Stochastic Processes 2

Probability Examples c-9

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Probability Examples c-9 – Stochastic Processes 2

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Contents

Introduction

This is the ninth book of examples from *Probability Theory.* The topic *Stochastic Processes* is so big that I have chosen to split into two books. In the previous (eighth) book was treated examples of *Random Walk* and Markov chains, where the latter is dealt with in a fairly large chapter. In this book we give examples of *Poisson processes, Birth and death processes, Queueing theory* and other types of stochastic processes.

The prerequisites for the topics can e.g. be found in the *Ventus: Calculus 2* series and the *Ventus: Complex Function Theory* series, and all the previous *Ventus: Probability c1-c7.*

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
27th October 2014

1 Theoretical background

1.1 The Poisson process

Given a sequence of independent events, each of them indicating the time when they occur. We assume

1. The probability that an event occurs in a time interval $I \subsetneqq [0, +\infty[$ does *only* depend on the length of the interval and not of where the interval is on the time axis.

2. The probability that there in a time interval of length t we have *at least one event*, is equal to

$$\lambda t + t\,\varepsilon(t),$$

where $\lambda > 0$ is a given positive constant.

3. The probability that we have *more than one event* in a time interval of length t is $t\,\varepsilon(t)$.

It follows that

4. The probability that there is *no event* in a time interval of length is given by

$$1 - \lambda t + t\varepsilon(t).$$

5. The probability that there is *precisely one event* in a time interval of length t is $\lambda t + t\,\varepsilon(t)$.

Here $\varepsilon(t)$ denotes some unspecified function, which tends towards 0 for $t \to 0$.

Given the assumptions on the previous page, we let $X(t)$ denote the number of events in the interval $]0,t]$, and we put

$$P_k(t) := P\{X(t)=k\}, \qquad \text{for } k \in \mathbb{N}_0.$$

Then $X(t)$ is a *Poisson distributed random variable of parameter* λt. The process

$$\{X(t) \mid t \in [0,+\infty[\}$$

is called a *Poisson process*, and the parameter λ is called the *intensity* of the Poisson process.

Concerning the Poisson process we have the following results:

1) If $t=0$, (i.e. $X(0)=0$), then

$$P_k = \begin{cases} 1, & \text{for } k=0, \\ 0, & \text{for } k \in \mathbb{N}. \end{cases}$$

2) If $t>0$, then $P_k(t)$ is a differentiable function, and

$$P_k'(t) = \begin{cases} \lambda\left\{P_{k-1}(t) - P_k(t)\right\}, & \text{for } k \in \mathbb{N} \text{ and } t>0, \\ -\lambda\, P_0(t), & \text{for } k=0 \text{ and } t>0. \end{cases}$$

When we solve these differential equations, we get

$$P_k(t) = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t}, \qquad \text{for } k \in \mathbb{N}_0,$$

proving that $X(t)$ is Poisson distributed with parameter λt.

Remark 1.1 Even if Poisson processes are very common, they are mostly applied in the theory of tele-traffic. $\Diamond$.

If $X(t)$ is a Poisson process as described above, then $X(s+t)-X(s)$ has the same distribution as $X(t)$, thus

$$P\{X(s+t)-X(s)\} = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t}, \text{for } k \in \mathbb{N}_0.$$

If $0 \le t_1 < t_2 \le t_3 < t_4$, then the two random variables $X(t_4) - X(t_3)$ and $X(t_2) - X(t_1)$ are *independent*. We say that the *Poisson process has independent and stationary growth.*

The *mean value function* of a Poisson process is

$$m(t) = E\{X(t)\} = \lambda t.$$

The *auto-covariance (covariance function)* is given by

$$C(s,t) = \operatorname{Cov}(X(s)\,,\,X(t)) = \lambda \min\{s,t\}.$$

The *auto-correlation* is given by

$$R(s,t) = E\{X(s) \cdot X(t)\} = \lambda \min(s,t) + \lambda^2 st.$$

The event function of a Poisson process is a step function with values in $\mathbb{N}_0$, each step of the size $+1$. We introduce the sequence of random variables $T_1, T_2, \dots$, which indicate the *distance in time* between two succeeding events in the Poisson process. Thus

$$Y_n = T_1 + T_2 + \cdots + T_n$$

is the time until the n-th event of the Poisson process.

Notice that T_1 is *exponentially distributed of parameter* λ, thus

$$P\{T_1 > t\} = P\{X(t) = 0\} = e^{-\lambda t}, \qquad \text{for } t > 0.$$

All random variables $T_1, T_2, \dots, T_n$ are mutually independent and exponentially distributed of parameter λ, hence

$$Y_n = T_1 + T_2 + \cdots + T_n$$

is *Gamma distributed*, $Y_n \in \Gamma\left(n\,,\,\frac{1}{\lambda}\right)$.

Connection with Erlang's B-formula. Since $Y_{n+1} > t$, if and only if $X(t) \le n$, we have

$$P\{X(t) \le n\} = P\{Y_{n+1} > t\},$$

from which we derive that

$$\sum_{k=1}^{n} \frac{(\lambda t)^k}{k!}\, e^{-\lambda t} = \frac{\lambda^{n+1}}{n!} \int_t^{+\infty} y^n\, e^{-\lambda y}\, dy.$$

We have in particular for $\lambda = 1$,

$$\sum_{k=0}^{n} \frac{t^k}{k!} = \frac{e^t}{n!} \int_t^{+\infty} y^n\, e^{-y}\, dy, \qquad n \in \mathbb{N}_0.$$

1.2 Birth and death processes

Let $\{X(t) \mid t \in [0,+\infty[\}$ be a stochastic process, which can be in the states $E_0, E_1, E_2, \dots$. The process can only move from one state to a neighbouring state in the following sense: If the process is in state E_k, and we receive a *positive signal*, then the process is transferred to E_{k+1}, and if instead we receive a *negative signal* (and $k \in \mathbb{N}$), then the process is transferred to E_{k-1}.

We assume that there are non-negative constants λ_k and μ_k, such that for $k \in \mathbb{N}$,

1) $P\{\text{one positive signal in }]\,t,t+h\,[\mid X(t)=k\} = \lambda_k\, h + h\,\varepsilon(h).$

2) $P\{\text{one negative signal in }]\,t,t+h\,[\mid X(t)=k\} = \mu_k\, h + h\,\varepsilon(h).$

3) $P\{\text{no signal in }]\,t,t+h\,[\mid X(t)=k\} = 1-(\lambda_k+\mu_k)\ h + h\,\varepsilon(h)$.

We call λ_k the *birth intensity* at state E_k, and μ_k is called the *death intensity* at state E_k, and the process itself is called a *birth and death process.* If in particular all $\mu_k = 0$, we just call it a *birth process*, and analogously a *death process*, if all $\lambda_k = 0$.

A simple analysis shows for $k \in \mathbb{N}$ and $h > 0$ that the event $\{X(t+h) = k\}$ is realized in on of the following ways:

- $X(t) = k$, and no signal in $]\,t,t+h\,[$.
- $X(t) = k-1$, and one positive signal in $]\,t,t+h\,[$.
- $X(t) = k+1$, and one negative signal in $]\,t,t+h\,[$.
- More signals in $]\,t,t+h\,[$.

We put

$$P_k(t) = P\{X(t) = k\}.$$

By a rearrangement and taking the limit $h \to 0$ we easily derive the *differential equations of the process*,

$$\begin{cases} P_0'(t) = -\lambda_0\,P_0(t) + \mu_1\,P_1(t), & \text{for } k=0, \\ P_k'(t) = -\left(\lambda_k+\mu_k\right)P_k(t) + \lambda_{k-1}\,P_{k-1}(t) + \mu_{k+1}\,P_{k+1}(t), & \text{for } k \in \mathbb{N}. \end{cases}$$

In the special case of a pure *birth process*, where all $\mu_k = 0$, this system is reduced to

$$\begin{cases} P_0'(t) = -\lambda_0\,P_0(t), & \text{for } k=0, \\ P_k'(t) = -\lambda_k\,P_k(t) + \lambda_{k-1}\,P_{k-1}(t), & \text{for } k \in \mathbb{N}. \end{cases}$$

If all $\lambda_k > 0$, we get the following iteration formula of the complete solution,

$$\begin{cases} P_0(t) = c_0\,e^{-\lambda_0\,t}, & \text{for } k=0, \\ P_k(t) = \lambda_{k-1}\,e^{-\lambda_k t}\int_0^t e^{\lambda_k\tau}\,P_{k-1}(\tau)\,d\tau + c_k\,e^{-\lambda_k t}, & \text{for } k \in \mathbb{N}. \end{cases}$$

From $P_0(t)$ we derive $P_1(t)$, etc.. Finally, if we know the initial distribution, we are e.g. at time $t = 0$ in state E_m, then we can find the values of the arbitrary constants c_k.

Let $\{X(t) \mid t \in [0,+\infty[\}$ be a birth and death process, where all λ_k and μ_k are positive, with the exception of $\mu_0 = 0$, and $\lambda_N = 0$, if there is a final state E_N. The process can be in any of the states, therefore, in analogy with the Markov chains, such a birth and death process is called *irreducible.* Processes like this often occur in queueing theory.

If there exists a state E_k, in which $\lambda_k = \mu_k$, then E_k is an *absorbing state*, because it is not possible to move away from E_k.

For the most common birth and death processes (including all irreducible processes) there exist non-negative constants p_k, such that

$$P_k(t) \to p_k \qquad \text{and} \qquad P_k'(t) \to 0 \qquad\qquad \text{for } t \to +\infty.$$

These constants fulfil the infinite system of equations,

$$\mu_{k+1}\, p_{k+1} = \lambda_k\, p_k, \qquad \text{for } k \in \mathbb{N}_0,$$

which sometimes can be used to find the p_k.

If there is a solution (p_k), which satisfies

$$p_k \geq 0 \quad \text{for all } k \in \mathbb{N}_0, \qquad \text{and} \qquad \sum_{k=0}^{+\infty} p_k = 1,$$

we say that the solution (p_k) is a *stationary distribution*, and the p_k are called the *stationary probabilities*. In this case we have

$$P_k(t) \to p_k \qquad \text{for } t \to +\infty.$$

If $\{X(t) \mid t \in [0, +\infty[\}$ is an *irreducible* process, then

$$p_k = \frac{\lambda_{k-1}\lambda_{k-2}\cdots\lambda_1\lambda_0}{\mu_k\mu_{k-1}\cdots\mu_2\mu_1} \cdot p_0 := a_k\, p_0, \qquad \text{for } k \in \mathbb{N}_0,$$

where all $a_k > 0$.

The condition of the existence of a stationary distribution is then reduced to that the series $\sum_k a_k$ is convergent of finite sum $a > 0$. In this case we have $p_0 = \dfrac{1}{a}$.

1.3 Queueing theory in general

Let $\{X(t) \mid t \in [0,+\infty[\}$ be a birth and death process as described in the previous section. We shall consider them as services in a service organization, where "birth" corresponds to the arrival of a new customer, and "death" correspond to the ending of the service of a customer. We introduce the following:

1) By the *arrival distribution (the arrival process)* we shall understand the distribution of the arrivals of the customers to the *service* (the shop). This distribution is often of Poisson type.

2) It the arrivals follow a *Poisson process of intensity* λ, then the random variable, which indicates the time difference between two succeeding arrivals *exponentially distributed of parameter* λ. We say that the arrivals follow an exponential distribution, and λ is called the *arrival intensity.*

3) *The queueing system* is described by the number of shop assistants or serving places, if there is the possibility of forming queues or not, and the way a queue is handled. The serving places are also called *channels.*

4) Concerning the *service times* we assume that if a service starts at time t, then the probability that it is ended at some time in the interval $]t, t+h[$ is equal to

$$\mu\, h + h\, \varepsilon(h), \qquad \text{where } \mu > 0.$$

Then the service time is *exponentially distributed of parameter* μ.
If at time t we are dealing with k (mutually independent) services, then the probability that one of these is ended in the interval $]t, t+h[$ equal to

$$k\, \mu\, h + h\, \varepsilon(h).$$

We shall in the following sections consider the three most common types of queueing systems. Concerning other types, cf. e.g. *Villy Bæk Iversen: Teletraffic Engineering and Network Planning Technical University of Denmark.*

1.4 Queueing system of infinitely many shop assistants

The model is described in the following way: Customers arrive to the service according a *Poisson process* of intensity λ, and they immediately go to a *free shop assistant*, where they are serviced according to an *exponential distribution* of parameter μ.

The process is described by the following birth and death process,

$$\{X(t) \mid t \in [0,+\infty[\} \qquad \text{med } \lambda_k = \lambda \text{ and } \mu_k = k\,\mu \quad \text{for alle } k.$$

The process is *irreducible*, and the differential equations of the system are given by

$$\begin{cases} P_0'(t) = -\lambda\, P_0(t) + \mu\, P_1(t), & \text{for } k = 0, \\ P_k'(t) = -(\lambda + k\,\mu) P_k(t) + \lambda\, P_{k-1}(t) + (k+1)\mu\, P_{k+1}(t), & \text{for } k \in \mathbb{N}. \end{cases}$$

The *stationary probabilities* exist and satisfy the equations

$$(k+1)\mu\, p_{k+1} = \lambda\, p_k, \qquad k \in \mathbb{N}_0,$$

of the solutions

$$p_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \exp\left(-\frac{\lambda}{\mu}\right), \qquad k \in \mathbb{N}_0.$$

These are the probabilities that there are k customers in the system, *when we have obtained equilibrium.*

The system of differential equations above is usually difficult to solve. One has, however, some partial results, e.g. the *expected number of customers* at time t, i.e.

$$m(t) := \sum_{k=1}^{+\infty} k\, P_k(t),$$

satisfies the simpler differential equation

$$m'(t) + \mu\, m(t) = \lambda.$$

If at time $t = 0$ there is no customer at the service, then

$$m(t) = \frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right), \qquad \text{for } t \geq 0.$$

1.5 Queueing system of a finite number of shop assistants, and with forming of queues

We consider the case where

1) the customers arrive according to a Poisson process of intensity λ,

2) the service times are exponentially distributed of parameter μ,

3) there are N shop assistants,

4) it is possible to form queues.

Spelled out, we have N shop assistants and a customer, who arrives at state E_k. If $k < N$, then the customer goes to a free shop assistant and is immediately serviced. If however $k = N$, thus all shop assistants are busy, then he joins a queue and waits until there is a free shop assistant. We assume here queueing culture.

With a slight change of the notation it follows that if there are N shop assistants and k *customers* (and not k states as above), where $k > N$, then there is a common queue for all shop assistants consisting of $k - N$ customers.

This process is described by the following birth and death process $\{X(t) \mid t \in [0, +\infty[\}$ of the parameters

$$\lambda_k = \lambda \qquad \text{and} \qquad \mu_k = \begin{cases} k\,\mu, & \text{for } k < N, \\ N\,\mu, & \text{for } k \geq N. \end{cases}$$

The process is *irreducible*. The equations of the stationary probabilities are

$$\begin{cases} (k+1)\mu\, p_{k+1} = \lambda\, p_k, & \text{for } k < N, \\ N\,\mu\, p_{k+1} = \lambda\, p_k, & \text{for } k \geq N. \end{cases}$$

We introduce the *traffic intensity* by

$$\varrho := \frac{\lambda}{N\,\mu}.$$

Then we get the stationary probabilities

$$p_k = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^k \cdot \dfrac{1}{k!}\, p_0 = \frac{\varrho^k \cdot N^k}{k!} \cdot p_0, & \text{for } k < N, \\ \left(\dfrac{\lambda}{\mu}\right)^k \cdot \dfrac{1}{N^{k-N} \cdot N!} \cdot p_0 = \dfrac{\varrho^k \cdot N^N}{N!} \cdot p_0, & \text{for } k \geq N. \end{cases}$$

Remark 1.2 Together with the *traffic intensity* one also introduce in teletraffic the *offer of traffic*. By this we mean the number of customers who at the average arrive to the system in a time interval of length equal to the mean service time. In the situation above the *offer of traffic* is $\dfrac{\lambda}{\mu}$. Both the traffic intensity and the offer of traffic are dimensionless. They are both measured in the unit Erlang.$\Diamond$

The condition that (p_k) become stationare probabilities is that the *traffic intensity* $\varrho < 1$, where

$$\sum_{k=N}^{+\infty} \frac{N^N}{N!}\,\varrho^k = \frac{(\varrho\, N)^N}{(1-\varrho)\cdot N!}.$$

If, however, $\varrho \geq 1$, it is easily seen that the queue is increasing towards infinity, and there does not exist a stationary distribution.

We assume in the following that $\varrho < 1$, so the stationary probabilities exist

1) If $N = 1$, then

$$p_k = \varrho^k(1-\varrho), \qquad \text{for } k \in \mathbb{N}_0.$$

2) If $N = 2$, then

$$p_k = \begin{cases} \dfrac{1-\varrho}{1+\varrho}, & \text{for } k = 0, \\ 2\varrho^k \cdot \dfrac{1-\varrho}{1+\varrho}, & \text{for } k \in \mathbb{N}. \end{cases}$$

3) If $N > 2$, the formulæ become somewhat complicated, so they are not given here.

The *average number of customers* at the service is under the given assumptions,

$$\begin{cases} \dfrac{\varrho}{1-\varrho}, & \text{for } N=1, \\ \sum_{k=1}^{+\infty} k\, p_k, & \text{generelt (naturligvis).} \end{cases}$$

The *average number of busy shop assistants* is

$$\begin{cases} \varrho, & \text{for } N=1, \\ \sum_{k=1}^{N-1} k\, p_k + N \sum_{k=N}^{+\infty} p_k, & \text{in general.} \end{cases}$$

The *waiting time* of a customer is defined as the time elapsed from his arrival to the service of him starts. The *staying time* is the time from his arrival until he leaves the system after the service of him. Hence we have the splitting

$$\text{staying time} \; = \; \text{waiting time} \; + \; \text{service time.}$$

The *average waiting time* is in general given by

$$V = \sum_{k=N}^{+\infty} \frac{k-N+1}{N\,\mu}\, p_k,$$

which by a computation is

$$V = \begin{cases} \dfrac{\varrho}{\mu(1-\varrho)}, & \text{for } N = 1, \\[2ex] \dfrac{\varrho^N \cdot N^{N-1}}{\mu \cdot N! \cdot (1-\varrho)^2} \cdot p_0, & \text{generelt.} \end{cases}$$

In the special case of $N = 1$ the *average staying time* is given by

$$\mathcal{O} = \frac{\varrho}{\mu(1-\varrho)} + \frac{1}{\mu} = \frac{1}{\mu - \lambda}.$$

The *average length of the queue* (i.e. the mean number of customers in the queue) is

$$\lambda V = \sum_{k=N+1}^{+\infty} (k-N) p_k = \frac{\varrho^{N+1} \cdot N^N}{N! \cdot (1-\varrho)^2} \cdot p_0.$$

1.6 Queueing systems with a finite number of shop assistants and without queues

We consider here the case where

1) the customers arrive according to a Poisson process of intensity λ,

2) the times of service are exponential distributed of parameter μ,

3) there are N shop assistants or channels,

4) it is not possible to form a queue.

The difference from the previous section is that if a customer arrives at a time when all shop assistants are busy, then he immediately leaves the system. Therefore, this is also called a *system of rejection*.

In this case the process is described by the following birth and death process $\{X(t) \mid t \in [0,+\infty[\}$ with a finite number of states $E_0, E_1, \dots, E_N$, where the intensities are given by

$$\lambda_k = \begin{cases} \lambda, & \text{for } k < N, \\ 0, & \text{for } k \geq N, \end{cases} \qquad \text{and} \qquad \mu_k = k\,\mu.$$

This process is also *irreducible*. The corresponding system of differential equations is

$$\begin{cases} P_0'(t) = -\lambda\, P_0(t) + \mu\, P_1(t), & \text{for } k = 0, \\ P_k'(t) = -(\lambda + k\,\mu) P_k(t) + \lambda\, P_{k-1}(t) + (k+1)\mu\, P_{k+1}(t), & \text{for } 1 \leq k \leq N-1, \\ P_N'(t) = -N\,\mu\, P_N(t) + \lambda\, P_{N-1}(t), & \text{for } k = N. \end{cases}$$

In general, this system is too complicated for a reasonable solution, so instead we use the *stationary probabilities*, which are here given by *Erlang's B-formula:*

$$p_k = \frac{\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k}{\sum_{j=0}^N \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j}, \qquad \text{for } k = 0,\, 1,\, 2,\, \dots,\, N.$$

The *average number of customers who are served*, is of course equal to the *average number of busy shop assistants, or channels*. The common value is

$$\sum_{k=1}^N k\, p_k = \frac{\lambda}{\mu}\,(1 - p_N)\,.$$

We notice that p_N can be interpreted as the *probability of rejection*. This probability p_N is large, when $\lambda >> \mu$. We get from

$$\sum_{j=0}^N \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j = \frac{\exp\left(\frac{\lambda}{\mu}\right)}{N!}\int_{\lambda/\mu}^{+\infty} y^N\, e^{-y}\, dy,$$

the *probability of rejection*

$$p_N = \frac{\frac{1}{N!}\left(\frac{\lambda}{\mu}\right)^N}{\sum_{j=0}^N \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j} = \frac{\left(\frac{\lambda}{\mu}\right)^N \exp\left(-\frac{\lambda}{\mu}\right)}{\int_{\lambda/\mu}^{+\infty} y^N\, e^{-y}\, dy}.$$

1.7 Some general types of stochastic processes

Given two stochastic processes, $\{X(t) \mid t \in T\}$ and $\{Y(s) \mid s \in T\}$, where we assume that all the moments below exist. We define

1) the *mean value function*,

$$m(t) := E\{X(t)\}, \qquad \text{for } t \in T,$$

2) the *auto-correlation*,

$$R(x,t) := E\{X(s)X(t)\}, \qquad \text{for } s,\, t \in T,$$

3) the *auto-covariance*,

$$C(s,t) := \text{ Cov}(X(s), X(t)), \qquad \text{for } s,\, t \in T,$$

4) the *cross-correlation*,

$$R_{XY}(s,t) := E\{X(s)Y(t)\}, \qquad \text{for } s,\, t \in T,$$

5) the *cross-covariance*,

$$C_{XY}(s,t) := \text{ Cov}(X(s), Y(t)), \qquad \text{for } s,\, t \in T.$$

A stochastic process $\{X(t) \mid t \in \mathbb{R}\}$ is *strictly stationary*, if the translated process $\{X(t+h) \mid t \in \mathbb{R}\}$ for every $h \in \mathbb{R}$ has the same distribution as $\{X(t) \mid t \in \mathbb{R}\}$.

In this case we have for all $n \in \mathbb{N}$, all $x_1, \dots, x_n \in \mathbb{R}$, and all $t_1, \dots, t_n \in \mathbb{R}$ that

$$P\left\{X\left(t_1+h\right) \le x_1 \,\wedge\, \cdots \,\wedge\, X\left(t_n+h\right) \le x_n\right\}$$

does not depend on $h \in \mathbb{R}$.

Since $P\{X(t) \le x\}$ does not depend on t for such a process, we have

$$m(t) = m,$$

and the auto-covariance $C(s,t)$ becomes a function in the real variable $s-t$. We therefore write in this case,

$$C(s,t) := C(s-t).$$

Analogously, the auto-correlation is also a function only depending on s and t, so we write

$$R(s,t) := R(s-t).$$

Conversely, if $m(t) = m$ and $C(s,t) = C(s-t)$, then we call the stochastic process $\{X(t) \mid t \in \mathbb{R}\}$ *weakly stationary*.

Let us consider a stochastic process $\{X(t) \mid t \in \mathbb{R}\}$ of mean 0 and auto-correlation

$$R(\tau) = E\{X(t+\tau)X(t)\}.$$

If $R(\tau)$ is absolutely integrable, we define the *effect spektrum* by

$$S(\omega) = \int_{-\infty}^{+\infty} e^{i\omega\tau}\, R(\tau)\, d\tau,$$

i.e. as the Fourier transformed of $R(\tau)$. Furthermore, if we also assume that $S(\omega)$ is absolutely integrable, then we can apply the *Fourier inversion formula* to reconstruct $R(\tau)$ from the effect spectrum,

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega\tau}\, S(\omega)\, d\omega.$$

In particular,

$$E\left\{|X(t)|^2\right\} = R(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega)\, d\omega.$$

A stochastic process $\{X(t) \mid t \in T\}$ is called a *normal process*, or a *Gaußiann process*, if for every $n \in \mathbb{N}$ and every $t_1, \dots, t_n \in T$ the distribution of $\{X(t_1), \dots, X(t_n)\}$ is an n-dimensional normal distribution. A *normal process* is always completely specified by its *mean value function* $m(t)$ and its *auto-covariance function* $C(s,t)$.

The most important *normal process* is the *Wiener process*, or the *Brownian movements*

$$\{W(t) \mid t \geq 0\}.$$

This is characterized by

1) $W(0) = 0$,

2) $m(t) = 0$,

3) $V\{W(t)\} = \alpha\, t$, where α is a positive constant,

4) mutually independent increments.

2 The Poisson process

Example 2.1 *Let $\{X(t),\, t \in [0,\infty[\}$ be a Poisson process of intensity λ, and let the random variable T denote the time when the first event occurs.*
Find the conditional distribution of T, given that at time t_0 precisely one event has occurred, thus find

$$P\left\{T \le t \mid X\left(t_0\right)=1\right\}.$$

When $t \in [0, t_0]$, then the conditional distribution is given by

$$\begin{aligned} P\left\{T \le t \mid X\left(t_0\right)=1\right\} &= \frac{P\left\{X(t)=1 \,\wedge\, X\left(t_0\right)=1\right\}}{P\left\{X\left(t_0\right)=1\right\}} = \frac{P\left\{X(t)=1 \,\wedge\, X\left(t_0\right)-X(t)=0\right\}}{P\left\{X\left(t_0\right)=1\right\}} \\ &= \frac{P\{X(t)=1\} \cdot P\left\{X\left(t_0\right)-X(t)=0\right\}}{P\left\{X\left(t_0\right)=1\right\}} = \frac{\lambda\, t\, e^{-\lambda t} \cdot e^{-\lambda\left(t_0-t\right)}}{\lambda\, t_0 e^{-\lambda\, t_0}} = \frac{t}{t_0}, \end{aligned}$$

because

$$P_k(t)=P\{X(t)=k\}=\frac{(\lambda\, t)^k}{k!}\, e^{-\lambda\, t}, \qquad k \in \mathbb{N},$$

and where we furthermore have applied that $X\left(t_0\right)-X(t)$ has the same distribution as $X\left(t_0-t\right)$.

The conditional distribution is a rectangular distribution over $]0, t_0[$.

Example 2.2 *Let* $\{X_1(t),\, t \geq 0\}$ *and* $\{X_2(t),\, t \geq 0\}$ *denote two independent Poisson processes of intensity* λ_1 *and* λ_2*, resp., and let the process* $\{Y(t),\, t \geq 0\}$ *be defined by*

$$Y(t) = X_1(t) + X_2(t).$$

Prove that $\{Y(t),\, t \geq 0\}$ *is a Poisson process.*

We first identify

$$P_n(t) = P\{X(t) = n\} = \frac{(\lambda_1 t)^n}{n!}\, e^{-\lambda_1 t},$$

and

$$Q_n(t) = P\{X(t) = n\} = \frac{(\lambda_2 t)^n}{n!}\, e^{-\lambda_2 t}.$$

We get from $X_1(t)$ and $X_2(t)$ being independent that

$$\begin{aligned}
P\{Y(t) = n\} &= P\left\{X_1(t) + X_2(t) = n\right\} \\
&= \sum_{j=0}^{n} P\left\{X_1(t) = j\right\} \cdot P\left\{X_2(t) = n-j\right\} = \sum_{j=0}^{n} \frac{(\lambda_1 t)^j}{j!}\, e^{-\lambda_1 t} \cdot \frac{(\lambda_2 t)^{n-j}}{(n-j)!}\, e^{-\lambda_2 t} \\
&= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}\, \lambda_1^j \cdot \lambda_2^{n-j} \cdot \frac{t^n}{n!}\, e^{-(\lambda_1+\lambda_2)t} = \sum_{j=0}^{n} \binom{n}{j} \lambda_1^j \lambda_2^{n-j} \cdot \frac{t^n}{n!}\, e^{-(\lambda_1+\lambda_2)t} \\
&= (\lambda_1 + \lambda_2)^n \cdot \frac{t^n}{n!} \cdot \exp\left(-\left(\lambda_1 + \lambda_2\right) t\right).
\end{aligned}$$

It follows that $\{Y(t),\, t \geq 0\}$ is also a Poisson process (of intensity $\lambda_1 + \lambda_2$).

Example 2.3 *A Geiger counter only records every second particle, which arrives to the counter. Assume that the particles arrive according to a Poisson process of intensity* λ*. Denote by* $N(t)$ *the number of particles recorded in* $]0,t]$*, where we assume that the first recorded particle is the second to arrive.*

1. *Find* $P\{N(t) = n\}$*,* $n \in \mathbb{N}_0$*.*

2. *Find* $E\{N(t)\}$*.*

Let T *denote the time difference between two succeeding recorded arrivals.*

3. *Find the frequency of* T*.*

4. *Find the mean* $E\{T\}$*.*

1. It follows from

$$P_n(t) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}, \qquad n \in \mathbb{N}_0,$$

that

$$\begin{aligned}
P\{N(t) = n\} &= P_{2n}(t) + P_{2n+1}(t) = \left\{\frac{(\lambda t)^{2n}}{(2n)!} + \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right\} e^{-\lambda t} \\
&= \frac{(\lambda t)^{2n}}{(2n+1)!}\, (2n + 1 + \lambda t) e^{-\lambda t}, \qquad n \in \mathbb{N}_0.
\end{aligned}$$

2. The mean is

$$\begin{aligned}
E\{N(t)\} &= \sum_{n=0}^{\infty} n\,P\{N(t)=n\} = e^{-\lambda t}\left\{\sum_{n=1}^{\infty}\frac{n(\lambda t)^{2n}}{(2n)!}+\sum_{n=1}^{\infty}\frac{n(\lambda t)^{2n+1}}{(2n+1)!}\right\} \\
&= e^{-\lambda t}\left\{\frac{\lambda t}{2}\sum_{n=0}^{\infty}\frac{(\lambda t)^{2n+1}}{(2n+1)!}+\sum_{n=1}^{\infty}\frac{(n+\frac{1}{2})(\lambda t)^{2n+1}}{(2n+1)!}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(\lambda t)^{2n+1}}{(2n+1)!}\right\} \\
&= e^{-\lambda t}\left\{\frac{\lambda t}{2}\cdot\sinh\lambda t+\frac{\lambda t}{2}(\cosh\lambda t-1)-\frac{1}{2}(\sinh\lambda t-\lambda t)\right\} \\
&= e^{-\lambda t}\left\{\frac{\lambda t}{2}\cdot e^{\lambda t}-\frac{1}{4}\left(e^{\lambda t}-e^{-\lambda t}\right)\right\} = \frac{\lambda t}{2}-\frac{1}{4}+\frac{1}{4}e^{-2\lambda t}.
\end{aligned}$$

3. & 4. It follows from $T = T_j + T_{j+1}$ that $T \in \Gamma\left(2\,,\,\frac{1}{\lambda}\right)$, thus the frequency is

$$f(x) = \begin{cases} \lambda^2 x\, e^{-\lambda x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}$$

and the mean is

$$E\{T\} = \frac{2}{\lambda}.$$

Example 2.4 *From a ferry port a ferry is sailing every quarter of an hour. Each ferry can carry N cars. The cars are arriving to the ferry port according to a Poisson process of intensity λ (measured in quarter^{-1}).*
Assuming that there is no car in the ferry port immediately after a ferry has sailed at 9^{00}, one shall

1) *find the probability that there is no car waiting at 9^{15} (immediately after the departure of the next ferry),*

2) *find the probability that no car is waiting at 9^{30} (immediately after the departure of the next ferry).*

3) *A motorist arrives at $p^{07\frac{1}{2}}$. What is the probability that he will not catch the ferry at p^{15}, but instead the ferry at 9^{30}?*

Measuring t in the unit quarter of an hour we have

$$P\{X(t)=n\} = \frac{(\lambda t)^n}{n!}\,e^{-\lambda t}, \qquad n \in \mathbb{N}_0.$$

1) From $t = 1$ follows that the wanted probability is

$$P\{X(1) \le N\} = \sum_{n=0}^{N}\frac{\lambda^n}{n!}\,e^{-\lambda}.$$

2) We have two possibilities:

a) Either there has arrived during the first quarter of an hour $\le N$ cars, which are all carried over, so we allow during the next quarter N cars to arrive,

b) or during the first quarter $N+j$ cars have arrived, $1 \le j \le N$, and at most $N-j$ cars in the second quarter.

We therefore get the probability

$$
\begin{aligned}
&P\{X(1) \le N\} \cdot P\{X(1) \le N\} + \sum_{j=1}^{N} P\{X(1) = N+j\} \cdot P\{X(1) \le N-j\} \\
&= \left\{\sum_{n=0}^{N} \frac{\lambda^n}{n!}\, e^{-\lambda}\right\}^2 + \sum_{j=1}^{N} \frac{\lambda^{N+j}}{(N+j)!}\, e^{-\lambda} \cdot \sum_{n=0}^{N-j} \frac{\lambda^n}{n!}\, e^{-\lambda} = e^{-2\lambda}\left\{\left(\sum_{n=0}^{N} \frac{\lambda^n}{n!}\right)^2 + \sum_{j=1}^{N}\sum_{n=0}^{N-j} \frac{\lambda^{N+j+n}}{n!(N+j)!}\right\} \\
&= e^{-2\lambda}\left\{\left(\sum_{n=0}^{N} \frac{\lambda^n}{n!}\right)^2 + \sum_{n=0}^{N-1}\sum_{j=1}^{N-n} \frac{\lambda^{N+j+n}}{n!(N+j)!}\right\}.
\end{aligned}
$$

3) Now the time $9^{07\frac{1}{2}}$ corresponds to $t = \frac{1}{2}$, so the probability is

$$
\sum_{j=0}^{N} P\left\{X\left(\frac{1}{2}\right) = N+j\right\} = \exp\left(-\frac{\lambda}{2}\right) \sum_{n=N}^{2N} \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n.
$$

Example 2.5 Paradox of waiting time.

Each morning Mr. Smith in X-borough takes the bus to his place of work. The busses of X-borough should according to the timetables run with an interval of 20 minutes. It is, however, well-known in X-borough that the busses mostly arrive at random times to the bus stops (meaning mathematically that the arrivals of the busses follow a Poisson process of intensity $\lambda = \frac{1}{20}\ min^{-1}$, because the average time difference between two succeeding busses is 20 minutes).

One day when Mr. Smith is waiting extraordinary long time for his bus, he starts reasoning about how long time he at the average must wait for the bus, and he develops two ways of reasoning:

1) The time distance between two succeeding buses is exponentially distributed of mean 20 minutes, and since the exponential distribution is "forgetful", de average waiting time must be 20 minutes.

2) He arrives at a random time between two succeeding busses, so by the "symmetry" the average waiting time is instead $\frac{1}{2} \cdot 20$ minutes = 10 minutes.

At this time Mr. Smith's bus arrives, and he forgets to think of this contradiction.

Can you decide which of the two arguments is correct and explain the mistake in the wrong argument?

The *argument of (1) is correct.* The mistake of (2) is that the length of the time interval, in which Mr. Smith arrives, is *not* exponentially distributed. In fact, there will be a tendency of Mr. Smith to arrive in one of the longer intervals.

This is more precisely described in the following way. Let t denote Mr. Smith's arrival time. Then

1)

$$P\{\text{wait in more than } x \text{ minutes}\} = P\{N(t+x) - N(t) = 0\} = P\{N(x) = 0\} = e^{-\lambda x}.$$

This shows that the waiting time is *exponentially distribution of the mean* $\dfrac{1}{\lambda} = 20$ minutes.

2) Let X_1, X_2, ..., denote the lengths of the succeeding intervals between the arrivals of the busses. By the assumptions, the X_k are mutually independent and exponentially distributed of parameter λ.
Put

$$S_n = \sum_{k=1}^{n} X_k.$$

The surprise is that the X_k, for which

$$S_k = \sum_{j=1}^{k} X_k < t < \sum_{j+1}^{k+1} X_j = S_{k+1},$$

have the frequency

$$(1) \quad f_t(x) = \begin{cases} \lambda^2 x \, e^{-\lambda x}, & 0 < x \le t, \\[2mm] \lambda(1+\lambda t)e^{-\lambda x}, & t < x. \end{cases}$$

We shall now prove (1). First notice that the frequencies of the S_n are given by

$$g_n(x) = \frac{\lambda^n}{(n-1)!}\, x^{n-1} e^{-\lambda x}, \qquad x > 0.$$

(a) First assume that $x < t$. Then the even occurs that the interval has the length $\le x$, if

$$S_n = y \qquad \text{and} \qquad t - y < X_{n+1} \le x,$$

for some combination of n and y, where $t - x < y \le t$.
Then

$$\begin{aligned} F_t(x) &= \sum_{n=1}^{\infty} \int_{t-x}^{t} g_n(y) \left\{ e^{-\lambda(t-y)} - e^{-\lambda x} \right\} dy = \int_{t-x}^{t} \left\{ \sum_{n=1}^{\infty} g_n(y) \right\} \cdot \left\{ e^{-\lambda(t-y)} - e^{-\lambda x} \right\} dy \\ &= \int_{t-x}^{t} \lambda \left\{ e^{-\lambda t} e^{\lambda y} - e^{-\lambda x} \right\} dy = \lambda e^{-\lambda t} \int_{t-x}^{t} e^{\lambda y} dy - \lambda x e^{-\lambda x} = 1 - e^{-\lambda x} - \lambda x e^{-\lambda x}, \end{aligned}$$

where we have used that

$$\sum_{n=1}^{\infty} g_n(y) = \lambda \sum_{n=1}^{\infty} \frac{(\lambda y)^{n-1}}{(n-1)!}\, e^{-\lambda y} = \lambda.$$

By a differentiation,

$$f_t(x) = \lambda^2 x e^{-\lambda x} \qquad \text{for } x \le t.$$

(b) Then let $x > t$. The event occurs that the interval has length $\le x$, if either

$$S_n = y \qquad \text{and} \qquad t - y < X_{n+1} \le x$$

for some combination of n and y, or if $S_1 \in [t, x]$.
Then

$$\begin{aligned} F_t(x) &= \sum_{n=1}^{\infty} \int_{0}^{t} g_n(y) \left\{ e^{-\lambda(t-y)} - e^{-\lambda x} \right\} dy + \left\{ e^{-\lambda t} - e^{-\lambda x} \right\} \\ &= \lambda \int_{0}^{t} \left\{ e^{-\lambda(t-y)} - e^{-\lambda x} \right\} dy + \left\{ e^{-\lambda t} - e^{-\lambda x} \right\} \\ &= 1 - e^{-\lambda t} - \lambda t e^{-\lambda x} + e^{-\lambda t} - e^{-\lambda x} = 1 - (1 + \lambda t) e^{-\lambda x}. \end{aligned}$$

By differentiation,

$$f_t(x) = \lambda(1 + \lambda t) e^{-\lambda x}, \qquad \text{for } x > t.$$

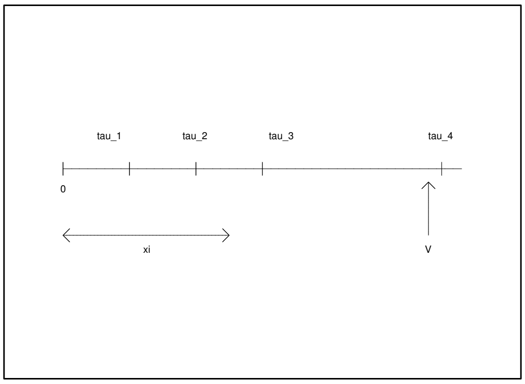
We have now found the distribution, so we can compute the *mean*

$$\begin{aligned}
\mu(t) &= \int_0^\infty x f_t(x)\,dx = \int_0^t \lambda^2 x^2 e^{-\lambda x}\,dx + \int_t^\infty \lambda x(1+\lambda t)e^{-\lambda x}\,dx \\
&= \left[-\lambda x^3 e^{-\lambda x}\right]_0^t + 2\int_0^t \lambda x e^{-\lambda x}\,dx + (1+\lambda t)\int_t^\infty \lambda x e^{-\lambda x} dx \\
&= \left[-\lambda x^2 e^{-\lambda x} - 2x e^{-\lambda x}\right]_0^1 + 2\int_0^t e^{-\lambda x}\,dx + (1+\lambda t)\left[-x e^{-\lambda x} - \frac{1}{\lambda}\,e^{-\lambda x}\right]_t^\infty \\
&= -\lambda t^2 e^{-\lambda t} - 2t e^{-\lambda t} + \frac{2}{\lambda}\left(1 - e^{-\lambda t}\right) + (1+\lambda t)\left(t e^{-\lambda t} + \frac{1}{\lambda}\,e^{-\lambda t}\right) \\
&= -\lambda t^2 e^{-\lambda t} - 2t e^{-\lambda t} + \frac{2}{\lambda} - \frac{2}{\lambda}\,e^{-\lambda t} + t e^{-\lambda t} + \frac{1}{\lambda}\,e^{-\lambda t} + \lambda t^2 e^{-\lambda t} + t e^{-\lambda t} \\
&= \frac{2}{\lambda} - \frac{1}{\lambda}\,e^{-\lambda t}.
\end{aligned}$$

An interpretation of this result is that for large values of t, i.e. when the Poisson process has been working for such a long time that some buses have arrived, then the mean is almost equal to $\frac{2}{\lambda}$, and definitely *not* $\frac{1}{\lambda}$, which Mr. Smith tacitly has used in his second argument.

Example 2.6 *Denote by* $\{X(t),\, t \geq 0\}$ *a Poisson process of intensity* a*, and let* ξ *be a fixed positive number. We define a random variable* V *by*

$$V = \inf\{v \geq \xi \mid \text{ there is no event from the Poisson process in the interval }]v-\xi, v]\}.$$



(On the figure the τ_i *indicate the times of the* i*-th event of the Poisson process,* V *the first time when we have had an interval of length* ξ *without any event).*

1) Prove that the distribution function $F(v)$ *of* V *fulfils*

$$(2)\;\; F(v) = \begin{cases} e^{-a\xi} + \int_0^\xi F(v-x)\, a\, e^{-ax}\, dx, & v \geq \xi, \\ 0, & v < \xi. \end{cases}$$

2) Prove that the Laplace transform of V *is given by*

$$L(\lambda) = \frac{(a+\lambda)e^{-(a+\lambda)\xi}}{\lambda + a\, e^{-(a+\lambda)\xi}}.$$

HINT: *Use that*

$$\int_0^\infty F(v)\, e^{-\lambda v}\, dv = \frac{1}{\lambda}\, L(\lambda) \qquad \textit{for } \lambda > 0.$$

3) Find the mean $E\{V\}$.

(In one-way single-track street cars are driving according to a Poisson process of intensity a*; a pedestrian needs the time* ξ *to cross the street; then* V *indicates the time when he has come safely across the street).*

The assumptions are

$$P\{X(t) = n\} = \frac{(a\,t)^n}{n!}\, e^{-at}, \qquad n \in \mathbb{N}_0,$$

and

$$P\{T_1 > t\} = P\{X(t) = 0\} = e^{-at}.$$

1) Clearly, $F(v) = 0$ if $v < \xi$. If $v = \xi$, then

$$F(v) = F(\xi) = P\left\{T_1 > \xi\right\} = P\{X(\xi) = 0\} = e^{-a\xi}.$$

If $v > \xi$, then $\tau_i = v - \xi$ and $v - x \in]v - \xi, c]$ for $x \in [0, \xi[$, and we are led to the following computation

$$\begin{aligned}
F(v) &= P\{V \le v\} = P\{V = \xi\} + P\{\xi < V \le v\} = e^{-a\xi} + P\{\xi < V \le v\} \\
(3) \qquad &= e^{-a\xi} + \int_{x=v-\xi}^{v} P\{V = x\}\, dP\{T > v - x\} \\
&= e^{-a\xi} + \int_{\xi}^{0} P\{V = v - x\}\, dP\{T > x\} = e^{-a\xi} + \int_{\xi}^{0} F(v - x)\, de^{-ax} \\
&= e^{-a\xi} + \int_{0}^{\xi} F(v - x)\, a\, e^{-ax}\, dx.
\end{aligned}$$

Here (3) is a generalized sum (i.e. an integral), where $V = x$ and $T > v - x$, which of course will contribute to $F(v)$.

2) If $L(\lambda) = \int_0^\infty f(v)\, e^{-\lambda v}\, dv$ then the Laplace transform of V is

$$\int_0^\infty F(v)\, e^{-\lambda v}\, dv = \frac{1}{\lambda} \int_0^\infty f(v)\, e^{-\lambda v}\, dv = \frac{1}{\lambda}\, L(\lambda) \qquad \text{for } \lambda > 0.$$

When we Laplace transform the result of (2), then

$$\begin{aligned}
\frac{1}{\lambda}\, L(\lambda) &= \frac{1}{\lambda}\, e^{-a\xi} e^{-a\lambda\xi} + \int_0^\infty \left\{ \int_0^\xi F(v - x)\, a\, e^{-ax}\, dx \right\} e^{-\lambda v}\, dv \\
&= \frac{1}{\lambda}\, e^{-(a+\lambda)\xi} + \int_0^\xi \left\{ \int_0^\infty F(v - x)\, e^{-\lambda v}\, dv \right\} a\, e^{-ax}\, dx \\
&= \frac{1}{\lambda}\, e^{-(a+\lambda)\xi} + \int_0^\xi \left\{ \int_x^\infty F(v - x)\, e^{-\lambda v}\, dv \right\} a\, e^{-ax}\, dx \\
&= \frac{1}{\lambda}\, e^{-(a+\lambda)\xi} + \int_0^\xi \left\{ \int_0^\infty F(v)\, e^{-\lambda v}\, dv \right\} e^{-\lambda x} \cdot a\, e^{-ax}\, dx \\
&= \frac{1}{\lambda}\, e^{-(a+\lambda)\xi} + \frac{1}{\lambda}\, L(\lambda) \cdot a \int_0^\xi e^{-(\lambda+a)x}\, dx \\
&= \frac{1}{\lambda}\, e^{-(a+\lambda)\xi} + \frac{1}{\lambda}\, L(\lambda) \cdot \frac{a}{a+\lambda} \left\{ 1 - e^{-(a+\lambda)\xi} \right\},
\end{aligned}$$

thus

$$e^{-(a+\lambda)\xi} = L(\lambda) \cdot \left\{ 1 - \frac{a}{a+\lambda} + \frac{a}{a+\lambda}\, e^{-(a+\lambda)\xi} \right\} = L(\lambda) \cdot \frac{\lambda + a\, e^{-(a+\lambda)\xi}}{a+\lambda},$$

and hence

$$L(\lambda) = \frac{(a+\lambda) e^{-(a+\lambda)\xi}}{\lambda + a\, e^{-(a+\lambda)\xi}}.$$

3) The mean is

$$
\begin{aligned}
E\{V\} &= -L'(0) \\
&= \lim_{\lambda\to 0+}\left\{\frac{e^{-(a+\lambda)\xi}-\xi(a+\lambda)e^{-(a+\lambda)\xi}}{\lambda+a\,e^{-(a+\lambda)\xi}}-\frac{(a+\lambda)e^{-(a+\lambda)\xi}\cdot\left(1-a\,\xi\,e^{-(a+\lambda)\xi}\right)}{\left(\lambda+a\,e^{-(a+\lambda)\xi}\right)^2}\right\} \\
&= -\left\{\frac{e^{-a\xi}-\xi\,a\,e^{-a\xi}}{a\,e^{-a\xi}}-\frac{a\,e^{-a\xi}\left(1-a\,\xi\,e^{-a\xi}\right)}{\left(a\,e^{-a\xi}\right)^2}\right\}=\frac{-e^{-a\xi}+\xi\,a\,e^{-a\xi}+1-a\,\xi\,e^{-a\xi}}{a\,e^{-a\xi}} \\
&= \frac{1-e^{-a\xi}}{a\,e^{-a\xi}}=\frac{1}{a}\left(e^{a\xi}-1\right).
\end{aligned}
$$

Example 2.7 *To a taxi rank taxis arrive from the south according to a Poisson process of intensity a, and independently there also arrive taxis from the north according to a Poisson process of intensity b.*
We denote by X the random variable which indicates the number of taxies, which arrive from the south in the time interval between two succeeding taxi arrivals from the north.
Find $P\{X=k\}$, $k\in\mathbb{N}_0$, as well as the mean and variance of X.

The length of the time interval between two succeeding arrivals from the north has the frequency

$$f(t)=b\,e^{-bt},\qquad t>0.$$

When this length is a (fixed) t, then the number of arriving taxies from the south is Poisson distributed of parameter $a\,t$. By the law of total probability,

$$
\begin{aligned}
P\{X=k\} &= \int_0^\infty \frac{(a\,t)^k}{k!}\,e^{-at}\cdot b\,e^{-kt}\,dt=\frac{b\,a^k}{k!}\int_0^\infty t^k e^{-(a+b)t}\,dt \\
&= \frac{b\,a^k}{k!}\cdot\frac{k!}{(a+b)^{k+1}}=\left(\frac{a}{a+b}\right)^k\cdot\frac{b}{a+b},\qquad k\in\mathbb{N}_0,
\end{aligned}
$$

so $X\in NB\left(1,\dfrac{b}{a+b}\right)$ is negative binomially distributed..

It follows by some formula in any textbook that

$$E\{X\}=1\cdot\frac{a}{b}=\frac{a}{b}\qquad\text{and}\qquad V\{X\}=\frac{a(a+b)}{b^2}=\frac{a}{b}\left(1+\frac{a}{b}\right).$$

Example 2.8 *The number of car accidents in a given region is assumed to follow a Poisson process* $\{X(t),\, t \in [0,\infty[\}$ *of intensity* λ*, and the number of persons involved in the i-th accident is a random variable*Y_i*, which is geometrically distributed,*

$$P\{Y_i = k\} = p\,q^{k-1}, \qquad k \in \mathbb{N},$$

where $p > 0$*,* $q > 0$ *and* $p + q = 1$*. We assume that the* Y_i *are mutually independent, and independent of* $\{X(t),\, t \geq 0\}$*.*

1. *Find the generating function of* $X(t)$*.*

2. *Find the generating function of* Y_i*.*

Denote by $Z(t)$ *the total number of persons involved in accidents in the time interval* $]0,t]$*.*

3. *Describe the generating function of* $Z(t)$ *expressed by the generating function of* Y_i *and the generating function of* $X(t)$*.*
HINT: *Use that*

$$P\{Z(t) = k\} = \sum_{i=0}^{\infty} P\{X(t) = i \wedge Y_1 + Y_2 + \cdots + Y_i = k\}.$$

4. *Compute* $E\{Z(t)\}$ *and* $V\{Z(t)\}$*.*

1) Since $X(t)$ is a Poisson process, we have

$$P\{X(t) = k\} = \frac{(\lambda\, t)^k}{k!}\, e^{-\lambda t}, \qquad k \in \mathbb{N}_0.$$

We find its generating function by using a table,

$$P_{X(t)}(s) = \exp(\lambda t(s-1)).$$

2) Also, by using a table, the generating function of Y_i is

$$P_{Y_i}(s) = \frac{p(s)}{1 - q\,s}.$$

The Y_i are mutually independent, so the generating function of $Y_1 + \cdots + Y_i$ is given by

$$\left(\frac{ps}{1-qs}\right), \qquad i \in \mathbb{N}.$$

3) The generating function of $Z(t)$ is

$$\begin{aligned}
P_{Z(t)}(s) &= \sum_{k=0}^{\infty} P\{Z(t) = k\}\, s^k = \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{\infty} P\{X(t) = i \wedge Y_1 + \cdots + Y_i = k\} \right\} s^k \\
&= \sum_{i=1}^{\infty} P\{X(t) = i\} \left(\sum_{k=0}^{\infty} P\{Y_1 + \cdots + Y_i = k)\, s^k \right) \\
&= \sum_{i=1}^{\infty} P\{X(t) = i\} \left(\frac{ps}{1-qs}\right) = P_{X(t)}\left(\frac{ps}{1-qs}\right) = \exp\left(\lambda t\left(\frac{ps}{1-qs} - 1\right)\right) \\
&= \exp\left(\lambda\, t\left(\frac{s-1}{1-qs}\right)\right) = \exp\left(\lambda t\left(\frac{p}{q} \cdot \frac{1}{1-qs} - 1\right)\right).
\end{aligned}$$

4) It follows from

$$P'_{Z(t)}(s) = \lambda t \cdot \frac{p}{(1-qs)^2}\, P_{Z(t)}(s) \qquad \text{med} \qquad P'_{Z(t)}(1) = \frac{\lambda t}{p},$$

and

$$P''_{Z(t)}(s) = \left(\lambda t \cdot \frac{p}{(1-qs)^2}\right)^2 P_{Z(t)}(s) + \lambda t \cdot \frac{2pq}{(1-qs)^3}\, P_{Z(t)}(s),$$

where

$$P''_{Z(t)}(1) = \left(\frac{\lambda t}{p}\right)^2 + \lambda t \cdot \frac{2q}{p^2},$$

that

$$E\{Z(t)\} = P'_{Z(t)}(1) = \frac{\lambda t}{p}$$

and

$$\begin{aligned} V\{Z(t)\} &= P''(1) + P'(1) - \left(P'(t)\right)^2 = \left(\frac{\lambda t}{p}\right)^2 + \lambda t \cdot \frac{2q}{p^2} + \frac{\lambda t}{p} - \left(\frac{\lambda t}{p}\right)^2 \\ &= \lambda t \cdot \frac{2q+p}{p^2} = \lambda t \cdot \frac{1+q}{p^2}. \end{aligned}$$

Example 2.9 (Continuation of Example 2.8).
Assume that the number of car accidents in a city follows a Poisson process $\{X(t),\, t \in [0,\infty[\}$ *of intensity 2 per day. The number of persons involved in one accident is assumed to be geometrically distributed with* $p = \frac{1}{2}$.
Find the mean and variance of the number of persons involved in car accidents in the city per week.

It follows from Example 2.8 that

$$E\{Z(t)\} = \frac{\lambda t}{p} \qquad \text{and} \qquad V\{Z(t)\} = \lambda t \cdot \frac{1+q}{p^2}.$$

In the specific case the intensity is $\lambda = 2$, and the time span is $t = 7$ days. Furthermore, $p = q = \frac{1}{2}$, thus

$$E\{Z(7)\} = \frac{2 \cdot 7}{\frac{1}{2}} = 28$$

and

$$V\{Z(7)\} = 2 \cdot 7 \cdot \frac{1+\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2 \cdot 7 \cdot 6 = 84.$$

Example 2.10 *Given a service to which customers arrive according to a Poisson process of intensity* λ *(measured in the unit minut*$^{-1}$*).*
Denote by I_1, I_2 *and* I_3 *three succeeding time intervals, each of the length of 1 minute.*

1. *Find the probability that there is no customer in any of the three intervals.*

2. *Find the probability that there is precisely one arrival of a customer in one of these intervals and none in the other two.*

3. *Find the probability that there are in total three arrivals in the time intervals* I_1, I_2 *and* I_3, *where precisely two of them occur in one of these intervals.*

4. *Find the value of* λ, *for which the probability found in* **3.** *is largest.*

Then consider 12 succeeding time intervals, each of length 1 minute. Let the random variable Z *denote the number of intervals, in which we have no arrival.*

5. *Find the distribution of* Z.

6. *For* $\lambda = 1$ *find the probability* $P\{Z = 4\}$ *(2 dec.).*

1) Let

$$I_1 =]0,1], \qquad I_2 =]1,2], \qquad I_3 =]2,3].$$

Then

$$P\{\text{no event in } I_1 \cup I_2 \cup I_3 =]0,3]\} = \left(e^{-\lambda}\right)^3 = e^{-3\lambda}.$$

2) By a rearrangement,

$$P\{\text{one event in one interval, none in the other two}\} = P\{\text{one event in } \left]0,3\right]\} = 3\lambda\, e^{-3\lambda}.$$

3) We have

$$\begin{aligned}
&P\{\text{two events in one interval, one in another one, and none in the remaining one}\}\\
&\qquad = P\{\text{two events in one interval, one in the remaining two intervals}\}\\
&\qquad = 3\cdot\frac{\lambda^2}{2}\cdot e^{-\lambda}\cdot 2\lambda\, e^{-2\lambda} = 3\lambda^3 e^{-3\lambda}.
\end{aligned}$$

4) We conclude from **3.** that $g(\lambda) = 3\lambda^3 e^{-3\lambda} > 0$ for $\lambda > 0$ with $g(\lambda) \to 0$ for $\lambda \to 0+$, and for $\lambda \to \infty$. By a differentiation,

$$g'(\lambda) = \left(9\lambda^2 - 9\lambda^3\right) e^{-3\lambda} = 9\lambda^2(1-\lambda)e^{-3\lambda} = 0 \quad \text{for } \lambda = 1 > 0,$$

thus the probability is largest for $\lambda = 1$ med $g(1) = 3\, e^{-3}$.

5) Assume now that we have 12 intervals. From

$$P\{\text{no arrival in an interval}\} = e^{-\lambda},$$

we get

$$P\{Z=k\} = \binom{12}{k} e^{-\lambda k}\left(1-e^{-\lambda}\right)^{12-k}, \qquad k=0,\,1,\,2,\,\dots,\,12,$$

thus $Z \in B\left(12, e^{-\lambda}\right)$.

6) By insertion of $\lambda = 1$ an $k=4$ into the result of **5.** we get

$$P\{Z=4\} = \binom{12}{4}\left\{e^{-1}\left(1-e^{-1}\right)^2\right\}^4 = 495\cdot\left(0.3679\cdot 0.6321^2\right)^4 = 0.2313 \approx 0.23.$$

Example 2.11 *A random variable X is Poisson distributed with parameter a.*

1. *Compute the characteristic function of X.*

2. *Prove for large values of a that X is approximately normally distributed of mean a and variance a (more precisely,*

$$\lim_{n\to\infty} P\left\{\frac{X-a}{\sqrt{a}} \le x\right\} = \Phi(x) \qquad \textit{for all } x\in\mathbb{R}).$$

To a service customers arrive according to a Poisson process of intensity $\lambda = 1$ minut^{-1}. Denote by X the number of customers who arrive in a time interval of length 100 minutes.

3. *Apply Chebyshev's inequality to find an lower bound of*

(4) $P\{80 < X < 120\}$.

4. *Find an approximate expression of (4) by using the result of* **2.**.

1) We get from

$$P\{X=k\} = \frac{a^k}{k!}\,e^{-a}, \qquad k\in\mathbb{N}_0,$$

the characteristic function

$$k_X(\omega) = \sum_{k=0}^{\infty} e^{i\omega k}\cdot\frac{a^k}{k!}\,e^{-a} = e^{-a}\sum_{k=0}^{\infty}\frac{1}{k!}\left(e^{i\omega}\,a\right)^k = e^{-a}\cdot\exp\left(a\cdot e^{i\omega}\right) = \exp\left(a\left(e^{i\omega}-1\right)\right).$$

2) Put

$$X_a = \frac{X-a}{\sqrt{a}}.$$

Then the characteristic function of X_a is given by

$$\begin{aligned} k_{X_a}(\omega) &= \sum_{k=0}^{\infty} \exp\left(i\omega \cdot \frac{k-a}{\sqrt{a}}\right) \frac{a^k}{k!}\, e^{-a} = e^{-i\omega\sqrt{a}} \cdot e^{-a} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ a \cdot \exp\left(i\,\frac{\omega}{\sqrt{a}}\right)\right\}^k \\ &= e^{-i\omega\sqrt{a}} \cdot e^{-a} \exp\left(a \cdot \exp\left(i\,\frac{\omega}{\sqrt{a}}\right)\right) = \exp\left(a\left\{\exp\left(\frac{i\omega}{\sqrt{a}}\right) - 1\right\} - i\omega\sqrt{a}\right). \end{aligned}$$

It follows from

$$\begin{aligned} a\left\{\exp\left(\frac{i\omega}{\sqrt{a}}\right) - 1\right\} - i\omega\sqrt{a} &= a\left\{1 + \frac{i\omega}{\sqrt{a}} - \frac{1}{2!}\,\frac{\omega^2}{a} + \frac{1}{a}\,\varepsilon\left(\frac{1}{a}\right) - 1\right\} - i\omega\sqrt{a} \\ &= -\frac{1}{2}\,\omega^2 + \varepsilon\left(\frac{1}{a}\right) \to -\frac{1}{2}\,\omega^2 \qquad \text{for } a \to \infty, \end{aligned}$$

that

$$k(\omega) = \lim_{a\to\infty} k_{X_a}(\omega) = \exp\left(-\frac{1}{2}\,\omega^2\right),$$

hence $k(\omega)$ is the characteristic function of a normally distributed random variable from $N(0,1)$. It follows that $\{X_a\}$ for $a \to \infty$ converges in distribution towards the normal distribution $N(0,1)$, thus

$$\lim_{a\to\infty} P\left\{\frac{X-a}{\sqrt{a}} \le x\right\} = \Phi(x) \qquad \text{for every } x \in \mathbb{R}.$$

3) If $t = 100$ and $\lambda = 1$ minut^{-1}, then

$$P\{X = n\} = \frac{100^n}{n!}\, e^{-100}, \qquad n \in \mathbb{N}_0,$$

hence $a = 100$ and $\sigma^2 = 100$. Then by Chebyshev's inequality

$$P\{|X-100| \ge 20\} \le \frac{100}{20^2} = \frac{1}{4},$$

so

$$P\{80 < X < 120\} = 1 - P\{|X-100| \ge 20\} \ge 1 - \frac{1}{4} = \frac{3}{4}.$$

4) An approximate expression of

$$P\{80 < X < 120\} = P\{|X-100| < 20\} = P\left\{\left|\frac{X-100}{10}\right| < 2\right\}$$

is then by **2.** given by

$$\Phi(2) - \Phi(-2) = 2\Phi(2) - 1 \approx 2 \cdot 0.9772 - 1 = 0.9544.$$

However, since X is an integer, we must here use the correction of continuity. Then the interval should be $80.5 < x < 119.5$. We get the improved approximate expression,

$$\begin{aligned} P\{80.5 < X < 119.5\} &= P\{|X-100| < 19.5\} = P\left\{\left|\frac{X-100}{10}\right| < 1.95\right\} \\ &= \Phi(1.95) - \Phi(1.95) = 2\Phi(1.95) - 1 \\ &\approx 2 \cdot 0.9744 - 1 = 0,9488. \end{aligned}$$

Remark 2.1 For comparison a long and tedious computation on a pocket calculator gives

$$P\{80 < X < 120\} \approx 0.9491. \qquad \Diamond$$

Example 2.12 *In a shop there are two shop assistants A and B. Customers may freely choose if they will queue up at A or at B, but they cannot change their decision afterwards. For all customers at A their serving times are mutually independent random variables of the frequency*

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x > 0, \\ 0, & x \le 0, \end{cases} \qquad (\lambda \textit{ is a positive constant}),$$

and for the customers at B the serving times are mutually independent random variables of frequency

$$g(y) = \begin{cases} 2\lambda\, e^{-2\lambda y}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

At a given time Andrew arrives and is queueing up at A, where there in front of him is only one customer, and where the service of this customer has just begun. We call the serving time of this customer X_1, while Andrew's serving time is called X_2.
At the same time Basil arrives and joins the queue at B, where there in front of him are two waiting customers, and where the service of the first customer has just begun. The service times of these two customers are denoted Y_1 and Y_2, resp..

1. *Find the frequencies of the random variables $X_1 + X_2$ and $Y_1 + Y_2$.*

2. *Express by means of the random variables Y_1, Y_2 and X_1 the event that the service of Basil starts after the time when the service of Andrew has started, and find the probability of this event.*

3. *Find the probability that the service of Basil starts after the end of the service of Andrew.*

Assume that the customers arrive to the shop according to a Poisson process of intensity α.

4. *Find the expected number of customers, who arrive to the shop in a time interval of length t.*

5. *Let N denote the random variable, which indicates the number of customers who arrive to the shop during the time when Andrew is in the shop (thus $X_1 + X_2$). Find the mean of N.*

1) Since $X_i \in \Gamma\left(1, \frac{1}{\lambda}\right)$ is exponentially distributed we have $X_1 + X_2 \in \Gamma\left(2, \frac{1}{\lambda}\right)$, thus

$$f_{X_1+X_2}(x) = \begin{cases} \lambda^2 x\, e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

Since $Y_i \in \Gamma\left(1, \frac{1}{2\lambda}\right)$, we have $Y_1 + Y_2 \in \Gamma\left(2, \frac{1}{2\lambda}\right)$ with the frequency

$$g_{Y_1+Y_2}(y) = \begin{cases} 4\lambda^2 y\, e^{-2\lambda y}, & y \ge 0, \\ 0, & y < 0. \end{cases}$$

2) The event is expressed by $X_1 < Y_1 + Y_2$. The probability of this event is

$$\begin{aligned}
P\left\{X_1 < Y_1 + Y_2\right\} &= \int\int_{\{0<x<y\}} \lambda\, e^{-\lambda x} \cdot 4\lambda^2 y\, e^{-2\lambda y}\, dx\, dy \\
&= \int_0^{\infty} 4\lambda^2 y\, e^{-2\lambda y} \left\{\int_0^y \lambda\, e^{-\lambda x}\, dx\right\} dy \\
&= \int_0^{\infty} 4\lambda^2 y\, e^{-2\lambda y} \left[-e^{-\lambda x}\right]_{x=0}^{y} dy \\
&= \int_0^{\infty} 4\lambda^2 y\, e^{-2\lambda y}\, dy - \int_0^{\infty} 4\lambda^2 y\, e^{-3\lambda y}\, dy \\
&= \int_0^{\infty} t\, e^{-t}\, dt - \frac{4}{9}\int_0^{\infty} t\, e^{-t}\, dt = \frac{5}{9}.
\end{aligned}$$

3) We must have in this case that $X_1 + X_2 < Y_1 + Y_2$. Hence the probability is

$$\begin{aligned}
&P\left\{X_1 + X_2 < Y_1 + Y_2\right\} = \int\int_{\{0<x<y\}} \lambda^2 x\, e^{-\lambda x} \cdot 4\lambda^2 y\, e^{-2\lambda y}\, dx\, dy \\
&= \int_0^{\infty} 4\lambda^2 y\, e^{-2\lambda y} \left(\int_0^y \lambda^2 x\, e^{-\lambda x}\, dx\right) dy = \int_0^{\infty} 4\lambda^2 y\, e^{-2\lambda y} \left\{\left[-\lambda x\, e^{-\lambda x}\right]_0^y + \int_0^y \lambda\, e^{-\lambda x}\, dx\right\} dy \\
&= \int_0^{\infty} 4\lambda^2 y\, e^{-2\lambda y} \int_0^y \lambda\, e^{-\lambda x}\, dx - \int_0^{\infty} 4\lambda^3 y^2 e^{-3\lambda y}\, dy \\
&= P\left\{X_1 < Y_1 + Y_2\right\} - \frac{4}{27}\int_0^{\infty} (3\lambda)^3 y^2 e^{-3\lambda y}\, dy = \frac{5}{9} - \frac{4}{27}\cdot 2 = \frac{15-8}{27} = \frac{7}{27}.
\end{aligned}$$

4) If $X(t)$ indicates the number of arrived customers in $]0,t]$, then

$$P\{X(t) = n\} = \frac{(\alpha t)^n}{n!}\, e^{-\alpha t}, \qquad n \in \mathbb{N}_0,$$

and

$$m(t) = E\{X(t)\} = \sum_{n=0}^{\infty} n\, \frac{(\alpha t)^n}{n!}\, e^{-\alpha t} = \alpha\, t.$$

5) Finally, (cf. **4.**),

$$E\{N\} = \alpha\, E\left\{X_1 + X_2\right\} = \alpha \left\{\frac{1}{\lambda} + \frac{1}{\lambda}\right\} = \frac{2\alpha}{\lambda}.$$

3 Birth and death processes

Example 3.1 *Consider a birth process $\{X(t), t \in [0,\infty[\}$ of states E_0, E_1, E_2, $\dots$ and positive birth intensities λ_k. The differential equations of the process are*

$$\begin{cases} P_0'(t) = -\lambda_0 P_0(t), \\ \\ P_k'(t) = -\lambda_k P_k(t) + \lambda_{k-1} P_{k-1}(t), \qquad k \in \mathbb{N}, \end{cases}$$

and we assume that the process at $t=0$ is in state E_0. It can be proved that the differential equations have a uniquely determined solution $(P_k(t))$ satisfying

$$P_k(t) \geq 0, \qquad \sum_{k=0}^{\infty} P_k(t) \leq 1.$$

One can also prove that either $\sum_{k=0}^{\infty} P_k(t) = 1$ for all $t > 0$, or $\sum_{k=0}^{\infty} P_k(t) < 1$ for all $t > 0$.
Prove that

$\sum_{k=0}^{\infty} P_k(t) = 1$ *for all $t > 0$, if and only if* $\sum_{k=0}^{\infty} \dfrac{1}{\lambda_k}$ *is divergent.*

HINT: *First prove that*

$$\frac{1}{\lambda_k}\, a(t) \leq \int_0^t P_k(s)\, ds \leq \frac{1}{\lambda_k}, \qquad k \in \mathbb{N}_0, \quad t > 0,$$

where $a(t) = 1 - \sum_{k=0}^{\infty} P_k(t)$.

We get by a rearrangement and recursion,

$$\lambda_k P_k(t) = -P_k'(t) + \lambda_{k-1} P_{k-1}(t) = -P_k'(t) - P_{k-1}'(t) + \lambda_{k-2} P_{k-2}(t) = \cdots = -\sum_{j=0}^{k} P_j'(t),$$

hence by integration,

$$\lambda_k \int_0^t P_k(s)\, ds = \left[-\sum_{j=0}^{k} P_j(s) \right]_0^t = \sum_{j=0}^{k} P_j(0) P_j(t) = 1 - \sum_{j=0}^{k} P_j(t),$$

because at time $t = 0$ we are in state E_0, so $P_0(0) = 0$, and $P_j(0) = 0$, $j \in \mathbb{N}$.
Thus we have the estimates

$$a(t) = 1 - \sum_{j=0}^{\infty} P_j(t) \leq 1 - \sum_{j=0}^{k} P_j(t) = \lambda_k \int_0^t P_k(s)\, ds \leq 1,$$

from which

$$\frac{1}{\lambda_k}\, a(t) \leq \int_0^t P_k(s)\, ds \leq \frac{1}{\lambda_k}.$$

Assume that $\sum_{k=0}^{\infty} P_k(t) = 1$. Applying the theorem of monotonous convergence (NB The Lebesgue integral!) it follows from the right hand inequality that

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_l} \geq \sum_{k=0}^{\infty} \int_0^t P_k(s)\,ds = \int_0^t \sum_{k=0}^{\infty} P_k(s)\,ds = \int_0^t 1\,dt = t \qquad \text{for alle } t \in \mathbb{R}_+,$$

proving that the series $\sum_{k=0}^{\infty} \dfrac{1}{\lambda_k}$ is divergent.

Then assume that $\sum_{k=0}^{\infty} P_k(t) < 1$, thus

$$a(t) = 1 - \sum_{k=0}^{\infty} P_k(t) > 0.$$

Using the theorem of monotonous convergence and the left hand inequality we get

$$\left(\sum_{k=0} \frac{1}{\lambda_k} \right) \cdot a(t) \leq \sum_{k=0}^{\infty} \int_0^t P_k(s)\,ds \leq t \qquad \text{for all } t \in \mathbb{R}_+.$$

Now $a(t) > 0$, so this implies that

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_k} \leq \frac{t}{a(t)} < \infty,$$

and the series $\sum_{k=0}^{\infty} \dfrac{1}{\lambda_k}$ is convergent.

Example 3.2 *To a carpark, cars arrive from* 9^{00} $(t=0)$ *following a Poisson process of intensity* λ. *There are in total* N *parking bays, and we assume that no car leaves the carpark. Let* E_n, $n = 0, 1, \dots, N$, *denote the state that* n *of the parking bays are occupied.*

1) Find the differential equations of the system.

2) Find $P_n(t)$, $n = 0, 1, \dots, N$.

3) Find the stationary probabilities p_n, $n = 0, 1, \dots, N$.

Put $\lambda = 1$ *minute*$^{-1}$ *and* $N = 5$. *Find the probability that a car driver who arrives at* 9^{03} *cannot find a vacant parking bay.*

1) This is a pure birth process with

$$\lambda_n = \begin{cases} \lambda & \text{for } n = 0,\, 1,\, \dots,\, N-1, \\ \\ 0 & \text{for } n = N, \end{cases}$$

and the system of differential equations

$$\begin{aligned} P_0'(t) &= -\lambda\, P_0(t), \\ P_n'(t) &= -\lambda\, P_n(t) + \lambda\, P_{n-1}(t), \qquad n = 1,\, 2,\, \dots,\, N-1, \\ P_N'(t) &= \lambda\, P_{N-1}(t), \end{aligned}$$

and initial conditions

$$P_n(0) = \begin{cases} 1 & \text{for } n = 0, \\ \\ 0 & \text{for } n > 0. \end{cases}$$

2) The system of **1.** can either be solved successively or by consulting a textbook,

$$P_n(t) = \begin{cases} e^{-\lambda t}\, \dfrac{(\lambda t)^n}{n!}, & n = 0,\, 1,\, 2,\, \dots,\, N-1, \\ \\ 1 - \sum_{n=0}^{N-1} \dfrac{(\lambda t)^n}{n!}\, e^{-\lambda t}, & n = n. \end{cases}$$

3) It follows immediately that

$$P_n(t) \to \begin{cases} 0, & n < N, \\ 1, & n = N, \end{cases} \qquad \text{for } t \to \infty,$$

thus

$$p_n = \begin{cases} 0, & n < N, \\ \\ 1, & n = n. \end{cases}$$

4) First identify

$$\lambda = 1 \text{ minute}^{-1}, \qquad t = 3 \quad \text{and} \quad N = 5.$$

Then by insertion,

$$P\left\{\text{no parking bay at } 9^{03}\right\} = P_5(3) = 1 - \sum_{n=0}^{4} P_n(3) = 1 - \sum_{n=0}^{4} \frac{3^n}{n!}\, e^{-3} = 0.1847 \approx 0.185.$$

Example 3.3 *Given a stochastic birth and death process $X(t), t \in [0,\infty[\}$, which can be in the states E_4, E_5, E_6 and E_7.*
Assume that the birth intensity λ_k is in state E_k given by

$$\lambda_k = \alpha k(7-k),$$

and that the death intensity μ_k in state E_k is equal to

$$\mu_k = \beta k(k-4),$$

where α and β are positive constants.
Find the stationary probabilities in each of the two cases below

1) $\beta = \alpha$,

2) $\beta = 2\alpha$.

The equations of equilibrium are here

$$\mu_{k+1} p_{k+1} = \lambda_k p_k \qquad \text{for } k = 4,\, 5,\, 6.$$

Thus

$$\begin{aligned}
p_5 &= \frac{\lambda_4}{\mu_5}\, p_4 = \frac{12\alpha}{5\beta}\, p_4 = \frac{12}{5}\left(\frac{\alpha}{\beta}\right)^1 p_4, \\
p_6 &= \frac{\lambda_5}{\mu_6}\, p_5 = \frac{10\alpha}{12\beta} \cdot \frac{12}{5}\, \frac{\alpha}{\beta} = 2\left(\frac{\alpha}{\beta}\right)^2 p_4, \\
p_7 &= \frac{\lambda_6}{\mu_7}\, p_6 = \frac{6\alpha}{21\beta} \cdot 2\left(\frac{\alpha}{\beta}\right)^2 = \frac{4}{7}\left(\frac{\alpha}{\beta}\right)^3 p_4.
\end{aligned}$$

Furthermore,

$$p_4 + p_5 + p_6 + p_7 = 1.$$

However, the exact values can first be found when we know the relationship between α and β.

1) If $\beta = \alpha$, then

$$1 = p_4\left\{1 + \frac{12}{5} + 2 + \frac{4}{7}\right\} = \frac{35 + 84 + 70 + 20}{35}\, p_4 = \frac{209}{35}\, p_4,$$

hence

$$p_4 = \frac{35}{209}, \qquad p_5 = \frac{12}{5} \cdot \frac{35}{209} = \frac{84}{209},$$

$$p_6 = \frac{70}{209}, \qquad p_7 = \frac{4}{7} \cdot \frac{35}{209} = \frac{20}{2+9},$$

so

$$\mathbf{p} = (p_4, p_5, p_6, p_7) = \frac{1}{209}\,(35, 84, 70, 20).$$

2) If $\beta = 2\alpha$, then $\dfrac{\alpha}{\beta} = \dfrac{1}{2}$, hence

$$p_5 = \frac{6}{5}\,p_4, \qquad p_6 = \frac{1}{2}\,p_4, \qquad p_7 = \frac{1}{14}\,p_4,$$

and

$$1 = p_4 + p_5 + p_6 + p_7 = p_4\left(1 + \frac{6}{5} + \frac{1}{2} + \frac{1}{14}\right) = \frac{70+84+35+5}{70}\,p_4 = \frac{97}{35}\,p_4,$$

from which

$$p_4 = \frac{35}{97}, \qquad p_5 = \frac{42}{97}, \qquad p_6 = \frac{35}{194}, \qquad p_7 = \frac{5}{194},$$

i.e.

$$\mathbf{p} = (p_4, p_5, p_6, p_7) = \frac{1}{194}\,(70, 84, 35, 5).$$

Example 3.4 *Given a birth and death process of the states E_0, E_1, E_2, ..., birth intensities λ_k and death intensities μ_k. Assume furthermore that*

a. $\lambda_k = \mu_k = k\,\alpha$, $k \in \mathbb{N}_0$, *(where a is a positive constant).*

b. $P_1(0) = 1$.

1. *Find the differential equations of the process.*

One may now without proof use that under the assumptions above,

$$P_1(t) = \frac{1}{(1+\alpha t)^2}.$$

2. *Find $P_0(t)$, $P_2(t)$ and $P_3(t)$.*

3. *Sketch the graph of $P_0(t) + P_1(t)$.*

4. *Sketch the graph of $P_2(t)$.*

5. *Find $\lim_{t\to\infty} P_n(t)$ for every $n \in \mathbb{N}_0$.*

1) We have

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) = \alpha\, P_1(t),$$

and

$$\begin{aligned} P_k'(t) &= -(\lambda_k+\mu_k)\,P_k(t) + \lambda_{k-1}P_{k-1}(t) + \mu_{k+1}P_{k+1}(t) \\ &= (k-1)\alpha P_{k-1}(t) - 2k\alpha P_k(t) + (k+1)\alpha P_{k+1}(t) \qquad \text{for } k \in \mathbb{N}. \end{aligned}$$

2) If $P_1(0) = 1$, then $P_k(0) = 0$ for $k \in \mathbb{N}_0 \setminus \{1\}$. It follows from

$$P_0'(t) = \alpha\, P_1(t) = \frac{\alpha}{(1+\alpha t)^2},$$

by an integration that

$$P_0(t) = \int_0^t \frac{\alpha\, d\tau}{(1+\alpha\tau)^2} = \left[-\frac{1}{1+\alpha\tau}\right]_0^t = 1 - \frac{1}{1+\alpha t} = \frac{\alpha t}{1+\alpha t}.$$

If $k = 1$, we get by a rearrangement,

$$\begin{aligned} P_2(t) &= \frac{1}{2\alpha}\left\{P_1'(t) - 0\cdot P_0(t) + 2\alpha\, P_1(t)\right\} = \frac{1}{2\alpha}\left\{-\frac{2\alpha}{(1+\alpha t)^3} + \frac{2\alpha}{(1+\alpha t)^2}\right\} \\ &= \frac{1}{(1+\alpha t)^2} - \frac{1}{(1+\alpha t)^3} = \frac{\alpha t}{(1+\alpha t)^3}. \end{aligned}$$

If $k=2$, we get by a rearrangement,

$$\begin{aligned}
P_3(t) &= \frac{1}{3\alpha}\left\{P_2'(t)-\alpha\,P_1(t)+4\alpha\,P_2(t)\right\}\\
&= \frac{1}{3\alpha}\left\{\frac{3\alpha}{(1+\alpha t)^4}-\frac{2\alpha}{(1+\alpha t)^3}-\frac{\alpha}{(1+\alpha t)^2}+\frac{4\alpha}{(1+\alpha t)^2}-\frac{4\alpha}{(1+\alpha t)^3}\right\}\\
&= \frac{1}{3\alpha}\left\{\frac{3\alpha}{(1+\alpha t)^4}-\frac{6\alpha}{(1+\alpha t)^3}+\frac{3\alpha}{(1+\alpha t)^2}\right\}\\
&= \frac{(1+\alpha t)^2-2(1+\alpha t)+1}{(1+\alpha t)^4}=\frac{\alpha^2t^2}{(1+\alpha t)^4}.
\end{aligned}$$

Summing up,

$$P_0(t)=\frac{\alpha t}{1+\alpha t},\qquad P_1(t)=\frac{1}{(1+\alpha t)^2},$$
$$P_2(t)=\frac{\alpha t}{(1+\alpha t)^3},\qquad P_3(t)=\frac{\alpha^2t^2}{(1+\alpha t)^4}.$$

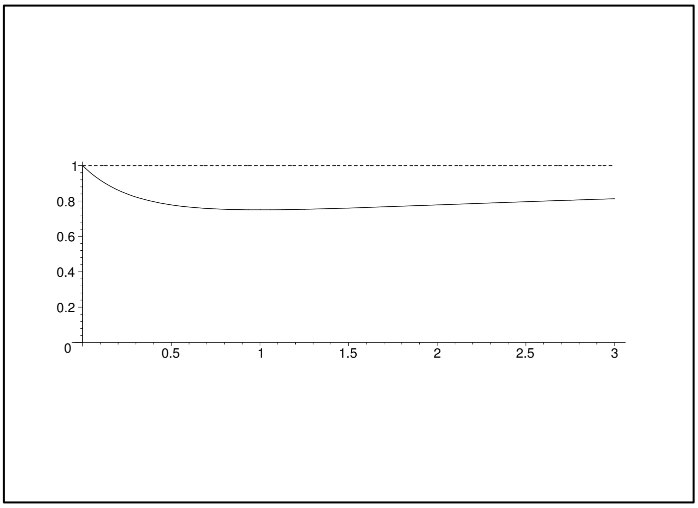


Figure 1: The graph of $1-\dfrac{x}{(1+x)^2}$ with $x=\alpha t$.

3) It follows that

$$P_0(t)+P_1(t)=\frac{\alpha t}{1+\alpha t}+\frac{1}{(1+\alpha t)^2}=\frac{1+\alpha t+\alpha^2t^2}{(1+\alpha t)^2}=1-\frac{\alpha t}{(1+\alpha t)^2}.$$

If we put $x=\alpha t$, we see that we shall only sketch

$$1-\frac{x}{(1+x)^2}=1-\frac{1}{1+x}+\frac{1}{(1+x)^2},$$

which has a minimum for $x=1$, and has $y=1$ as an asymptote.

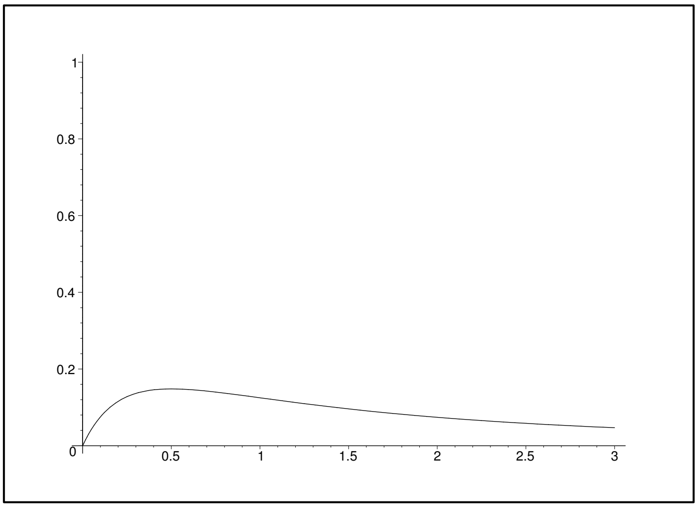


Figure 2: The graph of $\dfrac{x}{(1+x)^3}$ with $x = \alpha t$.

4) If we put $x = \alpha t$, it follows that we shall only sketch

$$\varphi(x) = \frac{x}{(1+x)^3}.$$

From

$$\varphi'(x) = \frac{1}{(1+x)^3} - \frac{3x}{(1+x)^4} = \frac{1-2x}{(1+x)^4},$$

follows that we have a maximum for $x = \dfrac{1}{2}$, corresponding to

$$\varphi\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\left(\frac{3}{2}\right)^3} = \frac{4}{27}.$$

5) Clearly,

$$\lim_{t\to\infty} P_0(t) = \lim_{t\to\infty} \frac{\alpha t}{1+\alpha t} = 1.$$

We conclude from

$$\sum_{n=0}^{\infty} P_n(t) = 1 \qquad \text{and} \qquad P_n(t) \geq 0,$$

that

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} P_n(t) = 0,$$

hence

$$\lim_{t\to\infty} P_n(t) = 0 \qquad \text{for alle } n \in \mathbb{N}.$$

Example 3.5 *A power station delivers electricity to N customers. If a customer at time t uses electricity there is the probability $\mu h + h\,\varepsilon(h)$ that he does not use electricity at time $t+h$, and probability $1-\mu h + h\,\varepsilon(h)$ that he is still using electricity at time $t+h$.*
However, if he to time t does not use electricity, then there is the probability $\lambda h + h\,\varepsilon(h)$ that he uses electricity at time $t+h$, and probability $1-\lambda h + h\,\varepsilon(h)$ that he does not do it.
The customers are using electricity mutually independently.
Denote by E_k the state that k consumers use electricity, $k = 0, 1, \dots, N$.
Find the differential equations of the system.
Find the stationary probabilities.

We put $X_k(t) = 1$, if the k-th customer uses electricity at time t, and $X_k(t) = 0$, if he does not do it. Let n and $j \in \{0, 1, \dots, N\}$, and assume that the system is in state E_j, i.e.

$$\sum_{k=1}^{N} X_k(t) = j \qquad \text{at time } t.$$

How can we realize that we are in state E_n at time $t+h$?

There must be an $m \in \{0, 1, \dots, j\}$, such that $j-m$ of the customers who were using electricity at time t, still are using electricity at time $t+h$.
Furthermore, $n-j+m$ of the customers, who did not use electricity at time t, must use electricity at time $t+h$, is we are in state E_n.
Thus we get the condition $m \geq j-n$, so

$$m \in \{\max\{0, j-n\}, \dots, \min\{j, N-n\}\}, \quad \text{and} \quad j \in \{0, 1, \dots, N\}.$$

Summing up, if the conditions above are fulfilled, then

1) m of the customers, who used electricity at time t, do not do it at time $t+h$.

2) $j-m$ use electricity both at time t and at time $t+h$.

3) $n-j+m$ did not use electricity at time t, but they do it at time $t+h$.

4) $N-n-m$ neither use electricity at time t nor at time $t+h$.

For fixed j this can be done of the probability

$$\sum_{m=\max\{0,j-n\}}^{\min\{j,N-n\}} \binom{j}{m} \{\mu h + h\varepsilon(h)\}^m \{1-\mu h + h\varepsilon(h)\}^{j-m}$$
$$\binom{N-j}{n-j+m} \{\lambda h + h\varepsilon(h)\}^{n-j+m} \{1-\lambda h + h\varepsilon(h)\}^{N-n-m}.$$

When we multiply this equation by $P_j(t)$ and then sum with respect to j, we get

$$\begin{aligned}(5)\ P_n(t+h) &= \sum_{j=0}^{N} P_j(t) \sum_{m=\max\{0,j-n\}}^{\min\{j,N-n\}} \binom{j}{m}\binom{N-j}{n-j+m} \times \\ &\qquad \times\{\mu h + h\varepsilon(h)\}^m\{1-\mu h + h\varepsilon(h)\}^{j-m} \times \\ &\qquad \times\{\lambda h + h\varepsilon(h)\}^{n-j+m}\{1-\lambda h + h\varepsilon(h)\}^{N-m-n}.\end{aligned}$$

If $m=0$ in the inner sum, then $j \le n$, and we isolate the term

$$\begin{aligned}&\binom{j}{0}\binom{N-j}{n-j}\{\mu h + h\varepsilon(h)\}^0\{1-\mu h + h\varepsilon(h)\}^j\{\lambda h + h\varepsilon(h)\}^{n-j}\{1-\lambda h + h\varepsilon(h)\}^{N-n} \\ &\quad = \binom{N-j}{n-j}\{1-\mu h + h\varepsilon(h)\}^j\{1-\lambda h + h\varepsilon(h)\}^{N-n} h^{n-j}\{\lambda + \varepsilon(h)\}^{n-j}.\end{aligned}$$

It follows clearly that if $j \ne n$, $n-1$, then we get terms of the type $h\varepsilon(h)$,
If furthermore $j=n$, then we get the term

$$\begin{aligned}&\binom{N-n}{0}\{1-\mu h + h\varepsilon(h)\}^n\{1-\lambda h + h\varepsilon(h)\}^{N-n}\cdot 1 \\ &\quad = (1-\mu h)^n(1-\lambda h)^{N-n} + h\varepsilon(h) = 1 - n\mu h + (N-n)\lambda h + h\,\varepsilon(h).\end{aligned}$$

If instead $j=n-1$, then we get the term

$$\begin{aligned}&\binom{N-n+1}{1}\{1-\mu h + h\varepsilon(h)\}^{n-1}\{1-\lambda h + h\varepsilon(h)\}^{N-1}\cdot h\cdot(\lambda + h\varepsilon(h)) \\ &\qquad\qquad = (N-n+1)h\lambda + h\varepsilon(h).\end{aligned}$$

If $m=1$ in the inner sum of (5), then

$$j-n \le n \le \min\{j, N-n\},$$

thus $1 \le j \le n+1$. For such j we get the contribution

$$\binom{j}{1}\binom{N-j}{n-j+1}\mu h(1-\mu h)^{j-1}(\lambda h)^{n-j+1}(1-\lambda h)^{N-n-m} + h\varepsilon(h).$$

It follows immediately that if $j \neq n+1$, then all these terms are of the type $h\varepsilon(h)$.
For $j = n+1$ we get the contribution

$$\binom{n+1}{1}\binom{N-n-1}{0}\mu h(1-\mu h)^n(1-\lambda h)^{N-n-m} + h\varepsilon(h) = (n+1)\mu h + h\varepsilon(h).$$

If $m \geq 2$, we only get terms of the type $h\varepsilon(h)$.

We now include ε functions. Then (5) is reduced by this analysis for $n = 1, \dots, N-1$, to

$$\begin{aligned} P_n(t+h) &= P_n\{1 - n\mu h - (N-n)\lambda h + h\varepsilon(h)\} + P_{n-1}(t)\cdot(N-n+1)h\lambda + h\varepsilon(h) \\ &\quad + P_{n+1}(t)\cdot(n+1)\cdot\mu h + h\varepsilon(h), \end{aligned}$$

thus by a rearrangement

$$\begin{aligned} &P_n(t+h) - P_n(t) \\ &\qquad = -h\{(n\mu + (N-n)\lambda)P_n(t)\} + h(N-n+1)\lambda P_{n-1}(t) + h(n+1)\mu P_{n+1}(t) + h\varepsilon(h), \end{aligned}$$

and hence dividing by h, followed by taking the limit $h \to 0$,

$$P_n'(t) = -\{n\mu + (N-n)\lambda\}P_n(t) + (N-n+1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t).$$

There are some modifications for $n = 0$ and $n = N$, in which cases we get instead

$$P_0'(t) = -N\,\lambda\,P_0(t) + \mu\,P_1(t),$$

and

$$P_N'(t) = -N\,\mu\,P_N(t) + \lambda\,P_{N-1}(t).$$

Then we have for the stationary probabilities,

$$\begin{aligned} 0 &= -N\,\lambda\,p_0 + \mu\,p_1, \\ 0 &= -\{n\mu + (N-n)\lambda\}p_n + (N-n+1)\lambda p_{n-1} + (n+1)\mu p_{n+1}, \qquad n = 1, \dots, N-1, \\ 0 &= -N\,\mu\,p_N + \lambda\,p_{N-1}, \end{aligned}$$

hence

$$\begin{cases} p_1 = N \cdot \dfrac{\lambda}{\mu}\,p_0 \\[2ex] p_{n+1} = \left(\dfrac{n}{n+1} + \dfrac{N-n}{n+1}\cdot\dfrac{\lambda}{\mu}\right)p_n - \dfrac{N-n+1}{n+1}\cdot\dfrac{\lambda}{\mu}\,p_{n-1} \qquad n = 1, \dots, N-1, \\[2ex] p_N = \dfrac{1}{N}\cdot\dfrac{\lambda}{\mu}\,p_{N-1}. \end{cases}$$

In order to find the pattern we compute p_2, i.e. we put $n = 1$ into the general formula

$$\begin{aligned} p_2 &= \left(\frac{1}{2} + \frac{N-1}{2}\cdot\frac{\lambda}{\mu}\right)p_1 - \frac{N}{2}\left(\frac{\lambda}{\mu}\right)p_0 = \frac{N}{2}\cdot\frac{\lambda}{\mu}\cdot p_0 + \frac{N(N-1)}{2}\left(\frac{\lambda}{\mu}\right)^2 p_0 - \frac{N}{2}\cdot\frac{\lambda}{\mu}\cdot p_0 \\ &= \binom{N}{2}\cdot\left(\frac{\lambda}{\mu}\right)^2 p_0. \end{aligned}$$

Now

$$p_1 = N \cdot \left(\frac{\lambda}{\mu}\right)^1 p_0 = \left(\begin{array}{c} N \\ 1 \end{array}\right) \cdot \left(\frac{\lambda}{\mu}\right)^1 p_0,$$

so we guess that we in general have

$$p_n = \left(\begin{array}{c} N \\ n \end{array}\right) \cdot \left(\frac{\lambda}{\mu}\right)^m p_0.$$

This is true for $n = 0, 1, 2$.
Assume that the claim holds for all indices up to n. If $n \leq N-1$, then

$$\begin{aligned} p_{n+1} &= \left(\frac{n}{n+1} + \frac{N-n}{n+1} \cdot \frac{\lambda}{\mu}\right) p_n - \frac{N-n+1}{n+1} \cdot \frac{\lambda}{\mu} p_{n-1} \\ &= \frac{n}{n+1} \cdot \frac{N!}{n!(N-n)!} \left(\frac{\lambda}{\mu}\right)^n + \frac{N-n}{n+1} \cdot \frac{N!}{n!(N-n)!} \left(\frac{\lambda}{\mu}\right)^{n+1} p_0 \\ &\qquad - \frac{N-n+1}{n+1} \cdot \frac{N!}{(n-1)!(N-n+1)!} \left(\frac{\lambda}{\mu}\right)^n p_0 \\ &= \frac{N!}{(n+1)\cdot(n-1)!(N-n)!} \left(\frac{\lambda}{\mu}\right)^n p_0 - \frac{N!}{(n+1)(n-1)!(N-n)!} \left(\frac{\lambda}{\mu}\right)^n p_0 \\ &\qquad + \frac{N!}{(n+1)!(N-n-1)!} \left(\frac{\lambda}{\mu}\right)^{n+1} p_0 \\ &= \left(\begin{array}{c} N \\ n+1 \end{array}\right) \left(\frac{\lambda}{\mu}\right)^{n+1} p_0, \end{aligned}$$

and the claim follows by induction. Then

$$1 = \sum_{n=0}^{N} p_n = p_0 \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array}\right) \left(\frac{\lambda}{\mu}\right)^n = p_0 \cdot \left(1 + \frac{\lambda}{\mu}\right)^N = p_0 \left(\frac{\lambda+\mu}{\mu}\right)^N,$$

hence

$$p_n = \left(\begin{array}{c} N \\ n \end{array}\right) \cdot \left(\frac{\lambda}{\mu}\right)^n \cdot \left(\frac{\mu}{\lambda+\mu}\right)^N = \left(\begin{array}{c} N \\ n \end{array}\right) \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right)^{N-n}.$$

The solution above is somewhat clumsy, though it follows the ordinary way one would solve problems of this type without too much training.

Alternatively we see that we have a birth and death process of states $E_0, E_1, \dots, E_N$, and intensities

$$\lambda_k = (N-k)\lambda, \qquad \mu_k = k\mu, \qquad k \in \{0, 1, \dots, N\}.$$

The corresponding system of differential equations becomes

$$\left\{ \begin{array}{l} P_0'(t) = -N\lambda P_0(t) + \mu P_1(t), \\ \\ P_k'(t) = -\{(N-k)\lambda + k\mu\} P_k(t) + (N-k+1)\lambda P_{k-1}(t) + (k+1)\mu P_{k+1}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq k \leq N-1, \\ P_N'(t) = -N\mu P_N(t) + \lambda P_{N-1}(t). \end{array} \right.$$

The stationary probabilities p_k are found from

$$\mu_k p_k = \lambda_{k-1} p_{k-1}, \qquad k = 1, 2, \dots, N,$$

thus

$$p_k = \frac{N-k+1}{k} \cdot \frac{\lambda}{\mu} \cdot p_{k-1}.$$

Then by recursion,

$$p_k = \frac{(N-k+1)(N-k+2)\cdot N}{k\cdot(k-1)\cdot 1} \cdot \left(\frac{\lambda}{\mu}\right)^k p_0 = \frac{N!}{k!(N-k)!}\left(\frac{\lambda}{\mu}\right)^k p_0 = \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k p_0.$$

Finally, it follows from

$$1 = \sum_{k=0}^{N} p_k = p_0 \sum_{k=0}^{N} \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k = p_0 \left\{\frac{\lambda}{\mu}+1\right\}^N = p_0\left(\frac{\lambda+\mu}{\mu}\right)^N$$

that

$$p_k = \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k p_0 = \binom{}{k}\left(\frac{\lambda}{\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^N = \binom{N}{k} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{N-k},$$

for $k = 0,\ 1,\ 2,\ \dots,\ N$, so we get a binomial distribution $B\left(N, \dfrac{\lambda}{\lambda+\mu}\right)$ of mean $N \cdot \dfrac{\lambda}{\lambda+\mu}$.

Example 3.6 *Given a stochastic process* $\{X(t),\, t \in [0,\infty[\}$ *by the following: At time* $t = 0$ *there are* N *cars in a carpark. No car arrives, and the cars leave the carpark mutually independently. If a car is staying at its parking bay at time* t, *then there is the probability* $\mu h + h\varepsilon(h)$ *[where* μ *is a positive constant] that it leaves the carpark in the time interval* $]t, t+h]$.
Put $X(t) = k$, $k = 0, 1, \dots, N$, *if there are* k *cars in the carpark at time* t, *and put*

$$P_k(t) = P\{X(t) = k\}.$$

1. *Prove that we have a death process with* $\mu_k = k\mu$, $k = 0, 1, \dots, N$.

2. *Find the differential equations of the system.*

3. *Find the stationary probabilities.*

4. *Prove that the mean value function*

$$m(t) = \sum_{k=1}^{N} k\, P_k(t)$$

is a solution of the differential equation

$$\frac{dx}{dt} + \mu x = 0,$$

and then find $m(t)$.

5. *Given that* $X(t)$ *is binomially distributed, find the probabilities* $P_k(t)$, $k = 0, 1, \dots, N$.

We introduce a random variable T *by putting* $T = t$, *if the last car leaves the carpark at time* t.

6. *Find the distribution function and the frequency of* T.

1) This follows e.g. from the fact that the probability that one of the k cars leaves the carpark in the time interval $]t, t+h]$ is

$$k\{\mu h + h\varepsilon(h)\} \cdot \{1 - \mu h + h\varepsilon(h)\}^{k-1} = k\mu h + h\varepsilon(h),$$

from which we conclude that $\mu_k = k\mu$.

2) The differential equations are immediately found to be

$$\begin{cases} P_k'(t) = -k\mu P_k(t) + (k+1)\mu P_{k+1}(t), & 0 \le k \le N-1, \\[2mm] P_N'(t) = -N\mu P_N(t). \end{cases}$$

3) The stationary probabilities become

$$k\, p_k = 0, \qquad k = 0, 1, \dots, N.$$

Since $\sum_{k=0}^{N} p_k = 1$, we get

$$p_k = 0 \qquad \text{for } k = 1, 2, \dots, N \qquad \text{and} \qquad p_0 = 1.$$

This result is of course obvious, because the carpark at last is empty.

4) If we multiply the k-th equation of **2.** by k, and then sum from 1 to N, we get

$$\begin{aligned}\sum_{k=1}^{N} k\,P_k'(t) &= -\mu\sum_{k=1}^{N} k^2 P_k(t)+\mu\sum_{k=1}^{N-1} k(k+1)P_{k+1}(t)\\ &= -\mu\sum_{k=1}^{N} k^2 P_k(t)+\mu\sum_{k=1}^{N}(k-1=jP_k(t) = -\mu\sum_{k=1}^{N} k\,P_k(t),\end{aligned}$$

which is also written

$$m'(t)+\mu\,m(t)=0, \qquad m(t)=\sum_{k=1}^{N} k\,P_k(t).$$

From $m(0)=N$ follows that $m(t)=N\,e^{-\mu t}$.

5) Since $X(t)$ is binomially distributed of parameter of numbers N, and since we also know the mean, we can find the probability parameter, thus

$$X(t)\in B\left(N,e^{-\mu t}\right),$$

and

$$P_k(t)=\binom{N}{k} e^{-k\mu t}\left(1-e^{-\mu t}\right)^{N-k}, \qquad k=0,\,1,\,\dots,\,N.$$

6) Now, $T\le t$, if and only if $X(t)=0$. Hence

$$F(t)=\begin{cases} P_0(t)=\left(1-e^{-\mu t}\right)^N, & \text{for } t\ge 0,\\ 0 & \text{for } t<0,\end{cases}$$

and finally by differentiation

$$f(t)=\begin{cases} N\left(1-e^{-\mu t}\right)^{N-1}\mu\,e^{-\mu t}, & \text{for } t\ge 0,\\ 0, & \text{for } t<0.\end{cases}$$

4 Queueing theory

Example 4.1 *Customers arrive to a shop by a Poisson process of intensity λ. There are 2 shop assistants and possibility of forming a queue. We assume that the service times are exponentially distributed of parameter μ.*
It is given that there are no customers in the shop in at the average 10 % of the time and that $\dfrac{1}{\lambda} = 11$.
Find $\dfrac{1}{\mu}$.
Then find the probability that both shop assistants are busy.

Here, $N = 2$ and $p_0 = \dfrac{1}{10}$ and $\dfrac{1}{\lambda} = 11$. In fact, it was given that $P_0(t) \to p_0 = 10$ % for $t \to \infty$.

The traffic intensity ϱ is for $N = 2$ given by

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{10}, \qquad \text{hvoraf } \varrho = \frac{9}{11}.$$

On the other hand, the traffic intensity is defined by

$$\varrho = \frac{\lambda}{N\mu} = \frac{\lambda}{2\mu} = \frac{1}{2 \cdot 11\,\mu} = \frac{9}{11}, \qquad \text{dvs. } \frac{1}{\mu} = 18.$$

Hence

$$p_1 = 2\varrho \cdot \frac{1-\varrho}{1+\varrho} = 2 \cdot \frac{9}{11} \cdot \frac{1}{10},$$

and therefore,

$$P\{\text{both shop assistants busy}\} = 1 - p_0 - p_1 = 1 - \frac{1}{10} - \frac{18}{110} = \frac{81}{110}.$$

Example 4.2 *Customers arrive to a shop following a Poisson process of intensity λ. We have 1 shop assistant and it is possible to form a queue. We assume that the service times are exponentially distributed of parameter μ. It is assumed that the traffic intensity is $\varrho = \frac{6}{5}$, where it is well-known that this implies that the system does not work properly (the queue increases indefinitely). Compare the advantages of the following two possibilities:*

1) Another shop assistant is hired (of the same service time distribution as the first one).

2) Improvement of the service, such that the average service time is lowered to its half.

We have a queueing system with possibility of forming a queue. The parameters are

$$N = 1, \qquad \varrho = \frac{6}{5} \quad \text{and} \quad \lambda, \quad \mu.$$

Since $\varrho = \frac{6}{5} > 1$, this system does not work properly.

1) If another shop assistant is hired, then the parameters are changed to

$$N = 2, \qquad \varrho = \frac{3}{5} \quad \text{and} \quad \lambda,\, \mu \text{ unchanged.}$$

Then

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{4}.$$

The average waiting time is

$$V_1 = \frac{\frac{1}{4} \cdot \left(\frac{3}{5}\right)^2 \cdot 2}{\mu \cdot 2 \left(\frac{2}{5}\right)^2} = \frac{9}{16} \cdot \frac{1}{\mu},$$

and the average staying time is

$$O_1 = \frac{9}{16} \cdot \frac{1}{\mu} + \frac{1}{\mu} = \frac{25}{16} \cdot \frac{1}{\mu}.$$

Remark 4.1 It should here be added that one can also find

$$\text{the average number of customers } = \frac{15}{8},$$

$$\text{the average number of busy shop assistants } = \frac{6}{5},$$

$$\text{the average length of the queue } = \frac{27}{40}. \qquad \diamond$$

2) If instead the service is improved as indicated, then the parameters become

$$N = 1, \qquad \varrho = \frac{3}{5}, \qquad \lambda \text{ unchanged}, \qquad \mu \text{ is doubled.}$$

The average waiting time is then

$$V_2 = \frac{\varrho}{2\mu(1-\varrho)} = \frac{12}{16} \cdot \frac{1}{\mu},$$

and the average staying time is

$$O_2 = \frac{12}{16} \cdot \frac{1}{\mu} + \frac{1}{2\mu} = \frac{20}{16} \cdot \frac{1}{\mu}.$$

Remark 4.2 Again we add for completeness,

$$\text{the average number of customers } = \frac{3}{5},$$

$$\text{ther average number of busy shop assistants } = \frac{3}{5},$$

$$\text{the average length of the queue } = \frac{9}{10}. \qquad \diamond$$

By comparing the two cases we get

$$V_1 < V_2, \qquad \text{and on the contrary} \qquad O_1 > O_2,$$

and the question does not have a unique answer.

The customer will prefer that the sum of waiting time and service time is as small as possible. Since

$$V_1 + O_1 = \frac{34}{16} \cdot \frac{1}{\mu} \qquad \text{and} \qquad V_2 + O_2 = \frac{32}{16} \cdot \frac{1}{\mu},$$

it follows that the customer will prefer the latter system, while it is far more uncertain what the shop would prefer, because we do not know the costs of each of the two possible improvements.

Example 4.3 *We consider an intersection which is not controlled by traffic lights. One has noticed that cars doing a left-hand turn are stopped and therefore delay the cars which are going straight on. Therefore, one plans to build a left-hand turn lane. Assuming that arrivals and departures of the cars doing the left-hand turn are exponentially distributed with the parameters λ and μ, where $\dfrac{\lambda}{\mu} = \dfrac{1}{2}$, one shall compute the smallest number of cars of the planned left-hand turn lane, if the probability is less than 5 % of the event that there are more cars than the new lane can contain.*

Here $N = 1$, so the capacity of the system is

$$\varrho = \frac{\lambda}{N\mu} = \frac{1}{2}.$$

The stationary probabilities are

$$p_k = \varrho^k(1-\varrho) = \left(\frac{1}{2}\right)^{k+1}, \qquad k \in \mathbb{N}_0.$$

Let n denote the maximum number of cars in the left turn lane. Then we get the condition

$$\sum_{k=n+1}^{\infty} p_k = \sum_{k=n+1}^{\infty} \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2^{n+1}} < 5\ \% \ = \frac{1}{20},$$

thus $\dfrac{1}{2^n} < \dfrac{1}{10}$, which is fulfilled for $n \geq 4$.

Example 4.4 *Given a queueing system of exponential distribution of arrivals and exponential distribution of service times (the means are called $\dfrac{1}{\lambda}$ and $\dfrac{1}{\mu}$, resp.). The number of service places is 2. We furthermore assume that it is possible to form a queue. Assuming that $\dfrac{1}{\lambda} = 1$ (minute) and $\dfrac{1}{\mu} = 1$ (minute),*

1. *find the average waiting time,*

2. *find the average staying time.*

For economic reasons the number of service places is cut down from 2 to 1, while the service at the same time is simplified (so the service time is decreased), such that the customer's average staying time is not prolonged. Assuming that the constant λ is unchanged,

3. *find the average service time $\dfrac{1}{\mu_1}$, such that the average staying time in the new system is equal to the average staying time in the previous mentioned system,*

4. *find in which of the two systems the probability is largest for a customer to wait.*

Here $N = 2$, $\dfrac{1}{\lambda} = 1$ and $\dfrac{1}{\mu} = 1$. This gives the traffic intensity

$$\varrho = \frac{\lambda}{N\mu} = \frac{1}{2 \cdot 1} = \frac{1}{2}, \qquad \text{and} \qquad p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{3}.$$

1) The average waiting time is

$$V = \frac{p_0 \cdot \varrho^N \cdot N^{N-1}}{\mu \cdot N!(1-\varrho)^2} = \frac{\frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 \cdot 2^1}{1 \cdot 2! \left(1-\frac{1}{2}\right)^2} = \frac{1}{3} \text{ minute.}$$

2) The staying time is the waiting time plus the serving time, so the average staying time is

$$\mathcal{O} = V + \frac{1}{\mu} = \frac{1}{3} + 1 = \frac{4}{3} \text{ minute.}$$

3) In the new system the traffic intensity is

$$\varrho_1 = \frac{\lambda}{N_1 \mu_1} = \frac{1}{\mu_1}, \qquad \text{idet } N_1 = 1.$$

The average waiting time is for N_1 given by some theoretical formula,

$$V_1 = \frac{\varrho_1}{\mu_1\left(1-\varrho_1\right)} = \frac{1}{\mu_1\left(\mu_1 - 1\right)},$$

and the average staying time is for $N_1 = 1$ given by

$$\mathcal{O}_1 = V_1 + \frac{1}{\mu_1} = \frac{1}{\mu_1 - 1}.$$

We want that $\mathcal{O}_1 = \mathcal{O} = \dfrac{4}{3}$. Hence, $\mu_1 - 1 = \dfrac{3}{4}$, i.e. $\mu_1 = \dfrac{7}{4}$, and

$$\frac{1}{\mu_1} = \frac{4}{7}.$$

4) The probability of waiting in the old system is

$$1 - p_0 - p_1 = 1 - \frac{1-\varrho}{1+\varrho} - 2\,\varrho\,\frac{1-\varrho}{1+\varrho} = 1 - \frac{1}{3} - 2 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}.$$

The probability of waiting in the new system is

$$1 - \tilde{p}_0 = 1 - (1 - \varrho_1) = \varrho_1 = \frac{1}{\mu_1} = \frac{4}{7}.$$

We see by comparison that there is largest probability of waiting in the new system.

Example 4.5 *Given a service (a shop) of which we assume:*

a. *There is only one shop assistant.*

b. *It is not possible to form a queue.*

c. *The customers arrive according to a Poisson process of intensity λ.*

d. *The service time is exponentially distributed of mean μ.*

1. *Find the differential equations of this system.*

2. *Solve these under the assumption that at time $t = 0$ there is no customer.*

Assume from now on that $\dfrac{\lambda}{\mu} = 6$.

3. *Find the stationary probabilities and the probability of rejection.*

Assuming that the probability of rejection is too large, we change the system, such that there are two shop assistants A and B, and the service is changed, such that a customer at his arrival goes to A and is served by him, if A is vacant at the arrival of the customer. If on the other hand A is busy, then the customer will turn to B in order to be serviced. If also B is busy, the customer is rejected. The assumptions of the arrivals and service times are the same as before. We want to compute in this system:

4. *The stationary probabilities and the probability of rejection.*

5. *The probability that A and B, res., are busy.*

6. *Finally, find the smallest number of shop assistants, for which the probability of rejection is smaller than $\dfrac{1}{2}$.*

1) Since $N = 1$, the differential equations of the system are

$$\begin{cases} P_0'(t) = -\lambda P_0(t) + \mu P_1(t), \\[2ex] P_1'(t) = \lambda P_0(t) - \mu P_1(t), \end{cases}$$

thus written in the form of a matrix equation,

$$\frac{d}{dt}\begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix} = \begin{pmatrix} -\lambda & \mu \\ \lambda & -\mu \end{pmatrix}\begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix}.$$

2) The characteristic polynomial (in R) is

$$\begin{vmatrix} -\lambda - R & \mu \\ \lambda & -\mu - R \end{vmatrix} = (R+\lambda)(R+\mu) - \lambda\mu = R^2 + (\lambda+\mu)R.$$

The roots are $R = 0$ and $R = -\lambda - \mu$.
For $R = 0$ we get the eigenvector (μ, λ).

For $R = -\lambda - \mu$ we get the eigenvector $(1, -1)$.
The complete solution is

$$\begin{pmatrix} P_0(t) \\ P_1(t) \end{pmatrix} = c_1 \begin{pmatrix} \mu \\ \lambda \end{pmatrix} + c_2 e^{-(\lambda+\mu)t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The initial conditions are $P_0(0) = 1$ and $P_1(0) = 0$, thus

$$\begin{cases} 1 = \mu c_1 + c_2, \\ \\ 0 = \lambda c_1 - c_2, \end{cases}$$

and hence

$$c_1 = \frac{1}{\lambda+\mu}, \qquad c_2 = \frac{\lambda}{\lambda+\mu},$$

and the solution becomes

$$\begin{cases} P_0(t) = \dfrac{\mu}{\lambda+\mu} + \dfrac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t}, \\ \\ P_1(t) = \dfrac{\lambda}{\lambda+\mu} - \dfrac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t}. \end{cases}$$

3) If $\dfrac{\lambda}{\mu} = 6$, then

$$\frac{\lambda}{\lambda+\mu} = \frac{\frac{\lambda}{\mu}}{\frac{\lambda}{\mu}+1} = \frac{6}{7} \qquad \text{and} \qquad \frac{\mu}{\lambda+\mu} = \frac{1}{7},$$

and $\lambda + \mu = 7\mu$, thus

$$\begin{cases} P_0(t) = \dfrac{1}{7} + \dfrac{6}{7}\exp(-7\mu t), \\ \\ P_1(t) = \dfrac{6}{7} - \dfrac{6}{7}\exp(-7\mu t), \end{cases} \qquad t \geq 0.$$

The stationary probabilities are obtained by letting $t \to \infty$, thus

$$p_0 = \frac{1}{7} \qquad \text{and} \qquad p_1 = \frac{6}{7}.$$

In particular, the probability of rejection is $p_1 = \dfrac{6}{7}$.

4) We have the following states:

E_0: No customer in the system.
E_1: A serves a customer, while B does not.
E_2: A is vacant, while B serves a customer.
E_3: Both A and B serve customers.

There is no change for A, so by **3.**,

$$\begin{cases} P_0(t)+P_2(t)=\dfrac{1}{7}+\dfrac{6}{7}\exp(-7\mu t), \\ \\ P_1(t)+P_3(t)=\dfrac{6}{7}-\dfrac{6}{7}\exp(-7\mu t), \end{cases} \qquad t \geq 0.$$

By taking the limit $t \to \infty$ we get

$$p_0+p_2=\frac{1}{7} \qquad \text{and} \qquad p_1+p_3=\frac{6}{7}.$$

We can realize $P_0(t+h)$ in the following ways, if the system at time t is in state

(i) E_0, and no customer arrives,

$$P_0(t)\cdot\{1-\lambda h+h\varepsilon(h)\}.$$

(ii) E_0, some customer arrive, and they are served until they are finished,

$$h\varepsilon(h).$$

(iii) E_1, and there is no customer coming, and A's customer is serviced to the end,

$$P_1(t)\cdot\{\mu h+h\varepsilon(h)\}.$$

(iv) E_1, and there arrive customers, who are served,

$$h\varepsilon(h).$$

(v) E_2, and no new customer is coming, and B's customer is served to the end,

$$P_2(t) \cdot \{\mu h + h\varepsilon(h)\}.$$

(vi) E_2 in all other cases,

$$h\varepsilon(h).$$

(vii) E_3 in general,

$$h\varepsilon(h).$$

By adding these we get

$$P_0(t+h) = P_0(t) \cdot \{1 - \lambda h + h\varepsilon(h)\} + \{P_1(t) + P_2(t)\} \cdot \{\mu h + h\varepsilon(h)\} + h\varepsilon(h).$$

Then compute the derivative in the usual way by taking the limit. This gives

$$P_0'(t) = \lim_{h \to 0} \{P_0(t+h) - P_0(t)\} = -\lambda P_0(t) + \mu \{P_1(t) + P_2(t)\}.$$

Then by taking the limit $t \to \infty$,

$$0 = -\lambda p_0 + \mu \{p_1 + p_2\} = -6\mu p_0 + \mu \{p_1 + p_2\},$$

hence

$$6p_0 = p_1 + p_2.$$

We are still missing one equation, when we want to find the stationary probabilities. We choose to realize $P_3(t+h)$. This can be done, if the system at time t is in state

(i) E_0, and at least two customers arrive,

$$h\varepsilon(h).$$

(ii) E_1, and at least one customer arrives, and neither A nor B finish their customers,

$$P_1(t) \cdot \{\lambda h + h\varepsilon(h)\} \cdot \{1 - \mu h + h\varepsilon(h)\}^2.$$

(iii) E_2, and at least one customer arrives, and neither A nor B finish their customers,

$$P_2(t) \cdot \{\lambda h + h\varepsilon(h)\} \cdot \{1 - \mu h + h\varepsilon(h)\}^2.$$

(iv) E_3, and neither A nor B finish their customers,

$$P_3(t) \cdot \{1 - \mu h + h\varepsilon(h)\}^2.$$

(v) Other, all of probability

$$h\varepsilon(h).$$

When we add these probabilities we get

$$\begin{aligned} P_3(t+h) &= \{P_1(t)+P_2(t)\}\cdot\{\lambda h+h\varepsilon(h)\}\cdot\{1-\mu h+h\varepsilon(h)\}^2 \\ &\quad +P_3(t)\cdot\{1-\mu h+h\varepsilon(h)\}^2+h\varepsilon(h). \end{aligned}$$

A rearrangement followed by a reduction gives

$$P_3(t+h)-P_3(t)=\lambda h\left\{P_1(t)+P_2(t)\right\}-2\mu h P_3(t)+h\varepsilon(h).$$

Then divide by h and let $h\to 0$. This will give us the differential equation

$$P_3'(t)=\lambda\left\{P_1(t)+P_2(t)\right\}-2\mu P_3(t),$$

hence by taking the limit $t\to\infty$,

$$0=\lambda\left(p_1+o_2\right)-2\mu p_3=6\mu\left(p_1+p_2\right)-2\mu p_3,$$

so

$$p_3=3\left(p_1+p_2\right)=18p_0.$$

Summing up we have obtained the four equations

$$\left\{\begin{array}{lcl} p_0+p_2 &=& \dfrac{1}{7}, \\[2mm] p_1+p_3 &=& \dfrac{6}{7}, \\[2mm] 6p_0 &=& p_1+p_2, \\[2mm] p_3 &=& 18p_0, \end{array}\right. \qquad \text{thus} \qquad \left\{\begin{array}{lcl} p_0+p_2 &=& \dfrac{1}{7}, \\[2mm] 18p_0+p_1 &=& \dfrac{6}{7}, \\[2mm] 6p_0-p_1-p_2 &=& 0, \\[2mm] p_3 &=& 18p_0. \end{array}\right.$$

By addition of the former three equations, we get $25p_0=1$, thus $p_0=\dfrac{1}{25}$. Then

$$p_1=\frac{6}{7}-\frac{18}{25}=\frac{6}{175}\,(25-21)=\frac{24}{175},$$

and

$$p_2=\frac{1}{7}-\frac{1}{25}=\frac{18}{175}, \qquad \text{and} \qquad p_3=\frac{18}{25},$$

so

$$\left(p_0,p_1,p_2,p_3\right)=\left(\frac{1}{25},\frac{24}{175},\frac{18}{175},\frac{18}{25}\right),$$

and the probability of rejection is

$$p_3=\frac{18}{25}.$$

5) The probability that A is busy is

$$p_1 + p_3 = \frac{6}{7}.$$

The probability that B is busy is

$$p_2 + p_3 = \frac{18}{175} + \frac{18}{25} = \frac{144}{175} \qquad \left(< \frac{6}{7} \right).$$

6) We have in the general case of N shop assistants, where E_j denotes that j customers are served, the system of differential equations

$$\begin{cases} P_0'(t) = -\lambda P_0(t) + \mu P_1(t), \\ \\ P_k'(t) = -(\lambda + k\mu)P_k(t) + \lambda P_{k-1}(t) + (k+1)\mu P_{k+1}(t), \qquad 1 \le k \le N-1, \\ \\ P_N'(t) = -N\mu P_N(t) + \lambda P_{N-1}(t). \end{cases}$$

Hence by taking the limit $t \to \infty$,

$$\begin{cases} 0 = -\lambda p_0 + \mu p_1, \\ \\ 0 = -(\lambda + k\mu)p_k + \lambda p_{k-1} + (k+1)\mu p_{k+1}, \qquad 1 \le k \le N-1, \\ \\ 0 = -N\mu p_N + \lambda p_{N-1}. \end{cases}$$

Since $\dfrac{\lambda}{\mu} = 6$, we get by a division by μ, followed by a rearrangement that

$$\begin{cases} 0 = 6p_0 - p_1, \\ \\ 6p_k - (k+1)p_{k+1} = 6p_{k-1} - kp_k, \qquad 1 \le k \le N-1, \\ \\ 0 = 6p_{N-1} - Np_N. \end{cases}$$

Then by recursion, $6p_{k-1} - k\,p_k = 0$, thus

$$kp_k = 6p_{k-1}, \qquad 1 \le k \le N.$$

The easiest way to solve this recursion formula is to multiply by

$$\frac{(k-1)!}{6^k} \neq 0,$$

and then do the recursion,

$$\frac{k!}{6^k}\,p_p = \frac{(k-1)!}{6^{k-1}}\,p_{k-1} = \cdots = \frac{0!}{6^0}\,p_0 = p_0, \qquad k = 0,\,1,\,\ldots,\,N,$$

thus

$$p_k = \frac{6^k}{k!}\,p_0, \qquad k = 0,\,1,\,\ldots,\,N.$$

Since $\mathbf{p}$ is a probability vector, we get the condition

$$1 = \sum_{k=0}^{N} p_k = p_0 \sum_{k=0}^{N} \frac{6^k}{k!}, \qquad \text{thus} \qquad p_0 = \frac{1}{\sum_{k=0}^{N} \frac{6^k}{k!}}.$$

The task is to find N, such that the probability of rejection $p_N \leq \frac{1}{2}$. Using

$$p_N = \frac{\frac{6^N}{N!}}{\sum_{k=0}^{N-1} \frac{6^k}{k!} + \frac{6^N}{N!}} \leq \frac{1}{2}, \qquad \text{if} \qquad \frac{6^N}{N!} \leq \sum_{k=0}^{N-1} \frac{6^k}{k!},$$

we compute the following table,

k	0	1	2	3	4
$\frac{6^k}{k!}$	1	6	18	36	54
$\sum_{j=0}^{k-1} \frac{6^j}{j!}$	$\star$	1	7	25	61

It follows that $N \geq 4$ gives $p_N \leq \frac{1}{2}$, so we shall at least apply 4 service places.

Example 4.6 *At a university there are two super computers A and B. Computer A is used for university tasks, while computer B is restricted to external tasks. Both systems allow forming queues, and the service times (i.e. the times used for computation of each task) is approximately exponentially distributed of mean* $\frac{1}{\mu} = 3$ *minutes. The university tasks arrive to computer A approximately as a Poisson process of intensity* $\lambda_A = \frac{1}{5}$ min^{-1}*, while the tasks of computer B arrive as a Poisson process of intensity* $\lambda_B = \frac{3}{10}$ min^{-1}*. Apply the stationary probabilities for the two computers A and B to compute*

1. *The fraction of time, A (resp. B) is vacant.*

2. *The average waiting time at A (resp. B).*

It is suggested to join the two systems to one, such that each computer can be used to university tasks as well external tasks. This means that we have a queueing system with two "shop assistants". Use again the stationary probabilities of this system to compute

3. *The fraction of time both computers are vacant.*

4. *The fraction of time both computers are busy.*

5. *The average waiting time.*

1) In both cases, $N = 1$.
For A we have the capacity

$$\varrho_A = \frac{\lambda_A}{N\mu_A} = \frac{3}{5}, \qquad \text{thus} \qquad p_{0,A} = 1 - \varrho_A = \frac{2}{5}.$$

For B we have the capacity

$$\varrho_B = \frac{\lambda_B}{N\mu_B} = \frac{9}{10}, \qquad \text{thus} \qquad p_{0,B} = 1 - \varrho_B = \frac{1}{10}.$$

These probabilities indicate the fraction of time, in which the given computer is vacant.

2) Since $N = 1$, the respective average waiting times are

$$V_A = \frac{\varrho_A}{\mu\,(1-\varrho_A)} = 3 \cdot \frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{9}{2} \text{ minutes,}$$

and

$$V_B = \frac{\varrho_B}{\mu\,(1-\varrho_B)} = 3 \cdot \frac{\frac{9}{10}}{1-\frac{9}{10}} = 27 \text{ minutes.}$$

3) The sum of two Poisson processes is again a Poisson process, here with the parameter

$$\lambda = \lambda_A + \lambda_B = \frac{1}{5} + \frac{3}{10} = \frac{1}{2}.$$

Hence the capacity

$$\varrho = \frac{\lambda}{N\mu} = \frac{1}{2} \cdot \frac{1}{2} \cdot 3 = \frac{3}{4}.$$

The fraction of time, in which none of the computers is busy, is

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1-\frac{3}{4}}{1+\frac{3}{4}} = \frac{1}{7}.$$

4) The probability that both computers are busy is

$$1 - p_0 - p_1 = 1 - \frac{1}{7} - 2\varrho\,\frac{1-\varrho}{1+\varrho} = 1 - \frac{1}{7} - 2 \cdot \frac{3}{4} \cdot \frac{1}{7} = \frac{14-2-3}{14} = \frac{9}{14}.$$

5) The average waiting time is

$$V = \frac{p_0 \varrho^N N^{N-1}}{\mu \cdot N!(1-\varrho)^2} = \frac{1}{7}\left(\frac{3}{4}\right)^2 \cdot 2^1 \cdot 3 \cdot \frac{1}{2!} \cdot \frac{1}{\left(1-\frac{3}{4}\right)^2} = \frac{1}{7} \cdot \frac{9}{16} \cdot 2 \cdot \frac{3}{2} \cdot 16 = \frac{27}{7} \text{ minutes.}$$

Example 4.7 *Given a birth and death process of the states E_0, E_1, E_2, ..., where the birth intensity λ_k in state E_k decreases in increasing k as follows,*

$$\lambda_k = \frac{\alpha}{k+1},$$

where α is a positive constant, while the death intensities μ_k are given by

$$\mu_k = \begin{cases} \mu, & k \in \mathbb{N}, \\ 0, & k = 0, \end{cases} \qquad \text{where } \mu > 0.$$

1. *Find the stationary probabilities.*

The above may be viewed as a model of a queueing process, where

a. *it is possible to form a queue,*

b. *there is only 1 channel,*

c. *the service time is exponentially distributed of mean $\frac{1}{\mu}$,*

d. *the arrival frequency decreases with increasing queue length according to the given formula. (Some customers will avoid a long queue and immediately leave the queue).*

2. *Compute for $\alpha = \mu$ the probability that there are at most 3 customers in the system (3 dec.).*

3. *Compare the probability of* **2.** *with the corresponding probability in the case of one shop assistant and $\lambda_k = \alpha$ constant and $\mu = 3\alpha$ (3 dec.).*

1) The system of differential equations for $\lambda_k = \frac{\alpha}{k+1}$ and $\mu > 0$ is given by

$$\begin{cases} P_0'(t) = -\alpha P_0(t) + \mu P_1(t), \\ P_k'(t) = -\left(\frac{\alpha}{k+1} + \mu\right) P_k(t) + \frac{\alpha}{k} P_{k-1}(t) + \mu P_{k+1}(t), \qquad k \in \mathbb{N}. \end{cases}$$

By taking the limit $t \to \infty$ we get

$$\begin{cases} 0 = -\alpha p_0 + \mu p_1, \\ 0 = -\left(\frac{\alpha}{k+1} + \mu\right) p_k + \frac{\alpha}{k} p_{k-1} + \mu p_{k+1}, \qquad k \in \mathbb{N}, \end{cases}$$

thus

$$-\frac{\alpha}{k+1} p_k + \mu p_{k+1} = -\frac{\alpha}{k} p_{k-1} + \mu p_k = \cdots = 0, \qquad k \in \mathbb{N},$$

and hence

$$\mu p_k = \frac{\alpha}{k} p_{k-1}, \qquad k \in \mathbb{N}.$$

When this equation is multiplied by

$$k!\,\frac{\mu^{k-1}}{\alpha^k} \neq 0,$$

it follows by a recursion that

$$k!\left(\frac{\mu}{\alpha}\right)^k p_k = (k-1)!\left(\frac{\mu}{\alpha}\right)^{k-1} p_{k-1} = \cdots = 0!\left(\frac{\mu}{\alpha}\right)^0 p_0 = p_0,$$

hence

$$p_k = \frac{1}{k!}\left(\frac{\alpha}{\mu}\right)^k p_0, \qquad k \in \mathbb{N}_0.$$

It follows from

$$1 = \sum_{k=0}^{\infty} p_k = p_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\alpha}{\mu}\right)^k = p_0 \exp\left(\frac{\alpha}{\mu}\right),$$

that

$$p_0 = \exp\left(-\frac{\alpha}{\mu}\right),$$

thus

$$p_k = \frac{1}{k!}\left(\frac{\alpha}{\mu}\right)^k \exp\left(-\frac{\alpha}{\mu}\right), \qquad k \in \mathbb{N}_0.$$

2) Put $\alpha = \mu$. The probability that there are at most 3 customers in the system is

$$p_0 + p_1 + p_2 + p_3 = \frac{1}{e}\left\{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right\} = \frac{16}{6e} \approx 0.9810.$$

3) The differential equations of the new system are

$$\begin{cases} P_0'(t) = -\alpha P_0(t) + 3\alpha P_1(t), \\ \\ P_k'(t) = -4\alpha P_k(t) + \alpha P_{k-1}(t) + 3\alpha P_{k+1}(t), \qquad k \in \mathbb{N}. \end{cases}$$

By taking the limit $t \to \infty$ we get the equations of the stationary probabilities,

$$\begin{cases} 0 = -\alpha p_0 + 3\alpha p_1, \\ \\ 0 = -4\alpha p_k + \alpha p_{k-1} + 3\alpha p_{k+1}, \qquad k \in \mathbb{N}. \end{cases}$$

We rewrite these and get by a reduction,

$$3p_{k+1} - p_k = 3p_k - p_{k-1} = \cdots = 3p_1 - p_0 = 0, \qquad k \in \mathbb{N},$$

thus $3p_k = p_{k-1}$. Multiply this equation by 3^{k-1} in order to get

$$3^k p_k = 3^{k-1} p_{k-1} = \cdots = 3^0 p_0 = 0_0,$$

hence

$$p_k = \frac{1}{3^k}\, p_0, \qquad k \in \mathbb{N}_0.$$

It follows from

$$1 = \sum_{k=0}^{\infty} p_k = p_0 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = p_0 \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{3}{2} \cdot p_0,$$

that $p_0 = \dfrac{2}{3}$, and the probability that there are at most three customers in this system is

$$p_0 + p_1 + p_2 + p_3 = p_0 \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3}\right) = \frac{2}{3} \cdot \frac{27+9+3+1}{27} = \frac{80}{81} \approx 0.9877.$$

There is a slightly higher probability in this case that there are at most three customers in this system than in the system which was considered in **2.**.

Example 4.8 *Given the following queueing model: M machines are working mutually independently of each other and they need no operation by men, except in the case when they break down. There are in total N service mechanics (where $N < M$) for making repairs. If a machine is working at time t, it is the probability $\lambda h + h\varepsilon(h)$ that it breaks down before time $t+h$, and probability $1 - \lambda h + h\varepsilon(h)$ that it is still working. Analogously, if it is repaired at time t, then there is the probability $\mu h + h\varepsilon(h)$ that it is working again before $t+h$, and probability $1 - \mu h + h\varepsilon(h)$ that it is not working. When a machine breaks down, it is immediately repaired by a service mechanic, if he is vacant. Otherwise, the machine is waiting in a queue, until a service mechanic becomes vacant. We define the* coefficient of loss *of a machine as*

$$\frac{1}{M} \cdot \text{average number of machines in the queue,}$$

and the coefficient of loss *of a service mechanic as*

$$\frac{1}{N} \cdot \text{average number of vacant service mechanics.}$$

Denote by E_k the state that k machines do not *work, $k = 0, 1, \dots, M$.*

1) Prove that the constants λ_k and μ_k are given by

$$\lambda_k = (M-k)\lambda, \qquad \mu_k = k\mu, \qquad 0 \le k \le N,$$

$$\lambda_k = (M-k)\lambda, \qquad \mu_k = N\mu, \qquad N \le k \le M.$$

2) Find a recursion formula for p_k (express p_{k+1} by p_k).

3) Find the average number of machines in the queue (expressed by the p_k-erne), and prove in particular that if $N = 1$ this can be written

$$M - \frac{\lambda+\mu}{\lambda}\left(1-p_0\right).$$

4) Find the probability that there are precisely 0, 1, 2, …, N vacant service mechanics.

5) Find the coefficients of loss of a machine and a service mechanics in the case of

$$\frac{\lambda}{\mu} = 0,1; \qquad M = 6; \qquad N = 1.$$

It should be mentioned for comparison that in the case when

$$\frac{\lambda}{\mu} = 0,1; \qquad M = 20; \qquad N = 3,$$

the coefficient of loss of a machine is 0.0169 and the coefficient of loss of a service mechanics is 0.4042. Which one of the two systems is best?

This problem of machines was first applied in the Swedish industry.

1) Let $0 \le k \le M$, and assume that we are in state E_k, thus k machines are being repaired or are waiting for reparation, and $M-k$ machines are working. The latter machines have each the probability

$$\lambda h + h\varepsilon(h)$$

of breaking down in the time interval $]t, t+h]$ of length h. Since $M-k$ machines are working, we get

$$\lambda_k = (M-k)\lambda \qquad \text{for } 0 \le k \le M.$$

If we are in state E_k, where $0 \le k \le N$, then all k machines are being repaired. Each of these have the probability

$$\mu h + h\varepsilon(h)$$

for being repaired before time $t+h$, thus

$$\mu_k = k\mu, \qquad \text{for } 0 \le k \le N.$$

If instead $N < k \le M$, then all service mechanics are working, so

$$\mu_k = N\mu, \qquad \text{for } N < k \le M.$$

2) By a known formula,

$$\mu_{k+1}p_{k+1} = \lambda_k p_k,$$

thus

$$p_{k+1} = \frac{\lambda_k}{\mu_{k+1}}\, p_n, \qquad \text{for } n = 0,\, 1,\, \dots,\, M-1.$$

When we insert the results of **1.**, we get

$$\begin{cases} p_{k+1} = \dfrac{(M-k)\lambda}{(k+1)\mu}\, p_k & \text{for } k = 0,\, 1,\, \dots,\, N-1, \\[2ex] p_{k+1} = \dfrac{(M-k)\lambda}{N\mu}\, p_k & \text{for } k = N,\, \dots,\, M-1. \end{cases}$$

When the first equation is multiplied by

$$\frac{1}{\binom{M}{k+1}} \left(\frac{\mu}{\lambda}\right)^{k+1},$$

we get

$$\begin{aligned} \frac{p_{k+1}}{\binom{M}{k+1}\left(\dfrac{\lambda}{\mu}\right)^{k+1}} &= \frac{(M-k)\lambda}{(k+1)\mu} \cdot \frac{\binom{M}{k}}{\binom{M}{k+1}} \cdot \frac{1}{\dfrac{\lambda}{\mu}} \cdot \frac{p_k}{\binom{M}{k}} \\ &= \frac{p_k}{\binom{M}{k}\left(\dfrac{\lambda}{\mu}\right)^{k}} = \cdots = \frac{p_0}{\binom{M}{0}\left(\dfrac{\lambda}{\mu}\right)^{0}} = p_0, \end{aligned}$$

hence

$$p_k = \binom{M}{k} \left(\frac{\lambda}{\mu}\right)^k p_0 \qquad \text{for } k = 0,\, 1,\, \dots,\, N.$$

We put $n = N + m$, $m = 0,\, 1,\, \dots,\, M - N - 1$, into the second equation. Then

$$\begin{aligned} p_{N+m+1} &= \frac{M-N-m}{M\mu} \cdot \frac{\lambda}{\mu}\, p_{N+m} = \frac{1}{N^{m+1}} \left(\frac{\lambda}{\mu}\right)^{m+1} \cdot (M-N-m) \cdots (M-N) p_N \\ &= \frac{1}{N^{m+1}} \left(\frac{\lambda}{\mu}\right)^{m+1} \cdot \frac{(M-N)!}{(M-N-m-1)!}\, p_N, \end{aligned}$$

hence

$$p_{N+m} = \frac{1}{N^m} \left(\frac{\lambda}{\mu}\right)^m \cdot \frac{(M-N)!}{(M-N-m)!}\, p_N = \frac{M!}{N!(M-N-m)!} \cdot \frac{1}{N^m} \left(\frac{\lambda}{\mu}\right)^{N+m} p_0,$$

for $m = 0,\, 1,\, \dots,\, M - N$.

3) The average number of machines in the queue is

$$\sum_{k=N+1}^{M} (k-N) p_k = \sum_{k=N}^{M} (k-N) p_k.$$

We get in particular for $N = 1$,

$$\sum_{k=1}^{M} (k-1) p_k = \sum_{k=1}^{M} k p_k - \sum_{k=1}^{M} p_k = \sum_{k=1}^{M} k p_k - (1 - p_0)\,.$$

Then by the recursion formula of **2.**,

$$p_{k+1} = (M-k)\, \frac{\lambda}{\mu}\, p_k = M\, \frac{\lambda}{\mu}\, p_k - \frac{\lambda}{\mu}\, p_k, \qquad k = 1,\, \dots,\, M-1.$$

Hence

$$\begin{aligned} \sum_{k=1}^{M} k p_k &= \sum_{k=1}^{M-1} k p_k + M p_M = M \sum_{k=1}^{M-1} p_k + M p_M - \frac{\mu}{\lambda} \sum_{k=1}^{M-1} p_{k+1} = M \sum_{k=0}^{M} p_k - M p_0 - \frac{\mu}{\lambda} \sum_{k=2}^{M} p_k \\ &= M\,(1-p_0) - \frac{\mu}{\lambda}\,(1 - p_0 - p_1) = M - \frac{\mu}{\lambda}\,(1-p_0) - M p_0 + \frac{\mu}{\lambda}\, p_1. \end{aligned}$$

It follows from

$$p_1 = \frac{M-0}{0+1} \cdot \frac{\lambda}{\mu}\, p_0 = M \cdot \frac{\lambda}{\mu}\, p_0,$$

by insertion that the average number of machines in the queue is for $N = 1$ given by

$$\begin{aligned} \sum_{k=1}^{M} (k-1) p_k &= \sum_{k=1}^{M} k p_k - (1-p_0) = M - \frac{\mu}{\lambda}\, m\,(1-p_0) - M p_0 + \frac{\mu}{\lambda} \cdot M \cdot \frac{\lambda}{\mu}\, p_0 - (1-p_0) \\ &= M - \left(\frac{\mu}{\lambda} + 1\right)(1-p_0) = M - \frac{\lambda+\mu}{\lambda}\,(1-p_0)\,. \end{aligned}$$

4) If there are $n \in \{1, 2, \dots, N\}$ vacant service mechanics, the system is in state E_{N-n}, so the probability is

$$p_{N-n} = \binom{M}{N-n} \left(\frac{\lambda}{\mu}\right)^{N-n} p_0, \qquad n = 1, 2, \dots, N.$$

If there is no vacant service mechanic, we get the probability

$$1 - \sum_{n=1}^{N} \binom{M}{N-n} \left(\frac{\lambda}{\mu}\right)^{N-n} p_0 = 1 - p_0 \sum_{n=0}^{N-1} \binom{M}{n} \left(\frac{\lambda}{\mu}\right)^{n}.$$

5) If $\dfrac{\lambda}{\mu} = \dfrac{1}{10}$, $M = 6$ and $N = 1$, then the coefficient of loss of the machine is by **3.** given by

$$\frac{1}{M} \cdot \left\{ M - \left(1 + \frac{\mu}{\lambda}\right)(1 - p_0) \right\} = 1 - \frac{1}{6}(1 + 10) \cdot (1 - p_0) = 1 - \frac{11}{6}(1 - p_0).$$

We shall only find p_0. We get by using the recursion formulae

$$p_1 = \frac{6}{10} p_0, \qquad p_2 = \frac{5}{10} p_1, \qquad p_3 = \frac{4}{10} p_2,$$

$$p_4 = \frac{3}{10} p_3, \qquad p_5 = \frac{2}{10} p_4, \qquad p_6 = \frac{1}{10} p_5,$$

hence

$$\begin{aligned} 1 &= \sum_{k=0}^{6} p_k = p_0 \left\{ 1 + \frac{6}{10}\left(1 + \frac{5}{10}\left(1 + \frac{4}{10}\left(1 + \frac{3}{10}\left(1 + \frac{2}{10}\left(1 + \frac{1}{10}\right)\right)\right)\right)\right) \right\} \\ &\approx p_0 \cdot 2.0639, \end{aligned}$$

so

$$p_0 \approx 0.4845.$$

We also get by insertion the coefficient of loss of the machine,

$$1 - \frac{11}{6}(1 - p_0) \approx 0.05049.$$

The loss coefficient of the service mechanic is

$$\frac{1}{N} \cdot p_0 = p_0 \approx 0.4845.$$

By comparison we see that the coefficients of loss are smallest in the system, where

$$\frac{\lambda}{\mu} = \frac{1}{10}, \qquad M = 20, \qquad N = 3,$$

so this system is the best.

Example 4.9 *In a shop the service time is exponentially distributed of mean* $\frac{1}{\mu}$*, thus the frequency is given by*

$$f(x)=\begin{cases}\mu\,e^{-\mu x}, & x>0,\\[2mm] 0, & x\le 0.\end{cases}$$

Let X_1*,* X_2*,* $\dots$ *denote the service times of customer number 1, 2,* $\dots$*. We assume that the* X_i *are mutually independent and that they all have the frequency* $f(x)$ *above.*
In total there arrive to the shop N *customers, where* N *is a random variable, which is independent of all the* X_i*, and* N *can have the values 1, 2,* $\dots$*, of the probabilities*

$$P\{N=k\}=p\,q^{k-1},\qquad k\in\mathbb{N},$$

where $p>0$*,* $q>0$*, and* $p+q=1$*.*

1) Prove that $Y_n=\sum_{i=1}^{n}X_i$ *has the frequency*

$$f_n(x)=\begin{cases}\mu\,\dfrac{(\mu x)^{n-1}}{(n-1)!}\,e^{-\mu x}, & x>0,\\[2mm] 0, & x\le 0.\end{cases}$$

2) Find the frequency and the distribution function of $Y=\sum_{i=1}^{N}X_i$ *by using that*

$$P\{Y\le x\}=\sum_{k=1}^{\infty}P\left\{N=k\,\wedge\,Y_k\le x\right\}.$$

3) Find mean and variance of Y*.*

1) Since $X_i\in\Gamma\left(1,\frac{1}{\mu}\right)$, it follows that

$$Y_n=\sum_{k=1}^{n}X_k\in\Gamma\left(n,\frac{1}{\mu}\right),$$

and the frequency is

$$f_n(x)=\begin{cases}\mu\,\dfrac{(\mu x)^{n-1}}{(n-1)!}\,e^{-\mu x}, & x>0,\\[2mm] 0, & x\le 0.\end{cases}$$

2) It follows immediately (without using generating functions),

$$P\{Y\le x\}=\sum_{k=1}^{\infty}P\left\{N=k,\,Y_k\le x\right\}=\sum_{k=1}^{\infty}P\{N=k\}\cdot P\left\{Y_k\le x\right\}=\sum_{n=1}^{\infty}pq^{n-1}\int_0^x f_n(t)\,dt.$$

Thus we get for $x > 0$ the frequency

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} p\, q^{n-1} f_n(x) = p \sum_{n=1}^{\infty} q^{n-1} \cdot \frac{\mu}{(n-1)!}\,(\mu x)^{n-1} e^{-\mu x} = p\mu \sum_{n=0}^{\infty} \frac{(q\mu x)^n}{n!}\, e^{-\mu x} \\ &= p\mu\, e^{+q\mu x} \cdot e^{-\mu x} = p\mu \cdot e^{-p\mu x}, \end{aligned}$$

so $Y \in \Gamma\left(1, \dfrac{1}{p\mu}\right)$ is exponentially distributed of frequency

$$g(x) = \begin{cases} p\mu\, e^{-p\mu x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}$$

and distribution function

$$G(x) = \begin{cases} 1 - e^{-p\mu x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

3) Since $Y \in \Gamma\left(1, \dfrac{1}{p\mu}\right)$, we have

$$E\{X\} = \frac{1}{p\mu} \qquad \text{og} \qquad V\{X\} = \frac{1}{p^2\mu^2}.$$

Example 4.10 *An old-fashioned shop with one shop assistant to serve the customers can be considered as a queuing system of one channel with the possibility of forming a queue. The customers arrive according to a Poisson process of intensity λ, and the service time is exponentially distributed of parameter μ. It has been noticed that when the system is in its equilibrium, then the shop assistant is in mean busy $\frac{3}{4}$ of the time, and the average staying time of customers is 10 minutes.*

1. *Prove that $\frac{1}{\lambda} = \frac{1}{18}$ hour and $\frac{1}{\mu} = \frac{1}{24}$ hour.*

2. *Find the probability that a customer is served immediately.*

3. *Find the average queue length.*

The shop is closed at 17^{30} and only the customers who are already in the shop are served by the shop assistant, before he leaves for his home.

4. *Find the probability that there at 17^{30} are 0, 1, 2, … customers in the shop.*

5. *Led the random variable T denote the time from 17^{30} until the shop assistant has served all customers. Find the distribution of T.*

It follows from $\lambda_k = \lambda$ and $\mu_k = \mu$ that

$$\mu p_{k+1} = \lambda p_k, \qquad n \in \mathbb{N}_0.$$

The traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{\lambda}{\mu},$$

which we assume satisfies $\varrho < 1$, so $p_0 = 1 - \varrho$. Thus

$$p_k = \frac{\lambda}{\mu}\, p_{k-1} = \cdots = \left(\frac{\lambda}{\mu}\right)^k p_0 = \varrho^k \cdot (1-\varrho).$$

1) The staying time is

$$\mathcal{O} = \frac{1}{\mu - \lambda} = 10 \text{ minutes } = \frac{1}{6} \text{ hour},$$

and the shop assistant is busy

$$\frac{3}{4} = 1 - p_0 = \varrho = \frac{\lambda}{\mu}.$$

Hence $\lambda = \frac{3}{4}\,\mu$ and $6 = \mu - \lambda = \frac{1}{4}\,\mu$, thus $\mu = 24$ and $\lambda = \frac{3}{4} \cdot 24 = 18$, corresponding to

$$\frac{1}{\lambda} = \frac{1}{18} \text{ hour} \qquad \text{and} \qquad \frac{1}{\mu} = \frac{1}{24} \text{ hour}.$$

2) A customer is immediately served if the system is in state E_0. The probability of this event is

$$p_0 = 1 - \varrho = 1 - \frac{3}{4} = \frac{1}{4}.$$

3) The average queue length is

$$\frac{\varrho^2}{1-\varrho} = \frac{\frac{9}{16}}{1-\frac{3}{4}} = \frac{9}{4}.$$

4) The probability that there are n customers in the shop at 17^{30} $(t \approx \infty)$ is

$$p_n = \varrho^n(1-\varrho) = \frac{1}{4}\cdot\left(\frac{3}{4}\right)^n.$$

5) Assume that there are k customers in the shop. Then the service time is Erlang distributed, $\Gamma\left(k, \frac{1}{\mu}\right)$, of frequency

$$\mu \cdot \frac{(\mu x)^{k-1}}{(k-1)!}\, e^{-\mu x}, \qquad x > 0, \quad k \in \mathbb{N}.$$

It follows that the distribution of T is given by

$$P\{T = 0\} = \frac{1}{4}$$

and

$$\begin{aligned} F_T'(x) &= \sum_{k=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^k \mu \cdot \frac{(\mu x)^{k-1}}{(k-1)!}\, e^{-\mu x} = \frac{1}{4}\cdot\frac{3}{4}\,\mu\cdot e^{-\mu x}\sum_{k=1}^{\infty}\left\{\frac{3}{4}\,\mu x\right\}^{k-1}\cdot\frac{1}{(k-1)!} \\ &= \frac{3}{16}\,\mu\, e^{-\mu x}\exp\left(\frac{3}{4}\,\mu x\right) = \frac{3}{16}\,\mu\cdot\exp\left(-\frac{1}{4}\,\mu x\right). \end{aligned}$$

Then by an integration,

$$P\{T \le x\} = \begin{cases} 1 - \dfrac{3}{4}\,\exp\left(-\dfrac{\mu}{4}\,x\right), & x \ge 0, \\[2mm] 0, & x < 0. \end{cases}$$

When we insert $\mu = 24$, found above, we get

$$P\{T \le x\} = \begin{cases} 1 - \dfrac{3}{4}\,e^{-6x}, & x \ge 0, \\[2mm] 0, & x < 0. \end{cases}$$

Alternatively, T has the Laplace transform

$$L_T(\lambda) = P(L(\lambda)),$$

where

$$L(\lambda) = \frac{\mu}{\lambda+\mu}$$

and

$$P(s)=\sum_{k=0}^{\infty}p_k s^k=\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\,s\right)^k=\frac{1}{4}\cdot\frac{1}{1-\dfrac{3}{4}\,s}=\frac{1}{4-3s}.$$

Hence by insertion,

$$L_T(\lambda)=\frac{1}{4-\dfrac{3\mu}{\lambda+\mu}}=\frac{\lambda+\mu}{4\lambda+\mu}=\frac{1}{4}\cdot 1+\frac{3}{4}\cdot\frac{\dfrac{1}{4}\,\mu}{\lambda+\dfrac{1}{4}\,\mu}.$$

We recognize this Laplace transform as corresponding to

$$F_T(x)=\begin{cases}1-\dfrac{3}{4}\,\exp\left(-\dfrac{\mu}{4}\,x\right), & x\geq 0,\\[2mm] 0, & x<0.\end{cases}$$

Example 4.11 *Given a service, where we assume:*

a. *There are two channels.*

b. *The customers arrive by a Poisson process of intensity 1 min^{-1}.*

c. *The service time is at each of the two channels exponentially distributed of mean 1 minute.*

d. *It is possible to form a queue.*

1. *Compute the average waiting time.*

2. *Find the fraction of time, in which both channels are vacant, and the fraction of time, in which both channels are busy.*

The flow of customers is then increased such that the customers now arrive according to a Poisson process of intensity $\lambda = 2$ min^{-1} (the other assumptions are unchanged).

3. *What is the impact of this change on the service?*

The service is then augmented by another channel of the same type as the old ones.

4. *Compute in this system for $\lambda = 2$ the average waiting time.*

1) The process is described by a birth and death process with

$$\lambda_k 1 \quad \text{and} \quad \mu_1 = 1, \quad \mu_k = 2 \text{ for } k \geq N = 2, \text{ thus } \mu = 1.$$

The traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{1}{2}.$$

We have

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{3} \qquad \text{og} \qquad p_k = 2\varrho^k \cdot \frac{1-\varrho}{1+\varrho} = \frac{1}{3}\left(\frac{1}{2}\right)^{k-1} \qquad \text{for } k \in \mathbb{N}.$$

The waiting time is given by

$$V = \frac{p_0 \cdot \varrho^N \cdot N^{N-1}}{\mu \cdot N!(1-\varrho)^2} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{1}{3}.$$

2) Both channels are vacant in the fraction of time

$$p_0 = \frac{1}{3}.$$

Both channels are busy in the fraction of time

$$\sum_{k=2}^{\infty} p_k = 1 - p_0 - p_1 = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}.$$

3) The only change in the new system is $\lambda_k = 2$, thus

$$\lambda_k = 2 \quad \text{and} \quad \mu_1 = 1, \quad \mu_k = 2 \text{ for } k \geq 2, \quad \text{and } \mu = 1.$$

The traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{2}{2 \cdot 1} = 1.$$

The queue will increase indefinitely.

4) Then we shift to $N = 3$ with $\lambda = 2$ and $\mu = 1$, so

$$\lambda_k = 2, \quad \mu_1 = 1, \quad \mu_2 = 2 \quad \text{and} \quad \mu_k = 3 \quad \text{for } k \geq 3.$$

The traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{2}{3 \cdot 1} = \frac{2}{3}.$$

It follows from

$$p_k = \begin{cases} \varrho^k \cdot \dfrac{1}{k!} N^k p_0, & k < N, \\[2ex] \varrho^k \cdot \dfrac{N^N}{N!}\, p_0, & k \geq N, \end{cases}$$

that

$$p_1 = \frac{2}{3} \cdot \frac{1}{1!} \cdot 3p_0 = 2p_0 \qquad \text{and} \qquad p_2 = \left(\frac{2}{3}\right)^2 \cdot \frac{3^2}{2!}\, p_0 = 2p_0,$$

and

$$p_k = \left(\frac{2}{3}\right)^k \cdot \frac{3^3}{3!}\, p_0 = 2\left(\frac{2}{3}\right)^{k-1} p_0 \qquad \text{for } k \geq 3.$$

The sum is

$$1 = \sum_{k=0}^{\infty} p_k = p_0 \left\{ 1 + 2 + 2 + 2\sum_{k=3}^{\infty} \left(\frac{2}{3}\right)^{k-2} \right\} = p_0 \left\{ 3 + 2\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^{k-2} \right\} = 9p_0,$$

from which $p_0 = \dfrac{1}{9}$. The waiting time is obtained by insertion,

$$V = \frac{p_0 \varrho^N \cdot N^{N-1}}{\mu \cdot N!(1-\varrho)^2} = \frac{1}{9} \cdot \frac{\left(\dfrac{2}{3}\right)^3 \cdot 3^2}{1 \cdot 3! \left(1 - \dfrac{2}{3}\right)^2} = \frac{\left(\dfrac{2}{3}\right)^3}{3 \cdot 2 \left(\dfrac{1}{3}\right)^2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Example 4.12 *Given a service for which*

a. *There are three channels.*

b. *The customers arrive according to a Poisson process of intensity* $1\ min^{-1}$.

c. *The service time for each channel is exponentially distributed of mean 1 minute.*

d. *It is possible to form a queue.*

1. *Prove that the stationary probabilities are given by*

$$p_k = \begin{cases} \dfrac{4}{11} \cdot \dfrac{1}{k!}, & k < 3, \\[2mm] \dfrac{2}{33} \cdot \left(\dfrac{1}{3}\right)^{k-3}, & k \geq 3. \end{cases}$$

2. *Find the fraction of time, in which all three channels are busy.*

3. *Compute the average length of the queue.*

Decrease the number of channels to two while the other assumptions are unchanged. Compute in this system,

4. *the stationary probabilities,*

5. *the fraction of time, in which both channels are busy,*

6. *the average length of the queue.*

Finally, decrease the number of channels to one, while the other assumptions are unchanged.

7. *How will this system function?*

1) The traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{1}{3 \cdot 1} = \frac{1}{3}.$$

It follows from

$$p_k = \begin{cases} \varrho^k \cdot \dfrac{1}{k!}\, N^k p_0, & k < N, \\[2mm] \dfrac{1}{N!}\, \varrho^k N^N p_0, & k \geq N, \end{cases}$$

that

$$p_k = \left(\frac{1}{3}\right)^k \cdot \frac{1}{k!}\, 3^k \cdot p_0 = \frac{1}{k!}\, p_0 \qquad \text{for } k = 0,\, 1,\, 2,\, 3,$$

and

$$p_k = \left(\frac{1}{3}\right)^k \cdot \frac{3^3}{3!}\, p_0 = \frac{1}{6}\left(\frac{1}{3}\right)^{k-3} p_0 \qquad \text{for } k \geq 3,$$

hence

$$1=\sum_{k=0}^{\infty}p_k=p_0\left\{1+1+\frac{1}{2}+\frac{1}{6}\sum_{k=3}^{\infty}\left(\frac{1}{3}\right)^{k-3}\right\}=p_0\left\{\frac{5}{2}+\frac{1}{6}\cdot\frac{1}{1-\frac{1}{3}}\right\}=p_0\left\{\frac{5}{2}+\frac{1}{4}\right\}=\frac{11}{4}\,p_0,$$

from which $p_0=\dfrac{4}{11}$, thus

$$p_k=\begin{cases}\dfrac{4}{11}\cdot\dfrac{1}{k!}, & k=0,\,1,\,2,\\[2mm] \dfrac{2}{33}\left(\dfrac{1}{3}\right)^{k-3}, & k\geq 3.\end{cases}$$

2) The fraction of time, in which all three channels are busy, is given by

$$\sum_{k=3}^{\infty}p_k=\frac{2}{33}\sum_{k=3}^{\infty}\left(\frac{1}{3}\right)^{k-3}=\frac{2}{33}\cdot\frac{1}{1-\frac{1}{3}}=\frac{2}{33}\cdot\frac{3}{2}=\frac{1}{11}.$$

ALTERNATIVELY, it is given by

$$1-p_0-p_1-p_2=1-\frac{4}{11}-\frac{4}{11}\,\frac{1}{1!}-\frac{4}{11}\,\frac{1}{2!}=\frac{1}{11}.$$

3) The average length of the queue is

$$\begin{aligned}\sum_{k=4}^{\infty}(k-3)p_k &= \sum_{k=4}^{\infty}(k-3)\cdot\frac{2}{33}\left(\frac{1}{3}\right)^{k-3} = \frac{2}{33}\sum_{k=1}^{\infty}k\left(\frac{1}{3}\right)^k \\ &= \frac{2}{33}\cdot\frac{1}{3}\cdot\frac{1}{\left(1-\frac{1}{3}\right)^2} = \frac{2}{33}\cdot\frac{1}{3}\cdot\frac{9}{4} = \frac{1}{22}.\end{aligned}$$

4) If $N=2$, then $\varrho=\dfrac{1}{2}$. The stationary probabilities are

$$p_k = \begin{cases} \dfrac{1-\frac{1}{2}}{1+\frac{1}{3}} = \dfrac{1}{3}, & k=0, \\[2ex] 2\left(\dfrac{1}{2}\right)^k\cdot\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}} = \dfrac{1}{3}\cdot\dfrac{1}{2^{k-1}}, & k\in\mathbb{N}.\end{cases}$$

5) The fraction of times, in which both channels are busy, is

$$1-p_0-p_1 = 1-\frac{1}{3}-\frac{1}{3} = \frac{1}{3}.$$

6) The average length of the queue is

$$\sum_{k=3}^{\infty}(k-2)p_k = \sum_{k=3}^{\infty}(k-2)\cdot\frac{2}{3}\cdot\frac{1}{2^k} = \frac{2}{3}\cdot\left(\frac{1}{2}\right)^3\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^{k-1} = \frac{1}{12}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{1}{3}.$$

7) If there is only one channel, the traffic intensity becomes $\varrho = 1$, and the queue is increasing indefinitely.

Example 4.13 *A shop serves M customers, and there is one shop assistant in the shop. It is possible to form a queue. We assume that the service time is exponentially distributed of mean $\frac{1}{\mu}$. Assume also that if a customer is not in the shop at time t, then there is the probability $\lambda h + h\varepsilon(h)$ [where λ is a positive constant] that this customer arrives to the shop before the time $t+h$. Finally, assume that the customers arrive to the shop mutually independent of each other. Thus we have a birth and death process $\{X(t),\, t \in [0,\infty[\}$ of the states E_0, E_1, ..., E_M, where E_k denotes the state that there are k customers in the shop, $k = 0$, 1, 2, ..., M.*

1) Prove that the birth intensities λ_k and death intensities μ_k, $k = 0$, 1, 2, ..., M, are given by

$$\lambda_k = (M-k)\lambda, \qquad \mu_k = \begin{cases} 0, & k = 0, \\ \mu, & k = 1, 2, \dots, M. \end{cases}$$

2) Find the equations of the stationary probabilities p_k, $k = 0$, 1, 2, ..., M.

3) Express the stationary probabilities p_k, $k = 0$, 1, 2, ..., M, by means of p_0.

4) Compute the stationary probabilities p_k, $k = 0$, 1, 2, ... M.

5) Find, expressed by the stationary probability p_0, the average number of customers, who are not *in the shop.*

6) Compute the stationary probabilities, first in the case, when $\frac{\lambda}{\mu} = 1$ and $M = 5$, and then in the case, when $\frac{\lambda}{\mu} = \frac{1}{2}$ and $M = 5$.

1) If we are in state E_k, then $M-k$ of the customers are not in the shop. They arrive to the shop before time $t+h$ of probability

$$(M-k)\{\lambda + \varepsilon(h)\}h,$$

(a time interval of length h, and we divide by h before we go to the limit $h \to 0$). Hence, the birth intensity is

$$\lambda_k = (M-k)\lambda, \qquad k = 0, 1, \dots, M.$$

If we are in state E_0, then no customer is served, so $\mu_0 = 0$.
In any other state precisely one customer is served with the intensity μ, so

$$\mu_k = \mu, \qquad k = 1, 2, \dots, M.$$

2) The equations of the stationary probabilities are

$$\mu_{k+1}p_{k+1} = \lambda_k p_k.$$

Thus, in the explicit case,

$$p_{k+1} = (M-k)\,\frac{\lambda}{\mu}\,p_k.$$

3) We get successively

$$p_0 = p_0, \qquad p_1 = M \cdot \frac{\lambda}{\mu}\,p_0, \qquad p_2 = M(M-1)\left(\frac{\lambda}{\mu}\right)^2 p_0,$$

and in general

$$p_k = \frac{M!}{(M-k)!}\left(\frac{\lambda}{\mu}\right)^k p_0, \qquad k = 0,\,1,\,2,\,\dots,\,M.$$

4) It follows from the equation

$$1 = \sum_{k=0}^{M} p_k = M!\sum_{k=0}^{M}\frac{1}{(M-k)!}\left(\frac{\lambda}{\mu}\right)^k p_0 = p_0 \cdot M!\left(\frac{\lambda}{\mu}\right)^M \sum_{k=0}^{M}\frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k p_0$$

that

$$p_0 = \frac{1}{M!\sum_{k=0}^{M}\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^{M-k}} = \frac{\left(\frac{\mu}{\lambda}\right)^M}{M!\sum_{k=0}^{M}\frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k},$$

and hence

$$\begin{aligned}
\mathbf{p} &= \frac{\left(\frac{\mu}{\lambda}\right)^M}{M!\sum_{k=0}^{M}\frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k}\left(1, M\,\frac{\lambda}{\mu}, M(M-1)\left(\frac{\lambda}{\mu}\right)^2, \cdots, \frac{M!}{(M-k)!}\left(\frac{\lambda}{\mu}\right)^k, \cdots, M!\left(\frac{\lambda}{\mu}\right)^M\right) \\
&= \frac{1}{M!\sum_{k=0}^{M}\frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k}\left(\frac{\left(\frac{\mu}{\lambda}\right)^M}{M!}, \frac{\left(\frac{\mu}{\lambda}\right)^{M-1}}{(M-1)!}, \cdots, \frac{\left(\frac{\mu}{\lambda}\right)^{M-k}}{(M-k)!}, \cdots, 1\right).
\end{aligned}$$

5) The average number of customers who are *not* in the shop is by e.g. **3.**,

$$\begin{aligned}
\sum_{k=0}^{M}(M-k)p_k &= \sum_{k=0}^{M-1}\frac{M!}{(M-k-1)!}\left(\frac{\lambda}{\mu}\right)^k p_0 = \sum_{k=1}^{M}\frac{M!}{(M-k)!}\left(\frac{\lambda}{\mu}\right)^{k-1} p_0 \\
&= \frac{\mu}{\lambda}\sum_{k=1}^{M}p_k = \frac{\mu}{\lambda}\,(1-p_0)\,.
\end{aligned}$$

6) If $\dfrac{\lambda}{\mu} = 1$ and $M = 5$, then

$$1 = \sum_{k=0}^{5} \frac{5!}{(5-k)!}\, p_0 = \{1 + 5 + 20 + 60 + 120 + 120\} p_0 = 326 p_0,$$

and

$$\mathbf{p} = \frac{1}{326}\,(1, 5, 20, 60, 120, 120).$$

7) Når $\dfrac{\lambda}{\mu} = \dfrac{1}{2}$ og $M = 5$, er

$$1 = \sum_{k=0}^{5} \frac{5!}{(5-k)!} \left(\frac{1}{2}\right)^k p_0 = \left\{1 + \frac{5}{2} + 5 + \frac{15}{2} + \frac{15}{2} + \frac{15}{4}\right\} p_0 = \frac{109}{4}\, p_0$$

and

$$\mathbf{p} = \frac{4}{109}\left(1, \frac{5}{2}, 5, \frac{15}{2}, \frac{15}{2}, \frac{15}{4}\right) = \frac{1}{109}\,(4, 10, 20, 30, 30, 15).$$

Example 4.14 *Given two queueing systems, A and B, which are mutually independent. We assume for each of the two systems:*

a. *there is one channel,*

b. *it is possible to form a queue,*

c. *the customers arrive according to a Poisson process of intensity λ,*

d. *the service times are exponentially distributed of parameter μ,*

e. *the traffic intensity is $\varrho = \dfrac{\lambda}{\mu} = \dfrac{1}{2}$.*

Denote by X_1 the random variable which indicates the number of customers in system A, and by X_2 the random variables which indicates the number of customers in system B.

1. *Compute by using the stationary probabilities,*

$$P\{X_1 = k\} \quad \textit{and} \quad P\{X_2 = k\}, \qquad k \in \mathbb{N}_0.$$

Let $Z = X_1 + X_2$ denote the total number of customers in the two systems.

2. *Compute $P\{Z = k\}$, $k \in \mathbb{N}_0$.*

3. *Compute the mean of Z*

Consider another queuing system C, in which we assume,

a. *there are two channels,*

b. *it is possible to form a queue,*

c. *the customers arrive according to a Poisson process of intensity 2λ,*

d. *the service times are exponentially distributed of the parameter μ,*

e. *the traffic intensity is $\varrho = \dfrac{2\lambda}{2\mu} = \dfrac{1}{2}$.*

Let the random variable Y denote the number of customers in system C.

4. *Compute by using the stationary probabilities,*

$$P\{Y = k\} \quad \textit{and} \quad P\{Y > k\}, \qquad k \in \mathbb{N}_0.$$

5. *Compute the mean of Y.*

6. *Prove for all $k \in \mathbb{N}_0$ that*

$$P\{Z > k\} > P\{Y > k\}.$$

HINT TO **6.**: *One may without proof use the formula,*

$$\sum_{i=N}^{\infty} i\, x^{i-1} = \frac{x^{N-1}\{N-(N-1)x\}}{(1-x)^2}, \qquad |x| < 1, \quad N \in \mathbb{N}.$$

1) The two queueing systems follow the same distribution, and $N = 1$ and $\varrho = \dfrac{1}{2}$, so we get by a known formula,

$$P\{X_1 = k\} = P\{X_2 = k\} = p_k = \varrho^k \cdot (1-\varrho) = \left(\frac{1}{2}\right)^{k+1}, \qquad k \in \mathbb{N}_0.$$

2) A straightforward computation gives

$$\begin{aligned} P\{Z = k\} &= \sum_{j=0}^{k} P\{X_1 = j\} \cdot P\{X_2 = k-j\} = \sum_{j=0}^{k} \left(\frac{1}{2}\right)^{j+1} \cdot \left(\frac{1}{2}\right)^{k-j+1} \\ &= (k+1) \cdot \left(\frac{1}{2}\right)^{k+2}, \qquad k \in \mathbb{N}_0. \end{aligned}$$

3) It follows from

$$E\{X_1\} = E\{X_2\} = \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^{k+1} = \frac{1}{4} \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{2}\right)^{k-1} = \frac{1}{4} \cdot \frac{1}{\left(1-\frac{1}{2}\right)^2} = 1,$$

that

$$\begin{aligned} E\{Z\} &= \sum_{k=1}^{\infty} k(k+1) \left(\frac{1}{2}\right)^{k+2} = \sum_{k=2}^{\infty} k(k-1) \left(\frac{1}{2}\right)^{k+1} = \frac{1}{8} \sum_{k=2}^{\infty} k(k-1) \left(\frac{1}{2}\right)^{k-2} \\ &= \frac{1}{8} \cdot \frac{2!}{\left(1-\dfrac{1}{2}\right)^3} = 2. \end{aligned}$$

4) Roughly speaking, A and B are joined to get C, so we have $N = 2$ and $\varrho = \dfrac{1}{2}$. Then it follows that

$$P\{Y = 0\} = p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{3},$$

and

$$P\{Y = k\} = 2\varrho^k \cdot \frac{1-\varrho}{1+\varrho} = \frac{1}{3}\left(\frac{1}{2}\right)^{k-1}, \qquad k \in \mathbb{N}.$$

Thus

$$P\{Y > k\} = \sum_{j=k+1}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{j-1} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{k} \cdot \frac{1}{1-\dfrac{1}{2}} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^{k-1}, \qquad k \in \mathbb{N}_0.$$

5) The mean is

$$E\{Y\} = \sum_{k=1}^{\infty} \frac{1}{3}\, k \left(\frac{1}{2}\right)^{k-1} = \frac{1}{3} \cdot \frac{1}{\left(1-\dfrac{1}{2}\right)^2} = \frac{4}{3}.$$

6) It follows from **2.** that

$$
\begin{aligned}
P\{Z>k\} &= \sum_{j=k+1}^{\infty}(j+1)\left(\frac{1}{2}\right)^{j+2}=\frac{1}{4}\sum_{j=k+2}^{\infty} j\left(\frac{1}{2}\right)^{j-1} \\
&= \frac{1}{4}\cdot\frac{\left(\frac{1}{2}\right)^{k+1}\left\{k+2-(k+1)\frac{1}{2}\right\}}{\left(1-\frac{1}{2}\right)^{2}}=\left(\frac{1}{2}\right)^{k+2}\cdot\{2k+4-k-1\} \\
&= \frac{k+3}{8}\cdot\left(\frac{1}{2}\right)^{k-1}>\frac{1}{3}\cdot\left(\frac{1}{2}\right)^{k-1}=P\{Y>k\}.
\end{aligned}
$$

We notice that $P\{Y=k\}=P\{Y>k\}$ for $k\in\mathbb{N}$, and that this is not true for $k=0$.

Example 4.15 *Given two mutually independent queueing systems A and B. We assume for each of the two systems,*

a. *there is one channel,*

b. *it is possible to form a queue,*

c. *customers arrive to A according to a Poisson process of intensity* $\lambda_A = \dfrac{1}{3}$ *minute*$^{-1}$*, and they arrive to B according to a Poisson process of intensity* $\lambda_B = \dfrac{2}{3}$ *minute*$^{-1}$*,*

d. *the service times of both A and B are exponentially distributed of the parameter* $\mu = 1$ *minute*$^{-1}$*.*

Let the random variable X_A *denote the number of customers in system A, and let the random variable* X_B *denote the number of customers in system B. Furthermore, we let* Y_A *and* Y_B*, resp., denote the number of customers in the queue at A and B, resp..*

1. *Find by using the stationary probabilities,*

$$P\{X_A = k\} \quad \text{and} \quad P\{X_B = k\}, \qquad k \in \mathbb{N}_0.$$

2. *Find the average waiting times at A and B, resp..*

3. *Find by using the stationary probabilities,*

$$P\{Y_A = k\} \quad \text{and} \quad P\{Y_B = k\}, \qquad k \in \mathbb{N}_0.$$

4. *Find the means* $E\{X_A + X_B\}$ *and* $E\{Y_A + Y_B\}$*.*

5. *Compute* $P\{X_A + X_B = k\}$*,* $k \in \mathbb{N}_0$*.*

The two queueing systems are now joined to one queueing system of two channels, where the customers arrive according to a Poisson process of intensity $\lambda = \lambda_A + \lambda_B$*, and where the serving times are exponentially distributed of parameter* $\mu = 1$ *minute*$^{-1}$*. Let X denote the number of customers in the system, and let Y denote the number of customers in the queue.*

6. *Find by using the stationary probabilities,*

$$P\{X = k\} \quad \text{and} \quad P\{Y = k\}, \qquad k \in \mathbb{N}_0.$$

7. *Find the means* $E\{X\}$ *and* $E\{Y\}$*.*

1A. Since $\lambda_A = \dfrac{1}{3}$ minute^{-1} and $\mu = 1$ minute^{-1}, and $N = 1$, we get the traffic intensity $\varrho_A = \dfrac{1}{3}$. The stationary probabilities are

$$P\{X_A = k\} = p_{A,k} = 2 \cdot \left(\frac{1}{3}\right)^{k+1}, \qquad k \in \mathbb{N}_0.$$

1B. Analogously, $\lambda_B = \dfrac{2}{3}$ minute^{-1} and $\mu = 1$ minute^{-1}, and $N = 1$, so $\varrho_B = \dfrac{2}{3}$, and

$$P\{X_B = k\} = p_{B,k} = \frac{1}{3}\left(\frac{2}{3}\right)^{k} = \frac{1}{2}\left(\frac{2}{3}\right)^{k+1}, \qquad k \in \mathbb{N}_0.$$

2A. The waiting time at A is given by

$$V_A = \frac{\varrho_A}{\mu\left(1-\varrho_A\right)} = \frac{\frac{1}{3}}{1 \cdot \frac{2}{3}} = \frac{1}{2}.$$

2B. Analogously, the waiting time at B is

$$V_B = \frac{\varrho_B}{\mu\left(1-\varrho_B\right)} = \frac{\frac{2}{3}}{1 \cdot \frac{1}{3}} = 2.$$

3A. Assume that there is no queue at A. Then either there is no customer at all in the system, or there is precisely one customer, who is served for the time being,

$$P\left\{Y_A = 0\right\} = P\left\{X_A = 0\right\} + P\left\{X_A = 1\right\} = 2 \cdot \left\{\frac{1}{3} + \frac{1}{9}\right\} = \frac{8}{9}.$$

If $k \in \mathbb{N}$, then

$$P\left\{Y_A = k\right\} = P\left\{X_A = k+1\right\} = 2 \cdot \left(\frac{1}{3}\right)^{k+2}.$$

3B. Analogously,

$$P\left\{Y_B = 0\right\} = P\left\{X_B = 0\right\} + P\left\{X_B = 1\right\} = \frac{1}{3}\left\{1 + \frac{2}{3}\right\} = \frac{5}{9}$$

and

$$P\left\{Y_B = k\right\} = P\left\{X_B = k+1\right\} = \frac{1}{2}\left(\frac{2}{3}\right)^{k+2}, \qquad k \in \mathbb{N}_0.$$

4. It follows from

$$E\left\{X_A\right\} = 2\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k+1} = \frac{2}{9}\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k-1} = \frac{2}{9} \cdot \frac{1}{\left(1-\frac{1}{3}\right)^2} = \frac{1}{2}$$

and

$$E\left\{X_B\right\} = \frac{2}{9}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1} = \frac{2}{9} \cdot \frac{1}{\left(1-\frac{2}{3}\right)^2} = 2,$$

that

$$E\left\{X_A + X_B\right\} = \frac{1}{2} + 2 = \frac{5}{2}.$$

It follows from

$$E\left\{Y_A\right\} = 2\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k+2} = \frac{2}{27}\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k-1} = \frac{2}{27} \cdot \frac{1}{\left(1-\frac{1}{3}\right)^2} = \frac{1}{6}$$

and

$$E\{Y_B\} = \frac{1}{2}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k+2} = \frac{4}{27}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1} = \frac{4}{27}\cdot\frac{1}{\left(1-\frac{2}{3}\right)^2} = \frac{4}{3},$$

then

$$E\{Y_A+Y_B\} = \frac{1}{6}+\frac{4}{3} = \frac{3}{2}.$$

5. If $k \in \mathbb{N}_0$, then

$$\begin{aligned}
P\{X_A+X_B=k\} &= \sum_{j=0}^{k} P\{X_A=j\}\cdot P\{X_B=k-j\} \\
&= \sum_{j=0}^{k} 2\cdot\left(\frac{1}{3}\right)^{j+1}\cdot\frac{1}{2}\cdot\left(\frac{2}{3}\right)^{k-j+1} = \sum_{j=0}^{k}\left(\frac{1}{3}\right)^{k+2}\cdot 2^{k-j+1} \\
&= \left(\frac{1}{3}\right)^{k+2}\sum_{n=1}^{k+1} 2^n = \left(2^{k+2}-2\right)\cdot\left(\frac{1}{3}\right)^{k+2} \\
&= \left(\frac{2}{3}\right)^{k+2} - 2\left(\frac{1}{3}\right)^{k+2} = \frac{2}{3^{k+2}}\left(2^{k+1}-1\right).
\end{aligned}$$

6. The traffic intensity is

$$\varrho = \frac{\lambda_A+\lambda_B}{N\mu} = \frac{\frac{1}{3}+\frac{2}{3}}{2\cdot 1} = \frac{\lambda}{2\mu} = \frac{1}{2}.$$

It follows that

$$P\{X=k\} = p_k = \begin{cases} \dfrac{1}{3}, & k=0, \\[2ex] \dfrac{2}{3}\left(\dfrac{1}{2}\right)^k, & k\in\mathbb{N}. \end{cases}$$

Since $Y=(X-2)\vee 0$, we get

$$P\{Y=0\} = P\{X=0\}+P\{X=1\}+P\{X=2\} = \frac{1}{3}+\frac{1}{3}+\frac{1}{6} = \frac{5}{6}$$

and

$$P\{Y=k\} = P\{X=k+2\} = \frac{2}{3}\left(\frac{1}{2}\right)^{k+2} = \frac{1}{6}\left(\frac{2}{3}\right)^k, \qquad k\in\mathbb{N}.$$

7. By a straightforward computation,

$$E\{X\} = \frac{2}{3}\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = \frac{1}{3}\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \frac{1}{3}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{4}{3}$$

and

$$E\{Y\}=\frac{1}{6}\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^k=\frac{1}{4}\,E\{X\}=\frac{1}{3}.$$

Example 4.16 *Consider a birth and death process E_0, E_1, E_2, $\dots$, where the birth intensities λ_k are given by*

$$\lambda_k=\frac{\alpha}{k+1},\qquad k\in\mathbb{N}_0,$$

where α is a positive constant, while the death intensities μ_k are given by

$$\mu_k=\begin{cases}0, & k=0,\\ \mu, & k=1,\\ 2\mu, & k\geq 2,\end{cases}$$

where $\mu>0$. We assume that $\dfrac{\alpha}{\mu}=8$.

1. *Find the equations of the stationary probabilities p_k, $k\in\mathbb{N}_0$.*

2. *Prove that*

$$p_k=2\cdot 4^k\cdot\frac{1}{k!}\,p_0,\qquad k\in\mathbb{N},$$

and find p_0.

The above can be viewed as a model of the forming of a queue in a shop, where

a. *there are two shop assistants,*

b. *the service time is exponentially distributed of mean $\dfrac{1}{\mu}$,*

c. *the frequency of the arrivals is decreasing with increasing number of customers according to the indicated formula.*

3. *Compute by means of the stationary probabilities the average number of customers in the shop. (3 dec.).*

4. *Compute by means of the stationary probabilities the average number of busy shop assistants. (3 dec.).*

5. *Compute by means of the stationary probabilities the probability that there are more than two customers in the shop. (3 dec.).*

1) We have

$$\mu_{k+1}p_{k+1}=\lambda_k p_k,\qquad k\in\mathbb{N}_0,$$

thus

$$p_1 = \frac{\lambda_0}{\mu_1}\, p_0 = \frac{\alpha}{\mu}\, p_0 = 8p_0$$

and

$$p_k = \frac{\lambda_{k-1}}{\mu_k}\, p_{k-1} = \frac{\alpha}{k} \cdot \frac{1}{2\mu}\, p_{k-1} = \frac{4}{k}\, p_{k-1} \qquad \text{for } k \geq 2.$$

2) If $k = 1$, then

$$p_1 = 8p_0 = 2 \cdot \frac{4^1}{1!}\, p_0,$$

and the formula is true for $k = 1$. Then assume that

$$p_{k-1} = 2 \cdot 4^{k-1} \cdot \frac{1}{(k-1)!}\, p_0.$$

Then

$$p_k = \frac{4}{k}\, p_{k-1} = 2 \cdot \frac{4^k}{k!}\, p_0,$$

and the formula follows by induction.

It follows from

$$1=\sum_{k=0}^{\infty} p_k = p_0\left\{1+2\sum_{k=1}^{\infty}\frac{4^k}{k!}\right\} = p_0\left(2e^4-1\right)$$

that

$$p_0=\frac{1}{2e^4-1}.$$

3) The task is now changed to queueing theory. Since p_k is the probability that there are k customers in the shop, the mean of the number of customers in the shop is

$$\sum_{k=1}^{\infty} kp_k = 2\cdot 4\cdot p_0\sum_{k=1}^{\infty}\frac{4^{k-1}}{(k-1)!}=\frac{8e^4}{2e^4-1}\approx 4.037.$$

4) The average number of busy shop assistants is

$$\begin{aligned}0\cdot p_0+1\cdot p_1+2\sum_{k=2}^{\infty}p_k = p_1+1\left(1-p_0-p_1\right) &= 2-2p_0-p_1=2-2p_0-8p_0\\ &= 2-10p_0=2-\frac{10}{2e^4-1}=1.908.\end{aligned}$$

5) The probability that there are more than two customers in the shop is

$$\sum_{k=3}^{\infty}p_k=1-p_0-p_1-p_2=1-p_0\left(1+8+\frac{32}{2}\right)=1-\frac{25}{2e^4-1}\approx 0.769.$$

Example 4.17 *Consider a birth and death process of the states E_0, E_1, E_2, $\dots$, where the birth intensities λ_k are given by*

$$\lambda_k = \begin{cases} 2\lambda, & k = 0, \\ \lambda, & k \in \mathbb{N}, \end{cases}$$

while the death intensities μ_k are given by

$$\mu_k = \begin{cases} 0, & k = 0, \\ \mu, & k \in \mathbb{N}. \end{cases}$$

Here, λ and μ are positive constants, and we assume everywhere that $\dfrac{\lambda}{\mu} = \dfrac{3}{4}$.

1. *Find the equations of the stationary probabilities, and prove that the stationary probabilities are given by*

$$p_k = 2 \cdot \left(\frac{3}{4}\right)^k p_0, \qquad k = 1, 2, 3, \dots,$$

and finally, find p_0.

The above can be considered as a model of forming queues in a shop, where

a. *there is one shop assistant,*

b. *the service time is exponentially distributed of mean $\dfrac{1}{\mu}$,*

c. *the customers arrive according to a Poisson process of intensity 2λ. However, if there already are customers in the shop, then half of the arriving customers will immediately leave the shop without being served.*

2. *Compute by means of the stationary probabilities the average number of customers in the shop.*

3. *Compute by means of the stationary probabilities the average number of customers in the queue.*

We now assume that instead of one shop assistant there are two shop assistants and that all arriving customers are served (thus we have the birth intensities $\lambda_k = 2\lambda$, $k \in \mathbb{N}_0$).

4. *Compute in this queueing system the stationary probabilities and then find the average number of customers in the queue.*

1) The equations of the stationary probabilities are

$$\mu_{k+1} p_{k+1} = \lambda_k p_k, \qquad k \in \mathbb{N}_0,$$

thus

$$p_1 = \frac{2\lambda}{\mu} p_0 = \frac{3}{2} p_0 = 2 \cdot \left(\frac{3}{4}\right)^1 p_0,$$

and

$$p_k = \frac{\lambda}{\mu} p_{k-1} = \frac{3}{4} p_{k-1}, \qquad k \geq 2,$$

hence by recursion,

$$p_k = \left(\frac{3}{4}\right)^{k-1} p_1 = 2 \cdot \left(\frac{3}{4}\right)^k p_0, \qquad k \geq 2.$$

We get

$$1 = \sum_{k=0}^{\infty} p_k = p_0 + p_0 \sum_{k=1}^{\infty} 2 \cdot \left(\frac{3}{4}\right)^k = p_0 \left\{ 1 + 2 \cdot \frac{3}{4} \cdot \frac{1}{1 - \frac{3}{4}} \right\} = p_0 \left\{ 1 + \frac{3}{2} \cdot 4 \right\} = 7p_0,$$

so

$$p_0 = \frac{1}{7} \qquad \text{and} \qquad p_k = \frac{2}{7} \cdot \left(\frac{3}{4}\right)^k, \quad k \in \mathbb{N}.$$

2) Since p_k is the probability that there are k customers in the shop, the average number of customer in the shop is

$$\sum_{k=1}^{\infty} k p_k = \frac{2}{7} \cdot \frac{3}{4} \sum_{k=1}^{\infty} k \cdot \left(\frac{3}{4}\right)^{k-1} = \frac{3}{14} \cdot \frac{1}{\left(1 - \frac{3}{4}\right)^2} = \frac{3}{14} \cdot 16 = \frac{24}{7}.$$

3) If there are k customers in the queue, there must also be 1 customer, who is being served, so the average is

$$\sum_{k=1}^{\infty} k p_{k+1} = \frac{2}{7} \cdot \frac{3}{4} \cdot \frac{3}{4} \sum_{k=1}^{\infty} k \left(\frac{3}{4}\right)^{k-1} = \frac{3}{28} \cdot 24 = \frac{18}{7},$$

where we have used the result of **2**.

4) The traffic intensity is $\varrho = \dfrac{2\lambda}{2 \cdot \mu} = \dfrac{3}{4}$, and since $N = 2$, we get

$$p_0 = \frac{1 - \varrho}{1 + \varrho} = \frac{1}{7} \qquad \text{and} \qquad p_k = \frac{2}{7} \left(\frac{3}{4}\right)^k, \quad k \in \mathbb{N}.$$

We see that they are identical with the stationary probabilities found in **1.**.
The average length of the queue is given by (end here we get to the divergence from the previous case)

$$\begin{aligned}\sum_{k=3}^{\infty} (k-2) p_k &= \frac{2}{7} \sum_{k=3}^{\infty} (k-2) \left(\frac{3}{4}\right)^k = \frac{2}{7} \sum_{k=1}^{\infty} k \left(\frac{3}{4}\right)^{k+2} = \frac{2}{7} \cdot \left(\frac{3}{4}\right)^3 \sum_{k=1}^{\infty} k \left(\frac{3}{4}\right)^{k-1} \\ &= \frac{2}{7} \cdot \left(\frac{3}{4}\right)^3 \cdot \frac{1}{\left(1 - \frac{3}{4}\right)^2} = \frac{2}{7} \cdot \frac{27}{4} = \frac{27}{14}.\end{aligned}$$

Example 4.18 *Consider a birth and death process of states E_0, E_1, E_2, ..., and with birth intensities λ_k given by*

$$\lambda_k = \begin{cases} \alpha, & k = 0,\, 1, \\ \dfrac{\alpha}{k}, & k \geq 2, \end{cases}$$

where α is a positive constant, and where the death intensities are given by

$$\mu_k = \begin{cases} 0, & k = 0, \\ \mu, & k \in \mathbb{N}, \end{cases}$$

where $\mu > 0$.
We assume in the following that $\dfrac{\alpha}{\mu} = 2$.

1. *Find the equations of the stationary probabilities p_k, $k \in \mathbb{N}_0$.*

2. *Prove that*

$$p_k = \frac{2^k}{(k-1)!}\, p_0, \qquad k \in \mathbb{N},$$

and find p_0.

The above can be considered as a model of forming a queue in a shop where

a. *there is one shop assistant,*

b. *the serving time is exponentially distribution of mean $\dfrac{1}{\mu}$,*

c. *the frequency of arrivals decreasis with increasing number of customers according to the formula for λ_k above.*

3. *Compute by means of the stationary probabilities the average length of the queue (3 dec.).*

4. *Compute by means of the stationary probabilities the average number of customers in the shop (3 dec.).*

1) We have

$$\mu_{k+1} p_{k+1} = \lambda_k p_k, \quad k \in \mathbb{N}_0, \qquad \text{and} \qquad \sum_{k=0}^{\infty} p_k = 1.$$

Hence, successively,

$$\mu p_1 = \alpha p_0, \qquad \mu p_2 = \alpha p_1, \quad \text{and} \quad \mu p_k = \frac{\alpha}{k-1}\, p_{k-1} \quad \text{for } k \geq 3.$$

It follows from $\dfrac{\alpha}{\mu} = 2$ that

$$(6) \;\; p_1 = 2p_0, \quad p_2 = 2p_1, \quad p_k = \frac{2}{k-1} p_{k-1}, \quad k \geq 3, \quad \text{and} \quad \sum_{k=0}^{\infty} p_k = 1.$$

2) We infer from (6) that $p_1 = 2p_0$ and $p_2 = 2p_2 = 4p_0$, and for $k \geq 3$,

$$p_k = \frac{2}{k-1}\, p_{k-1} = \frac{2^2}{(k-1)(k-2)}\, p_{k-2} = \cdots = \frac{2^{k-2}}{(k-1)!}\, p_2 = \frac{2^2}{(k-1)!}\, p_0.$$

A check shows that the latter formula is also true for $k = 1$ and $k = 2$, thus

$$p_k = \frac{2^k}{(k-1)!}\, p_0, \qquad k \in \mathbb{N}.$$

Then we find p_0 from

$$1 = \sum_{k=0}^{\infty} p_k = p_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{2^k}{(k-1)!} \right\} = p_0 \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{2^{k-1}}{(k-1)!} \right\} = p_0 \left(1 + 2e^2 \right),$$

thus

$$p_0 = \frac{1}{1+2e^2} \quad (\approx 0.0634), \qquad \text{and} \qquad p_k = \frac{2^k}{(k-1)!} \cdot \frac{1}{1+2e^2}, \quad k \in \mathbb{N}.$$

3) The average length of the queue is (notice that since 1 customer is served, we have here $k-1$ instead of k),

$$\sum_{k=2}^{\infty}(k-1)p_k = \sum_{k=2}^{\infty}\frac{k-1}{(k-1)!}\,2^k\,p_0 = 4\sum_{k=2}^{\infty}\frac{2^{k-2}}{(k-2)!}\,p_0 = 4e^2p_0 = \frac{4e^2}{1+2e^2} \approx 1.873.$$

4) The average number of customers is

$$\begin{aligned}\sum_{k=1}^{\infty}kp_k &= \sum_{k=1\,(2)}^{\infty}(k-1)p_k + \sum_{k=1}^{\infty}p_k = 4e^2p_0 + (1-p_0)\\ &= \frac{4e^2}{1+2e^2} + \frac{2e^2}{1+2e^2} = \frac{6e^2}{1+2e^2} \approx 2.810.\end{aligned}$$

Example 4.19 *Given a queueing system, for which*

a. *there is one shop assistant,*

b. *it is possible to form a queue,*

c. *the customers arrive according to a Poisson process of intensity* λ*,*

d. *the serving times are exponentially distributed of parameter* μ*,*

e. *the traffic intensity* $\dfrac{\lambda}{\mu}$ *is* $\dfrac{2}{3}$.

Let the random variable X denote the number of customers in the system, and let Y denote the number of customers in the queue.

1. *Find by means of the stationary probabilities,*

$$P\{X=k\} \quad \textit{and} \quad P\{Y=k\}, \qquad k \in \mathbb{N}_0.$$

2. *Find the means $E\{X\}$ and $E\{Y\}$.*

The system is changed by introducing another shop assistant, whenever there are 3 or more customers in the shop; this extra shop assistant is withdrawn after ending his service, if the number of customers then is smaller than 3. The other assumptions are unchanged.

3. *Explain why this new system can be described by a birth and death process of states E_0, E_1, E_2, …, birth intensities $\lambda_k = \lambda$, $k \in \mathbb{N}_0$, and death intensities μ_k given by*

$$\mu_k = \begin{cases} 0, & k=0,\\ \mu, & k=1,\,2,\\ 2\mu, & k=3,\,4,\,\dots. \end{cases}$$

4. *Find the stationary probabilities p_k of this system.*

5. *Find the average number of customers in the system,*

$$\sum_{k=1}^{\infty}kp_k.$$

1) Since $N=1$, it follows that

$$p_k = \frac{1}{3}\left(\frac{2}{3}\right)^k, \qquad k \in \mathbb{N}_0,$$

thus

$$P\{X=k\} = p_k = \frac{1}{3}\left(\frac{2}{3}\right)^k, \qquad k \in \mathbb{N}_0,$$

and

$$\begin{aligned} P\{Y=0\} &= P\{X=0\}+P\{X=1\} = \frac{1}{3}\left(1+\frac{2}{3}\right) = \frac{5}{9}, \\ P\{Y=k\} &= P\{X=k+1\} = \frac{1}{3}\left(\frac{2}{3}\right)^{k+1}, \qquad k \in \mathbb{N}. \end{aligned}$$

2) The means are

$$E\{X\} = \sum_{k=1}^{\infty} k p_k = \frac{1}{3}\cdot\frac{2}{3}\sum_{k=1}^{\infty} k\cdot\left(\frac{2}{3}\right)^{k-1} = 2,$$

and

$$E\{Y\} = \sum_{k=1}^{\infty} \frac{1}{3}\left(k\,\frac{2}{3}\right)^{k+1} = \frac{2}{3}\,E\{X\} = \frac{4}{3}.$$

3) The birth intensities $\lambda_k = \lambda$, $k \in \mathbb{N}_0$, are clearly not changed, and $\mu_0 = 0$, $\mu_1 = \mu_2 = \mu$. When $k \geq 3$, another shop assistant is also serving the customers, so $\mu_k = 2\mu$ for $k \geq 3$.

4) We have

$$\mu_{k+1}p_{k+1} = \lambda_k p_k.$$

Thus we get the equations

$$p_1 = \frac{\lambda}{\mu}\,p_0 = \frac{2}{3}\,p_0, \qquad p_2 = \frac{\lambda}{\mu}\,p_1 = \frac{2}{3}\,p_1,$$

and

$$p_{k+1} = \frac{\lambda}{2\mu}\,p_k = \frac{1}{3}\,p_k, \qquad k \geq 2.$$

Hence

$$p_1 = \frac{2}{3}\,p_0, \qquad p_2 = \frac{4}{9}\,p_0,$$

and

$$p_k = \left(\frac{1}{3}\right)^{k-2} p_2 = 3\left(\frac{1}{3}\right)^k p_0 \qquad \text{for } k \geq 3.$$

It follows from

$$\begin{aligned}
1 &= \sum_{k=0}^{\infty} p_0 = p_0 \left\{1 + \frac{2}{3} + \frac{4}{9} + 4 \sum_{k=3}^{\infty} \left(\frac{1}{3}\right)^k\right\} = p_0 \left\{\frac{5}{3} + \frac{4}{9} \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j\right\} = p_0 \left\{\frac{5}{3} + \frac{4}{9} \cdot \frac{1}{1 - \frac{1}{3}}\right\} \\
&= p_0 \left\{\frac{5}{3} + \frac{4}{9} \cdot \frac{3}{2}\right\} = p_0 \left\{\frac{5}{3} + \frac{2}{3}\right\} = \frac{7}{3}\, p_0,
\end{aligned}$$

that

$$p_0 = \frac{3}{7}, \qquad p_1 = \frac{2}{7}, \qquad p_2 = \frac{4}{21},$$

and

$$p_k = \frac{4}{7} \cdot \left(\frac{1}{3}\right)^{k-1}, \qquad k \geq 3.$$

5) The average number of customers is

$$\begin{aligned}
\sum_{k=1}^{\infty} k p_k &= \frac{2}{7} + \frac{8}{21} + \frac{4}{7} \sum_{k=3}^{\infty} k \cdot \left(\frac{1}{3}\right)^{k-1} = \frac{6+8}{221} + \frac{4}{7} \left\{\sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^{k-1} - 1 - \frac{2}{3}\right\} \\
&= \frac{2}{3} + \frac{4}{7} \left\{\frac{1}{\left(1 - \frac{1}{3}\right)^2} - \frac{5}{3}\right\} = \frac{2}{3} + \frac{4}{7} \left\{\frac{9}{4} - \frac{5}{3}\right\} = \frac{2}{3} + \frac{4}{7} \cdot \frac{27 - 20}{4 \cdot 3} = \frac{2}{3} + \frac{1}{3} = 1.
\end{aligned}$$

Example 4.20 *Given a queueing for which*

a. *there is one channel,*

b. *there is the possibility of an (unlimited) queue,*

c. *the customers arrive according to a Poisson process of intensity* λ*,*

d. *the service times are exponentially distributed of parameter* μ*,*

e. *the traffic intensity* $\dfrac{\lambda}{\mu}$ *is* $\dfrac{4}{5}$.

Let the random variable X *denote the number of customers in the system.*

1. *Find by using the stationary probabilities,*

$$P\{X = k\} \quad \text{and} \quad P\{X > k\}, \qquad k \in \mathbb{N}_0.$$

2. *Find the mean* $E\{X\}$.

We then change the system, such that there is only room for at most 3 waiting customers, thus only room for 4 customers in total in the system (1 being served and 3 waiting). The other conditions are unchanged. This system can be described by a birth and death process of the states E_0, E_1, E_2, E_3, E_4 *and*

$$\text{birth intensities:} \quad \lambda_k = \begin{cases} \lambda, & k = 0,\, 1,\, 2,\, 3, \\ 0, & k = 4, \end{cases}$$

$$\text{death intensities:} \quad \mu_k = \begin{cases} 0, & k = 0, \\ \mu, & k = 1,\, 2,\, 3,\, 4. \end{cases}$$

Let the random variable Y *denote the number of customers in this system.*

3. *Find by means of the stationary probabilities,*

$$P\{Y = k\}, \qquad k = 0,\, 1,\, 2,\, 3,\, 4, \qquad (3 \text{ dec.}).$$

4. *Find the means* $E\{Y\}$ *(3 dec.).*

Now the intensity of arrivals λ *is doubled, while the other assumptions are the same as above. This will imply that the probability of rejection becomes too big, so one decides to hire another shop assistant. Then the system can be described by a birth and death process with states* E_0, E_1, E_2, E_3, E_4, E_5, *(where* E_5 *corresponds to 2 customers being served and 3 waiting).*

5. *Find the equations of this system of the stationary probabilities* p_0, p_1, p_2, p_3, p_4, p_5.

6. *find the stationary probabilities (3 dec.).*

1) We have

$$P\{X = k\} = p_k = \varrho^k(1 - \varrho) = \frac{1}{5}\left(\frac{4}{5}\right)^k, \qquad k \in \mathbb{N}_0,$$

hence

$$P\{X>k\}=\sum_{j=k+1}^{\infty}\frac{1}{5}\left(\frac{4}{5}\right)^j=\frac{1}{5}\cdot\frac{\left(\frac{4}{5}\right)^{k+1}}{1-\frac{4}{5}}=\left(\frac{4}{5}\right)^{k+1},\qquad k\in\mathbb{N}_0.$$

2) The mean is

$$E\{X\}=\frac{1}{5}\cdot\frac{4}{5}\sum_{k=1}^{\infty}k\left(\frac{4}{5}\right)^{k-1}=\frac{4}{25}\cdot\frac{1}{\left(1-\frac{4}{5}\right)^2}=4.$$

3) It follows from

$$\mu_{k+1}p_{k+1}=\lambda_k p_k,$$

that

$$p_1=\frac{\lambda}{\mu}\,p_0=\frac{4}{5}\,p_0,\qquad p_2=\left(\frac{4}{5}\right)^2 p_0,$$

$$p_3=\left(\frac{4}{5}\right)^3 p_0,\qquad p_4=\left(\frac{4}{5}\right)^4 p_0,$$

hence

$$1=p_0\left\{1+\frac{4}{5}+\left(\frac{4}{5}\right)^2+\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^4\right\}=p_0\cdot\frac{1-\left(\frac{4}{5}\right)^5}{1-\frac{4}{5}}=p_0\left\{5-4\cdot\left(\frac{4}{5}\right)^4\right\},$$

and

$$\begin{aligned}
P\{Y=0\} &= p_0 = \frac{1}{5-4\left(\frac{4}{5}\right)^4} &&\approx 0.297,\\
P\{Y=1\} &= p_1 = \frac{4}{5}\,p_0 &&\approx 0.238,\\
P\{Y=2\} &= p_2 = \frac{4}{5}\,p_1 &&\approx 0.190,\\
P\{Y=3\} &= p_3 = \frac{4}{5}\,p_2 &&\approx 0.152,\\
P\{Y=4\} &= p_4 = \frac{4}{5}\,p_3 &&\approx 0.122.
\end{aligned}$$

4) The mean is

$$E\{Y\}=1\cdot p_1+2p_2+3p_3+4p_4=\left\{\frac{4}{5}+2\left(\frac{4}{5}\right)^2+3\left(\frac{4}{5}\right)^3+4\left(\frac{4}{5}\right)^4\right\}p_0\approx 1.563.$$

5) The birth intensities are

$$\lambda_k = \begin{cases} 2\lambda, & k = 0,\, 1,\, 2,\, 3,\, 4, \\ 0, & k = 5, \end{cases}$$

and the death intensities are

$$\mu_k = \begin{cases} 0, & k = 0, \\ \mu, & k = 1, \\ 2\mu, & k = 2,\, 3,\, 4,\, 5. \end{cases}$$

It follows from

$$\mu_{k+1} p_{k+1} = \lambda_k p_k,$$

that

$$p_1 = \frac{2\lambda}{\mu}\, p_0 = \frac{8}{5},$$

and

$$p_k = \frac{2\lambda}{2\mu}\, p_{k-1} = \frac{4}{5}\, p_{k-1} \qquad \text{for } k = 2,\, 3,\, 4,\, 5.$$

6) Now

$$p_k = 2\left(\frac{4}{5}\right)^k p_0 \qquad \text{for } k = 1,\, 2,\, 3,\, 4,\, 5,$$

thus

$$\begin{aligned} 1 &= p_0\left\{1+\frac{8}{5}\left(1+\frac{4}{5}+\left(\frac{4}{5}\right)^2+\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^4\right)\right\} = p_0\left\{1+\frac{8}{5}\cdot\frac{1-\left(\frac{4}{5}\right)^5}{1-\frac{4}{5}}\right\} \\ &= p_0\left\{9-8\left(\frac{4}{5}\right)^5\right\}, \end{aligned}$$

and hence

$$
\begin{aligned}
p_0 &= \frac{1}{9-8\left(\frac{4}{5}\right)^5} &&\approx 0,157, \\
p_1 &= 2\cdot\frac{4}{5}\,p_0 &&\approx 0.251, \\
p_2 &= \frac{4}{5}\,p_1 &&\approx 0.201, \\
p_3 &= \frac{4}{5}\,p_2 &&\approx 0.161, \\
p_4 &= \frac{4}{5}\,p_3 &&\approx 0.128, \\
p_5 &= \frac{4}{5}\,p_4 &&\approx 0.103.
\end{aligned}
$$

Example 4.21 *Given two queueing systems A and B, which are independent of each other. We assume for each of the systems,*

a. *there is one shop assistant,*

b. *It is possible to have a queue,*

c. *customers are arriving to A according a Poisson process of intensity* $\lambda_A = \dfrac{3}{4}$ *minute*$^{-1}$*, and to B according to a Poisson process of intensity* $\lambda_B = \dfrac{1}{2}$ *minute*$^{-1}$*,*

d. *the service times at both A and B are exponentially distributed of parameter* $\mu = 1$ *minute*$^{-1}$*.*

Let the random variable X_A *denote the number of customers in system A, and let* X_B *denote the number of customers in system B.*

1. *Find by means of the stationary probabilities,*

$$P\{X_A = k\} \quad \text{and} \quad P\{X_B = k\}, \qquad k \in \mathbb{N}_0.$$

2. *Find the average waiting times at A and B, resp..*

3. *Compute the probabilities* $P\{X_B > k\}$, $k \in \mathbb{N}_0$*, and then find*

$$P\{X_A < X_B\}.$$

The arrivals of the customers at A is now increased, such that the customers arrive according to a Poisson process of intensity 1 minute^{-1}*. For that reason the two systems are joined to one queueing system with two shop assistants, thus the customers now arrive according to a Poisson process of intensity*

$$\lambda = \left(1 + \frac{1}{2}\right) \text{ minute}^{-1} = \frac{3}{2} \text{ minute}^{-1},$$

and the service times are still exponentially distributed with the parameter

$$\mu = 1 \text{ minut}^{-1}.$$

Let Y denote the number of customers in this new system.

4. *Find by means of the stationary probabilities,*

$$P\{Y = k\}, \qquad k \in \mathbb{N}_0.$$

5. *Prove that the average number of customers in the new system,* $E\{Y\}$*, is smaller than* $E\{X_A + X_B\}$*.*

1A. We get from $\varrho_A = \dfrac{\lambda_A}{\mu} = \dfrac{3}{4}$ and $N = 1$ that

$$P\{X_A = k\} = p_{A,k} = \frac{1}{4}\left(\frac{3}{4}\right)^k, \qquad k \in \mathbb{N}_0.$$

1B. Analogously, $\varrho_B = \frac{1}{2}$, so

$$P\{X_B = k\} = p_{B,k} = \left(\frac{1}{2}\right)^{k+1}, \qquad k \in \mathbb{N}_0.$$

2. Since $N = 1$, the waiting times are

$$V_A = \frac{\varrho_A}{\mu(1-\varrho_A)} = \frac{\frac{3}{4}}{1 \cdot \frac{1}{4}} = 3 \qquad \text{and} \qquad V_B = \frac{\varrho_B}{\mu(1-\varrho_B)} = 1.$$

3. We get

$$P\{X_B > k\} = \sum_{j=k+1}^{\infty} \left(\frac{1}{2}\right)^{j+1} = \frac{\left(\frac{1}{2}\right)^{k+2}}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^{k+1}, \qquad k \in \mathbb{N}_0,$$

so

$$\begin{aligned} P\{X_A < X_B\} &= \sum_{k=0}^{\infty} P\{X_A = k\} \cdot P\{X_B > k\} = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^k \cdot \left(\frac{1}{2}\right)^{k+1} \\ &= \frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{3}{8}\right)^k = \frac{1}{8} \cdot \frac{1}{1-\frac{3}{8}} = \frac{1}{5}. \end{aligned}$$

The new traffic intensity is

$$\varrho = \frac{\lambda}{2\mu} = \frac{\frac{3}{2}}{2 \cdot 1} = \frac{3}{4},$$

and since $N = 2$, we get

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{7}, \qquad p_k = 2\varrho^k \cdot \frac{1-\varrho}{1+\varrho} = \frac{2}{7}\left(\frac{3}{4}\right)^k, \quad k \in \mathbb{N},$$

thus

$$P\{Y = 0\} = \frac{1}{7} \quad \text{and} \quad P\{Y = k\} = \frac{2}{7}\left(\frac{3}{4}\right)^k, \quad k \in \mathbb{N}.$$

Then

$$E\{Y\} = \frac{2}{7} \cdot \frac{3}{4}\sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^{k-1} = \frac{3}{14} \cdot \frac{1}{\left(1-\frac{3}{4}\right)^2} = \frac{3 \cdot 16}{14} = \frac{24}{7},$$

and

$$E\{X_A\} = \frac{1}{4} \cdot \frac{3}{4}\sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^{k-1} = \frac{3}{16} \cdot 16 = 3,$$

and

$$E\{X_B\} = \frac{1}{4}\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \frac{1}{4}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = 1,$$

hence

$$E\{X_A+X_B\} = 3+1 = 4 > \frac{24}{7} = E\{Y\}.$$

Example 4.22 *Given two independent queueing systems A and B, where we assume for each of them,*

a. *there is one shop assistant,*

b. *it is possible to create a queue,*

c. *the customers arrive according to a Poisson process of intensity* $\lambda = \frac{3}{5}$ *min*$^{-1}$*,*

d. *the service times are exponentially distributed of parameter* $\mu = 1$ *min*$^{-1}$*.*

Let the random variable X_A *denote the number of customers in system A, and let* X_B *denote the number of customers in system B, and put* $Z = X_A + X_B$*.*

1. *Compute by means of the stationary probabilities,*

$$P\{X_A = k\} \qquad \text{and} \qquad P\{X_B = k\}, \quad k \in \mathbb{N}_0.$$

2. *Find the means* $E\{X_A\}$*,* $E\{X_B\}$ *and* $E\{Z\}$*.*

3. *Compute* $P\{Z = k\}$*,* $k \in \mathbb{N}_0$*.*

The number of arrivals of customers to A is increased, so the customers are arriving according to a Poisson process of intensity 1 minute^{-1}*. Therefore, the two systems are joined to one system with two shop assistants, so the customers now arrive according to a Poisson process of intensity* $\left(1+\frac{3}{5}\right)$ *minute*$^{-1}$*, and the service times are still exponentially distributed of parameter* $\mu = 1$ *minute*$^{-1}$*.*
Let Y denote the number of customers in this system.

4. *Compute by means of the stationary probabilities,*

$$P\{Y = k\} \qquad \text{and} \qquad P\{Y > k\}, \quad k \in \mathbb{N}_0.$$

5. *Find the mean* $E\{Y\}$*.*

1) The traffic intensities are

$$\varrho_A = \varrho_B = \frac{\lambda}{N\cdot\mu} = \frac{3}{5},$$

and since $N = 1$, we get

$$P\{X_A = k\} = P\{X_B = k\} = \frac{2}{5}\left(\frac{3}{5}\right)^k, \qquad k \in \mathbb{N}_0.$$

2) The means are

$$E\left\{X_A\right\}=E\left\{X_B\right\}=\frac{2}{5}\cdot\frac{3}{5}\sum_{k=1}^{\infty}k\left(\frac{3}{5}\right)^{k-1}=\frac{6}{25}\cdot\frac{1}{\left(1-\frac{3}{5}\right)^2}=\frac{6}{4}=\frac{3}{2},$$

thus

$$E\{Z\}=E\left\{X_A\right\}+E\left\{X_B\right\}=3.$$

3) The probabilities are

$$\begin{aligned}P\{Z=k\} &= \sum_{j=0}^{k}P\left\{X_A=j\right\}\cdot P\left\{X_B=k-j\right\}=\sum_{j=0}^{k}\frac{2}{5}\left(\frac{3}{5}\right)^j\cdot\frac{2}{5}\left(\frac{3}{5}\right)^{k-j}\\ &= \frac{4}{25}\left(k+1\right)\left(\frac{3}{5}\right)^k, \qquad k\in\mathbb{N}_0.\end{aligned}$$

4) The traffic intensity of the new system is

$$\varrho = \frac{\lambda}{N \cdot \mu} = \frac{1 + \frac{3}{5}}{2 \cdot 1} = \frac{4}{5},$$

and since $N = 2$, we get

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{9} \quad \text{and} \quad p_k = 2\varrho^k \, \frac{1-\varrho}{1+\varrho} = \frac{2}{9}\left(\frac{4}{5}\right)^k, \quad k \in \mathbb{N}.$$

Thus

$$P\{Y=0\} = \frac{1}{9} \quad \text{and} \quad P\{Y=k\} = \frac{2}{9}\left(\frac{4}{5}\right)^k, \quad k \in \mathbb{N},$$

and hence

$$P\{Y > k\} = \sum_{j=k+1}^{\infty} P\{Y=j\} = \frac{2}{9} \sum_{j=k+1}^{\infty} \left(\frac{4}{5}\right)^j = \frac{2}{9} \cdot \frac{\left(\frac{4}{5}\right)^{k+1}}{1 - \frac{4}{5}} = \frac{8}{9}\left(\frac{4}{5}\right)^k.$$

5) The mean is

$$E\{Y\} = \frac{2}{9} \cdot \frac{4}{5} \sum_{k=1}^{\infty} k \left(\frac{4}{5}\right)^{k-1} = \frac{8}{45} \cdot \frac{1}{\left(1-\frac{4}{5}\right)^2} = \frac{40}{9}.$$

Example 4.23 *Given a queueing system, for which*

a. *There are two shop assistants.*

b. *The customers arrive according to a Poisson process of intensity $\lambda = 3$ min^{-1}.*

c. *The service times are exponentially distributed of parameter $\mu = 2$ min^{-1}.*

d. *It is possible to queue up.*

1. *Find the stationary probabilities.*

2. *Find by means of the stationary probabilities the probability that we have more than two customers in the shop.*

3. *Find by means of the stationary probabilities the average length of the queue.*

Then chance the system, such that it becomes a rejection system, while the other assumptions **a.–c.** *are unchanged.*

4. *Find the probability of rejection of this system.*

1) We get from

$$\lambda = 3, \qquad \mu = 2 \quad \text{and} \quad N = 2,$$

that the traffic intensity is

$$\varrho = \frac{\lambda}{N \cdot \mu} = \frac{3}{2 \cdot 2} = \frac{3}{4}.$$

From $N = 2$ we find the p_k by a known formula,.

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{7} \quad \text{and} \quad p_k = 2\varrho^k \cdot \frac{1-\varrho}{1+\varrho} = \frac{2}{7} \cdot \left(\frac{3}{4}\right)^k, \qquad k \in \mathbb{N}.$$

In particular,

$$p_1 = \frac{2}{7} \cdot \frac{3}{4} = \frac{3}{14} \qquad \text{and} \qquad p_2 = \frac{2}{7} \cdot \frac{9}{16} = \frac{9}{56}.$$

2) The probability that there are more than two customers in the shop is

$$\sum_{k=3}^{\infty} p_k = 1 - p_0 - p_1 - p_2 = 1 - \frac{8+12+9}{56} = 1 - \frac{29}{56} = \frac{27}{56}.$$

ALTERNATIVELY,

$$\sum_{k=3}^{\infty} p_k = \frac{2}{7} \sum_{k=3}^{\infty} \left(\frac{3}{4}\right)^k = \frac{2}{7} \left(\frac{3}{4}\right)^3 \cdot \frac{1}{1-\frac{3}{4}} = \frac{2}{7} \cdot \frac{3 \cdot 3 \cdot 3 \cdot 4}{4 \cdot 4 \cdot 4} = \frac{27}{56}.$$

3) The average length of the queue is again given by a known formula,

$$\sum_{k=3}^{\infty}(k-2)p_k = \frac{2}{7}\cdot\left(\frac{3}{4}\right)^3\sum_{k=3}^{\infty}(k-2)\left(\frac{3}{4}\right)^{k-3} = \frac{2}{7}\cdot\left(\frac{3}{4}\right)^3\cdot\frac{1}{\left(1-\frac{3}{4}\right)^2} = \frac{27}{14}.$$

4) The probability of rejection is p_2, because $N = 2$. It is given by some known formula in any textbook,

$$p_2 = \frac{\left(\frac{3}{2}\right)^2\cdot\frac{1}{2!}}{\sum_{j=0}^{2}\frac{1}{j!}\left(\frac{3}{2}\right)^j} = \frac{\frac{9}{8}}{1+\frac{3}{2}+\frac{9}{8}} = \frac{9}{8+12+9} = \frac{9}{29}.$$

Example 4.24 *Given a queueing system, for which*

a. *There are two shop assistants.*

b. *The customers arrive according to a Poisson process of intensity* $\lambda = 5\ \left(quarter^{-1}\right)$.

c. *The service times are exponentially distributed of parameter* $\mu = 3\ \left(quarter^{-1}\right)$.

d. *It is possible for queue up.*

1. *Prove that the stationary probabilities are given by*

$$p_k = \begin{cases} \dfrac{1}{11}, & k = 0, \\[2mm] \dfrac{2}{11}\left(\dfrac{5}{6}\right)^k, & k > 0. \end{cases}$$

2. *Find by means of the stationary probabilities the average waiting time.*

3. *Find by means of the stationary probabilities the average length of the queue.*

Then the service is rationalized, such that the average service time is halved. At the same time one removes one of the shop assistants for other work in the shop.

4. *Check if the average waiting time is bigger or smaller in the new system than in the old system.*

1) It follows from $N = 2$, $\lambda = 5$ and $\mu = 3$ that the traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{5}{2\cdot 3} = \frac{5}{6}.$$

Since $N = 2$, we may use a known formula, so

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{11} \quad\text{and}\quad p_k = 2\varrho^k p_0 = \frac{1}{11}\left(\frac{5}{6}\right)^k,$$

and hence

$$p_k = \begin{cases} \dfrac{1}{11}, & k = 0, \\[2mm] \dfrac{2}{11} \cdot \left(\dfrac{5}{6}\right)^k, & k \in \mathbb{N}. \end{cases}$$

2) The average waiting time V is again found by some known formula,

$$V = \frac{p_0 \varrho^N \cdot N^{N-1}}{\mu \cdot N!(1-\varrho)^2} = \frac{\dfrac{1}{11} \cdot \left(\dfrac{5}{6}\right)^2 \cdot 2^1}{3 \cdot 2 \cdot \left(\dfrac{1}{6}\right)^2} = \frac{5^2 \cdot 2}{11 \cdot 3 \cdot 2} = \frac{25}{33} \text{ quarter.}$$

3) Also the average length of the queue is found by a given formula,

$$\begin{aligned} \sum_{k=3}^{\infty}(k-2)p_k &= \sum_{k=3}^{\infty}(k-2)\left(\frac{5}{6}\right)^k \cdot \frac{2}{11} = \frac{2}{11} \cdot \left(\frac{5}{6}\right)^3 \sum_{k=1}^{\infty} k \left(\frac{5}{6}\right)^{k-1} \\ &= \frac{2}{11} \cdot \left(\frac{5}{6}\right)^3 \cdot \frac{1}{\left(1-\dfrac{5}{6}\right)^2} = \frac{2 \cdot 5^3}{11 \cdot 6} = \frac{125}{33} \quad (= \lambda V). \end{aligned}$$

4) We have in the new system that $N = 1$, $\lambda = 5$, $\mu = 6$ and $\varrho = \dfrac{5}{6}$.
Then the average waiting time is because $N = 1$ given by a known formula,

$$V = \frac{\varrho}{\mu(1-\varrho)} = \frac{\frac{5}{6}}{6 \cdot \frac{1}{5}} = \frac{5}{6} \text{ quarter.}$$

It is seen that the average waiting time is larger in the new system than in the old one.

Example 4.25 *Given a queueing system, for which*

a. *There are two shop assistants.*

b. *The customers arrive according to a Poisson process of intensity* $\lambda = 8 \left(quarter^{-1}\right)$.

c. *The service times are exponentially distributed of parameter* $\mu = 6 \left(quarter^{-1}\right)$.

d. *It is possible to queue up.*

1. *Prove that the stationary probabilities are given by*

$$P_k = \begin{cases} \dfrac{1}{5}, & k = 0, \\[2ex] \dfrac{2}{5}\left(\dfrac{2}{3}\right)^k, & k \in \mathbb{N}. \end{cases}$$

2. *Find by means of the stationary probabilities the average number of customers in the shop.*

3. *Find by means of the stationary probabilities the average waiting time.*

4. *Find by means of the stationary probabilities the probability that both shop assistants are busy.*

5. *Find the median in the stationary distribution.*

1) The traffic intensity is

$$\varrho = \frac{\lambda}{N\mu} = \frac{8}{2 \cdot 6} = \frac{2}{3}.$$

Then by a known formula,

$$p_0 = \frac{1-\varrho}{1+\varrho} = \frac{1}{5}, \qquad p_k = 2\varrho^k p_0 = \frac{2}{5}\left(\frac{2}{3}\right)^k, \qquad k \in \mathbb{N}.$$

2) By computing the mean it follows that the average number of customers is

$$\sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} k \cdot \frac{2}{5}\left(\frac{2}{3}\right)^k = \frac{2}{5} \cdot \frac{2}{3} \sum_{k=1}^{\infty} k \cdot \left(\frac{2}{3}\right)^{k-1} = \frac{4}{15} \cdot \frac{1}{\left(1-\frac{2}{3}\right)^2} = \frac{4}{15} \cdot \frac{9}{1} = \frac{12}{5}.$$

3) The average waiting time is also found by a standard formula,

$$V=\frac{p_0\varrho^2\cdot 2}{\mu\cdot 2\cdot(1-\varrho)^2}=\frac{\frac{1}{5}\cdot\left(\frac{2}{3}\right)^2\cdot 2}{6\cdot 2\cdot\left(\frac{1}{3}\right)^2}=\frac{4}{5\cdot 6}=\frac{2}{15}\text{ quarter}\quad(=2\text{ minutes}).$$

Supplement. The average length of queue is also easily found by some known formula,

$$\begin{aligned}\sum_{k=3}^{\infty}(k-2)p_k &= \sum_{k=3}^{\infty}(k-2)\cdot\frac{2}{5}\cdot\left(\frac{2}{3}\right)^k=\sum_{\ell=1}^{\infty}\ell\left(\frac{2}{3}\right)^{\ell-1}\cdot\frac{2}{5}\left(\frac{2}{3}\right)^3=\frac{1}{\left(1-\frac{2}{3}\right)^2}\cdot\frac{2}{5}\cdot\left(\frac{2}{3}\right)^3\\ &= \frac{2\cdot 2^3\cdot 5^2}{5\cdot 3^3}=\frac{16}{15}=\lambda V=8\cdot\frac{2}{15}.\end{aligned}$$

4) The complementary event: Both shop assistants are busy with the probability

$$1-(p_0+p_1)=1-\left(\frac{1}{5}+\frac{4}{14}\right)=1-\frac{7}{15}=\frac{8}{15}.$$

Alternatively, the probability is given by

$$\sum_{k=2}^{\infty}p_k=\sum_{k=2}^{\infty}\frac{2}{5}\left(\frac{2}{3}\right)^k=\frac{2}{5}\cdot\left(\frac{2}{3}\right)^3\cdot 3=\frac{8}{15}.$$

5) The distribution is discrete, and

$$\sum_{k=2}^{\infty}p_k=\frac{8}{15}>\frac{1}{2},$$

cf. **4.**. Thus

$$p_0=\frac{1}{5},\qquad p_1=\frac{4}{15},\qquad p_2=\frac{8}{45}.$$

Finally,

$$P\{X\geq 2\}=\sum_{k=2}^{\infty}p_k=\frac{8}{15}>\frac{1}{2},$$

and

$$P\{X\leq 2\}=p_0+p_1+p_2=\frac{1}{5}+\frac{4}{15}+\frac{8}{45}=\frac{9+12+8}{45}=\frac{29}{45}>\frac{1}{2}.$$

Since both probabilities are $\geq\frac{1}{2}$, the median is $(X)=2$.

5 Other types of stochastic processes

Example 5.1 *An aeroplane has 4 engines (2 on each wing), and it can carry through a flight if just 1 motor from each wing is working. At start ($t = 0$) all 4 engines are intact, but they may break down during the flight. We assume (as a crude approximation) that the operating times of the 4 engines are mutually independent and exponentially distributed of mean $\frac{1}{\lambda}$ (which hopefully is much larger than the flight time). The system can be described as a Markov process of 4 states:*

E_4**:** *all 4 engines are working,*

E_3**:** *3 engines are working,*

E_2**:** *1 engine in each wing is working,*

E_1**:** *the aeroplane has crashed.*

1. *Derive the system of differential equations of the probabilities*

$$P_i(t) = P\{\text{the process is in state } E_i \text{ at time } t\}, \qquad i = 1,\, 2,\, 3,\, 4.$$

(Notice that this is not *a birth and death process, because the probability of transition from E_3 to E_1 in a small time interval of length h is almost proportional to h.)*

2. *Find $P_i(t)$, $i = 1$, 2, 3, 4.*

1) It follows from the diagram

$$E_4 \quad \stackrel{4\lambda}{\longrightarrow} \quad E_3 \quad \stackrel{2\lambda}{\longrightarrow} \quad E_2 \quad \stackrel{2\lambda}{\longrightarrow} \quad E_1$$

$$E_3 \qquad\qquad \stackrel{\lambda}{\to} \qquad\qquad E_1$$

that we have the conditions

$$\begin{aligned}
P_4(t+h) &= (1-4\lambda h)P_4(t) + h\varepsilon(h),\\
P_3(t+4) &= (1-3\lambda h)P_3(t) + 4\lambda h P_4(t) + h\varepsilon(h),\\
P_2(t+h) &= (1-2\lambda h)P_2(t) + 2\lambda h P_3(t) + h\varepsilon(h),\\
P_1(t+h) &= P(t) + 2\lambda h P_2(t) + \lambda h P_3(t) + h\varepsilon(h),
\end{aligned}$$

hence by a rearrangement and taking the limit $h \to 0$ we get the system of differential equations,

$$\begin{cases}
P_4'(t) = -4\lambda P_4(t), & P_4(0) = 1,\\[2mm]
P_3'(t) = -3\lambda P_3(t) + 4\lambda P_4(t), & P_3(0) = 0,\\[2mm]
P_2'(t) = -2\lambda P_2(t) + 2\lambda P_3(t), & P_2(0) = 0,\\[2mm]
P_1'(t) = 2\lambda P_2(t) + \lambda P_3(t), & P_1(0) = 0.
\end{cases}$$

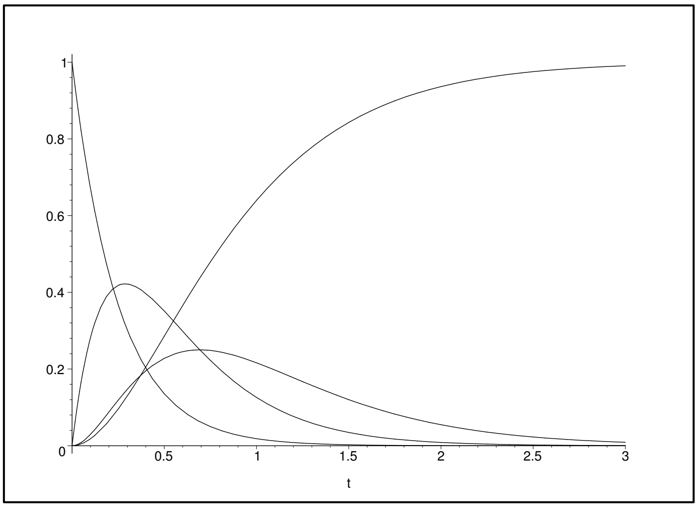


Figure 3: The graphs of $P_1(t), \dots, P_4(t)$ for $\lambda = 1$.

2) It follows immediately that

$$P_4(t) = e^{-4\lambda t}.$$

By insertion into the next differential equation we get

$$P_3'(t) + 3\lambda P_3(t) = 4\lambda e^{-4\lambda t},$$

hence

$$P_3(t) = e^{-3\lambda t}\int_0^t e^{3\lambda\tau}\cdot 4\lambda e^{-4\lambda\tau}\,d\tau = e^{-3\lambda t}\int_0^t 4\lambda e^{-\lambda\tau}\,d\tau = e^{-3\lambda t}\left(4-4e^{-\lambda t}\right) = 4e^{-3\lambda t} - 4e^{-4\lambda t}.$$

Then by insertion into the next equation and a rearrangement,

$$P_2'(t) + 2\lambda P_2(t) = 8\lambda e^{-3\lambda t} - 8\lambda e^{-4\lambda t},$$

the solution of which is

$$\begin{aligned} P_2(t) &= e^{-2\lambda t}\int_0^t e^{2\lambda\tau}\left\{8\lambda e^{-3\lambda\tau} - 8\lambda e^{-4\lambda\tau}\right\}d\tau = e^{-2\lambda t}\int_0^t \left\{8\lambda e^{-\lambda\tau} - 8\lambda e^{-2\lambda\tau}\right\}d\tau \\ &= e^{-2\lambda t}\left(4 - 8e^{-\lambda t} + 4e^{-\lambda t}\right) = 4e^{-2\lambda t} - 8e^{-3\lambda t} + 4e^{-4\lambda t}. \end{aligned}$$

Finally, $P_1(t)$ is found from the condition

$$\sum_{k=1}^4 P_k(t) = 1, \qquad \text{thus} \qquad P_1(t) = 1 - P_2(t) - P_3(t) - P_4(t),$$

and we get summing up,

$$\begin{aligned} P_4(t) &= e^{-4\lambda t}, \\ P_3(t) &= 4e^{-3\lambda t} - 4e^{-4\lambda t}, \\ P_2(t) &= 4e^{-2\lambda t} - 8e^{-3\lambda t} + 4e^{-4\lambda t}, \\ P_1(t) &= 1 - 4e^{-2\lambda t} + 4e^{-3\lambda t} - e^{-4\lambda t}. \end{aligned}$$

Example 5.2 *Let Y and Z be independent $N(0,1)$ distributed random variables, and let the process $\{X(t),\, t \in \mathbb{R}\}$ be defined by*

$$X(t) = Y\,\cos t + Z\,\sin t.$$

Find the mean value function $m(t)$ and the autocorrelation $R(s,t)$.

The mean value function is

$$m(t) = E\{X(t)\} = E\{Y\,\cos t\} + E\{Z\,\sin t\} = \cos t \cdot E\{Y\} + \sin t \cdot E\{Z\} = 0.$$

The autocorrelation is

$$\begin{aligned}
R(s,t) &= E\{X(s)X(t)\} = E\{(Y\,\cos s + Z\,\sin s)(Y\,\cos t + Z\,\sin t)\} \\
&= \cos s \cdot \cos t \cdot E\left\{Y^2\right\} + \sin s \cdot \sin t \cdot E\left\{Z^2\right\} + (\cos s \cdot \sin t + \sin s \cdot \cos t)E\{YZ\} \\
&= \cos s \cdot \cos t \cdot E\left\{Y^2\right\} + \sin s \cdot \sin t \cdot E\{Y^2\} + 0\left(E\left\{Z^2\right\} = E\left\{Y^2\right\}\right) \\
&= \cos(s-t)\,\left(V\{Y\} + (E\{Y\})^2\right) = \cos(s-t).
\end{aligned}$$

Example 5.3 *Let $\{X(t),\, t \geq 0\}$ denote a Poisson process of intensity a, and let $\{Y(t),\, t \geq 0\}$ be given by*

$$Y(t) = X(t+1) - X(t).$$

Compute the mean value function and the autocovariance of $\{Y(t),\, t \geq 0\}$.

We have

$$P\{X(t) = n\} = \frac{(at)^n}{n!}\, e^{-at}, \qquad n \in \mathbb{N}_0.$$

The mean value function is obtained by first noticing that

$$P\{T(t) = n\} = P\{X(t+1) - X(t) = n\} = P\{X(1) = n\} = \frac{a^n}{n!}\, e^{-a},$$

thus $Y(t) = X(1)$, (The Poisson process is "forgetful") and

$$m(t) = E\{Y(t)\} = \sum_{n=1}^{\infty} n\, \frac{a^n}{n!}\, e^{-\lambda} = a.$$

If $s \leq t$, then

$$\begin{aligned} \operatorname{Cov}(Y(s), Y(t)) &= \operatorname{Cov}(X(s+1) - X(s), X(t+1) - X(t)) = a \cdot (s+1-\min\{s+1,t\} - s + s) \\ &= a\,(s+1-\min\{s+1,t\}). \end{aligned}$$

If therefore $s + 1 \leq t$, then

$$\operatorname{Cov}(Y(s), Y(t)) = 0,$$

and if $s + 1 > 1$, then

$$\operatorname{Cov}(Y(s), T(t)) = a\{s+1-t\}.$$

Summing up,

$$\operatorname{Cov}(Y(s), Y(t)) = \begin{cases} a\{1 - |s-t|\}, & \text{for } |s-t| < 1, \\ 0, & \text{for } |s-t| \geq 1. \end{cases}$$

Example 5.4 *Let X_1 and X_2 be independent random variables, both normally distributed of mean 0 and variance σ^2. We define a stochastic process $\{X(t),\, t \in \mathbb{R}\}$ by*

$$X(t) = X_1 \sin t + X_2 \cos t.$$

1) Find the mean value function $m(t)$ and the autocorrelation $R(s,t)$.

2) Prove that the process $\{X(t),\, t \in \mathbb{R}\}$ is weakly stationary.

3) Find the values of $s-t$, for which the random variables $X(s)$ and $X(t)$ are non-correlated.

4) Given the random variables $X(s)$ and $X(t)$, where $s-t$ is fixed as above. Are $X(s)$ and $X(t)$ independent?

1) The mean value function is

$$m(t) = E\{X(t)\} = \sin t \cdot E\{X_1\} + \cos t \cdot E\{X_2\} = 0.$$

The autocorrelation is

$$\begin{aligned} R(s,t) &= E\{X(s)X(t)\} = E\{(X_1 \sin s + X_2 \cos s)(X_1 \sin t + X_2 \cos t)\} \\ &= \sin s \cdot \sin t \cdot E\left\{X_1^2\right\} + \cos s \cdot \cos t \cdot E\left\{X_2^2\right) + (\cdots) \cdot E\{X_1 X_2\} \\ &= \sin s \cdot \sin t \left(V\{X_1\} + \left(E\left\{X_1^2\right\}\right)\right) + \cos s \cdot \cos t \left(V\{X_2\} + E\left\{X_2^2\right\}\right) + 0 \\ &= (\cos s \cdot \cos t + \sin s \cdot \sin t)\sigma^2 = \cos(s-t) \cdot \sigma^2. \end{aligned}$$

2) A stochastic process is weakly stationary, if $m(t) = m$ is constant, and $C(s,t) = C(s-t)$. In the specific case,

$$m(t) = 0 = m,$$

and

$$\begin{aligned} C(s,t) &= \operatorname{Cov}\{X(s), X(t)\} = E\{X(s)X(t)\} - E\{X(s)\} \cdot E\{X(t)\} \\ &= R(s,t) - m(s)m(t) = \sigma^2 \cos(s-t), \end{aligned}$$

and we have proved that the process is weakly stationary.

3) It follows from

$$\operatorname{Cov}\{X(s), X(t)\} = C(s,t) = \sigma^2 \cos(s-t),$$

that $X(s)$ and $X(t)$ are non-correlated, if

$$s = t + \frac{\pi}{2} + p\pi, \qquad p \in \mathbb{Z},$$

i.e. if

$$s - t = \frac{\pi}{2} + p\pi, \qquad p \in \mathbb{Z}.$$

4) Since $(X(s), X(t))$ with $s - t = \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$, follows a two-dimensional normal distribution, and $X(s)$ and $X(t)$ are non-correlated, we conclude that they are independent.

Example 5.5 *Let $\{X(t),\, t \in \mathbb{R}\}$ be a stationary process of mean 0, autocorrelation $R(\tau)$ and effect spectrum $S(\omega)$.*
Let $\{Y(t),\, t \in \mathbb{R}\}$ be defined by

$$Y(t) = X(t+a) - X(t-a), \qquad \textit{where } a > 0.$$

Express the autocorrelation and the effect spectrum of $\{Y(t)\}$ by the corresponding expressions of $\{X(t)\}$ (and a).

The assumptions are

$$m(t) = 0, \qquad R(\tau) = E\{X(t+\tau)X(t)\} \quad \text{and} \quad S(\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau} R(\tau)\, d\tau.$$

Hence for $Y(t) = X(t+a) - X(t-a)$, $a > 0$,

$$\begin{aligned}
R_Y(\tau) &= E\{Y(t+\tau)Y(t)\} = E\{[X(t+\tau+a) - X(t+\tau-a)] \cdot [X(t+a) - X(t-a)]\} \\
&= E\{X(t+\tau+a)X(t+a)\} - E\{X(t+\tau+a)X(t-a)\} \\
&\qquad -E\{X(t+\tau-a)X(t+a)\} + E\{X(t+\tau-a)X(t-a)\} \\
&= R_X(\tau) - R_X(\tau+2a) - R_X(\tau-2a) + R_X(\tau) \\
&= 2R_X(\tau) - R_X(\tau+2a) - R_X(\tau-2a),
\end{aligned}$$

so

$$\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty} e^{i\omega\tau} R_Y(\tau)\, d\tau \\
&= 2\int_{-\infty}^{\infty} e^{i\omega\tau} R_X(\tau)\, d\tau - \int_{-\infty}^{\infty} e^{i\omega\tau} R_X(\tau+2a)\, d\tau - \int_{-\infty}^{\infty} e^{i\omega\tau} R_X(\tau-2a)\, d\tau \\
&= 2S_X(\omega) - e^{-2ia\omega} S_X(\omega) - e^{2ia\omega} S_X(\omega) = 2\{1 - \cos 2a\omega\} S_X(\omega) \\
&= 4\sin^2 a\omega\, S_X(\omega).
\end{aligned}$$

Example 5.6 *Let $\{X(t),\, t \in \mathbb{R}\}$ be a stationary process of mean 0 and effect spectrum $S(\omega)$, and let*

$$Y = \frac{1}{n} \sum_{k=1}^{n} X(kT), \qquad \textit{hvor } T > 0.$$

Prove that

$$E\left\{Y^2\right\} = \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} S(\omega) \cdot \frac{\sin^2\left(\frac{1}{2}\, n\omega T\right)}{\sin^2\left(\frac{1}{2}\,\omega T\right)}\, d\omega.$$

HINT:

$$\frac{\sin^2\left(\frac{1}{2}\, n\omega T\right)}{\sin^2\left(\frac{1}{2}\,\omega T\right)} = \sum_{m=-(n-1)}^{n-1} (n - |m|) e^{-i\omega m T}.$$

First compute

$$\begin{aligned}
E\left\{Y^2\right\} &= \frac{1}{n^2}\, E\left\{ \sum_{k=1}^{n} \sum_{m=1}^{n} X(kT)X(mT) \right\} \\
&= \frac{1}{n^2}\, E\left\{ \sum_{k=1}^{n} X(kT)X(kT) + 2 \sum_{k=1}^{n-1} \sum_{m=k+1}^{n} X(kT)X(mT) \right\} \\
&= \frac{1}{n^2} \sum_{k=1}^{n} R(0) + \frac{2}{n^2} \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} E\{X(kT)X((k+m)T)\} \\
&= \frac{n}{n^2}\, R(0) + \frac{2}{n^2} \sum_{k=1}^{n-1} \sum_{m=1}^{n-k} R(mT) = \frac{n}{n^2}\, R(0) + \frac{2}{n^2} \sum_{m=1}^{n-1} \sum_{k=1}^{n-m} R(mT) \\
&= \frac{n}{n^2}\, R(0) + \frac{2}{n^2} \sum_{m=1}^{n-1} (n-m) R(mT) = \frac{1}{n^2} \sum_{m=-(n-1)}^{n-1} (n - |m|) R(|m|T).
\end{aligned}$$

Using

$$R(-mT) = E\{X(kT)X((k-m)T)\} = E\{X(kT)X((k+m)T)\} = R(mT),$$

and the hint and the inversion formula we get

$$\begin{aligned}
E\left\{Y^2\right\} &= \frac{1}{n^2} \sum_{m=-(n-1)}^{n-1} (n - |m|) R(mT) = \frac{1}{n^2} \sum_{m=-(n-1)}^{n-1} (n - |m|) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-im\omega T} S(\omega)\, d\omega \\
&= \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} S(\omega) \sum_{m=-(n-1)}^{n-1} (n - |m|) e^{-i\omega m T}\, d\omega = \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} S(\omega) \cdot \frac{\sin^2\left(\frac{1}{2}\, n\omega T\right)}{\sin^2\left(\frac{1}{2}\,\omega T\right)}\, d\omega,
\end{aligned}$$

and the formula is proved.

Example 5.7 *Let $\{W(t),\, t \geq 0\}$ be a Wiener process..*

1) Find the autocorrelation $R(s,t)$ and the autocovariance $C(s,t)$, $s,\, t \in \mathbb{R}_+$.

2) Let $0 < s < t$. Find the simultaneous frequency of the two-dimensional random variable $\{W(s), W(t)\}$.

The Wiener process is a normal process $\{W(t),\, t \geq 0\}$ with

$$W(0) = 0, \qquad m(t) = 0, \qquad V\{W(t)\} = \alpha\, t \quad (\alpha > 0),$$

and of independent increments. It follows from $m(t) = 0$ that

$$C(s,t) = \operatorname{Cov}\{W(s), W(t)\} = R(s,t) - m(s)m(t) = R(s,t).$$

1) If $0 < s < t$, then

$$\begin{aligned} R(s,t) &= C(s,t) = \operatorname{Cov}\{W(s), W(t)\} = \operatorname{Cov}\{W(s), W(s) + [W(t) - W(s)]\} \\ &= \operatorname{Cov}\{W(s), W(s)\} + \operatorname{Cov}\{W(s), W(t) - W(s)\} \\ &= V\{W(s)\} + 0 \qquad\qquad \text{(independent increments)} \\ &= \alpha \cdot s. \end{aligned}$$

Analogously, $R(s,t) = C(s,t) = \alpha \cdot t$, if $0 < t < s$, thus

$$R(s,t) = C(s,t) = \alpha \cdot \min\{s,t\} = \begin{cases} \alpha s, & \text{if } 0 < s < t, \\ \alpha t, & \text{if } 0 < t < s. \end{cases}$$

2) If $0 < s < t$, then $(W(s), W(t) - W(s))$ has the simultaneous frequency

$$f(x,y) = \frac{1}{\sqrt{2\pi\alpha s}} \exp\left(-\frac{1}{2}\frac{x^2}{\alpha s}\right) \cdot \frac{1}{\sqrt{2\pi\alpha(t-s)}} \exp\left(-\frac{1}{2}\frac{y^2}{\alpha(t-s)}\right)$$

for $(x,y) \in \mathbb{R}^2$. Finally, it follows that

$$(W(s), W(t)) = (W(s), \{W(t) - W(s)\} + W(s))$$

has the frequency

$$g(x,y) = f(x, y-x) = \frac{1}{2\pi\alpha\sqrt{s(t-s)}} \exp\left(-\frac{1}{2}\left\{\frac{x^2}{\alpha s} + \frac{(y-x)^2}{\alpha(t-s)}\right\}\right), \qquad (x,y) \in \mathbb{R}^2.$$

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