Real Functions in Several Variables: Volume VIII

Line Integrals and Surface Integrals

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Real Functions in Several Variables: Volume VIII Line Integrals and Surface Integrals
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Contents

Preface

The topic of this series of books on *"Real Functions in Several Variables"* is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of $\mathbb{R}^3$ to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to $\mathbb{R}^3$ alone. Some motions may be rectilinear, so only $\mathbb{R}$ is needed to describe their movements on a line segment. This opens up for also dealing with $\mathbb{R}^2$, when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use $\mathbb{R}^n$ as our abstract model, and then restrict ourselves in examples mainly to $\mathbb{R}^2$ and $\mathbb{R}^3$.

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in $\mathbb{R}^2$ the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to $\mathbb{R}^3$. Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n-dimensional Euclidean space E^n by $\mathbb{R}^n$. There is a subtle difference between E^n and $\mathbb{R}^n$, although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space $\mathbb{R}^n$ we can use ordinary calculus, which in principle is not possible in E^n. In order to stress this point, we call E^n the "abstract space" (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the "abstract space", in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of $\mathbb{R}$ and $\mathbb{R}^2$, resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in $\mathbb{R}^n$. This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral.*

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I**. *Implementation.* However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

Introduction to volume VIII, The line integral and the surface integral

This is the eighth volume in the series of books on *Real Functions in Several Variables.*

We investigate in Chapter 26 the line integrals in space, i.e. there is given a curve $\mathcal{K}$ in space and a continuous function $f : \mathcal{K} \to \mathbb{R}$. For mathematical reasons we assume that f is continuous on a slightly larger closed and bounded set A, which contains $\mathcal{K}$. Then we define the line integral

$$\int_{\mathcal{K}} f \, \mathrm{d}s,$$

and set up some theorems and procedures of how to calculate the actual value of this symbol. We can visualize this by attaching to each point of the curve the value of f at this point as a density, and then we stretch out the curve and lie it down following the x-axis in the plane. This gives us a function $\tilde{f} : B \to \mathbb{R}$, where $B \subset RR$, which can be integrated in the well-known way. The price of this straightening out the curve is – not surprisingly – a weight function, which is added as a factor to the integrand. This weight function is specified by the curve. It usually contains a square root, which means that applications of mathematical programs like MAPLE and MATHEMATICA may be more difficult, and one should use special packages. Since the examples have been designed as simple as possible, such that one can calculate everything by hand, we have not put much labour into MAPLE in this volume. One always first has to perform a geometrical analysis, before one can set up the integral, and before one can apply MAPLE, and then it does not make sense to emphasize the use of MAPLE, because it is not in focus here.

In Chapter 27 we go a step further, by replacing the line above by a surface $\mathcal{F}$ in $\mathbb{R}^3$. If f is a continuous function defined on $\mathcal{F}$ (or on a slightly larger set), then we introduce the *surface integral*

$$\int_{\mathcal{F}} f \, \mathrm{d}S,$$

and analyze the area element $\mathrm{d}S$ in order to obtain a reduction formula to e.g. a double integral in the parameter domain $E \subset \mathbb{R}^2$. Again the prize for this straightening out the surface to a subset of the plane is the introduction of a weight function as a factor. Given describing parameters (u, v) of the surface these define two systems of parameter curves, by which we construct a field of normal vectors to the surface. The weight function at a point is then the length of the corresponding normal vector. Since the length in $\mathbb{R}^3$ always involves a square root, we are in the same situation as above for the line integral. Programs like MAPLE do not like these problems, unless we use some special extra packages. In order not to focus too much on MAPLE problems we shall only occasionally apply MAPLE.

26 The line integral

26.1 Introduction

The idea of replacing an abstract integral over a set by an ordinary rectangular integration over some parameter space at the cost of adding a weight function as a factor to the integrand can also be applied in other situations.

We start with the 1-dimensional case, i.e. consider a C^1-curve $\mathcal{K}$ in the general space $\mathbb{R}^m$. We shall in this Chapter 26 explain the *line integral*, notated by

$$\int_{\mathcal{C}} f(\mathbf{x}) \, \mathrm{d}s.$$

We shall find the weight function in the chosen parameter domain, when the type of coordinate system has been chosen.

This idea is then extended in Chapter 27 to surfaces in two dimensions, i.e. we want to define the abstract *surface integral*

$$\int_{\mathcal{F}} f(\mathbf{x}) \, \mathrm{d}S,$$

where $\mathcal{S}$ is some 2-dimensional C^1-surface, typically in $\mathbb{R}^3$, but higher dimensional spaces $\mathbb{R}^m$ are not excluded. We find in the chosen parameters (u,v) used in the description of the surface the relevant weight function, which is used in the reduction theorem.

More general, we consider in Chapter 28 the problem of how to change variables in the previously considered plane and space integrals. The structure is the same as above. We first find the relevant weight function and then integrate in the new variables.

Finally, we add for completeness Chapter 29 on improper integrals, where the domain is either not bounded or closed.

26.2 Reduction theorem of fhe line integral

Let $\mathcal{C}$ be a C^1-curve of parametric description

$$\mathcal{C}: \quad \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{x} = \mathbf{r}(t), \, t \in [a,b]\}, \qquad \mathbf{r} \in C^1.$$

Any $m \in \mathbb{N}$ is possible here, but we shall mostly only consider $m = 2$ (plane curves) and $m = 3$ (space curves).

Then (cf. Figure 26.1)

$$\mathbf{r}(t + \Delta t) - \mathbf{r}(t) \approx \mathbf{r}'(t)\Delta t,$$

which is a small *vector*. The corresponding infinitesimal *length* (the line element) at the point $\mathbf{x} = \mathbf{r}(t)$ must therefore have the structure

$$\mathrm{d}s = \|\mathbf{r}'(t)\| \, \mathrm{d}t.$$

It should not come as a surprise that this can indeed be proved (we shall of course skip the proof here), so we can formulate the following

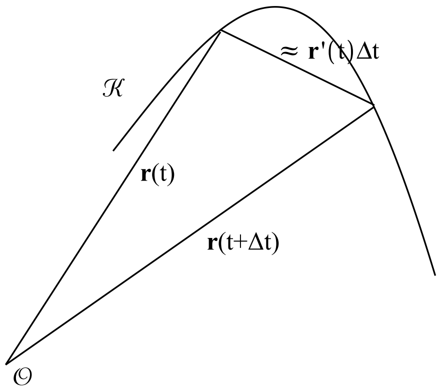

Figure 26.1: Analysis of the line element $\mathrm{d}s \approx \|\mathbf{r}'(t)\|\Delta t$.

Theorem 26.1 Reduction theorem for a line integral. *Given a C^1-curve $\mathcal{C}$ of parametric description (a function) $\mathbf{r} : [a,b] \to \mathbb{R}^m$, where $\mathbf{r}$ is injective almost everywhere. Let A be a closed and bounded set, such that $\mathcal{C} \subseteq A \subset \mathbb{R}^m$, and let $f : A \to \mathbb{R}$ be a continuous function. Then the line integral is reduced in the following way,*

$$\int_{\mathcal{C}} f(\mathbf{x})\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t.$$

One can visualize the process above as if we are straightening out the curve $\mathcal{C}$ to a straight line, represented by the interval $[a, b]$ and the additional weight function $\|\mathbf{r}'(t)\|$. The latter has the same structure as if we were changing variables in $\mathbb{R}$, only this is not taking place between two line segments, but between a curve and a line segment. If we use the curve length as a parameter, then the picture becomes even more clear. Form the curve by some wire and then stretch it out, before it is laid down on the x-axis, and let the function values follow this stretching. Clearly, we must therefore introduce the curve length, so we put $f(\mathbf{x}) \equiv 1$ above to get

Theorem 26.2 The length of a curve. *Let $\mathcal{C}$ be a piecewise C^1-curve of parametric description $\mathbf{r} : [a, b] \to \mathbb{R}^m$, where $\mathbf{r}$ is injective almost everywhere. Then the* length *of $\mathcal{C}$ is given by*

$$\ell(\mathcal{C}) = \int_a^b \|\mathbf{r}'(t)\| \, \mathrm{d}t.$$

Of special importance are the plane curves, where $m = 2$, so for later reference we here include an overview of the line elements, when the parametric description is given by either a function (i.e. graph) in rectangular coordinates (x, y), or a function (i.e. a graph) in polar coordinates (ϱ, φ) in the plane.

1) **Rectangular coordinates**. The curve is given as the graph of the function

$$y = Y(x), \qquad x \in [a, b], \qquad Y \in C^1.$$

Then the line element is

$$\mathrm{d}s = \sqrt{1 + (Y'(x))^2}\, \mathrm{d}x.$$

2) **Polar coordinates, first version**. The curve is in polar coordinates given as the graph of the equation

$$\varrho = P(\varphi), \qquad \varphi \in [\alpha, \beta], \qquad P \in C^1.$$

The line element is given by

$$\mathrm{d}s = \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2}\, \mathrm{d}\varphi.$$

3) *Polar coordinates, second version.* The curve is in polar coordinates given as the graph of the equation

$$\varphi = \Phi(\varrho), \qquad \varrho \in [a, b], \qquad \Phi \in C^1.$$

The line element is given by

$$\mathrm{d}s = \sqrt{1 + \{\varrho \cdot \Phi'(\varrho)\}^2}\, \mathrm{d}\varrho.$$

26.2.1 Natural parametric description

Let $\mathcal{C}$ be a curve as above wiith the parametric description $\mathbf{x} = \mathbf{r}/t)$, $\mathbf{r} \in C^1$. Choose some fixed $t_0 \in [a,b]$, and define

$$S(t) := \int_{t_0}^{t} \|\mathbf{r}'(\tau)\| \, \mathrm{d}\tau \qquad \text{for } t \in [a,b].$$

Then the *signed length of the curve* $\mathcal{C}$ *measured from* $\mathbf{r}\,(t_0)$ is defined as the function

$$s = S(t), \qquad t \in [a,b], \qquad\qquad \text{where } S'(t) = \|\mathbf{r}'(t)\|\,.$$

If $\|\mathbf{r}'(t)\| > 0$ for all $t \in [a,b]$, then the function $S(t)$ has an inverse function, so there exists a (unique) function T, such that

$$s = S(t), \qquad \text{if and only if} \qquad t = T(s),$$

Then by insertion,

$$\mathbf{x} = \mathbf{r}(t) = \mathbf{r}(T(s)) := \mathbf{r}_N(s), \qquad \text{i.e. } \mathbf{r}_N := \mathbf{r} \circ T.$$

We call $\mathbf{x} = \mathbf{r}_N(s)$ the *natural parametric description* of the curve $\mathcal{C}$, and the curve length s iis called the *natural parameter of the curve*. In spite of its name, one shall in practice not use the natural parameter, because the expressions in general become more complicated than in other descriptions. It is mentioned here for historical reasons, and anyway, the geometric interpretation of the natural parametric description is very nice.

26.3 Procedures for reduction of a line integral

This is a 1-dimensional case where the domain of integration is not an interval, but a curve in the plane or the space. The reduction formula becomes

$$\int_{\mathcal{C}} f(\mathbf{x})\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,\mathrm{d}t.$$

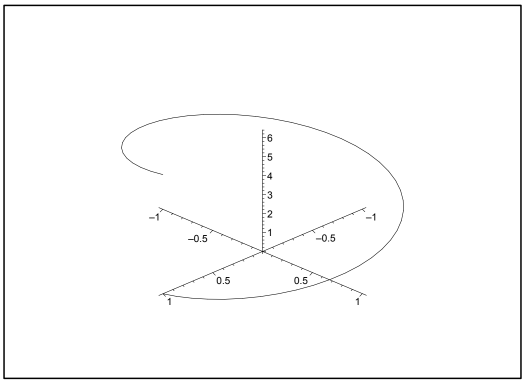

Figure 26.2: The curve of parameter representation $\mathbf{r}(t) = (\cos t, \sin t, t)$.

Procedure:

1) Write down a *rectangular* parameter representation for the curve $\mathcal{C}$:

   In the plane: $(x, y) = \mathbf{r}(t), \quad t \in [a, b]$.

   In the space: $(x, y, z) = \mathbf{r}(t), \quad t \in [a, b]$.

2) Calculate the *curve element* (the weight is $\|\mathbf{r}'(t)\|$)

   $\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t.$

   Mnemonic rule, Pythagoras,

   $\mathrm{d}s =$ "$\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2}$".

3) Insert the result and calculate the right hand side of

$$\int_{\mathcal{C}} f(\mathbf{x})\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,\mathrm{d}t.$$

Special case:

1) *Curve length:*

$$\ell(\mathcal{C}) = \int_a^b \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}r_1}{\mathrm{d}t}\right)^2 + \cdots + \left(\frac{\mathrm{d}r_k}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

2) *Graph*, $y = Y(x)$, $x \in [a,b]$, *rectangular*. The curve element is

$$\mathrm{d}s = \sqrt{1 + \{Y'(x)\}^2}\,\mathrm{d}x.$$

3) *Graph*, $\varrho = P(\varphi)$, $\varphi \in [\alpha,\beta]$, *polar, first variant*. The curve element is

$$\mathrm{d}s = \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2}\,\mathrm{d}\varphi.$$

4) *Graph*, $\varphi = \Phi(\varrho)$, $\varrho \in [a,b]$, *polar, second variant*. The curve element is

$$\mathrm{d}s = \sqrt{1 + \{\varrho\,\Phi'(\varrho)\}^2}\,\mathrm{d}\varrho.$$

Remark 26.1 We see that the square root is quite natural here. Since even programmable pocket calculators are not too happy with square roots, they will usually stop, when they are not given some help from the user. Therefore, we cannot expect to get any final result by applying pocket calculators to problems of this type. Note in particular that

$$\sqrt{g(t)^2} = |g(t)|.$$

(In other words: *Remember* the numerical signs!) $\diamondsuit$

26.4 Examples of the line integral in rectangular coordinates

Example 26.1

A. *A space curve $\mathcal{C}$ is given in a rectangular parametric representation*

$$\mathbf{r}(t) = (x,y,z) = \left(\ln t, \sqrt{2}\cdot t, \frac{1}{2}\,t^2\right), \qquad t \in [1,2].$$

Find the arc length $\ell(\mathcal{C})$, and the line integral

$$I = \int_{\mathcal{C}} e^{-x}\left(y^2 + 2z\right)\,\mathrm{d}s.$$

D. Find first the line element $\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t$.

I. We get from $\mathbf{r}(t) = \left(\ln t, \sqrt{2}\cdot t, \frac{1}{2}\,t^2\right)$ that $\mathbf{r}'(t) = \left(\frac{1}{t}, \sqrt{2}, t\right)$, hence

$$\|\mathbf{r}'(t)\|^2 = \left(\frac{1}{t}\right)^2 + 2 + t^2 = \left(\frac{1}{t} + t\right)^2.$$

Thus we get the line element

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \left|\frac{1}{t} + t\right|\,\mathrm{d}t = \left(\frac{1}{t} + t\right)\,\mathrm{d}t, \qquad \text{da } t > 0.$$

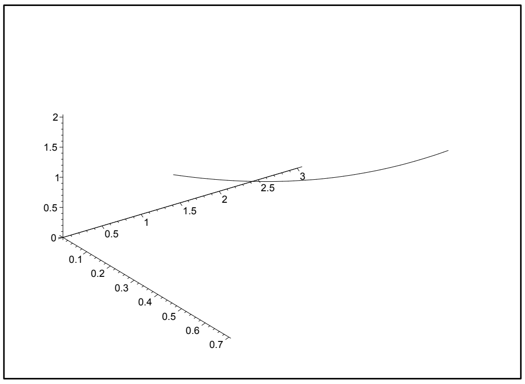

Figure 26.3: The space curve $\mathbf{x} = \mathbf{r}(t)$.

The arc length is

$$\ell(\mathcal{C}) = \int_{\mathcal{C}} \mathrm{d}s = \int_1^2 \left(\frac{1}{t} + t\right) dt = \left[\ln t + \frac{1}{2}\,t^2\right]_1^2 = \ln 2 + \frac{3}{2}.$$

The line integral becomes

$$\begin{aligned} I &= \int_1^2 \frac{1}{t} \cdot \left\{(\sqrt{2}\cdot t)^2 + 2\left(\frac{1}{2}\,t^2\right)\right\} \cdot \left(\frac{1}{t} + t\right) \mathrm{d}t \\ &= \int_1^2 \frac{1}{t}\,\left(2t^2 + t^2\right) \cdot \left(\frac{1}{t} + t\right) \mathrm{d}t = \int_1^2 (3 + 3t^2)\,dt = 3 + \left[t^3\right]_1^2 = 10. \qquad \Diamond \end{aligned}$$

Example 26.2

A. Let a, $h > 0$. Consider the *helix*

$$\mathbf{r}(t) = (x, y, z) = (a\,\cos t, a\,\sin t, h\,t), \qquad t \in \mathbb{R}.$$

This is lying on the cylinder $x^2 + y^2 = a^2$.

Find the *natural parametric representaion* of the curve from $(a, 0, 0)$, corresponding to $t = 0$.

D. Find the arc length $s = s(t)$ as a function of the parameter t. Solve this equation $t = t(s)$, and put the result into the parametric representation above.

I. Let us first find the *line element* $\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t$. Since

$$\mathbf{r}'(t) = (-a\,\sin t, a\,\cos t, h),$$

we have

$$\|\mathbf{r}'(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + h^2} = \sqrt{a^2 + h^2},$$

hence the arc length is

$$s = s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\,\mathrm{d}\tau = \int_0^t \sqrt{a^2 + h^2}\,\mathrm{d}\tau = \sqrt{a^2 + t^2} \cdot t.$$

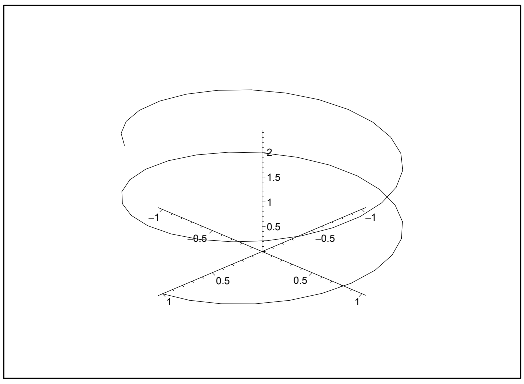

Figure 26.4: The helix for $a = 1$ and $h = \frac{1}{5}$.

By solving after t we get

$$t = t(s) = \frac{s}{\sqrt{a^2 + h^2}}.$$

When this is put into the parametric representation of the helix, we get

$$\begin{aligned} (x, y, z) &= (a \cos t, a \sin t, h\, t) \\ &= \left(a \cos\left(\frac{s}{\sqrt{a^2+h^2}}\right), a \sin\left(\frac{s}{\sqrt{a^2+h^2}}\right), \frac{h\, s}{\sqrt{a^2+h^2}}\right), \qquad s \in \mathbb{R}, \end{aligned}$$

which is the natural parametric representation of the helix. $\diamond$

Example 26.3 *Calculate in each of the following cases the given line integral, where the curve* $\mathcal{C}$ *is given by the parametric description*

$$\mathcal{C}: \quad \{\mathbf{x} \in \mathbb{R}^k \mid \mathbf{x} = \mathbf{r}(t), \quad t \in I\}, \qquad k = 2 \text{ or } k = 3.$$

1) *The line integral* $\int_{\mathcal{C}} \mathrm{d}s$, *where*

$$\mathbf{r}(t) = (a(1-\cos t), a(t-\sin t)), \qquad t \in [0, 4\pi].$$

2) *The line integral* $\int_{\mathcal{C}} \sqrt{x}\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = (a(1-\cos t), a(t-\sin t)), \qquad t \in [0, 4\pi].$$

3) *The line integral* $\int_{\mathcal{C}} z\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = \left(t, 3t^2, 6t^3\right), \qquad t \in [0, 2].$$

4) *The line integral* $\int_{\mathcal{C}} \frac{1}{1+6y}\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = \left(t, 3t^2, 6t^3\right), \qquad t \in [0, 2].$$

5) *The line integral* $\int_{\mathcal{C}} (x + e^z)\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = (\cos t, \sin t, \ln \cos t), \qquad t \in \left[0, \frac{\pi}{4}\right].$$

[Cf. **Example 26.15.6**.]

6) *The line integral* $\int_{\mathcal{C}} (x^2 + y^2 + z^2)\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = (e^t \cos t, e^t \sin t, e^t), \qquad t \in [0, 2].$$

7) *The line integral* $\int_{\mathcal{C}} \frac{x+y}{z^2}\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = \frac{1}{\sqrt{3}}\left(e^t, e^t \sin t, e^t\right), \qquad t \in [0, u].$$

[Cf. **Example 26.15.7**.]

8) *The line integral* $\int_{\mathcal{C}} (x^2 + y^2)\,\mathrm{d}s$, *where*

$$\mathbf{r}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), \qquad t \in \mathbb{R}.$$

9) *The line integral* $\int_{\mathcal{C}} \mathrm{d}s$, *where*

$$\mathbf{r}(t) = \left(2\,\mathrm{Arcsin}\; t, \ln(1-t^2), \ln\frac{1+t}{1-t}\right), \qquad t \in \left[0, \frac{1}{\sqrt{2}}\right].$$

10) The line integral $\int_{\mathcal{C}} xe^y \, \mathrm{d}s$*, where*

$$\mathbf{r}(t) = \left(2\,\mathrm{Arcsin}\, t, \ln(1-t^2), \ln\frac{1+t}{1-t}\right), \qquad t \in \left[0, \frac{1}{\sqrt{2}}\right].$$

11) The line integral $\int_{\mathcal{C}} \frac{1}{\sqrt{1+3x^2+z^2}} \, \mathrm{d}s$*, where*

$$\mathbf{r}(t) = (\cos t, 2\sin t, e^t), \qquad t \in [-1,1].$$

A Line integrals.

D First find $\|\mathbf{r}'(t)\|$ in each case. Then compute the line integral.

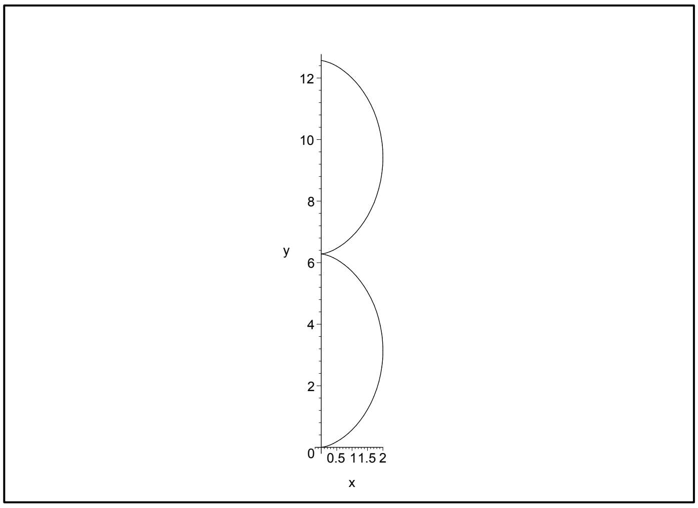

Figure 26.5: The plane curve $\mathcal{C}$ of **Example 26.3.1** and **Example 26.3.2** for $a=1$.

I 1) Here,

$$\mathbf{r}'(t) = a(\sin t, 1-\cos t),$$

so

$$\|\mathbf{r}'(t)\| = a\sqrt{\sin^2 t + (1-\cos t)^2} = a\sqrt{2-2\cos t} = a\sqrt{4\sin^2\frac{t}{2}} = 2a\left|\sin\frac{t}{2}\right|.$$

Then accordingly,

$$\int_{\mathcal{C}} \mathrm{d}s = \int_0^{4\pi} 2a\left|\sin\frac{t}{2}\right| \mathrm{d}t = 4a\int_0^{2\pi} |\sin u|\,\mathrm{d}u = 8a\int_0^{\pi}\sin u\,\mathrm{d}u = 16a.$$

2) It follows from 1) that

$$\|\mathbf{r}'(t)\| = a\sqrt{2(1-\cos t)},$$

thus

$$\int_{\mathcal{C}} \sqrt{x}\,\mathrm{d}s = \int_0^{4\pi} \sqrt{a(1-\cos t)} \cdot a\sqrt{2(1-\cos t)}\,\mathrm{d}t$$
$$= a\sqrt{2a}\int_0^{4\pi} |1-\cos t|\,\mathrm{d}t = a\sqrt{2a}\int_0^{4\pi}(1-\cos t)\,\mathrm{d}t = 4\sqrt{2}\,\pi\,a\sqrt{a}.$$

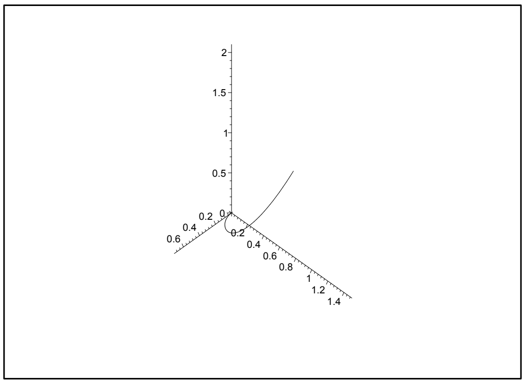

Figure 26.6: The curve $\mathcal{C}$ for $t \in [0, \frac{7}{10}]$. It is used in **Example 26.3.3** and **Example 26.3.4** for $t \in [0, 2]$.

3) It follows from $\mathbf{r}'(t) = (1, 6t, 18t^2)$ that

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 36t^2 + 324t^4} = \sqrt{(1 + 18t^2)^2} = 1 + 18t^2,$$

hence

$$\int_{\mathcal{C}} z\,\mathrm{d}s = \int_0^2 6t^3(1 + 18t^2)\,\mathrm{d}t = \left[\frac{6}{4}\,t^4 + \frac{6 \cdot 18}{6}\,t^6\right]_0^2 = \frac{3}{2} \cdot 16 + 18 \cdot 64 = 24 + 1152 = 1176.$$

4) It follows from 3) above that $\|\mathbf{r}'(t)\| = 1 + 18t^2$, so

$$\int_{\mathcal{C}} \frac{1}{1 + 6y}\,\mathrm{d}s = \int_0^2 \frac{1 + 18t^2}{1 + 18t^2}\,\mathrm{d}t = 2.$$

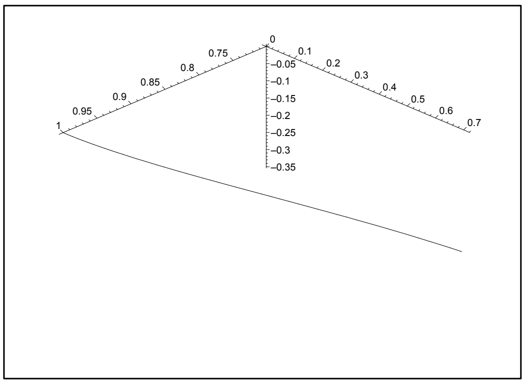

Figure 26.7: The curve $\mathcal{C}$ of **Example 26.3.5**.

5) It follows from

$$\mathbf{r}'(t) = \left(-\sin t, \cos t, -\frac{\sin t}{\cos t}\right),$$

that

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{\sin^2 t}{\cos^2 t}} = \sqrt{1 + \frac{\sin^2 t}{\cos^2 t}} = \frac{1}{|\cos t|},$$

hence

$$\int_{\mathcal{C}} (x + e^z)\,\mathrm{d}s = \int_0^{\frac{\pi}{4}} \left(\cos t + e^{\ln \cos t}\right) \frac{1}{\cos t}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} 2\,\mathrm{d}t = \frac{\pi}{2}.$$

6) It follows from

$$\mathbf{r}'(t) = \left(e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t\right),$$

that

$$\|\mathbf{r}'(t)\| = e^t\,\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = \sqrt{3}\,e^t,$$

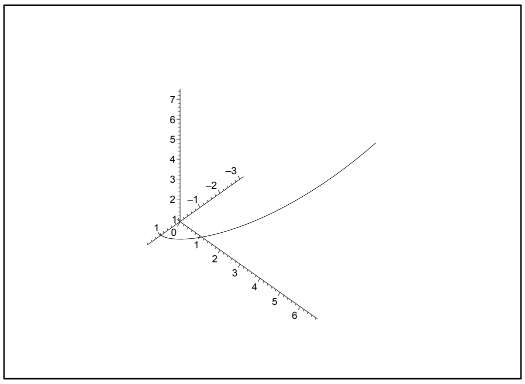

Figure 26.8: The curve $\mathcal{C}$ of **Example 26.3.6** and – apart from the factor $1/\sqrt{3}$ – of **Example 26.3.7**.

thus

$$\int_{\mathcal{C}}(x^2+y^2+z^2)\,\mathrm{d}s = \int_0^2 e^{2t}\left(\cos^2 t+\sin^2 t+1\right)\cdot\sqrt{3}\,e^t\,\mathrm{d}t = 2\sqrt{3}\int_0^2 e^{3t}\,dt = \frac{2\sqrt{3}}{3}\left(e^6-1\right).$$

7) If we first divide by $\sqrt{3}$, we get by **Example 26.3.6** the more nice expression $\|\mathbf{r}'(t)\| = e^t$. Then the line integral becomes

$$\int_{\mathcal{C}}\mathrm{d}s = \int_0^u e^t\,\mathrm{d}t = e^u-1.$$

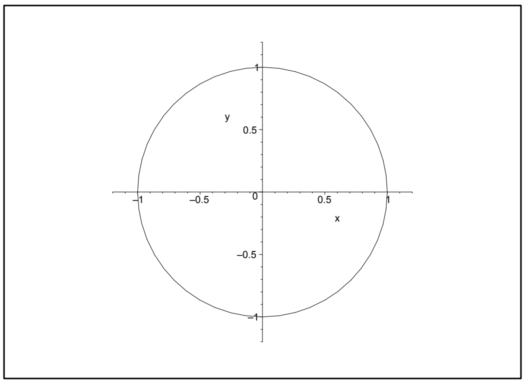

Figure 26.9: The curve $\mathcal{C}$ of **Example 26.3.8**, i.e. a circle except for the point $(-1,0)$.

8) We get by just computing

$$\mathbf{r}'(t) = \left(\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2},\frac{2(1+t^2)-2t\cdot 2t}{(1+t^2)^2}\right) = \left(-\frac{4t}{(1+t^2)^2},\frac{2(1-t^2)}{(1+t^2)^2}\right) = \frac{1}{(1+t^2)^2}\,(-2t,1-t^2),$$

hence

$$\|\mathbf{r}'(t)\| = \frac{1}{(1+t^2)^2}\sqrt{4t^2+(1-t^2)^2} = \frac{1}{(1+t^2)^2}\sqrt{(1+t^2)^2} = \frac{1}{1+t^2}.$$

Then finally,

$$\begin{aligned}\int_{\mathcal{C}} (x^2+y^2)\,\mathrm{d}s &= \int_{-\infty}^{+\infty} \frac{1}{(1+t^2)^2}\left\{(1-t^2)^2+4t^2\right\}\cdot\frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} \frac{(1+t^2)^2}{(1+t^2)^2}\cdot\frac{2}{1+t^2}\,\mathrm{d}t = [2\,\mathrm{Arctan}\,t]_{-\infty}^{+\infty} = 2\pi.\end{aligned}$$

ALTERNATIVE. The computation above was a little elaborated. However, the line integral is independent of the chosen parametric description, and $\mathcal{C}$ is a circle with the exception of the point $(-1,0)$, which is of no importance for the integration. Therefore, we can apply the simpler parametric description

$$\mathbf{r}(t) = (\cos t, \sin t), \qquad t \in]-\pi,\pi[,$$

where

$$\mathbf{r}'(t) = (-\sin t, \cos t) \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t+\cos^2 t} = 1.$$

Then the line integral becomes almost trivial,

$$\int_{\mathcal{C}} (x^2+y^2)\,\mathrm{d}s = \int_{-\pi}^{\pi} 1^2\,\mathrm{d}t = 2\pi.$$

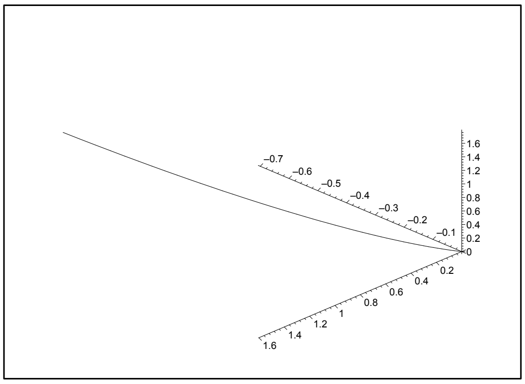

Figure 26.10: The curve $\mathcal{C}$ of **Example 26.3.9** and **Example 26.3.10**.

9) Here

$$\mathbf{r}'(t) = \left(\frac{2}{\sqrt{1-t^2}}, -\frac{2t}{1-t^2}, \frac{1}{1+t}+\frac{1}{1-t}\right) = \frac{2}{1-t^2}\left(\sqrt{1-t^2}, -t, 1\right),$$

hence

$$\|\mathbf{r}'(t)\| = \frac{2}{1-t^2}\sqrt{1-t^2+t^2+1} = \frac{2\sqrt{2}}{1-t^2} = \sqrt{2}\left(\frac{1}{1+t}-\frac{1}{t-1}\right).$$

The line integral is

$$\begin{aligned}\int_{\mathcal{C}} \mathrm{d}s &= \int_0^{\frac{1}{\sqrt{2}}} \sqrt{2}\left(\frac{1}{1+t}-\frac{1}{t-1}\right)\,\mathrm{d}t = \sqrt{2}\left[\ln\frac{1-t}{1-t}\right]_0^{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) = \sqrt{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2}\,\ln(\sqrt{2}+1).\end{aligned}$$

10) We consider the same curve as in **Example 26.3.9**, so we can reuse that

$$\|\mathbf{r}'(t)\| = \frac{2\sqrt{2}}{1-t^2} = \sqrt{2}\left(\frac{1}{1+t}-\frac{1}{t-1}\right), \qquad t \in \left[0, \frac{1}{\sqrt{2}}\right],$$

and the line integral becomes

$$\begin{aligned}\int_{\mathcal{C}} x\,e^y\,\mathrm{d}s &= \int_0^{\frac{1}{\sqrt{2}}} 2\,\mathrm{Arcsin}\,t\cdot(1-t^2)\cdot\frac{2\sqrt{2}}{1-t^2}\,\mathrm{d}t = 4\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\mathrm{Arcsin}\,t\,dt \\ &= 4\sqrt{2}\int_0^{\frac{\pi}{4}} u\,\cos u\,\mathrm{d}u = 4\sqrt{2}[u\,\sin u+\cos u]_0^{\frac{\pi}{4}} = 4\sqrt{2}\left\{\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right\} \\ &= 4+\pi-4\sqrt{2} = \pi-4(\sqrt{2}-1).\end{aligned}$$

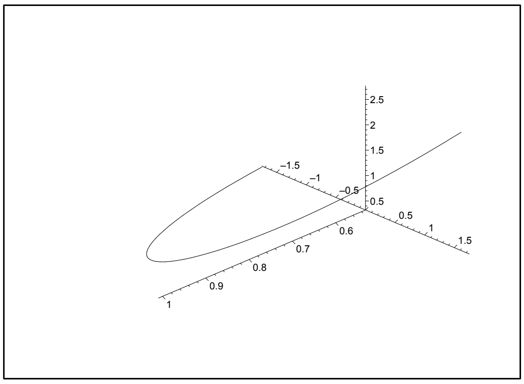

Figure 26.11: The curve $\mathcal{C}$ of **Example 26.3.11**.

11) Here

$$\mathbf{r}'(t) = \left(-\sin t, 2\cos t, e^t\right),$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4\cos^2 + e^{2t}} = \sqrt{1 + 3cos^2 + 2\,e^{2t}}.$$

The parametric description of the integrand restricted to the curve is

$$\sqrt{1+3x^2+z^2} = \sqrt{1+3\cos^2 t + e^{2t}},$$

so the line integral becomes easy to compute

$$\int_{\mathcal{C}} \frac{1}{\sqrt{1+3x^2+z^2}}\,\mathrm{d}s = \int_{-1}^{1} \frac{1}{\sqrt{1+3\cos^2 t + e^{2t}}}\,\sqrt{1+3\cos^2 t + e^{2t}}\,\mathrm{d}t = \int_{-1}^{1} 1\,\mathrm{d}t = 2.$$

Example 26.4 *Calculate in each of the following cases the given line integral along the given plane curve $\mathcal{C}$ of the equation $y = Y(x)$, $x \in I$.*

1) *The line integral* $\int_{\mathcal{C}} x^2 \, \mathrm{ds}$ *along the curve*

$$y = Y(x) = \ln x, \qquad x \in [1, 2\sqrt{2}].$$

2) *The line integral* $\int_{\mathcal{C}} \dfrac{1}{1+4y} \, \mathrm{ds}$ *along the curve*

$$y = Y(x) = x^2, \qquad x \in [0, 1].$$

3) *The line integral* $\int_{\mathcal{C}} y^2 \, \mathrm{ds}$ *along the curve*

$$y = Y(x) = x, \qquad x \in [1, 2].$$

4) *The line integral* $\int_{\mathcal{C}} \dfrac{1}{\sqrt{2-y^2}} \, \mathrm{ds}$ *along the curve*

$$y = Y(x) = \sin x, \qquad x \in [0, \pi].$$

5) *The line integral* $\int_{\mathcal{C}} \dfrac{1}{\sqrt{2+y^2}} \, \mathrm{ds}$ *along the curve*

$$y = Y(x) = \sinh x, \qquad x \in [0, 2].$$

6) *The line integral* $\int_{\mathcal{C}} y \, e^x \, \mathrm{ds}$ *along the curve*

$$y = Y(x) = e^x, \qquad x \in [0, 1].$$

A Line integrals along plane curves.

D Sketch if possible the den plane curve. Compute the weight function $\sqrt{1+Y'(x)^2}$ and finally reduce the line integral to an ordinary integral.

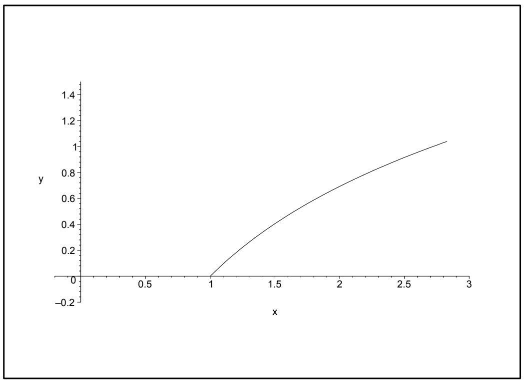

Figure 26.12: The curve $\mathcal{C}$ of **Example 26.4.1**.

I 1) It follows from $Y'(x) = \dfrac{1}{x}$ that

$$\sqrt{1+Y'(x)^2} = \frac{1}{x}\sqrt{1+x^2}, \qquad x \in [1, 2\sqrt{2}].$$

Thus we get the line integral

$$\begin{aligned}\int_{\mathcal{C}} x^2\,\mathrm{d}s &= \int_1^{2\sqrt{2}} x^2 \cdot \frac{1}{x^2}\sqrt{1+x^2}\,\mathrm{d}x = \int_0^{2\sqrt{2}} \sqrt{1+x^2}\cdot x\,\mathrm{d}x \\ &= \left[\frac{1}{2}\cdot\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]_1^{2\sqrt{2}} = \frac{1}{3}\left\{9^{\frac{3}{2}} - 2^{\frac{3}{2}}\right\} = \frac{1}{3}(27 - 2\sqrt{2}) = 9 - \frac{2}{3}\sqrt{2}.\end{aligned}$$

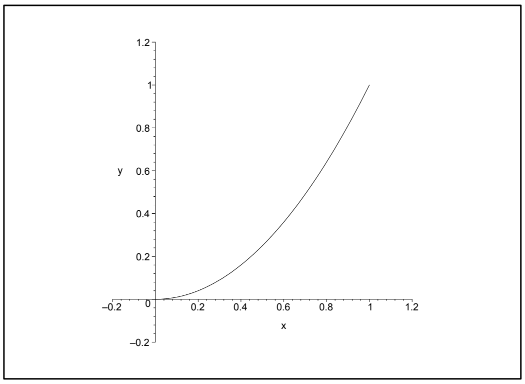

Figure 26.13: The curve $\mathcal{C}$ of **Example 26.4.2**.

2) From $Y'(x) = 2x$ follows that

$$\sqrt{1+Y'(x)^2} = \sqrt{1+4x^2},$$

and hence

$$\begin{aligned}\int_{\mathcal{C}} \frac{1}{1+4y}\,\mathrm{d}s &= \int_0^1 \frac{\sqrt{1+4x^2}}{1+4x^2}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1+(2x)^2}}\,\mathrm{d}x = \frac{1}{2}\int_0^2 \frac{1}{\sqrt{1+t^2}}\,\mathrm{d}t \\ &= \frac{1}{2}\left[\ln\left(t+\sqrt{1+t^2}\right)\right]_0^2 = \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1}\right) = \frac{1}{2}\ln(2+\sqrt{5}).\end{aligned}$$

3) Here clearly $\sqrt{1+Y'(x)^2} = \sqrt{1+1^2} = \sqrt{2}$, so

$$\int_{\mathcal{C}} y^2\,\mathrm{d}s = \int_1^2 x^2\sqrt{2}\,\mathrm{d}x = \frac{\sqrt{2}}{3}\left[x^3\right]_1^2 = \frac{7}{3}\sqrt{2}.$$

4) We get by differentiation of $Y(x) = \sin x$ that $Y'(x) = \cos x$, hence the weight function is

$$\sqrt{1+Y'(x)^2} = \sqrt{1+\cos^2 x} = \sqrt{2-\sin^2 x}.$$

We finally get the line integral by insertion

$$\int_{\mathcal{C}} \frac{1}{\sqrt{2-y^2}}\,\mathrm{d}s = \int_0^{\pi} \frac{1}{\sqrt{2-\sin^2 x}}\sqrt{2-\sin^2 x}\,\mathrm{d}x = \int_0^{\pi}\mathrm{d}x = \pi.$$

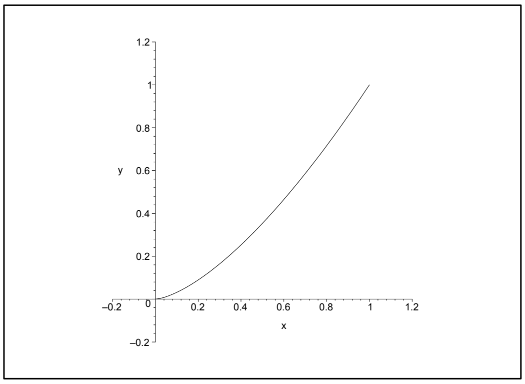

Figure 26.14: The curve $\mathcal{C}$ of **Example 26.4.3**.

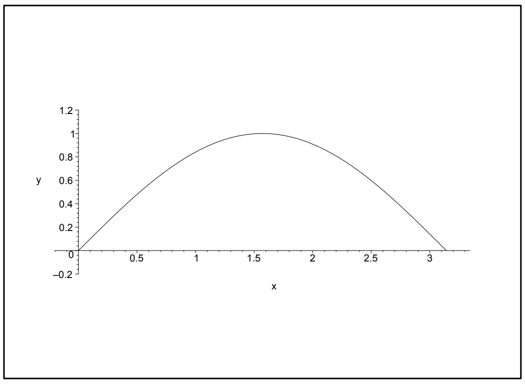

Figure 26.15: The curve $\mathcal{C}$ of **Example 26.4.4**.

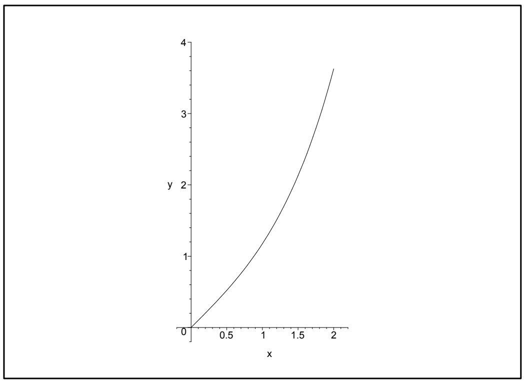

Figure 26.16: The curve $\mathcal{C}$ of **Example 26.4.5**.

5) When $Y(x) = \sinh x$, then $Y'(x) = \cosh x$, and the weight function becomes

$$\sqrt{1 + Y'(x)^2} = \sqrt{1 + \cosh^2 x} = \sqrt{2 + \sinh^2 x}.$$

We finally get the line integral by insertion

$$\int_{\mathcal{C}} \frac{1}{\sqrt{2+y^2}}\,\mathrm{d}s = \int_0^2 \frac{1}{\sqrt{2+\sinh^2 x}}\sqrt{2+\sinh^2 x}\,\mathrm{d}x = \int_0^2 \mathrm{d}x = 2.$$

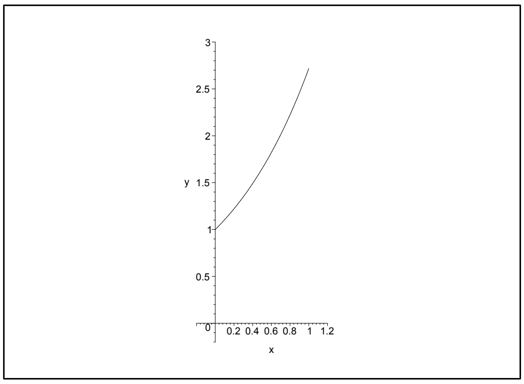

Figure 26.17: The curve $\mathcal{C}$ of **Example 26.4.6**.

6) When $Y(x) = e^x$ then also $Y'(x) = e^x$, so the weight function becomes

$$\sqrt{1+Y'(x)^2} = \sqrt{1+e^{2x}}.$$

We get the line integral by insertion

$$\begin{aligned}\int_{\mathcal{C}} ye^x\,\mathrm{d}s &= \int_0^1 e^x \cdot e^x \cdot \sqrt{1+e^{2x}}\,\mathrm{d}x = \int_0^1 \sqrt{1+e^{2x}} \cdot e^{2x}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{x=0}^1 \sqrt{1+e^{2x}}\,\mathrm{d}(1+e^{2x}) = \frac{1}{2}\cdot\frac{2}{3}\left[\left(1+e^{2x}\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}\left\{(1+e^2)^{\frac{3}{2}} - 1\right\}.\end{aligned}$$

Example 26.5 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(e^t, t\sqrt{2}, e^{-t}\right), \qquad t \in [0, \ln 3].$$

Prove that the curve element $\mathrm{d}s$ *is given by* $(e^t + e^{-t})\,\mathrm{d}t$, *and then find the value of the line integral*

$$\int_{\mathcal{C}} x^3 z\,\mathrm{d}s.$$

A Curve element and line integral.

D Follow the guidelines.

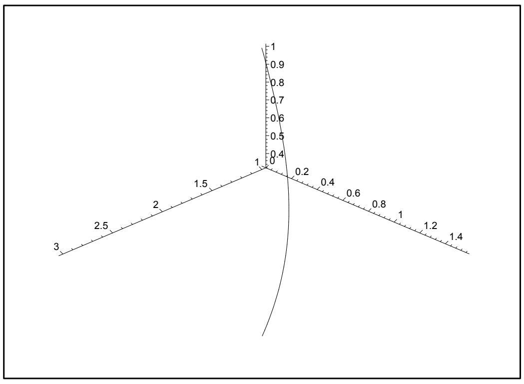

Figure 26.18: The curve $\mathcal{C}$.

I The curve is clearly of class C^∞. Furthermore,

$$\|\mathbf{r}'(t)\|^2 = \left(e^t\right)^2 + (\sqrt{2})^2 + \left(e^{-t}\right) = e^{2t} + 2 + e^{-2t} = \left(e^t + e^{-t}\right)^2,$$

and we get the curve element

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \left(e^t + e^{-t}\right), a\,\mathrm{d}t$$

with respect to the given parametric description.

Then compute the line integral,

$$\begin{aligned}\int_{\mathcal{C}} x^3 z \, \mathrm{d}s &= \int_0^{\ln 3} e^{3t} e^{-t} \left(e^t + e^{-t}\right) \, \mathrm{d}t = \int_0^{\ln 3} \left(e^{3t} + e^t\right) \, \mathrm{d}t \\ &= \left[\frac{1}{3}\, e^{3t} + e^t\right]_0^{\ln 3} = \frac{1}{3} \cdot 3^3 + 3 - \frac{1}{3} - 1 = \frac{3}{2},\end{aligned}$$

where we alternatively first can apply the change of variables $u = e^t$, from which

$$\int_{\mathcal{C}} x^3 z \, \mathrm{d}s = \int_1^3 (u^2 + 1) \, \mathrm{d}u = \left[\frac{1}{3}\, u^3 + u\right]_1^3 = 9 + 3 - \frac{1}{3} - 1 = \frac{32}{3}.$$

Example 26.6 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(t + 4\cos t, \frac{4}{3}t - 3\cos t, 5\sin t\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

Find the value of the line integral

$$\int_{\mathcal{C}} x \, \mathrm{d}s.$$

A Line integral.

D First find the curve element $\mathrm{d}s = \|\mathbf{r}'(t)\| \, \mathrm{d}t$.

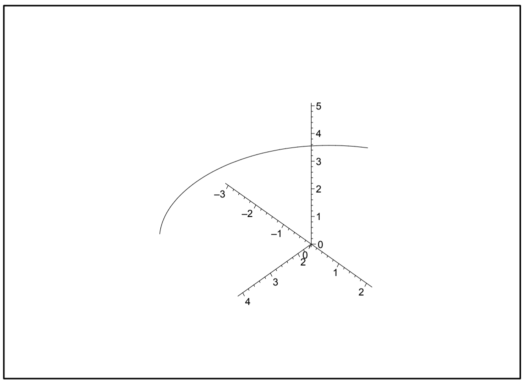

Figure 26.19: The space curve $\mathcal{C}$.

I From

$$\mathbf{r}'(t) = \left(1 - 4\sin t, \frac{4}{3} + 3\sin t, 5\cos t\right),$$

follows that

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= (1-4\sin t)^2 + \left(\frac{4}{3}+3\sin t\right)^2 + 25\cos^2 t \\ &= 1-8\sin t+16\sin^2 t+\frac{16}{9}+8\sin t+9\sin^2 t+25\cos^2 t = \frac{25}{9}+25 = \frac{25}{9}\cdot 10,\end{aligned}$$

thus

$$\mathrm{d}s = \|\mathbf{r}'(t)\| \, \mathrm{d}t = \frac{5}{3}\sqrt{10}\, \mathrm{d}t.$$

The line integral is

$$\begin{aligned}\int_{\mathcal{C}} x \, \mathrm{d}s &= \int_0^{\frac{\pi}{2}} (t + 4\cos t) \cdot \frac{5}{3}\sqrt{10}\, \mathrm{d}t = \frac{5}{3}\sqrt{10}\left[\frac{t^2}{2} + 4\sin t\right]_0^{\frac{\pi}{2}} = \frac{5}{3}\sqrt{10}\left(\frac{\pi^2}{8}+4\right) \\ &= \frac{5\pi^2\sqrt{10}}{24} + \frac{20\sqrt{10}}{3}.\end{aligned}$$

Example 26.7 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(t^2, e^{2t}, \frac{1}{2}\, t^4 \right), \qquad t \in [1,2].$$

Find the value of the line integral

$$\int_{\mathcal{C}} \frac{1}{\sqrt{x+2xz+y^2}}\, \mathrm{d}s.$$

A Line integral.

D First calculate the weight function $\|\mathbf{r}'(t)\|$.

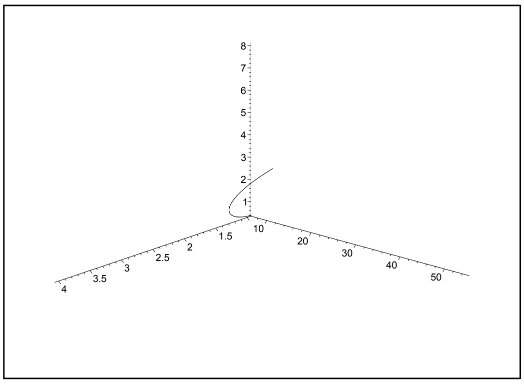

Figure 26.20: The curve $\mathcal{C}$. Note the different scales on the axes.

I We get from

$$\mathbf{r}'(t) = \left(2t, 2e^{2t}, 2t^3\right) = 2\left(t, e^{2t}, t^3\right)$$

that

$$\|\mathbf{r}'(t)\| = 2\sqrt{e^{4t}+t^2+t^6}.$$

Then by insertion and reduction,

$$\int_{\mathcal{C}} \frac{1}{\sqrt{x+2xz+y^2}}\, \mathrm{d}s = \int_1^2 \frac{1}{\sqrt{t^2+2t^2\cdot\frac{1}{2}\,t^4+e^{4t}}} \cdot 2\sqrt{e^{4t}+t^2+t^6}\, dt = 2.$$

Example 26.8 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(\ln t, t^2, \frac{1}{2}\,t^4 \right), \qquad t \in [1,2].$$

Prove that the curve element ds is given by $\left(\frac{1}{t} + 2t^3\right)$ dt, and then compute the line integral

$$\int_{\mathcal{C}} \frac{y\,e^x}{z}\,\mathrm{d}s.$$

A Line integral.

D Follow the guidelines.

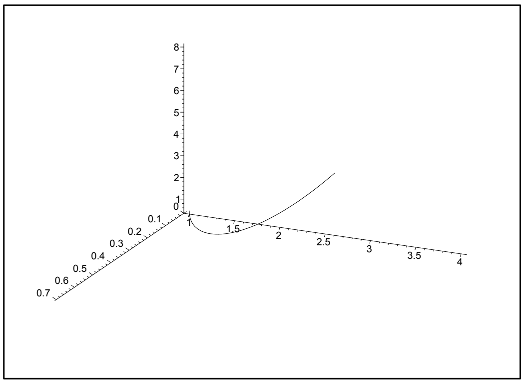

Figure 26.21: The curve $\mathcal{C}$.

I Clearly, $\mathbf{r}(t)$ is of class C^∞ for $t \in]1,2[$. Then

$$\mathbf{r}'(t) = \left(\frac{1}{t}, 2t, 2t^3 \right), \qquad t \in]1,2[,$$

implies that

$$\|\mathbf{r}'(t)\|^2 = \frac{1}{t^2} + 4t^2 + 4t^6 = \left(\frac{1}{t} + 2t^3 \right)^2,$$

so

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \left| \frac{1}{t} + 2t^3 \right| \mathrm{d}t = \left(\frac{1}{t} + 2t^3 \right) \mathrm{d}t \quad \text{for } t \in]1,2[.$$

We get by insertion of the parametric description,

$$\begin{aligned} \int_{\mathcal{C}} \frac{y\,e^x}{z}\,\mathrm{d}s &= \int_1^2 \frac{t^2 \cdot t}{\frac{1}{2}\,t^4} \cdot \left(\frac{1}{t} + 2t^3 \right) \mathrm{d}t = 2 \int_1^2 \left(\frac{1}{t^2} + 2t^2 \right) \mathrm{d}t \\ &= 2 \left[-\frac{1}{t} + \frac{2}{3}\,t^3 \right]_1^2 = 2 \left(-\frac{1}{2} + \frac{16}{3} + 1 - \frac{2}{3} \right) = \frac{31}{3}. \end{aligned}$$

Example 26.9 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(\ln t, t^2, 2t\right), \qquad t \in \left[\frac{1}{2}, \frac{3}{2}\right].$$

1) Find a parametric description of the tangent to $\mathcal{C}$ at the point $\mathbf{r}(1)$.

2) Prove that the curve element $\mathrm{d}s$ is given by $\left(\dfrac{1}{t} + 2t\right)\,\mathrm{d}t$.

3) Compute the value of the line integral

$$\int_{\mathcal{C}} \left(e^x + \sqrt{y} + 2z\right)\,\mathrm{d}s.$$

A Space curve; tangent; curve element; line integral.

D Find $\mathbf{r}'(t)$, and apply that $ds = \|\mathbf{r}'(t)\|\,\mathrm{d}t$.

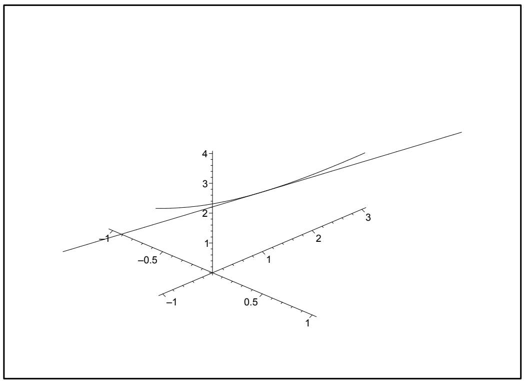

Figure 26.22: The curve $\mathcal{C}$ and its tangent at $(0, 1, 2)$.

I 1) Since $\mathbf{r}(1) = \left(\ln 1, 1^2, 2 \cdot 1\right) = (0, 1, 2)$, and

$$\mathbf{r}'(t) = \left(\frac{1}{t}, 2t, 2\right), \qquad \mathbf{r}'(1) = (1, 2, 2),$$

a parametric description of the tangent is given by

$$(x(u), y(u), z(u)) = (0, 1, 2) + u\,(1, 2, 2) = (u, 2u+1, 2u+2), \qquad u \in \mathbb{R}.$$

2) Since

$$\mathbf{r}'(t)\|^2 = \frac{1}{t^2} + 4t^2 + 4 = \left(\frac{1}{t} + 2t\right)^2,$$

we get for $t \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right]$ that

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \left|\frac{1}{t} + 2t\right|\,\mathrm{d}t = \left(\frac{1}{t} + 2t\right)\,\mathrm{d}t.$$

3) Then by insertion and computation we get the line integral

$$\int_{\mathcal{C}} \left(e^x+\sqrt{y}+2z\right)\,\mathrm{d}s = \int_{\frac{1}{2}}^{\frac{3}{2}} \left(e^{\ln t}+\sqrt{t^2}+2\cdot 2t\right)\cdot\left(\frac{1}{t}+2t\right)\,\mathrm{d}t$$

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} (t+t+4t)\cdot\frac{1}{t}\left(1+2t^2\right)\mathrm{d}t = 6\int_{\frac{1}{2}}^{\frac{3}{2}}\left(1+2t^2\right)\mathrm{d}t$$

$$= \left[6t+4t^3\right]_{\frac{1}{2}}^{\frac{3}{2}} = 6+4\left\{\left(\frac{3}{2}\right)^3-\left(\frac{1}{2}\right)^3\right\} = 6+\frac{1}{2}\,(27-1) = 19.$$

26.5 Examples of the line integral in polar coordinates

Example 26.10 *Compute in each of the following cases the given line integral along the plane curve $\mathcal{C}$ which is given by an equation in polar coordinates.*

1) *The line integral* $\int_{\mathcal{C}} (x^2+y^2)\,\mathrm{ds}$ *along the curve given by*

$$\varrho = e^{\varphi}, \qquad \varphi \in [0,4].$$

2) *The line integral* $\int_{\mathcal{C}} y\,\mathrm{ds}$ *along the curve given by*

$$\varrho = a(1-\cos\varphi), \qquad \varphi \in [0,\pi].$$

3) *The line integral* $\int_{\mathcal{C}} \sqrt{y}\,\mathrm{ds}$ *along the curve given by*

$$\varphi = \text{ Arcsin } \varrho, \qquad \varrho \in [0,1].$$

4) *The line integral* $\int_{\mathcal{C}} \frac{y}{\sqrt{4a-3\varrho}}\,\mathrm{ds}$ *along the curve given by*

$$\varrho = a\cos^2\varphi, \qquad \varphi \in \left[0,\frac{\pi}{2}\right].$$

5) *The line integral* $\int_{\mathcal{C}} \frac{1}{x^2+y^2}\,\mathrm{ds}$ *along the curve given by*

$$\varrho = \frac{a}{\cos\varphi}, \qquad \varphi \in \left[0,\frac{\pi}{4}\right].$$

6) *The line integral* $\int_{\mathcal{C}} \left(\sqrt{x^2+y^2}-1\right)\ a\,\mathrm{ds}$ *along the curve given by*

$$\varphi = \varrho - \ln\varrho, \qquad \varrho \in [1,2].$$

7) *The line integral* $\int_{\mathcal{C}} (x^2+y^2)\,\mathrm{ds}$ *along the curve given by*

$$\varphi = \varrho, \qquad \varrho \in [1,2].$$

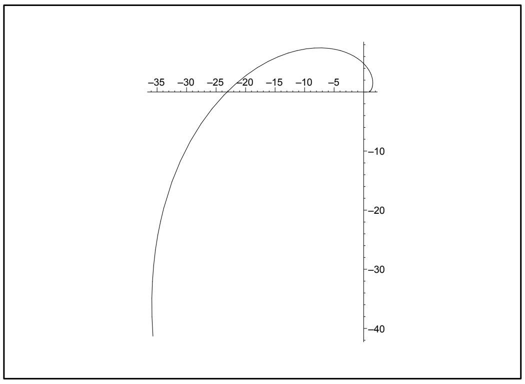

Figure 26.23: The curve $\mathcal{C}$ of **Example 26.10.1**.

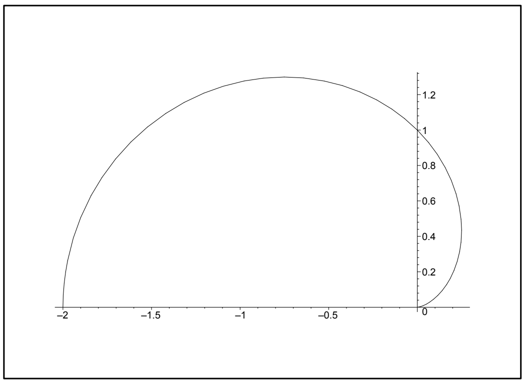

Figure 26.24: The curve $\mathcal{C}$ of **Example 26.10.2**.

A Line integral in polar coordinates.

D First compute the weight function $\sqrt{\varrho^2+\left(\frac{d\varrho}{d\varphi}\right)}$ $\left(\text{possibly } \sqrt{1+\left(\varrho\,\frac{\mathrm{d}\varphi}{\mathrm{d}\varrho}\right)^2}\right)$, and then the line integral.

I 1) From $\dfrac{\mathrm{d}\varrho}{\mathrm{d}\varphi}=e^{\varphi}=\varrho$ follows that

$$\sqrt{\varrho^2+\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2}=\sqrt{2}\cdot\varrho=\sqrt{2}\cdot e^{\varphi},$$

and thus

$$\int_{\mathcal{C}}\left(x^2+y^2\right)\mathrm{d}s=\int_0^4\varrho^2\sqrt{2}\cdot e^{\varphi}\,\mathrm{d}\varphi=\sqrt{2}\int_0^4 e^{3\varphi}\,\mathrm{d}\varphi=\frac{\sqrt{2}}{3}\left(e^{12}-1\right).$$

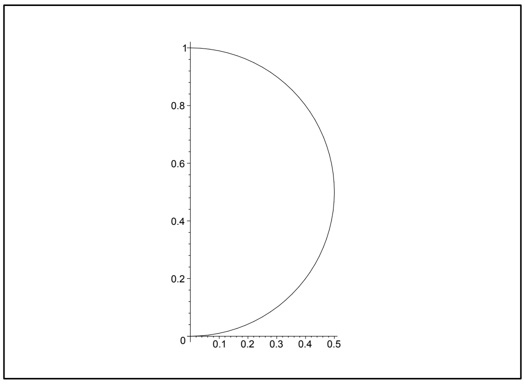

Figure 26.25: The curve $\mathcal{C}$ of **Example 26.10.3**.

2) From $\dfrac{d\varrho}{d\varphi} = a \sin\varphi$, follows that

$$\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2 = a^2\left\{(1-\cos\varphi)^2 + \sin^2\varphi\right\} = a^2 \cdot 2(1-\cos\varphi),$$

hence

$$\begin{aligned}\int_{\mathcal{C}} y\, ds &= \int_0^{\pi} \varrho(\varphi)\sin\varphi \cdot a\sqrt{2}\cdot\sqrt{1-\cos\varphi}\, d\varphi = a^2\sqrt{2}\int_0^{\pi}(1-\cos\varphi)^{\frac{3}{2}}\sin\varphi\, d\varphi \\ &= a^2\sqrt{2}\left[\frac{2}{5}(1-\cos\varphi)^{\frac{5}{2}}\right]_0^{\pi} = a^2\sqrt{2}\cdot\frac{2}{5}\cdot 2^{\frac{5}{2}} = \frac{16a^2}{5}.\end{aligned}$$

3) It follows from

$$\sqrt{1+\left\{\varrho\,\frac{d\varphi}{d\varrho}\right\}^2} = \sqrt{1+\left\{\varrho\cdot\frac{1}{\sqrt{1-\varrho^2}}\right\}^2} = \frac{1}{\sqrt{1-\varrho^2}},$$

that

$$\begin{aligned}\int_{\mathcal{C}} \sqrt{y}\, ds &= \int_0^1 \sqrt{\varrho\cdot\sin\varphi(\varrho)}\cdot\frac{1}{\sqrt{1-\varrho^2}}\, d\varrho = \int_0^1 \sqrt{\varrho^2}\cdot\frac{1}{\sqrt{1-\varrho^2}}\, d\varrho \\ &= \int_0^1 \frac{\varrho}{\sqrt{1-\varrho^2}}\, d\varrho = \left[-\sqrt{1-\varrho^2}\right]_0^1 = 1.\end{aligned}$$

ALTERNATIVELY, $\varrho = \sin\varphi$, $\varphi \in \left[0, \frac{\pi}{2}\right]$ and

$$\sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2} = \sqrt{\sin^2\varphi + \cos^2\varphi} = 1,$$

thus

$$\int_{\mathcal{C}} \sqrt{y}\, ds = \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\varphi}\cdot 1\, d\varphi = \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi = 1.$$

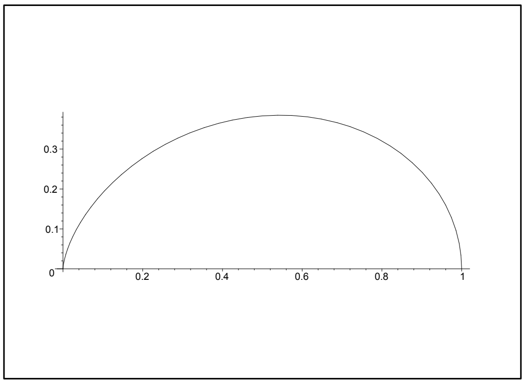

Figure 26.26: The curve $\mathcal{C}$ of **Example 26.10.4**.

4) It follows from

$$\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi} = -2a\,\sin\varphi \cdot \cos\varphi,$$

that

$$\begin{aligned}\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2 &= a^2\cos^4\varphi + 4a^2\sin^2\varphi\cdot\cos^2\varphi = a^2\cos^2\varphi\left\{\cos^2\varphi + 4\sin^2\varphi\right\}\\ &= a^2\cos^2\left\{4 - 3\cos^2\varphi\right\}.\end{aligned}$$

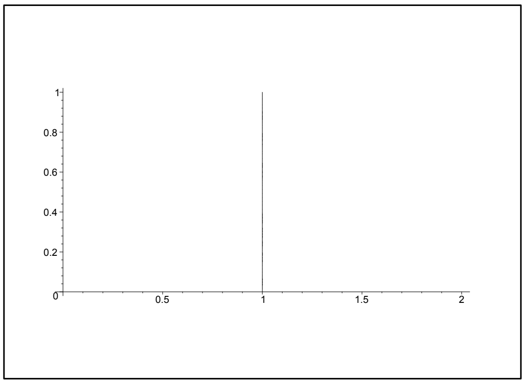

Figure 26.27: The curve $\mathcal{C}$ of **Example 26.10.5**. (In fact, a segment of the line $x = a$.

Now, $\cos\varphi \geq 0$ for $\varphi \in \left[0, \frac{\pi}{2}\right]$, so

$$\sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2} = a\,\cos\varphi\,\sqrt{4 - 3\cos^2\varphi},$$

and the line integral becomes

$$\begin{aligned}\int_{\mathcal{C}} \frac{y}{\sqrt{4a-3\varrho}}\,ds &= \int_{\mathcal{C}} \frac{\varrho\,\sin\varphi}{\sqrt{4a-3\varrho}}\,ds = \int_0^{\frac{\pi}{2}} \frac{a\cos^2\varphi\cdot\sin\varphi}{\sqrt{4a-3a\cos^2\varphi}}\cdot a\cos\varphi\sqrt{4-3\cos^2\varphi}\,d\varphi \\ &= a\sqrt{a}\int_0^{\frac{\pi}{2}}\cos^3\varphi\cdot\sin\varphi\,d\varphi = a\sqrt{a}\left[-\frac{\cos^4\varphi}{4}\right]_0^{\frac{\pi}{2}} = \frac{a\sqrt{a}}{4}.\end{aligned}$$

5) If $\varrho = \dfrac{a}{\cos\varphi}$, then

$$\frac{d\varrho}{d\varphi} = \frac{a\sin\varphi}{\cos^2\varphi},$$

hence

$$\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2 = a^2\left\{\frac{1}{\cos^2\varphi} + \frac{\sin^2\varphi}{\cos^4\varphi}\right\} = \frac{a^2}{\cos^4\varphi},$$

where

$$\sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2} = \frac{a}{\cos^2\varphi}.$$

The line integral is obtained by insertion,

$$\int_{\mathcal{C}} \frac{1}{x^2+y^2}\,ds = \int_{\mathcal{C}} \frac{1}{\varrho^2}\,ds = \int_0^{\pi} \frac{\cos^2\varphi}{a^2}\cdot\frac{a}{\cos^2\varphi}\,d\varphi = \frac{\pi}{4a}.$$

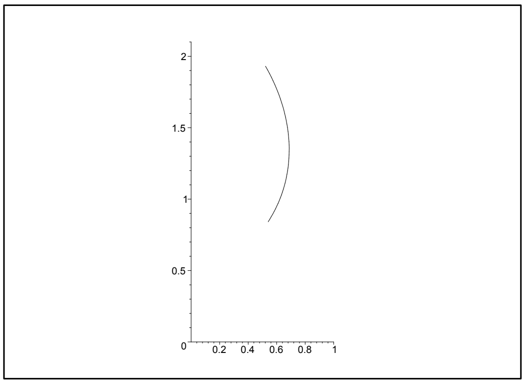

Figure 26.28: The curve $\mathcal{C}$ of **Example 26.10.6**.

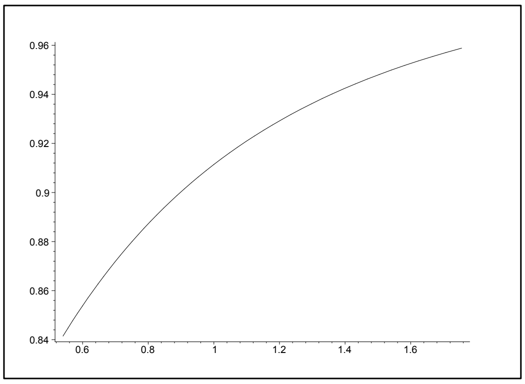

Figure 26.29: The curve $\mathcal{C}$ of **Example 26.10.7**.

6) If $\varphi = \varrho - \ln \varrho$, then

$$\frac{d\varphi}{d\varrho} = 1 - \frac{1}{\varrho} = \frac{\varrho - 1}{\varrho},$$

hence

$$\sqrt{1 + \left(\varrho \frac{d\varphi}{d\varrho}\right)^2} = \sqrt{1 + (\varrho - 1)^2}.$$

Finally, we get the line integral by insertion,

$$\begin{aligned} \int_{\mathcal{C}} \left(\sqrt{x^2+y^2} - 1\right) ds &= \int_{\mathcal{C}} (\varrho - 1)\, ds = \int_1^2 \sqrt{1 + (\varrho - 1)^2} \cdot (\varrho - 1)\, d\varrho \\ &= \frac{1}{3}\left[\left\{1 + (\varrho - 1)^2\right\}^{\frac{3}{2}}\right]_1^2 = \frac{1}{3}(2\sqrt{2} - 1). \end{aligned}$$

7) If $\varphi = \dfrac{1}{\varrho}$, then $\dfrac{d\varphi}{d\varrho} = -\dfrac{1}{\varrho^2}$, so

$$\sqrt{1+\left(\varrho\,\frac{d\varphi}{d\varrho}\right)^2} = \sqrt{1+\frac{1}{\varrho^2}} = \frac{\sqrt{1+\varrho^2}}{\varrho}.$$

We get the line integral by insertion

$$\int_{\mathcal{C}} (x^2+y^2)\,ds = \int_1^2 \varrho^2 \cdot \frac{\sqrt{1+\varrho^2}}{\varrho}\,d\varrho = \frac{1}{3}\left[\left\{1+\varrho^2\right\}^{\frac{3}{2}}\right]_1^2 = \frac{1}{3}\{5\sqrt{5}-2\sqrt{2}\}.$$

Example 26.11 **A.** *Find the curve length from* $(0,0)$ *of any finite piece* $(0 \le \varphi \le \alpha)$ *of the Archimedes's spiral,* given in polar coordinates by

$$\varrho = a\,\varphi, \qquad 0 \le \varphi < +\infty, \qquad \text{where } a > 0,$$

i.e. calculate the line integral

$$\ell = \int_{\varphi=0}^{\alpha} ds.$$

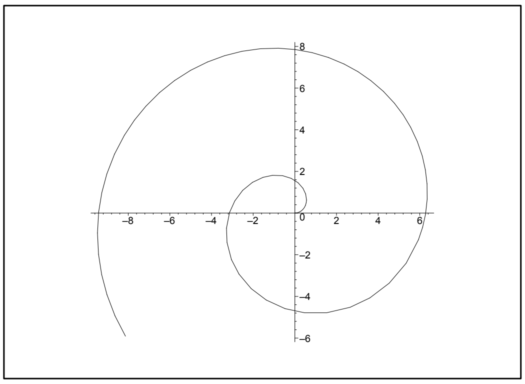

Figure 26.30: A piece of the Archimedes's spiral for $a = 1$.

D. First find the line element ds expressed by means of φ and $d\varphi$.

We shall here meet a very unpleasant integral, which we shall calculate in four different ways:

1) by a substitution,
2) by using partial integration,
3) by using a pocket calculator,
4) by using MAPLE.

I. Since $\varrho = P(\varphi) = a\,\varphi$, and since we have a description of the curve in *polar* coordinates, the line element is

$$\mathrm{d}s = \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2}\,\mathrm{d}\varphi = \sqrt{(a\,\varphi)^2 + a^2}\,\mathrm{d}\varphi = a\sqrt{1+\varphi^2}\,\mathrm{d}\varphi.$$

Then by a reduction,

$$\ell = \int_{\varphi=0}^{\alpha} \mathrm{d}s = \int_0^{\alpha} a\sqrt{1+\varphi^2}\,\mathrm{d}\varphi = a\int_0^{\alpha}\sqrt{1+\varphi^2}\,\mathrm{d}\varphi.$$

1) Since $1+\sinh^2 t = \cosh^2 t$, we have $\sqrt{1+\sinh^2 t} = +\cosh t$, because both sides of the equation sign must be positive. Thus we can remove the square root of the integrand by using the *monotonous* substitution,

$$\varphi = \sinh t, \qquad \mathrm{d}\varphi = \cosh t\,\mathrm{d}t, \qquad t = \operatorname{Arsinh}\varphi = \ln\left(\varphi + \sqrt{1+\varphi^2}\right).$$

Since t can be expressed uniquely by φ, the substitution must be monotonous.

Then

$$\begin{aligned}
\ell &= a\int_0^\alpha \sqrt{1+\varphi^2}\,\mathrm{d}\varphi = a\int_{\varphi=0}^\alpha \sqrt{1+\sinh^2 t}\cdot\cosh t\,\mathrm{d}t \\
&= a\int_{\varphi=0}^\alpha \cosh^2 t\,\mathrm{d}t = a\cdot\frac{1}{2}\int_{\varphi=0}^\alpha \{\cosh 2t+1\}\,\mathrm{d}t \\
&= \frac{a}{2}\left[\frac{1}{2}\sinh 2t+t\right]_{\varphi=0}^\alpha = \frac{a}{2}\left[(t+\sinh t\cdot\cos t)\right]_{\varphi=0}^\alpha \\
&= \frac{a}{2}\left[t+\sinh t\cdot\sqrt{1+\sinh^2 t}\right]_{\varphi=\sinh t=0}^\alpha \\
&= \frac{a}{2}\left[\ln\left(\varphi+\sqrt{1+\varphi^2}\right)+\varphi\cdot\sqrt{1+\varphi^2}\right]_0^\alpha \\
&= \frac{a}{2}\left\{\alpha\sqrt{1+\alpha^2}+\ln\left(\alpha+\sqrt{1+\alpha^2}\right)\right\}.
\end{aligned}$$

2) If we instead apply *partial integration*, then

$$\begin{aligned}
\ell &= a\int_0^\alpha \sqrt{1+\varphi^2}\,\mathrm{d}\varphi = a\int_0^\alpha 1\cdot\sqrt{1+\varphi^2}\,\mathrm{d}\varphi \\
&= a\left[\varphi\sqrt{1+\varphi^2}\right]_0^\alpha - a\int_0^\alpha \varphi\cdot\frac{\varphi}{\sqrt{1+\varphi^2}}\,\mathrm{d}\varphi \\
&= a\left\{\alpha\sqrt{1+\alpha^2}-\int_0^\alpha \frac{(\varphi^2+1)-1}{\sqrt{1+\varphi^2}}\,\mathrm{d}\varphi\right\} \\
&= a\left\{\alpha\sqrt{1+\alpha^2}-\int_0^\alpha \sqrt{1+\varphi^2}\,\mathrm{d}\varphi+\int_0^\alpha \frac{\mathrm{d}\varphi}{\sqrt{1+\varphi^2}}\right\} \\
&= -a\int_0^\alpha \sqrt{1+\varphi^2}\,\mathrm{d}\varphi + a\left\{\alpha\sqrt{1+\alpha^2}+\ln\left(\alpha+\sqrt{1+\alpha^2}\right)\right\}.
\end{aligned}$$

The first term is $-a\int_0^\alpha \sqrt{1+\varphi^2}\,\mathrm{d}\varphi = -\ell$, so we get by adding ℓ and dividing by 2 that

$$\ell = \frac{a}{2}\left\{\alpha\sqrt{1+\alpha^2}+\ln\left(\alpha+\sqrt{1+\alpha^2}\right)\right\}.$$

3) This is an example where a *pocket calculator* will give an equivalent, though different answer, so it is easy to see, whether a pocket calculator has been applied or not. It is here illustrated by the use of a TI-89, where the *command* is given by

$$\mathtt{a}\star\int(\sqrt{(}\mathtt{1+t\hat{\ }2}),\mathtt{t},\mathtt{0},\mathtt{b}),$$

because neither φ nor α are natural. Then the answer of the pocket calculator is

$$(26.1)\quad a\cdot\left(\frac{\ln(\sqrt{b^2+1}+b)}{2}+\frac{b\sqrt{b^2+1}}{2}\right),$$

followed by writing α again instead of b.

However, if one does *not* apply a pocket calculator, but instead uses the standard methods of integration, one would *never* state the result in the form (26.1). The reason for this discrepancy is that the programs of the pocket calculator are created from specialists in Algebra, and they do not always speak the same mathematical language as the specialists in Calculus or Mathematical Analysis. In Calculus the priority of the terms would be ($b = \alpha$)

$$\frac{a}{2}\left\{\alpha\sqrt{1+\alpha^2}+\ln\left(\alpha+\sqrt{1+\alpha^2}\right)\right\},$$

because one would try to put as many factors as possible outside the parentheses and then order the rest of the terms, such that the simplest is also the first one. Obviously, this is not the structure of (26.1).

The morale of this story is that even if a pocket calculator may give the right result, this result does not have to be put in a practical form. It is even worse by applications of e.g. MAPLE where the result is sometimes given in a form using functions which are not known by students of Calculus.

Note also that pocket calculators in general do not like the operations $|\cdot|$ and $\sqrt{\cdot}$, and cases where we have got two parameters. The latter is not even one of the favorites of MAPLE either, and it is in fact possible to obtain some very strange results by using MAPLE on even problems from this part of Calculus. I shall therefore give the following warning: *Do not use pocket calculators and computer programs like MAPLE or Mathematica uncritically!* Since they exist, they should of course be applied, but do it with care. $\Diamond$

4) For completeness we include an application of MAPLE. Without further help we just get

$$\mathrm{Int}\left(a\cdot\sqrt{1+t^2},t=0..\alpha\right)$$

$$\int_0^{\alpha} a\sqrt{t^2+1}\,\mathrm{d}t$$

Apparently one should use some additional package from MAPLE in order to get this right and not like here just use the most "obvious".

Example 26.12

A. Find the value of the line integral $I = \int_{\mathcal{C}} |y|\,\mathrm{d}s$, where $\mathcal{C}$ is the cardioid *given in polar coordinates by*

$$\varrho = P(\varphi) = a(1+\cos\varphi), \qquad -\pi \le \varphi \le \pi.$$

Examination of dimensions: Since $\varrho \sim a$, we get $\int_{\mathcal{C}} \cdots \mathrm{d}s \sim a$, and since $y \sim a$, The result must be of the form $c\cdot a\cdot a = c\cdot a^2$.

Due to the *numerical sign* in the integrand we must be very careful. In particular, a pocket calculator will be in big trouble here, if one does not give it a hand from time to time during the calculations.

D. First find the line element $\mathrm{d}s$.

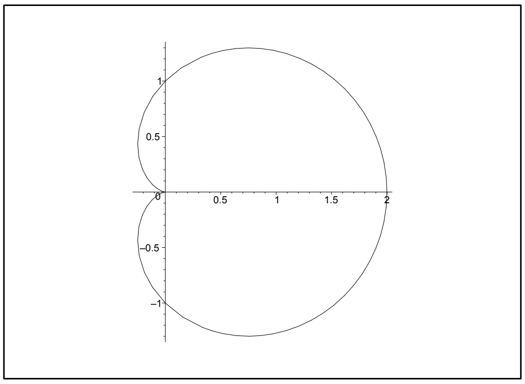

Figure 26.31: The cardioid for $a = 1$; ($\kappa\alpha\varrho\delta\iota\alpha$ = heart).

I. The line element is seen to be

$$\begin{aligned}
ds &= \sqrt{\{P(\varphi)\}^2 + \{P'(\varphi)\}^2}\, d\varphi = \sqrt{\{a(1+\cos\varphi)\}^2 + (-a\sin\varphi)^2}\, d\varphi \\
&= a\sqrt{(1+2\cos\varphi+\cos^2\varphi)+\sin^2\varphi}\, d\varphi = a\sqrt{2}\cdot\sqrt{1+\cos\varphi}\, d\varphi.
\end{aligned}$$

By a reduction we get

$$\begin{aligned}
\int_{\mathcal{C}} |y|\, ds &= \int_{-\pi}^{\pi} |P(\varphi)\sin\varphi| \cdot a\sqrt{2}\cdot\sqrt{1+\cos\varphi}\, d\varphi \\
&= \int_{-\pi}^{\pi} a(1+\cos\varphi)\cdot|\sin\varphi|\cdot a\sqrt{2}\cdot\sqrt{1+\cos\varphi}\, d\varphi \\
&= a^2\sqrt{2}\int_{-\pi}^{\pi}(1+\cos\varphi)^{\frac{3}{2}}|\sin\varphi|\, d\varphi \\
&= a^2\sqrt{2}\cdot 2\int_0^{\pi}(1+\cos\varphi)^{\frac{3}{2}}\sin\varphi\, d\varphi \\
&= 2\sqrt{2}a^2\int_{\varphi=0}^{\pi}(1+\cos\varphi)^{\frac{3}{2}}\cdot(-1)\, d\cos\varphi \\
&= -2\sqrt{2}\,a^2\left[\frac{2}{5}\,(1+\cos\varphi)^{\frac{5}{2}}\right]_0^{\pi} \\
&= -\frac{4\sqrt{2}}{5}\,a^2\left\{0 - 2^{\frac{5}{2}}\right\} = \frac{32a^2}{5}.
\end{aligned}$$

Without using some additional help MAPLE does not return the result.

C. *Weak control.* The result is of the correct dimension a^2. Furthermore, the integrand is positive almost everywhere, so the result must also be positive, which it is here. ◊

26.6 Examples of arc lengths and parametric descriptions by the arc length

Example 26.13 *Compute in each of the following cases the arc length of the plane curve $\mathcal{C}$ given by an equation of the form $y = Y(x)$, $x \in I$.*

1) *The arc length* $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve*

$$y = Y(x) = \frac{x^4 + 48}{24x}, \qquad x \in [2, 4].$$

2) *The arc length* $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve*

$$y = Y(x) = a \cosh \frac{x}{a}, \qquad x \in [-a, a].$$

[Cf. **Example 26.16.1** and **Example 32.7.8**.]

3) *The arc length* $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve*

$$y = Y(x) = \ln \frac{e^x - 1}{e^x + 1}, \qquad x \in [2, 4].$$

4) *The arc length* $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve*

$$y = Y(x) = x^{\frac{3}{2}}, \qquad x \in [0, 1].$$

5) *The arc length* $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve*

$$y = Y(x) = x^{\frac{2}{3}}, \qquad x \in [0, 1].$$

A Arc lengths of plane curves.

D Sketch the plane curve. Calculate the weight function $\sqrt{1 + Y'(x)^2}$ and reduce the line integral of integrand 1 to an ordinary integral.

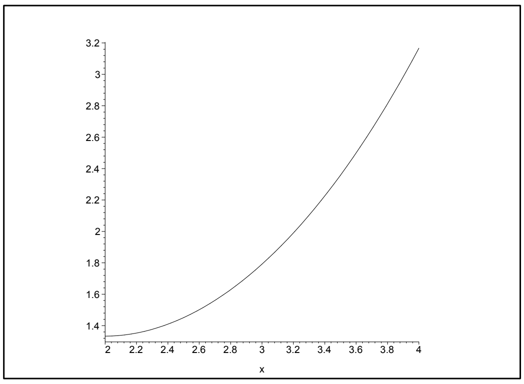

Figure 26.32: The curve $\mathcal{C}$ of **Example 26.13.1**.

I 1) It follows from

$$Y(x)=\frac{x^3}{24}+\frac{2}{x},\quad \text{thus}\quad Y'(x)=\frac{x^2}{8}-\frac{2}{x^2}=\frac{x^4-16}{8x^2},$$

that

$$\sqrt{1+Y'(x)^2}=\frac{1}{8x^2}\sqrt{64x^4+(x^4-16)^2}=\frac{1}{8x^2}\left(x^4+16\right)=\frac{x^2}{8}+\frac{2}{x^2}.$$

We get the arc length by insertion,

$$\begin{aligned}\int_{\mathcal{C}}\mathrm{d}s &= \int_2^4\sqrt{1+Y'(x)^2}\,\mathrm{d}x=\int_2^4\left\{\frac{x^2}{8}+\frac{2}{x^2}\right\}dx=\left[\frac{x^3}{24}-\frac{2}{x}\right]_2^4\\ &= \frac{64-8}{24}-\frac{2}{4}+\frac{2}{2}=\frac{56}{24}-\frac{1}{2}+1=\frac{7}{3}+\frac{1}{2}=\frac{17}{6}.\end{aligned}$$

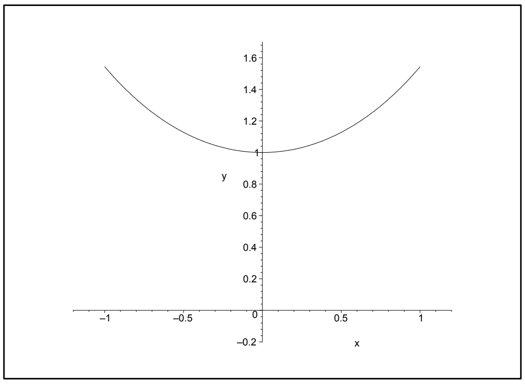

Figure 26.33: The curve $\mathcal{C}$ of **Example 26.13.2** for $a = 1$.

2) From $Y'(x) = \sinh \dfrac{x}{a}$ follows that

$$\sqrt{1+Y'(x)^2} = \sqrt{1+\sinh^2\left(\frac{x}{a}\right)} = \cosh \frac{h}{a}.$$

The arc length is

$$\int_{\mathcal{C}} \mathrm{d}s = \int_{-a}^{a} \cosh \frac{x}{a}\,\mathrm{d}x = a\left[\sinh \frac{x}{a}\right]_{-a}^{a} = 2a\sinh 1 = \frac{a}{e}\left(e^2-1\right).$$

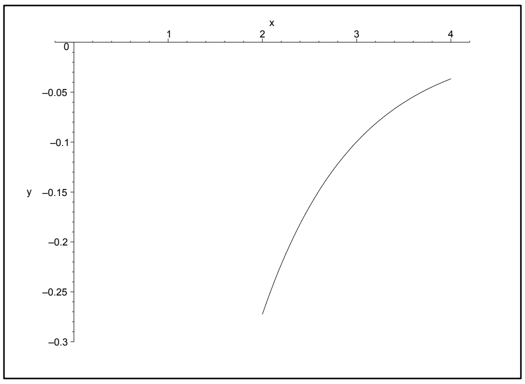

Figure 26.34: The curve $\mathcal{C}$ of **Example 26.13.3**.

3) It follows from

$$Y'(x) = \frac{e^x}{e^x-1} - \frac{e^x}{e^x+1} = \frac{2e^x}{e^{2x}-1},$$

that

$$\sqrt{1+Y'(x)^2} = \frac{1}{e^{2x}-1}\sqrt{\left(e^{2x}-1\right)^2+4e^{2x}} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{\cosh x}{\sinh x},$$

so the arc length becomes

$$\begin{aligned}\int_{\mathcal{C}} \mathrm{d}s &= \int_2^4 \frac{\cosh x}{\sinh x}\,\mathrm{d}x = [\ln\sinh x]_2^4 = \ln\left(\frac{\sinh 4}{\sinh 2}\right) = \ln\left(\frac{2\sinh 2\cosh 2}{\sinh 2}\right) \\ &= \ln(2\cosh 2) = \ln\left(e^2+e^{-2}\right) = \ln(e^4+1)-2.\end{aligned}$$

4) Here, $Y'(x) = \dfrac{3}{2}\sqrt{x}$, so

$$\sqrt{1+Y'(x)^2} = \sqrt{1+\frac{9}{4}\,x}.$$

The arc length is

$$\int_{\mathcal{C}} \mathrm{d}s = \int_0^1 \sqrt{1+\frac{9}{4}\,x}\,\mathrm{d}x = \frac{4}{9}\cdot\frac{2}{3}\left[\left(1+\frac{9}{4}\,x\right)^{\frac{3}{2}}\right]_0^1 = \frac{8}{27}\left\{\left(\frac{13}{4}\right)^{\frac{3}{2}}-1\right\} = \frac{1}{27}\,\{13\sqrt{13}-8\}.$$

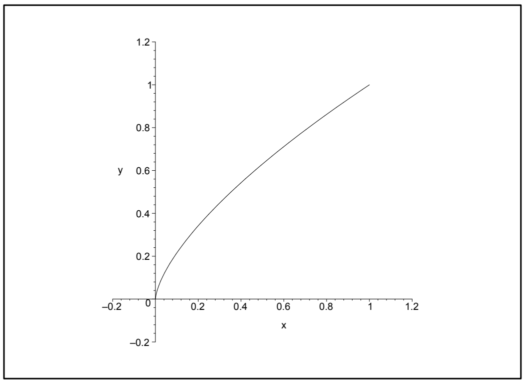

Figure 26.35: The curve $\mathcal{C}$ of **Example 26.13.5**.

5) Since the arc length of $y = x^{\frac{2}{3}}$, $x \in [0,1]$, is equal to the arc length of $x = y^{\frac{3}{2}}$, it follows from **Example 26.13.4** that

$$\int_{\mathcal{C}} \mathrm{d}s = \frac{1}{27}\,\{13\sqrt{13}-9\}.$$

Alternatively, $Y'(x) = \dfrac{2}{3}\,x^{-\frac{1}{3}}$, thus

$$\begin{aligned}\int_{\mathcal{C}} \mathrm{d}s &= \int_0^1 \sqrt{1+\frac{4}{9}\,x^{-\frac{2}{3}}}\,\mathrm{d}x = \int_0^1 x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}}+\frac{4}{9}}\,\mathrm{d}x = \frac{3}{2}\int_0^1 \sqrt{t+\frac{4}{9}}\,\mathrm{d}t \\ &= \left[\left(t+\frac{4}{9}\right)^{\frac{3}{2}}\right]_0^1 = \left(\frac{13}{9}\right)^{\frac{3}{2}} - \left(\frac{4}{9}\right)^{\frac{3}{2}} = \frac{13\sqrt{13}}{27} - \frac{8}{27} = \frac{1}{27}\,\{13\sqrt{13}-8\}.\end{aligned}$$

I

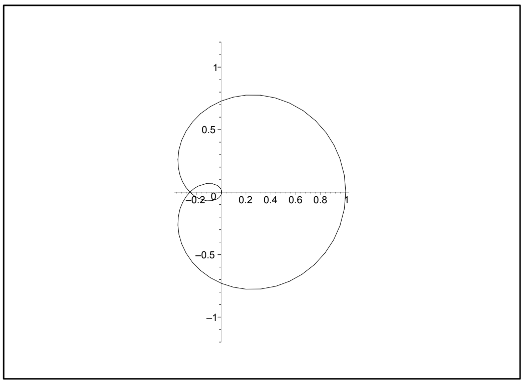

Figure 26.36: The curve $\mathcal{C}$ of **Example 26.14.1**.

Example 26.14 *Compute in each of the following cases the arc length of the given plane curve $\mathcal{K}$ by an equation in polar coordinates.*

1) The arc length $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve given by*

$$\varrho = a\,\cos^4 \frac{\varphi}{4}, \qquad \varphi \in [0, 4\pi].$$

2) The arc length $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve given by*

$$\varrho = a(1+\cos\varphi), \qquad \varphi \in [0, 2\pi].$$

3) The arc length $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve given by*

$$\varphi = \ln\varrho, \qquad \varrho \in [1, \mathrm{e}].$$

4) The arc length $\int_{\mathcal{C}} \mathrm{d}s$ *of the curve given by*

$$\varrho = a\sin^3 \frac{\varphi}{3}, \qquad \varphi \in [0, 3\pi].$$

A Arc lengths in polar coordinates.

D First calculate the weight function $\sqrt{\varrho^2 + \left(\dfrac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)}$ $\left(\text{possibly } \sqrt{1+\left(\varrho\,\dfrac{\mathrm{d}\varphi}{\mathrm{d}\varrho}\right)^2}\right)$, and then the line integral.

1) Since

$$\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi} = -a\cdot\cos^3\frac{\varphi}{4}\cdot\sin\frac{\varphi}{4},$$

the weight function is given by

$$\varrho^2 + \left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 = a^2\,\cos^8\frac{\varphi}{4} + a^2\,\cos^6\frac{\varphi}{4}\cdot\sin^2\frac{\varphi}{4} = a^2\,\cos^6\frac{\varphi}{4},$$

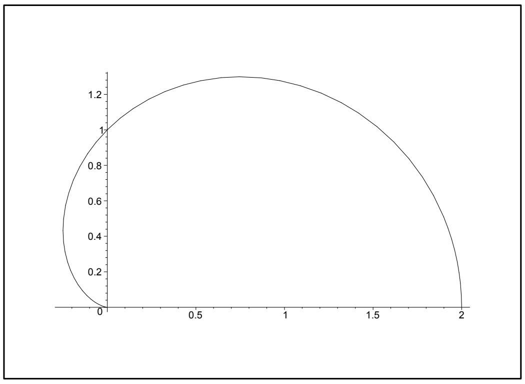

Figure 26.37: The curve $\mathcal{C}$ of **Example 26.14.2**.

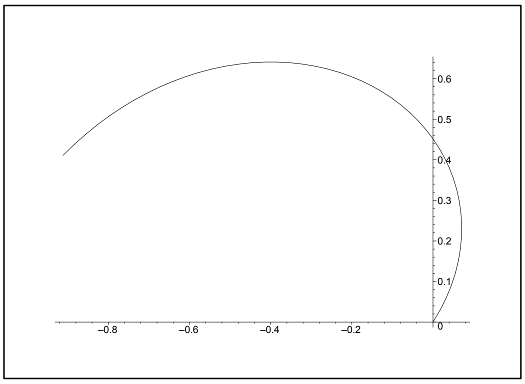

Figure 26.38: The curve $\mathcal{C}$ of **Example 26.14.3**. (Part of the curve of **Example 26.10.2**).

hence

$$\begin{aligned}\int_{\mathcal{C}} \mathrm{d}s &= \int_0^{4\pi} \sqrt{\varrho^2 + \left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2}\,\mathrm{d}\varphi = \int_0^{4\pi} a\left|\cos^3\frac{\varphi}{4}\right|\,\mathrm{d}\varphi = 4a\int_0^{\pi} |\cos^3 t|\,\mathrm{d}t \\ &= 8a\int_0^{\frac{\pi}{2}} \cos^3 t\,\mathrm{d}t = 8a\int_0^{\frac{\pi}{2}} (1-\sin^2 t)\cos t\,\mathrm{d}t = 8a\left[\sin t - \frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}} = \frac{16a}{3}.\end{aligned}$$

2) In this case,

$$\sqrt{\varrho^2 + \left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2} = a\sqrt{(1+\cos\varphi)^2 + \sin^2\varphi} = a\sqrt{2(1+\cos\varphi)} = a\sqrt{4\cos^2\frac{\varphi}{2}} = 2a\left|\cos\frac{\varphi}{2}\right|,$$

so

$$\int_{\mathcal{C}} \mathrm{d}s = \int_0^{2\pi} 2a\left|\cos\frac{\varphi}{2}\right|\,\mathrm{d}\varphi = 4a\int_0^{\pi} |\cos t|\,dt = 8a\int_0^{\frac{\pi}{2}} \cos t\,\mathrm{d}t = 8a.$$

3) From

$$\sqrt{1+\left\{\varrho\,\frac{d\varphi}{< da\varrho}\right\}^2}=\sqrt{1+\left\{\varrho\cdot\frac{1}{\varrho}\right\}^2}=\sqrt{2},$$

follows that

$$\int_{\mathcal{C}} ds=\int_1^e \sqrt{2}\,d\varrho=\sqrt{2}\,(e-1).$$

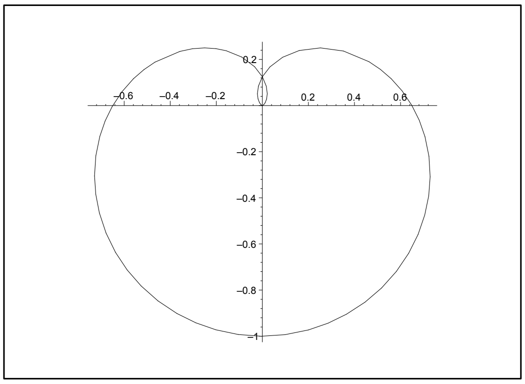

Figure 26.39: The curve $\mathcal{C}$ of **Example 26.14.4**.

ALTERNATIVELY, $\varrho = e^{\varphi}$, $\varphi \in [0,1]$, so (cf. **Example 26.10.1**)

$$\sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2} = \sqrt{2}\, e^{\varphi},$$

hence

$$\int_{\mathcal{C}} ds = \int_0^1 \sqrt{2}\, e^{\varphi}\, d\varphi = \sqrt{2}(e-1).$$

4) Here $\dfrac{d\varrho}{d\varphi} = a \cdot \sin^2 \dfrac{\varphi}{3} \cdot \cos \dfrac{\varphi}{3}$, so

$$\sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2} = a\sqrt{\sin^6 \frac{\varphi}{3} + \sin^4 \frac{\varphi}{3} \cdot \cos^2 \frac{\varphi}{3}} = a \cdot \sin^2 \frac{\varphi}{3},$$

thus

$$\int_{\mathcal{C}} ds = \int_0^{3\pi} a\, \sin^2 \frac{\varphi}{3}\, d\varphi = 3a \int_0^{\pi} \sin^2 t\, dt = \frac{3a}{2} \int_0^{\pi} (1 - \cos 2t)\, dt = \frac{3a\pi}{2}.$$

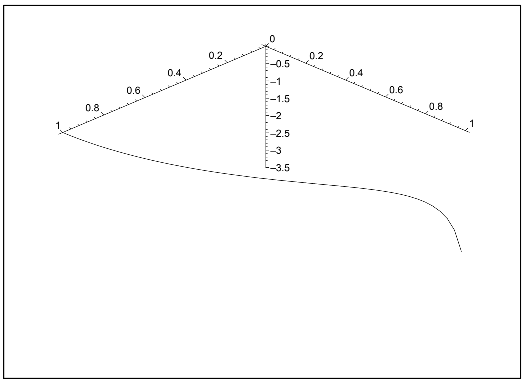

Figure 26.40: The space curve of **Example 26.15.1**.

Example 26.15 *Below are given some space curves by their parametric descriptions* $\mathbf{x} = \mathbf{r}(t)$*,* $t \in I$*. Express for each of the curves there parametric description with respect to arc length from the point of the parametric value* t_0*.*

1) The curve $\mathbf{r}(t) = (\cos t, \sin t, \ln \cos t)$*, from* $t_0 = 0$ *in the interval* $I = \left[0, \frac{\pi}{2}\right[$*.*

2) The curve $\mathbf{r}(t) = \frac{1}{\sqrt{3}}\left(e^t \cos t, e^t \sin t, e^t\right)$ *from* $t_0 = 0$ *in the interval* $I = \mathbb{R}$*.*
[Cf. **Example 26.3.7**.]

3) The curve $\mathbf{r}(t) = (\ln \cos t, \ln \sin t, \sqrt{2}\, t)$ *from* $t_0 = \frac{\pi}{4}$ *in the interval* $I = \left]0, \frac{\pi}{2}\right[$*.*

4) The curve $\mathbf{r}(t) = (7t + \cos t, 7t - \cos t, \sqrt{2}\, \sin t)$ *from* $t_0 = \frac{\pi}{2}$ *in the interval* $I = \mathbb{R}$*.*

5) The curve $\mathbf{r}(t) = (\cos(2t), \sin(2t), 2\cosh t)$ *from* $t_0 = 0$ *in the interval* $I = \mathbb{R}$*.*

6) The curve $\mathbf{r}(t) = (\cos t, \sin t, \ln \cos t)$ *from* $t_0 = 0$ *in the interval* $I = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$*.*
[Cf. **Example 26.3.5**.]

A Parametric description by the arc length.

D Find $s'(t) = \|\mathbf{r}'(t)\|$ and then $s = s(t)$ and $t = \tau(s)$, where we integrate from t_0. Finally, insert in $\mathbf{x} = \mathbf{r}(t) = \mathbf{r}(\tau(s))$.

I 1) From

$$\mathbf{r}'(t) = \left(-\sin t, \cos t, -\frac{\sin t}{\cos t}\right), \qquad t \in \left[0, \frac{\pi}{2}\right[,$$

follows that

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{\sin^2 t}{\cos^2 t}} = \frac{1}{\cos t},$$

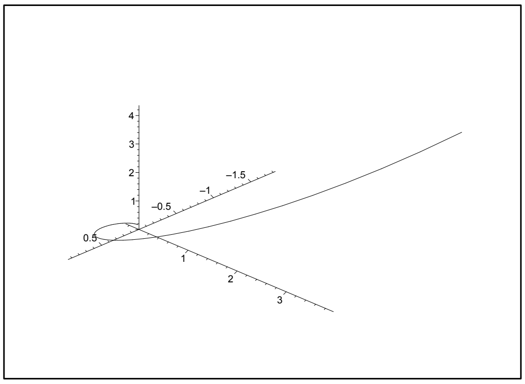

Figure 26.41: The space curve of **Example 26.15.2**.

hence

$$\begin{aligned} s(t) &= \int_0^t \frac{1}{\cos u}\,\mathrm{d}u = \int_0^t \frac{\cos u}{1-\sin^2 u}\,\mathrm{d}u = \int_0^t \frac{1}{2}\left(\frac{1}{1+\sin u}+\frac{1}{1-\sin u}\right)\cos u\,\mathrm{d}u \\ &= \frac{1}{2}\left[\ln\left(\frac{1+\sin u}{1-\sin u}\right)\right]_0^t = \frac{1}{2}\,\ln\left(\frac{1+\sin t}{1-\sin t}\right). \end{aligned}$$

Then

$$\frac{1+\sin t}{1-\sin t} = e^{2s}, \quad \text{i.e.} \quad \sin t = \frac{e^{2s}-1}{e^{2s}+1} = \tanh s, \quad s \geq 0.$$

Note that it follows from $t \in \left[0, \frac{\pi}{2}\right[$ that

$$\cos t = \frac{2e^s}{e^{2s}+1} = \frac{1}{\cosh s}.$$

Thus

$$t = \operatorname{Arcsin}\left(\frac{e^{2s}-1}{e^{2s}+1}\right)\ \operatorname{Arcsin}(\tanh s), \qquad s \geq 0,$$

and the parametric description by the arc length is

$$\begin{aligned} \mathbf{r}(s) &= (\cos t, \sin t, \ln\cos t) = \left(\frac{2e^s}{e^{2s}+1}, \frac{e^{2s}-1}{e^{2s}+1}, \ln\left(\frac{2e^s}{e^{2s}+1}\right)\right) \\ &= \left(\frac{1}{\cosh s}, \tanh s, -\ln\cosh s\right), \qquad s \geq 0. \end{aligned}$$

2) Here

$$\mathbf{r}'(t) = \frac{1}{\sqrt{3}}\,e^t\,(\cos t - \sin t, \cos t + \sin t, 1),$$

so

$$s'(t) = \|\mathbf{r}'(t)\| = \frac{1}{\sqrt{3}}\,e^t\,\sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1} = e^t,$$

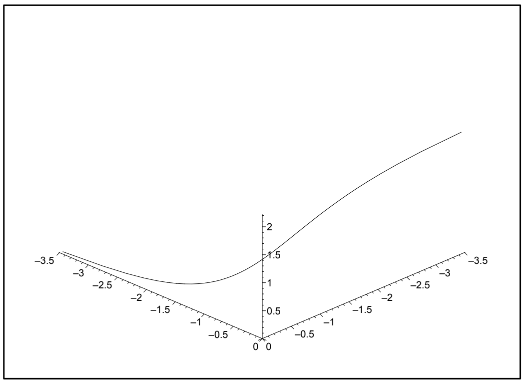

Figure 26.42: The space curve of **Example 26.15.3**.

hence

$$s(t)=\int_0^t e^u\,\mathrm{d}u=e^t-1,\quad \text{and}\quad t=\ln(s+1),\quad s>-1.$$

Finally, we get the parametric description by the arc length,

$$\begin{aligned}\mathbf{r}(s) &= \frac{1}{\sqrt{3}}\left((s+1)\cos(\ln(s+1)),(s+1)\sin(\ln(s+1)),s+1\right)\\ &= \frac{s+1}{\sqrt{3}}\left(\cos(\ln(s+1)),\sin(\ln(s+1)),1\right), \qquad s>-1.\end{aligned}$$

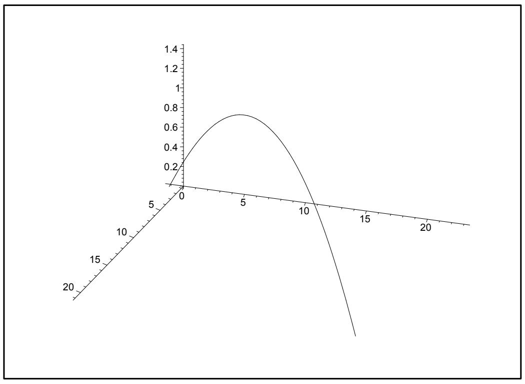

Figure 26.43: The space curve of **Example 26.15.4**.

3) From

$$\mathbf{r}'(t) = \left(-\frac{\sin t}{\cos t}, \frac{\cos t}{\sin t}, \sqrt{2}\right), \qquad t \in \left]0, \frac{\pi}{2}\right[,$$

follows that

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{\frac{\sin^2 t}{\cos^2 t} + 2 + \frac{\cos^2 t}{\sin^2 t}} = \frac{\sin t}{\cos t} + \frac{\cos t}{\sin t} = \frac{1}{\cos t \sin t},$$

as $t \in \left]0, \frac{\pi}{2}\right[$. Then

$$s(t) = \int_{\frac{\pi}{4}}^{t} \frac{1}{\cos u \sin u}\,\mathrm{d}u = \int_{\frac{\pi}{4}}^{t} \left(\frac{\sin u}{\cos u} + \frac{\cos u}{\sin u}\right)\mathrm{d}u = \ln \frac{\sin t}{\cos t} = \ln \tan t,$$

and thus $s \in \mathbb{R}$ and $\tan t = e^s$, and

$$\cos t = \frac{+1}{\sqrt{1+\tan^2 t}} = \frac{1}{\sqrt{1+e^{2s}}}, \quad \text{and} \quad \sin t = \frac{e^s}{\sqrt{1+e^{2s}}}.$$

The parametric description by the arc length is

$$\mathbf{r}(s) = \left(-\frac{1}{2}\ln(1+e^{2s}), s - \frac{1}{2}\ln(1+e^{2s}), \sqrt{2}\,\mathrm{Arctan}(e^s)\right), \quad s \in \mathbb{R}.$$

4) Here,

$$\mathbf{r}'(t) = (7 - \sin t, 7 + \sin t, \sqrt{2}\,\cos t),$$

so

$$\begin{aligned} s'(t) &= \|\mathbf{r}'(t)\| = \sqrt{(7-\sin t)^2 + (7+\sin t)^2 + 2\cos^2 t} \\ &= \sqrt{2\cdot 49 + 2\sin^2 t + 2\cos^2 t} = \sqrt{98+2} = 10, \end{aligned}$$

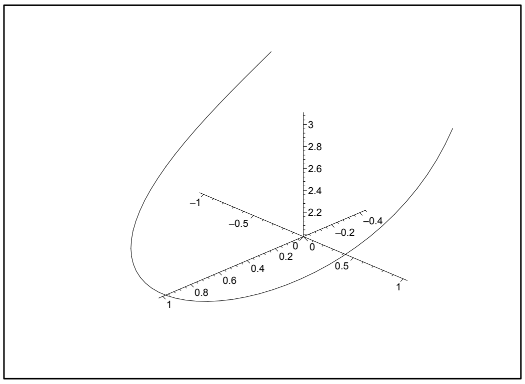

Figure 26.44: The space curve of **Example 26.15.5**.

and thus

$$s(t) = \int_{\frac{\pi}{2}}^{t} 10\,\mathrm{d}u = 10\left(t - \frac{\pi}{2}\right), \quad \text{hence} \quad t = \frac{s}{10} + \frac{\pi}{2}, \quad s \in \mathbb{R},$$

and the parametric description with the arc length as parameter from the point $\left(\frac{7\pi}{2}, \frac{7\pi}{2}, \sqrt{2}\right)$ is

$$\mathbf{r}(s) = \left(\frac{7s+35\pi}{10} - \sin\left(\frac{s}{10}\right), \frac{7s+35\pi}{10} + \sin\left(\frac{s}{10}\right), \sqrt{2}\,\cos\left(\frac{s}{10}\right)\right),$$

for $s \in \mathbb{R}$.

5) It follows from

$$\mathbf{r}'(t) = (-2\sin 2t, 2\cos 2t, 2\sinh t),$$

that

$$s'(t) = \|\mathbf{r}'(t)\| = 2\sqrt{\sin^2(2t) + \cos^2(2t) + \sinh^2 t} = 2\cosh t,$$

hence

$$s(t) \int_0^t 2\cosh u\,\mathrm{d}u = 2\sinh t,$$

so $s \in \mathbb{R}$ and

$$t = \operatorname{Arsinh}\left(\frac{s}{2}\right) = \ln\left(\frac{1}{2}\left(s + \sqrt{s^2+4}\right)\right), \qquad s \in \mathbb{R}.$$

The parametric description with the arc length as parameter is

$$\begin{aligned}\mathbf{r}(s) &= \left(\cos\left(2\operatorname{Arsinh}\left(\frac{s}{2}\right)\right), \sin\left(2\operatorname{Arsinh}\left(\frac{s}{2}\right)\right), 2\sqrt{1+\frac{s^2}{4}}\right) \\ &= \left(\cos\left(2\operatorname{Arsinh}\left(\frac{s}{2}\right)\right), \sin\left(2\operatorname{Arsinh}\left(\frac{s}{2}\right)\right), \sqrt{4+s^2}\right),\end{aligned}$$

for $s \in \mathbb{R}$.

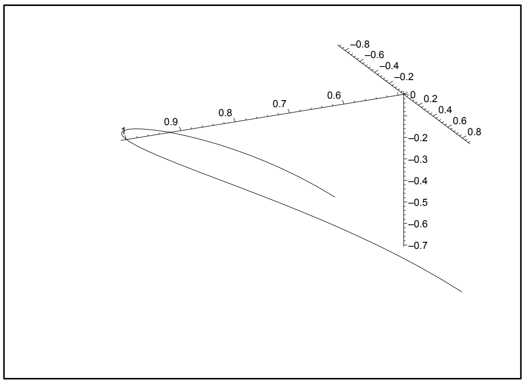

Figure 26.45: The space curve of **Example 26.15.6**; cf. **Example 26.15.1**.

6) This is an extension of the curve of **Example 26.15.1**, with the same parametric description evaluated from the same point $t_0 = 0$. We can therefore reuse this example, since the only change is that $s \in \mathbb{R}$,

$$\mathbf{r}(s) = \left(\frac{1}{\cosh s}, \tanh s, -\ln\cosh s \right), \qquad \text{for } s \in \mathbb{R}.$$

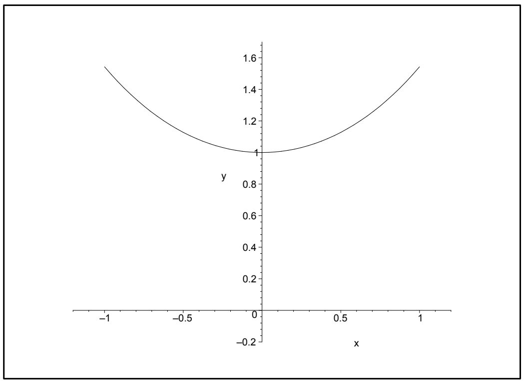

Figure 26.46: The chain curve for $a = 1$, cf. **Example 26.16.1**.

Example 26.16 *Find for every one of the given plane curves below an equation of the form*

(26.2) $\psi = \Psi(s)$,

where the signed arc length s is computed from a fixed point P_0 on the curve, while ψ is the angle between the oriented tangents at P_0 and at the point P on the curve given by s.

1) The chain curve given by $y = a\cosh\dfrac{x}{a}$, from P_0 given by $x = 0$.

[Cf. **Example 26.4.2**.]

2) The asteroid given by

$$\mathbf{r}(t) = a\left(-\cos^3 t, \sin^3 t\right), \qquad t \in \left[0, \frac{\pi}{2}\right],$$

from P_0 given by $t = 0$.

3) The winding of a circle given by

$$\mathbf{r}(t) = a(\cos t + t\sin t, \sin t - t\cos t), \qquad t \in \mathbb{R}_+,$$

from P_0 given by $t = 0$.

4) The cycloid given by

$$\mathbf{r}(t) = a(t - \sin t, 1 - \cos t), \qquad t \in [0, 2\pi],$$

from P_0 given by $t = \pi$.

It can by proved that (26.2) determines the curve uniquely with exception of its placement in the plane. Therefore, (26.2) is also called the natural equation of the curve.

A Natural equation.

D Find the arc length s, and then ψ by a geometrical analysis.

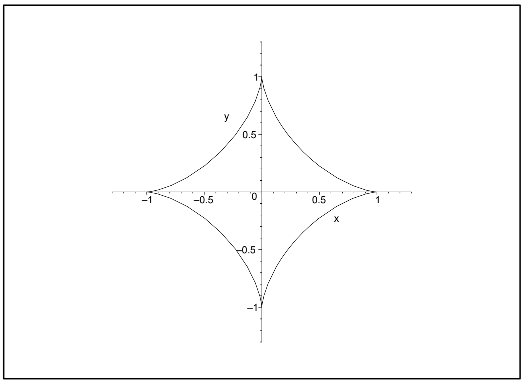

Figure 26.47: The asteroid of **Example 26.16.2**.

1) The point P_0 has the coordinates $(0, a)$. A parametric description of the chain curve is e.g.

$$\mathbf{r}(t) = \left(t, a\,\cosh\frac{t}{a}\right),$$

hence

$$\mathbf{r}'(t) = \left(1, \sinh\frac{t}{a}\right), \qquad \text{where } \mathbf{r}'(0) = (1, 0),$$

and thus $\psi = \text{Arctan}\left(\sinh\dfrac{t}{a}\right)$.

From

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{1 + \sinh^2\frac{t}{a}} = \cosh\frac{t}{a},$$

follows that

$$s(t) = \int_0^t \cosh\frac{u}{a}\,\mathrm{d}y = a\left[\sinh\frac{u}{a}\right]_0^t = a\,\sinh\left(\frac{t}{a}\right).$$

The natural equation is

$$\psi = \Psi(s) = \ \text{Arctan}\left(\frac{s}{a}\right).$$

2) The point P_0 has the coordinates $(-a, 0)$, and

$$\mathbf{r}'(t) = a\left(3\cos^2 t \cdot \sin t, 3\sin^2 t \cdot \cos t\right) = 3a\cos t \cdot \sin t(\cos t, \sin t).$$

For $t \to 0+$ we get $\mathbf{r}'(0) = \mathbf{0}$, and by considering a figure we may conclude that we have a horizontal half tangent. Then it follows that $\psi = t$.

It follows from

$$s'(t) = \|\mathbf{r}'(t)\| = 3a\,\cos t \cdot \sin t,$$

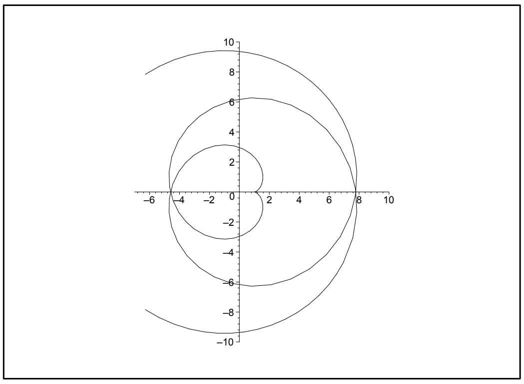

Figure 26.48: The winding of the circle of **Example 26.16.3**.

that

$$s(t)=\int_0^t 3a\,\cos u\cdot\sin u\,\mathrm{d}u=\frac{3a}{2}\int_0^t\sin 2u\,\mathrm{d}u=\frac{3a}{4}\{1-\cos 2t\},$$

hence

$$\cos 2t=1-\frac{4s}{3a},$$

and

$$\psi=\Psi(s)=t=\frac{1}{2}\,\text{Arccos}\left(1-\frac{4s}{3a}\right).$$

3) The point P_0 has the coordinates $(a,0)$, and

$$\mathbf{r}'(t)=a(-\sin t+\sin t+t\cos t,\cos t-\cos t+t\sin t)=at(\cos t,\sin t).$$

It follows that $\psi=t$.
As $t>0$ we have

$$s'(t)=\|\mathbf{r}'(t)\|=at,$$

thus

$$s(t)=a\int_0^t u\,\mathrm{d}u=\frac{a}{2}\,t^2,\qquad\text{hence } t=\sqrt{\frac{2s}{a}},$$

and hence

$$\psi=\Psi(s)=\sqrt{\frac{2s}{a}}.$$

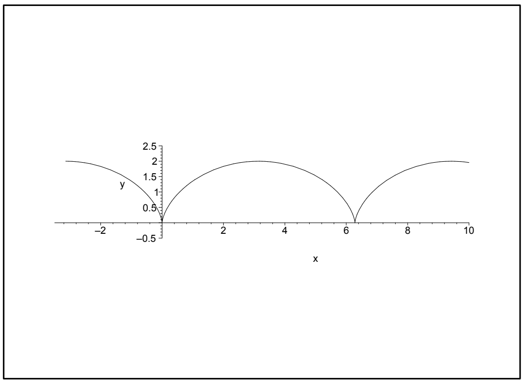

Figure 26.49: The cycloid of **Example 26.16.4**.

4) The point P_0 is described by $\mathbf{r}(\pi) = a(\pi, 0)$. The curve has a vertical half tangent at P_0. From

$$\mathbf{r}'(t) = a(1 - \cos t, \sin t) = a\left(2\sin^2\frac{t}{2}, 2\sin\frac{t}{2}\,\cos\frac{t}{2}\right) = 2a\sin\frac{t}{2}\left(\sin\frac{t}{2}, \cos\frac{t}{2}\right),$$

follows that

$$s'(t) = \|\mathbf{r}'(t)\| = 2a\sin\frac{t}{2},$$

so

$$s(t)=\int_{\pi}^{t} 2a\sin\frac{u}{2}\,\mathrm{d}u = 4a\left[-\cos\frac{u}{2}\right]_{\pi}^{t} = 4a\left\{-\cos\frac{t}{2}\right\},$$

and hence

$$\cos\frac{t}{2}=-\frac{s}{4a},\quad \text{hence}\quad t=2\,\mathrm{Arccos}\left(-\frac{s}{4a}\right).$$

Since ψ must have the form $at+b$, it is easy to derive that

$$\psi=\pi-t=\pi-2\,\mathrm{Arccos}\left(-\frac{s}{4a}\right).$$

Example 26.17 *A plane curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t)=\left(a\int_0^t \sin\left(u^2\right)\,\mathrm{d}u\,,\,a\int_0^t\cos\left(u^2\right)\,\mathrm{d}u\right),\qquad t\in\mathbb{R}.$$

The signed arc length from the point $(0,0)$ is called s.

1. *Find s, and find the parametric description of the curve given by the arc length.*

It is proved in Differential Geometry that any plane curve has a curvature

$$\kappa(t)=\frac{\{\mathbf{e}_z\times\mathbf{r}'(t)\}\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3},$$

where we let the plane of the curve be the (X,Y)-plane in the space.

2. *Prove that κ is proportional to s for $\mathcal{C}$.*

The curve under consideration has many names: the clothoid, Euler's spiral, Cornu's spiral*.*

Remark. "Clothoid" means in koiné, i.e. Ancient Greek: $\kappa\lambda\omega\theta\omega$ = I spin. $\Diamond$

A Parametric description with respect to arc length, curvature.

D Find ds and then compute.

I 1) As $s'(t)=\|\mathbf{r}'(t)\|$ and $\mathbf{r}'(t)=a\left(\sin\left(t^2\right),\cos\left(t^2\right)\right)$, we get

$$s'(t)=a\sqrt{\sin^2\left(t^2\right)+\cos^2\left(t^2\right)}=a,$$

so

$$s(t)=at\qquad\text{and}\qquad t(s)=\frac{1}{a}\,s.$$

The parametric description using the arc length is

$$\mathbf{x}=\mathbf{r}(t)=a\left(\int_0^t\sin\left(u^2\right)\,\mathrm{d}u\,,\int_0^t\cos\left(u^2\right)\,\mathrm{d}u\right)=a\left(\int_0^{\frac{s}{a}}\sin\left(u^2\right)\,\mathrm{d}u\,,\int_0^{\frac{s}{a}}\cos\left(u^2\right)\,\mathrm{d}u\right).$$

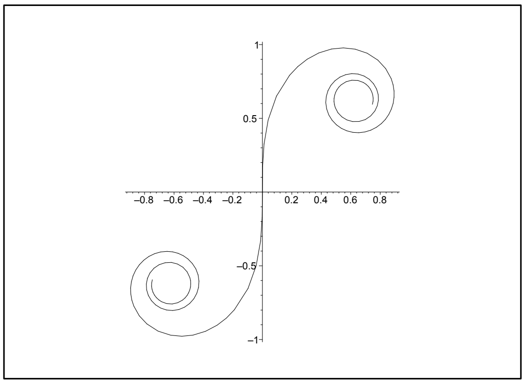

Figure 26.50: The clothoid for $a=1$ and $s \in [-4,4]$.

2) From

$$\mathbf{r}'(t) = a\left(\sin\left(t^2\right), \cos\left(t^2\right)\right) \sim a\left(\sin\left(t^2\right), \cos\left(t^2\right), 0\right),$$

and

$$\mathbf{r}''(t) = 2ta\left(\cos\left(t^2\right), -\sin\left(t^2\right)\right) \sim 2ta\left(\cos\left(t^2\right), -\sin\left(t^2\right), 0\right),$$

follows that

$$\begin{aligned}\{\mathbf{e}_z \times \mathbf{r}'(t)\}\cdot\mathbf{r}''(t) &= \begin{vmatrix} 2ta\cos\left(t^2\right) & -2ta\sin\left(t^2\right) & 0 \\ 0 & 0 & 1 \\ a\sin\left(t^2\right) & a\cos\left(t^2\right) & 0 \end{vmatrix} = -\begin{vmatrix} 2ta\cos\left(t^2\right) & -2ta\sin\left(t^2\right) \\ a\sin\left(t^2\right) & a\cos\left(t^2\right) \end{vmatrix} \\ &= -2ta^2.\end{aligned}$$

As $\|\mathbf{r}'(t)\| = a$, we finally get

$$\kappa = \frac{\{\mathbf{e}_z \times \mathbf{r}'(t)\}\cdot\mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3} = -\frac{2ta^2}{a^3} = -2\,\frac{t}{a} = -\frac{2s}{a^2}.$$

Example 26.18 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(\frac{1}{2}\,t^2 - \ln t, 2\sin t, 2\cos t\right), \qquad t \in [1,2].$$

Prove that $\|\mathbf{r}'(t)\| = t + \dfrac{1}{t}$, and find the length of $\mathcal{C}$.

A Arc length.

D Compute $\|\mathbf{r}'(t)\|$ og $\ell = \int_0^1 \|\mathbf{r}'(t)\|\,\mathrm{d}t$.

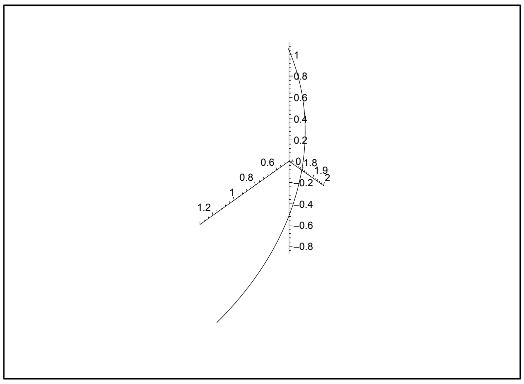

Figure 26.51: The curve $\mathcal{C}$.

I It follows from

$$\mathbf{r}'(t) = \left(t - \frac{1}{t}, 2\cos t, -2\sin t\right), \qquad t \in [1,2],$$

that

$$\|\mathbf{r}'(t)\|^2 = \left(t - \frac{1}{t}\right)^2 + 4\cos^2 t + 4\sin^2 t = t^2 - 2 + \frac{1}{t^2} = \left(t + \frac{1}{t}\right)^2,$$

thus

$$\|\mathbf{r}'(t)\| = \left|t + \frac{1}{t}\right| = t + \frac{1}{t},$$

and accordingly,

$$\ell = \int_1^2 \|\mathbf{r}'(t)\|\,dt = \int_1^2 \left(1 + \frac{1}{t}\right)\,dt = \left[\frac{t^2}{2} + \ln t\right]_1^2 = \frac{3}{2} + \ln 2.$$

Example 26.19 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(e^{3t}, e^{-3t}, \sqrt{18}\,t\right), \qquad t \in [-1,1].$$

Prove that $\|\mathbf{r}'(t)\| = 3\left(e^{3t} + 3^{-3t}\right)$, *and find the arc length of* $\mathcal{C}$.

A Arc length.

D Find $\mathbf{r}'(t)$.

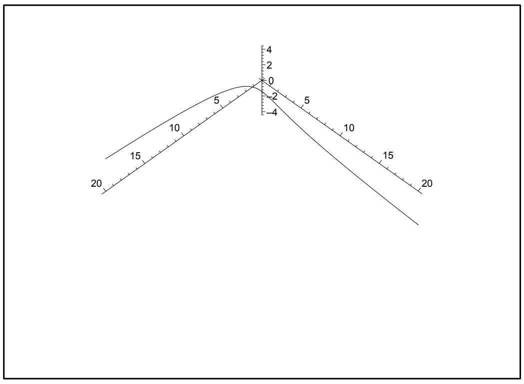

Figure 26.52: The curve $\mathcal{C}$.

I We get by differentiation

$$\mathbf{r}'(t) = \left(3\,e^{3t}, -3\,e^{-3t}, 3\sqrt{2}\right) = 3\left(e^{3t}, -e^{-3t}, \sqrt{2}\right),$$

thus

$$\|\mathbf{r}'(t)\| = 3\sqrt{(e^{3t})^2 + (-e^{-3t})^2 + 2} = 3\sqrt{(e^{3t} + e^{-3t})^2} = 3\left(e^{3t} + e^{-3t}\right),$$

and we get the arc length

$$\begin{aligned} \ell(\mathcal{C}) &= \int_{-1}^{1} \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_{-1}^{1} 3\left(e^{3t} + e^{-3t}\right)\,\mathrm{d}t \\ &= 2\int_0^1 3 \cdot 2\cosh 3t\,dt = 4[\sinh 3t]_0^1 = 4\sinh 3. \end{aligned}$$

Example 26.20 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(\frac{1}{3}\,t^3 - t, \frac{1}{3}\,t^3 + t, t^2\right), \qquad t \in [-1, 1].$$

Find the arc length of $\mathcal{C}$.

A Arc length.

D Find $\|\mathbf{r}'(t)\|$.

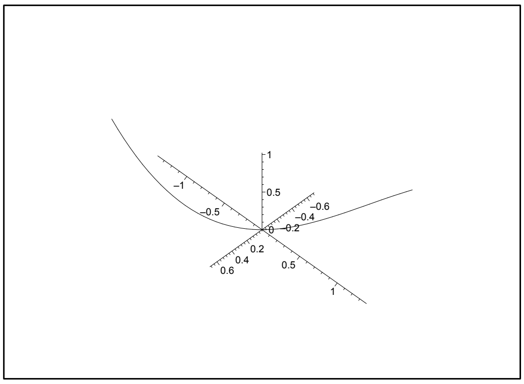

Figure 26.53: The curve $\mathcal{C}$.

I It follows from

$$\mathbf{r}'(t) = (t^2 - 1, t^2 + 1, 2t),$$

that

$$\|\mathbf{r}'(t)\|^2 = (t^2-1)^2 + (t^2+1)^2 + 4t^2 = 2t^4 + 2 + 4t^2 = 2(t^2+1)^2,$$

hence

$$\ell(\mathcal{C}) = \int_{-1}^{1} \|\mathbf{r}'(t)\|\,\mathrm{d}t = 2\int_0^1 \sqrt{2}(t^2+1)\,\mathrm{d}t = 2\sqrt{2}\left(\frac{1}{3} + 1\right) = \frac{8\sqrt{2}}{3}.$$

Example 26.21 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = \left(6t^2, 4\sqrt{2}\,t^3, 3t^4\right), \qquad t \in [-1,1].$$

Explain why the curve is symmetric with respect to the (X,Z)-plane. Then find the arc length of $\mathcal{C}$.

A Arc length.

D Replace t by $-t$. Then find $\mathbf{r}'(t)$.

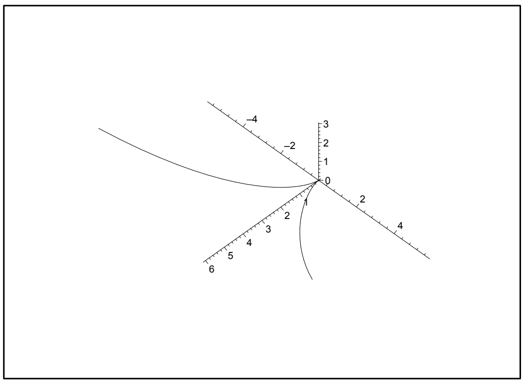

Figure 26.54: The curve $\mathcal{C}$.

I It follows from $\mathbf{r}(-t) = \left(6t^2, -4\sqrt{2}\,t^3, 3t^4\right)$, that the curve is symmetric with respect to the (X,Z)-plane.

From

$$\mathbf{r}'(t) = \left(12t, 12\sqrt{2}\,t^2, 12t^3\right) = 12t\left(1, \sqrt{2}\,t, t^2\right)$$

follows that

$$\|\mathbf{r}'(t)\| = 12|t| \cdot \sqrt{1+2t^2+t^4} = 12|t| \cdot (1+t^2).$$

Finally, when we exploit the symmetry above and put $u = t^2$, we find the arc length

$$\ell(\mathcal{C}) = 2\int_0^1 \|\mathbf{r}'(t)\|\,\mathrm{d}t = 2\int_0^1 12t(1+t^2)\,\mathrm{d}t = 12\int_0^1 (1+u)\,\mathrm{d}u = 12\left(1+\frac{1}{2}\right) = 18.$$

Example 26.22 *A space curve $\mathcal{C}$ is given by the parametric description*

$$\mathbf{r}(t) = (t + \sin t, \sqrt{2}\,\cos t, t - \sin t), \qquad t \in [-1, 1].$$

1) Find a parametric description of the tangent of $\mathcal{K}$ at the point corresponding to

$$t = 0.$$

2) Compute the arc length of $\mathcal{C}$.

A Space curve.

D Follow the standard method.

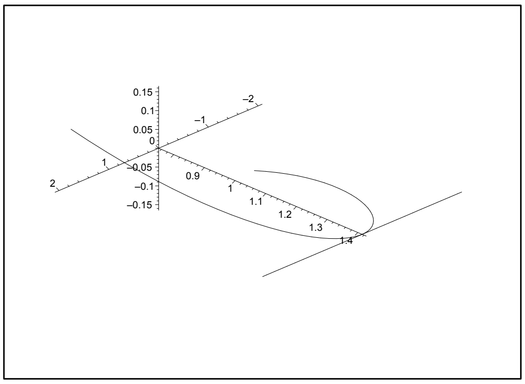

Figure 26.55: The curve $\mathcal{C}$ and its tangent at $(0, \sqrt{2}, 0)$.

I 1) As $\mathbf{r}(0) = (0, \sqrt{2}, 0)$, and

$$\mathbf{r}'(t) = (1 + \cos t, -\sqrt{2}\,\sin t, 1 - \cos t), \qquad \mathbf{r}'(0) = (2, 0, 0),$$

it follows that a parametric description of the tangent of $\mathcal{C}$ at $(0, \sqrt{2}, 0)$ is given by

$$\mathbf{x}(u) = (0, \sqrt{2}, 0) + (2u, 0, 0) = (2u, \sqrt{2}, 0), \qquad u \in \mathbb{R}.$$

2) The arc length of $\mathcal{C}$ is

$$\begin{aligned}\int_{-1}^{1} \|\mathbf{r}'(t)\|\,\mathrm{d}t &= \int_{-1}^{1} \sqrt{(1+\cos t)^2 + 2\sin^2 t + (1-\cos t)^2}\,\mathrm{d}t \\ &= \int_{-1}^{1} \sqrt{2 + 2\cos^2 t + 2\sin^2 t}\,\mathrm{d}t = \int_{-1}^{1} \sqrt{4}\,\mathrm{d}t = 2 \cdot 2 = 4.\end{aligned}$$

27 The surface integral

27.1 The reduction theorem for a surface integral

Given a two-dimensional (piecewise) C^1-surface $\mathcal{F}$ in $\mathbb{R}^2$. We shall in this chapter see, how we can integrate a continuous function f, defined on $\mathcal{F}$, over the surface $\mathcal{F}$. We shall use the notation

$$\int_{\mathcal{F}} f(\mathbf{x})\,\mathrm{d}S$$

for this surface integral. In the following we make sense of this abstract symbol.

Roughly speaking, the introduction of the surface integral should in some sense follow that of the line integral in Chapter 26. The present complication is of course that we are now dealing with two dimensions instead of just one, and two-dimensional connected sets may have a far more complicated boundary than a one-dimensional connected set, the boundary of which only consists of at most two points. Furthermore, if $\mathcal{F}$ is a C^1-surface (of dimension 2) in $\mathbb{R}^3$, then clearly $\mathcal{F}$ does not have interior points in $\mathbb{R}^3$, and yet we have a sense of the existence of an *intrinsic boundary* of $\mathcal{F}$, which we shall denote by $\delta\mathcal{F}$. Obviously, $\delta\mathcal{F}$ must not have a too complicated geometrical structure.

Besides the geometry of the surface the *area* and the *area element* also play key roles in the definition of the surface integral. For later use we denote the *area* of the surface $\mathcal{F}$ by $\text{area}(\mathcal{F})$, though we still have not given the slightest clue of how to find $\text{area}(\mathcal{F})$.

We shall first analyze the *area element* $\mathrm{d}S$. In order to do that we assume that $\mathcal{F}$ is a C^1-surface in $\mathbb{R}^3$, given by

$$\mathcal{F} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(u,v) \text{ for } (u,v) \in E\right\},$$

where $E \subseteq \mathbb{R}^2$ is a *parameter domain* in the usual plane, and $\mathbf{r} : E \to \mathbb{R}^3$ is of class C^1.

The idea is to approximate a small surface area element ΔS by a small parallelogram in the neighbourhood, because we can calculate the area of the parallelogram. The price is that this parallelogram does not have to lie in $\mathcal{F}$, and seldom does, with the exception of the reference point. This construction is done by introducing the *parameter curves* through the point P under consideration.

Let $P \in \mathcal{F}$ be the point corresponding to the parameters $(u,v) \in E^\circ$, i.e. in the interior of E. Since E° is open, we have for small Δu, $\Delta v \neq 0$ that $(u+\Delta u, v)$, $(u, v+\Delta v) \in E$, and we can define the two small vectors

$$\mathbf{U} := \mathbf{r}(u+\Delta u, v) - \mathbf{r}(u,v) \qquad \text{and} \qquad \mathbf{V} = \mathbf{r}(u, v+\Delta v) - \mathbf{r}(u,v),$$

cf. Figure 27.1, where the parallelogram spanned by $\mathbf{U}$ and $\mathbf{V}$ approximates the small area element ΔS lying on $\mathcal{F}$.

We note that

$$\mathbf{U} \simeq \mathbf{r}'_u(u,v)\Delta u \qquad \text{and} \qquad \mathbf{V} \simeq \mathbf{r}'_v(u,v)\Delta v.$$

Since $\mathbf{U}$ and $\mathbf{V}$ are vectors in $\mathbb{R}^3$, we can form the *vector product* $\mathbf{N} = \mathbf{U} \times \mathbf{V}$, which is perpendicular to both $\mathbf{U}$ and $\mathbf{V}$, hence a *normal* to the surface $\mathcal{F}$ at P. Furthermore, it is well-known from *Linear Algebra* that the length $\|\mathbf{N}\| = \|\mathbf{U} \times \mathbf{V}\|$ is equal to the *area* of the parallelogram defined by the two vectors $\mathbf{U}$ and $\mathbf{V}$. We therefore get

$$\Delta S \simeq \mathbf{U} \times \mathbf{V}\| \simeq \|\mathbf{N}(u,v)\|\Delta u\Delta v = \|\mathbf{r}'_u \times \mathbf{r}'_v\|\,\Delta u\Delta v.$$

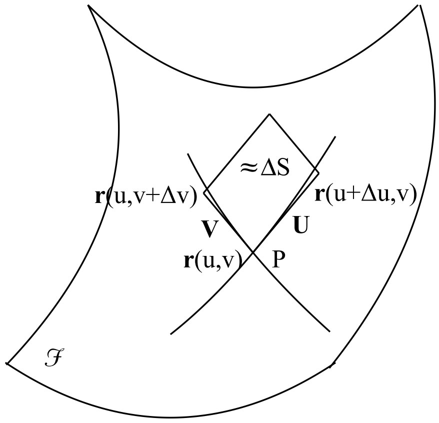

Figure 27.1: Approximation of the area element ΔS.

Hence, we may expect that we in the limit may write

$$\mathrm{d}S = \|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v = \|\mathbf{r}_u' \times \mathbf{r}_v'\|\,\mathrm{d}u\,\mathrm{d}v.$$

This is correct in all the cases we shall consider in the following. Here we only mention that it is possible to create some geometric examples where this construction fails. These unexpected examples will not be relevant in this series of books.

We formulate without proof

Theorem 27.1 The reduction theorem for a surface integral *Assume that $\mathcal{F}$ is a C^1-surface in $\mathbb{R}^3$ of parametric description $\mathbf{r} : E \to \mathbb{R}^3$, where the parameter domain $E \subset \mathbb{R}^3$ is bounded and closed. We assume that $\mathbf{r}$ is injective almost everywhere, and also that*

$$\mathbf{N} = \mathbf{r}'_u \times \mathbf{r}'_v \neq \mathbf{0} \qquad \textit{almost everywhere.}$$

Let $\mathcal{F} \subseteq A \subseteq \mathbb{R}^3$, and assume that $f : A \to \mathbb{R}$ is a continuous function. Then the surface integral is reduced to an ordinary plane integral by the formula

$$\int_{\mathcal{F}} f(\mathbf{x})\,\mathrm{d}S = \int_E f(\mathbf{r}(u,v))\|\mathbf{N}(u,v)\|\,\mathrm{d}u\,dv.$$

Theorem 27.1 reduces a surface integral to an ordinary plane integral at the cost of an additional factor, the *weight function* $\|\mathbf{N}(u,v)\|$, which is the length of the *normal vector* with respect to the given parametric description $\mathbf{r} : E \to \mathbb{R}^3$. This (abstract) plane integral is then again in the next step reduced to a double integral by using some of the previous reduction theorems from Chapter 20. So in principal we calculate a surface integral by the following scheme,

surface integral $\to$ plane integral $\to$ double integral.

If we choose $f \equiv 1$, we of course get the *area* of $\mathcal{F}$,

Theorem 27.2 The area of a surface $\mathcal{F}$. *Let $\mathcal{F}$ be a C^1-surface of parametric description $\mathbf{r} : E \to \mathbb{R}^3$, where $E \subset \mathbb{R}^2$ is a closed and bounded domain, and $\mathbf{r}$ is injective almost everywhere, as well as the normal $\mathbf{N} \neq \mathbf{0}$ almost everywhere. Then the area of $\mathcal{F}$ is given by the weighted plane integral*

$$\textit{area}(\mathcal{F}) = \int_E \|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v.$$

In the following sections we shall give reduction formulæ in some important special cases.

27.1.1 The integral over the graph of a function in two variables

Consider a surface $\mathcal{F}$ which in rectangular coordinates is the graph of the equation

$$z = Z(x,y), \qquad \text{for } (x,y) \in E,$$

where $Z \in C^1(E)$. Then the parametric description of $\mathcal{F}$ is given by

$$\mathbf{x} = (x,y,z) = \mathbf{r}(x,y) = (x,y,Z(x,y)),$$

so

$$\mathbf{r}'_x(x,y) = (1,0,Z'_x(x,y)) \qquad \text{and} \qquad \mathbf{r}'_y(x,y) = \left(0,1,Z'_y(x,y)\right),$$

hence by a method known from *Linear Algebra*,

$$\mathbf{N}(x,y) = \mathbf{r}'_x \times \mathbf{r}'_y = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & Z'_x \\ 0 & 1 & Z'_y \end{vmatrix} = \left(-Z'_x(x,y), -Z'_y(x,y), 1\right).$$

The length of $\mathbf{N}$ is

$$\|\mathbf{N}(x,y)\| = \sqrt{1 + \{Z_x'(x,y)\}^2 + \{Z_y'(x,y)\}^2},$$

so we have proved,

Theorem 27.3 Reduction theorem for the surface integral over a graph. *Let $\mathcal{F}$ be a C^1-graph of the function*

$$z = Z(x,y), \qquad \textit{for } (x,y) \in E,$$

where $E \subset \mathbb{R}^2$ is a closed and bounded domain. Let f be a continuous function on $\mathcal{F}$. Then the surface integral of f over the graph of Z is reduced in the following way,

$$\int_{\mathcal{F}} f(x,y,z)\, \mathrm{d}S = \int_E f(x,y,Z(x,y)) \cdot \sqrt{1 + \{Z_x'(x,y)\}^2 + \{Z_y'(x,y)\}^2}\, \mathrm{d}S.$$

When we as in Theorem 27.3 are considering a graph of $z = Z(x,y)$, then also

$$\|\mathbf{N}\| = \frac{1}{\cos\vartheta},$$

where ϑ is the angle between the normal $\mathbf{N}$ and the z-axis. Note that $0 \le \vartheta < \frac{\pi}{2}$, because if $\vartheta = \frac{\pi}{2}$, the $\mathcal{F}$ would not be a graph. In particular, $\cos\vartheta > 0$.

Concerning the area of the graph we have the following theorem.

Theorem 27.4 Area of a graph. *Let $\mathcal{F}$ be the C^1-graph of the function*

$$z = Z(x,y) \qquad \textit{for } (x,y) \in E,$$

where E is a closed and bounded domain in $\mathbb{R}^2$. Then the area of $\mathcal{F}$ is given by

$$\operatorname{area}(\mathcal{F}) = \int_E \sqrt{1 + \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2}\, \mathrm{d}S = \int_E \frac{1}{\cos\vartheta(x,y)}\, \mathrm{d}S.$$

27.1.2 The integral over a cylindric surface

For convenience we shall assume that the *cylindric surface* $\mathcal{C}$ is given by the parametric description

$$\mathcal{C}: \quad \mathbf{r}(t,z) = (X(t), Y(t), z), \qquad t \in [a,b], \quad z \in [Z_1(t), Z_2(t)]\,.$$

Then the normal $\mathbf{N}$ with respect to the given parametric description $\mathbf{r}$ is given by

$$\mathbf{N}(t,z) = (Y'(t), -X'(t), 0).$$

We therefore get in rectangular coordinates,

Theorem 27.5 Reduction theorem of an integral over a cylindric surface *Let $\mathcal{C}$ be a cylindric surface, given by*

$$\mathbf{r}(t,z) = (X(t), Y(t), z), \qquad t \in [a,b], \quad z \in [Z_1(t), Z_2(t)]\,.$$

Then the integral over $\mathcal{C}$ is reduced in the following way as a double integral,

$$\int_{\mathcal{C}} f(x,y,z)\,\mathrm{d}S = \int_a^b \left\{ \int_{Z_1(t)}^{Z_2(t)} f(X(t), Y(t), z)\,\mathrm{d}z \right\} \sqrt{\{X'(t)\}^2 + \{Y'(t)\}^2}\,\mathrm{d}t.$$

If we instead use semi-polar coordinates, so the cylinder is perpendicular to the plane curve

$$\mathcal{L}: \quad \varrho = P(t), \quad \varphi = \Phi(t), \qquad \text{for } t \in [a,b],$$

then the (surface) integral over $\mathcal{C}$ is reduced in the following way,

$$\int_{\mathcal{C}} f(x,y,z)\,\mathrm{d}S = \int_a^b \left\{ \int_{Z_1(t)}^{Z_2(t)} f(P(t)\cos\Phi(t), P(t)\sin\Phi(t), x)\,\mathrm{d}z \right\} \sqrt{\{P'(t)\}^2 + \{P(t)\Phi'(t)\}^2}\,\mathrm{d}t.$$

27.1.3 The integral over a surface of revolution

For convenience we assume that the surface $\mathcal{O}$ of revolution has the z-axis as its axis. The *meridian curve* is denoted $\mathcal{M}$. We use that

$$\mathbf{r}(t,\varphi) = (P(t)\cos\varphi, P(t)\sin\varphi, Z(t)), \qquad t \in [a,b], \quad \varphi \in [0,2\pi],$$

is a parametric description of $\mathcal{O}$.

The *meridian curve* $\mathcal{M}$ is in semi-polar coordinates given by

$$\varrho = P(t) \quad \text{and} \quad z = Z(t) \qquad \text{for } t \in [a,b].$$

We have previously found – or it is easy to calculate again – that the normal vector is then given by

$$\mathbf{N}(t,\varphi) = P(t)\{-Z'(t)\cos\varphi, -Z'(t)\sin\varphi, P'(t)\}.$$

Then the reduction formular for the surface integral becomes

$$\int_{\mathcal{O}} f(x,y,z)\,\mathrm{d}S = \int_a^b \left\{\int_0^{2\pi} f(P(t)\cos\varphi, P(t)\sin\varphi, Z(t))\,\mathrm{d}\varphi\right\} P(t)\sqrt{\{P'(t)\}^2+\{Z'(t)\}^2}\,\mathrm{d}t.$$

If we instead use *spherical coordinates*, then the *meridian curve* $\mathcal{M}$ is given by

$$r = R(t) \quad \text{and} \quad \theta = \Theta(t) \qquad \text{for } t \in [a,b].$$

Since

$$\varrho = r\sin\theta \qquad \text{and} \qquad z = r\cos\theta,$$

we get

$$P(t) = R(t)\sin\Theta(t) \qquad \text{and} \qquad Z(t) = R(t)\cos\Theta(t),$$

so the reduction formula for the surface integral in spherical coordinates is given by

$$\int \mathcal{C} f(x,y,z)\,\mathrm{d}S = \int_a^b \left\{\int_0^{2pi} F(t,\varphi)\,\mathrm{d}\varphi\right\} R(t)\sin\Theta(t)\sqrt{\{R'(t)\}^2+\{R(t)\Theta'(t)\}^2}\,\mathrm{d}t,$$

where we for short have written

$$F(t,\varphi) := f(R(t)\sin\Theta(t)\cos\varphi, R(t)\sin\Theta(t)\sin\varphi, R(t)\cos\Theta(t)).$$

The latter equation is also written in the following more compact (and abstract) form

$$\int_{\mathcal{C}} f(x,y,z)\,\mathrm{d}S = \int_{\mathcal{M}} \left\{\int_0^{2\pi} f(\varrho\cos\varphi, \varrho\sin\varphi, z)\,\mathrm{d}\varphi\right\} \varrho\,\mathrm{d}s.$$

All these formulæ may at a first glance look scaring, but in practical applications it is usually easy to see what we should do. The worst complication is probably that when we take the length of the normal vector, then we almost always will get a square root in the new integrand, and functions like square roots and absolute values are always difficult to integrate.

27.2 Procedures for reduction of a surface integral

We consider 2-dimensional surfaces imbedded in $\mathbb{R}^3$. The idea is to pull the integration over the surface $\mathcal{F}$ back to a *plane integral* over *the parameter domain* E, where we can use one of the methods from Chapter 20. This procedure has its price because we must add some *weight function* as a factor to the integrand.

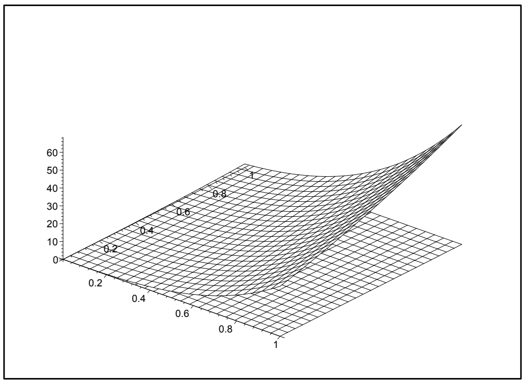

Figure 27.2: Example of a surface $\mathcal{F}$ with a corresponding parameter domain E in the (x, y)-plane.

Procedure:

1) Write down a *rectangular* parameter representation of the surface $\mathcal{F}$:

$$(x, y, z) = \mathbf{r}(u, v), \qquad (u, v) \in E.$$

The parameter domain $E \subseteqq \mathbb{R}^2$ is then identified and sketched (a set lying in the plane).

Remark 27.1 It is not always possible to sketch $\mathcal{F}$ in the space, but this does not matter much, because the real calculations are taking place in the *parameter domain* E. $\Diamond$

2) Determine the *weight function:* Calculate the vectors

$$\mathbf{r}'_u(u, v) \qquad \text{and} \qquad \mathbf{r}'_v(u, v),$$

and the corresponding *normal vector* to the surface $\mathcal{F}$ in this parameter representation,

$$\mathbf{N}(u, v) = \mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

The wanted *weight function* is $\|\mathbf{N}(u, v)\|$ (calculate it), and the *surface element* is

$$\mathrm{d}S = \|\mathbf{N}(u, v)\| \, \mathrm{d}u \, \mathrm{d}v.$$

3) Insert the parameter representation and the surface element, and calculate the right hand side by applying one of the methods from Chapter refch20,

$$\int_{\mathcal{F}} f(x,y,z)\,\mathrm{d}S = \int_E f(\mathbf{r}(u,v))\,\|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v.$$

If $f(x,y,z) \equiv 1$, the surface integral is interpreted as the *area of the surface* $\mathcal{F}$. In this case we get

Theorem 27.6 Surface area:

$$\mathit{area}(\mathcal{F}) = \int_{\mathcal{F}} \mathrm{d}S = \int_E \|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v.$$

It is of course possible also to use some known area formulæ instead of calculating the cumbersome integral above. If for instance $\mathcal{F}$ is the surface of a sphere of radius r, then it is well-known that

$$\mathrm{area}(\mathcal{F}) = \mathrm{area}(\partial B[0,r]) = 4\pi r^2.$$

Special cases:

In the following special cases we reduce 2) in the procedure by inserting the given area element $\mathrm{d}S$.

1) **Integral over a graph** for $z = f(x,y)$, *rectangular:*

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

(Compare with Section 26.3 on the line integral, the case of a graph.)

2) **Integral over a cylindric surface** $\mathbf{r}(t,z) = (X(t), Y(t), z)$, *rectangular:*

$$\mathrm{d}S = \sqrt{\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}Y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t\,\mathrm{d}z = \mathrm{d}s\,\mathrm{d}z,$$

where $\mathrm{d}s$ is the *curve element* for $\tilde{\mathbf{r}}(t) = (X(t), Y(t))$ in the *plane*, cf. Chapter 26. Line integral.

3) **Integral over a rotational surface**

$$\mathbf{r}(t,\varphi) = (P(t)\,\cos\varphi, P(t)\,\sin\varphi, Z(t)),$$

semi-polar

$$\mathrm{d}S = P(t)\sqrt{\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t\,\mathrm{d}\varphi = \varrho\,\mathrm{d}s\,\mathrm{d}\varphi.$$

The latter abstract form

$$\mathrm{d}S = \varrho\,\mathrm{d}s\,\mathrm{d}\varphi,$$

can often be of some help when one is using the geometry (sketch a figure) when one sets up the reduced plane integral.

27.3 Examples of surface integrals

Example 27.1

A. *Find the surface integral* $I = \int_{\mathcal{F}} |z| \, \mathrm{d}S$, *where* $\mathcal{F}$ *is given by the parametric representation*

$$(x, y, z) = \mathbf{r}(u, v) = (u \, \sin v, u \, \cos v, u \, v) = u \, (\sin v, \cos v, v),$$

where $-1 \leq u \leq 1$, $0 \leq v \leq 1$.

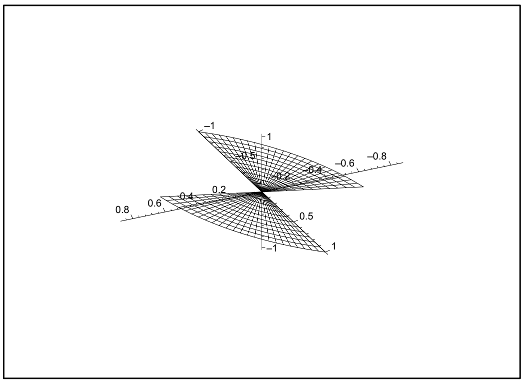

Figure 27.3: The surface $\mathcal{F}$ has two components.

If we keep $u = 1$ fixed and let v vary, then we get an arc of the helix with $a = h = 1$, cf. Example 26.2. When $(0, 0, 0)$ is removed, the surface is split into its two components $\mathcal{F}_1$ and $\mathcal{F}_2$, which are symmetric with respect to the point $(0, 0, 0)$. The surface $\mathcal{F}_1$ is obtained by drawing all lines from $(0, 0, 0)$ to a point on the helix.

D. The area element is given by $\mathrm{d}S = \|\mathbf{N}(u, v)\| \, \mathrm{d}u \, \mathrm{d}v$. We first calculate the normal vector $\mathbf{N}(u, v)$ corresponding to the given parametric representation.

I. It follows from $\mathbf{r}(u, v) = u \, (\sin v, \cos v, v)$ that

$$\frac{\partial \mathbf{r}}{\partial u} = (\sin v, \cos v, v), \qquad \frac{\partial \mathbf{r}}{\partial v} = u \, (\cos v, -\sin v, 1),$$

hence the normal vector is

$$\begin{aligned} \mathbf{N}(u, v) &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \sin v & \cos v & v \\ u \, \cos v & -u \, \sin v & u \end{vmatrix} \\ &= u \, (\cos v + v \, \sin v, v \, \cos v - \sin v, -1) \\ &= u \{ (\cos v, -\sin v, -1) + v \, (\sin v, \cos v, 0) \}. \end{aligned}$$

Now

$$(\cos v, -\sin v, -1) \cdot (\sin v, \cos v, 0) = 0,$$

so the two vectors are perpendicular. Then we get by Pythagoras's theorem

$$\begin{aligned}
\|\mathbf{N}(u,v)\|^2 &= u^2\left\{\|(\cos v, -\sin v, -1)\|^2 + v^2\|(\sin v, \cos v, 0)\|^2\right\} \\
&= u^2\left\{\left(\cos^2 v + \sin^2 + 1\right) + v^2\left(\sin^2 v + \cos^2 + 0^2\right)\right\} \\
&= u^2\left\{2 + v^2\right\}.
\end{aligned}$$

Note that $-1 \leq u \leq 1$ shows that u may be *negative*. When we take the square root we get the area element

$$\mathrm{d}S = \|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v = |u|\,\sqrt{2+v^2}\,\mathrm{d}u\,\mathrm{d}v.$$

Putting $D = [-1,1] \times [0,1]$ we get by the reduction formula

$$\begin{aligned}
I &= \int_{\mathcal{F}} |z|\,\mathrm{d}S = \int_D |u\,v| \cdot |u|\,\sqrt{2+v^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-1}^{1}\left\{\int_0^1 |u|^2|v|\,\sqrt{2+v^2}\,\mathrm{d}v\right\}\mathrm{d}u = \int_{-1}^{1} u^2\,\mathrm{d}u \cdot \int_0^1 \sqrt{2+v^2}\cdot v\,\mathrm{d}v \\
&= \left[\frac{1}{3}u^3\right]_{-1}^{1} \cdot \int_2^3 \sqrt{t}\cdot\frac{1}{2}\,dt = \frac{2}{3}\cdot\left(\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_2^3\right) \\
&= \frac{2}{9}(3\sqrt{3} - 2\sqrt{2}). \qquad \diamond
\end{aligned}$$

Example 27.2

A. *Let $\mathcal{F}$ be the surface given by the graph representation*

$$0 \le x \le \sqrt{3}, \qquad 0 \le y \le \sqrt{1+x^2}, \qquad z = xy.$$

Calculate the surface integral $\int_{\mathcal{F}} z \, \mathrm{d}S$.

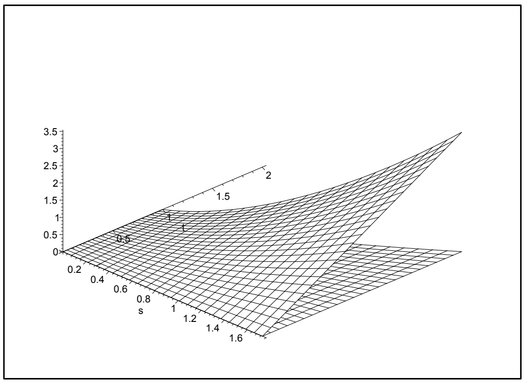

Figure 27.4: The surface $\mathcal{F}$ with its projection E.

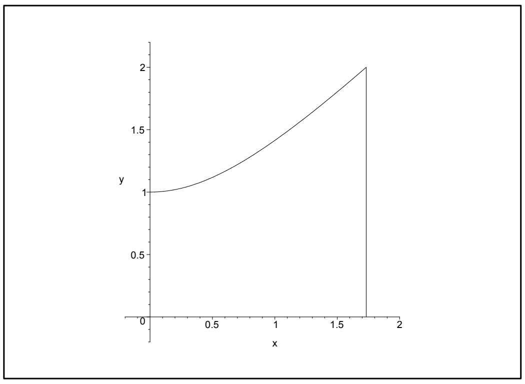

Figure 27.5: The projection E of $\mathcal{F}$ in the (x,y)-plane.

D. The usual procedure is to consider $\mathcal{F}$ as a graph of the function

$$z = f(x,y) = xy, \qquad (x,y) \in E.$$

We shall not do this here, but instead *alternatively* introduce a rectangular parametric representation $(x,y,z) = \mathbf{r}(u,v)$. Then afterwards we shall find the weight function $\|\mathbf{N}(u,v)\|$.

I. The parameters u and v are for obvious reasons not given above. They are introduced by duplicating (x,y) by the trivial formula

$$(x,y) = (u,v),$$

i.e. we choose the parametric representation

$$\mathbf{r}(u,v) = (x,y,z) = (u,v,uv), \quad 0 \le u \le \sqrt{3}, \quad 0 \le v \le \sqrt{1+u^2},$$

so we can distinguish between (x,y) as the first two coordinates on the *surface* in the 3-dimensional space, and $(u,v) \in E$ in the parametric domain. By experience it is always difficult to understand why we use this duplication, until one realizes that we in this way can describe two different aspects (as described above) of the same coordinates. This will be very useful in the following.

Since

$$\frac{\partial \mathbf{r}}{\partial u} = (1,0,v) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = (0,1,u),$$

the corresponding normal vector becomes

$$\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v,-u,1).$$

Hence

$$\|\mathbf{N}(u,v)\| = \sqrt{1+u^2+v^2}.$$

When $\mathrm{d}S$ denotes the area element on $\mathcal{F}$, and $\mathrm{d}S_1$ denotes the area element on E, then we have the correspondence

$$\mathrm{d}S = \sqrt{1+u^2+v^2}\,\mathrm{d}S_1.$$

The abstract *surface* integral over $\mathcal{F}$ is therefore reduced to the abstract *plane* integral over E by

$$\int_{\mathcal{F}} z\,\mathrm{d}S = \int_E u\,v\,\sqrt{1+u^2+v^2}\,\mathrm{d}S.$$

Then we reduce the abstract *plane* integral over E in rectangular coordinates, where the v-integral is the inner one,

$$\int_{\mathcal{F}} z\,\mathrm{d}S = \int_E u\,v\,\sqrt{1+u^2+v^2}\,\mathrm{d}S = \int_0^{\sqrt{3}} u \left\{ \int_0^{\sqrt{1+u^2}} \sqrt{1+u^2+v^2}\,v,\,\mathrm{d}v \right\} \mathrm{d}u.$$

Then calculate the inner integral by means of the substitution

$$t = v^2, \qquad \mathrm{d}t = 2v\,\mathrm{d}v.$$

From this we get

$$\begin{aligned}
\int_0^{\sqrt{1+u^2}} \sqrt{1+u^2+v^2}\,v\,\mathrm{d}v &= \int_0^{1+u^2} \sqrt{1+u^2+t}\cdot\frac{1}{2}\,\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{2}{3}\left(1+u^2+t\right)^{\frac{3}{2}}\right]_{t=0}^{1+u^2} = \frac{1}{3}\left\{\left(2(1+u^2)\right)^{\frac{3}{2}} - (1+u^2)^{\frac{3}{2}}\right\} \\
&= \frac{1}{3}\,(2\sqrt{2}-1)\cdot\left(1+u^2\right)^{\frac{3}{2}}.
\end{aligned}$$

By insertion and by the substitution $t = u^2$, $\mathrm{d}t = 2u\,\mathrm{d}u$ we finally get

$$\begin{aligned}
\int_{\mathcal{F}} z\,\mathrm{d}S &= \int_0^{\sqrt{3}} u \cdot \frac{1}{3}\,(2\sqrt{2}-1)\cdot\left(1+u^2\right)^{\frac{3}{2}}\,\mathrm{d}u = \frac{1}{3}\,(2\sqrt{2}-1)\int_0^{\sqrt{3}} \left(1+u^2\right)^{\frac{3}{2}}\,u\,\mathrm{d}u \\
&= \frac{1}{3}\,(2\sqrt{2}-1)\int_0^{3}(1+t)^{\frac{3}{2}}\,\frac{1}{2}\,\mathrm{d}t = \frac{1}{3}\,(2\sqrt{2}-1)\cdot\frac{1}{2}\left[\frac{2}{5}\,(1+t)^{\frac{5}{2}}\right]_0^3 \\
&= \frac{1}{15}\,(2\sqrt{2}-1)\cdot\left\{4^{\frac{5}{2}}-1\right\} = \frac{31(2\sqrt{2}-1)}{15}. \qquad \Diamond
\end{aligned}$$

Example 27.3

A. *A surface of revolution $\mathcal{O}$ is obtained by revolving the meridian curve $\mathcal{M}$ given by*

$$r = a(1+\sin\theta), \qquad 0 \le \theta \le \frac{\pi}{2}, \quad a > 0,$$

where θ is the angle measured from the z-axis and r is the distance to $(0,0)$ (an arc of a cardioid, *cf. Example 26.12). Calculate the surface integral*

$$I = \int_{\mathcal{O}} \frac{z}{x^2+y^2+z^2}\,\mathrm{d}S.$$

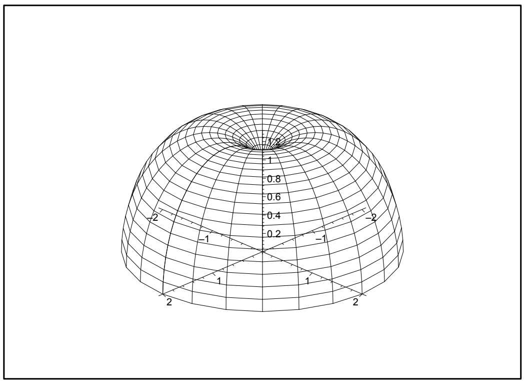

Figure 27.6: The surface $\mathcal{O}$ for $a = 1$.

An examination of the dimensions shows that x, y, $z \sim a$ and $\int_{\mathcal{O}} \cdots \,\mathrm{d}S \sim a^2$, thus

$$\int_{\mathcal{O}} \frac{z}{x^2+y^2+z^2}\,\mathrm{d}S \sim \frac{a}{a^2}\cdot a^2 = a.$$

The final result must therefore be of the form $c \cdot a$, where c is the constant, we are going to find.

D. When we look at surfaces (or bodies) of revolution one should always try either *semi-polar* or *spherical* coordinates. Since the parametric representation of the meridian curve $\mathcal{M}$ is given in a way which is very similar to the *spherical* coordinates, it is quite reasonable to expect that one should use spherical coordinates.

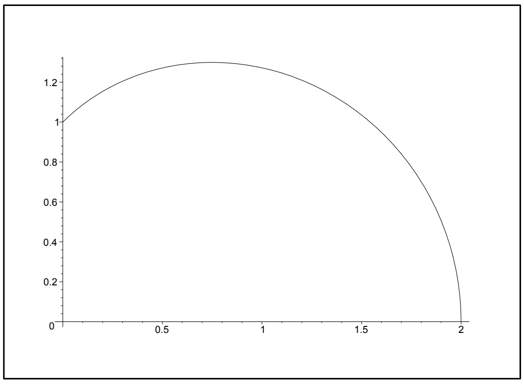

Figure 27.7: The meridian curve $\mathcal{M}$ for $a = 1$.

Although it is here possible to solve the problem by a very nasty trick, it is far better for pedagogical reasons to follow the way which most people would go. Let us analyze the reduction formula

$$\int_{\mathcal{O}} f(x,y,z)\,\mathrm{d}S = \int_a^b \left\{ \int_0^{2\pi} F(t,\varphi)\,\mathrm{d}\varphi \right\} R(t)\,\sin\Theta(t)\sqrt{\{R'(t)\}^2+\{R(t)\Theta'(t)\}^2}\,\mathrm{d}t,$$

where

$$F(t,\varphi) := f(R(t)\sin\Theta(t), \cos\varphi, R(t)\sin\Theta(t)\sin\varphi, R(r)\cos\Theta(t)).$$

There is no t in **A**, so we must start by introducing the parameter t in a convenient form. Then we shall identify the transformed function $F(t, \varphi)$ as well as the weight function, and finally we shall carry out all the integrations.

I. 1) *The introduction of the parameter* t. The most obvious thing is to put $\theta = t$, i.e. $\mathcal{M}$ is described by

$$r = R(t) = a\,(1+\sin t), \qquad \theta = \Theta(t) = t, \qquad 0 \le t \le \frac{\pi}{2}.$$

By doing this we split the different aspects: θ belongs to the curve $\mathcal{M}$, and t belongs to the parametric interval

$$\left[0, \frac{\pi}{2}\right] = [a, b].$$

2) *Identification of* $F(t, \varphi)$ *and the weight function.* Since

$$z = R(t)\cos\Theta(t) = a(1+\sin t)\cos t \qquad \text{on } \mathcal{M},$$

and

$$x^2+y^2+z^2 = r^2 = R(t)^2 = a^2(1+\sin t)^2 \qquad \text{on } \mathcal{M},$$

we obtain the *integrand*

$$f(x,y,z) = \frac{z}{x^2+y^2+z^2} = \frac{a(1+\sin t)\cos t}{a^2(1+\sin t)^2} = \frac{\cos t}{a(1+\sin t)} = F(t,\varphi),$$

which is independent of φ. Since the weight function and the boundaries of do not depend on t either, the φ-integral becomes trivial, and we can put $\int_0^{2\pi} d\varphi = 2\pi$ outside the integral as a factor.

Then we calculate the *weight function*,

$$\begin{aligned} &R(t)\sin\Theta(t)\sqrt{\{R'(t)\}^2 + \{R(t)\Theta'(t)\}^2} \\ &\quad = a(1+\sin t)\cdot\sin t\cdot\sqrt{\{a\,\cos t\}^2 + \{a(1+\sin t)\cdot 1\}^2} \\ &\quad = a(1+\sin t)\cdot\sin t\cdot a\sqrt{\cos^2 t + (1+2\sin t+\sin^2 t)} \\ &\quad = a^2(1+\sin t)\cdot\sin t\cdot\sqrt{2(1+\sin t)} \\ &\quad = \sqrt{2}\,a^2(1+\sin t)^{\frac{3}{2}}\cdot\sin t. \end{aligned}$$

3) *Integration by reduction.* First we note that the parametric domain is 2-dimensional,

$$D = \left\{(\varphi, t) \;\middle|\; 0 \le \varphi \le 2\pi,\, 0 \le t \le \frac{\pi}{2}\right\}.$$

In fact, dimension corresponds to dimension, and since $\mathcal{F}$ is a C^∞-*surface*, the parametric domain D must necessarily be 2-dimensional. (If not we have made an error, so start from the very beginning!)

We have now identified all functions, so we get by the reduction formula that

$$
\begin{aligned}
\int_{\mathcal{O}} \frac{z}{x^2+y^2+z^2}\,\mathrm{d}S &= \int_D \frac{\cos t}{a(1+\sin t)} \cdot \sqrt{2}\,a^2\,(1+\sin t)^{\frac{3}{2}}\,\sin t\,\mathrm{d}\varphi\,\mathrm{d}t \\
&= \sqrt{2}\cdot a\cdot 2\pi \int_0^{\frac{\pi}{2}} \sin t(1+\sin t)^{\frac{1}{2}}\cos t\,\mathrm{d}t \\
&= 2\sqrt{2}\,\pi\,a \int_0^1 u(1+u)^{\frac{1}{2}}\,\mathrm{d}u \\
&= 2\sqrt{2}\,\pi\,a \int_0^1 (1+u-1)\,(1+u)^{\frac{1}{2}}\,\mathrm{d}u \\
&= 2\sqrt{2}\,\pi a \int_0^1 \left\{(1+u)^{\frac{3}{2}}-(1+u)^{\frac{1}{2}}\right\}\mathrm{d}u \\
&= 2\sqrt{2}\,\pi\,a \left[\frac{2}{5}\,(1+u)^{\frac{5}{2}}-\frac{2}{3}\,(1+u)^{\frac{3}{2}}\right]_0^1 \\
&= 2\sqrt{2}\,\pi\,a\cdot\frac{2}{15}\left[3(1+u)^{\frac{5}{2}}-5(1+u)^{\frac{3}{2}}\right]_0^1 \\
&= \frac{4\pi a}{15}\cdot\sqrt{2}\left\{3\left(2^{\frac{5}{2}}-1\right)-5\left(2^{\frac{3}{2}}-1\right)\right\} \\
&= \frac{4\pi a}{15}\,\sqrt{2}\,\{3(4\sqrt{2}-1)-5(2\sqrt{2}-1)\} \\
&= \frac{4\pi a}{15}\,\sqrt{2}\,\{2\sqrt{2}+2\} = \frac{8\pi a}{15}\,\sqrt{2}\,(\sqrt{2}+1) \\
&= \frac{8\pi(2+\sqrt{2})a}{15}.
\end{aligned}
$$

C. *Weak control.* The result has the form $c\cdot a$, in agreement with **A**.

Since $z \geq 0$ on $\mathcal{O}$ [cf. the figure], the result must be ≥ 0. We see that this is also the case here. $\Diamond$

Example 27.4

A. *A surface of revolution $\mathcal{O}$ has an arc of a parabola $\mathcal{M}$ as its meridian curve, given by the equation*

$$z = \frac{\varrho^2}{a}, \qquad 0 \le \varrho \le a, \qquad a > 0.$$

Compute the surface integral

$$I = \int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2+4az}}\,\mathrm{d}S.$$

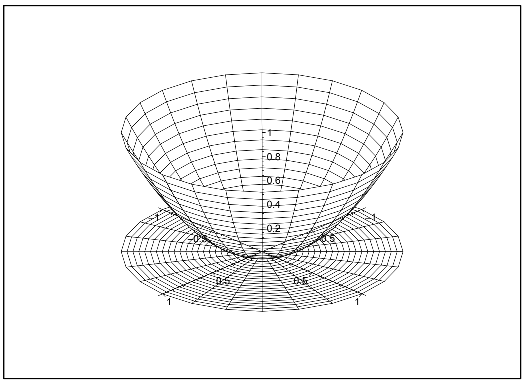

Figure 27.8: The surface $\mathcal{O}$ and its projection onto the (x, y)-plane for $a = 1$.

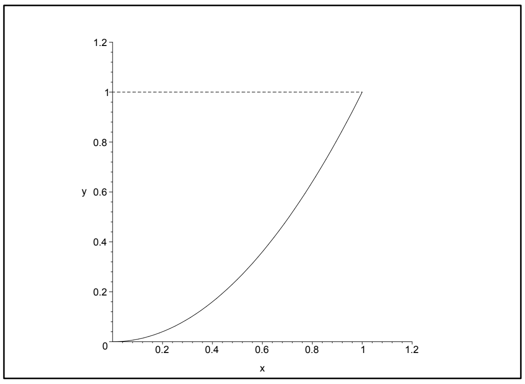

Figure 27.9: The meridian curve $\mathcal{M}$ for $a = 1$.

Examination of the dimensions. It follows from x,y, $z \sim a$, that

$$\frac{x^2}{\sqrt{a^2+4az}} \sim \frac{a^2}{\sqrt{a^2}} = a.$$

Since $\int_{\mathcal{O}} \cdots \,\mathrm{d}S \sim a^2$, we get all together

$$\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2+4az}}\,\mathrm{d}S \sim a \cdot a^2 = a^3,$$

i.e. the result must have the form

$$\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2+4az}}\,\mathrm{d}S = c\cdot a^3,$$

where the constant c must be positive, because the integrand is ≥ 0.

D. The description invites to *semi-polar* coordinates **I 1**. For the matter of training we also add **I 2**. *Rectangular coordinates*, which give a slightly different variant, although we in the end are forced back to (semi-)polar coordinates.

I 1. *Semi-polar coordinates.* We introduce t as a parameter by

$$\varrho = P(t) = t.$$

Then

$$z = Z(t) = \frac{1}{a}\,t^2, \qquad 0 \leq t \leq a.$$

Since

$$x = P(t)\cos\varphi = t\cos\varphi, \quad y = P(t)\sin\varphi = t\sin\varphi, \quad z = Z(t) = \frac{1}{a}\,t^2,$$

we get the following interpretation of the *integrand*,

$$f(x,y,z) = \frac{x^2}{\sqrt{a^2+4az}} = \frac{t^2\cos^2\varphi}{\sqrt{a^2+4t^2}},$$

and the *weight function* is

$$P(t)\sqrt{\{P'(t)\}^2+\{Z'(t)\}^2} = t\sqrt{1^2+\left(\frac{2}{a}\,t\right)^2} = \frac{t}{a}\,\sqrt{a^2+4t^2}.$$

The parametric domain is

$$D = \{(t,\varphi) \mid 0 \le t \le a,\, 0 \le \varphi \le 2\pi\} = [0,a]\times[0,2\pi].$$

Hence we get by a reduction

$$\begin{aligned}
\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2+4az}}\,\mathrm{d}S &= \int_D \frac{t^2\cos^2\varphi}{\sqrt{a^2+4t^2}}\cdot\frac{t}{a}\,\sqrt{a^2+4t^2}\,\mathrm{d}t\,\mathrm{d}\varphi \\
&= \frac{1}{a}\int_D t^3\cos^2\varphi\,\mathrm{d}t\,\mathrm{d}\varphi = \frac{1}{a}\int_0^a t^3\,\mathrm{d}t\cdot\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi \\
&= \frac{1}{a}\left[\frac{1}{4}\,t^4\right]_0^a\cdot\int_0^{2\pi}\frac{1+\cos 2\varphi}{2}\,d\varphi = \frac{\pi a^3}{4}.
\end{aligned}$$

C 1. *Weak control.* The result has the right dimension $[a^3]$, and it is positive, cf. **A**.

I 2. The *rectangular* version. In this case we interpret the surface as the *graph* of the function

$$z = f(x,y) = \frac{1}{a}\,(x^2+y^2) \qquad \text{for } (x,y)\in E,$$

where the parametric domain is the disc

$$E = \{(x,y) \mid x^2+y^2 \le a^2\}.$$

The *weight function* is

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\left(\frac{2x}{a}\right)^2+\left(\frac{2y}{a}\right)^2} = \frac{1}{a}\,\sqrt{a^2+4x^2+4y^2}.$$

We have found everything which is needed for an application of the reduction theorem:

$$\begin{aligned}
\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2+4az}}\,\mathrm{d}S &= \int_E \frac{x^2}{\sqrt{a^2+4a\cdot\frac{1}{a}(x^2+y^2)}}\cdot\frac{1}{a}\,\sqrt{a^2+4x^2+4y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{a}\int_E \frac{x^2}{\sqrt{a^2+4x^2+4y^2}}\cdot\sqrt{a^2+4x^2+4y^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{a}\int_E x^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

From this point it is again most natural to change to *polar coordinates*,

$$\begin{aligned}\int_{\mathcal{O}} \frac{x^2}{\sqrt{a^2+4az}}\,\mathrm{d}S &= \frac{1}{a}\int_E x^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{a}\int_0^{2\pi}\left\{\int_0^a (\varrho\cos(\varphi)^2\cdot\varrho\,\mathrm{d}\varrho)\right\}\mathrm{d}\varphi \\ &= \frac{1}{a}\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi\cdot\int_0^a \varrho^3\,\mathrm{d}\varrho = \frac{1}{a}\cdot\pi\cdot\frac{a^4}{4} = \frac{\pi a^3}{4}. \qquad \Diamond\end{aligned}$$

Example 27.5 *Calculate in each of the following cases the given surface integral over a surface $\mathcal{F}$, which is the graph of a function in two variables, thus*

$$\mathcal{F} = \{(x,y,z) \mid (x,y) \in E,\, z = Z(x,y)\}.$$

1) The surface integral $\int_{\mathcal{F}} \sqrt{1+(x+y+1)^2}\,\mathrm{d}S$, *where*

$$Z(x,y) = \frac{1}{\sqrt{2}}\ln(1+x+y), \qquad (x,y) \in [0,1]\times[0,1].$$

2) The surface integral $\int_{\mathcal{F}} \sqrt{x^2+y^2}\,\mathrm{d}S$, *where*

$$Z(x,y) = 2-x^2-y^2, \qquad \textit{for } x^2+y^2 \le 2.$$

3) The surface integral $\int_{\mathcal{F}} z\,\mathrm{d}S$, *where*

$$Z(x,y) = 2-x^2-y^2, \qquad \textit{for } x^2+y^2 \le 2.$$

4) The surface integral $\int_{\mathcal{F}} x^2\sqrt{1+x^2+y^2}\,\mathrm{d}S$, *where*

$$Z(x,y) = xy, \qquad \textit{for } x^2+y^2 \le 1.$$

5) The surface integral $\int_{\mathcal{F}}(a+z)\,\mathrm{d}S$, *where*

$$Z(x,y) = \frac{x^2-y^2}{a}, \qquad \textit{for } x^2+y^2 \le 2a^2.$$

6) The surface integral $\int_{\mathcal{F}} \frac{1}{\sqrt{a^2+4x^2+4y^2}}\,\mathrm{d}S$, *where*

$$Z(x,y) = \frac{x^2-y^2}{a}, \qquad \textit{for } x^2+y^2 \le 2a^2.$$

7) The surface integral $\int_{\mathcal{F}} \sqrt{a^2+4x^2+4y^2}\,\mathrm{d}S$, *where*

$$Z(x,y) = \frac{x^2-y^2}{a}, \qquad \textit{for } x^2+y^2 \le 2a^2.$$

8) The surface integral $\int_{\mathcal{F}} z^3\,\mathrm{d}S$, *where*

$$Z(x,y) = \sqrt{2a^2-x^2-y^2} \qquad \textit{for } -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4} \;\textit{ og }\; 0 \le \varrho \le a\,\cos(2\varphi).$$

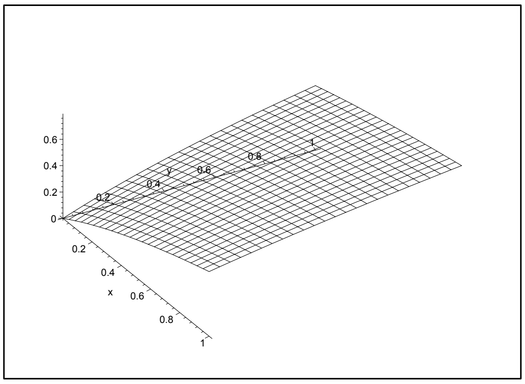

Figure 27.10: The surface of **Example 27.5.1**.

A Surface integrals in rectangular coordinates.

D Find the weight function

$$\|\mathbf{N}\| = \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2} = \sqrt{1+\|\bigtriangledown g\|^2},$$

and then calculate the surface integral.

I 1) We get from $g(x,y)=\dfrac{1}{\sqrt{2}}\ln(1+x+y)$ that

$$\bigtriangledown g = \frac{1}{\sqrt{2}}\cdot\frac{1}{1+x+y}\,(1,1),$$

and as $x,\,y\geq 0$,

$$\sqrt{1+\|\bigtriangledown f\|^2} = \sqrt{1+\frac{1}{(1+x+y)^2}} = \frac{\sqrt{1+(1+x+y)^2}}{1+x+y},$$

hence

$$\begin{aligned}
\int_{\mathcal{F}}\sqrt{1+(x+y+1)^2}\,\mathrm{d}S &= \int_E \frac{1+(x+y+1)^2}{1+x+y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\left\{\int_0^1\left\{\frac{1}{1+x+y}+x+y+1\right\}\mathrm{d}y\right\}\mathrm{d}x \\
&= \int_0^1\left[\ln(1+x+y)+\frac{1}{2}(x+y+1)^2\right]_{y=0}^1\mathrm{d}x \\
&= \int_0^1\left\{\ln(x+2)+\frac{1}{2}(x+2)^2-\ln(x+1)-\frac{1}{2}(x+1)^2\right\}\mathrm{d}x \\
&= \left[(x+2)\ln(x+2)-(x+1)\ln(x+1)+\frac{1}{6}(x+2)^3-\frac{1}{6}(x+1)^3\right]_0^1 \\
&= 3\ln 3-2\ln 2+\frac{1}{6}\cdot 3^3-\frac{1}{6}\cdot 2^3-2\ln 2-\frac{1}{6}\cdot 2^3+\frac{1}{6} \\
&= 3\ln 3-4\ln 2+\frac{1}{6}\{27-8-8+1\} = 3\ln 3-4\ln 2+2 = \ln\frac{27}{16}+2.
\end{aligned}$$

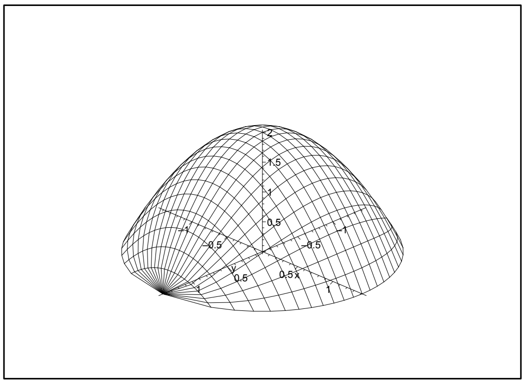

Figure 27.11: The surface of **Example 27.5.2** and **Example 27.5.3**.

2) We get from $g(x,y) = 2 - x^2 - y^2$ that

$$\nabla g = (-2x, -2y) = -2(x,y),$$

hence

$$\sqrt{1 + \|\nabla g\|^2} = \sqrt{1 + 4(x^2+y^2)}.$$

The method here is that we first transform from the surface $\mathcal{F}$ to the domain of integration E in rectangular coordinates. Then we continue by transforming the integral into polar coordinates,

$$\begin{aligned}
\int_{\mathcal{F}} \sqrt{x^2+y^2}\,\mathrm{d}S &= \int_E \sqrt{x^2+y^2} \cdot \sqrt{1+4(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = 2\pi \int_0^{\sqrt{2}} \varrho^2 \sqrt{1+4\varrho^2}\,\mathrm{d}\varrho \\
&= 2\pi \int_0^{\mathrm{Arsinh}(2\sqrt{2})} \frac{1}{8} \sinh^2 t \cdot \cosh^2 t\,\mathrm{d}t = \frac{\pi}{16} \int_0^{\mathrm{Arsinh}(2\sqrt{2})} \sinh^2(2t)\,dt \\
&= \frac{\pi}{32} \int_0^{\mathrm{Arsinh}(2\sqrt{2})} \{\cosh(4t) - 1\}\,dt = \frac{\pi}{32} \left[\frac{1}{4}\sinh(4t) - t\right]_0^{\mathrm{Arsinh}(2\sqrt{2})} \\
&= \frac{\pi}{32} \left[\frac{1}{2}\sinh(2t)\cosh(2t)\right]_0^{\mathrm{Arsinh}(2\sqrt{2})} - \frac{\pi}{32} \ln\left(2\sqrt{2} + \sqrt{1+(2\sqrt{2})^2}\right) \\
&= \frac{\pi}{32} \left[\sinh t \cdot \cosh t (1 + 2\sinh^2 t)\right]_0^{\mathrm{Arsinh}(2\sqrt{2})} - \frac{\pi}{32} \ln(3+2\sqrt{2}) \\
&= \frac{\pi}{32} \cdot 2\sqrt{2} \cdot \sqrt{1+(2\sqrt{2})^2} \cdot \left\{1 + 2 \cdot (2\sqrt{2})^2\right\} - \frac{\pi}{32} \ln\left\{(1+\sqrt{2})^2\right\} \\
&= \frac{\pi}{32} \cdot 2\sqrt{2} \cdot 3 \cdot (1 + 2 \cdot 8) - \frac{\pi}{16} \ln(1+\sqrt{2}) = \frac{\pi}{16}(51\sqrt{2} - \ln(1+\sqrt{2})).
\end{aligned}$$

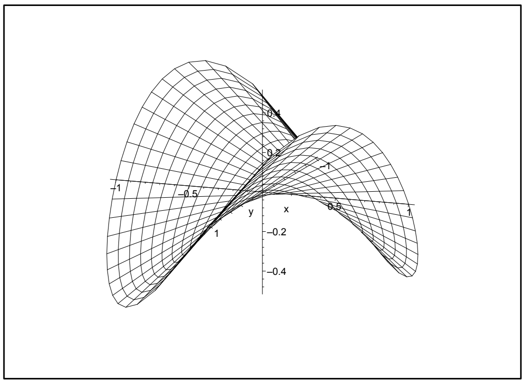

Figure 27.12: The surface of **Example 27.5.4**.

3) We shall here integrate over the same surface as in **Example 27.5.2**. We can therefore reuse the previous result

$$\sqrt{1+\|\bigtriangledown g\|^2}=\sqrt{1+4(x^2+y^2)}.$$

If we put $t = 4\varrho^2 + t$, then we get the surface integral

$$\int_{\mathcal{F}} z\,\mathrm{d}S = \int_E (2 - x^2 - y^2)\sqrt{1 + 4(x^2 + y^2)}\,\mathrm{d}x\,\mathrm{d}y = 2\pi \int_0^{\sqrt{2}} (2 - \varrho^2)\sqrt{1 + 4\varrho^2} \cdot \varrho\,\mathrm{d}\varrho$$

$$= \frac{2\pi}{8} \int_1^9 \left\{2 - \frac{1}{4}(t-1)\right\} \sqrt{t}\,\mathrm{d}t = \frac{\pi}{4} \int_1^9 \left\{\frac{9}{4}\,t^{\frac{1}{2}} - \frac{1}{4}\,t^{\frac{3}{2}}\right\} \mathrm{d}t = \frac{\pi}{16}\left[9 \cdot \frac{2}{3}\,t^{\frac{3}{2}} - \frac{2}{5}\,t^{\frac{5}{2}}\right]_1^9$$

$$= \frac{\pi}{8}\left[3t^{\frac{3}{2}} - \frac{1}{5}\,t^{\frac{5}{2}}\right]_1^9 = \frac{\pi}{8}\left\{3 \cdot 27 \cdot \frac{1}{5} \cdot 243 - 3 + \frac{1}{5}\right\} = \frac{\pi}{8}\left(78 - \frac{242}{5}\right)$$

$$= \frac{\pi}{4}\left\{39 - \frac{121}{5}\right\} = \frac{\pi}{20}(195 - 121) = \frac{\pi}{20} \cdot 74 = \frac{37\pi}{10}.$$

4) It follows immediately that $\bigtriangledown g = (y, x)$, so the weight function is

$$\sqrt{1 + \| \bigtriangledown g\|^2} = \sqrt{1 + (x^2 + y^2)}.$$

Then we compute the surface integral,

$$\int_{\mathcal{F}} x^2\sqrt{1 + x^2 + y^2}\,\mathrm{d}S = \int_E x^2\sqrt{1 + x^2 + y^2} \cdot \sqrt{1 + x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_E x^2(1 + x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \left\{\int_0^1 \varrho^2 \cos^2\varphi \cdot (1 + \varrho^2)\varrho\,\mathrm{d}\varrho\right\} \mathrm{d}\varphi$$

$$= \int_0^{2\pi} \cos^2\varphi\,\mathrm{d}\varphi \cdot \frac{1}{2}\int_0^1 t(1+t)\,\mathrm{d}t = \pi \cdot \frac{1}{2}\left[\frac{1}{2}\,t^2 + \frac{1}{3}\,t^3\right]_0^1 = \frac{5\pi}{12}.$$

5) Here $\bigtriangledown g = \dfrac{2}{a}\,(x, -y)$, hence the weight function is

$$\sqrt{1 + \| \bigtriangledown g\|^2} = \sqrt{1 + \frac{4}{a^2}\,(x^2 + y^2)} = \frac{1}{a}\,\sqrt{a^2 + 4(x^2 + y^2)}.$$

Then we get the surface integral,

$$\int_{\mathcal{F}} (a + z)\,\mathrm{d}S = \int_E \left(a + \frac{x^2 - y^2}{a}\right) \frac{1}{a}\,\sqrt{a^2 + 4(x^2 + y^2)}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{a^2}\int_E (a^2 + x^2 - y^2)\sqrt{a^2 + 4(x^2 + y^2)}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{a^2}\int_0^{2\pi} \left\{\int_0^{\sqrt{2}a} (a^2 + \varrho^2[\cos^2\varphi - \sin^2\varphi])\sqrt{a^2 + 4\varrho^2}\,\varrho\,\mathrm{d}\varrho\right\} \mathrm{d}\varphi$$

$$= 2\pi \int_0^{\sqrt{2}a} \sqrt{a^2 + 4\varrho^2} \cdot \varrho\,\mathrm{d}\varrho + \frac{1}{a^2}\int_0^{2\pi} \cos 2\varphi\,\mathrm{d}\varphi \int_0^{\sqrt{2}a} \varrho^2\sqrt{a^2 + 4\varrho^2}\,\varrho\,\mathrm{d}\varrho$$

$$= \frac{2\pi}{8}\int_{\varrho=0}^{\sqrt{2}a} (a^2 + 4\varrho^2)^{\frac{1}{2}} d(a^2 + 4\varrho^2) + 0 = \frac{\pi}{4} \cdot \frac{2}{3}\,\left[(a^2 + 4\varrho^2)^{\frac{3}{2}}\right]_{\varrho=0}^{\sqrt{2}a}$$

$$= \frac{\pi}{6}\left\{(a^2 + 4 \cdot 2a^2)^{\frac{3}{2}} - a^3\right\} = \frac{13\pi}{3}\,a^3.$$

6) The surface is the same as in **Example 27.5.5**. Therefore, we get the weight function

$$\sqrt{1 + \| \bigtriangledown g\|^2} = \frac{1}{a}\,\sqrt{a^2 + 4(x^2 + y^2)},$$

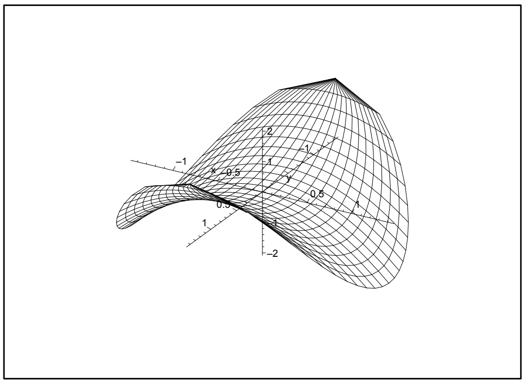

Figure 27.13: The surface of **Example 27.5.5, Example 27.5.6** and **Example 27.5.7**.

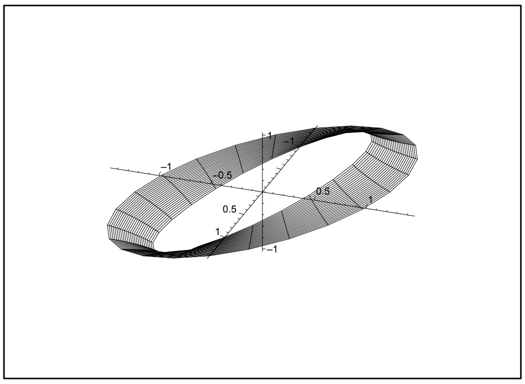

Figure 27.14: The surface of **Example 27.5.10**.

and the surface integral is

$$\int_{\mathcal{F}} \frac{1}{\sqrt{a^2+4x^2+4y^2}}\,\mathrm{d}S = \int_E \frac{1}{a}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{a}\,\mathrm{area}(E) = \frac{1}{a}\cdot\pi\cdot 2a^2 = 2\pi a.$$

7) The surface is the same as in **Example 27.5.5**, so the weight function is

$$\sqrt{1+\|\bigtriangledown g\|^2} = \frac{1}{a}\sqrt{a^2+4(x^2+y^2)},$$

and the surface integral becomes

$$\begin{aligned}\int_{\mathcal{F}} \sqrt{a^2+4x^2+4y^2}\,\mathrm{d}S &= \int_E \frac{1}{a}\left(a^2+4(x^2+y^2)\right)\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2\pi}{a}\int_0^{\sqrt{2}a}(a^2+4\varrho^2)\varrho\,\mathrm{d}\varrho = \frac{2\pi}{a}\left[\frac{1}{2}\,a^2\varrho^2+\varrho^4\right]_{\varrho=0}^{\sqrt{2}a} \\ &= \frac{2\pi}{a}\left\{\frac{1}{2}\,a^2\cdot 2a^2+4a^4\right\} = \frac{2\pi}{a}\cdot 5a^4 = 10\pi a^3.\end{aligned}$$

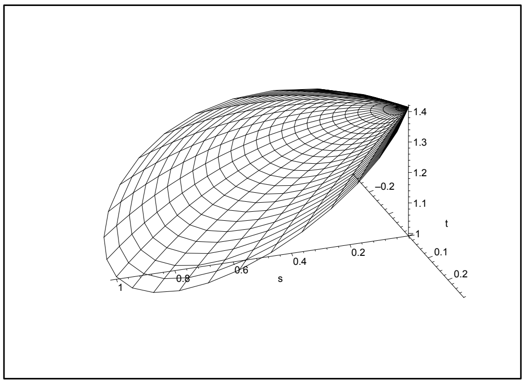

Figure 27.15: The surface of **Example 27.5.11** for $a = 1$.

8) Here

$$\nabla g = \left(-\frac{2xy}{(x^2+y^2)^2}, \frac{x^2-y^2}{(x^2+y^2)^2}\right) = \frac{1}{(x^2+y^2)^2}\,(-2xy, x^2-y^2),$$

hence

$$\| \nabla g\|^2 = \frac{1}{(x^2+y^2)^4}\left(4x^2y^2+(x^2-y^2)^2\right) = \frac{1}{(x^2+y^2)^2}.$$

The surface integral is

$$\begin{aligned}
\int_{\mathcal{F}} \mathrm{d}S &= \int_E \frac{1}{x^2+y^2}\sqrt{1+(x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y = 2\pi\int_0^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^2}\,\varrho\,\mathrm{d}\varrho \\
&= \frac{2\pi}{4}\int_1^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^4}\cdot 4\varrho^3\,\mathrm{d}\varrho = \frac{\pi}{2}\int_1^4 \frac{1+t}{t}\,\mathrm{d}t = \frac{\pi}{2}\int_{\sqrt{2}}^{\sqrt{5}} \frac{u\cdot 2u}{u^2-1}\,\mathrm{d}u \\
&= \pi\int_{\sqrt{2}}^{\sqrt{5}} \left\{1+\frac{1}{2}\,\frac{1}{u-1}-\frac{1}{2}\,\frac{1}{u+1}\right\}\mathrm{d}u = \pi\left[u+\frac{1}{2}\ln\frac{u-1}{u+1}\right]_{\sqrt{2}}^{\sqrt{5}} \\
&= \pi\left\{\sqrt{5}-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right\} \\
&= \pi\left\{\sqrt{5}-\sqrt{2}+\ln\left(\frac{(\sqrt{5}-1)(\sqrt{2}+1)}{2}\right)\right\} \\
&= \pi\{\sqrt{5}-\sqrt{2}+\ln(\sqrt{5}-1)+\ln(\sqrt{2}+1)-\ln 2.
\end{aligned}$$

9) It follows from $g(x,y) = \sqrt{2a^2-x^2-y^2}$ that

$$\nabla g = \frac{1}{\sqrt{2a^2-x^2-y^2}}\,(-x-y),$$

hence

$$\sqrt{1+\|\nabla g\|^2} = \sqrt{1+\frac{x^2+y^2}{2a^2-x^2-y^2}} = \frac{\sqrt{2}\cdot a}{\sqrt{2a^2-x^2-y^2}}.$$

If we use polar coordinates in the parameter domain, we get

$$\begin{aligned}
\int_{\mathcal{F}} z^3\,\mathrm{d}S &= \int_E \left(\sqrt{2a^2-x^2-y^2}\right)^3 \cdot \frac{\sqrt{2}\cdot a}{\sqrt{2a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{2}\,a\int_E (2a^2-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left\{\int_0^{a\cos 2\varphi}(2a^2-\varrho^2)\varrho\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= \sqrt{2}\,a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[a^2\varrho^2-\frac{1}{4}\,\varrho^4\right]_0^{a\cos 2\varphi}\mathrm{d}\varphi = \sqrt{2}\,a^5\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\cos^2 2\varphi-\frac{1}{4}\cos^4 2\varphi\right)\mathrm{d}\varphi \\
&= 2\sqrt{2}\,a^5\int_0^{\frac{\pi}{4}}\left\{\frac{1}{2}+\frac{1}{2}\cos 4\varphi-\frac{1}{4}\left(\frac{1+\cos 4\varphi}{2}\right)^2\right\}\mathrm{d}\varphi \\
&= 2\sqrt{2}\,a^5\left[\frac{1}{2}\,\varphi+\frac{1}{8}\sin 4\varphi\right]_0^{\frac{\pi}{4}} - \frac{2\sqrt{2}}{16}\,a^5\int_0^{\frac{\pi}{4}}\left(1+2\cos 4\varphi+\cos^2 4\varphi\right)\mathrm{d}\varphi \\
&= \frac{\sqrt{2}\pi}{4}\,a^5 - \frac{2\sqrt{2}}{16}\cdot\frac{\pi}{4}\,a^5 - \frac{2\sqrt{2}}{16}\cdot\frac{1}{2}\,a^5\cdot\frac{\pi}{4} = \frac{\sqrt{2}\pi\,a^5}{64}\,(16-2-1) \\
&= \frac{13\sqrt{2}}{64}\,\pi\,a^5.
\end{aligned}$$

Example 27.6 *Calculate in each of the following cases the given surface integral over a cylinder surface $\mathcal{C}$, which is given by the plane curve $\mathcal{L}$ in the (X,Y)-plane, and the interval, in which z lies, when (x,y) is a point of the curve. Note that $\mathcal{L}$ can either be given by an equation in rectangular or in polar coordinates, or by a parametric description.*

1) *The surface integral* $\int_{\mathcal{C}}(y^2z+x^2z+y)\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $x^2+y^2=2x$, *and where* $z\in\left[0,\sqrt{x^2+y^2}\right]$.

2) *The surface integral* $\int_{\mathcal{C}} z^2\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $x^2+y^2=4$, *and where* $z\in[-2,x]$.

3) *The surface integral* $\int_{\mathcal{C}}(z^2+x^2)\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $x^2+y^2=1$, *and where* $z\in[0,2]$.

4) *The surface integral* $\int_{\mathcal{C}} z\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $y=x^2$ *for* $x\in[0,1]$, *and where* $z\in[0,x]$.

5) *The surface integral* $\int_{\mathcal{C}} z\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $\mathbf{r}(t)=(a\cos^3 t, a\sin^3 t)$ *for* $t\in\left[0,\frac{\pi}{2}\right]$, *and where* $z\in[0,y]$.

6) *The surface integral* $\int_{\mathcal{C}} \frac{1}{x}\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $\varrho=e^{\varphi}$ *for* $\varphi\in[0,1]$, *and where* $z\in[0,x]$.

7) *The surface integral* $\int_{\mathcal{C}} \frac{z}{x^2}\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $\varrho=a\cos^2\frac{\varphi}{2}$ *for* $\varphi\in\left[0,\frac{\pi}{2}\right]$, *and where* $z\in[0,xy]$.

8) *The surface integral* $\int_{\mathcal{C}} xz\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $x^2+y^2=ax$, *and where* $z\in\left[0,\sqrt{a^2-x^2-y^2}\right]$.

9) *The surface integral* $\int_{\mathcal{C}} \mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $y=\ln\sin x$ *for* $x\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$, *and where* $z\in\left[0,\frac{\cos^2 x}{\sin x}\right]$.

10) *The surface integral* $\int_{\mathcal{C}} \cosh\frac{z}{a}\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $y=a\cosh\frac{x}{a}$ *for* $x\in[0,a]$, *and where* $z\in 0,x]$.

11) *The surface integral* $\int_{\mathcal{C}} z^2\,\mathrm{d}S$, *where the curve* $\mathcal{L}$ *is given by* $y=x^3$ *for* $x\in[0,1]$, *and where* $z\in[0,x]$.

A Surface integral over a cylinder surface.

D Reduce to a line integral by first integrating in the direction of the Z-axis. Find the line element and calculate the line integral.

I 1) The curve is the circle of centrum $(1,0)$ and radius 1, thus in polar coordinates

$$\varrho(\varphi)=2\cos\varphi,\qquad \varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$

and the line element is

$$\mathrm{d}s=\sqrt{\varrho^2+\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2}\,\mathrm{d}\varphi=\sqrt{4\cos^2\varphi+4\sin^2\varphi}\,\mathrm{d}\varphi=2\,\mathrm{d}\varphi.$$

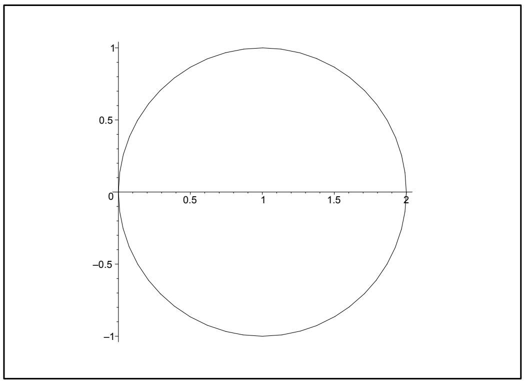

Figure 27.16: The curve $\mathcal{L}$ of **Example 27.6.1**.

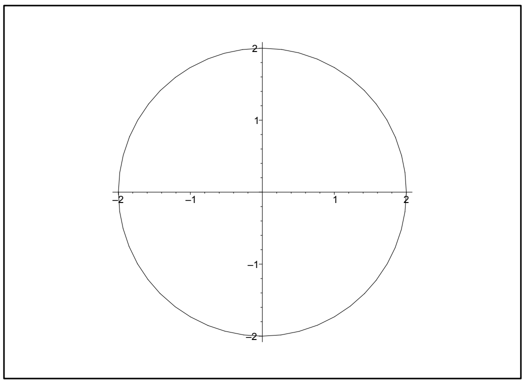

Figure 27.17: The curve $\mathcal{L}$ of **Example 27.6.2**.

Hence

$$\int_{\mathcal{C}} (y^2 z + x^2 z + y)\,\mathrm{d}S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{\varrho(\varphi)} \{z\varrho(\varphi)\sin\varphi\}\,\mathrm{d}z \right\} 2\,\mathrm{d}\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{2} z^2\varrho(\varphi)^2 + \varrho(\varphi) z \sin\varphi\right]_{z=0}^{2\cos\varphi} \cdot 2\,\mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{16\cos^4\varphi + 8\cos^2\varphi\cdot\sin\varphi\right\}\mathrm{d}\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4(1+\cos 2\varphi)^2\,\mathrm{d}\varphi + 0 = 8\int_0^{\frac{\pi}{2}} \left(1 + 2\cos 2\varphi + \frac{1}{2} + \frac{1}{2}\cos 4\varphi\right)\mathrm{d}\varphi = 8\cdot\frac{3}{2}\cdot\frac{\pi}{2} = 6\pi.$$

2) The curve is the circle of centrum $(0,0)$ and radius 2. It is described in polar coordinates by

$$\varrho = 2, \qquad \varphi \in [0, 2\pi],$$

hence the line element is

$$\mathrm{d}s = \sqrt{\varrho^2 + \left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2}\,\mathrm{d}\varphi = 2\,\mathrm{d}\varphi.$$

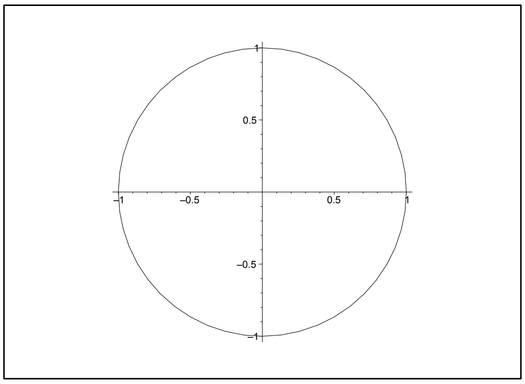

Figure 27.18: The curve $\mathcal{L}$ of **Example 27.6.3**.

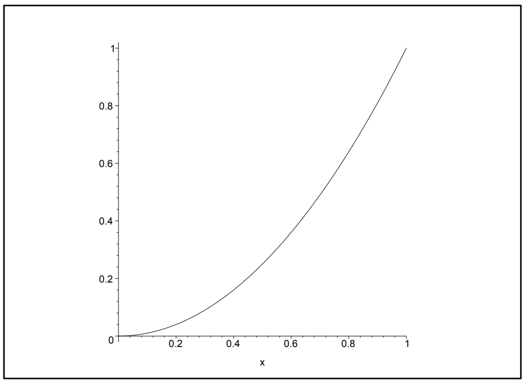

Figure 27.19: The curve $\mathcal{L}$ of **Example 27.6.4**.

Hence

$$\begin{aligned}\int_{\mathcal{C}} z^2\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_{-2}^{2\cos\varphi} z^2\,\mathrm{d}z\right\}2\,\mathrm{d}\varphi = \frac{2}{3}\int_0^{2\pi}\left\{8\cos^3\varphi-(-2)^3\right\}\,\mathrm{d}\varphi \\ &= \frac{16}{3}\int_0^{2\pi}\left\{\cos^3\varphi+1\right\}\,\mathrm{d}\varphi = \frac{32\pi}{3}+0=\frac{32\pi}{3}.\end{aligned}$$

3) The curve is the unit circle given in polar coordinates by

$$\varrho = 1, \qquad \varphi\in[0,2\pi].$$

Thus $\mathrm{d}s=\mathrm{d}\varphi$, and

$$\begin{aligned}\int_{\mathcal{C}}(z^2+x^2)\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^2(z^2+\cos^2\varphi)\,\mathrm{d}z\right\}\,\mathrm{d}\varphi \\ &= \frac{8}{3}\cdot 2\pi+2\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi = \frac{8}{3}\cdot 2\pi+2\pi=\frac{22\pi}{3}.\end{aligned}$$

4) The curve is an arc of a parabola. It follows by putting $y = g(x) = x^2$ that the line element is

$$\mathrm{d}s = \sqrt{1 + g'(x)^2}\, \mathrm{d}x = \sqrt{1 + 4x^2}\, \mathrm{d}x,$$

hence

$$\int_{\mathcal{C}} z \, \mathrm{d}S = \int_0^1 \left\{ \int_0^x z \, \mathrm{d}z \right\} \sqrt{1 + 4x^2}\, \mathrm{d}x = \frac{1}{2} \int_0^1 x^2 \sqrt{1 + 4x^2}\, \mathrm{d}x.$$

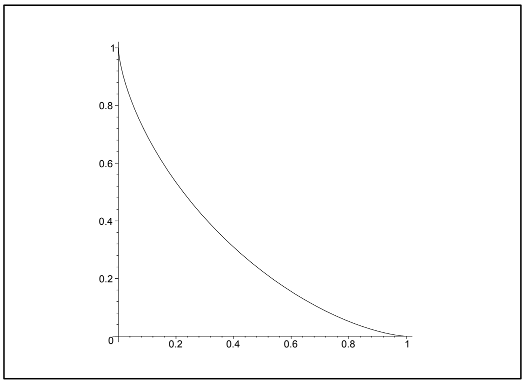

Figure 27.20: The curve $\mathcal{L}$ of **Example 27.6.5** for $a = 1$.

Then we get by the substitution $x = \frac{1}{2}\sinh t$, $t = \text{Arsinh}(2t)$ that

$$\begin{aligned}\int_{\mathcal{C}} z\,\mathrm{d}S &= \frac{1}{2}\int_0^{\text{Arsinh } 2} \frac{1}{4}\sinh^2 t\cdot\cosh t\cdot\frac{1}{2}\cosh t\,\mathrm{d}t = \frac{1}{16}\int_0^{\text{Arsinh } 2}\left(\frac{1}{2}\sinh 2t\right)^2\mathrm{d}t \\ &= \frac{1}{64}\int_0^{\text{Arsinh } 2}\frac{1}{2}(\cosh 4t - 1)\,\mathrm{d}t = \frac{1}{512}\left[\sinh 4t\right]_0^{\text{Arsinh } 2} - \frac{1}{128}\,\text{Arsinh } 2 \\ &= \frac{1}{512}\left[4\sinh t\cdot\sqrt{1+\sinh^2 t}\cdot(1+2\sinh^2 t)\right]_0^{\text{Arsinh } 2} - \frac{1}{128}\ln(2+\sqrt{5}) \\ &= \frac{1}{128}\cdot 2\sqrt{5}\cdot(1+2\cdot 4) - \frac{1}{128}\ln(2+\sqrt{5}) = \frac{9\sqrt{5}}{64} - \frac{1}{128}\ln(2+\sqrt{5}).\end{aligned}$$

5) We have in the given interval, $\cos t\cdot\sin t \geq 0$, so we do not need the absolute sign in the latter equality,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= a\sqrt{(-3\cos^2 t\,\sin t)^2 + (3\sin^2 t\,\cos t)^2} \\ &= 3a\sqrt{\cos^2 t\left\{\cos^2 t\,\sin^2 t\right\} + \sin^2 t\left\{\cos^2 t\,\sin^2 t\right\}} = 3a\,\cos t\,\sin t,\end{aligned}$$

hence the line element becomes

$$\mathrm{d}s = 3a\,\cos t\,\sin t\,\mathrm{d}t, \qquad t \in \left[0, \frac{\pi}{2}\right].$$

Then

$$\begin{aligned}\int_{\mathcal{C}} z\,\mathrm{d}S &= \int_0^{\frac{\pi}{2}}\left\{\int_0^{a\sin^3 t} z\,\mathrm{d}z\right\}3a\,\cos t\,\sin t\,\mathrm{d}t = \frac{3a^2}{2}\int_0^{\frac{\pi}{2}}\sin^7 t\,\cos t\,\mathrm{d}t \\ &= \frac{3}{16}\,a^3\left[\sin^8 t\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{16}.\end{aligned}$$

6) The line element along the curve is

$$\mathrm{d}s = \sqrt{\varrho^2 + \left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2}\,\mathrm{d}\varphi = \sqrt{2}\,e^{\varphi}\,\mathrm{d}\varphi, \qquad \varphi \in [0,1],$$

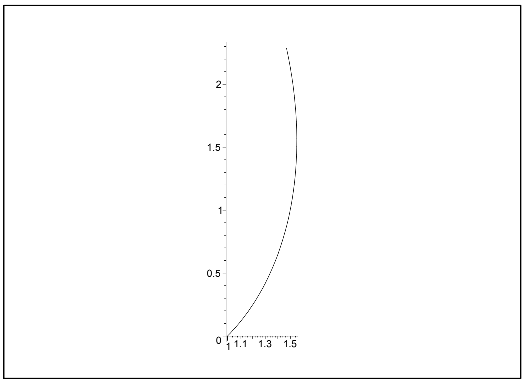

Figure 27.21: The curve $\mathcal{L}$ of **Example 27.6.6**.

and we get the surface integral

$$\int_{\mathcal{C}} \frac{1}{x}\,\mathrm{d}S = \int_0^1 \frac{1}{x}\cdot x\,\sqrt{2}\,e^{\varphi}\,\mathrm{d}\varphi = \sqrt{2}(e-1).$$

7) The line element is

$$\begin{aligned}
\mathrm{d}s &= \sqrt{\varrho^2+\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2}\,\mathrm{d}\varphi = \sqrt{a^2\cos^4\frac{\varphi}{2}+a^2\left(-2\cos\frac{\varphi}{2}\cdot\sin\frac{\varphi}{2}\cdot\frac{1}{2}\right)^2}\,\mathrm{d}\varphi \\
&= a\,\cos\frac{\varphi}{2}\,\mathrm{d}\varphi \qquad \text{for } \varphi\in\left[0,\frac{\pi}{2}\right],
\end{aligned}$$

hence

$$\begin{aligned}
\int_{\mathcal{C}} \frac{z}{x^2}\,\mathrm{d}S &= \int_0^{\frac{\pi}{2}} \frac{1}{x^2}\left\{\int_0^{xy} z\,\mathrm{d}z\right\} a\,\cos\frac{\varphi}{2}\,\mathrm{d}\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(xy)^2}{x^2}\cdot a\,\cos\frac{\varphi}{2}\,\mathrm{d}\varphi \\
&= \frac{a}{2}\int_0^{\frac{\pi}{2}} \varrho(\varphi)^2\sin^2\varphi\cdot\cos\frac{\varphi}{2}\,\mathrm{d}\varphi = \frac{a}{2}\int_0^{\frac{\pi}{2}} a^2\cos^4\frac{\varphi}{2}\cdot 4\sin^2\frac{\varphi}{2}\cdot\cos^2\frac{\varphi}{2}\cdot\cos\frac{\varphi}{2}\,\mathrm{d}\varphi \\
&= 2a^3\int_0^{\frac{\pi}{2}} \cos^6\frac{\varphi}{2}\cdot\sin^2\frac{\varphi}{2}\cdot\cos\frac{\varphi}{2}\,\mathrm{d}\varphi \\
&= 4a^3\int_0^{\frac{\pi}{2}} \left\{1-\sin^2\frac{\varphi}{2}\right\}^3\cdot\sin^2\frac{\varphi}{2}\cdot\left(\frac{1}{2}\cos\frac{\varphi}{2}\right)\mathrm{d}\varphi \\
&= 4a^3\int_{\varphi=0}^{\frac{\pi}{2}} \left\{\sin^2\frac{\varphi}{2}-3\sin^4\frac{\varphi}{2}+3\sin^6\frac{\varphi}{2}-\sin^8\frac{\varphi}{2}\right\}\mathrm{d}\left(\sin\frac{\varphi}{2}\right) \\
&= 4a^3\left[\frac{1}{3}t^3-\frac{3}{5}t^5+\frac{3}{7}t^7-\frac{1}{9}t^9\right]_0^{\frac{1}{\sqrt{2}}} = \frac{4a^3}{2\sqrt{2}}\left\{\frac{1}{3}-\frac{3}{5}\cdot\frac{1}{2}+\frac{3}{7}\cdot\frac{1}{4}-\frac{1}{9}\cdot\frac{1}{8}\right\} \\
&= \frac{a^3\sqrt{2}}{2520}(840-756+270-35) = \frac{319\sqrt{2}}{2520}\,a^3.
\end{aligned}$$

8) The curve is in polar coordinates given by

$$\varrho = a\,\cos\varphi, \qquad \varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$

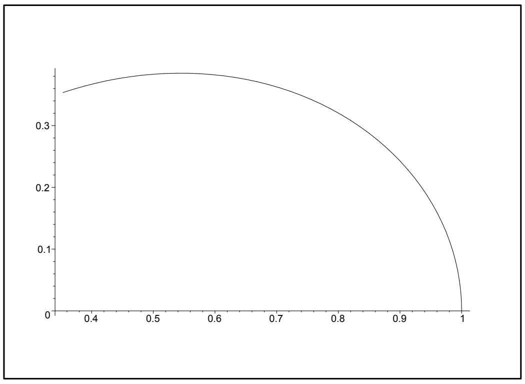

Figure 27.22: The curve $\mathcal{L}$ of **Example 27.6.7** for $a = 1$.

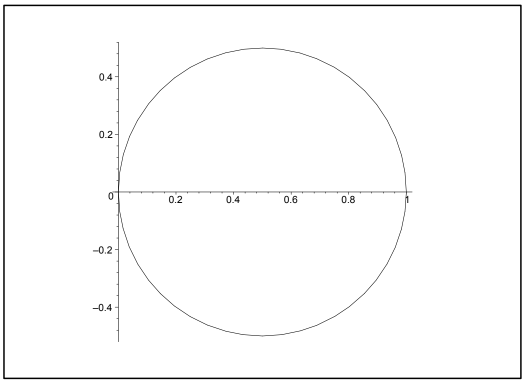

Figure 27.23: The curve $\mathcal{L}$ of **Example 27.6.8** for $a = 1$.

hence

$$ds = \sqrt{\varrho^2 + \left(\frac{d\varrho}{d\varphi}\right)^2}\, d\varphi = a\, d\varphi,$$

and

$$\begin{aligned}\int_{\mathcal{C}} xz\, dS &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{\sqrt{a^2-a^2\cos^2\varphi}} a\,\cos^2\varphi \cdot z\, dz \right\} a\, d\varphi \\ &= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi\,(1-\cos^2\varphi)a^2\, d\varphi = \frac{a^4}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2}\sin 2\varphi\right)^2 d\varphi \\ &= \frac{a^4}{2}\cdot\frac{1}{4}\cdot 2\int_0^{\frac{\pi}{2}} \sin^2 2\varphi\, d\varphi = \frac{a^4}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4\varphi)\, d\varphi = \frac{a^4\pi}{16}.\end{aligned}$$

9) We conclude from $y = g(x) = \ln\sin x$, $x \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ that the line element is

$$ds = \sqrt{1+g'(x)^2}\, dx = \sqrt{1+\left(\frac{\cos x}{\sin x}\right)^2}\, dx = \frac{1}{\sin x}\, dx,$$

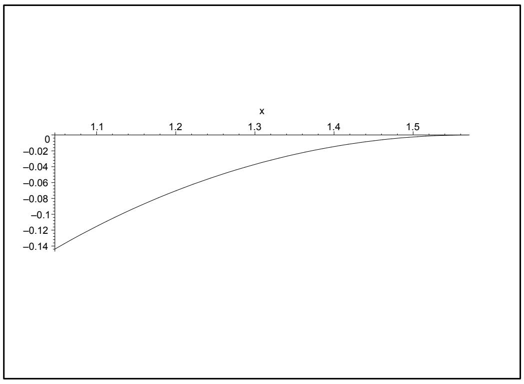

Figure 27.24: The curve $\mathcal{L}$ of **Example 27.6.9**.

and hence

$$\begin{aligned}\int_{\mathcal{C}} \mathrm{d}S &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x} \cdot \frac{1}{\sin x}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2 x} - 1 \right) \mathrm{d}x \\ &= \left[-\cot x - x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \cot \frac{\pi}{3} - \frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{\pi}{6}.\end{aligned}$$

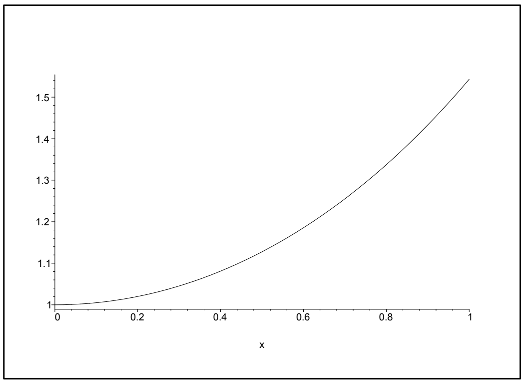

Figure 27.25: The curve $\mathcal{L}$ of **Example 27.6.10** for $a = 1$.

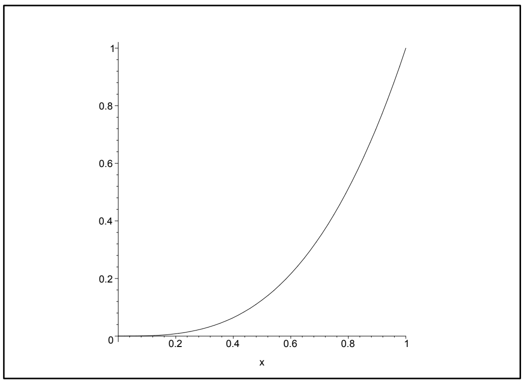

Figure 27.26: The curve $\mathcal{L}$ of **Example 27.6.11**.

10) When the curve is given by $y = g(x) = a\,\cosh\dfrac{x}{a}$, we obtain the line element

$$\mathrm{d}s = \sqrt{1 + g'(x)^2}\,\mathrm{d}x = \sqrt{1 + \sinh^2\frac{x}{a}}\,\mathrm{d}x = \cosh\frac{x}{a}\,\mathrm{d}x,$$

so

$$\begin{aligned}
\int_{\mathcal{C}} \cosh\frac{z}{a}\,\mathrm{d}S &= \int_0^a \left\{\int_0^x \cosh\frac{z}{a}\,\mathrm{d}z\right\}\cdot\cosh\frac{x}{a}\,\mathrm{d}x \\
&= a\int_0^a \sinh\frac{x}{a}\cdot\cosh\frac{x}{a}\,\mathrm{d}x = \frac{a^2}{2}\cdot\sinh^2 1 \\
&= \frac{a^2}{2}\left(\frac{e-e^{-1}}{2}\right)^2 = \frac{a^2}{8e^2}\left(e^2-1\right)^2 = \frac{a^2}{8e^2}\left(e^4-2e^2+1\right).
\end{aligned}$$

11) For the curve given by $y = g(x) = x^3$, the line element is

$$\mathrm{d}s = \sqrt{1 + g'(x)^2}\,\mathrm{d}x = \sqrt{1 + 9x^4}\,\mathrm{d}x,$$

hence

$$\begin{aligned}\int_{\mathcal{C}} z^2\,\mathrm{d}S &= \int_0^1 \left\{\int_0^x z^2\,\mathrm{d}z\right\}\sqrt{1+9x^4}\,\mathrm{d}x = \frac{1}{3}\int_0^1 \sqrt{1+9x^4}\cdot x^3\,\mathrm{d}x \\ &= \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{9}\int_0^1 \sqrt{1+9x^4}\cdot 9\,\mathrm{d}\left(x^4\right) = \frac{1}{108}\cdot\frac{2}{3}\left[\left(\sqrt{1+9x^4}\right)^3\right]_0^1 \\ &= \frac{1}{162}\,(10\sqrt{10}-1).\end{aligned}$$

Example 27.7 *Calculate in each of the following cases the given surface integral over a surface of revolution $\mathcal{O}$ which is given by a meridian curve $\mathcal{M}$ in the meridian half plane, in which ϱ and z are rectangular coordinates.*

1) *The surface integral $\int_{\mathcal{O}}(x^2+y^2)\,\mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z = \dfrac{\varrho^2}{2a}$ for $\varrho \le a$.*

2) *The surface integral $\int_{\mathcal{O}}(x^2+y^2)\,\mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z = \dfrac{h\varrho}{a}$ for $\varrho \le a$.*

3) *The surface integral $\int_{\mathcal{O}} \dfrac{e^z}{\varrho}\,\mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z = \ln \in \varrho$ for $\varrho \in \left[\dfrac{\pi}{3},\dfrac{2\pi}{3}\right]$.*

4) *The surface integral $\int_{\mathcal{O}} x^2\,\mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z^2+\varrho^2 = az$.*

5) *The surface integral $\int_{\mathcal{O}} |x|e^{-x}\,\mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z = -\ln\cos\varrho$ for $\varrho \in \left[0,\dfrac{\pi}{3}\right]$.*

6) *The surface integral $\int_{\mathcal{O}} \dfrac{y^2}{z}\,\mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z = a\,\cosh\dfrac{\varrho}{a}$ for $\varrho \in [0,a]$.*

A Surface integral over a surface of revolution.

D Use either semi-polar or spherical coordinates and the area element $\varrho\,\mathrm{d}\varphi\,\mathrm{d}s$, where $\mathrm{d}s$ is the curve element, i.e. if e.g. $z = g(\varrho)$, then

$$\mathrm{d}s = \sqrt{1+g'(\varrho)^2}\,\mathrm{d}\varrho,$$

and similarly.

I 1) Here $\mathrm{d}s = \sqrt{1+\left(\dfrac{\varrho}{a}\right)^2}\,\mathrm{d}\varrho$, hence

$$\begin{aligned}\int_{\mathcal{O}}(x^2+y^2)\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^a \varrho^2\cdot\varrho\sqrt{1+\left(\frac{\varrho}{a}\right)^2}\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi = 2\pi\cdot\frac{a^4}{2}\int_0^1 t\sqrt{1+t}\,\mathrm{d}t \\ &= \pi a^4\int_0^1\left\{(1+t)^{\frac{3}{2}}-(1+t)^{\frac{1}{2}}\right\}\mathrm{d}t = \pi a^4\left[\frac{2}{5}(1+t)^{\frac{5}{2}}-\frac{2}{3}(1+t)^{\frac{3}{2}}\right]_0^1 \\ &= \pi a^4\left\{\frac{2}{5}\left(2^{\frac{5}{2}}-1\right)-\frac{2}{3}\left(2^{\frac{3}{2}}-1\right)\right\} = \frac{\pi a^4}{15}\{6(4\sqrt{2}-1)-10(2\sqrt{2}-1)\} \\ &= \frac{\pi a^4}{15}\{24\sqrt{2}-6-20\sqrt{2}+10\} = \frac{\pi a^4}{15}\{4\sqrt{2}+4\} = \frac{4\pi a^4}{15}(\sqrt{2}+1).\end{aligned}$$

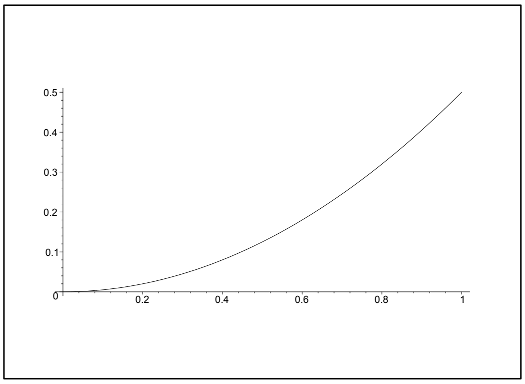

Figure 27.27: The meridian curve $\mathcal{M}$ of **Example 27.7.1** for $a = 1$.

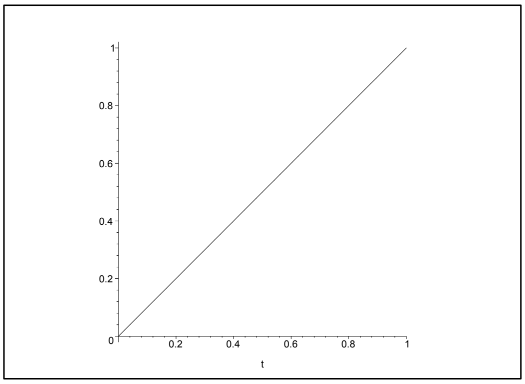

Figure 27.28: The meridian curve $\mathcal{M}$ of **Example 27.7.2** for $a = 1$ and $h = 1$.

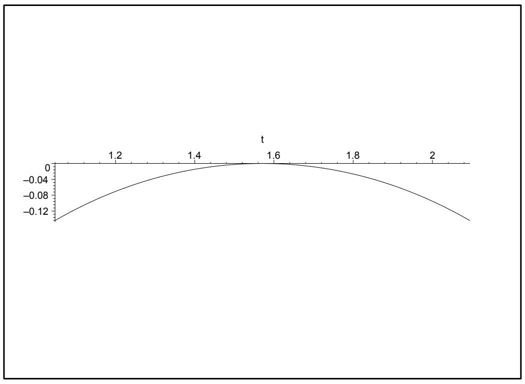

Figure 27.29: The meridian curve $\mathcal{M}$ of **Example 27.7.3**.

2) Here

$$\mathrm{d}s = \sqrt{1+\frac{h^2}{a^2}}\,\mathrm{d}\varrho = \frac{1}{a}\,\sqrt{a^2+h^2}\,\mathrm{d}\varrho,$$

hence

$$\begin{aligned}\int_{\mathcal{O}} \left(x^2+y^2\right)\mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^a \varrho^2\cdot\varrho\cdot\frac{1}{a}\,\sqrt{a^2+h^2}\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\ &= 2\pi\cdot\frac{1}{a}\,\sqrt{a^2+h^2}\cdot\frac{1}{4}\,a^4 = \frac{\pi}{2}\,a^3\,\sqrt{a^2+h^2}.\end{aligned}$$

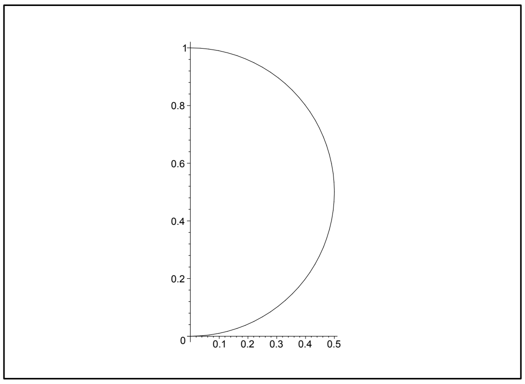

Figure 27.30: The meridian curve $\mathcal{M}$ of **Example 27.7.4** and **Example 27.7.5** for $a = 1$.

3) From $z = \ln \sin \varrho$ follows that $\dfrac{dz}{d\varrho} = \dfrac{\cos \varrho}{\sin \varrho}$, hence

$$\sqrt{1+\left(\frac{dz}{d\varrho}\right)^2} = \sqrt{1+\frac{\cos^2\varrho}{\sin^2\varrho}} = \frac{1}{|\sin\varrho|} = \frac{1}{\sin\varrho} \qquad \text{for } \varrho \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right].$$

The area element is

$$\varrho \, d\varphi \, ds = \frac{\varrho}{\sin\varrho} \, d\varrho \, d\varphi = dS,$$

hence by insertion

$$\int_{\mathcal{O}} \frac{e^z}{\varrho} \, dS = \int_0^{2\pi} \left\{ \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin\varrho}{\varrho} \cdot \frac{\varrho}{\sin\varrho} \, d\varrho \right\} d\varphi = 2\pi \left(\frac{2\pi}{3} - \frac{\pi}{3} \right) = \frac{2\pi^2}{3}.$$

4) The figure shows that the meridian curve is a half circle of radius $\dfrac{a}{2}$. Hence, the integral $\int_{\mathcal{O}} dS$ is equal to the surface area of the sphere, i.e.

$$\int_{\mathcal{O}} dS = 4\pi \left(\frac{a}{2}\right)^2 = \pi a^2$$

where we have used the result of **Example 27.7.6** with $a = b$.
ALTERNATIVELY,

$$\varrho = \sqrt{\left(\frac{a}{2}\right)^2 - \left(z - \frac{a}{2}\right)^2}, \qquad \text{for } z \in [0, a],$$

in rectangular coordinates, so

$$ds = \sqrt{1 + \frac{\left(z-\frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2 - \left(z-\frac{a}{2}\right)^2}} \, dz = \frac{a}{2} \, \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(z-\frac{a}{2}\right)^2}} \, dz.$$

Hence

$$\int_{\mathcal{O}} \mathrm{d}S = 2\pi \int_0^a \sqrt{\left(\frac{a}{2}\right)^2} \cdot \frac{a}{2} \cdot \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2 - \left(z-\frac{a}{2}\right)^2}}\,\mathrm{d}z = 2\pi \cdot \frac{a}{2} \cdot a = \pi a^2.$$

ALTERNATIVELY we have $r = a\cos\theta$, $\theta \in \left[0, \frac{\pi}{2}\right]$, in spherical coordinates, and $\varrho = r\sin\theta = a\sin\theta\cos\theta$, and

$$\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = a\,\mathrm{d}\theta,$$

and we get

$$\int_{\mathcal{O}} \mathrm{d}S = 2\pi \int_0^{\frac{\pi}{2}} a\sin\theta\cos\theta \cdot a\,\mathrm{d}\theta = a^2\pi\left[\sin^2\theta\right]_0^{\frac{\pi}{2}} = a^2\pi.$$

5) Since $x = \varrho\cos\varphi$ in semi-polar coordinates we get from **Example 27.7.4** that

$$\begin{aligned}\int_{\mathcal{O}} x^3\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^a \left\{\left(\frac{a}{2}\right)^2 - \left(z-\frac{a}{2}\right)^2\right\}\cos^2\varphi \cdot \frac{a}{2} \cdot \frac{\sqrt{(\frac{a}{2})^2 - (z-\frac{a}{2})^2}}{\sqrt{(\frac{a}{2})^2 - (z-\frac{a}{2})^2}}\,\mathrm{d}z\right\}\mathrm{d}\varphi \\ &= \frac{a}{2}\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi \int_0^a (az - z^2)\,\mathrm{d}z = \frac{a}{2}\cdot\pi\left[\frac{a}{2}z^2 - \frac{1}{3}z^3\right]_0^a = \frac{a^4\pi}{12}.\end{aligned}$$

ALTERNATIVELY,

$$x = r\sin\theta\cos\varphi = a\cos\theta\cos\varphi$$

in spherical coordinates, cf. **Example 27.7.4**, so accordingly

$$\begin{aligned}\int_{\mathcal{O}} \mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\sin^2\theta\cos^2\varphi \cdot a\sin\theta\cos\theta\,a\,\mathrm{d}\theta\right\}\mathrm{d}\varphi \\ &= a^4\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\sin^3\theta\cdot(1-\sin^2\theta)\cos\theta\,\mathrm{d}\theta = a^4\pi\left[\frac{1}{4}-\frac{1}{6}\right] = \frac{a^4\pi}{12}.\end{aligned}$$

6) As

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\sin\varrho}{\cos\varrho}\right)^2}\,\mathrm{d}\varrho = \frac{1}{\cos\varrho}\,\mathrm{d}\varrho, \quad \text{for } \varrho \in \left[0, \frac{\pi}{3}\right],$$

we get

$$\begin{aligned}\int_{\mathcal{O}} |x|e^{-z}\,\mathrm{d}S &= \int_0^{2\pi}\left\{\int_0^{\frac{\pi}{3}} \varrho|\cos\varphi| \cdot \cos\varrho \cdot \frac{\varrho}{\cos\varrho}\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\ &= 4\int_0^{\frac{\pi}{2}}\cos\varphi\,\mathrm{d}\varphi \cdot \int_0^{\frac{\pi}{3}}\varrho^2\,\mathrm{d}\varrho = 4\cdot\frac{1}{3}\left(\frac{\pi}{3}\right)^3 = \frac{4\pi^3}{81}.\end{aligned}$$

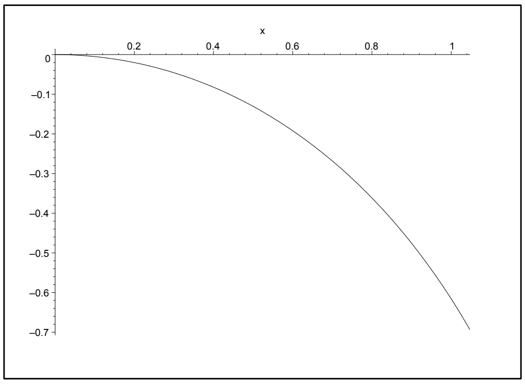

Figure 27.31: The meridian curve $\mathcal{M}$ of **Example 27.7.6**.

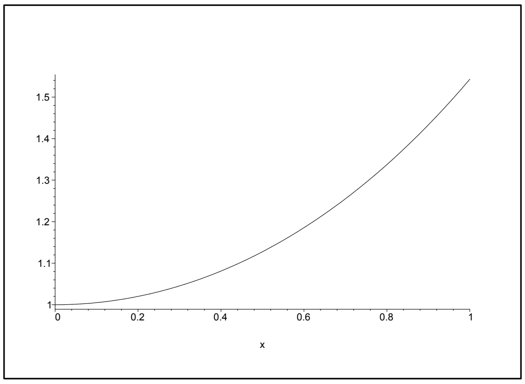

Figure 27.32: The meridian curve $\mathcal{M}$ of **Example 27.7.7** for $a = 1$.

7) We get from $z = g(\varrho) = a\,\cosh\dfrac{\varrho}{a}$ that $g'(\varrho) = \sinh\dfrac{\varrho}{a}$ and

$$ds = \sqrt{1+g'(\varrho)^2}\,d\varrho = \sqrt{1+\sinh^2\frac{\varrho}{a}}\,d\varrho = \cosh\frac{\varrho}{a}\,d\varrho,$$

hence

$$\begin{aligned}\int_{\mathcal{O}} \frac{y^2}{z}\,dS &= \int_0^{2\pi}\left\{\int_0^a \frac{\varrho^2\sin^2\varphi}{a\cosh\frac{\varrho}{a}}\cdot\varrho\cdot\cosh\frac{\varrho}{a}\,d\varrho\right\}d\varphi \\ &= \frac{1}{a}\int_0^{2\pi}\sin^2\varphi\,d\varphi\cdot\int_0^a \varrho^3\,d\varrho = \frac{1}{a}\cdot\pi\cdot\frac{1}{4}\,a^4 = \frac{\pi a^3}{4}.\end{aligned}$$

Example 27.8 *Calculate in each of the following cases the given surface integral over the surface given by a parametric description*

$$\mathcal{F} = \left\{x \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{r}(u,v),\, (u,v) \in E\right\}.$$

First find the normal vector of the surface $\mathbf{N}(u,v)$.

1) The surface integral $\int_{\mathcal{F}} xz^2 \,\mathrm{d}S$, *where the surface* $\mathcal{F}$ *is given by*

$$\mathbf{x} = \mathbf{r}(u,v) = (u\,\cos v, u\,\sin v, hv), \quad \textit{for } 0 \le u \le 1,\, 0 \le v \le 2\pi.$$

2) The surface integral $\int_{\mathcal{F}} z^2 \,\mathrm{d}S$, *where the surface* $\mathcal{F}$ *is given by*

$$\mathbf{x} = \mathbf{r}(u,v) = (\sqrt{u}\,\cos v, \sqrt{u}\,\sin v, e^v) \quad \textit{for } 1 \le u \le 2,\, \frac{\ln u}{2} \le v \le \frac{\ln(2u)}{2}.$$

3) The surface integral $\int_{\mathcal{F}} (x^2+y^2) \,\mathrm{d}S$, *where the surface* $\mathcal{F}$ *is given by*

$$\mathbf{x} = \mathbf{r}(u,v) = \left(\sqrt{u}\,\cos v, \sqrt{u}\,\sin v, v^{\frac{3}{2}}\right) \quad \textit{for } 1 \le u \le 2,\, 0 \le v \le u.$$

4) The surface integral $\int_{\mathcal{F}} (x^3+2z-3xy) \,\mathrm{d}S$, *where the surface* $\mathcal{F}$ *is given by*

$$\mathbf{x} = \mathbf{r}(u,v) = (u+v, u^2+v^2, u^3+v^3) \quad \textit{for } u+v \le 0,\, u^2+v^2 \le 5.$$

A Surface integrals, where the surface is given by a parametric description.

D First find the normal vector $\mathbf{N}(u,v)$. Then compute the weight function $\|\mathbf{N}(u,v)\|$ as a function of the parameters $(u,v) \in E$.

I 1) The normal vector is

$$\mathbf{N}(u,v) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos v & \sin v & 0 \\ -u\,\sin v & u\,\cos v & h \end{vmatrix} = (h\,\sin v, -h\,\cos v, u),$$

and we find accordingly the weight function

$$\|\mathbf{N}(u,v)\| = \sqrt{h^2+u^2}.$$

Then we get the following reduction of the surface integral,

$$\begin{aligned}
\int_{\mathcal{F}} xz^2 \,\mathrm{d}S &= \int_0^1 \left\{ \int_0^{2\pi} u\,\cos v \cdot h^2 v^2 \sqrt{h^2+u^2}\,\mathrm{d}v \right\} \mathrm{d}u \\
&= h^2 \int_0^1 u\,\sqrt{h^2+u^2}\,\mathrm{d}u \cdot \int_0^1 v^2 \cos v \,\mathrm{d}v \\
&= h^2 \left[\frac{1}{2}\cdot\frac{2}{3}(h^2+u^2)^{\frac{3}{2}}\right]_0^1 \cdot \left[v^2 \sin v + 2v\cos v - 2\sin v\right]_0^{2\pi} \\
&= \frac{1}{3}h^2 \left\{ \left(h^2+1\right)^{\frac{3}{2}} - h^3 \right\} \cdot 4\pi = \frac{4\pi}{3}\,h^2 \left\{ \left(h^2+1\right)\sqrt{h^2+1} - h^3 \right\}.
\end{aligned}$$

2) The normal vector is

$$\mathbf{N}(u,v)=\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}=\left|\begin{array}{ccc}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{1}{2}\,\frac{1}{\sqrt{u}}\,\cos v & \frac{1}{2}\,\frac{1}{\sqrt{u}}\,\sin v & 0 \\ -\sqrt{u}\,\sin v & \sqrt{u}\,\cos v & e^v\end{array}\right|=\left(\frac{1}{2}\,\frac{e^v}{\sqrt{u}}\,\sin v,-\frac{1}{2}\,\frac{e^v}{\sqrt{u}}\,\cos v,\frac{1}{2}\right),$$

so the weight function becomes

$$\|\mathbf{N}(u,v)\|=\sqrt{\frac{1}{4}\,\frac{e^{2v}}{u}+\frac{1}{4}}=\frac{1}{2\sqrt{u}}\,\sqrt{e^{2v}+u}.$$

Then we have the following reduction of the surface integral

$$\begin{aligned}\int_{\mathcal{F}} z^2\,\mathrm{d}S &= \int_1^2\left\{\int_{\frac{1}{2}\ln u}^{\frac{1}{2}\ln(2u)} e^{2v}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{u}}\cdot\sqrt{e^{2v}+u}\,\mathrm{d}v\right\}\mathrm{d}u \\ &= \frac{1}{2}\cdot\frac{1}{2}\int_1^2\left\{\int_{v=\frac{1}{2}\ln u}^{\frac{1}{2}\ln(2u)}\frac{1}{\sqrt{u}}\,\sqrt{e^{2v}+u}\,\mathrm{d}\left(e^{2v}\right)\right\}\mathrm{d}u \\ &= \frac{1}{4}\int_1^2\frac{1}{\sqrt{u}}\cdot\frac{2}{3}\left[(t+u)^{\frac{3}{2}}\right]_{t=u}^{t=2u}\mathrm{d}u=\frac{1}{6}\int_1^2\frac{1}{\sqrt{u}}\left\{(3u)^{\frac{3}{2}}-(2u)^{\frac{3}{2}}\right\}\mathrm{d}u \\ &= \frac{1}{6}\,(3\sqrt{3}-2\sqrt{2})\int_1^2 u\,\mathrm{d}u=\frac{1}{12}\,(3\sqrt{3}-2\sqrt{2}\left[u^2\right]_1^2=\frac{1}{4}\,(3\sqrt{3}-2\sqrt{2}).\end{aligned}$$

3) The normal vector is

$$\mathbf{N}(u,v)=\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}=\left|\begin{array}{ccc}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{1}{2}\,\frac{1}{\sqrt{u}}\,\cos v & \frac{1}{2}\,\frac{1}{\sqrt{u}}\,\sin v & 0 \\ -\sqrt{u}\,\sin v & \sqrt{u}\,\cos v & \frac{3}{2}\,\sqrt{v}\end{array}\right|=\left(\frac{3}{4}\sqrt{\frac{v}{u}}\,\sin v,-\frac{3}{4}\sqrt{\frac{v}{u}}\,\cos v,\frac{1}{2}\right),$$

and the weight function is

$$\|\mathbf{N}(u,v)\|=\sqrt{\frac{9}{16}\,\frac{v}{u}+\frac{1}{4}}=\frac{3}{4}\sqrt{\frac{v}{u}+\frac{4}{9}}.$$

Finally, we get the following reduction of the surface integral

$$\begin{aligned}
\int_{\mathcal{F}} (x^2+y^2)\,\mathrm{d}S &= \int_1^2 \left\{ \int_0^{u(u\,\cos^2 + u\,\sin^2 v)\cdot \frac{3}{4}\sqrt{\frac{v}{u}+\frac{4}{9}}\,\mathrm{d}v} \right\} \mathrm{d}u \\
&= \frac{3}{4}\int_1^2 \left\{ \int_0^{uu\sqrt{\frac{v}{u}+\frac{4}{9}}\,\mathrm{d}v} \right\} \mathrm{d}u = \frac{3}{4}\int_1^2 \left\{ \int_0^u \sqrt{v+\frac{4}{9}\,u}\cdot\sqrt{u}\,\mathrm{d}v \right\} \mathrm{d}u \\
&= \frac{3}{4}\cdot\frac{2}{3}\int_1^2 \sqrt{u}\left[\left(v+\frac{4}{9}\,u\right)^{\frac{3}{2}}\right]_0^u \mathrm{d}u = \frac{1}{2}\int_1^2 \sqrt{u}\left\{\left(\frac{13}{9}\,u\right)^{\frac{3}{2}} - \left(\frac{4}{9}\,u\right)^{\frac{3}{2}}\right\} \mathrm{d}u \\
&= \frac{1}{2}\cdot\frac{1}{27}\,(13\sqrt{13}-8)\int_1^2 u^2\,\mathrm{d}u = \frac{7}{162}\,(13\sqrt{13}-8).
\end{aligned}$$

4) The normal vector is

$$\begin{aligned}
\mathbf{N}(u,v) &= \frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 2u & 3u^2 \\ 1 & 2v & 3v^2 \end{vmatrix} = \left(6uv^2-6u^2v, 3u^2-3v^2, 2v-2u\right) \\
&= (6uv(vu), 3(u+v)(u-v), 2(v-u)) = (v-u)(6uv, -3(u+v), 2).
\end{aligned}$$

Hence the weight function

$$\|\mathbf{N}(u,v)\| = |v-u|\sqrt{36u^2v^2+9(u^2+2uv+v^2)+4}.$$

This expression looks very impossible, so we can only hope for that some factor of the integrand cancels the unfortunate square root.

The integrand is given in the parameters of the surface by

$$\begin{aligned}
&x^3+2z-3xy = (u+v)^3+2(u^3+v^3)-3(u+v)(u^2+v^2) \\
&\quad = u^3+3u^2v+3uv^2+v^3+2u^3+2v^3-3u^3-3u^2v-3uv^2-3v^3 \\
&\quad = 0.
\end{aligned}$$

Luckily, the surface of integration $\mathcal{F}$ is a zero surface of the integrand, so there is nothing to worry about,

$$\int_{\mathcal{F}} (x^3+2z-3xy)\,\mathrm{d}S = 0.$$

Example 27.9 *Let* $\mathcal{F}$ *be the sphere of centrum* $(0,0,0)$ *and radius* a*, and let*

$$f(x,y,z) = \alpha(x^2+y^2-2z^2)+\beta xy,$$

where α *and* β *are constants. Calculate the surface integrals*

$$Q = \int_{\mathcal{F}} f(x,y,z)\,\mathrm{d}S \quad \text{and} \quad \mathbf{P} = \int_{\mathcal{F}} (x,y,z) f(x,y,z)\,\mathrm{d}S.$$

A Surface integral.

D Exploit the symmetry of the sphere, since this is far easier than just to insert into some formula. Note that there are several possibilities of insertion into standard formulæ, though none of them looks promising.

I It follows by the symmetry that

$$\int_{\mathcal{F}} x^2\,\mathrm{d}S = \int_{\mathcal{F}} y^2\,\mathrm{d}S = \int_{\mathcal{F}} z^2\,\mathrm{d}S,$$

and that

$$\int_{\mathcal{F}} xy\,\mathrm{d}S = 0.$$

Then it is immediate that

$$Q = \alpha\left(\int_{\mathcal{F}} x^2\,\mathrm{d}S + \int_{\mathcal{F}} y^2\,\mathrm{d}S - 2\int_{\mathcal{F}} z^2\,\mathrm{d}S\right) + \beta\int_{\mathcal{F}} xy\,\mathrm{d}S = 0.$$

A similar symmetric consideration shows that if $g(x,y,z)$ is a homogeneous polynomial of odd degree, then

$$\int_{\mathcal{F}} g(x,y,z)\,\mathrm{d}S = 0.$$

Split $\mathcal{F}$ into the eight surfaces occurring by the intersections by the three coordinate planes. By assuming that $g(x,y,z)$ is *odd*, it follows by the symmetry of the sphere that the surfaces can be paired in such a way that the sum of the surface integrals over each pair is zero. (The details are left to the reader).

Since $x\,f(x,y,z)$, $y\,f(x,y,z)$ and $z\,f(x,y,z)$ all are homogeneous of degree 3, we conclude that

$$\mathbf{P} = \mathbf{0}.$$

REMARK. We shall for obvious reasons skip the traditional variants which give a lot of tedious computations. The reason for including this example is of course to demonstrate that one in some cases may benefit from the symmetry. $\Diamond$

Example 27.10 *Let $\mathcal{F}$ be the sphere given by $r = a$ and let R denote the distance from the point (x,y,z) on the sphere to the point $(0,0,w)$ on the Z-axis. Calculate*

$$U(w) = \int_{\mathcal{F}} \frac{1}{R}\,\mathrm{d}S.$$

One may assume that $w \geq 0$. The cases $w = a$ and $w = 0$, however, must be treated separately.

A Surface integral.

D We may for symmetric reasons assume that $w \geq 0$. We shall first check where $\dfrac{1}{R}$ is harmonic. To this end we use the mean value theorem, whenever possible. Then proceed by calculating $U(w)$ directly. We get some special cases, when either $w = a$ or $w = 0$. We have an improper integral in the former case and lots of symmetry in the latter one.

I Clearly,

$$\frac{1}{R} = \frac{1}{\sqrt{x^2+y^2+(z-w)^2}} = \left\{x^2+y^2+(z-w)^2\right\}^{-\frac{1}{2}}.$$

It follows immediately for $w = 0$ that

$$U(0) = \int_{\mathcal{F}} \frac{1}{a}\,\mathrm{d}S = \frac{1}{a}\,\mathrm{area}(\mathcal{F}) = 4\pi a.$$

REMARK. It can be mentioned aside that we get by using a so-called *Riesz transformation* that

$$U(w) = U(0) = 4\pi a \qquad \text{for } -a < w < a.$$

However, *Riesz-transformations* cannot be assumed for most readers, so we shall here give a straight proof instead. $\Diamond$

It follows from the expression of $\dfrac{1}{R}$ that $U(-w) = U(w)$, and we have again explained why we can choose $w \geq 0$.

First attempt. We first check if $\dfrac{1}{R}$ is harmonic for $(x,y,z) \neq (0,0,w)$. We find

$$\frac{\partial}{\partial x}\left(\frac{1}{R}\right) = -x\left(\frac{1}{R}\right)^3,$$

and

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{R}\right) = -\left(\frac{1}{R}\right)^3 - 3x\left(\frac{1}{R}\right)\cdot\left\{-x\left(\frac{1}{R}\right)^3\right\} = -\left(\frac{1}{R}\right)^3 + 3x^2\left(\frac{1}{R}\right)^5.$$

Then by the symmetry,

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{1}{R}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{R}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right) &= -3\left(\frac{1}{R}\right)^3 + 3\left(\frac{1}{R}\right)^5\left\{x^2+y^2+(z-w)^2\right\} \\ &= -3\left(\frac{1}{R}\right)^3 + 3\left(\frac{1}{R}\right)^5 \cdot R^2 = 0,\end{aligned}$$

and the function is harmonic for $(x,y,z) \neq (0,0,w)$.

It follows when $w > a$ from the mean value theorem that

$$U(w) = \int_{\mathcal{F}} \frac{1}{R}\,\mathrm{d}S = \frac{1}{R(0,0,0)}\,\text{area}(\mathcal{F}) = \frac{4\pi a^2}{w}, \qquad w > a,$$

hence in general

$$U(w) = \frac{4\pi a^2}{|w|} \qquad \text{for } |w| > a.$$

Note that when $|w| < a$, we cannot use the argument above because of the singularity at $(0,0,w)$ for $\dfrac{1}{R}$ which then lies inside K.

Second attempt. Split the surface $\mathcal{F}$ into an upper surface $\mathcal{F}_1$ and a lower surface $\mathcal{F}_2$. Then

$$z = \sqrt{a^2-x^2-y^2} \text{ on } \mathcal{F}_1, \qquad z = -\sqrt{a^2-x^2-y^2} \text{ on } \mathcal{F}_2.$$

The surface element is in rectangular coordinates given by

$$\mathrm{d}S = \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y, \qquad x^2+y^2 < a^2,$$

and we have

$$R = \sqrt{x^2+y^2+(z-w)^2} = \sqrt{a^2-(a^2-x^2-y^2)+(\pm\sqrt{a^2-x^2-y^2}-w)^2},$$

where the sign $+$ is used on $\mathcal{F}_1$, and the sign $-$ on $\mathcal{F}_2$.

Let S be the disc $\varrho^2 = x^2 + y^2 < a^2$. Then

$$
\begin{aligned}
U(w) &= \int_{\mathcal{F}} \frac{1}{R}\,\mathrm{d}S = \int_{\mathcal{F}_1} \frac{1}{R}\,\mathrm{d}S + \int_{\mathcal{F}_2} \frac{1}{R}\,\mathrm{d}S \\
&= \int_0^{2\pi} \left\{ \int_0^a \frac{1}{\sqrt{a^2-(a^2-\varrho^2)+(\sqrt{a^2-\varrho}-w)^2} \cdot \frac{a\varrho}{\sqrt{a^2-\varrho^2}}\,\mathrm{d}\varrho} \right\} \mathrm{d}\varphi \\
&\quad + \int_0^{2\pi} \left\{ \int_0^a \frac{1}{\sqrt{a^2-(a^2-\varrho^2)+(\sqrt{a^2-\varrho}+w)^2}} \cdot \frac{a\varrho}{\sqrt{a^2-\varrho^2}}\,\mathrm{d}\varrho \right\} \mathrm{d}\varphi \\
&= 2\pi a \int_0^a \left\{ \frac{1}{\sqrt{a^2-t^2+(t-w)^2} + \dfrac{1}{\sqrt{a^2-t^2+(t+w)^2}}} \right\} dt \\
&= 2\pi a \int_0^a \left\{ \frac{1}{\sqrt{a^2+w^2-2tw}} + \frac{1}{\sqrt{a^2+w^2+2tw}} \right\} dt \\
&= 2\pi a \left[\frac{\sqrt{a^2+w^2-2tw}}{-w} + \frac{\sqrt{a^2+w^2+2tw}}{w} \right]_0^a \\
&= \frac{2\pi a}{w} \left\{ -\sqrt{a^2+w^2-2aw} + \sqrt{a^2+w^2+2aw} + \sqrt{a^2} - \sqrt{a^2} \right\} \\
&= \frac{2\pi a}{w} \{|a+w| - |a-w|\}.
\end{aligned}
$$

For $w = 0$ we get instead (cf. the above)

$$
U(0) = 2\pi a \int_0^a \left\{ \frac{1}{\sqrt{a^2}} + \frac{1}{\sqrt{a^2}} \right\} \mathrm{d}t = 2\pi a \cdot \frac{2}{a} \cdot a = 4\pi a,
$$

in agreement with the previous result.

If $0 < w < a$, then

$$
U(w) = \frac{2\pi a}{w}\,(a+w-a+w) = 4\pi a,
$$

cf. the previous remark about the *Riesz transformation*.
When $w = a$, then $U(a) = 4\pi a$.
When $w > a$, then

$$
U(w) = \frac{2\pi a}{w}\,(a+w+a-w) = \frac{4\pi a^2}{w},
$$

cf. the result on harmonic functions.

Summarizing,

$$U(w)=\begin{cases}4\pi a & \text{for } |w|\le a,\\[2mm] \dfrac{4\pi a^2}{|w|} & \text{for } |w|>a.\end{cases}$$

Example 27.11 *A surface of revolution $\mathcal{F}$ is given in semi-polar coordinates (ϱ, φ, z) by*

$$z = \varrho^2, \qquad \varrho \in [0,2], \qquad \varphi \in [0, 2\pi].$$

Sketch the meridian curve $\mathcal{M}$, and calculate the surface integral

$$\int_{\mathcal{F}} \frac{1}{\sqrt{1+4z}}\,\mathrm{d}S.$$

A Surface integral.

D Follow the guidelines.

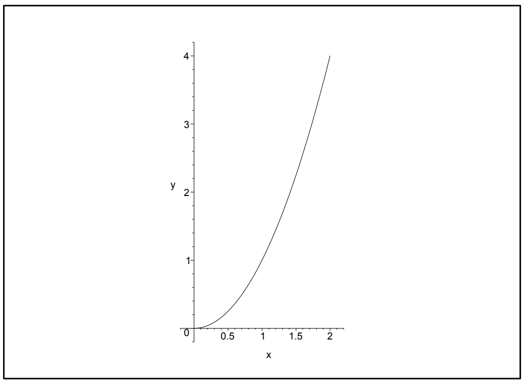

Figure 27.33: The meridian curve $\mathcal{M}$.

I The surface element is $\mathrm{d}S = P\,\mathrm{d}\varphi\,ds$, where $P = \varrho(z) = \sqrt{z}$ and

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}\varrho}{\mathrm{d}z}\right)^2}\,\mathrm{d}z = \sqrt{1 + \left(\frac{1}{2}\cdot\frac{1}{\sqrt{z}}\right)^2}\,\mathrm{d}z,$$

hence

$$\begin{aligned}\int_{\mathcal{F}} \frac{1}{\sqrt{1+4z}}\,\mathrm{d}S &= 2\pi \int_0^4 \frac{1}{1+4z}\cdot\sqrt{z}\cdot\sqrt{1+\frac{1}{4z}}\,\mathrm{d}z \\ &= 2\pi \int_0^4 \frac{\sqrt{z}}{1+4z}\cdot\sqrt{\frac{1+4z}{4z}}\,dz = \frac{2\pi}{2}\int_0^4 \mathrm{d}z = 4\pi.\end{aligned}$$

Example 27.12 *A surface of revolution $\mathcal{F}$ is in semi-polar coordinates (ϱ, φ, z) given by*

$$z = \varrho^3, \qquad \varrho \in \left[0, \frac{1}{2}\right], \qquad \varphi \in [0, 2\pi].$$

Sketch the meridian curve $\mathcal{M}$, and find the line element $\mathrm{d}s$ on this curve. Then calculate the surface integral

$$\int_{\mathcal{F}} \frac{\varrho^2}{1+9z\varrho}\,\mathrm{d}S.$$

A Surface integral.

D Follow the guidelines.

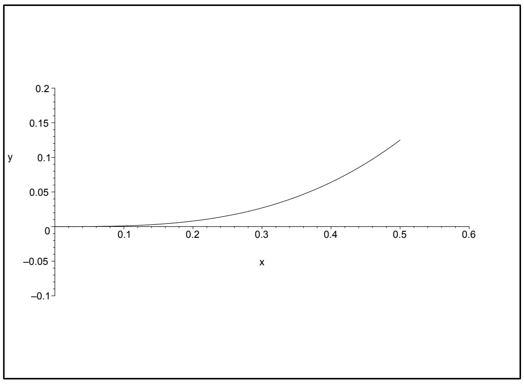

Figure 27.34: The meridian curve $\mathcal{M}$.

I It follows from $\dfrac{\mathrm{d}z}{\mathrm{d}\varrho} = 3\varrho^2$ that the line element is

$$\mathrm{d}s = \sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}\varrho}\right)^2}\,\mathrm{d}\varrho = \sqrt{1+9\varrho^4}\,\mathrm{d}\varrho, \qquad \varrho \in \left[0, \frac{1}{2}\right],$$

and accordingly the surface element

$$\mathrm{d}S = \varrho\sqrt{1+9\varrho^4}\,\mathrm{d}\varrho\,\mathrm{d}\varphi, \qquad \varrho \in \left[0, \frac{1}{2}\right], \quad \varphi \in [0, 2\pi].$$

We have $z = \varrho^3$ on $\mathcal{F}$, so by insertion into the surface integral,

$$\begin{aligned}\int_{\mathcal{F}} \frac{\varrho^2}{1+9z\varrho}\,\mathrm{d}S &= 2\pi\int_0^{\frac{1}{2}} \frac{\varrho^2}{1+9\varrho^4}\cdot\varrho\sqrt{1+9\varrho^4}\,\mathrm{d}\varrho = 2\pi\int_0^{\frac{1}{2}} \frac{\varrho^3}{\sqrt{1+9\varrho^4}}\,\mathrm{d}\varrho = \frac{2\pi}{4\cdot 9}\int_1^{\frac{25}{16}} \frac{\mathrm{d}u}{\sqrt{u}} \\ &= \frac{2\pi}{36}\left[2\sqrt{u}\right]_1^{(5/4)^2} = \frac{4\pi}{36}\left(\frac{5}{4}-1\right) = \frac{\pi}{36}.\end{aligned}$$

Example 27.13 *The surface $\mathcal{F}$ is given by*

$$z = g(x,y) = \frac{y^2}{x} + \frac{3}{4}\,x, \qquad (x,y) \in E,$$

where

$$E = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 2,\, 0 \le y \le x^2\}.$$

Prove that

$$\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \left(\frac{y}{x}\right)^2 + \frac{5}{4},$$

and then calculate the surface integral $\int_{\mathcal{F}} x\,\mathrm{d}S$.

A Surface integral.

D Follow the guidelines.

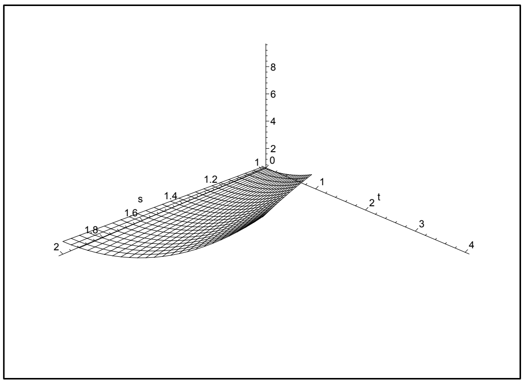

Figure 27.35: The surface $\mathcal{F}$.

I It follows from

$$\frac{\partial g}{\partial x} = -\frac{y^2}{x^2} + \frac{3}{4}, \qquad \frac{\partial g}{\partial y} = 2\,\frac{y}{x}$$

that

$$\begin{aligned}
1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 &= 1 + \left\{-\frac{y^2}{x^2} + \frac{3}{4}\right\}^2 + 4 \cdot \frac{y^2}{x^2} \\
&= 1 + \left(\frac{y}{x}\right)^4 - \frac{3}{2}\left(\frac{y}{x}\right) + \frac{9}{16} + 4\left(\frac{y}{x}\right)^2 = \left\{\left(\frac{y}{x}\right)^2\right\}^2 + \frac{5}{2}\left(\frac{y}{x}\right)^2 + \frac{25}{16} \\
&= \left\{\left(\frac{y}{x}\right)^2\right\}^2 + 2 \cdot \left(\frac{y}{x}\right) \cdot \frac{5}{4} + \left(\frac{5}{4}\right)^2 = \left\{\left(\frac{y}{x}\right)^2 + \frac{5}{4}\right\}^2,
\end{aligned}$$

hence

$$\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}=\left(\frac{y}{x}\right)^2+\frac{5}{4}.$$

Then by the usual reduction of the surface integral to a plane integral,

$$\begin{aligned}
\int_{\mathcal{F}} x\,\mathrm{d}S &= \int_E x\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y=\int_E\left\{\left(\frac{y}{x}\right)^2+\frac{5}{4}\right\}x\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_1^2\left\{\int_0^{x^2}\left(\frac{y^2}{blackx}+\frac{5}{4}x\right)\mathrm{d}y\right\}\mathrm{d}x=\int_1^2\left[\frac{y^3}{3x}+\frac{5}{4}xy\right]_{y=0}^{x^2}\mathrm{d}x\\
&= \int_1^2\left\{\frac{x^6}{3x}+\frac{5}{4}x^3\right\}\mathrm{d}x=\int_1^2\left\{\frac{1}{3}x^5+\frac{5}{4}x^3\right\}\mathrm{d}x\\
&= \left[\frac{1}{18}x^6+\frac{5}{16}x^4\right]_1^2=\frac{64}{18}+\frac{5}{16}\cdot 16-\frac{1}{18}-\frac{5}{16}=\frac{63}{18}+\frac{75}{16}=\frac{7}{2}+\frac{75}{16}=\frac{131}{16}.
\end{aligned}$$

Example 27.14 *A plane curve $\mathcal{L}$ is given by the parametric description*

$$(x,y) = (\cos t, -2\ln\sin t), \qquad t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

1. *Show that the line element ds is given by*

$$\mathrm{d}s = \frac{2-\sin^2 t}{\sin t}\,\mathrm{d}t.$$

A cylinder surface $\mathcal{C}$ with $\mathcal{L}$ as its leading curve is given in the following way:

$$x = \cos t, \quad y = -2\ln\sin t, \quad z \in [0, \sin t], \quad t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

2. *Calculate the surface integral $\int_{\mathcal{C}} xz\,\mathrm{d}S$.*

A Curve element and surface integral.

D Follow the guidelines; apply the formula of the surface integral over a cylinder surface.

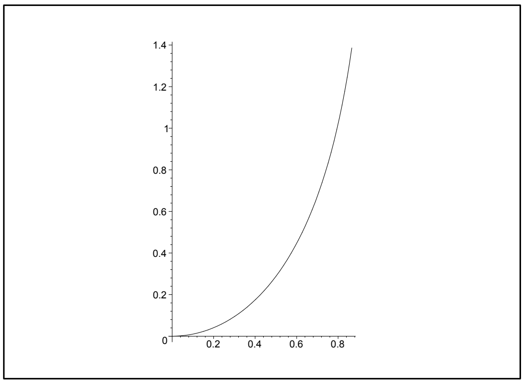

Figure 27.36: The leading curve $\mathcal{L}$.

I 1) From

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2\,\frac{\cos t}{\sin t},$$

follows that

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \sin^2 t + \frac{4\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}\left\{(\sin^2 t)^2 - 4\sin^2 t + 4\right\} \\ &= \left\{\frac{2-\sin^2 t}{\sin t}\right\}^2,\end{aligned}$$

hence

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \left|\frac{2-\sin^2 t}{\sin t}\right|\,\mathrm{d}t = \frac{2-\sin^2 t}{\sin t}\,dt, \qquad t \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

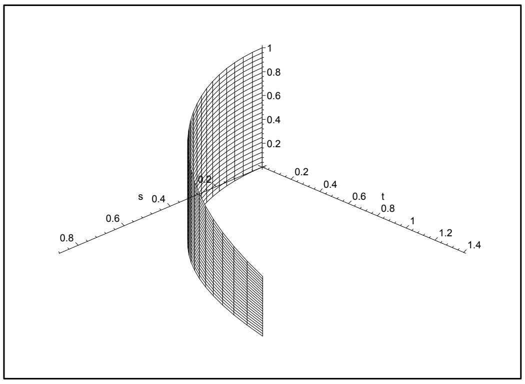

Figure 27.37: The surface $\mathcal{F}$.

2) Then the surface integral is calculated by means of the formula of an integral over a cylinder surface,

$$\begin{aligned}
\int_{\mathcal{C}} xz\,\mathrm{d}S &= \int_{\mathcal{L}} \left\{ \int_0^{\sin t} \cos t \cdot z\,\mathrm{d}z \right\} \mathrm{d}s = \int_{\mathcal{L}} \cos t \cdot \left[\frac{z^2}{2} \right]_0^{\sin t} \mathrm{d}s \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \cdot \frac{1}{2} \sin^2 t \cdot \frac{2 - \sin^2 t}{\sin t}\,\mathrm{d}t = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{ 2 \sin t - \sin^3 t \right\} \cos t\,\mathrm{d}t \\
&= \frac{1}{2} \left[\sin^2 t - \frac{1}{4} \sin^4 t \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2} \left\{ 1 - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} \left(\frac{1}{2} \right)^4 \right\} \\
&= \frac{1}{128} \{64 - 16 - 32 + 1\} = \frac{17}{128}.
\end{aligned}$$

Example 27.15 *Let $\mathcal{F}$ denote the surface given by the parametric description*

$$\mathbf{r}(u,v) = ((a+u)\cos v, (a+u)\sin v, av), \qquad (u,v) \in E,$$

where

$$E = \{(u,v) \in \mathbb{R}^2 \mid 0 \le u \le a,\, 0 \le v \le 2u\},$$

and where $a \in \mathbb{R}_+$ is a given constant.
Calculate the surface integral

$$\int_{\mathcal{F}} \frac{z^2}{\sqrt{a^2+x^2+y^2}}\,\mathrm{d}S.$$

A Surface integral.

D First find the weight function, i.e. the length of each normal vector in the normal vector field.

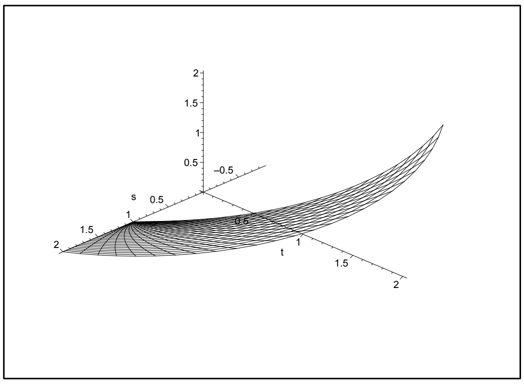

Figure 27.38: The surface $\mathcal{F}$ for $a=1$.

I It follows from

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos v, \sin v, 0), \qquad \frac{\partial \mathbf{r}}{\partial v} = (-(a+u)\sin v, (a+u)\cos v, a),$$

that the normal vector is given by

$$\mathbf{N}(u,v) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \cos v & \sin v & 0 \\ -(a+u)\sin v & (a+u)\cos v & a \end{vmatrix} = (a\,\sin v, a\,\cos v, a+u),$$

hence

$$\|\mathbf{N}(u,v)\| = \sqrt{a^2+(a+u)^2}.$$

Then we can calculate the surface integral,

$$\begin{aligned}
\int_{\mathcal{F}} \frac{z^2}{\sqrt{a^2+x^2+y^2}}\,\mathrm{d}S &= \int_E \frac{a^2v^2}{\sqrt{a^2+(a+u)^2}} \cdot \|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v \\
&= a^2 \int_0^a \left\{ \int_0^{2u} v^2\,\mathrm{d}v \right\} \mathrm{d}u = a^2 \int_0^a \left[\frac{v^3}{3} \right]_0^{2u} \mathrm{d}u \\
&= \frac{8}{3}\,a^2 \int_0^a u^3\,\mathrm{d}u = \frac{2}{3}\,a^2 \cdot a^4 = \frac{2}{3}\,a^6.
\end{aligned}$$

Example 27.16 *A surface of revolution $\mathcal{O}$ is in semi-polar coordinates (ϱ, φ, z) given by*

$$\varrho \in [0, 2a], \qquad \varphi \in [0, 2\pi], \qquad z = \sqrt{a^2 + \varrho^2},$$

where $a \in \mathbb{R}_+$ is some given constant.

1) Sketch the meridian curve $\mathcal{M}$.

2) Show that the line element $\mathrm{d}s$ on $\mathcal{M}$ is given by

$$\mathrm{d}s = \sqrt{\frac{a^2 + 2\varrho^2}{a^2 + \varrho^2}}\,\mathrm{d}\varrho.$$

3) Calculate the line integral

$$\int_{\mathcal{M}} z\varrho\,\mathrm{d}s.$$

4) Calculate the surface integral

$$\int_{\mathcal{O}} \frac{1}{z^2\sqrt{z^2 + \varrho^2}}\,\mathrm{d}S.$$

A Surface of revolution, line integral and surface integral.

D Standard example.

I

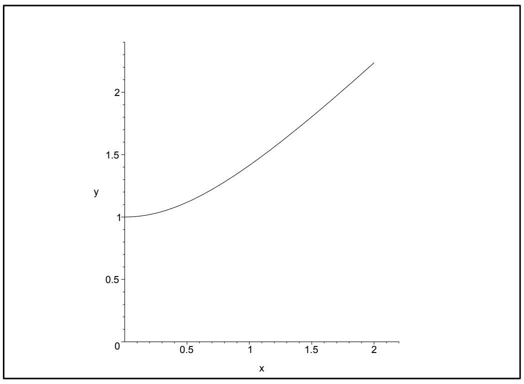

Figure 27.39: The meridian curve $\mathcal{M}$ for $a = 1$.

It follows from

$$\mathrm{d}z = \frac{\varrho}{\sqrt{a^2 + \varrho^2}}\,\mathrm{d}\varrho,$$

that

$$\mathrm{d}s = \sqrt{(\,\mathrm{d}\varrho)^2 + (\,\mathrm{d}z)^2} = \sqrt{1 + \frac{\varrho^2}{a^2 + \varrho^2}}\,\mathrm{d}\varrho = \sqrt{\frac{a^2 + 2\varrho^2}{a^2 + \varrho^2}}\,\mathrm{d}\varrho.$$

2) We conclude from 2) that

$$\begin{aligned}\int_{\mathcal{M}} z\varrho\,\mathrm{d}s &= \int_0^{2a}\sqrt{a^2+\varrho^2}\cdot\varrho\cdot\sqrt{\frac{a^2+2\varrho^2}{a^2+\varrho^2}}\,\mathrm{d}\varrho = \int_0^{2a}\sqrt{a^2+2\varrho^2}\cdot\varrho\,\mathrm{d}\varrho \\ &= \frac{1}{4}\cdot\frac{2}{3}\left[\left(a^2+2\varrho^2\right)^{\frac{3}{2}}\right]_{\varrho=0}^{2a} = \frac{1}{6}\left\{\left(9a^2\right)^{\frac{3}{2}}\right\} = \frac{1}{6}\,a^3\,\left(3^3-1\right) = \frac{26}{6}\,a^3 = \frac{13}{3}\,a^3.\end{aligned}$$

3) Again we get by first applying the result of 2),

$$\begin{aligned}\int_{\mathcal{O}}\frac{1}{z^2\sqrt{z^2+\varrho^2}}\,\mathrm{d}S &= 2\pi\int_0^{2\pi}\frac{1}{\left(a^2+\varrho^2\right)\sqrt{a^2+2\varrho^2}}\cdot\varrho\sqrt{\frac{a^2+2\varrho^2}{a^2+\varrho^2}}\,\mathrm{d}\varrho \\ &= 2\pi\int_0^{2a}\left(a^2+\varrho^2\right)^{-\frac{3}{2}}\varrho\,\mathrm{d}\varrho = 2\pi\left[-\frac{1}{\sqrt{a^2+\varrho^2}}\right]_{\varrho=0}^{2a} \\ &= 2\pi\left(\frac{1}{a}-\frac{1}{\sqrt{5}}\cdot\frac{1}{a}\right) = \frac{2\pi}{a}\left(1-\frac{1}{\sqrt{5}}\right) = \frac{2(5-\sqrt{5})\pi}{5a}.\end{aligned}$$

Example 27.17 *A surface of revolution $\mathcal{O}$ is in semi-polar coordinates given by*

$$\varrho \in [a, 2a], \qquad \varphi \in [0, 2\pi], \qquad z = 2a - \frac{\varrho^2}{a},$$

where $a \in \mathbb{R}_+$ is some given constant.

1) Sketch the meridian curve $\mathcal{M}$, and show that the line element $\mathrm{d}s$ on $\mathcal{M}$ is given by

$$\mathrm{d}s = \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\,\mathrm{d}\varrho.$$

2) Calculate the line integral

$$\int_{\mathcal{M}} \sqrt{2 - \frac{z}{a}}\,\mathrm{d}s.$$

3) Calculate the surface integral

$$\int_{\mathcal{O}} \frac{1}{az + 9\varrho^2}\,\mathrm{d}S.$$

A Line integral and surface integral.

D Apply the standard methods.

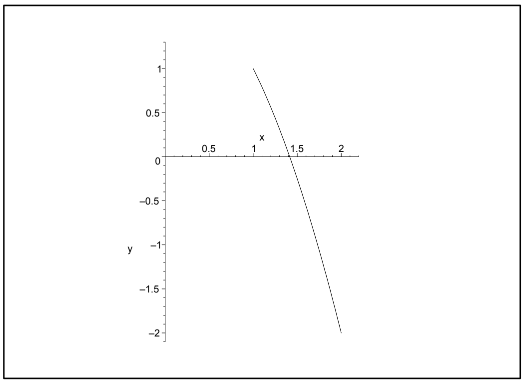

Figure 27.40: The meridian curve $\mathcal{M}$ for $a = 1$.

I 1) When we use the parametric description

$$\mathcal{M}: \quad (\varrho, z) = \left(\varrho, 2a - \frac{\varrho^2}{a}\right), \qquad \varrho \in [a, 2a],$$

the square of the weight function becomes

$$\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varrho}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\varrho}\right)^2 = 1 + \left(-\frac{2\varrho}{a}\right)^2 = \frac{1}{a^2}\left(a^2 + 4\varrho^2\right),$$

hence

$$\mathrm{d}s = \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\,\mathrm{d}\varrho.$$

2) Then by 1) and the substitution $t = 4\varrho^2$,

$$\begin{aligned}\int_{\mathcal{M}} \sqrt{2 - \frac{z}{a}}\,\mathrm{d}s &= \int_a^{2a} \sqrt{2 - \left(2 - \frac{\varrho^2}{a^2}\right)} \cdot \frac{1}{a}\sqrt{a^2 + 4\varrho^2}\,\mathrm{d}\varrho \\ &= \int_a^{2a} \frac{\varrho}{a^2}\sqrt{a^2+4\varrho^2}\,\mathrm{d}\varrho = \frac{1}{8a^2}\int_{4a^2}^{16a^2} \sqrt{a^2+t}\,\mathrm{d}t \\ &= \frac{1}{8a^2}\left[\frac{2}{3}\left(a^2+t\right)^{\frac{3}{2}}\right]_{4a^2}^{16a^2} = \frac{1}{12a^2}\left\{\left(17a^2\right)^{\frac{3}{2}} - \left(5a^2\right)^{\frac{3}{2}}\right\} \\ &= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\,a.\end{aligned}$$

3) By first intersecting the surface $\mathcal{O}$ with the planes $z = \text{constant}$, we get

$$\begin{aligned}\int_{\mathcal{O}} \frac{1}{az+9\varrho^2}\,\mathrm{d}S &= \int_{\mathcal{M}} \frac{2\pi\varrho}{a\left(2a-\dfrac{\varrho^2}{a}\right)+9\varrho^2}\,\mathrm{d}s = \int_a^{2a} \frac{2\pi\varrho}{2a^2+8\varrho^2}\cdot\frac{1}{a}\sqrt{a^2+4\varrho^2}\,\mathrm{d}\varrho \\ &= \frac{\pi}{a}\int_a^{2a} \frac{\varrho}{\sqrt{a^2+4\varrho^2}}\,\mathrm{d}\varrho = \frac{\pi}{a}\left[\frac{1}{4}\sqrt{a^2+4\varrho^2}\right]_a^{2a} = \frac{\pi}{4a}\left\{\sqrt{17a^2}-\sqrt{5a^2}\right\} \\ &= \frac{\pi}{4}\left(\sqrt{17}-\sqrt{5}\right).\end{aligned}$$

Example 27.18 *A surface of revolution $\mathcal{O}$ is in semi-polar coordinates (ϱ, φ, z) given by*

$$\varrho = \sqrt{z^2 + 2az}, \qquad z \in [a, 2a], \qquad \varphi \in [0, 2\pi],$$

where a is some positive constant. The meridian curve of the surface is denoted by $\mathcal{M}$.

1) Explain why $\mathcal{M}$ is a subset of a conic section, and indicate its type and centrum. Then sketch $\mathcal{M}$.

2) Show that the line element $\mathrm{d}s$ on $\mathcal{M}$ is given by

$$\mathrm{d}s = \sqrt{\frac{2z^2 + 4az + a^2}{z^2 + 2az}}\,\mathrm{d}z.$$

3) Calculate the surface integral

$$\int_{\mathcal{O}} \frac{|x|(z+a)}{\sqrt{x^2+y^2}}\,\mathrm{d}S.$$

4) Explain why $\mathcal{O}$ is a subset of a surface of a conic section. Find its type and centrum.

A Conic sections, meridian curve, surface integral.

D If only the surface integral is calculated in semi-polar coordinates, the rest is purely standard.

I 1) We get by a squaring and a rearrangement that $\mathcal{M}$ is a subset of the point set given by

$$(z+a)^2 - \varrho^2 = a^2.$$

This describes in the whole PZ-plane an hyperbola of centrum $(0, -a)$ and half axes a and a.

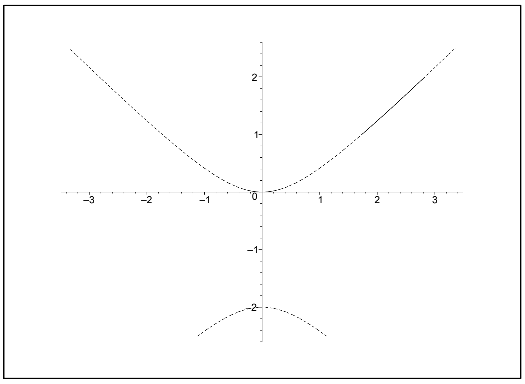

Figure 27.41: The meridian curve $\mathcal{M}$ and the corresponding conic section (dotted) for $a = 1$.

2) The line element on $\mathcal{M}$ is given by

$$\begin{aligned} ds &= \sqrt{1+\left(\frac{d\varrho}{dz}\right)^2}\,dz = \sqrt{1+\left(\frac{2z+2a}{2\sqrt{z^2+2az}}\right)^2}\,dz \\ &= \sqrt{1+\frac{z^2+2az+a^2}{z^2+2az}}\,dz = \sqrt{\frac{2z^2+4az+a^2}{z^2+2az}}\,dz. \end{aligned}$$

3) First express the integrand in semi-polar coordinates on the surface:

$$f(x,y,z) = \frac{|x|(z+a)}{\sqrt{x^2+y^2}} = \frac{\varrho|\cos\varphi|\,(z+a)}{\varrho} = |\cos\varphi|(z+a).$$

Then the surface integral becomes

$$\begin{aligned} \int_{\mathcal{O}} \frac{|x|(z+a)}{\sqrt{x^2+y^2}}\,dS &= \int_a^{2a}\left\{\int_0^{2\pi}|\cos\varphi|(z+a)\varrho(z)\,d\varphi\right\}\sqrt{\frac{2z^2+4az+a^2}{z^2+2az}}\,dz \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\,d\varphi\cdot\int_a^{2a}(z+a)\sqrt{z^2+2az}\cdot\sqrt{\frac{2z^2+4aza^2}{z^2+2az}}\,dz \\ &= 4\int_a^{2a}(z+a)\sqrt{2z^2+4az+a^2}\,dz \\ &= \int_{z=a}^{2a}\sqrt{2z^2+4az+a^2}\,d\left(2z^2+4az+a^2\right) \\ &= \frac{2}{3}\left[(2z^2+4az+a^2)^{\frac{3}{2}}\right]_{z=a}^{2a} \\ &= \frac{2}{3}\left\{\left(8a^2+8a^2+a^2\right)^{\frac{3}{2}}-\left(2a^2+4a^2+a^2\right)^{\frac{3}{2}}\right\} \\ &= \frac{2}{3}\left\{\left(17a^2\right)^{\frac{3}{2}}-\left(7a^2\right)^{\frac{3}{2}}\right\} = \frac{2}{3}\{17\sqrt{17}-7\sqrt{7}\}a^3. \end{aligned}$$

4) The curve $\mathcal{M}$ is a part of an hyperbola, cf. 1), and the axis of rotation intersects the foci of the hyperbola. We therefore conclude that $\mathcal{O}$ is a subset of an hyperboloid of revolution with two nets and centrum $(0,0,-a)$.

We get the equation of the hyperboloid of revolution by replacing ϱ^2 by x^2+y^2 in the expression from 1),

$$(z+a)^2 - x^2 - y^2 = a^2,$$

or in its standard form,

$$\left(\frac{z+a}{a}\right)^2 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{a}\right)^2 = 1.$$

The surface $\mathcal{O}$ it the subset which lies between the planes $z=a$ and $z=2a$.

Example 27.19 *A surface of revolution $\mathcal{O}$ is in semi-polar coordinates (ϱ, φ, z) given by*

$$0 \le \varphi \le 2\pi, \qquad a \le \varrho \le 2a, \qquad z = a \ln \frac{\varrho}{a},$$

where a is some positive constant.

1) Sketch the meridian curve $\mathcal{M}$, and find the line element $\mathrm{d}s$ on $\mathcal{M}$.

2) Calculate the line integral

$$\int_{\mathcal{M}} \frac{1}{\sqrt{a^2+\varrho^2}}\,\mathrm{d}s.$$

3) Calculate the surface integral

$$\int_{\mathcal{O}} \left(x + a\,\exp\frac{z}{a}\right)\mathrm{d}S.$$

A Surface of revolution, meridian curve, line integral, surface integral.

D Standard example.

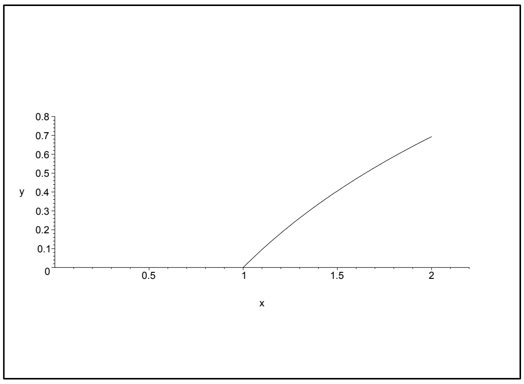

Figure 27.42: The meridian curve $\mathcal{M}$ for $a = 1$.

I 1) The line element $\mathrm{d}s$ on $\mathcal{M}$ is given by

$$\mathrm{d}s = \sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}\varrho}\right)^2}\,\mathrm{d}\varrho = \sqrt{1+\left(\frac{a}{\varrho}\right)^2}\,\mathrm{d}\varrho = \frac{\sqrt{a^2+\varrho^2}}{\varrho}\,\mathrm{d}\varrho.$$

2) By using ϱ as variable it follows from 1) that

$$\int_{\mathcal{M}} \frac{1}{\sqrt{a^2+\varrho^2}}\,\mathrm{d}s = \int_a^{2a} \frac{1}{\sqrt{a^2+\varrho^2}} \cdot \frac{\sqrt{a^2+\varrho^2}}{\varrho}\,\mathrm{d}\varrho = \int_a^{2a} \frac{\mathrm{d}\varrho}{\varrho} = [\ln \varrho]_a^{2a} = \ln 2.$$

3) The surface element on $\mathcal{O}$ is given by

$$\mathrm{d}S = \varrho\,\mathrm{d}\varphi\,\mathrm{d}s = \varrho\,\frac{\sqrt{a^2+\varrho^2}}{\varrho}\,\mathrm{d}\varphi\,\mathrm{d}\varrho = \sqrt{a^2+\varrho^2}\,\mathrm{d}\varrho\,\mathrm{d}\varphi,$$

so accordingly the surface integral

$$\begin{aligned}
\int_{\mathcal{O}} \left\{x + a\,\exp\left(\frac{z}{a}\right)\right\} \mathrm{d}S &= \int_0^{2\pi} \left\{\int_a^{2a} \left(\varrho\cos\varphi + a\exp\left(\frac{a\ln\frac{\varrho}{a}}{a}\right)\right)\sqrt{a^2+\varrho^2}\,\mathrm{d}\varrho\right\}\mathrm{d}\varphi \\
&= 0 + 2\pi\int_a^{2a} \varrho\sqrt{a^2+\varrho^2}\,\mathrm{d}\varrho = \pi\int_{\varrho=a}^{2a}\left(a^2+\varrho^2\right)^{\frac{1}{2}}\,\mathrm{d}\left(a^2+\varrho^2\right) \\
&= \pi\cdot\frac{2}{3}\left[\left(a^2+\varrho^2\right)^{\frac{3}{2}}\right]_{\varrho=a}^{2a} = \frac{2\pi}{3}\left\{\left(5a^2\right)^{\frac{3}{2}} - \left(2a^2\right)^{\frac{3}{2}}\right\} \\
&= \frac{2\pi}{3}\,(5\sqrt{5}-2\sqrt{2})\,a^3.
\end{aligned}$$

Example 27.20 *A surface $\mathcal{F}$ is given by the parametric description*

$$\mathbf{r}(u,v) = (e^u, e^v, u+v)\,, \qquad u^2+v^2 \le 1.$$

1) Show that the normal vector of the surface is given by

$$\mathbf{N}(u,v) = \left(-e^x, -e^u, e^{u+v}\right).$$

2) Find an equation of the tangent plane of $\mathcal{F}$ at the point $\mathbf{r}(0,0)$.

3) Calculate the surface integral

$$\int_{\mathcal{F}} \frac{1}{\sqrt{x^2+y^2+e^{2z}}}\,\mathrm{d}S.$$

A Surface integral.

D Use that $\mathrm{d}S = \|\mathbf{N}(u,v)\|\,\mathrm{d}u\,\mathrm{d}v$.

I 1) We conclude from

$$\frac{\partial\mathbf{r}}{\partial u} = (e^u, 0, 1) \quad \text{and} \quad \frac{\partial\mathbf{r}}{\partial v} = (0, e^v, 1)\,,$$

that

$$\mathbf{N}(u,v) = \frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ e^u & 0 & 1 \\ 0 & e^v & 1 \end{vmatrix} = \left(-e^v, -e^u, e^{u+v}\right).$$

2) From $\mathbf{r}(0,0) = (1,1,0)$ and the normal vector $\mathbf{N}(0,0) = (-1,-1,1)$ we get the equation of the tangent plane

$$0 = \mathbf{N}(0,0,0)\cdot(x-1, y-1, z) = (-1,-1,1)\cdot(x-1,y-1,z) = -x+1-y+1+z,$$

hence by a rearrangement

$$x + y - z = 2.$$

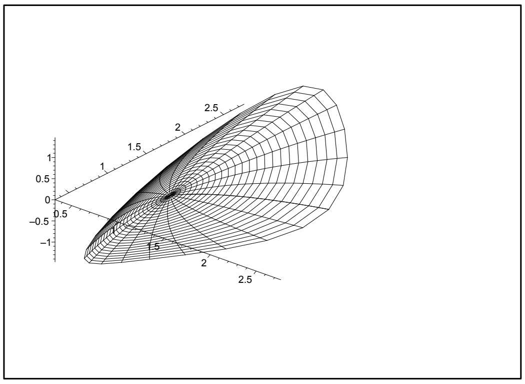

Figure 27.43: The surface $\mathcal{F}$.

3) From $\|\mathbf{N}(u,v)\|^2 = e^{2v} + e^{2u} + e^{2u+2v}$ follows that

$$\int_{\mathcal{F}} \frac{1}{\sqrt{x^2+y^2+e^{2z}}}\,\mathrm{d}S = \int_{u^2+v^2\le 1} \frac{\sqrt{e^{2u}+e^{2v}+e^{2u+2v}}}{\sqrt{e^{2u}+e^{2v}+e^{2u+2v}}}\,\mathrm{d}u\,\mathrm{d}v = \pi\cdot 1^2 = \pi.$$

I

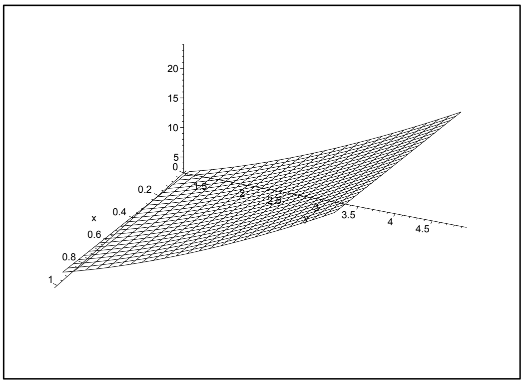

Figure 27.44: The surface of **Example 27.21.1**.

27.4 Examples of surface area

Example 27.21 *Calculate in each of the following cases the surface area of a surface $\mathcal{F}$, which is the graph of a function in two variables, hence*

$$\mathcal{F} = \{(x, y, z) \mid (x, y) \in E,\, z = Z(x, y)\}.$$

1) The surface integral $\int_{\mathcal{F}} \mathrm{d}S$, where

$$Z(x, y) = 1 + 2x + 2y\sqrt{y}, \qquad (x, y) \in [0, 1] \times \left[\frac{11}{9}, \frac{44}{9}\right].$$

2) The surface integral $\int_{\mathcal{F}} \mathrm{d}S$, where

$$Z(x, y) = \frac{x^2}{2} + 3y, \qquad \textit{where } -1 \le x \le 1 \textit{ and } -\frac{1}{6}\,x^2 \le y \le 1.$$

3) The surface integral $\int_{\mathcal{F}} \mathrm{d}S$, where

$$Z(x, y) = \frac{y}{x^2 + y^2}, \textit{ where } 1 \le x^2 + y^2 \le 2.$$

A Surface area in rectangular coordinates.

D Find the weight function

$$\|\mathbf{N}\| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \sqrt{1 + \|\bigtriangledown g\|^2},$$

and then compute the surface integral with the integrand 1.

Here $\bigtriangledown g = (2, 3\sqrt{y})$, so the weight function is

$$\sqrt{1 + \|\bigtriangledown g\|^2} = \sqrt{1 + 4 + 9y} = \sqrt{5 + 9y},$$

1)

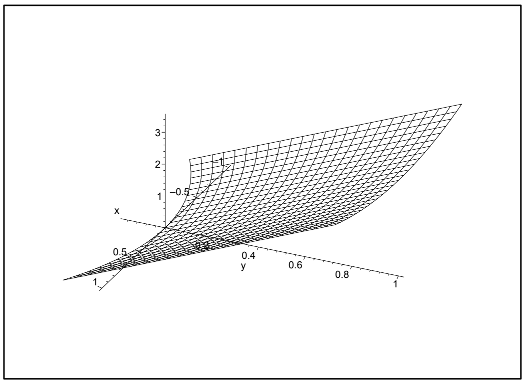

Figure 27.45: The surface of **Example 27.21.2**.

and we can setup the surface integral

$$\begin{aligned}\int_{\mathcal{F}} \mathrm{d}S &= \int_E \sqrt{5+9y}\,\mathrm{d}x\,\mathrm{d}y = \int_{\frac{11}{9}}^{\frac{44}{9}} \sqrt{5+9y}\,\mathrm{d}y = \frac{1}{9}\cdot\frac{2}{3}\left[(5+9y)^{\frac{3}{2}}\right]_{\frac{11}{9}}^{\frac{44}{9}} \\ &= \frac{2}{27}\left\{49^{\frac{3}{2}}-16^{\frac{3}{2}}\right\})\frac{2}{27}\left\{7^3-4^3\right\} = \frac{2}{27}(343-64) = \frac{2}{27}\cdot 279 = \frac{2\cdot 31}{3} = \frac{62}{3}.\end{aligned}$$

2) We get from $\bigtriangledown g = (x,3)$ that $\sqrt{1+\|\bigtriangledown g\|^2} = \sqrt{10+x^2}$. The surface area is

$$\begin{aligned}\int_{\mathcal{F}} \mathrm{d}S &= \int_E \sqrt{10+x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{1}\left\{\int_{-\frac{x^2}{6}}^{1} \sqrt{10+x^2}\,\mathrm{d}y\right\}\mathrm{d}x \\ &= \int_{-1}^{1}\left(1+\frac{x^2}{6}\right)\sqrt{10+x^2}\,\mathrm{d}x = \frac{2}{6}\int_0^1 (6+x^2)\sqrt{10+x^2}\,\mathrm{d}x.\end{aligned}$$

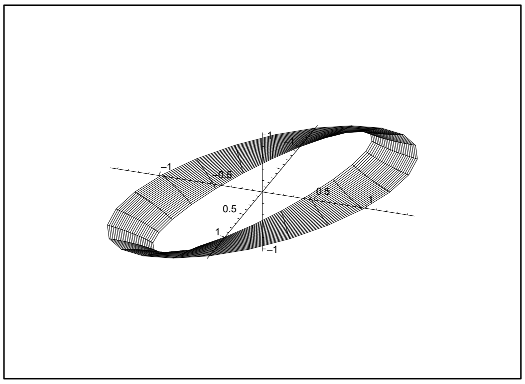

Figure 27.46: The surface of **Example 27.21.3**.

Then by the substitution $x = \sqrt{10}\sinh t$, $t = \operatorname{Arsinh}\left(\dfrac{x}{\sqrt{10}}\right)$,

$$\begin{aligned}
\int_{\mathcal{F}} \mathrm{d}S &= \frac{1}{3}\int_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} (6+10\sinh^2 t)\cdot\sqrt{10}\cosh t\cdot\sqrt{10}\cosh t\,\mathrm{d}t \\
&= \frac{20}{3}\int_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} (3+5\sinh^2 t)\cosh^2 t\,\mathrm{d}t \\
&= \frac{20}{3}\int_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} \left\{\frac{3}{2}(1+\cosh 2t)+\frac{5}{4}\sinh^2 2t\right\}\,\mathrm{d}t \\
&= \frac{5}{3}\int_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} \left\{6+6\cosh 2t+\frac{5}{2}(\cosh 4t-1)\right\}\,\mathrm{d}t \\
&= \frac{5}{6}\int_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} \{7+12\cosh 2t+5\cosh 4t\}\,\mathrm{d}t \\
&= \frac{5}{6}\left[7t+6\sinh 2t+\frac{5}{4}\sinh 4t\right]_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} \\
&= \frac{5}{6}\left[7t+12\sinh t\sqrt{1+\sinh^2 t}+5\sinh t\sqrt{1+\sinh^2 t}\cdot(1+2\sinh^2 t)\right]_0^{\operatorname{Arsinh}(\frac{1}{\sqrt{10}})} \\
&= \frac{5}{6}\left\{7\ln\left(\frac{1}{\sqrt{10}}+\sqrt{\frac{11}{10}}\right)+12\cdot\frac{1}{\sqrt{10}}\cdot\sqrt{\frac{11}{10}}+5\cdot\frac{1}{\sqrt{10}}\cdot\sqrt{\frac{11}{10}}\cdot\left(1+\frac{2}{10}\right)\right\} \\
&= \frac{5}{6}\left\{7\ln\left(\frac{1+\sqrt{11}}{\sqrt{10}}\right)+\frac{12}{10}\cdot\sqrt{11}+\frac{6}{10}\cdot\sqrt{11}\right\} = \frac{35}{12}\ln\left(\frac{6+\sqrt{11}}{5}\right)+\frac{3}{2}\cdot\sqrt{11}.
\end{aligned}$$

3) Here

$$\nabla g = \left(-\frac{2xy}{(x^2+y^2)^2},\frac{x^2-y^2}{(x^2+y^2)^2}\right) = \frac{1}{(x^2+y^2)^2}\,(-2xy,x^2-y^2),$$

so

$$\|\nabla g\|^2 = \frac{1}{(x^2+y^2)^4}\left(4x^2y^2+(x^2-y^2)^2\right) = \frac{1}{(x^2+y^2)^2}.$$

The surface area is

$$
\begin{aligned}
\int_{\mathcal{F}} \mathrm{d}S &= \int_E \frac{1}{x^2+y^2}\sqrt{1+(x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y = 2\pi\int_0^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^2}\,\varrho\,\mathrm{d}\varrho \\
&= \frac{2\pi}{4}\int_1^{\sqrt{2}} \frac{\sqrt{1+\varrho^4}}{\varrho^4}\cdot 4\varrho^3\,\mathrm{d}\varrho = \frac{\pi}{2}\int_1^4 \frac{1+t}{t}\,\mathrm{d}t = \frac{\pi}{2}\int_{\sqrt{2}}^{\sqrt{5}} \frac{u\cdot 2u}{u^2-1}\,\mathrm{d}u \\
&= \pi\int_{\sqrt{2}}^{\sqrt{5}}\left\{1+\frac{1}{2}\,\frac{1}{u-1}-\frac{1}{2}\,\frac{1}{u+1}\right\}\mathrm{d}u = \pi\left[u+\frac{1}{2}\ln\frac{u-1}{u+1}\right]_{\sqrt{2}}^{\sqrt{5}} \\
&= \pi\left\{\sqrt{5}-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right\} \\
&= \pi\left\{\sqrt{5}-\sqrt{2}+\ln\left(\frac{(\sqrt{5}-1)(\sqrt{2}+1)}{2}\right)\right\} \\
&= \pi\{\sqrt{5}-\sqrt{2}+\ln(\sqrt{5}-1)+\ln(\sqrt{2}+1)-\ln 2.
\end{aligned}
$$

I

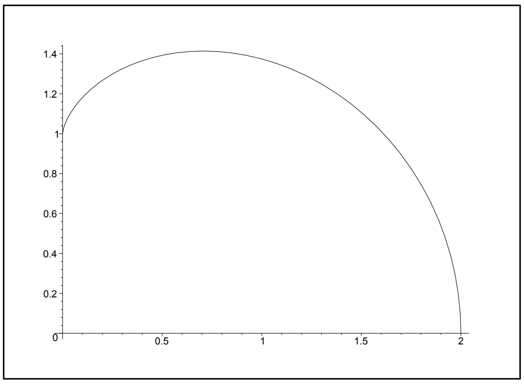

Figure 27.47: The meridian curve $\mathcal{M}$ of **Example 27.22.1**.

Example 27.22 *Calculate in each of the following cases the surface area of a surface of revolution $\mathcal{O}$, which is given by a meridian curve $\mathcal{M}$ in the meridian half plane, in which ϱ and z are the rectangular coordinates.*

1) *The surface area $\int_{\mathcal{O}} \mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by the parametric description*

$$(\varrho, z) = \left(2\sin^3 t, 3\cos t - 2\cos^3 t\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

2) *The surface area $\int_{\mathcal{O}} \mathrm{d}S$, where the meridian curve $\mathcal{N}$ is given by the parametric description*

$$(\varrho, z) = \left(a\,\sin^3 t, a\,\cos^3 t\right), \qquad t \in [0, \pi].$$

3) *The surface area $\int_{\mathcal{O}} \mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by the parametric description*

$$(\varrho, z) = (b\,\sin t, a\,\cos t), \qquad t \in [0, \pi].$$

4) *The surface area $\int_{\mathcal{O}} \mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $z^2 + \varrho^2 = az$.*

5) *The surface area $\int_{\mathcal{O}} \mathrm{d}S$, where the meridian curve $\mathcal{M}$ is given by $\varrho = z^3$ for $x \in [0, 1]$.*

A Surface area of a surface of revolution.

D Use either semi-polar or spherical coordinates and the area element $\varrho\,\mathrm{d}\varphi\,\mathrm{d}s$, where $\mathrm{d}s$ is the line element, thus if e.g. $z = g(\varrho)$, then

$$\mathrm{d}s = \sqrt{1 + g'(\varrho)^2}\,\mathrm{d}\varrho,$$

and similarly.

We get from

$$\mathbf{r}(t) = \left(2\sin^3 t, 3\cos t - 2\cos^3 t\right)$$

that

$$\mathbf{r}'(t) = \left(6\sin^2 t \cdot \cos t, -3\sin t + 6\cos^2 t \cdot \sin t\right),$$

1)

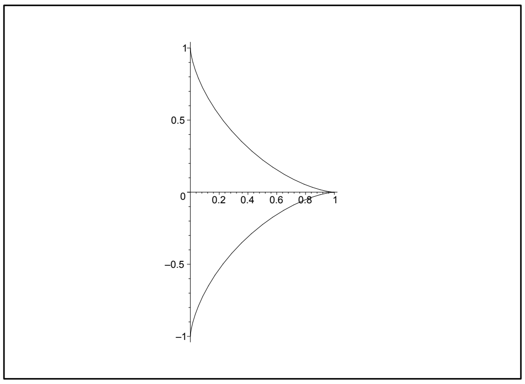

Figure 27.48: The meridian curve $\mathcal{M}$ of **Example 27.22.2** for $a = 1$.

hence

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= \left(6\sin^2 t\cdot\cos t\right)^2+\left(-3\sin t+6\cos^2 t\cdot\sin t\right)^2\\ &= 36\sin^4 t\cdot\cos^2 + 9\sin^2 t\left(2\cos^2 t-1\right)^2\\ &= 9\sin^2 t\left(\sin^2 2t+\cos^2 2t\right) = 9\sin^2 t,\end{aligned}$$

and accordingly

$$ds = \|\mathbf{r}'(t)\|\,dt = 3|\sin t|\,dt = 3\sin t\,dt \quad \text{for } t\in\left[0,\frac{\pi}{2}\right].$$

Then

$$\begin{aligned}\int_{\mathcal{O}} dS &= \int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}} 2\sin^3 t\cdot 3\sin t\,dt\right\}d\varphi = 2\pi\cdot 6\int_0^{\frac{\pi}{2}}\sin^4 t\,dt\\ &= 3\pi\int_0^{\frac{\pi}{2}}\left\{2\sin^2 t\right\}^2 dt = 3\pi\int_0^{\frac{\pi}{2}}(1-\cos 2t)^2\,dt\\ &= 3\pi\int_0^{\frac{\pi}{2}}\left\{1-2\cos 2t+\frac{1}{2}+\frac{1}{2}\cos 4t\right\}dt = 3\pi\cdot\frac{3}{2}\cdot\frac{\pi}{2} = \frac{9\pi^2}{4}.\end{aligned}$$

2) It follows from $\mathbf{r}(t) = a\left(\sin^3 t, \cos^3 t\right)$ that

$$\mathbf{r}'(t) = a\left(3\sin^2 t\,\cos t, -3\cos^2 t\,\sin t\right) = 3a\sin t\cdot\cos t(\sin t, -\cos t),$$

hence

$$\|\mathbf{r}'(t)\| = 3a\sin t\cdot|\cos t|, \qquad t\in[0,\pi].$$

(Remember the absolute value). The line element is given by

$$ds = \|\mathbf{r}'(t)\|\,dt = 3a\sin t\,|\cos t|\,dt.$$

Finally, it follows from $\varrho\,d\varphi = a\sin^3 t\,d\varphi$ that

$$\begin{aligned}\int_{\mathcal{O}} dS &= \int_0^{2\pi}\left\{\int_0^{black\pi} a\sin^3 t\cdot 3a\sin t\,|\cos t|\,dt\right\}d\varphi\\ &= 2\pi\cdot 3a^2\cdot 2\int_0^{\frac{\pi}{2}}\sin^4 t\,\cos t\,dt = \frac{2\pi}{5}a^2.\end{aligned}$$

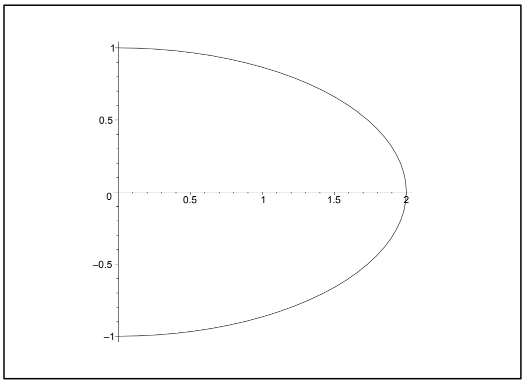

Figure 27.49: The meridian curve $\mathcal{M}$ of **Example 27.22.3** for $a = 1$ og $b = 2$.

3) Here

$$\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \sqrt{b^2\cos^2 t + a^2\sin^2 t}\,\mathrm{d}t = \sqrt{a^2 + (b^2-a^2)\cos^2 t}\,\mathrm{d}t, \qquad t \in [0,\pi],$$

hence

$$\int_{\mathcal{O}} \mathrm{d}S = \int_0^{2\pi}\left\{\int_0^{\pi} b\,\sin t\,\sqrt{a^2+(b^2-a^2)\cos^2 t}\,\mathrm{d}t\right\}\mathrm{d}\varphi = 4\pi b\int_0^1 \sqrt{a^2+(b^2-a^2)u^2}\,\mathrm{d}u.$$

We shall here consider three different cases.

a) If $a = b$, then

$$\int_{\mathcal{O}} \mathrm{d}S = 4\pi a\int_0^1 a\,\mathrm{d}u = 4\pi a^2,$$

and the surface area of the sphere is $4\pi a^2$.

b) If $0 < b < a$, then

$$\int_{\mathcal{O}} \mathrm{d}S = 4\pi ba\int_0^1 \sqrt{1-\left(1-\frac{b^2}{a^2}\right)u^2}\,\mathrm{d}u.$$

Then by the substitution $\sqrt{1-\dfrac{b^2}{a^2}}\,u = \sin v$,

$$
\begin{aligned}
\int_{\mathcal{O}} \mathrm{d}S &= 4\pi ab \int_0^{\operatorname{Arcsin}\left(\sqrt{1-\frac{b^2}{a^2}}\right)} \sqrt{1-\sin^2 v}\cdot \frac{1}{\sqrt{1-\dfrac{b^2}{a^2}}}\,\cos v\,\mathrm{d}v \\
&= \frac{4\pi ab}{\sqrt{1-\dfrac{b^2}{a^2}}} \int_0^{\operatorname{Arccos}\left(\frac{b}{a}\right)} \cos^2 v\,\mathrm{d}v = \frac{2\pi ab}{\sqrt{1-\dfrac{b^2}{a^2}}} \int_0^{\operatorname{Arccos}\left(\frac{b}{a}\right)} (1+\cos 2v)\,\mathrm{d}v \\
&= \frac{2\pi ab}{\sqrt{1-\dfrac{b^2}{a^2}}} \left\{ \operatorname{Arccos}\left(\frac{b}{a}\right) + \left[\frac{1}{2}\sin 2v\right]_0^{\operatorname{Arccos}\left(\frac{b}{a}\right)} \right\} \\
&= \frac{2\pi ab}{\sqrt{1-\dfrac{b^2}{a^2}}} \left\{ \operatorname{Arccos}\left(\frac{b}{a}\right) + \sqrt{1-\frac{b^2}{a^2}}\cdot\frac{b}{a} \right\} = \frac{2\pi ab}{\sqrt{1-\dfrac{b^2}{a^2}}} \operatorname{Arccos}\left(\frac{b}{a}\right) + 2\pi b^2.
\end{aligned}
$$

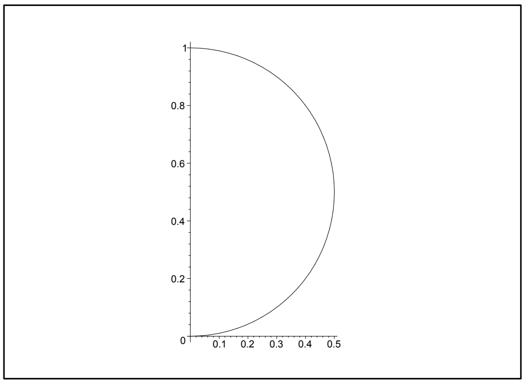

Figure 27.50: The meridian curve $\mathcal{M}$ of **Example 27.22.4** and **Example 27.7.8** for $a = 1$.

c) If $0 < a < b$, then

$$\int_{\mathcal{O}} \mathrm{d}S = 4\pi ab \int_0^1 \sqrt{1 + \left(\frac{b^2}{a^2} - 1\right) u^2}\, \mathrm{d}u.$$

Then by the substitution $\sqrt{\dfrac{b^2}{a^2} - 1}\, u = \sinh v$,

$$\begin{aligned}
\int_{\mathcal{O}} \mathrm{d}S &= 4\pi ab \int_0^{\mathrm{Arsinh}\left(\sqrt{\frac{b^2}{a^2}-1}\right)} \sqrt{1 + \sinh^2 v} \cdot \frac{1}{\sqrt{\frac{b^2}{a^2} - 1}} \cosh v\, \mathrm{d}v \\
&= \frac{4\pi ab}{\sqrt{\frac{b^2}{a^2}}} \int_0^{\ln\left(\frac{b}{a}+\sqrt{\frac{b^2}{a^2}-1}\right)} \cosh^2 v\, \mathrm{d}v = \frac{2\pi ab}{\sqrt{\frac{b^2}{a^2}}} \int_0^{\ln\left(\frac{b}{a}+\sqrt{\frac{b^2}{a^2}-1}\right)} (\cosh 2v + 1)\, \mathrm{d}v \\
&= \frac{2\pi ab}{\sqrt{\frac{b^2}{a^2} - 1}} \left\{ \ln\left(\frac{b}{a} + \sqrt{\frac{b^2}{a^2} - 1}\right) + \sqrt{\frac{b^2}{a^2} - 1} \cdot \frac{b}{a} \right\} \\
&= \frac{2\pi ab}{\sqrt{\frac{b^2}{a^2} - 1}} \ln\left(\frac{b}{a} + \sqrt{\frac{b^2}{a^2} - 1}\right) + 2\pi b^2.
\end{aligned}$$

4) It follows from the figure that the meridian curve is a half circle of radius $\dfrac{a}{2}$. Thus the integral $\int_{\mathcal{O}} \mathrm{d}S$ is equal to the surface area of the sphere, i.e.

$$\int_{\mathcal{O}} \mathrm{d}S = 4\pi \left(\frac{a}{2}\right)^2 = \pi a^2$$

according to **Example 27.22.3** with $a = b$.
Alternatively,

$$\varrho = \sqrt{\left(\frac{a}{2}\right)^2 - \left(z - \frac{a}{2}\right)^2}, \qquad \text{for } z \in [0, a],$$

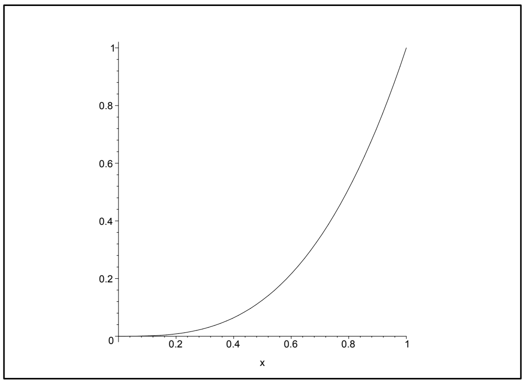

Figure 27.51: The meridian curve $\mathcal{M}$ of **Example 27.22.5**.

in rectangular coordinates, so

$$\mathrm{d}s = \sqrt{1+\frac{\left(z-\frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2-\left(z-\frac{a}{2}\right)^2}}\,\mathrm{d}z = \frac{a}{2}\,\frac{1}{\sqrt{\left(\frac{a}{2}\right)^2-\left(z-\frac{a}{2}\right)^2}}\,\mathrm{d}z.$$

Hence

$$\int_{\mathcal{O}} \mathrm{d}S = 2\pi\int_0^a \sqrt{\left(\frac{a}{2}\right)^2}\cdot\frac{a}{2}\cdot\frac{1}{\sqrt{\left(\frac{a}{2}\right)^2-\left(z-\frac{a}{2}\right)^2}}\,\mathrm{d}z = 2\pi\cdot\frac{a}{2}\cdot a = \pi a^2.$$

ALTERNATIVELY, $r = a\,\cos\theta$, $\theta\in\left[0,\frac{\pi}{2}\right]$, in spherical coordinates, and

$$\varrho = r\,\sin\theta = a\,\sin\theta\,\cos\theta,$$

and

$$\mathrm{d}s = \sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = a\,\mathrm{d}\theta,$$

hence

$$\int_{\mathcal{O}} \mathrm{d}S = 2\pi\int_0^{\frac{\pi}{2}} a\,\sin\theta\,\cos\theta\cdot a\,\mathrm{d}\theta = a^2\pi\left[\sin^2\theta\right]_0^{\frac{\pi}{2}} = a^2\pi.$$

5) Since $\mathrm{d}s = \sqrt{1+9z^4}\,\mathrm{d}z$, we get

$$\begin{aligned}\int_{\mathcal{O}} \mathrm{d}S &= 2\pi\int_0^1 z^3\sqrt{1+9z^4}\,\mathrm{d}z = \frac{2\pi}{4}\int_0^1 \sqrt{1+9t}\,dt \\ &= \frac{\pi}{2}\cdot\frac{1}{9}\cdot\frac{2}{3}\left[(1+9t)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{27}\left(10\sqrt{10}-1\right).\end{aligned}$$

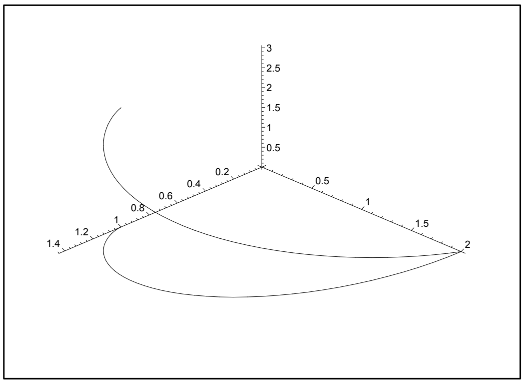

Figure 27.52: The space curve $\mathcal{K}$ and its projection onto the (X, Y)-plane.

Example 27.23 *Consider the space curve $\mathcal{K}$ given by the parametric description*

$$\mathbf{r}(t) = \left(3\cos t - 2\cos^3 t, 2\sin^3 t, 3\cos t\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

1. *Show that the curve has a tangent at the points of the curve corresponding to $t \in \left]0, \frac{\pi}{2}\right]$.*

2. *Show that the curve has a tangent at the point corresponding to $t = 0$.*

3. *Find the length of $\mathcal{K}$.*

The curve $\mathcal{K}$ is projected onto the (X, Y)-plane in a curve $\mathcal{K}^*$. Let $\mathcal{O}$ denote the surface of revolution which is obtained by rotating the curve $\mathcal{K}^*$ once around the X-axis; and $\mathcal{C}$ denotes the cylinder surface which has $\mathcal{K}^*$ as its leading curve and the Z-axis as its direction of generators, and which is lying between the curve $\mathcal{K}$ and the plane $z = -x$.

4. *Find the area of $\mathcal{O}$.*

5. *Find the area of $\mathcal{C}$.*

A Length of a space curve; area of a surface of revolution and a cylinder surface.

D Calculate $\mathbf{r}'(t)$ and show that $\mathbf{r}'(t) \neq \mathbf{0}$ in $\left]0, \frac{\pi}{2}\right[$. Check what happens for $t \to 0$. Find $\|\mathbf{r}'(t)\|$. Finally, calculate the surface areas.

I 1) We get by a differentiation

$$\begin{aligned}
\mathbf{r}'(t) &= \left(-3\sin t + 6\cos^2 t \, \sin t, 6\sin^2 t \, \cos t, -3\sin t\right) \\
&= 3\sin t\left(2\cos^2 t - 1, 2\sin t \, \cos t, -1\right) \\
&= 3\sin t\,(cos2t, \sin 2t, -1).
\end{aligned}$$

Clearly, $\mathbf{r}'(t) \neq \mathbf{0}$ for $t \in \left]0, \frac{\pi}{2}\right[$, hence the curve has a tangent in each of the points corresponding to $t \in \left]0, \frac{\pi}{2}\right[$.

2) It follows from

$$\frac{1}{3\sin t}\,\mathbf{r}'(t) = (\cos 2t, \sin 2t, -1) \to (1,0,-1) \neq (0,0,0) \qquad \text{for } t \to 0,$$

that the curve has a tangent (actually a "half tangent") at the point corresponding to $t = 0$.

3) From

$$\|\mathbf{r}'(t)\|^2 = (3\sin t)^2 \cdot \left\{\cos^2 2t + \sin^2 2t + 1\right\} = (3\sqrt{2}\sin t)^2,$$

follows that the length of the curve $\mathcal{K}$ is

$$\ell = 3\sqrt{2}\int_0^{\frac{\pi}{2}} \sin t\,\mathrm{d}t = 3\sqrt{2}[-\cos t]_0^{\frac{\pi}{2}} = 3\sqrt{2}.$$

The projection of the curve onto the (X,Y)-plane has the parametric description

$$\tilde{\mathbf{r}}'(t) = \left(\cos t\{3 - 2\cos^2 t\}, 2\sin^3 t, 0\right), \qquad t \in \left[0, \frac{\pi}{2}\right].$$

By glancing at 1) we get

$$\tilde{\mathbf{r}}'(t) = 3\sin t\,(\cos 2t, \sin 2t, 0) \quad \text{and} \quad \|\tilde{\mathbf{r}}'(t)\| = 3\sin t.$$

4) The surface area of $\mathcal{O}$ is

$$\begin{aligned}\text{area}(\mathcal{O}) &= \int_0^{\frac{\pi}{2}} 2\pi\tilde{y}(t)\cdot\|\tilde{\mathbf{r}}'(t)\|\,\mathrm{d}t\\ &= 2\pi\int_0^{\frac{\pi}{2}} 2\sin^3 t\cdot 3\sin t\,\mathrm{d}t = 3\pi\int_0^{\frac{\pi}{2}}\left(2\sin^2 t\right)^2\,\mathrm{d}t\\ &= 3\pi\int_0^{\frac{\pi}{2}}(1-\cos 2t)^2\,\mathrm{d}t = 3\pi\int_0^{\frac{\pi}{2}}(1-2\cos 2t+\cos^2 2t)\,\mathrm{d}t\\ &= \frac{3\pi^2}{2} - 3\pi[\sin 2t]_0^{\frac{\pi}{2}} + \frac{3\pi}{2}\int_0^{\frac{\pi}{2}}(1+\cos 4t)\,\mathrm{d}t = \frac{3\pi^2}{2}+\frac{3\pi^2}{4} = \frac{9\pi^2}{4}.\end{aligned}$$

5) The surface area of $\mathcal{C}$ is

$$\begin{aligned}\text{area}(\mathcal{C}) &= \int_0^{\frac{\pi}{2}}\{3\cos t + x(t)\}\cdot\|\tilde{\mathbf{r}}'(t)\|\,\mathrm{d}t = \int_0^{\frac{\pi}{2}}\{6\cos t - 2\cos^3 t)\cdot 3\sin t\,\mathrm{d}t\\ &= 3\int_0^{\frac{\pi}{2}}\{3-\cos^2 t\}\cdot\sin 2t\,\mathrm{d}t = \frac{3}{2}\int_0^{\frac{\pi}{2}}(5-\cos 2t)\cdot\sin 2t\,\mathrm{d}t\\ &= \frac{15}{2}\int_0^{\frac{\pi}{2}}\sin 2t\,\mathrm{d}t - \frac{3}{4}\int_0^{\frac{\pi}{2}}\sin 4t\,\mathrm{d}t = \frac{15}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\frac{\pi}{2}} + \frac{3}{16}[\cos 4t]_0^{\frac{\pi}{2}} = \frac{15}{2}.\end{aligned}$$

Example 27.24 .

1. *Find the length of the curve $\mathcal{K}$ given by the parametric description*

$$\mathbf{r}(t) = \left(3\left(1-t^2\right)^2, 8t^3, 0\right), \qquad t\in[0,1].$$

Choose $\mathcal{K}$ as the leading curve for a cylinder surface $\mathcal{C}$ with the Z-axis as its direction of the generators.

2. *Find the area of that part of $\mathcal{C}$, which lies between the curve $\mathcal{K}$ and the plane of equation $z = 1+y$.*

A Curve length; surface area.

D Find $\|\mathbf{r}'(t)\|$ and integrate. Then find the surface area.

I 1) We get from

$$\mathbf{r}'(t) = \left(-12t\left(1-t^2\right), 24t^2, 0\right) = 12t\left(t^2-1, 2t, 0\right)$$

that

$$\|\mathbf{r}'(t)\|^2 = (12t)^2\cdot\left\{t^4-2t^2+1+4t^2\right\} = (12t)^2\left(t^2+1\right)^2,$$

and thus

$$\|\mathbf{r}'(t)\| = 12t\left(t^2+1\right).$$

Hence, the arc length is

$$\ell = \int_0^1\|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^1 12t\left(t^2+1\right)\,\mathrm{d}t = 6\int_{u=t^2=0}^1(u+1)\,\mathrm{d}u = \left[3u^2+6u\right]_0^1 = 9.$$

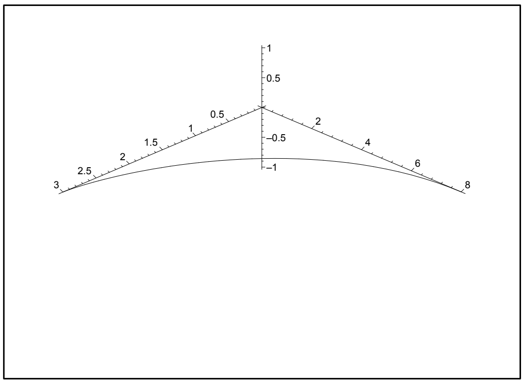

Figure 27.53: The space curve $\mathcal{K}$.

2) The surface area is

$$\begin{aligned} A &= \int_0^1 [1+y]_{y=8t^3} \cdot \|\mathbf{r}'(t)\| \, dt = \int_0^1 \left(1+8t^3\right) \cdot 12t \left(t^2+1\right) \, dt \\ &= \ell + 96 \int_0^1 \left(t^6+t^4\right) \, dt = 9 + 96\left(\frac{1}{7}+\frac{1}{5}\right) = \frac{1467}{35}. \end{aligned}$$

Example 27.25 *Find the area of that part $\mathcal{C}$ of the cylinder surface of equation $x^2+y^2=9$, which is bounded by the plane $z=0$ and the surface of equation $z=1+x^2$.*

A Area of a part of a cylinder surface.

D Just compute.

I When we integrate along the curve

$$\mathcal{K}: \quad (x,y) = (3\cos\varphi, 3\sin\varphi),$$

we get

$$\text{area}(\mathcal{C}) = \int_{\mathcal{K}} \left(1+x^2\right) ds = \int_0^{2\pi} \left(1+9\cos^2\varphi\right) \cdot 3 \, d\varphi = 6\pi + 27\pi = 33\pi.$$

Example 27.26 *Given a curve $\mathcal{K}$ in the (X,Z)-plane by*

$$z = \left(x - \frac{4}{9}\right)^{\frac{3}{2}}, \qquad x \in [1,2].$$

1) Find the length of $\mathcal{K}$.

2) Find the area of that surface $\mathcal{F}$, which is created when $\mathcal{K}$ is rotated once around the Z-axis.

A Curve length, surface area.

D Find the line element

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)}\,\mathrm{d}x$$

and calculate $\int_{\mathcal{K}} \mathrm{d}s$ and $2\pi \int_{\mathcal{K}} x\,\mathrm{d}s$.

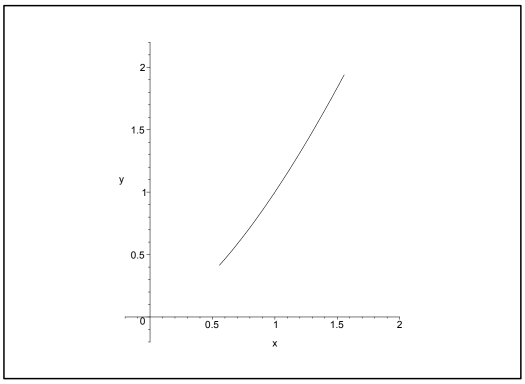

Figure 27.54: The curve $\mathcal{K}$.

I 1) We get from

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{3}{2}\sqrt{x - \frac{4}{9}},$$

the line element

$$\mathrm{d}s = \sqrt{1 + \frac{9}{4}\left(x - \frac{4}{9}\right)}\,\mathrm{d}x = \frac{3}{2}\sqrt{x}\,\mathrm{d}x,$$

and the curve length becomes

$$\ell = \frac{3}{2}\int_1^2 \sqrt{x}\,\mathrm{d}x = \left[x\sqrt{x}\right]_1^2 = 2\sqrt{2} - 1.$$

2) The surface area is according to a formula

$$\text{area}(\mathcal{F}) = 2\pi \int_{\mathcal{K}} x\,\mathrm{d}s = 2\pi \cdot \frac{3}{2}\int_1^2 x\sqrt{x}\,\mathrm{d}x = 2\pi \cdot \frac{3}{2}\cdot\frac{2}{5}\left[x^2\sqrt{x}\right]_1^2 = \frac{6\pi}{5}\left(4\sqrt{2} - 1\right).$$

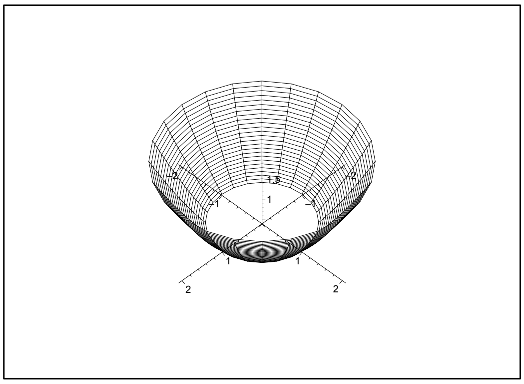

Figure 27.55: The surface of revolution $\mathcal{F}$.

Example 27.27 *A cylinder surface $\mathcal{C}$ has its generators parallel to the Z-axis and its leading curve $\mathcal{K}$ in the (X, Y)-plane is given by the parametric description*

$$\mathbf{r}(t) = \left(t^2 - t, t^2 + t\right), \qquad t \in \left[0, \frac{\sqrt{3}}{2}\right].$$

Find the area of that part $\mathcal{F}$ of $\mathcal{C}$, which is bounded by the plane $z = 0$ and the plane $z = 8y - 8x$.

A Surface area.

D First find $\mathbf{r}'(t)$.

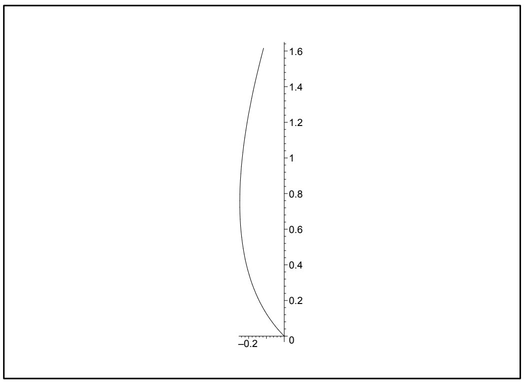

Figure 27.56: The curve $\mathcal{K}$.

I First note that $z = 8y - 8x = 16t \geq 0$ on $\mathcal{K}$. Then

$$\mathbf{r}'(t) = (2t - 1, 2t + 1), \qquad \|\mathbf{r}'(t)\| = \sqrt{2} \cdot \sqrt{4t^2 + 1}.$$

When we insert the above into the formula of the area of a cylinder surface with a leading curve,

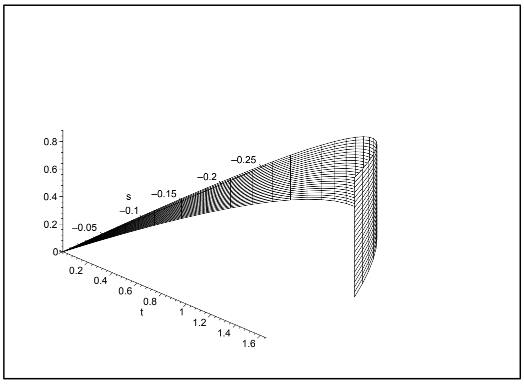

Figure 27.57: The surface $\mathcal{F}$.

then

$$\begin{aligned}\operatorname{area}(\mathcal{F}) &= \int_{\mathcal{K}} (8y - 8x)\,\mathrm{d}s = \sqrt{2}\int_0^{\frac{\sqrt{3}}{2}} 16t\cdot\sqrt{4t^2+1}\,\mathrm{d}t = 2\sqrt{2}\left[\frac{2}{3}\left(\sqrt{4t^2+1}\right)^3\right]_0^{\frac{\sqrt{3}}{2}} \\ &= \frac{4\sqrt{2}}{3}\left\{\left(\sqrt{4\cdot\frac{3}{4}+1}\right)^3 - 1\right\} = \frac{28\sqrt{2}}{3}.\end{aligned}$$

Example 27.28 *Find an equation of the tangent plane of the graph of the function*

$$g(x,y)=\sqrt{2xy},\qquad (x,y)\in[1,4]\times[1,4]$$

at the point $(x,y)=(2,2)$*. Find the area of the graph.*

A Tangent plane and surface area.

D Find the approximating polynomial of at most first degree at the point of contact.

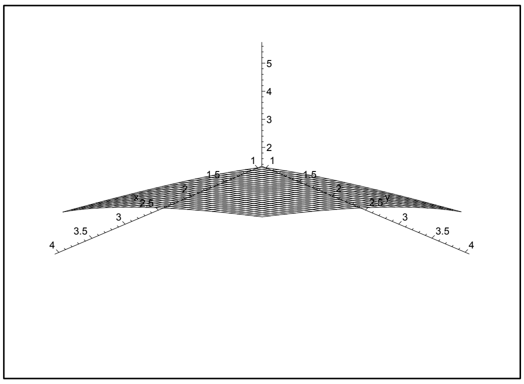

Figure 27.58: The graph of f.

An equation of the tangent plane of $z=g(x,y)$ is

$$\begin{aligned} z &= P_1(x,y)=g(2,2)+\bigtriangledown g(2,2)\cdot(x-2,y-2)\\ &= 2\sqrt{2}+\left(\sqrt{\frac{y}{2x}},\sqrt{\frac{x}{2y}}\right)_{(x,y)=(2,2)}\cdot(x-2,y-2)\\ &= 2\sqrt{2}+\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\cdot(x-2,y-2)=2\sqrt{2}+\frac{1}{\sqrt{2}}(x+y-4)=\frac{1}{\sqrt{2}}\,x+\frac{1}{\sqrt{2}}\,y, \end{aligned}$$

hence

$$x+y-\sqrt{2}\,z=0.$$

Then according to some formula, the area of the graph is

$$\begin{aligned} &\int_E \sqrt{1+\|\bigtriangledown g\|^2}\,\mathrm{d}x\,\mathrm{d}y=\int_1^4\left\{\int_1^4\sqrt{1+\frac{y}{2x}+\frac{x}{2y}}\,\mathrm{d}x\right\}\mathrm{d}y\\ &= \int_1^4\left\{\int_1^4\sqrt{\frac{1}{2xy}\left(2xy+y^2+x^2\right)}\,\mathrm{d}x\right\}\mathrm{d}y=\int_1^4\left\{\int_1^4\sqrt{\frac{(x+y)^2}{2xy}}\,\mathrm{d}x\right\}\mathrm{d}y\\ &= \int_1^4\frac{1}{\sqrt{2y}}\left\{\int_1^4\left(x^{\frac12}+y\,x^{-\frac12}\right)\mathrm{d}x\right\}\mathrm{d}y=\frac{1}{\sqrt{2}}\int_1^4\frac{1}{\sqrt{y}}\left[\frac{2}{3}\,x^{\frac32}+2y\,x^{\frac12}\right]_{redx=1}^4\mathrm{d}y\\ &= \frac{1}{\sqrt{2}}\int_1^4\frac{1}{\sqrt{y}}\left\{\frac{2}{3}(8-1)+2y(2-1)\right\}\mathrm{d}y=\frac{1}{\sqrt{2}}\int_1^4\left\{\frac{14}{3}\,y^{-\frac12}+2y^{\frac12}\right\}\mathrm{d}y\\ &= \frac{1}{\sqrt{2}}\left[\frac{28}{3}\,y^{\frac12}+\frac{4}{3}\,y^{\frac32}\right]_1^4=\frac{4}{3\sqrt{2}}\{7(2-1)+(8-1)\}=\frac{28\sqrt{2}}{3}. \end{aligned}$$

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28 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart.*

28.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{array}{ll}
(a+b)^2 = a^2+b^2+2ab, & a^2+b^2+2ab = (a+b)^2, \\
(a-b)^2 = a^2+b^2-2ab, & a^2+b^2-2ab = (a-b)^2, \\
(a+b)(a-b) = a^2-b^2, & a^2-b^2 = (a+b)(a-b), \\
(a+b)^2 = (a-b)^2+4ab, & (a-b)^2 = (a+b)^2-4ab.
\end{array}$$

28.2 Powers etc.

Logarithm:

$$\begin{array}{rcl}
\ln|xy| = & \ln|x|+\ln|y|, & x,y\neq 0, \\
\ln\left|\dfrac{x}{y}\right| = & \ln|x|-\ln|y|, & x,y\neq 0, \\
\ln|x^r| = & r\ln|x|, & x\neq 0.
\end{array}$$

Power function, fixed exponent:

$$(xy)^r = x^r\cdot y^r, x,y>0 \quad \text{(extensions for some } r\text{)},$$

$$\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}, x,y>0 \qquad \text{(extensions for some } r\text{)}.$$

Exponential, fixed base:

$$a^x\cdot a^y = a^{x+y}, \quad a>0 \quad \text{(extensions for some } x,\, y\text{)},$$
$$(a^x)^y = a^{xy}, a>0 \qquad \text{(extensions for some } x,\, y\text{)},$$

$$a^{-x} = \frac{1}{a^x}, a>0, \qquad \text{(extensions for some } x\text{)},$$

$$\sqrt[n]{a} = a^{1/n}, a\geq 0, \qquad n\in\mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \qquad x\in\mathbb{R}.$$

Remark 28.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: "If you can master the square root, you can master everything in mathematics!" Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* $\Diamond$

28.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x)\left\{\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.
If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{\frac{f(x)}{g(x)}\right\} = \frac{d}{dx}\left\{f(x) \cdot \frac{1}{g(x)}\right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)}\left\{\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \qquad f(x) \neq 0,$$

we also name these the *logarithmic derivatives.*

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule.*

28.4 Special derivatives.

Power like:

$$\frac{d}{dx}(x^\alpha) = \alpha \cdot x^{\alpha-1}, \qquad \text{for } x > 0, \quad \text{(extensions for some } \alpha\text{)}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad \text{for } x \neq 0.$$

Exponential like:

$$\frac{d}{dx}\exp x = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x, \qquad \text{for } x \in \mathbb{R} \text{ and } a > 0.$$

Trigonometric:

$$\frac{d}{dx}\sin x = \cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cos x = -\sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z},$$

$$\frac{d}{dx}\cot x = -\left(1 + \cot^2 x\right) = -\frac{1}{\sin^2 x}, \qquad \text{for } x \neq p\pi, p \in \mathbb{Z}.$$

Hyperbolic:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\coth x = 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, \qquad \text{for } x \neq 0.$$

Inverse trigonometric:

$$\frac{d}{dx}\ \text{Arcsin}\ x = \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \left]-1,1\right[,$$

$$\frac{d}{dx}\ \text{Arccos}\ x = -\frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in \left]-1,1\right[,$$

$$\frac{d}{dx}\ \text{Arctan}\ x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\ \text{Arccot}\ x = \frac{1}{1+x^2}, \qquad \text{for } x \in \mathbb{R}.$$

Inverse hyperbolic:

$$\frac{d}{dx}\ \text{Arsinh}\ x = \frac{1}{\sqrt{x^2+1}}, \qquad \text{for } x \in \mathbb{R},$$

$$\frac{d}{dx}\ \text{Arcosh}\ x = \frac{1}{\sqrt{x^2-1}}, \qquad \text{for } x \in \left]1,+\infty\right[,$$

$$\frac{d}{dx}\ \text{Artanh}\ x = \frac{1}{1-x^2}, \qquad \text{for } |x| < 1,$$

$$\frac{d}{dx}\ \text{Arcoth}\ x = \frac{1}{1-x^2}, \qquad \text{for } |x| > 1.$$

Remark 28.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. $\Diamond$

28.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\}\, dx = \int f(x)\, dx + \lambda \int g(x)\, dx, \qquad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are "inverses to each other", i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x)\, dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}'\, dx = \int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x)\, dx = f(x)g(x) - \int f(x)g'(x)\, dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 28.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. $\Diamond$

Remark 28.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. $\Diamond$

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x)\, dx = \text{“} \int f(\varphi(x))\, d\varphi(x)\text{''} = \int_{y=\varphi(x)} f(y)\, dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y-interval *one-to-one* onto the x-interval, then

$$\int f(x)\,dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y)\,dy.$$

Remark 28.5 This rule is usually used when we have some "ugly" term in the integrand $f(x)$. The idea is to put this ugly term equal to $y=\varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form $\sqrt{x}$, we put $y=\varphi^{-1}(x)=\sqrt{x}$, hence $x=\varphi(y)=y^2$ and $\varphi'(y)=2y$. $\Diamond$

28.6 Special antiderivatives

Power like:

$$\int \frac{1}{x}\,dx = \ln|x|, \qquad \text{for } x \neq 0. \quad \text{(Do not forget the numerical value!)}$$

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1}x^{\alpha+1}, \qquad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2}\,dx = \text{ Arctan } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Artanh } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2}\,dx = \text{ Arcoth } x, \qquad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \text{ Arcsin } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\text{ Arccos } x, \qquad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{ Arsinh } x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln(x+\sqrt{x^2+1}), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{ Arcosh } x, \qquad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln|x+\sqrt{x^2-1}|, \qquad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \qquad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \qquad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln|\cos x|, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln|\sin x|, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right), \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right), \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \qquad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \qquad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln|\sinh x|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln\left(\frac{\cosh x - 1}{\cosh x + 1}\right), \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x}\,dx = \ln\left|\frac{e^x-1}{e^x+1}\right|, \qquad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x, \qquad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x, \qquad \text{for } x \neq 0.$$

28.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u, cf. figure A.1. This geometrical interpretation is used from time to time.

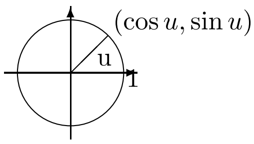

Figure 28.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \qquad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0,0)$, i.e. it is lying on the boundary of the circle of centre $(0,0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(\mathrm{i}\,u) := \cos u + \mathrm{i}\ \sin u.$$

It can be checked that the usual functional equation for exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(\mathrm{i}\,u)$ and $\exp(-\mathrm{i}\,u)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(\mathrm{i}\,u) + \exp(-\mathrm{i}\,u)),$$

$$\sin u = \frac{1}{2i}(\exp(\mathrm{i}\,u) - \exp(-\mathrm{i}\,u)),$$

.

Moivre's formula: We get by expressing $\exp(\mathrm{i}nu)$ in two different ways:

$$\exp(\mathrm{i}nu) = \cos nu + \mathrm{i}\,\sin nu = (\cos u + \mathrm{i}\,\sin u)^n.$$

Example 28.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned}\cos(3u) + \mathrm{i}\,\sin(3u) &= (\cos u + \mathrm{i}\,\sin u)^3 \\ &= \cos^3 u + 3\mathrm{i}\,\cos^2 u \cdot \sin u + 3\mathrm{i}^2 \cos u \cdot \sin^2 u + \mathrm{i}^3 \sin^3 u \\ &= \{\cos^3 u - 3\cos u \cdot \sin^2 u\} + \mathrm{i}\{3\cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4\cos^3 u - 3\cos u\} + \mathrm{i}\{3\sin u - 4\sin^3 u\}\end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4\cos^3 u - 3\cos u, \qquad \sin 3u = 3\sin u - 4\sin^3 u. \quad \Diamond$$

Addition formulæ:

$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v,$$

$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v,$$

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v,$$

$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v.$$

Products of trigonometric functions to a sum:

$$\sin u\,\cos v = \frac{1}{2}\sin(u+v) + \frac{1}{2}\sin(u-v),$$

$$\cos u\,\sin v = \frac{1}{2}\sin(u+v) - \frac{1}{2}\sin(u-v),$$

$$\sin u\,\sin v = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v),$$

$$\cos u\,\cos v = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v).$$

Sums of trigonometric functions to a product:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2\sin u\,\cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\,\cos^2 u - 1 = 1 - 2\,\sin^2 u,$$

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} \qquad \text{followed by a discussion of the sign,}$$

28.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right), \qquad \sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

"Moivre's formula":

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \qquad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\operatorname{Arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right), \qquad x \in \mathbb{R},$$

$$\operatorname{Arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1,$$

$$\operatorname{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad |x| < 1,$$

$$\operatorname{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

28.9 Complex transformation formulæ

$$\cos(\mathrm{i}x) = \cosh(x), \qquad \cosh(\mathrm{i}x) = \cos(x),$$
$$\sin(\mathrm{i}x) = \mathrm{i}\,\sinh(x), \qquad \sinh(\mathrm{i}x) = \mathrm{i}\,\sin x.$$

28.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{1\cdot 2\cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. $= 1$ for the standard series) and*exponential like* (the radius of convergency is infinite).
Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j, \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad |x| < 1,$$

$$\mathrm{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \qquad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \qquad x \in \mathbb{R}.$$

28.11 Magnitudes of functions

We often have to compare functions for $x \to 0+$, or for $x \to \infty$. The simplest type of functions are therefore arranged in an hierarchy:

1) logarithms,

2) power functions,

3) exponential functions,

4) faculty functions.

When $x \to \infty$, a function from a higher class will always dominate a function form a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \to \infty$:

$$\frac{(\ln x)^{\beta}}{x^{\alpha}} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \to \infty$:

$$\frac{x^{\alpha}}{a^{x}} \to 0 \qquad \text{for } x \to \infty, \quad \alpha,\, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \to \infty$:

$$\frac{a^n}{n!} \to 0, \qquad n \to \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \to 0+$ we also have that a *power function* dominates the *logarithm:*

$$x^{\alpha} \ln x \to 0-, \qquad \text{for } x \to 0+, \quad \alpha > 0.$$

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