Y_2\right\}\right)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}.$$

3) Since X_1 and X_2 only have values between 0 and 2, it follows that $Y_1 = X_1 + X_2$ has only values between 0 and 4, and the frequency of Y_1 is given by the convolution integral

$$f_{Y_1}\left(y_1\right) = \int_{-\infty}^{\infty} f_{X_1}(x)\, f_{X_2}\left(y_1 - x\right)\, dx.$$

This expression is only $\neq 0$, when $0 < x < 2$ and $0 < y_1 - x < 2$, so the restrictions are

$$0 < x < 2 \qquad \text{and} \qquad y_1 - 2 < x < y_1.$$

a) If $0 < y_1 \leq 2$, the restrictions are reduced to $0 < x < y_1$, hence

$$f_{Y_1}\left(y_1\right) = \int_0^{y_1} \frac{1}{2}\,x\cdot\frac{1}{2}\,dx = \frac{1}{8}\,y_1^2.$$

b) If $2 < y_1 < 4$, the restrictions are reduced to $y_1 - 2 < x < 2$, hence

$$f_{Y_1}\left(y_1\right) = \int_{y_1-2}^{2} \frac{1}{4}\,x\,dx = \left[\frac{1}{8}\,x^2\right]_{y_1-2}^{2} = \frac{1}{2} - \frac{1}{8}\left(y_1-2\right)^2 = \frac{1}{2}\,y_1 - \frac{1}{8}\,y_1^2.$$

Summing up,

$$f_{Y_1}\left(y_1\right) = \begin{cases} \dfrac{1}{8}\,y_1^2, & \text{for } 0 < y_1 \leq 2,\\[2mm] \dfrac{1}{2}\,y_1 - \dfrac{1}{8}\,y_1^2, & \text{for } 2 < y_1 < 4,\\[2mm] 0, & \text{otherwise.}\end{cases}$$

4) Analogously, $Y_2 = \dfrac{1}{2}\,X_1X_2 = X_1\cdot\left(\dfrac{1}{2}\,X_2\right)$ has only values between 0 and 2. The rewriting is convenient, because $\dfrac{1}{2}\,X_2$ is rectangularly distributed over $]0,1[$ of the frequency

$$g\left(x_2\right) = \begin{cases} 1, & \text{for } x \in]0,1[,\\[2mm] 0, & \text{otherwise.}\end{cases}$$

If $0 < y_2 < 2$, then the frequency of Y_2 is given by

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{X_1}(x)\, g\left(\frac{y_2}{x}\right) \frac{1}{|x|}\, dx.$$

the integrand is $\neq 0$, when $0 < x < 2$ and $0 < \dfrac{y_2}{x} < 1$, so we get the restrictions

$$0 < x < 2 \quad \text{and} \quad 0 < y_2 < x, \qquad \text{thus} \qquad y_2 < x < 2.$$

Hence,

$$f_{Y_2}(y_2) = \int_{y_2}^{2} \frac{1}{2}\, x \cdot 1\, \frac{dx}{x} = \int_{y_2}^{2} \frac{1}{2}\, dx = 1 - \frac{1}{2}\, y_2,$$

and summing up,

$$f_{Y_2}(y_2) = \begin{cases} 1 - \dfrac{1}{2}\, y_2 & \text{for } 0 < y_2 < 2, \\ \\ 0 & \text{otherwise.} \end{cases}$$

5) Since X_1 and C_2 aer independent, we get

$$E\{Z\} = E\left\{X_2 \cdot \frac{1}{X_1}\right\} = E\{X_2\} \cdot E\left\{\frac{1}{X_1}\right\} = 1 \cdot \int_0^2 \frac{1}{x_1} \cdot \frac{1}{2}\, x_1\, dx_1 = \left[\frac{1}{2}\, x_1\right]_0^2 = 1.$$

In particular, the mean exists.

Remark 5.1 It is possible, though far more difficult first to solve the questions 3 and 4, from which questions 1 and 2 can be derived. These computations are far bigger than the computations above. $\Diamond$

Index

2-dimensional random variable, 5

almost everywhere, 7

causal distribution, 4
Čebyšev's inequality, 13
conditional distribution, 11
conditional distribution function, 11
conditional probability, 11
continuous distribution, 5, 6
continuous random variable, 5, 6
convergence in distribution, 16
convergence in probability, 16
correlation, 15
correlation coefficient, 21
covariance, 15

discrete distribution, 4, 6
discrete random variable, 4, 6
distribution function, 4

expectation, 11

frequency, 5, 6

Helly-Bray's lemma, 16

independent random variables, 7

Jacobian, 10

law of total probability, 11, 18

marginal distribution, 5
marginal frequency, 6
mean, 11
median, 4
moment, 12

null-set, 7

Poisson distribution, 58
polar coordinates, 42
probability field, 4

quantile, 4

random variable, 4

rectangular distribution, 42, 76, 79, 85, 96, 111, 112

simultaneous distribution, 5
simultaneous distribution function, 6
skewness, 21

transformation theorem, 8

weak law of large numbers, 16