

Real Functions in Several Variables: Volume XI

Vector Fields II

Leif Mejlbro



Leif Mejlbro

Real Functions in Several Variables

Volume XI Vector Fields II

Stokes's Theorem Nabla Calculus



Real Functions in Several Variables: Volume XI Vector Fields II

Stokes's Theorem Nabla Calculus

2nd edition

© 2015 Leif Mejlbro & bookboon.com

ISBN 978-87-403-0918-8

Contents

Volume I, Point Sets in \mathbb{R}^n	1
Preface	15
Introduction to volume I, Point sets in \mathbb{R}^n. The maximal domain of a function	19
1 Basic concepts	21
1.1 Introduction	21
1.2 The real linear space \mathbb{R}^n	22
1.3 The vector product	26
1.4 The most commonly used coordinate systems	29
1.5 Point sets in space	37
1.5.1 Interior, exterior and boundary of a set	37
1.5.2 Starshaped and convex sets	40
1.5.3 Catalogue of frequently used point sets in the plane and the space	41
1.6 Quadratic equations in two or three variables. Conic sections	47
1.6.1 Quadratic equations in two variables. Conic sections	47
1.6.2 Quadratic equations in three variables. Conic sectional surfaces	54
1.6.3 Summary of the canonical cases in three variables	66
2 Some useful procedures	67
2.1 Introduction	67
2.2 Integration of trigonometric polynomials	67
2.3 Complex decomposition of a fraction of two polynomials	69
2.4 Integration of a fraction of two polynomials	72
3 Examples of point sets	75
3.1 Point sets	75
3.2 Conics and conical sections	104
4 Formulae	115
4.1 Squares etc.	115
4.2 Powers etc.	115
4.3 Differentiation	116
4.4 Special derivatives	116
4.5 Integration	118
4.6 Special antiderivatives	119
4.7 Trigonometric formulae	121
4.8 Hyperbolic formulae	123
4.9 Complex transformation formulae	124
4.10 Taylor expansions	124
4.11 Magnitudes of functions	125
Index	127

Volume II, Continuous Functions in Several Variables	133
Preface	147
Introduction to volume II, Continuous Functions in Several Variables	151
5 Continuous functions in several variables	153
5.1 Maps in general	153
5.2 Functions in several variables	154
5.3 Vector functions	157
5.4 Visualization of functions	158
5.5 Implicit given function	161
5.6 Limits and continuity	162
5.7 Continuous functions	168
5.8 Continuous curves	170
5.8.1 Parametric description	170
5.8.2 Change of parameter of a curve	174
5.9 Connectedness	175
5.10 Continuous surfaces in \mathbb{R}^3	177
5.10.1 Parametric description and continuity	177
5.10.2 Cylindric surfaces	180
5.10.3 Surfaces of revolution	181
5.10.4 Boundary curves, closed surface and orientation of surfaces	182
5.11 Main theorems for continuous functions	185
6 A useful procedure	189
6.1 The domain of a function	189
7 Examples of continuous functions in several variables	191
7.1 Maximal domain of a function	191
7.2 Level curves and level surfaces	198
7.3 Continuous functions	212
7.4 Description of curves	227
7.5 Connected sets	241
7.6 Description of surfaces	245
8 Formulae	257
8.1 Squares etc.	257
8.2 Powers etc.	257
8.3 Differentiation	258
8.4 Special derivatives	258
8.5 Integration	260
8.6 Special antiderivatives	261
8.7 Trigonometric formulæ	263
8.8 Hyperbolic formulæ	265
8.9 Complex transformation formulæ	266
8.10 Taylor expansions	266
8.11 Magnitudes of functions	267
Index	269

Volume III, Differentiable Functions in Several Variables	275
Preface	289
Introduction to volume III, Differentiable Functions in Several Variables	293
9 Differentiable functions in several variables	295
9.1 Differentiability	295
9.1.1 The gradient and the differential	295
9.1.2 Partial derivatives	298
9.1.3 Differentiable vector functions	303
9.1.4 The approximating polynomial of degree 1	304
9.2 The chain rule	305
9.2.1 The elementary chain rule	305
9.2.2 The first special case	308
9.2.3 The second special case	309
9.2.4 The third special case	310
9.2.5 The general chain rule	314
9.3 Directional derivative	317
9.4 C^n -functions	318
9.5 Taylor's formula	321
9.5.1 Taylor's formula in one dimension	321
9.5.2 Taylor expansion of order 1	322
9.5.3 Taylor expansion of order 2 in the plane	323
9.5.4 The approximating polynomial	326
10 Some useful procedures	333
10.1 Introduction	333
10.2 The chain rule	333
10.3 Calculation of the directional derivative	334
10.4 Approximating polynomials	336
11 Examples of differentiable functions	339
11.1 Gradient	339
11.2 The chain rule	352
11.3 Directional derivative	375
11.4 Partial derivatives of higher order	382
11.5 Taylor's formula for functions of several variables	404
12 Formulæ	445
12.1 Squares etc.	445
12.2 Powers etc.	445
12.3 Differentiation	446
12.4 Special derivatives	446
12.5 Integration	448
12.6 Special antiderivatives	449
12.7 Trigonometric formulæ	451
12.8 Hyperbolic formulæ	453
12.9 Complex transformation formulæ	454
12.10 Taylor expansions	454
12.11 Magnitudes of functions	455
Index	457

Volume IV, Differentiable Functions in Several Variables	463
Preface	477
Introduction to volume IV, Curves and Surfaces	481
13 Differentiable curves and surfaces, and line integrals in several variables	483
13.1 Introduction	483
13.2 Differentiable curves	483
13.3 Level curves	492
13.4 Differentiable surfaces	495
13.5 Special C^1 -surfaces	499
13.6 Level surfaces	503
14 Examples of tangents (curves) and tangent planes (surfaces)	505
14.1 Examples of tangents to curves	505
14.2 Examples of tangent planes to a surface	520
15 Formulæ	541
15.1 Squares etc.	541
15.2 Powers etc.	541
15.3 Differentiation	542
15.4 Special derivatives	542
15.5 Integration	544
15.6 Special antiderivatives	545
15.7 Trigonometric formulæ	547
15.8 Hyperbolic formulæ	549
15.9 Complex transformation formulæ	550
15.10 Taylor expansions	550
15.11 Magnitudes of functions	551
Index	553
Volume V, Differentiable Functions in Several Variables	559
Preface	573
Introduction to volume V, The range of a function, Extrema of a Function in Several Variables	577
16 The range of a function	579
16.1 Introduction	579
16.2 Global extrema of a continuous function	581
16.2.1 A necessary condition	581
16.2.2 The case of a closed and bounded domain of f	583
16.2.3 The case of a bounded but not closed domain of f	599
16.2.4 The case of an unbounded domain of f	608
16.3 Local extrema of a continuous function	611
16.3.1 Local extrema in general	611
16.3.2 Application of Taylor's formula	616
16.4 Extremum for continuous functions in three or more variables	625
17 Examples of global and local extrema	631
17.1 MAPLE	631
17.2 Examples of extremum for two variables	632
17.3 Examples of extremum for three variables	668

17.4 Examples of maxima and minima	677
17.5 Examples of ranges of functions	769
18 Formulæ	811
18.1 Squares etc.	811
18.2 Powers etc.	811
18.3 Differentiation	812
18.4 Special derivatives	812
18.5 Integration	814
18.6 Special antiderivatives	815
18.7 Trigonometric formulæ	817
18.8 Hyperbolic formulæ	819
18.9 Complex transformation formulæ	820
18.10 Taylor expansions	820
18.11 Magnitudes of functions	821
Index	823
Volume VI, Antiderivatives and Plane Integrals	829
Preface	841
Introduction to volume VI, Integration of a function in several variables	845
19 Antiderivatives of functions in several variables	847
19.1 The theory of antiderivatives of functions in several variables	847
19.2 Templates for gradient fields and antiderivatives of functions in three variables	858
19.3 Examples of gradient fields and antiderivatives	863
20 Integration in the plane	881
20.1 An overview of integration in the plane and in the space	881
20.2 Introduction	882
20.3 The plane integral in rectangular coordinates	887
20.3.1 Reduction in rectangular coordinates	887
20.3.2 The colour code, and a procedure of calculating a plane integral	890
20.4 Examples of the plane integral in rectangular coordinates	894
20.5 The plane integral in polar coordinates	936
20.6 Procedure of reduction of the plane integral; polar version	944
20.7 Examples of the plane integral in polar coordinates	948
20.8 Examples of area in polar coordinates	972
21 Formulæ	977
21.1 Squares etc.	977
21.2 Powers etc.	977
21.3 Differentiation	978
21.4 Special derivatives	978
21.5 Integration	980
21.6 Special antiderivatives	981
21.7 Trigonometric formulæ	983
21.8 Hyperbolic formulæ	985
21.9 Complex transformation formulæ	986
21.10 Taylor expansions	986
21.11 Magnitudes of functions	987
Index	989

Volume VII, Space Integrals	995
Preface	1009
Introduction to volume VII, The space integral	1013
22 The space integral in rectangular coordinates	1015
22.1 Introduction	1015
22.2 Overview of setting up of a line, a plane, a surface or a space integral	1015
22.3 Reduction theorems in rectangular coordinates	1021
22.4 Procedure for reduction of space integral in rectangular coordinates	1024
22.5 Examples of space integrals in rectangular coordinates	1026
23 The space integral in semi-polar coordinates	1055
23.1 Reduction theorem in semi-polar coordinates	1055
23.2 Procedures for reduction of space integral in semi-polar coordinates	1056
23.3 Examples of space integrals in semi-polar coordinates	1058
24 The space integral in spherical coordinates	1081
24.1 Reduction theorem in spherical coordinates	1081
24.2 Procedures for reduction of space integral in spherical coordinates	1082
24.3 Examples of space integrals in spherical coordinates	1084
24.4 Examples of volumes	1107
24.5 Examples of moments of inertia and centres of gravity	1116
25 Formulæ	1125
25.1 Squares etc.	1125
25.2 Powers etc.	1125
25.3 Differentiation	1126
25.4 Special derivatives	1126
25.5 Integration	1128
25.6 Special antiderivatives	1129
25.7 Trigonometric formulæ	1131
25.8 Hyperbolic formulæ	1133
25.9 Complex transformation formulæ	1134
25.10 Taylor expansions	1134
25.11 Magnitudes of functions	1135
Index	1137
Volume VIII, Line Integrals and Surface Integrals	1143
Preface	1157
Introduction to volume VIII, The line integral and the surface integral	1161
26 The line integral	1163
26.1 Introduction	1163
26.2 Reduction theorem of the line integral	1163
26.2.1 Natural parametric description	1166
26.3 Procedures for reduction of a line integral	1167
26.4 Examples of the line integral in rectangular coordinates	1168
26.5 Examples of the line integral in polar coordinates	1190
26.6 Examples of arc lengths and parametric descriptions by the arc length	1201

27 The surface integral	1227
27.1 The reduction theorem for a surface integral	1227
27.1.1 The integral over the graph of a function in two variables	1229
27.1.2 The integral over a cylindric surface	1230
27.1.3 The integral over a surface of revolution	1232
27.2 Procedures for reduction of a surface integral	1233
27.3 Examples of surface integrals	1235
27.4 Examples of surface area	1296
28 Formulæ	1315
28.1 Squares etc.	1315
28.2 Powers etc.	1315
28.3 Differentiation	1316
28.4 Special derivatives	1316
28.5 Integration	1318
28.6 Special antiderivatives	1319
28.7 Trigonometric formulæ	1321
28.8 Hyperbolic formulæ	1323
28.9 Complex transformation formulæ	1324
28.10 Taylor expansions	1324
28.11 Magnitudes of functions	1325
Index	1327



www.sylvania.com

We do not reinvent the wheel we reinvent light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA



Click on the ad to read more

Volume IX, Transformation formulæ and improper integrals	1333
Preface	1347
Introduction to volume IX, Transformation formulæ and improper integrals	1351
29 Transformation of plane and space integrals	1353
29.1 Transformation of a plane integral	1353
29.2 Transformation of a space integral	1355
29.3 Procedures for the transformation of plane or space integrals	1358
29.4 Examples of transformation of plane and space integrals	1359
30 Improper integrals	1411
30.1 Introduction	1411
30.2 Theorems for improper integrals	1413
30.3 Procedure for improper integrals; bounded domain	1415
30.4 Procedure for improper integrals; unbounded domain	1417
30.5 Examples of improper integrals	1418
31 Formulae	1447
31.1 Squares etc.	1447
31.2 Powers etc.	1447
31.3 Differentiation	1448
31.4 Special derivatives	1448
31.5 Integration	1450
31.6 Special antiderivatives	1451
31.7 Trigonometric formulæ	1453
31.8 Hyperbolic formulæ	1455
31.9 Complex transformation formulæ	1456
31.10 Taylor expansions	1456
31.11 Magnitudes of functions	1457
Index	1459
Volume X, Vector Fields I; Gauß's Theorem	1465
Preface	1479
Introduction to volume X, Vector fields; Gauß's Theorem	1483
32 Tangential line integrals	1485
32.1 Introduction	1485
32.2 The tangential line integral. Gradient fields.	1485
32.3 Tangential line integrals in Physics	1498
32.4 Overview of the theorems and methods concerning tangential line integrals and gradient fields	1499
32.5 Examples of tangential line integrals	1502
33 Flux and divergence of a vector field. Gauß's theorem	1535
33.1 Flux	1535
33.2 Divergence and Gauß's theorem	1540
33.3 Applications in Physics	1544
33.3.1 Magnetic flux	1544
33.3.2 Coulomb vector field	1545
33.3.3 Continuity equation	1548
33.4 Procedures for flux and divergence of a vector field; Gauß's theorem	1549
33.4.1 Procedure for calculation of a flux	1549
33.4.2 Application of Gauß's theorem	1549
33.5 Examples of flux and divergence of a vector field; Gauß's theorem	1551
33.5.1 Examples of calculation of the flux	1551
33.5.2 Examples of application of Gauß's theorem	1580

34 Formulæ	1619
34.1 Squares etc.	1619
34.2 Powers etc.	1619
34.3 Differentiation	1620
34.4 Special derivatives	1620
34.5 Integration	1622
34.6 Special antiderivatives	1623
34.7 Trigonometric formulæ	1625
34.8 Hyperbolic formulæ	1627
34.9 Complex transformation formulæ	1628
34.10 Taylor expansions	1628
34.11 Magnitudes of functions	1629
Index	1631
Volume XI, Vector Fields II; Stokes's Theorem	1637
Preface	1651
Introduction to volume XI, Vector fields II; Stokes's Theorem; nabla calculus	1655
35 Rotation of a vector field; Stokes's theorem	1657
35.1 Rotation of a vector field in \mathbb{R}^3	1657
35.2 Stokes's theorem	1661
35.3 Maxwell's equations	1669
35.3.1 The electrostatic field	1669
35.3.2 The magnostatic field	1671
35.3.3 Summary of Maxwell's equations	1679
35.4 Procedure for the calculation of the rotation of a vector field and applications of Stokes's theorem	1682



Deloitte.

Discover the truth at www.deloitte.ca/careers

© Deloitte & Touche LLP and affiliated entities.



Click on the ad to read more

35.5 Examples of the calculation of the rotation of a vector field and applications of Stokes's theorem	1684
35.5.1 Examples of divergence and rotation of a vector field	1684
35.5.2 General examples	1691
35.5.3 Examples of applications of Stokes's theorem	1700
36 Nabla calculus	1739
36.1 The vectorial differential operator ∇	1739
36.2 Differentiation of products	1741
36.3 Differentiation of second order	1743
36.4 Nabla applied on \mathbf{x}	1745
36.5 The integral theorems	1746
36.6 Partial integration	1749
36.7 Overview of Nabla calculus	1750
36.8 Overview of partial integration in higher dimensions	1752
36.9 Examples in nabla calculus	1754
37 Formulæ	1769
37.1 Squares etc.	1769
37.2 Powers etc.	1769
37.3 Differentiation	1770
37.4 Special derivatives	1770
37.5 Integration	1772
37.6 Special antiderivatives	1773
37.7 Trigonometric formulæ	1775
37.8 Hyperbolic formulæ	1777
37.9 Complex transformation formulæ	1778
37.10 Taylor expansions	1778
37.11 Magnitudes of functions	1779
Index	1781
Volume XII, Vector Fields III; Potentials, Harmonic Functions and Green's Identities	1787
Preface	1801
Introduction to volume XII, Vector fields III; Potentials, Harmonic Functions and Green's Identities	1805
38 Potentials	1807
38.1 Definitions of scalar and vectorial potentials	1807
38.2 A vector field given by its rotation and divergence	1813
38.3 Some applications in Physics	1816
38.4 Examples from Electromagnetism	1819
38.5 Scalar and vector potentials	1838
39 Harmonic functions and Green's identities	1889
39.1 Harmonic functions	1889
39.2 Green's first identity	1890
39.3 Green's second identity	1891
39.4 Green's third identity	1896
39.5 Green's identities in the plane	1898
39.6 Gradient, divergence and rotation in semi-polar and spherical coordinates	1899
39.7 Examples of applications of Green's identities	1901
39.8 Overview of Green's theorems in the plane	1909
39.9 Miscellaneous examples	1910

40 Formulæ	1923
40.1 Squares etc.	1923
40.2 Powers etc.	1923
40.3 Differentiation	1924
40.4 Special derivatives	1924
40.5 Integration	1926
40.6 Special antiderivatives	1927
40.7 Trigonometric formulæ	1929
40.8 Hyperbolic formulæ	1931
40.9 Complex transformation formulæ	1932
40.10 Taylor expansions	1932
40.11 Magnitudes of functions	1933
Index	1935

**WHY
WAIT FOR
PROGRESS?**

DARE TO DISCOVER

Discovery means many different things at Schlumberger. But it's the spirit that unites every single one of us. It doesn't matter whether they join our business, engineering or technology teams, our trainees push boundaries, break new ground and deliver the exceptional. If that excites you, then we want to hear from you.

Dare to discover.
careers.slb.com/recentgraduates



S
chlumberger

Preface

The topic of this series of books on “*Real Functions in Several Variables*” is very important in the description in e.g. *Mechanics* of the real 3-dimensional world that we live in. Therefore, we start from the very beginning, modelling this world by using the coordinates of \mathbb{R}^3 to describe e.g. a motion in space. There is, however, absolutely no reason to restrict ourselves to \mathbb{R}^3 alone. Some motions may be rectilinear, so only \mathbb{R} is needed to describe their movements on a line segment. This opens up for also dealing with \mathbb{R}^2 , when we consider plane motions. In more elaborate problems we need higher dimensional spaces. This may be the case in *Probability Theory* and *Statistics*. Therefore, we shall in general use \mathbb{R}^n as our abstract model, and then restrict ourselves in examples mainly to \mathbb{R}^2 and \mathbb{R}^3 .

For rectilinear motions the familiar *rectangular coordinate system* is the most convenient one to apply. However, as known from e.g. *Mechanics*, circular motions are also very important in the applications in engineering. It becomes natural alternatively to apply in \mathbb{R}^2 the so-called *polar coordinates* in the plane. They are convenient to describe a circle, where the rectangular coordinates usually give some nasty square roots, which are difficult to handle in practice.

Rectangular coordinates and polar coordinates are designed to model each their problems. They supplement each other, so difficult computations in one of these coordinate systems may be easy, and even trivial, in the other one. It is therefore important always in advance carefully to analyze the geometry of e.g. a domain, so we ask the question: Is this domain best described in rectangular or in polar coordinates?

Sometimes one may split a problem into two subproblems, where we apply rectangular coordinates in one of them and polar coordinates in the other one.

It should be mentioned that in *real life* (though not in these books) one cannot always split a problem into two subproblems as above. Then one is really in trouble, and more advanced mathematical methods should be applied instead. This is, however, outside the scope of the present series of books.

The idea of polar coordinates can be extended in two ways to \mathbb{R}^3 . Either to *semi-polar* or *cylindric coordinates*, which are designed to describe a cylinder, or to *spherical coordinates*, which are excellent for describing spheres, where rectangular coordinates usually are doomed to fail. We use them already in daily life, when we specify a place on Earth by its longitude and latitude! It would be very awkward in this case to use rectangular coordinates instead, even if it is possible.

Concerning the contents, we begin this investigation by modelling point sets in an n -dimensional Euclidean space E^n by \mathbb{R}^n . There is a subtle difference between E^n and \mathbb{R}^n , although we often identify these two spaces. In E^n we use *geometrical methods* without a coordinate system, so the objects are independent of such a choice. In the coordinate space \mathbb{R}^n we can use ordinary calculus, which in principle is not possible in E^n . In order to stress this point, we call E^n the “abstract space” (in the sense of calculus; not in the sense of geometry) as a warning to the reader. Also, whenever necessary, we use the colour black in the “abstract space”, in order to stress that this expression is theoretical, while variables given in a chosen coordinate system and their related concepts are given the colours blue, red and green.

We also include the most basic of what mathematicians call *Topology*, which will be necessary in the following. We describe what we need by a function.

Then we proceed with limits and continuity of functions and define continuous curves and surfaces, with parameters from subsets of \mathbb{R} and \mathbb{R}^2 , resp..

Continue with (partial) differentiable functions, curves and surfaces, the chain rule and Taylor's formula for functions in several variables.

We deal with maxima and minima and extrema of functions in several variables over a domain in \mathbb{R}^n . This is a very important subject, so there are given many worked examples to illustrate the theory.

Then we turn to the problems of integration, where we specify four different types with increasing complexity, *plane integral, space integral, curve (or line) integral and surface integral*.

Finally, we consider *vector analysis*, where we deal with vector fields, Gauß's theorem and Stokes's theorem. All these subjects are very important in theoretical Physics.

The structure of this series of books is that each subject is *usually* (but not always) described by three successive chapters. In the first chapter a brief theoretical theory is given. The next chapter gives some practical guidelines of how to solve problems connected with the subject under consideration. Finally, some worked out examples are given, in many cases in several variants, because the standard solution method is seldom the only way, and it may even be clumsy compared with other possibilities.

I have as far as possible structured the examples according to the following scheme:

A Awareness, i.e. a short description of what is the problem.

D Decision, i.e. a reflection over what should be done with the problem.

I Implementation, i.e. where all the calculations are made.

C Control, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

From high school one is used to immediately to proceed to **I. Implementation**. However, examples and problems at university level, let alone situations in real life, are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be executed.

This is unfortunately not the case with **C Control**, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of \wedge I shall either write "and", or a comma, and instead of \vee I shall write "or". The arrows \Rightarrow and \Leftrightarrow are in particular misunderstood by the students, so they should be totally avoided. They are not telegram short hands, and from a logical point of view they usually do not make sense at all! Instead, write in a plain language what you mean or want to do. This is difficult in the beginning, but after some practice it becomes routine, and it will give more precise information.

When we deal with multiple integrals, one of the possible pedagogical ways of solving problems has been to colour variables, integrals and upper and lower bounds in blue, red and green, so the reader by the colour code can see in each integral what is the variable, and what are the parameters, which

do not enter the integration under consideration. We shall of course build up a hierarchy of these colours, so the order of integration will always be defined. As already mentioned above we reserve the colour black for the theoretical expressions, where we cannot use ordinary calculus, because the symbols are only shorthand for a concept.

The author has been very grateful to his old friend and colleague, the late Per Wennerberg Karlsson, for many discussions of how to present these difficult topics on real functions in several variables, and for his permission to use his textbook as a template of this present series. Nevertheless, the author has felt it necessary to make quite a few changes compared with the old textbook, because we did not always agree, and some of the topics could also be explained in another way, and then of course the results of our discussions have here been put in writing for the first time.

The author also adds some calculations in MAPLE, which interact nicely with the theoretic text. Note, however, that when one applies MAPLE, one is forced first to make a geometrical analysis of the domain of integration, i.e. apply some of the techniques developed in the present books.

The theory and methods of these volumes on "Real Functions in Several Variables" are applied constantly in higher Mathematics, Mechanics and Engineering Sciences. It is of paramount importance for the calculations in *Probability Theory*, where one constantly integrate over some point set in space.

It is my hope that this text, these guidelines and these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
March 21, 2015

SIMPLY CLEVER



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



Click on the ad to read more

Introduction to volume XI, Vector Fields II; Stokes's Theorem; Nabla calculus

This is the eleventh volume in the series of books on *Real Functions in Several Variables*.

It is also the second volume on Vector Fields. It was necessary to split the material into three volumes because the material is too big for one volume, and even these three volumes are large. In the first volume we dealt with the *tangential line integral*, which e.g. can be used to describe the work of a particle when it is forced along a given curve by some force. It was natural to introduce the *gradient fields*, where the tangential line integral only depends on the initial and the terminal points of the curve and not of the curve itself. Such gradients fields are describing *conservative forces* in Physics.

Tangential line integrals are one-dimensional in nature. In case of two dimensions we consider the *flux* of a flow through a surface. When the surface $\partial\Omega$ is surrounding a three dimensional body Ω , this leads to *Gauß's theorem*, by which we can express the flux of a vector field \mathbf{V} through $\partial\Omega$, which is a surface integral, by a space integral over Ω of the *divergence* of the vector field \mathbf{V} . This theorem works both ways. Sometimes, and most frequently, the surface integral is expressed as space integral, other times we express a space integral as a flux, i.e. a surface integral. Applications are obvious in *Electro-Magnetic Field Theory*, though other applications can also be found.

In this volume we shall study *Stokes's theorem*. By using a more advanced mathematical formalism from modern *Differential Geometry* it is possible to show that *Gauß's theorem* and *Stokes's theorem* as presented here in Volume X and Volume XI can be considered as special cases of the same *general Gauß's theorem*. This is difficult to see in the terminology chosen here. However, the pattern is similar in both cases. An integration in n dimensions over a domain Ω is transformed into an integration in $n - 1$ dimensions over the intrinsic boundary $\delta\Omega$, and the integrand is changed appropriately during this transformation.

Gauß's and *Stokes's* theorems have always been considered as extremely difficult to understand for the student. Therefore we have included a section on *Maxwell's equations* from Physics, where these two theorems are constantly been applied. We also give lots of examples of worked out problems.

We also include a chapter on nabla calculus, which in three dimensions uses a formalism with the cross product and the dot product known from *Linear Algebra*. Some formulæ become easier to comprehend than the traditional ones using the notations **grad**, **div** and **rot**, of some authors also written **curl**. This is actually the first step towards the unification of *Gauß's* and *Stokes's theorems* in the *general Gauß's theorem* mentioned above. However, we shall not go into the full generality in n dimensions. The following Volume XII is the third one concerning these vector fields. Here we shall conclude with introducing vector potentials, harmonic functions and Green's theorems.

35 Rotation of a vector field; Stokes's theorem

35.1 Rotation of a vector field in \mathbb{R}^3

We considered in Volume X the *flux* of a vector field \mathbf{V} through a closed surface \mathcal{F} in the sense that $\mathcal{F} = \partial\Omega$ is the boundary of a three dimensional body Ω . We shall in this chapter instead consider a closed curve \mathcal{K} in the space \mathbb{R}^3 (i.e. coinciding initial and final point of the curve), which then can be considered as the boundary curve $\delta\mathcal{G}$ of some surface \mathcal{G} in \mathbb{R}^3 . Note that $\delta\mathcal{G}$ means the *intrinsic boundary* of \mathcal{G} , because in \mathbb{R}^3 we have $\partial\mathcal{G} = \mathcal{G}$ for every continuous and piecewise C^1 two dimensional surface \mathcal{G} .

Remark 35.1 When we compare with *Gauß's theorem*, we see that we here replace the three dimensional body Ω with a two dimensional surface \mathcal{F} , and the boundary surface $\partial\Omega$ is replaced with an in principle one dimensional closed curve $\mathcal{K} = \delta\mathcal{G}$. At the first glance the reader may feel a little uneasy, because there are many piecewise C^1 surfaces \mathcal{G} which satisfy the requirement that its boundary curve is equal to the given curve \mathcal{K} , so \mathcal{G} is not uniquely determined, in contrast to the body Ω in Gauß's theorem. This is probably the reason for why *Stokes's theorem* below intuitively is felt to be "difficult". We shall in this chapter try to explain what lies behind. ◇

The idea is that given a vector field \mathbf{V} and a surface \mathcal{F} of the closed (intrinsic) boundary curve $\delta\mathcal{F}$, then it should be possible to express the circulation of \mathbf{V} along the closed curve $\delta\mathcal{F}$, i.f.

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds,$$

as a *flux* of some vector field \mathbf{W} , depending only on \mathbf{V} , through the surface \mathcal{F} , i.e.

$$(35.1) \quad \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{W} \cdot \mathbf{n} \, dS,$$

where we shall find the relationship between \mathbf{W} and \mathbf{V} .

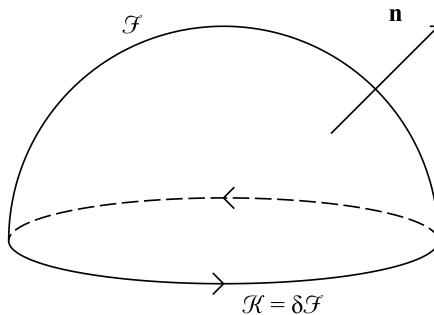


Figure 35.1: Coupled orientation of a surface \mathcal{F} and its intrinsic boundary curve $\delta\mathcal{F}$.

The first problem is of course that there are two orientations of the intrinsic boundary curve $\delta\mathcal{F}$ and two ways to define the unit normal field \mathbf{n} on \mathcal{F} . We therefore need to describe how these two ways of orientation are coupled.

- 1) If the continuous normal vector field \mathbf{n} is given on \mathcal{F} , then the orientation of $\delta\mathcal{F}$ is determined in the following way. Choose an \mathbf{n} close to the boundary curve $\delta\mathcal{F}$. Grasp this \mathbf{n} with your right hand, such that your thumb is pointing in the same direction as \mathbf{n} . Then the other four fingers will indicate the orientation of $\delta\mathcal{F}$, cf. Figure 35.1.
- 2) If instead the orientation of $\delta\mathcal{F}$ is given, then put your right hand, such that your four fingers are pointing in the direction of the tangent of $\delta\mathcal{F}$. Then your thumb will indicate the direction of the unit normal vector field \mathbf{n} on \mathcal{F} in the neighbourhood of $\delta\mathcal{F}$. Finally, extend \mathbf{n} by continuity to all of \mathcal{F} .

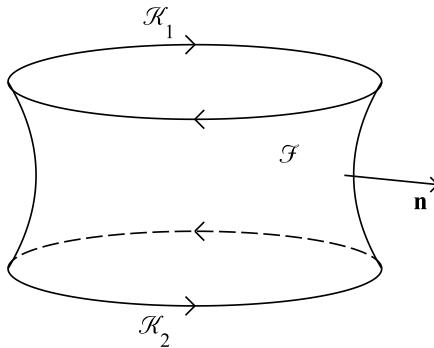


Figure 35.2: The coupling between \mathbf{n} on \mathcal{F} and the orientations of the two (intrinsically) closed boundary curves \mathcal{K}_1 and \mathcal{K}_2 , when the intrinsic boundary has two components.

A connected surface \mathcal{F} may have several intrinsic boundary curves. If \mathbf{n} is a continuous normal vector field on \mathcal{F} , then procedure 2) above defines the orientations on all boundary curves. An example is shown on Figure 35.2, where we note that we have a sense of that “ \mathcal{K}_1 and \mathcal{K}_2 are given opposite orientations”. This is, however, due to the chosen convention.

Remark 35.2 The northern hemisphere of the Earth satisfies the convention described above. If you are on the North Pole and let your right thumb point along the axis of rotation away from the surface, then the other four fingers will indicate the direction of the rotation of the Earth, i.e. eastwards. Clearly, this rule does not apply on the South Pole. \diamond

Let us return to the problem of determining \mathbf{W} in (35.1), i.e.

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{W} \cdot \mathbf{n} \, dS,$$

where we assume that the orientations of $\delta\mathcal{F}$ and \mathcal{F} are linked together as described above. Let us first start with the simple case, where \mathcal{F} is a plane domain E in the (x, y) -plane, cf. Figure 35.3.

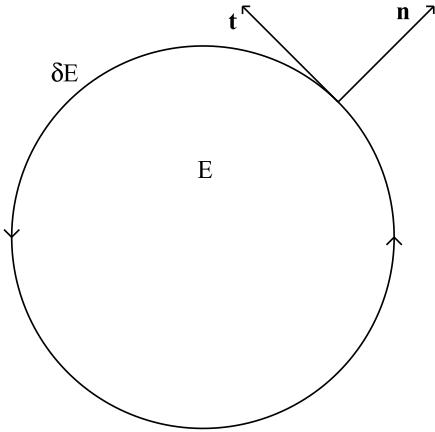


Figure 35.3: Analysis of the circulation along a closed curve δE surrounding a plane set E .

Note that we have changed the notation on Figure 35.3. Since δE is given the positive orientation in the plane, the normal vector field in the whole \mathbb{R}^3 space is represented by the normed vector \mathbf{e}_z , which points towards you from the paper. If \mathbf{t} is the unit tangent field along δE , and \mathbf{n} is the normal field to the curve *in the plane*, then it is well-known that

$$\mathbf{n} = \mathbf{t} \times \mathbf{e}_z, \quad \text{i.e.} \quad n_x = t_y \quad \text{and} \quad n_y = -t_x.$$

The circulation is given by

$$C = \int_{\partial E} (V_x t_x + V_y t_y) \, ds = \int_{\partial E} (-V_x n_y + V_y n_x) \, ds = \int_{\partial E} (V_y, -V_x) \cdot (n_x, n_y) \, ds.$$

Then use Gauß's theorem for a plane vector field, cf. Volume X, to conclude that this is equal to

$$= \int_E \operatorname{div} (V_y, -V_x) \, dS = \int_E \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \, dS.$$

It follows that we have proved that the circulation is

$$\oint_{\partial E} (V_x t_x + V_y t_y) \, ds = \int_E \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \, dS = \int_E \mathbf{W} \cdot \mathbf{e}_z \, dS = \int_E W_z \, dS,$$

because in this setup \mathbf{e}_z is the unit normal field.

If instead E is lying in the (y, z) -plane with \mathbf{e}_x as its normal vector field, then a similar analysis shows that

$$\oint_{\partial E} (V_y t_y + V_z t_z) \, ds = \int_E \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \, dS = \int_E W_x \, dS,$$

and when E lies in the (z, x) -plane in this order (defining the orientation from z towards x), then \mathbf{e}_y is its normal vector, so we obtain by similar calculations

$$\oint_{\partial E} (V_z t_z + V_x t_x) \, ds = \int_E \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \, dS = \int_E W_y \, dS.$$

Note that if we instead had chosen the (x, z) -plane with the orientation from x towards z , then the result would change its sign, but the same would be true for the normal vector field, when it satisfies the right convention, so we get the same result.

Before we proceed we first collect the above in a convenient definition of the *rotation*, because the expression of \mathbf{W} derived above occurs over and over again in these calculations.

Definition 35.1 Let \mathbf{V} be a differentiable vector field in the ordinary space \mathbb{R}^3 . Then we define the rotation of V , written $\text{rot}\mathbf{V}$ by

$$\text{rot}\mathbf{V} := \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathbf{e}_z,$$

where \mathbf{e}_x , \mathbf{e}_y and \mathbf{e}_z are orthonormal unit vectors.

In some books, **rot** is instead written **curl**.

I joined MITAS because
I wanted **real responsibility**

The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com

Month 16

I was a construction supervisor in the North Sea advising and helping foremen solve problems

Real work
International opportunities
Three work placements

Month 16

I was a construction supervisor in the North Sea advising and helping foremen solve problems

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

↗

<div style="position: absolute; bottom: 0; left: 50%; background-color: green; border-radius

There is an easier way to remember this formula, namely as a formal determinant, using a similar determinant as when we calculate the cross product of two vectors, namely

$$\text{rot} \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}.$$

This structure invites one to also use the notation $\nabla \times$ instead of **rot**. This formal determinant is calculating by developing the determinant after the first row and then always let the differential operator work on the function.

35.2 Stokes's theorem

We proved in Section 35.1 that the only candidate \mathbf{W} of the solution of the problem

$$\oint_{\partial\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{W} \cdot \mathbf{n} \, dS,$$

in some special cases is $\mathbf{W} = \text{rot} \mathbf{V}$. It can be proved that this is true in general. As usual in these matters the proof of this statement is too complicated to be brought here, so following the usual style of these books we shall only formulate without given a correct proof the important

Theorem 35.1 Stokes's theorem. *Let \mathcal{F} be an oriented, piecewise C^1 -surface with a continuous normal vector field \mathbf{n} , defined almost everywhere.*

- 1) *Assume that its boundary curve $\delta\mathcal{F}$ is a closed and piecewise C^1 -curve without double points, and with a tangent almost everywhere.*
- 2) *Let the orientations of \mathcal{F} and $\delta\mathcal{F}$ be linked by the right hand convention as described above.*
- 3) *Let $\mathbf{V} : A \rightarrow \mathbb{R}^3$ be a C^1 -vector field, where $\mathcal{F} \subseteq A \subseteq \mathbb{R}^3$.*

Then

$$\oint_{\partial\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot} \mathbf{V} \, dS.$$

As in the case of *Gauß's theorem* there is also a two dimensional version of this theorem. This was more or less proved in Section 35.1, when we derived the possible structure of *Stokes's theorem* in the three dimensional case. This two dimensional version is also called *Green's theorem in the plane*.

Theorem 35.2 Green's theorem in the plane. *Let $E \subset \mathbb{R}^2$ be a bounded plane set with a boundary ∂E which is a piecewise C^1 -curve with no double points with a tangent field almost everywhere. Let the unit normal vector field \mathbf{n} on ∂E always point away from E . Then*

$$\oint_{\partial E} \{V_x \, dx + V_y \, dy\} = \int_E \left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\} \, dS.$$

Stokes's theorem (or *Green's theorem*) can be used in two ways. We mention the procedure in one of them below. The task is to find the circulation of a C^1 -vector field \mathbf{V} along a closed curve \mathcal{K} , then we shall choose a surface \mathcal{F} , such that

- 1) The surface \mathcal{F} has \mathcal{K} as its boundary curve, $\delta\mathcal{F} = \mathcal{K}$.
- 2) Depending on the structure of $\text{rot}\mathbf{V}$, which we calculate at this step, we should furthermore choose \mathcal{F} , such that the surface integral

$$\int_{\mathcal{F}} \mathbf{n} \cdot \text{rot}\mathbf{V} \, dS$$

becomes easy to handle. This is not always an easy task.

We say that a vector field \mathbf{V} is *rotational free*, if $\text{rot}\mathbf{V} = \mathbf{0}$ in all the domain of \mathbf{V}

We have the following important simple results

Theorem 35.3 *A C^1 -gradient field, $\text{grad}f$, is always rotational free.*

PROOF. In fact, $f \in C^2$, so the order of differentiation can be interchanged in the following. We get straightforward

$$\begin{aligned} \text{rot}(\text{grad}f) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \\ &= (0, 0, 0) = \mathbf{0}. \quad \diamond \end{aligned}$$

One can prove the following converse statement, which is not formulated in its full generality

Theorem 35.4 *If \mathbf{v} is a rotational free vector field in a starshaped domain, then $\mathbf{V} = \text{grad}f$ is a gradient field, i.e.*

$\text{rot}\mathbf{V} = \mathbf{0}$ implies the existence of f , such that $\mathbf{V} = \text{grad}f$.

PROOF. Just calculate,

$$\begin{aligned} \text{div}(\text{rot}\mathbf{V}) &= \frac{\partial}{\partial x} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \\ &= \left(\frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_y}{\partial x \partial z} \right) + \left(\frac{\partial^2 V_x}{\partial y \partial z} - \frac{\partial^2 V_z}{\partial y \partial x} \right) + \left(\frac{\partial^2 V_y}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z \partial y} \right) \\ &= \left(\frac{\partial^2 V_x}{\partial y \partial z} - \frac{\partial^2 V_x}{\partial z \partial y} \right) + \left(-\frac{\partial^2 V_y}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial z \partial x} \right) + \left(\frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_z}{\partial y \partial x} \right) = 0. \quad \diamond \end{aligned}$$

We also have a theorem, which states when a divergence free vector field is a rotational field. We postpone the formulation of this result to a later section, where it fits into the context.

Remark 35.3 As mentioned previously we also write

$$\mathbf{grad} f = \nabla f, \quad \operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} \quad \text{and} \quad \mathbf{rot} \mathbf{V} = \nabla \times \mathbf{V},$$

where nabla, ∇ , is the three dimensional differential operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Then the results above are also written

$$\nabla \times \nabla f = \mathbf{0} \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{V}) = 0,$$

which should remind one of similar rules in *Linear Algebra*, when we are dealing with vectors in \mathbb{R}^3 .

We shall here include some simple examples, which illustrate the theorems above. In a later section we supply these with others which may go into various other directions. \diamond

Example 35.1 We shall find the circulation C of the given vector field

$$\mathbf{V}(x, y, z) = (z^2 x, x^2 y, y^2 z), \quad \text{for } (x, y, z) \in \mathbb{R}^3,$$

along the oriented curve \mathcal{K} on Figure 35.4, consisting of three circular arcs, all of centrum $\mathbf{0}$ and radius $a > 0$, and which furthermore lie in the planes given by $z = 0$, $y = -x$ and $x = y\sqrt{3}$.

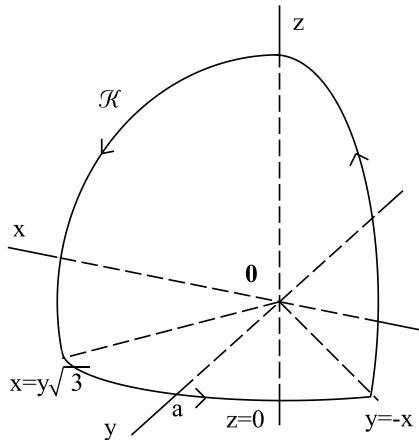


Figure 35.4: The curve \mathcal{K} in Example 35.1.

If we should directly calculate the circulation by using the definition using line integrals, then we should deal with three different line integrals. It is, of course, possible. However, it is easier here to apply *Stokes's theorem*, because all three curves lie on the same spherical surface of centrum $\mathbf{0}$ and radius a . Therefore, we choose \mathcal{F} as the spherical triangle on this sphere, bounded by the curve \mathcal{K} . The orientation of \mathcal{K} and the right hand convention forces the unit normal vector field always to point away from $\mathbf{0}$, so the unit normal vector field is given by

$$\mathbf{n} = \frac{1}{a} (x, y, z), \quad \text{for } (x, y, z) \in \mathcal{F}, \quad \text{i.e. } x^2 + y^2 + z^2 = a^2.$$

We first compute rotV to see if it has a simple structure,

$$\text{rotV}(x, y, z) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2x & z^2y & y^2z \end{vmatrix} = (2yz - 0, 2zx + 0, 2xy - 0) = 2(yz, zx, xy),$$

so restricted to \mathcal{F} we get

$$\mathbf{n} \cdot \text{rotV} = \frac{1}{a} (x, y, z) \cdot 2(yz, zx, xy) = \frac{6}{a} xyz, \quad \text{where } x^2 + y^2 + z^2 = a^2.$$

The advertisement features a man in a green jacket walking down a city street at night, looking towards the right. In the background, there are tall office buildings. On the left, the IE business school logo is visible. On the right, a speech bubble contains the hashtag #gobeyond. Above the man, a graphic shows the Financial Times ranking of IE Business School as the #1 European Business School in 2013. Below the man, a banner reads "MASTER IN MANAGEMENT". The text "Because achieving your dreams is your greatest challenge." is followed by a list of benefits: "Choose your area of specialization.", "Customize your master through the different options offered.", and "Global Immersion Weeks in locations such as London, Silicon Valley or Shanghai.". A tagline "Because you change, we change with you." is at the bottom. Social media icons for Facebook, Twitter, LinkedIn, YouTube, and Instagram are at the very bottom.

www.ie.edu/master-management | mim.admissions@ie.edu | [Facebook](#) [Twitter](#) [LinkedIn](#) [YouTube](#) [Instagram](#)

The surface \mathcal{F} defined above is in spherical coordinates given by

$$\mathcal{F} : \quad \left\{ (r, \varphi, \theta) \mid r = a, \varphi \in \left[\frac{\pi}{6}, \frac{3\pi}{4} \right], \theta \in \left[0, \frac{\pi}{2} \right] \right\},$$

cf. also Figure 35.4, so we finally get by Stokes's theorem,

$$\begin{aligned} C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{n} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{V} \, dS = \frac{6}{a} \int_{\mathcal{F}} xyz \, dS \\ &= \frac{6}{a} \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{ \int_0^{\frac{\pi}{2}} (a^3 \cos \theta \sin^2 \theta \cos \varphi \sin \varphi) a^2 \sin \theta \, d\theta \right\} d\varphi \\ &= 6a^4 \left[\frac{1}{4} \sin^4 \theta \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{1}{2} \sin^2 \varphi \right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \\ &= \frac{3}{4} a^4 \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{3}{16} a^4. \quad \diamond \end{aligned}$$

Example 35.2 Clearly, $\mathbf{V}(x, y, z) = (x, y, z)$ is a *rotational free* vector field, $\text{rot } \mathbf{V} = \mathbf{0}$, so it follows from *Stokes's theorem* that the circulation along any closed curve \mathcal{K} is

$$\oint_{\mathcal{K}} \mathbf{x} \cdot \mathbf{t} \, ds = 0. \quad \diamond$$

Example 35.3 We cannot in advance predict the direction of $\text{rot } \mathbf{V}$ for given \mathbf{V} . We shall in this example show that in some cases $\text{rot } \mathbf{V}$ and \mathbf{V} may be parallel to each other, and in other cases they are perpendicular to each other. To illustrate the first case we choose the vector field

$$\mathbf{V}(x, y, z) = (\cos y, \cos x, \sin y - \sin x), \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

Then

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & \cos x & \sin y - \sin x \end{vmatrix} = (\cos y, \cos x, \sin y - \sin x) = \mathbf{V},$$

so $\text{rot } \mathbf{V} = \mathbf{V}$, and the rotation of \mathbf{V} is even equal to \mathbf{V} itself, and they are trivially parallel.

Concerning the other statement, we choose the vector field

$$\mathbf{U}(x, y, z) = (r, r, r), \quad \text{where } r = \sqrt{x^2 + y^2 + z^2}.$$

We note that, when $r \neq 0$, then

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r},$$

so we get for $r \neq 0$,

$$\text{rot}\mathbf{U} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r & r & r \end{vmatrix} = \left(\frac{y-z}{r}, \frac{z-x}{r}, \frac{x-y}{r} \right),$$

and it follows that

$$\mathbf{U} \cdot \text{rot}\mathbf{U} = (y-z) + (z-x) + (x-y) = 0,$$

which shows that $\mathbf{U} \perp \text{rot}\mathbf{U}$. In other words, \mathbf{U} and $\text{rot}\mathbf{U}$ are perpendicular to each other, and the claim is proved. \diamond

Example 35.4 We shall in this example demonstrate how we may choose the surface \mathcal{F} with a given closed curve \mathcal{K} as its intrinsic boundary $\delta\mathcal{F} = \mathcal{K}$, when we apply *Stokes's theorem*.

- 1) *Specification of the curve \mathcal{K} .* We let the curve \mathcal{K} be defined as the intersection of the circular cylindric surface of the equation

$$x^2 + y^2 = ax,$$

with the surface, given by the equation

$$z = \sqrt{4a^2 - ax}.$$

The latter is one half of a parabolic cylindric surface.

We choose the orientation on \mathcal{K} , such that the direction of the z -axis is the direction of the bounded part of the circular cylindric surface above of boundary \mathcal{K} .

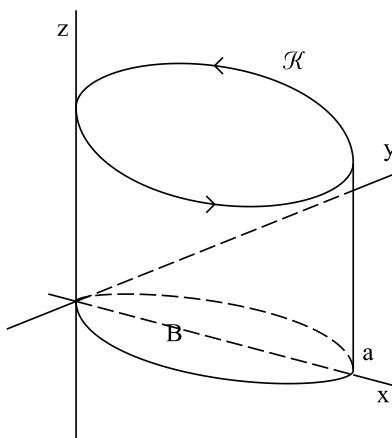


Figure 35.5: The curve \mathcal{K} in Example 35.4.

2) *Specification of the vector field \mathbf{V} and its rotation.* We choose

$$\mathbf{V}(x, y, z) = (3xy, 2x^2, -yz) \quad \text{for } (x, y, z) \in \mathbb{R}^3,$$

and we shall find the circulation C of \mathbf{V} along the curve \mathcal{K} above with the chosen orientation. Since we shall apply *Stokes's theorem*, we start by calculating the rotation of \mathbf{V} ,

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & 2x^2 & -yz \end{vmatrix} = (-z, 0, x).$$

3) *Choice of surface \mathcal{F} , such that $\delta\mathcal{F} = \mathcal{K}$.* First note that the projection of \mathcal{K} onto the (x, y) -plane is the circle of centrum $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$, so it cuts the plane into two open domains, of which only the bounded domain B is of interest for us, so

$$B := \left\{ (x, y) \mid \left(x - \frac{a}{2}\right)^2 + y^2 \leq \left(\frac{a}{2}\right)^2 \right\},$$

so the surface \mathcal{F} , we are going to choose, should be parametrized over B .

Since \mathcal{K} is the intersection of two surfaces, it lies on both of them, so it is obvious to choose

$$\mathcal{F}_1 : \quad z = Z(x, y) := \sqrt{4a^2 - ax}, \quad \text{for } (x, y) \in B.$$

However, since \mathcal{K} is defined by

$$x^2 + y^2 = ax, \quad \text{and} \quad z^2 = 4a^2 - ax, \quad z > 0,$$

it follows by addition that \mathcal{K} also lies on the surface

$$\mathcal{F}_2 : \quad x^2 + y^2 + z^2 = 4a^2, \quad z > 0,$$

which is a part of a sphere.

a) First consider \mathcal{F}_1 , which is the graph of the function $Z(x, y) = \sqrt{4a^2 - ax}$. It therefore follows from Section 13.5 that its field of normal vectors is given by

$$\mathbf{N}(x, y) = \mathbf{r}'_x \times \mathbf{r}'_y = (-Z'_x, -Z'_y, 1) = \left(\frac{a}{2z}, 0, 1\right),$$

which has a positive z -component, so the orientation is correct. It therefore follows from *Stokes's theorem* that the circulation is

$$\begin{aligned} C &= \int_{\mathcal{F}_1} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} dS = \int_B \mathbf{N} \cdot \operatorname{rot} \mathbf{V} dx dy = \int_B \left(\frac{a}{2z}, 0, 1\right) \cdot (-z, 0, x) dx dy \\ &= \int_B \left(-\frac{a}{2} + x\right) dx dy = -\frac{a}{2} \pi \left(\frac{a}{2}\right)^2 + \int_B x dx dy, \end{aligned}$$

where

$$\begin{aligned}
 \int_B x \, dx \, dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{a \cos \varphi} \mathbf{r} \cdot \cos \varphi \cdot \mathbf{r} \, dr \right\} d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^4 \varphi \cdot a^3}{3} d\varphi \\
 &= \frac{a^3}{3 \cdot 4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2 \cos 2\varphi + \cos^2 2\varphi) d\varphi = \frac{a^3}{12} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + \frac{\cos 4\varphi}{2} \right) d\varphi \\
 &= \frac{a^3}{12} \cdot \frac{3}{2} \cdot \pi = \frac{a^3}{8} \pi,
 \end{aligned}$$

so summing up,

$$C = -\frac{a}{2} \pi \left(\frac{a}{2}\right)^2 + \frac{a^3}{8} \pi = 0.$$

b) Then consider \mathcal{F}_2 , i.e. the part of the sphere $x^2 + y^2 + z^2 = 4a^2$, which lies above B . Then,

$$\mathbf{n} = \frac{1}{2a} (x, y, z),$$

because the normal vector is always pointing away from $\mathbf{0}$, and furthermore,

$$\mathbf{n} \cdot \text{rot} \mathbf{V} = \frac{1}{2a} (x, y, z) \cdot (-z, 0, x) = 0,$$

and it follows trivially that

$$C = \int_{\mathcal{F}_2} \mathbf{n} \cdot \text{rot} \mathbf{V} \, dS = 0. \quad \diamond$$

**"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



Click on the ad to read more

Example 35.5 When we apply *Green's theorem in the plane* on the vector fields $\mathbf{V}_1 = (0, x)$ and $\mathbf{V}_2 = (-y, 0)$ along a closed curve ∂E , which is the boundary of the domain E , we get

$$\oint_{\partial E} V_{x,1} dx + V_{y,1} dy = \oint_{\partial E} x dy = \int_E \left(\frac{\partial V_{y,1}}{\partial x} - \frac{\partial V_{x,1}}{\partial y} \right) dS = \int_E dS = \text{area}(E),$$

and

$$\oint_{\partial E} V_{x,2} dx + V_{y,2} dy = \oint_{\partial E} (-y) dy = \int_E \left(\frac{\partial V_{y,2}}{\partial x} - \frac{\partial V_{x,2}}{\partial y} \right) dS = \int_E dS = \text{area}(E),$$

hence

$$\text{area}(E) = \oint_{\partial E} x dy = \oint_{\partial E} (-y) dx = \frac{1}{2} \oint_{\partial E} x dy - y dx,$$

where ∂E has been given the positive orientation in the plane.

Formulæ of this type are applied in e.g. the theory of materials of magnetic hysteresis. \diamond

35.3 Maxwell's equations

Gauß's and *Stokes's* equations are in particular applied in the theory of electro-magnetic fields. We shall here briefly sketch the connection.

35.3.1 The electrostatic field

We consider the space \mathbb{R}^3 containing a set Ω of electrically charged particles. The force on any one particle is proportional to the strength of its own charge. The collection of all charged particles defines a vector field, the *electrostatic field* \mathbf{E} in \mathbb{R}^3 , where the vector $\mathbf{E}(x, y, z)$ at one particular point of the coordinates (x, y, z) is defined as the ratio of the force (a vector) on a test particle at this point to the strength of charge (a scalar) of the test particle.

Consider for the time being the electrostatic field \mathbf{E}_q created by one single particle of charge q , which we may assume lies at origo $\mathcal{O} : (0, 0, 0)$. Then experience has shown that the force at a point (x, y, z) in absolute value is inverse proportional to the square of the distance to $(0, 0, 0)$, and proportional to the charge q . We get for $r > 0$

$$|\mathbf{E}_q(x, y, z)| = \frac{1}{\varepsilon} \frac{|q|}{r^2}, \quad \text{where } r^2 = x^2 + y^2 + z^2.$$

The constant ε , which for convenience has been put into the denominator, is characteristic of the medium and is called the *dielectric constant* of the medium. The direction of the vector field \mathbf{E}_q is given by

$$\pm \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right), \quad \text{where } r = \sqrt{x^2 + y^2 + z^2}.$$

Here + is chosen if $q > 0$, and - when $q < 0$, because the force is repulsive (away from $\mathbf{0}$), when $q > 0$, and attractive (towards $\mathbf{0}$), when $q < 0$, when applied to a test particle of charge +1. Hence,

$$\mathbf{E}_q(x, y, z) = \frac{q}{\varepsilon r^3} (x, y, z), \quad \text{for } (x, y, z) \neq (0, 0, 0).$$

This is a conservative vector field of the same structure as the *Coulomb vector field* considered previously in Section 33.3.2, so we get immediately that if Ω is an open set containing $\mathbf{0}$, then – cf. the calculations of Section 33.3.2,

$$\int_{\partial\Omega} \mathbf{E}_q \cdot \mathbf{n} dS = \frac{q}{\varepsilon} \cdot 4\pi = \frac{4\pi}{\varepsilon} q,$$

because the only difference from the Coulomb vector field is the constant factor $\frac{q}{\varepsilon}$. We note that we in Section 33.3.2 applied *Gauß's theorem* to derive this result.

We have derived for a single particle that the flux through any closed surface surrounding $\mathbf{0}$ is equal to $4\pi/\varepsilon$ times the charge q . It should be equal to $\operatorname{div} \mathbf{E}_q$, but it is not, because \mathbf{E}_q is not defined at $\mathbf{0}$. We shall now repair this. If the open set Ω contains many points of charge q_n , say, then we should add all these contributions. It is, however, customary instead to replace their contributions by a smooth distribution of charge of *density* $\varrho(x, y, z, t)$. Then the vector field \mathbf{E} is also smooth, and we get by *Gauß's theorem* the usual connection between the flux through a closed surface and the divergence of \mathbf{E} . In particular, when this density is independent of time, and Ω is a small axiparallel parallelepipedum of edge lengths dx, dy, dz , and containing the point $(x, y, z) \in \Omega$ under consideration, then the net flux out of this infinitesimal volume element Ω is

$$\oint_{\partial\Omega} \mathbf{E} \cdot d\mathbf{A} = \oint_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} dS \approx \operatorname{div} \mathbf{E}(x, y, z) dx dy dz,$$

where we have used the (mathematical) *mean value theorem* on the right hand side. Hence, by taking the limit $\operatorname{diam} \Omega \rightarrow 0$,

$$\operatorname{div} \mathbf{E} = \frac{4\pi}{\varepsilon} \varrho.$$

This is called *Gauß's law* in Physics. It is also the first of *Maxwell's four equations*. The constant 4π is the area of the unit sphere, and it occurred only in the derivation, so instead one often writes

$$\varepsilon_0 = \frac{\varepsilon}{4\pi}$$

in books on electro-magnetic field theory. We shall here use both notations.

Gauß's law is usually given in two equivalent versions. The one above, which we have just derived, is called *Gauß's law as a differential equation*, because div is a differential operator,

$$\operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_0} \quad \left(= \frac{4\pi}{\varepsilon} \varrho \right).$$

The other version is *Gauß's law as an integral equation*,

$$\int_{\partial E} \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\varepsilon_0} \int_{\Omega} \varrho d\Omega, \quad \text{where } d\mathbf{A} := \mathbf{n} dS.$$

This was actually used in the derivation of Gauß's law as a differential equation, and it follows immediately by insertion and an application of Gauß's (mathematical) theorem, that if the first law is given, then the second one follows. So the two versions are equivalent.

When the *dielectric constant* ε is not a constant in the medium, it is often better instead to use the so-called *displacement field* \mathbf{D} , which is given by the two equations,

$$\operatorname{div} \mathbf{D} = 4\pi \varrho, \quad \text{and} \quad \mathbf{D} = \varepsilon \mathbf{E}.$$

35.3.2 The magnostatic field

Ferromagnetic materials *behave* as if they were charged with a “magnetic fluid” analogous to the electrical fluid, so one would expect that the magnetic “charge” would be described by a *magnetic vector field* \mathbf{H} , just like we obtained the electrical vector field \mathbf{E} . In analogy with the dielectric constant ε above we would expect another constant μ , called the *permeability*, such that the displacement vector $\mathbf{D} = \varepsilon \mathbf{E}$ is analogous to the *magnetic induction* $\mathbf{B} = \mu \mathbf{H}$. This construction, however, requires the existence of a unit positive magnetic charge, and no such magnetic charge (a so-called *monopole*) has ever been found. So we must use another procedure.

Since it does not look like that a magnetic monopole exists, we instead of an analogue of the equations of the displacement field above, introduce the following equation for the *induction field* \mathbf{B} ,

$$\operatorname{div} \mathbf{B} = 0.$$

This describes precisely that there is no magnetic monopole. This is actually *Gauß's law for magnetism* formulated as a differential equation,

$$\operatorname{div} \mathbf{B} = 0.$$

Then let Ω be a body of surface $\partial\Omega$. According to Gauß's (mathematical) theorem from Chapter 33 the magnetic flux of \mathbf{B} through the closed surface $\partial\Omega$ is

$$\int_{\partial\Omega} \mathbf{B} \cdot \mathbf{n} dS = \int_{\Omega} \operatorname{div} \mathbf{B} d\Omega = 0,$$

and we have derived *Gauß's law for magnetism as an integral equation*,

$$\int_{\partial\Omega} \mathbf{B} \cdot d\mathbf{A} = 0, \quad d\mathbf{A} := \mathbf{n} dS.$$

This is the second one of Maxwell's equations in its two equivalent formulations.

We saw above that \mathbf{B} is a divergence free vector field, $\operatorname{div} \mathbf{B} = 0$.

Let \mathbf{V} be any C^1 divergence free vector field in \mathbb{R}^3 , and let $\Omega \subset \mathbb{R}^2$ be a domain, such that its boundary $\partial\Omega$ is cut into two surfaces, \mathcal{F}_1 and \mathcal{F}_2 , by a closed C^1 curve \mathcal{K} without double points, cf. Figure 35.6. Choose the normal vector field \mathbf{n} , such that it points into Ω on the surface \mathcal{F}_1 , and away from Ω on the surface \mathcal{F}_2 . Let Φ_1 and Φ_2 denote the fluxes on \mathcal{F}_1 and \mathcal{F}_2 , resp.. Then it follows from Gauß's theorem that the outgoing flux (seen from Ω) is

$$\Phi_2 + (-\Phi_1) = \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS + \int_{\mathcal{F}_1} \mathbf{V} \cdot (-\mathbf{n}) dS = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = 0,$$

so we get by a rearrangement that $\Phi_2 = \Phi_1$. It therefore follows that we can talk about the flux of \mathbf{V} passing through the loop formed by the closed curve \mathcal{K} , because the flux is the same on all (admissible) bounded surfaces having \mathcal{K} as its boundary curve.

This applies in particular to the magnetic field \mathbf{B} , and it is therefore possible to talk about a *magnetic flux* surrounded by a closed curve \mathcal{K} .

Consider a current I through a closed curve \mathcal{K} as above. Then it produces a magnetic field, which is governed by the *Ampère-Laplace law*,

$$\mathbf{B} = K I \oint_{\mathcal{K}} \frac{1}{r^2} \mathbf{t} \times \mathbf{u} ds,$$

where K is a constant only depending on the chosen physical units, and \mathbf{t} is the unit tangent field on the curve \mathcal{K} , and \mathbf{u} is the unit vector which points from the point on the curve towards the point P under consideration.

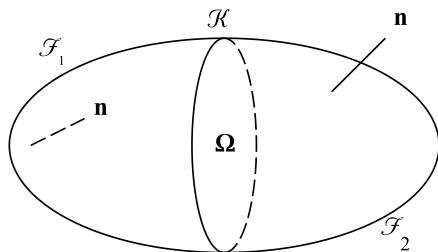


Figure 35.6: When $\operatorname{div} \mathbf{V} = 0$, then the flux of \mathbf{V} through a surface depends only on its boundary curve \mathcal{K} .

The law above was obtained *after many experiments and measurements on closed curves of different shapes*. It is therefore a mathematical model which gives a good mathematical description of the magnetic field. Note in particular that it is not derived from some mathematical theorem.

Excellent Economics and Business programmes at:

 university of
groningen



“The perfect start
of a successful,
international career.”

www.rug.nl/feb/education

CLICK HERE
to discover why both socially
and academically the University
of Groningen is one of the best
places for a student to be

It is customary to write $K = \mu_0/4\pi$, where μ_0 is called the *magnetic permeability of vacuum*. Thus we shall in the following write the *Ampère-Laplace law* in the following way,

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \oint_{\mathcal{K}} \frac{1}{r^2} \mathbf{t} \times \mathbf{u} ds.$$

It follows in particular that a magnetic field is produced when we move electric charges.

If we stretch \mathcal{K} towards an “infinite loop” we get in the limit the magnetic field produced by an infinite rectilinear current I . In fact, choose e.g. \mathcal{K}_n in the (x, z) -space, such that \mathcal{K}_n is composed of the line segments

$$\{0\} \times [-n, n], \quad [0, n] \times \{n\}, \quad \{n\} \times [-n, n], \quad [0, n] \times \{-n\},$$

cf. Figure 35.7.

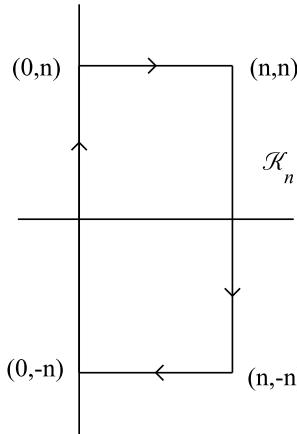


Figure 35.7: The curve \mathcal{K}_n .

The line integrals along the three latter line segments are all of size $\sim n^{-1}$, so their contributions tend to zero, when $n \rightarrow +\infty$. Therefore, the magnetic field generated by a current $I > 0$ along the infinitely thin z -axis is a point $(x, y, 0)$ in the (x, y) -plane given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} I \int_{-\infty}^{+\infty} \frac{1}{r^2} (0, 0, 1) \times \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) dz = \frac{\mu_0}{4\pi} I (-y, x, 0) \int_{-\infty}^{+\infty} \frac{1}{r^3} dz,$$

where

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2} \cdot \sqrt{1 + \left(\frac{z}{\sqrt{x^2 + y^2}} \right)^2}.$$

If we put

$$z = \sqrt{x^2 + y^2} \cdot \sinh t, \quad dz = \sqrt{x^2 + y^2} \cdot \cosh t dt,$$

we get by this substitution,

$$\mathbf{B} = \frac{\mu_0}{4\pi} I (-y, x, 0) \frac{2}{x^2 + y^2} = \frac{\mu_0}{2\pi r} I \left(-\frac{y}{r}, \frac{x}{r}, 0 \right) = \frac{\mu_0 I}{2\pi r} \mathbf{t} = \frac{\mu_0}{2\pi (x^2 + y^2)} (-y, x, 0),$$

where $r = \sqrt{x^2 + y^2}$ and $\|\mathbf{t}\| = \|(-y, x, 0)/r\| = 1$.

Then we compute the circulation of \mathbf{B} around a circular path of radius $r > 0$. It follows from the above that the magnetic field \mathbf{B} is tangent to this circular path, so

$$\mathbf{B} \cdot \mathbf{t} \, ds = B \, ds, \quad \text{where} \quad B := \|\mathbf{B}\| = \frac{\mu_0 I}{2\pi r},$$

which is constant in magnitude. The circle C of radius r has the length $\ell(C) = 2\pi r$, so the *magnetic circulation* $\Lambda_{\mathbf{B}}$ is

$$\Lambda_{\mathbf{B}} = \oint_C \mathbf{B} \cdot \mathbf{t} \, ds = \oint_C B \, ds = B \int_C \, ds = B\ell(C) = 2\pi r B = 2\pi r \cdot \frac{\mu_0 I}{2\pi r} = \mu_0 I.$$

This shows that the magnetic circulation $\Lambda_{\mathbf{B}}$ (also called the *magnetic force*, mmf) is proportional to the electric current I , and it is independent of the radius $r > 0$. *Locally* \mathbf{B} is a gradient field,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \nabla \operatorname{Arctan} \left(\frac{y}{x} \right) \quad \text{for } x > 0 \text{ or } x < 0,$$

or

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \left(\frac{\pi}{2} - \nabla \operatorname{Arccot} \left(\frac{x}{y} \right) \right) \quad \text{for } y > 0 \text{ or } y < 0,$$

i.e. a gradient field in each of the four open half-planes. If we therefore consider any closed loop C winding once around the infinite current I along the z -axis, then we can deform C into a circle in the plane, supplied with some additional loops, which lie in one of the four half-planes above or can be projected into one of them. In particular, these additional loops do not surround the z -axes, so their total contribution is 0. This argument shows that the magnetic circulation corresponding to a rectilinear current along the infinitely thin z -axis is independent of the closed path, as long as it does not go through points on the z -axis.

We have in the simple example of Figure 35.8 illustrated the technique. It follows in this particular case that

$$\begin{aligned} \oint_C \mathbf{B} \cdot \mathbf{t} \, ds &= \int_{C_1} \mathbf{B} \cdot \mathbf{t} \, ds + \int_{C_2} \mathbf{B} \cdot \mathbf{t} \, ds + \left(\int_{\Gamma_1} \mathbf{B} \cdot \mathbf{t} \, ds + \int_{\Gamma_2} \mathbf{B} \cdot \mathbf{t} \, ds \right) \\ &= \int_{C_1 + \Gamma_1} \mathbf{B} \cdot \mathbf{t} \, ds + \int_{C_2 + \Gamma_2} \mathbf{B} \cdot \mathbf{t} \, ds = \mu_0 I + 0 = \mu_0 I. \end{aligned}$$

We say that a closed curve C links a current I , if the path of I traverses every bounded C^1 -surface \mathcal{F} , which has C as its intrinsic boundary.

Using a similar, though more sophisticated technique it is possible to prove that the result is true for any shape of the path of the current, or even currents, so we have derived

Theorem 35.5 Ampère's law for the magnetic field. *The circulation of the magnetic field \mathbf{B} along a closed path C , which links the currents I_1, I_2, \dots , is*

$$\Lambda_{\mathbf{B}} = \oint_C \mathbf{B} \cdot \mathbf{t} \, ds = \mu_0 (I_1 + I_2 + \dots) = \mu_0 I,$$

where $I = I_1 + I_2 + \dots$ denotes the total current linked to the path C .

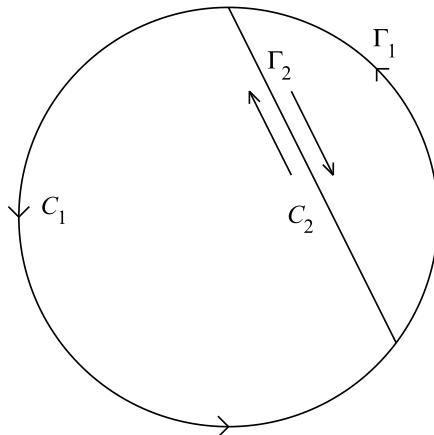


Figure 35.8: By adding the line integral of ∇F along $\Gamma = \gamma_1 + \Gamma_2$, we just add 0. On the other hand, the line integrals along C_2 and Γ_2 cancel each other, because they have the same integrand and opposite orientations. Similarly for other closed plane curves as well as closed space curves.

In the next step we shall establish a connection between the electric field \mathbf{E} and the magnetic field \mathbf{B} . The electric field is defined as the force per unit charge, hence the tangential line integral along a curve C is equal to the work done when we move one unit of charge along the curve C .

When C is closed, this tangential line integral becomes the circulation of the electric field,

$$V_{\mathbf{E},C} = \oint_C \mathbf{E} \cdot \mathbf{t} \, ds.$$

Then assume that given an electric conductor which forms a closed path C . We place it in a region in which a magnetic field exists.

Let \mathcal{F} be any (admissible) bounded surface which has the closed curve C as its intrinsic boundary. If the magnetic flux $\Phi_{\mathbf{B}} = \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, dS$ through \mathcal{F} varies with time, a current is observed in circuit, while the flux is varying, and this current again produces an electric field \mathbf{E} , which is called the *induced electric field*.

Also, in this case the physicists have been forced to rely on experiments to set up a model. Measurements of this induced electric field have shown that it depends on the rate of change $d\Phi_{\mathbf{E}}/dt$. One observed also that the greater the rate of change of the flux, the larger the induced electrical field. Also, the direction in which the induced electrical field acts depends on whether the magnetic flux is increasing or decreasing. One can use the right-hand rule to determine the direction of the act of the induced electric field. If the right-hand thumb points in the direction of the magnetic field, then the induced electrical field acts in the opposite/same direction as the fingers, when the flux increases/decreases. A simple analysis shows that if $d\Phi_{\mathbf{E}}/dt$ is positive, then the induced electrical field $V_{\mathbf{E}} = \oint_C \mathbf{E} \cdot \mathbf{t} \, ds$ acts in the negative sense, and *vice versa*, so they have always opposite signs.

More detailed measurements in experiments have shown that if we choose the physical units right, then a mathematical model is as simple as

$$V_{\mathbf{E}} = \oint_C \mathbf{E} \cdot \mathbf{t} \, ds = -\frac{d\Phi_{\mathbf{E}}}{dt} = -\frac{d}{dt} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, dS,$$

and we can – under the assumption of the chosen model above – formulate the following

Theorem 35.6 The Faraday-Henry law of electromagnetic induction. *Let \mathbf{B} be a dynamic magnetic field. Then an electric field \mathbf{E} is induced in any closed circuit. The induced electrical field \mathbf{E} is equal to the negative of the time rate of the magnetic flux through the circuit, i.e.*

$$\oint_C \mathbf{E} \cdot \mathbf{t} \, ds = -\frac{d}{dt} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, dS.$$

When we apply *Stokes's* (mathematical) *theorem* on the left hand side of the *Faraday-Henry law* of electromagnetic induction, where we use the same surface \mathcal{F} in the resulting surface integral, we get

$$\int_{\mathcal{F}} \operatorname{rot} \mathbf{E} \cdot \mathbf{n} \, dS = -\frac{d}{dt} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} \, dS, \quad \text{i.e.} \quad \int_{\mathcal{F}} \left(\operatorname{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} \, dS = 0.$$

This relation must hold for every (admissible) surface \mathcal{F} , so using the usual argument we easily conclude that we must have

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{E} = \mathbf{0}, \quad \text{i.e.} \quad \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Conversely, this equation immediately implies Theorem 35.6, so the two results are equivalent.

This result in its two versions is also called *Maxwell's third equation*.

American online LIGS University

is currently enrolling in the
Interactive Online BBA, MBA, MSc,
DBA and PhD programs:

- ▶ enroll by September 30th, 2014 and
- ▶ save up to 16% on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit www.ligsuniversity.com to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.
[More info here.](#)



Click on the ad to read more

Finally, by inspecting the first three of Maxwell's equations above we get the feeling that there should also exist a result containing the circulation of the magnetic field \mathbf{B} along a closed curve C , i.e. $\oint_C \mathbf{B} \cdot \mathbf{t} ds$, when the field is not static. At this step this is only a hunch guided by some vague sense of symmetry, but we shall see in the following that our guess is actually giving us the right equation.

Let us start by analyzing the *Faraday-Henry law*, Theorem 35.6, above, i.e.

$$\int_C \mathbf{E} \cdot \mathbf{t} ds = -\frac{d}{dt} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} dS.$$

A symmetric statement, where \mathbf{E} and \mathbf{B} have been interchanged, should therefore contain the time rate of change of flux of the electric, i.e. we may expect that

$$\frac{d}{dt} \int_{\mathcal{F}} \mathbf{E} \cdot \mathbf{n} dS$$

enters. However, when we look at the static case, i.e. *Ampère's law*, Theorem 35.5, which should hold in the limit, we get instead

$$\oint_C \mathbf{B} \cdot \mathbf{t} ds = \mu_0 I,$$

which apparently has nothing to do with a time rate of change of flux, so something must be missing.

The trick is the following: Consider some closed surface \mathcal{F} and cut it by some closed oriented curve C on \mathcal{F} into two. We let \mathcal{F}_C denote the part of \mathcal{F} , which is given the right-hand orientation induced by the orientation of C . We keep \mathcal{F} fixed and let C shrink to a point. Then in some obvious sense $\mathcal{F}_C \rightarrow \mathcal{F}$, and clearly,

$$\lim_{C \rightarrow 0} \oint_C \mathbf{B} \cdot \mathbf{t} ds = 0.$$

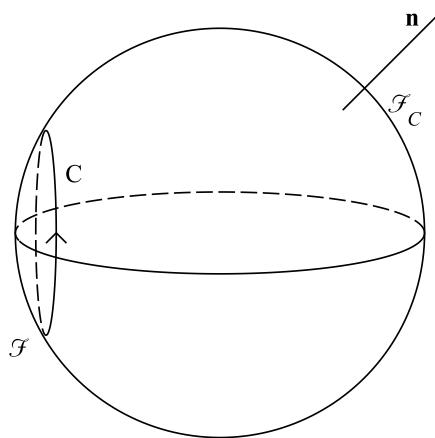


Figure 35.9: The closed surface \mathcal{F} is cut into two by the closed oriented curve C , where we choose the part \mathcal{F}_C , for which the normal vector field induced by the orientation of C is pointing outwards from $\mathcal{F} \supset \mathcal{F}_C$.

Let $q(t)$ in the dynamic case denote the net charge inside the bounded closed surface \mathcal{F} at a given time t . Then the net outgoing charge flux passing through \mathcal{F} per unit time is a current $I_{\mathcal{F}}$, so

$$I_{\mathcal{F}} = -\frac{dq}{dt},$$

with a minus sign, because when $q(t)$ decreases in t , then the current $I_{\mathcal{F}}$ is leaving \mathcal{F} in its positive direction.

Given the dynamic electric field $\mathbf{E}(t)$, the total charge within \mathcal{F} is expressed by the surface integral

$$q(t) = \varepsilon_0 \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS,$$

hence

$$\frac{dq}{dt} = \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS = \varepsilon_0 \int_{\mathcal{F}} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{n} dS \quad (= -I_{\mathcal{F}}).$$

It therefore follows that

$$I_{\mathcal{F}} + \varepsilon_0 \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS = 0.$$

Then let $\mathcal{G} \subset \mathcal{F}$ be any sub-surface of the bounded closed surface \mathcal{F} with the closed curve C as its intrinsic boundary, and then replace I in *Ampère's law*

$$\oint_C \mathbf{B} \cdot \mathbf{t} ds = \mu_0 I$$

by

$$I = I_{\mathcal{G}} + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{G}} \mathbf{E}(t) \cdot \mathbf{n} dS.$$

By this qualified *guess* we replace *Ampère's law* by

$$\oint_C \mathbf{B}(t) \cdot \mathbf{t} ds = \mu_0 \left\{ I_{\mathcal{G}} + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{G}} \mathbf{E}(t) \cdot \mathbf{n} dS \right\}.$$

We see that we keep the structure of *Ampère's law* and at the same time obtain that if C shrinks towards a point, then we indeed get 0 as requested.

We coin the above model in the following

Theorem 35.7 The Ampère-Maxwell law. *Let $\mathbf{E}(t)$ be a dynamic electric field with the corresponding current density \mathbf{J} , and let \mathcal{F} be an oriented surface with the oriented closed curve C as its intrinsic boundary. Then the current I through \mathcal{F} is given by*

$$I = \int_{\mathcal{F}} \mathbf{J} dS.$$

Furthermore, a dynamic magnetic field \mathbf{B} is induced, and

$$\oint_C \mathbf{B} \cdot \mathbf{t} ds = \mu_0 \left\{ I + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS \right\} = \mu_0 \left\{ \int_{\mathcal{F}} \mathbf{J} dS + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS \right\}.$$

This is the fourth of the so-called *Maxwell's equations*.

Then note that according to *Stokes's theorem*, the left hand side is also equal to

$$\begin{aligned}\int_{\mathcal{F}} \operatorname{rot} \mathbf{B} \cdot \mathbf{n} dS &= \left(\oint_C \mathbf{B} \cdot \mathbf{t} ds \right) = \mu_0 \left\{ I + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS \right\} \\ &= \mu_0 \left\{ \int_{\mathcal{F}} \mathbf{J} dS + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS \right\}.\end{aligned}$$

Hence, by a rearrangement,

$$0 = \int_{\mathcal{F}} \left\{ \operatorname{rot} \mathbf{B} - \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \right\} \cdot \mathbf{n} dS,$$

This equation holds for every choice of an admissible surface \mathcal{F} . This is only possible, if the first factor of the integrand is 0,

$$\operatorname{rot} \mathbf{B} - \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0,$$

and we have shown the following pointwise result,

$$\operatorname{rot} \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

Conversely, it is obvious that Theorem 35.7 follows from this pointwise equation, so the two results are equivalent.

Note that Theorem 35.7 so far is only a model derived under the assumption that we can extend the static Ampère's law

$$\oint_C \mathbf{B}(t) \cdot \mathbf{t} ds = \mu_0 I$$

to the dynamic case by replacing I by adding a term concerning the time rate of the flux of the electric field,

$$I + \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E}(t) \cdot \mathbf{n} dS.$$

It has therefore afterwards been necessary through the years to verify this model experimentally, and it turned up that it really describes the actual situation found in nature.

35.3.3 Summary of Maxwell's equations

It is customary to collect all Maxwell's equations in a separate section, so this is also done here. First we make some preparations. The notion of an *electromagnetic field* is characterized by a vector consisting of the *electric field* \mathbf{E} and the *magnetic field* \mathbf{B} . They produce a force on an electric charge q , which is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

where the charge q is moving with the velocity \mathbf{v} . This mathematical model has been verified by many experiments and measurements. It does not follow from mathematics alone.

Then we shall use the *nabla notation*, i.e.

$$\mathbf{grad}\mathbf{V} = \nabla\mathbf{V}, \quad \operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V}, \quad \text{and} \quad \mathbf{rot}\mathbf{V} = \nabla \times \mathbf{V},$$

suggesting the strong connection with Linear Algebra and Geometry in the Euclidean space $E_3 \sim \mathbb{R}^3$. In some books **rot** is written **curl** instead.

We then turn to Maxwell's equations.

..... Alcatel-Lucent 

www.alcatel-lucent.com/careers

What if you could build your future and create the future?

One generation's transformation is the next's status quo.
In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".

 Click on the ad to read more

Maxwell's first law = Gauß's law.

- 1) As integrals, where q denotes the density of the charge,

$$\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} dS = \int_{\Omega} \nabla \cdot \mathbf{E} dV = \frac{1}{\varepsilon_0} \int_{\Omega} \varrho dV.$$

- 2) Pointwise,

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0}.$$

Maxwell's second law = Gauß's law on magnetism. We assume that no magnetic monopole exists.

- 1) As a surface integral, where \mathcal{F} is a closed surface,

$$\int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} dS = 0.$$

- 2) Pointwise,

$$\nabla \cdot \mathbf{B} = 0.$$

Maxwell's third law = Faraday-Henry's law of electromagnetic induction.

- 1) As integrals, where C is a closed curve, and \mathcal{F} is any surface having C as its intrinsic boundary,

$$\oint_C \mathbf{E} \cdot \mathbf{t} ds = -\frac{d}{dt} \int_{\mathcal{F}} \mathbf{B} \cdot \mathbf{n} dS.$$

- 2) Pointwise,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Maxwell's fourth law = Ampère-Maxwell's law. Let \mathbf{J} denote the *vectorial current density*, i.e. the current I through a surface \mathcal{F} is given by

$$I = \int_{\mathcal{F}} \mathbf{J} \cdot \mathbf{n} dS.$$

- 1) As integrals,

$$\oint_C \mathbf{B} \cdot \mathbf{t} ds = \mu_0 \int_{\mathcal{F}} \mathbf{J} \cdot \mathbf{n} dS + \mu_0 \varepsilon_0 \frac{d}{dt} \int_{\mathcal{F}} \mathbf{E} \cdot \mathbf{n} dS.$$

- 2) Pointwise,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon \frac{\partial \mathbf{E}}{\partial t}.$$

Maxwell's four equations together with the above mentioned law of the force on a particle of charge q and of velocity \mathbf{v} ,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}),$$

form the basis of the electromagnetic theory. In the description above some effort has been made to explain, when a result follows from mathematics, in particular from Gauß's and Stokes's theorems, and when some experimental measurements have been necessary to set up the right mathematical model. In particular, in the derivation of *Ampère-Maxwell's law* we first made some reasonable mathematical assumptions, though in the beginning were not justified. By using some mathematics we deduced a possible model, and finally this model was indeed verified through the years by lots of measurements on experiments.

35.4 Procedure for calculation of the rotation of a vector field and application of Stokes's theorem

Let us first mention the most practical way of calculating the *rotation* of a 3-dimensional C^1 vector field by means of a formal determinant,

$$\text{rot } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}, \quad \mathbf{V} = (V_x, V_y, V_z).$$

Note that this rule is only valid *when rectangular coordinates are used!*

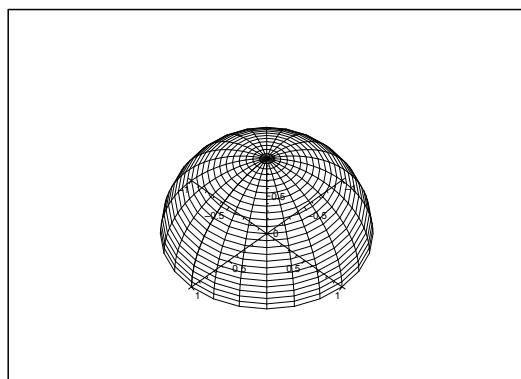


Figure 35.10: As an example we choose again the surface of the upper unit half sphere \mathcal{F} , where the boundary curve $\delta\mathcal{F}$ is the unit circle in the (x, y) -plane.

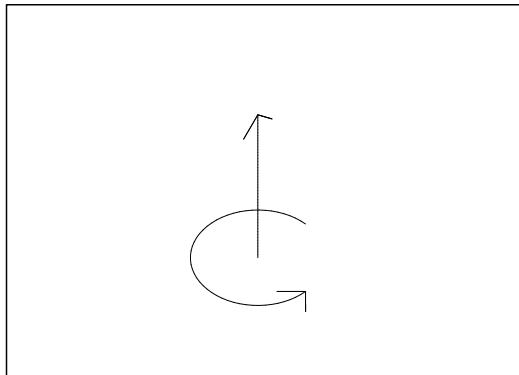


Figure 35.11: The connection between a vector and the corresponding curl which fixes the orientation.

The fixing of the *orientation of a surface* (i.e. the *direction of the normal vector*), when the *direction of the run through of the boundary curve*, is given by:

- **The convention of orientation:** Let the *normal vector field* on the surface \mathcal{F} be supplied with a *curl* around the foot of the normal vector. The *direction* of the curl is by continuity fixed by the *direction* of the run through of the boundary curve. If you put your right hand along the normal vector with the four fingers in the direction of the curl, then your thumb will point in the direction of the normal. Cf. also Figure 35.11.

Whenever *circulation of a vector field* is mentioned, one should think of an application of **Stokes's theorem**:

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS,$$

where the left hand side is the circulation of \mathbf{V} (1 dimension), and the right hand side is a flux of the vector field $\operatorname{rot} \mathbf{V}$ (2 dimensions).

In exercises the student will often in an earlier question have calculated $\operatorname{rot} \mathbf{V}$, so the task is reduced to the choice of a convenient surface \mathcal{F} for the given closed (boundary) curve $\delta\mathcal{F}$. As long as the student is learning these ideas, the surface \mathcal{F} will typically either be flat or a part of a sphere.

Remark 35.4 Stokes's theorem can also be applied from the right to the left. If e.g. one shall find the *flux* $\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{U} \, dS$ of a *rotational field* $\mathbf{U} = \operatorname{rot} \mathbf{V}$, and \mathbf{V} is *fairly easy to find*, then

$$\int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{U} \, dS = \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS = \oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds. \quad \diamond$$

35.5 Examples of calculation of rotation of a vector field and application of Stokes's theorem

35.5.1 Examples of divergence and rotation of a vector field

Example 35.6 Find coordinate expressions of the vector fields $\text{rot}(\text{rot } \mathbf{V})$ and $\text{grad}(\text{div } \mathbf{V})$. Then prove the formula

$$\text{grad}(\text{div } \mathbf{V}) - \text{rot}(\text{rot } \mathbf{V}) = (\text{div}(\text{grad } V_x), \text{div}(\text{grad } V_y), \text{div}(\text{grad } V_z)).$$

A Calculations using nabla.

D The results can be obtained by very mechanical calculations. It is a matter of taste whether one prefers the notation above or

$$\text{grad} = \nabla, \quad \text{div} = \nabla \cdot, \quad \text{rot} = \nabla \times.$$

We shall here use the latter, thereby keeping the formal connection to the geometric relationships that the operations are describing.

I Let \mathbf{V} be a vector field of class C^2 . Then

$$\text{rot}\mathbf{V} = \nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \mathbf{e}_z.$$



Maastricht University *Leading in Learning!*

Join the best at
the Maastricht University
School of Business and
Economics!

Visit us and find out why we are the best!
Master's Open Day: 22 February 2014

Top master's programmes

- 33rd place Financial Times worldwide ranking: MSc International Business
- 1st place: MSc International Business
- 1st place: MSc Financial Economics
- 2nd place: MSc Management of Learning
- 2nd place: MSc Economics
- 2nd place: MSc Econometrics and Operations Research
- 2nd place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

Maastricht
University is
the best specialist
university in the
Netherlands
(Elsevier)

www.mastersopenday.nl



Click on the ad to read more

By repeating this pattern we get for the double rotation that

$$\begin{aligned}\text{rot}(\text{rot } \mathbf{V}) &= \nabla \times (\nabla \times \mathbf{V}) \\ &= \left(\frac{\partial}{\partial y} \left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\} - \frac{\partial}{\partial z} \left\{ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right\} \right) \mathbf{e}_x \\ &\quad + \left(\frac{\partial}{\partial z} \left\{ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right\} - \frac{\partial}{\partial x} \left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\} \right) \mathbf{e}_y \\ &\quad + \left(\frac{\partial}{\partial x} \left\{ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right\} - \frac{\partial}{\partial y} \left\{ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right\} \right) \mathbf{e}_z,\end{aligned}$$

thus

$$\begin{aligned}\text{rot}(\text{rot } \mathbf{V}) &= \nabla \times (\nabla \mathbf{V}) \\ &= \left(-\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial x \partial y} \right) \mathbf{e}_x \\ &\quad + \left(-\frac{\partial^2 V_y}{\partial x^2} - \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial^2 V_y}{\partial z \partial y} + \frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial y \partial z} \right) \mathbf{e}_y \\ &\quad + \left(-\frac{\partial^2 V_z}{\partial x^2} - \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial x \partial z} \right) \mathbf{e}_z \\ &= -(\nabla^2 V_x, \nabla^2 V_y, \nabla^2 V_z) \\ &\quad + \frac{\partial}{\partial x} \left\{ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right\} \mathbf{e}_x \\ &\quad + \frac{\partial}{\partial y} \left\{ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right\} \mathbf{e}_y \\ &\quad + \frac{\partial}{\partial z} \left\{ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right\} \mathbf{e}_z \\ &= -\nabla^2 \mathbf{V} + \nabla(\nabla \cdot \mathbf{V}) \\ &= -(\text{div}(\text{grad } V_x), \text{div}(\text{grad } V_y), \text{div}(\text{grad } V_z)) + \text{grad}(\text{div } \mathbf{V}),\end{aligned}$$

and the formula follows by a rearrangement.

REMARK 1. Note that the formula can also be written

$$\nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V}) = \nabla^2 \mathbf{V}. \quad \diamond$$

REMARK 2. We note for completeness

$$\begin{aligned}\nabla(\nabla \cdot \mathbf{V}) &= \nabla \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) \\ &= \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z} \right) \mathbf{e}_x + \left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z} \right) \mathbf{e}_y \\ &\quad + \left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2} \right) \mathbf{e}_z. \quad \diamond\end{aligned}$$

Example 35.7 Find $\operatorname{div} \mathbf{V}$ and $\operatorname{rot} \mathbf{V}$ for each of the following vector fields on \mathbb{R}^3 .

- 1) $\mathbf{V}(x, y, z) = (xz, -y^2, 2x^2y).$
- 2) $\mathbf{V}(x, y, z) = (z + \sin y, -z + \cos y, 0).$
- 3) $\mathbf{V}(x, y, z) = (e^{xy}, \cos(xy), \cos(xz^2)).$
- 4) $\mathbf{V}(x, y, z) = (x^2 + yz, y^2 + xz, z^2 + xy).$
- 5) $\mathbf{V}(x, y, z) = (x + \operatorname{Arctan} y, 3x - z, 2^{yz}).$
- 6) $\mathbf{V}(x, y, z) = (xz^3, -2x^2yz, 2yz^4).$
- 7) $\mathbf{V}(x, y, z) = (\sinh(xyz), z, x).$
- 8) $\mathbf{V}(\operatorname{Arctan} z, \operatorname{Arctan} x, \operatorname{Arctan} y).$

A This is just a simple exercise in finding the divergence and the rotation.

D Insert into the formulæ

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} \quad \text{and} \quad \operatorname{rot} \mathbf{V} = \nabla \times \mathbf{V}.$$

I 1) We get for $\mathbf{V} = (xz, -y^2, 2x^2y),$

$$\operatorname{div} \mathbf{V} = z - 2y + 0 = z - 2y$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y^2 & 2x^2y \end{vmatrix} = (2x^2, x - 4xy, 0).$$

2) We get for $\mathbf{V} = (z + \sin y, -z + \cos y, 0),$

$$\operatorname{div} \mathbf{V} = 0 - \sin y + 0 = -\sin y$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z + \sin y & -z + \cos y & 0 \end{vmatrix} = (1, 1, -\cos y).$$

3) We get for $\mathbf{V} = (e^{xy}, \cos(xy), \cos(xz^2)),$

$$\operatorname{div} \mathbf{V} = y e^{xy} - x \sin(xy) - 2xz \sin(xz^2)$$

and

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \cos(xy) & \cos(xz^2) \end{vmatrix} = (0, z^2 \sin(xz^2), -y \sin(xy) - x e^{xy}).$$

4) We get for $\mathbf{V} = (x^2 + yz, y^2 + xz, z^2 + xy)$,

$$\text{div } \mathbf{V} = 2(x + y + z)$$

and

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + xz & z^2 + xy \end{vmatrix} = (x - x, y - y, z - z) = (0, 0, 0) = \mathbf{0}.$$

5) When

$$\mathbf{V} = (x + \text{Arctan } y, 3x - z, e^{yz}) = (x + \text{Arctan } y, 3x - z, e^{\ln 2 \cdot yz})$$

we find

$$\text{div } \mathbf{V} = 1 + 0 + \ln 2 \cdot y \cdot e^{\ln 2 \cdot yz} = 1 + \ln 2 \cdot y \cdot 2^{yz}$$

and

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + \text{Arctan } y & 3x - z & e^{\ln 2 \cdot yz} \end{vmatrix} = \left(\ln 2 \cdot z \cdot 2^{yz} + 1, 0, 3 - \frac{1}{1+y^2} \right).$$

6) If $\mathbf{V} = (xz^3, -2x^2yz, 2yz^4)$, then

$$\text{div } \mathbf{V} = z^3 - 2x^2z + 8yz^3$$

and

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y, 3xz^2, -4xyz).$$

7) If $\mathbf{V} = (\sinh(xyz), z, x)$, then

$$\operatorname{div} \mathbf{V} = yz \cosh(xyz)$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sinh(xyz) & z & x \end{vmatrix} = (-1, xy \cosh(xyz) - 1, -xz \cos(xyz)).$$

8) If $\mathbf{V} = (\operatorname{Arctan} z, \operatorname{Arctan} x, \operatorname{Arctan} y)$ then

$$\operatorname{div} \mathbf{V} = 0$$

and

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \operatorname{Arctan} z & \operatorname{Arctan} x & \operatorname{Arctan} y \end{vmatrix} = \left(\frac{1}{1+y^2}, \frac{1}{1+z^2}, \frac{1}{1+x^2} \right).$$

e-learning for kids

The number 1 MOOC for Primary Education
Free Digital Learning for Children 5-12
15 Million Children Reached

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.



Click on the ad to read more

Example 35.8 Find the divergence and the rotation of the vector field (the so-called Coulomb vector field),

$$\mathbf{V}(x, y, z) = \frac{1}{r^3} (x, y, z), \quad (x, y, z) \neq (0, 0, 0), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

[Cf. Example 33.14]

A Divergence and rotation.

D Compute $\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V}$ and $\operatorname{rot} \mathbf{V} = \nabla \times \mathbf{V}$.

I First note that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{og} \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

These are easy rules of calculations, by which

$$\begin{aligned} \operatorname{div} \mathbf{V} &= \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) \\ &= \left(\frac{1}{r^3} - \frac{3x}{r^4} \frac{\partial}{\partial x} \right) + \left(\frac{1}{r^3} - \frac{3y}{r^4} \frac{\partial}{\partial y} \right) + \left(\frac{1}{r^3} - \frac{3z}{r^4} \frac{\partial}{\partial z} \right) \\ &= \frac{3}{r^3} - \frac{3}{r^4} \cdot \frac{1}{r} (x^2 + y^2 + z^2) = \frac{3}{r^3} - \frac{3}{r^3} = 0, \end{aligned}$$

and

$$\begin{aligned} \operatorname{rot} \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} \\ &= \left\{ z \frac{\partial}{\partial y} \left(\frac{1}{r^3} \right) - y \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) \right\} \mathbf{e}_x + \left\{ x \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) - z \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) \right\} \mathbf{e}_y \\ &\quad + \left\{ y \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) - x \frac{\partial}{\partial y} \left(\frac{1}{r^3} \right) \right\} \mathbf{e}_z \\ &= -\frac{3}{r^4} \left(z \frac{\partial r}{\partial y} - y \frac{\partial r}{\partial z}, x \frac{\partial r}{\partial z} - z \frac{\partial r}{\partial x}, y \frac{\partial r}{\partial x} - x \frac{\partial r}{\partial y} \right) \\ &= -\frac{3}{r^4} \left(z \cdot \frac{y}{r} - y \cdot \frac{z}{r}, x \cdot \frac{z}{r} - z \cdot \frac{x}{r}, y \cdot \frac{x}{r} - x \cdot \frac{y}{r} \right) = (0, 0, 0). \end{aligned}$$

REMARK. A variant is

$$\begin{aligned}\text{rot } \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) & \frac{\partial}{\partial y} \left(\frac{1}{r^3} \right) & \frac{\partial}{\partial z} \left(\frac{1}{r^3} \right) \\ x & y & z \end{vmatrix} \\ &= -\frac{3}{r^4} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ x & y & z \end{vmatrix} = -\frac{3}{r^5} \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & x \\ x & y & z \end{vmatrix} = -\frac{3}{r^5} \mathbf{x} \times \mathbf{x} = \mathbf{0}. \quad \diamond\end{aligned}$$

Example 35.9 Choose the constants α and β , such that the vector field

$$\mathbf{V}(x, y, z) = (xyz)^\beta (x^\alpha, y^\alpha, z^\alpha), \quad (x, y, z) \in \mathbb{R}_+^3,$$

has zero rotation.

A Rotation free vector field.

D Compute $\text{rot } \mathbf{V}$.

I We get by a calculation,

$$\begin{aligned}\text{rot } \mathbf{V} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^{\alpha+\beta} y^\beta z^\beta & x^\beta y^{\alpha+\beta} & x^\beta y^\beta z^{\alpha+\beta} \end{vmatrix} \\ &= \beta (x^\beta y^{\beta-1} z^{\alpha+\beta} - x^\beta y^{\alpha+\beta} z^{\beta-1}) \mathbf{e}_x + \beta (x^{\alpha+\beta} y^\beta z^{\beta-1} - x^{\beta-1} y^\beta z^{\alpha+\beta}) \mathbf{e}_y \\ &\quad + \beta (x^{\beta-1} y^{\alpha+\beta} z^\beta - x^{\alpha+\beta} y^{\beta-1} z^\beta) \mathbf{e}_z \\ &= \beta (xyz)^\beta (y^{-1} z^\alpha - y^\beta z^{-1}, x^\alpha z^{-1} - x^{-1} z^\alpha, x^{-1} y^\alpha - x^\alpha y^{-1}).\end{aligned}$$

If $\beta = 0$, then the factor outside the vector is 0, and the vector field becomes rotation free in \mathbb{R}_+^3 . This corresponds to the vector field

$$\mathbf{V}(x, y, z) = (x^\alpha, y^\alpha, z^\alpha), \quad \alpha \in \mathbb{R},$$

where the condition $(x, y, z) \in \mathbb{R}_+^3$ assures that the vector is always defined.

The second possibility is that the vector is

$$(y^{-1} z^\alpha - y^\alpha z^{-1}, x^\alpha z^{-1} - x^{-1} z^\alpha, x^{-1} y^\alpha - x^\alpha y^{-1}) = \mathbf{0}.$$

This gives the condition $\alpha = -1$, in which case the vector field becomes

$$\mathbf{V}(x, y, z) = (x^{\beta-1}y^\beta z^\beta, x^\beta y^{\beta-1}z^\beta, x^\beta y^\beta z^{\beta-1}) = (xyz)^\beta \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right)$$

which is also free of rotation.

35.5.2 General examples

Example 35.10

A. Find the circulation C of the vector field

$$\mathbf{V}(x, y, z) = (z^2x, x^2y, y^2z), \quad (x, y, z) \in \mathbb{R}^3,$$

along the curve \mathcal{K} on the figure which is composed of three circle arcs of centre $\mathbf{0}$ and radius a , and which lie in the planes $z = 0$, $y = -x$ and $x = y\sqrt{3}$, respectively.

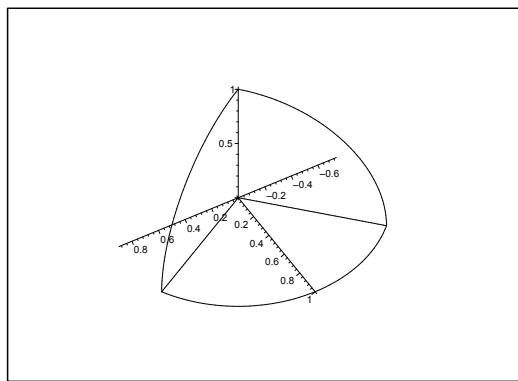


Figure 35.12: The curve \mathcal{K} for $a = 1$.

D. A circulation along a closed curve can either be calculated by using its definition as an ordinary *line integral*, or it can be transformed by means of *Stokes's theorem* to a surface integral. Most students are at their first encounter with this problem inclined to preferring the line integral, because it should now be better known, even when applications of *Stokes's theorem* very often give much simpler calculations. For that reason we shall here demonstrate both variants.

I 1. The circulation as a line integral.

Let us first give the parametric representations of the three arcs, which \mathcal{K} can be composed into in a natural way:

$$\begin{aligned} \mathcal{K}_1 : \quad & (x, y, z) = a(\cos \varphi, \sin \varphi, 0), & \varphi \in \left[\frac{\pi}{6}, \frac{3\pi}{4} \right], \\ \mathcal{K}_2 : \quad & (x, y, z) = a \left(-\frac{1}{\sqrt{2}} \cos \theta, \frac{1}{\sqrt{2}} \cos \theta, \sin \theta \right), & \theta \in \left[0, \frac{\pi}{2} \right], \\ \mathcal{K}_3 : \quad & (x, y, z) = a \left(\frac{\sqrt{3}}{2} \sin \theta, \frac{1}{2} \sin \theta, \cos \theta \right), & \theta \in \left[0, \frac{\pi}{2} \right]. \end{aligned}$$

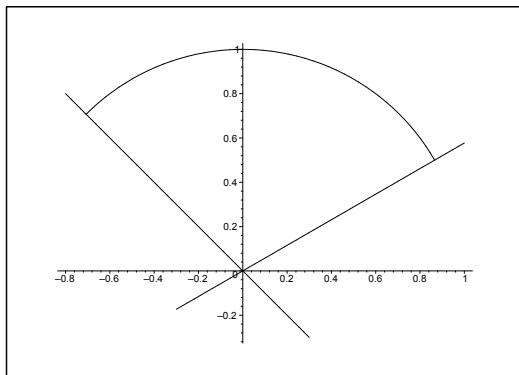


Figure 35.13: The projection onto the (x, y) -plane for $a = 1$.

Check here that all three curves are circular arcs of radius a and centre $\mathbf{0}$. Then check that the initial and the end points are the right ones, i.e. that the direction of the run through is correct. (Here this is left to the reader).

In order to avoid too many complications in the calculations we rewrite the integrand in the following way,

$$\begin{aligned}\mathbf{V} \cdot d\mathbf{x} &= z^2 x dx + x^2 y dy + y^2 z dz \\ &= \frac{1}{2} \{z^2 d(x^2) + x^2 d(y^2) + y^2 d(z^2)\}.\end{aligned}$$

BI NORWEGIAN BUSINESS SCHOOL

Empowering People. Improving Business.

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

www.bi.edu/master



Click on the ad to read more

By insertion we get the following confusing picture of formulæ, where there are lots of possibilities of making errors,

$$\begin{aligned}
 C &= \frac{1}{2} \int_{\mathcal{K}_1} + \frac{1}{2} \int_{\mathcal{K}_2} + \frac{1}{2} \int_{\mathcal{K}_3} (z^2 d(x^2) + x^2 d(y^2) + y^2 d(z^2)) \\
 &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{ 0 \cdot \frac{d(x^2)}{d\varphi} + a^2 \cos^2 \varphi \cdot 2a^2 \sin \varphi \cdot \cos \varphi + y^2 \cdot 0 \right\} d\varphi \\
 &\quad + \frac{1}{2} \int_0^{\frac{\pi}{2}} \left\{ a^2 \sin^2 \theta \cdot \frac{a^2}{2} (-2 \cos \theta \sin \theta) + \frac{a^2}{2} \cos^2 \theta \cdot \frac{a^2}{2} (-2 \cos \theta \sin \theta) \right. \\
 &\quad \left. + \frac{a^2}{2} \cos^2 \theta \cdot a^2 \cdot 2 \sin \theta \cos \theta \right\} d\theta \\
 &\quad + \frac{1}{2} \int_0^{\frac{\pi}{2}} \left\{ a^2 \cos^2 \theta \cdot \frac{3a^2}{4} \cdot 2 \cos \theta \sin \theta + \frac{3a^2}{4} \sin^2 \theta \cdot \frac{a^2}{4} \cdot 2 \cos \theta \sin \theta \right. \\
 &\quad \left. + \frac{a^2}{4} \sin^2 \theta \cdot a^2 (-2 \sin \theta \cos \theta) \right\} d\theta \\
 &= a^4 \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos^3 \varphi \sin \varphi d\varphi + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left(-\sin^3 \theta \cos \theta - \frac{1}{2} \cos^3 \theta \sin \theta + \cos^3 \theta \sin \theta \right) d\theta \\
 &\quad + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left\{ \frac{3}{2} \cos^3 \theta \sin \theta + \frac{3}{8} \sin^3 \theta \cos \theta - \frac{1}{2} \sin^3 \theta \cos \theta \right\} d\theta,
 \end{aligned}$$

i.e.

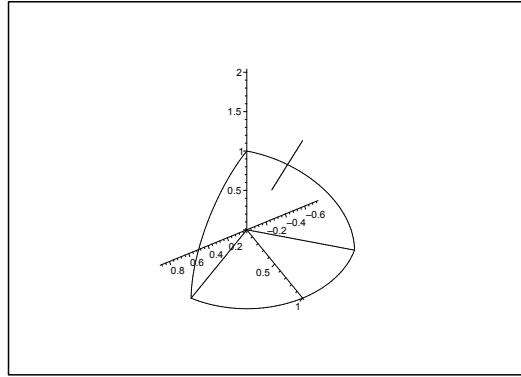
$$\begin{aligned}
 C &= a^4 \left[-\frac{1}{4} \cos^4 \varphi \right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left\{ \left(-1 + \frac{3}{8} - \frac{1}{2} \right) \sin^3 \theta \cos \theta \right. d\theta \\
 &= \frac{a^4}{4} \left\{ \left(\frac{\sqrt{3}}{2} \right)^4 - \left(-\frac{1}{\sqrt{2}} \right)^4 \right\} + \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \left\{ -\frac{9}{8} \sin^3 \theta \cos \theta + 2 \cos^3 \theta \sin \theta \right\} d\theta \\
 &= \frac{a^4}{4} \left(\frac{9}{16} - \frac{1}{4} \right) + \frac{a^4}{2} \left[-\frac{9}{8} \cdot \frac{1}{4} \sin^4 \theta - 2 \cdot \frac{1}{4} \cos^4 \theta \right]_0^{\frac{\pi}{2}} \\
 &= \frac{5a^4}{64} + \frac{a^4}{2} \left(-\frac{9}{32} - 0 + 0 + \frac{1}{2} \right) = \frac{a^4}{64} (5 - 9 + 16) = \frac{12a^4}{64} = \frac{3a^4}{16}.
 \end{aligned}$$

The calculations can be carried out, though they are far from simple.

I 2. Circulation by means of Stokes's theorem.

We shall now demonstrate that the calculations by an application of Stokes's theorem in this case is far simpler. First note that since all sub-curves are circular arcs of radius a and centre $\mathbf{0}$, they all lie on the sphere of radius a and centre $\mathbf{0}$. They bound a part \mathcal{F} of the sphere, where the orientation of \mathcal{K} forces the normal vector \mathbf{n} on \mathcal{F} to point away from $\mathbf{0}$ (the right hand convention), i.e.

$$\mathbf{n} = \frac{1}{a} (x, y, z), \quad \text{where } a = \sqrt{x^2 + y^2 + z^2}.$$

Figure 35.14: The surface \mathcal{F} for $a = 1$ with a single normal vector \mathbf{n} .

Furthermore,

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2x & x^2y & y^2z \end{vmatrix} = (2yz, 2zx, 2xy).$$

It is on the sphere most natural to use *spherical* coordinates. Then by Stokes's theorem and a reduction,

$$\begin{aligned} C &= \int_{\mathcal{K}} \mathbf{V} \cdot d\mathbf{x} = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} dS \\ &= \int_{\mathcal{F}} \frac{1}{a} (x, y, z) \cdot (2yz, 2zx, 2xy) dS = \frac{6}{a} \int_{\mathcal{F}} xyz dS \\ &= \frac{6}{a} \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \left\{ \int_0^{\frac{\pi}{2}} a \sin \theta \cos \varphi \cdot a \sin \theta \sin \varphi \cdot a \cos \theta a^2 \sin \theta d\theta \right\} d\varphi \\ &= 6a^4 \int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos \varphi \cdot \sin \varphi d\varphi \cdot \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos \theta d\theta \\ &= 6a^4 \left[\frac{1}{2} \sin^2 \varphi \right]_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cdot \left[\frac{1}{4} \sin^4 \theta \right]_0^{\frac{\pi}{2}} \\ &= 6a^4 \cdot \frac{1}{2} \left\{ \frac{1}{2} - \frac{1}{4} \right\} \cdot \frac{1}{4} = \frac{6a^4}{32} = \frac{3a^4}{16}, \end{aligned}$$

i.e. far easier calculations than by the line integral. \diamond

Example 35.11

A. An oriented curve \mathcal{K} is given as the intersection curve between

- the cylindric surface $x^2 + y^2 = ax$,

and

- half of the parabolic cylindric surface

$$z = \sqrt{4a^2 - ax}.$$

The orientation of \mathcal{K} obeys the right hand convention with the positive z -axis. Find the circulation along \mathcal{K} of the vector field

$$\mathbf{V}(x, y, z) = (3xy, 2x^2, -yz), \quad (x, y, z) \in \mathbb{R}^3.$$

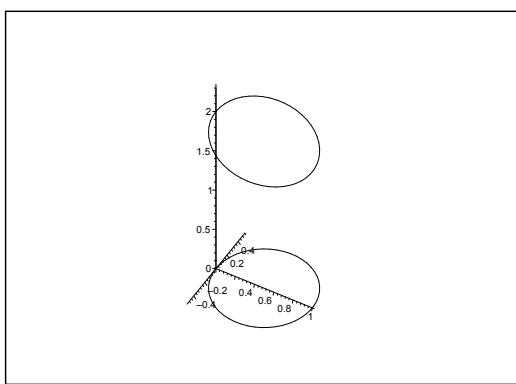


Figure 35.15: The curve \mathcal{K} and its projection onto the (x, y) -plane.

D. We shall here demonstrate three variants:

- 1) The line integral (the calculations are worse than those of Example 35.10).
- 2) Stokes's theorem by the surface

$$\mathcal{F}_1 : \quad z = \sqrt{4a^2 - ax}, \quad (x, y) \in B,$$

- 3) Stokes' sætning by the surface

$$\mathcal{F}_2 : \quad x^2 + y^2 + z^2 = (2a)^2, \quad (x, y) \in B, \quad z > 0.$$

I 1. As a general warning we shall first give the solution calculated as a *line integral*.

When \mathcal{K} is projected onto the (x, y) -plane we get the curve \mathcal{L} of the parametric representation

$$(x, y, z) = (\varrho \cos \varphi, \varrho \sin \varphi, 0) = a (\cos^2 \varphi, \cos \varphi \sin \varphi, 0), \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

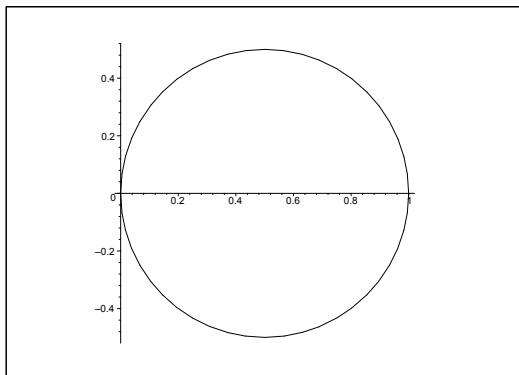


Figure 35.16: The parametric domain B is the disc in the (x, y) -plane of centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$. Here we have chosen $a = 1$.

Now $z = \sqrt{4a^2 - ax}$, so \mathcal{K} has the parametric representation

$$\mathcal{K} : \quad \mathbf{r}(\varphi) = (x, y, z) = a \left(\cos^2 \varphi, \cos \varphi \sin \varphi, \sqrt{4 - \cos^2 \varphi} \right), \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

Then we get the values of the vector field along the curve \mathcal{K} ,

$$\begin{aligned} \mathbf{V}(x, y, z) &= (3xy, 2x^2, -yz) \\ &= \left(3a \cos^2 \varphi \cdot a \cos \varphi \sin \varphi, 2a^2 \cos^4 \varphi, -a \cos \varphi \sin \varphi \cdot a \sqrt{4 - \cos^2 \varphi} \right) \\ &= a^2 \left(3 \cos^3 \varphi \sin \varphi, 2 \cos^4 \varphi, -\cos \varphi \cdot \sin \varphi \cdot \sqrt{4 - \cos^2 \varphi} \right). \end{aligned}$$

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now



Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Hence,

$$\mathbf{r}'(\varphi) = a \left(-2 \sin \varphi \cdot \cos \varphi, \cos^2 \varphi - \sin^2 \varphi, \frac{\cos \varphi \cdot \sin \varphi}{\sqrt{4 - \cos^2 \varphi}} \right).$$

In order to avoid too big calculations we first calculate the *integrand* separately,

$$\begin{aligned} \mathbf{V} \cdot \mathbf{r}'(\varphi) &= a^3 (3 \cos^3 \varphi \sin \varphi \cdot (-2 \sin \varphi \cos \varphi) + 2 \cos^4 \varphi \cdot (\cos^2 \varphi - \sin^2 \varphi) - \cos^2 \varphi \sin^2 \varphi) \\ &= a^3 \left(3 \cos^2 \varphi \cos \varphi \sin \varphi \cdot (-\sin 2\varphi) + 2(\cos^2 \varphi)^2 \cdot \cos 2\varphi - \frac{1}{4} (2 \cos \varphi \sin \varphi)^2 \right) \\ &= a^3 \left\{ 3 \frac{1+\cos 2\varphi}{2} \frac{\sin 2\varphi}{2} \cdot (-\sin 2\varphi) + 2 \left(\frac{1+\cos 2\varphi}{2} \right)^2 \cos 2\varphi - \frac{1}{4} \sin^2 2\varphi \right\} \\ &= \frac{a^3}{4} \{ -3(1 + \cos 2\varphi) \sin^2 2\varphi + 2(1 + \cos 2\varphi)^2 \cos 2\varphi - \sin^2 2\varphi \} \\ &= \frac{a^3}{4} \{ -3 \sin^2 2\varphi - 3 \cos 2\varphi \sin^2 2\varphi + 2(1 + 2 \cos 2\varphi + \cos^2 2\varphi) \cos 2\varphi - \sin^2 2\varphi \} \\ &= \frac{a^3}{4} \{ -4 \sin^2 2\varphi - 3 \cos 2\varphi \sin^2 2\varphi + 2 \cos 2\varphi + 4 \cos^2 2\varphi + 2 \cos^2 2\varphi \cdot \cos 2\varphi \} \\ &= \frac{a^3}{4} \{ 4(\cos^2 2\varphi - \sin^2 2\varphi) - 5 \cos 2\varphi \sin^2 2\varphi + 2 \cos 2\varphi (\sin^2 2\varphi + \cos^2 2\varphi) + 2 \cos 2\varphi \}, \end{aligned}$$

i.e.

$$\mathbf{V} \cdot \mathbf{r}'(\varphi) = \frac{a^3}{4} \{ 4 \cos 4\varphi - 5 \cos 2\varphi \sin^2 2\varphi + 4 \cos 2\varphi \},$$

which is a fairly tough calculation. At the same time we see why we do not immediately insert the expression into the integral.

However, after this reduction the circulation becomes easy to calculate,

$$\begin{aligned} C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{V} \cdot \mathbf{r}'(\varphi) \, d\varphi \\ &= \frac{a^3}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ 4 \cos 4\varphi - 5 \cos 2\varphi \sin^2 2\varphi + 4 \cos 2\varphi \} \, d\varphi \\ &= \frac{a^3}{4} \left\{ [\sin 4\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 5 \left[\frac{1}{2} \cdot \frac{1}{3} \sin^3 2\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + [2 \sin 2\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right\} \\ &= 0, \end{aligned}$$

and we see that after so much trouble we only get zero as our result.

- I.** As an introduction to the applications of *Stokes's theorem* we first calculate

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & 2x^2 & -yz \end{vmatrix} = (-z, 0, 4x - 3x) = (-z, 0, x),$$

which looks promising, because $\text{rot } \mathbf{V}$ is much simpler than \mathbf{V} .

Then we shall choose a surface which has \mathcal{K} as its boundary curve. This surface is of course not unique.

I 2. The most obvious possibility is to choose the parabolic cylindric surface

$$\mathcal{F}_1 : z = \sqrt{4a^2 - ax}, \quad (x, y) \in B,$$

where the parametric domain B is best described in *polar* coordinates,

$$B = \left\{ (\varrho, \varphi) \mid \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], 0 \leq \varrho \leq a \cos \varphi \right\},$$

while (x, y, z) still denotes the *rectangular* coordinates.

Because \mathcal{F}_1 is given as a *graph*, the parametric representation is

$$\mathbf{r}(x, y) = (x, y, z) = \left(x, y, \sqrt{4a^2 - ax} \right), \quad (x, y) \in B,$$

from which we get the tangential vector fields

$$\mathbf{r}'_x = \left(1, 0, \frac{1}{2} \frac{-a}{\sqrt{4a^2 - ax}} \right) = \left(1, 0, -\frac{a}{2z} \right), \quad \text{and} \quad \mathbf{r}'_y = (0, 1, 0),$$

and the corresponding normal vector field

$$\mathbf{N}(x, y) = \mathbf{r}'_x \times \mathbf{r}'_y = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{a}{2z} \\ 0 & 1 & 0 \end{vmatrix} = \left(\frac{a}{2z}, 0, 1 \right).$$

We see that $\mathbf{N}(x, y)$ and the orientation of \mathcal{K} satisfy the right hand convention $\mathbf{N} \cdot \mathbf{e}_z = 1$, so we have the correct orientation.

An application of *Stokes's theorem* then gives

$$\begin{aligned} C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}_1} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS = \int_B \left(\frac{a}{2z}, 0, 1 \right) \cdot (-z, 0, x) \, dx \, dy \\ &= \int_B \left(-\frac{1}{2}a + x \right) \, dx \, dy = -\frac{a}{2} \text{area}(B) + \int_B x \, dx \, dy \\ &= -\frac{a}{2} \cdot \pi \left(\frac{a}{2} \right)^2 + \int_{-\frac{\pi}{2}}^{\pi} 2 \left\{ \int_0^{a \cos \varphi} \varrho \cos \varphi \cdot \varrho \, d\varrho \right\} \, d\varphi \\ &= -\frac{\pi a^3}{8} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi \cdot \left[\frac{1}{3} \varrho^3 \right]_0^{a \cos \varphi} \, d\varphi = -\frac{\pi a^3}{8} + \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \varphi \, d\varphi \\ &= -\frac{\pi a^3}{8} + \frac{2a^3}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\varphi}{2} \right)^2 \, d\varphi \\ &= -\frac{\pi a^3}{8} + \frac{2a^3}{3} \cdot \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 + 2\cos 2\varphi + \cos^2 2\varphi) \, d\varphi \\ &= -\frac{\pi a^3}{8} + \frac{a^3}{6} \left\{ \frac{\pi}{2} + 0 + \frac{\pi}{4} \right\} = 0. \end{aligned}$$

I 3. The calculations of **I 2.** were much simpler than the calculations of **I 1.**, although they still gave some trigonometric problems. The question is now, if it is possible to choose another surface \mathcal{F} with \mathcal{K} as its boundary curve, such that the calculations become even more easy. We shall now show that this is possible, though far from obvious.

First we note that since K satisfies the two conditions

$$\text{a) } x^2 + y^2 = ax \quad \text{and} \quad \text{b) } z^2 = 4a^2 - ax,$$

the parametric representation of \mathcal{F} must also satisfy

$$(x^2 + y^2) + z^2 = ax + (4a^2 - ax) = 4a^2 = (2a)^2, \quad z > 0,$$

i.e. \mathcal{K} lies on the half sphere of centre $\mathbf{0}$ and radius $2a$ and $z > 0$. We choose \mathcal{F}_2 as that part of this half sphere which lies above the parametric domain B :

$$\mathcal{F}_2 : \quad x^2 + y^2 + z^2 = (2a)^2, \quad (x, y) \in B, \quad z \geq 0.$$

The sphere of centre $\mathbf{0}$ and radius $2a$ has the unit normal vector

$$\mathbf{n} = \frac{1}{2a} (x, y, z),$$

hence we have on \mathcal{F}_2

$$\mathbf{n} \cdot \operatorname{rot} \mathbf{V} = \frac{1}{2a} (x, y, z) \cdot (-z, 0, x) = 0.$$

Brain power

By 2020, wind could provide one-tenth of our planet's electricity needs. Already today, SKF's innovative know-how is crucial to running a large proportion of the world's wind turbines.

Up to 25 % of the generating costs relate to maintenance. These can be reduced dramatically thanks to our systems for on-line condition monitoring and automatic lubrication. We help make it more economical to create cleaner, cheaper energy out of thin air.

By sharing our experience, expertise, and creativity, industries can boost performance beyond expectations.

Therefore we need the best employees who can meet this challenge!

The Power of Knowledge Engineering

Plug into The Power of Knowledge Engineering.
Visit us at www.skf.com/knowledge

SKF



Click on the ad to read more

By an application of *Stokes's theorem* the calculation of the circulation is now reduced to a triviality,

$$C = \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}_2} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS = \int_{\mathcal{F}_2} 0 \, dS = 0,$$

in which we shall not even insert a parametric representation followed by some reduction theorem!
 ◇

35.5.3 Examples of applications of Stokes's theorem

Example 35.12 Apply in each of the following cases Stokes's theorem to find the circulation of the given vector field $\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ along the given closed curve \mathcal{K} , we one shall indicate the direction of the curve on a figure.

1) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y \sinh(xy) + z^2, x \sinh(xy) + z^2 + x, 2x^2 + 2y^2)$$

along the closed curve \mathcal{K} given by $x^2 + y^2 = 1, z = 1$.

2) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y^2 + z^4, (x - a)^2 + z^4, x^2 + y^2)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = bx, \quad z^4 = a^2 - x^2 - y^2, \quad \text{where } b < a \text{ and } z > 0.$$

3) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y, x - yz, x^2)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = 1, \quad z = 4 - 2x^2 - y^2.$$

4) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (yz - 2y, xz + 4x, xy)$$

along the closed curve \mathcal{K} given by

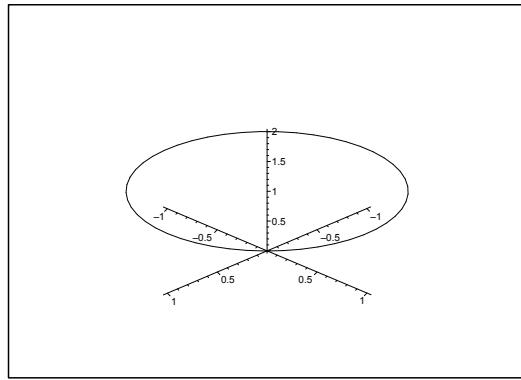
$$\varrho = 1 + \cos \varphi, \quad z = \sqrt{4 - \varrho^2} \quad \text{for } \varphi \in [-\pi, \pi].$$

5) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (y^2 - 2xy, 2xy, 2az + 3a^2)$$

along the closed curve \mathcal{K} given by

$$x^2 + y^2 = ax \quad z = a - \sqrt{x^2 + y^2}.$$

Figure 35.17: The curve \mathcal{K} of **Example 35.12.1.**

6) The circulation of the vector field $\mathbf{V}(x, y, z) = (z, x, y)$ along the boundary of the triangle \mathcal{K} of vertices $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$.

7) The circulation of the vector field $\mathbf{V}(x, y, z) = (y, z, x)$ along the closed curve \mathcal{K} given by

$$x^2 + y^2 + z^2 = a^2, \quad z = y - x.$$

8) The circulation of the vector field $\mathbf{V}(x, y, z) = (y + \sin z, x, x \cos z)$ along the closed curve \mathcal{K} given by

$$x^2 + y^2 + z^2 = 1, \quad z = x.$$

9) The circulation of the vector field $\mathbf{V}(x, y, z) = (z^2, ax + z^2, 2x^2 + 2y^2)$ along the closed curve \mathcal{K} given by

$$x^2 + y^2 = a^2, \quad z = a.$$

10) The circulation of the vector field

$$\mathbf{V}(x, y, z) = (-y(x^2 + 2z^2), x(x^2 + 4y^2 + 2z^2), z^3)$$

along the closed curve \mathcal{K} given by

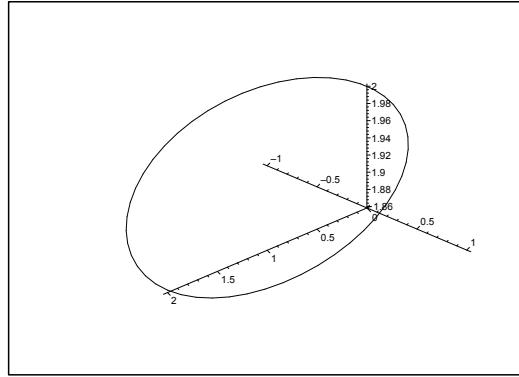
$$x^2 + y^2 = a^2, \quad z = a.$$

A Circulation of vector fields.

D Sketch the curve and choose an orientation of it. Compute $\text{rot } \mathbf{V}$, and choose the surface \mathcal{F} . Finally, apply Stokes's theorem.

I 1) The most obvious choice of the surface is

$$\mathcal{F} = \{(x, y, 1) \mid x^2 + y^2 \leq 1\}$$

Figure 35.18: The curve \mathcal{K} of **Example 35.12.2** for $a = 4$ and $b = 2$.

where the orientation is given by the normal vector $\mathbf{n} = (0, 0, 1)$. Hence

$$\mathbf{n} \cdot \text{rot } \mathbf{V} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \sinh(xy) + xy \cosh(xy) + 1 - \sinh(xy) - xy \cosh(xy) = 1.$$

Then according to Stokes's theorem the circulation is

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS = 1 \cdot \text{area}(\mathcal{F}) = \pi.$$

- 2) If we choose the orientation of \mathcal{K} , such that the projection of the curve onto the XY -plane has a positive orientation, it is quite natural to choose the corresponding surface

$$\mathcal{F} = \left\{ (x, y, \sqrt[4]{z^2 - x^2 - y^2}) \mid x^2 + y^2 \leq bx \right\},$$

with the normal vector $\mathbf{n} \cdot \mathbf{e}_z > 0$. We first find

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^4 & (x-a)^2 + z^4 & x^2 + y^2 \end{vmatrix} = (2y - 4z^3, 4z^3 - 2x, 2(x-a) - 2y).$$

Then calculate the normal vector of the surface \mathcal{F} ,

$$\begin{aligned} \mathbf{N}(x, y) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{1}{2}x(\sqrt[4]{a^2 - x^2 - y^2})^{-3} \\ 0 & 1 & -\frac{1}{2}y(\sqrt[4]{a^2 - x^2 - y^2})^{-3} \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{1}{2}\frac{x}{z^3} \\ 0 & 1 & -\frac{1}{2}\frac{y}{z^3} \end{vmatrix} \\ &= \left(\frac{1}{2}\frac{x}{z^3}, \frac{1}{2}\frac{y}{z^3}, 1 \right), \end{aligned}$$

hence

$$\begin{aligned}\mathbf{n} \cdot \operatorname{rot} \mathbf{V} &= \frac{1}{\|\mathbf{N}(x, y)\|} \left\{ \frac{1}{2} \frac{x}{z^3} (2y - 4z^3) + \frac{1}{2} \frac{y}{z^3} (4z^3 - 2x) + 2(x - a) - 2y \right\} \\ &= \frac{1}{\|\mathbf{N}(x, y)\|} \left\{ \frac{xy}{z^3} - 2x + 2y - \frac{yx}{z^3} + 2x - 2a - 2y \right\} = -\frac{2a}{\|\mathbf{N}(x, y)\|}.\end{aligned}$$



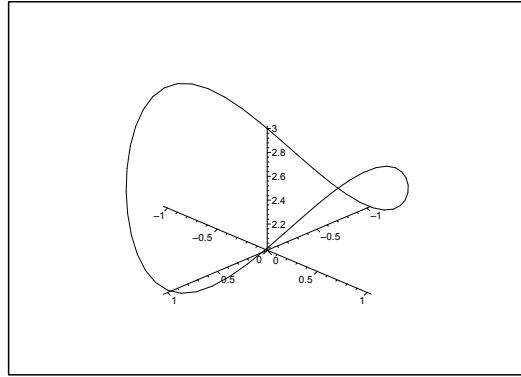
What do you want to do?

No matter what you want out of your future career, an employer with a broad range of operations in a load of countries will always be the ticket. Working within the Volvo Group means more than 100,000 friends and colleagues in more than 185 countries all over the world. We offer graduates great career opportunities – check out the Career section at our web site www.volvologroup.com. We look forward to getting to know you!

VOLVO
AB Volvo (publ)
www.volvologroup.com

VOLVO TRUCKS | RENAULT TRUCKS | MACK TRUCKS | VOLVO BUSES | VOLVO CONSTRUCTION EQUIPMENT | VOLVO PENTA | VOLVO AERO | VOLVO IT
VOLVO FINANCIAL SERVICES | VOLVO 3P | VOLVO POWERTRAIN | VOLVO PARTS | VOLVO TECHNOLOGY | VOLVO LOGISTICS | BUSINESS AREA ASIA



Figure 35.19: The curve \mathcal{K} of Example 35.12.3

Choose the parameter domain

$$B = \{(x, y) \mid x^2 + y^2 \leq bx\}.$$

According to Stokes's theorem the circulation of \mathbf{V} along the curve \mathcal{K} is given by

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS = \int_B \frac{-2a}{\|\mathbf{N}(x, y)\|} \|\mathbf{N}(x, y)\| \, dx \, dy \\ &= -2a \operatorname{area}(B) = -2a \cdot \pi \left(\frac{b}{2}\right)^2 = -\frac{\pi}{2} ab^2. \end{aligned}$$

- 3) Here we choose the surface

$$\mathcal{F} = \{(x, y, 2 - 2x^2 - y^2) \mid x^2 + y^2 \leq 1\},$$

where the boundary curve \mathcal{K} is oriented such that it is positive in the XY -plane. Then $\mathbf{n} \cdot \mathbf{e}_z > 0$ for the normal vector on \mathcal{F} .

Then by computing,

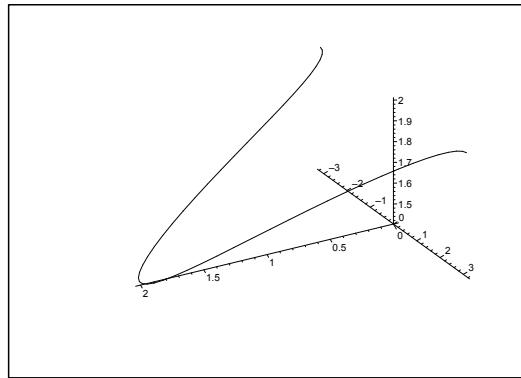
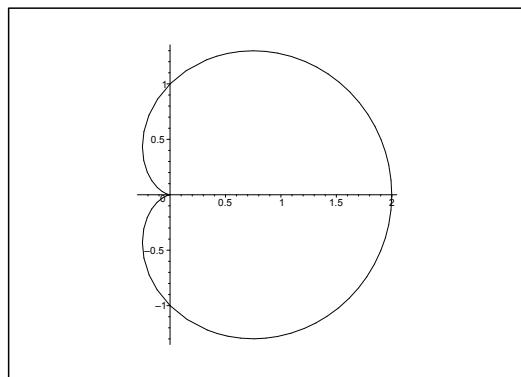
$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - yz & x^2 \end{vmatrix} = (y, -2x, 1 - 1) = (y, -2x, 0).$$

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -4x \\ 0 & 1 & -2y \end{vmatrix} = (4x, 2y, 1),$$

hence

$$\mathbf{n} \cdot \operatorname{rot} \mathbf{V} = \frac{1}{\|\mathbf{N}(x, y)\|} (2x, 2y, 1) \cdot (y, -2x, 0) = 0.$$

Figure 35.20: The curve \mathcal{K} of Example 35.12.4Figure 35.21: The parameter domain B of Example 35.12.4.

Then it is easy to find the circulation,

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} dS = 0.$$

- 4) Choose the surface which is given in semi polar coordinates by

$$\mathcal{F} = \left\{ (\varrho, \varphi, \sqrt{4 - \varrho^2}) \mid 0 \leq \varrho \leq 1 + \cos \varphi, \varphi \in [-\pi, \pi] \right\} = \{(\varrho, \varphi, \sqrt{4 - \varrho^2}) \mid (\varrho, \varphi) \in B\},$$

where the parameter domain

$$B = \{(\varrho, \varphi) \mid 0 \leq \varrho \leq 1 + \cos \varphi, \varphi \in [-\pi, \pi]\}$$

lies inside the cardioid.

Choose the orientation of \mathcal{K} such that the projection of \mathcal{K} onto the cardioid is run through in

the the positive sense of the plane. Then by a computation,

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz - 2y & xz + 4x & xy \end{vmatrix} = (x - x, y - y, z + 4 - z + 2) = (0, 0, 6).$$

The surface \mathcal{F} is described in rectangular coordinates (though in polar parameters) by

$$(x, y, z) = (\varrho \cos \varphi, \varrho \sin \varphi, \sqrt{4 - \varrho^2}).$$

This rectangular description is necessary when we compute the normal vector by the usual method,

$$\mathbf{N}(\varrho, \varphi) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos \varphi & \sin \varphi & -\frac{\varrho}{\sqrt{4 - \varrho^2}} \\ -\varrho \sin \varphi & \varrho \cos \varphi & 0 \end{vmatrix} = \left(\frac{\varrho^2 \cos \varphi}{\sqrt{4 - \varrho^2}}, \frac{\varrho^2 \sin \varphi}{\sqrt{4 - \varrho^2}}, \varrho \right),$$

hence

$$\mathbf{n} \cdot \text{rot } \mathbf{V} = \frac{6\varrho}{\|\mathbf{B}(\varrho, \varphi)\|}.$$

The circulation along \mathcal{K} is

$$\begin{aligned} \int_{\mathcal{F}} \frac{6\varrho}{\|\mathbf{N}(\varrho, \varphi)\|} dS &= \int_B 6\varrho d\varrho d\varphi = \int_{-\pi}^{\pi} \left\{ \int_0^{1+\cos \varphi} 6\varrho d\varrho \right\} d\varphi \\ &= \int_{-\pi}^{\pi} 3(1 + \cos \varphi)^2 d\varphi = 3 \int_{-\pi}^{\pi} (1 + 2\cos \varphi + \cos^2 \varphi) d\varphi \\ &= 3 \int_{-\pi}^{\pi} \{1 + \cos^2 \varphi\} d\varphi + 0 = \frac{3}{2} \cdot 3 \cdot 2\pi = 9\pi. \end{aligned}$$

5) Choose the surface

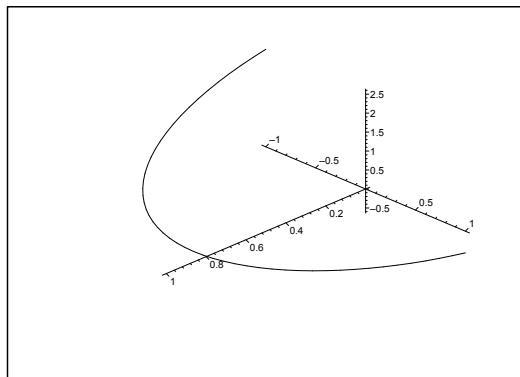
$$\mathcal{F} = \{(x, y, a - \sqrt{x^2 + y^2}) \mid x^2 + y^2 \leq ax\} = \{(x, y, a - \sqrt{x^2 + y^2}) \mid (x, y) \in B\},$$

where the parameter domain B is described in polar coordinates of the plane,

$$B = \left\{ (\varrho, \varphi) \mid \varrho \leq a \cos \varphi, \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}.$$

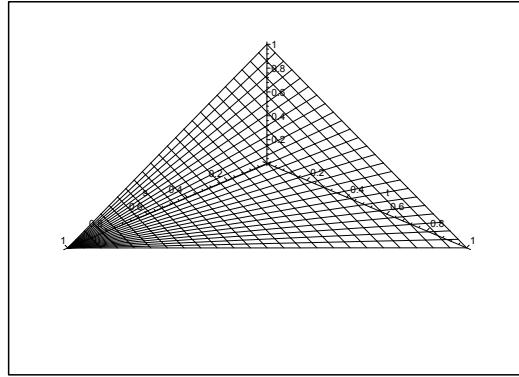
Then

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - 2xy & 2xy & 2az + 3a^2 \end{vmatrix} = (0, 0, 2y - 2y + 2x) = (0, 0, 2x),$$

Figure 35.22: The curve \mathcal{K} of Example 35.12.5 for $a = 1$.

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -\frac{x}{\sqrt{x^2 + y^2}} \\ 0 & 1 & -\frac{y}{\sqrt{x^2 + y^2}} \end{vmatrix} = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 1 \right).$$

Figure 35.23: The surface \mathcal{F} of Example 35.12.6.

The circulation of \mathbf{V} along \mathcal{K} is then by Stokes's theorem,

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \mathbf{V} \, dx \, dy = \int_B 2x \, dx \, dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{a \cos \varphi} 2\rho \cos \varphi \cdot \rho \, d\rho \right\} d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3} \rho^3 \right]_0^{a \cos \varphi} \cos \varphi \, d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3} a^3 \cos^4 \varphi \, d\varphi = \frac{2}{3} a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\varphi}{2} \right)^2 \, d\varphi \\ &= \frac{1}{6} a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2 \cos 2\varphi + \frac{1 + \cos 4\varphi}{2} \right) \, d\varphi = \frac{1}{6} a^3 \cdot \frac{3}{2} \pi = \frac{\pi}{4} a^3. \end{aligned}$$

6) First calculate

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1, 1, 1).$$

We choose naturally the surface \mathcal{F} in the following way

$$\mathcal{F} = \{(x, y, 1 - x - y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$$

with the normal vector

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1).$$

The circulation of the vector field along \mathcal{K} is then

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \text{rot } \mathbf{V} \, dx \, dy = 3 \text{ area}(B) = \frac{3}{2}.$$

7) Choose \mathcal{F} as the plane surface (a disc)

$$\mathcal{F} = \{(x, y, y - x) \mid x^2 + y^2 + (y - x)^2 \leq a^2\}$$

with $\mathbf{N} \cdot \mathbf{e}_z > 0$ by the chosen orientation. We get

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1)$$

and

$$\mathbf{N}(x, y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (1, -1, 1),$$

hence

$$\mathbf{N} \cdot \text{rot } \mathbf{V} = (-1, -1, -1) \cdot (1, -1, 1) = -1.$$

The projection B of \mathcal{F} onto the XY -plane is given by

$$a^2 \geq x^2 + y^2 + (y - x)^2 = x^2 + y^2 - \frac{1}{2}(x - y)^2 + \frac{3}{2}(x - y)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2 + 3\left(\frac{x-y}{\sqrt{2}}\right)^2,$$

which describes the interior of an elliptic disc with the directions of the axes

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

and the half axes a and $\frac{a}{\sqrt{3}}$.

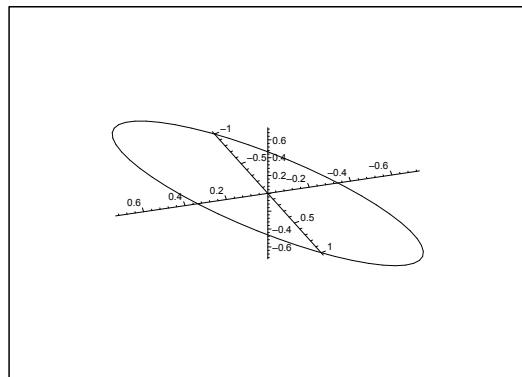
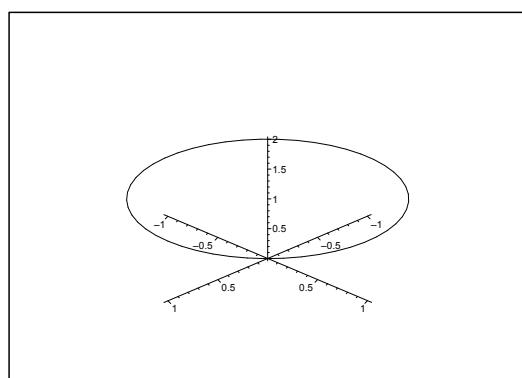
The circulation is

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \text{rot } \mathbf{V} \, dx \, dy = -\text{area}(B) = -\pi \cdot \frac{a^2}{\sqrt{3}}.$$

8) Since

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + \sin z & x & x \cos z \end{vmatrix} = (0, \cos z - \cos z, 1 - 1) = (0, 0, 0),$$

the circulation is trivially 0 by Stokes's theorem.

Figure 35.24: The curve \mathcal{K} of **Example 35.12.8**.Figure 35.25: The curve \mathcal{K} of **Example 35.12.9** and **Example 35.12.10** for $a = 1$.

9) Here

$$\text{rot } \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & ax + z^2 & 2x^2 + 2y^2 \end{vmatrix} = (4y - 2z, 2z - 4x, a).$$

We have $\mathbf{n} = (0, 0, 1)$ in the chosen orientation of \mathcal{K} , so the circulation becomes

$$\int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} ds = \int_{\mathcal{F}} \mathbf{n} \cdot \text{rot } \mathbf{V} dS = \int_B a dx dy = a \cdot \text{area}(B) = \pi a^3.$$

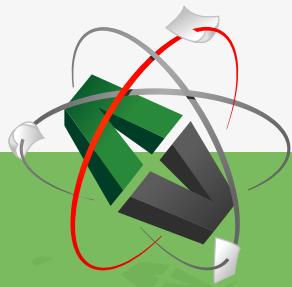
ALTERNATIVELY it is here also easy to compute the circulation as a line integral. We choose the parametric description

$$(x, y, z) = (a \cos \varphi, a \sin \varphi, a), \quad \varphi \in [0, 2\pi],$$

for \mathcal{K} . Then we get the tangent vector field

$$\mathbf{t} ds = (-a \sin \varphi, a \cos \varphi, 0) d\varphi,$$

This e-book
is made with
SetaPDF



PDF components for **PHP** developers

www.setasign.com



hence

$$\begin{aligned}
 \int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds &= \int_0^{2\pi} (-a \sin \varphi, a \cos \varphi, 0) \cdot (a^2, a^2 \cos \varphi + a^2, 2a^2) \, d\varphi \\
 &= a^3 \int_0^{2\pi} (-\sin \varphi + \cos^2 \varphi + \cos \varphi + 0) \, d\varphi \\
 &= a^3 \int_0^{2\pi} \cos^2 \varphi \, d\varphi = a^3 \int_0^{2\pi} \sin^2 \varphi \, d\varphi \\
 &= a^3 \int_0^{2\pi} \frac{\cos^2 \varphi + \sin^2 \varphi}{2} \, d\varphi = \frac{a^3}{2} \int_0^{2\pi} 1 \, d\varphi = \pi a^3.
 \end{aligned}$$

- 10) The surface \mathcal{F} is the same as in **Example 35.12.9**, so we can reuse $\mathbf{n} = \mathbf{N} = (0, 0, 1)$ and

$$\mathbf{n} \cdot \operatorname{rot} \mathbf{V} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 3x^2 + 4y^2 + 2z^2 + x^2 + 2z^2 = 4(x^2 + y^2 + z^2).$$

Since $z = a$ on \mathcal{F} , the circulation becomes

$$\begin{aligned}
 \int_{\mathcal{K}} \mathbf{t} \cdot \mathbf{V} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS = 4 \int_B (x^2 + y^2 + a^2) \, dx \, dy \\
 &= 4 \int_0^{2\pi} \left\{ \int_0^a \varrho^2 \cdot \varrho \, d\varrho \right\} \, d\varphi + 4a^2 \cdot \pi a^2 = 2\pi a^4 + 4\pi a^4 = 6\pi a^4.
 \end{aligned}$$

Example 35.13 In each of the following cases apply Stokes's theorem to compute the flux

$$\int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS$$

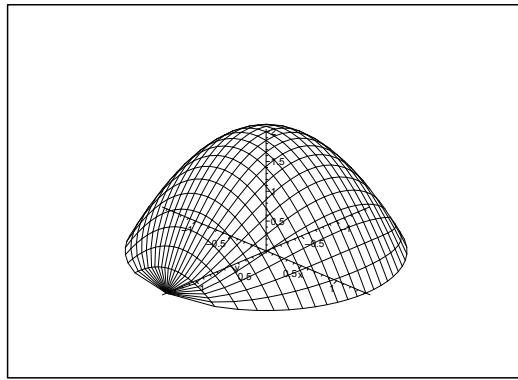
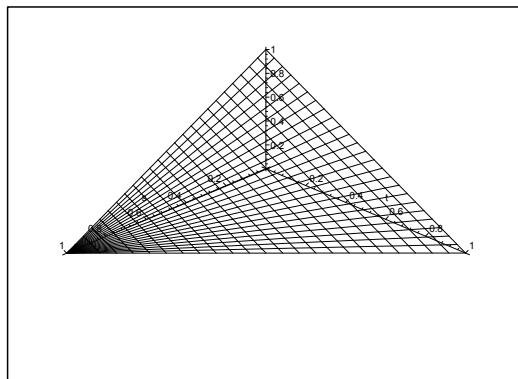
of the rotation of the given vector field $\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ through the surface \mathcal{F} , where we shall choose an orientation, which is indicated on a figure.

- 1) The flux of $\mathbf{V}(x, y, z) = (y^2, x - 2xz, -xy)$ through the surface \mathcal{F} given by $z = \sqrt{a^2 - x^2 - y^2}$ for $x^2 + y^2 \leq a^2$.
- 2) The flux of $\mathbf{V}(x, y, z) = (2y^3, x^2 + yz, x)$ through the triangle \mathcal{F} with the vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.
- 3) The flux of $\mathbf{V}(x, y, z) = (y + z^2, z \ln(1 - x^2 + y^2), \operatorname{Arctan}(xyz))$ through the surface \mathcal{F} given by $z = 1 - x^2 - y^2$ for $x^2 + y^2 \leq 1$.

A Flux computed by means of Stokes's theorem.

D Sketch the surface \mathcal{F} and the boundary curve \mathcal{K} and choose an orientation. (It has not been possible for me to sketch the orientation of the figures). Finally, exploit that the flux according to Stokes's theorem is given by

$$(35.2) \quad \int_{\mathcal{F}} (\nabla \times \mathbf{V}) \cdot \mathbf{n} \, dS = \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds, \quad \text{where } \mathcal{K} = \text{"}\partial\mathcal{F}\text{"}.$$

Figure 35.26: The surface \mathcal{F} of **Example 35.13.1** for $a = \sqrt{2}$.Figure 35.27: The surface \mathcal{F} of **Example 35.13.2**

I 1) The boundary curve is the circle in the XY -plane of centrum $(0, 0)$ and radius a . Choose the parametric description

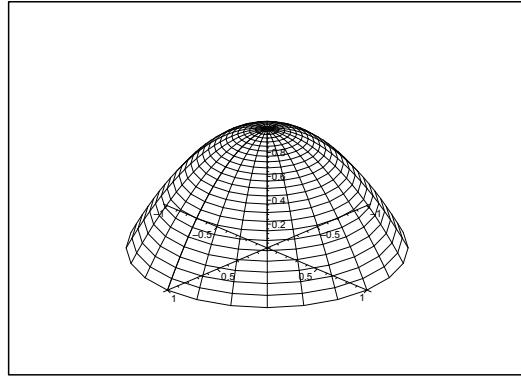
$$(x, y, z) = (a \cos \varphi, a \sin \varphi, 0), \quad \varphi \in [0, 2\pi],$$

for \mathcal{K} in \mathbb{R}^3 corresponding to a positive orientation. Since $\mathbf{t} = (-\sin \varphi, \cos \varphi, 0)$ and $ds = a d\varphi$, the flux is according to (35.2) given by

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds &= \int_0^{2\pi} (a^2 \sin^2 \varphi, a \cos \varphi, -a^2 \sin \varphi \cos \varphi) \cdot (-\sin \varphi, \cos \varphi, 0) a d\varphi \\ &= \int_0^{2\pi} \{-a^3 \sin^3 \varphi + a^2 \cos^2 \varphi + 0\} d\varphi = 0 + a^2 \cdot \frac{1}{2} \cdot 2\pi + 0 = \pi a^2. \end{aligned}$$

2) The boundary curve is the boundary of the triangle with e.g. the parametric description

$$\left\{ \begin{array}{l} \mathcal{K}_1 : (x, y, z) = (1-t, t, 0), \quad t \in [0, 1], \\ \mathcal{K}_2 : (x, y, z) = (0, 1-t, t), \quad t \in [0, 1], \\ \mathcal{K}_3 : (x, y, z) = (t, 0, 1-t), \quad t \in [0, 1], \end{array} \right.$$

Figure 35.28: The surface \mathcal{F} of Example 35.13.3.

where $ds = \sqrt{2} dt$ on each of the three sub-curves. According to (35.2) the flux is given by

$$\begin{aligned}
\int_{\mathcal{F}} (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS &= \int_{\mathcal{K}_1} \mathbf{V} \cdot \mathbf{t} ds + \int_{\mathcal{K}_2} \mathbf{V} \cdot \mathbf{t} ds + \int_{\mathcal{K}_3} \mathbf{V} \cdot \mathbf{t} ds \\
&= \int_0^1 (2t^3, (1-t)^2, 1-t) \cdot (-1, 1, 0) dt \\
&\quad + \int_0^1 (2(1-t)^3, (1-t)t, 0) \cdot (0, -1, 1) dt \\
&\quad + \int_0^1 (0, t^2, t) \cdot (1, 0, -1) dt \\
&= \int_0^2 \{-2t^3 + (1-t)^2\} dt + \int_0^1 \{-(1-t)t\} dt + \int_0^1 (-t) dt \\
&= \int_0^1 \{-2t^3 + (1-t)^2 + (1-t)^2\} dt = \int_0^1 \{-2t^3 + 2(t-1)^2 - 1\} dt \\
&= \left[-\frac{1}{2}t^4 + \frac{2}{3}(t-1)^3 - t \right]_0^1 \\
&= -\frac{1}{2} - 1 + \frac{2}{3} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}.
\end{aligned}$$

- 3) The boundary curve \mathcal{K} is the unit circle in the XY -plane. Choose the orientation corresponding to the parametric description

$$(x, y, z) = (\cos \varphi, \sin \varphi, 0), \quad \varphi \in [0, 2\pi],$$

for \mathcal{K} . Then

$$\mathbf{t} = (-\sin \varphi, \cos \varphi, 0) \quad \text{and} \quad ds = da\varphi,$$

and the flux through \mathcal{F} is then according to (35.13) given by

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds = \int_0^{2\pi} (\sin \varphi, 0, 0) \cdot (-\sin \varphi, \cos \varphi, 0) d\varphi = - \int_0^{2\pi} \sin^2 \varphi d\varphi = -\frac{1}{2} \cdot 2\pi = -\pi.$$

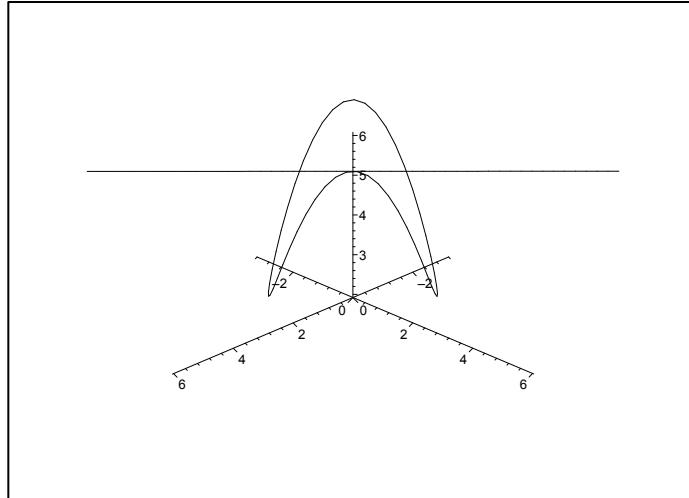


Figure 35.29: The space curve \mathcal{K} and its tangent at $\mathbf{r}\left(\frac{\pi}{4}\right)$.

Example 35.14 A space curve \mathcal{K} is given by the parametric description

$$\mathbf{r}(t) = (2 \cos t, 2 \sin t, 4 + 2 \sin(2t)), \quad t \in \mathbb{R}.$$

1. Find a parametric description of the tangent of \mathcal{K} at the point $\mathbf{r}\left(\frac{\pi}{4}\right)$.

2. Show that \mathcal{K} lies on the surface \mathcal{F} , given by the equation $z = 4 + xy$.

Let \mathcal{K}_1 be the restriction of \mathcal{K} corresponding to the parameter interval $[0, 2\pi]$, where its orientation is corresponding to increasing t . Furthermore, we have given the vector field

$$\mathbf{V}(x, y, z) = (y, x, y^2 + 2z), \quad (x, y, z) \in \mathbb{R}^3.$$

3. Find the circulation of the vector field along the curve \mathcal{K}_1 .

A Space curve; circulation along a closed curve.

D Find $\mathbf{r}'(t)$ and the tangent corresponding to $t = \frac{\pi}{4}$.

Put $(x, y, z) = \mathbf{r}(t)$ into the equation of \mathcal{F} .

Try to apply Stokes's theorem. Alternatively, compute directly the circulation.

I 1) It follows from

$$\mathbf{r}'(t) = (-2 \sin t, 2 \cos t, 4 \cos 2t)$$

that

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 4 + 2\right) = (\sqrt{2}, \sqrt{2}, 6)$$

and

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = \left(-\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 0\right) = (-\sqrt{2}, \sqrt{2}, 0),$$

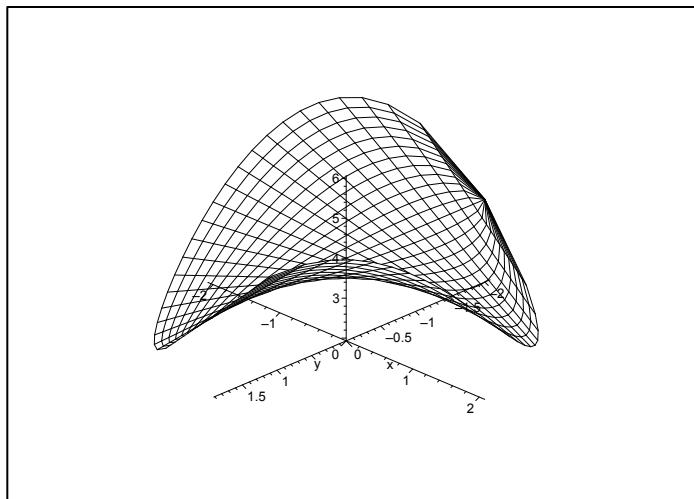
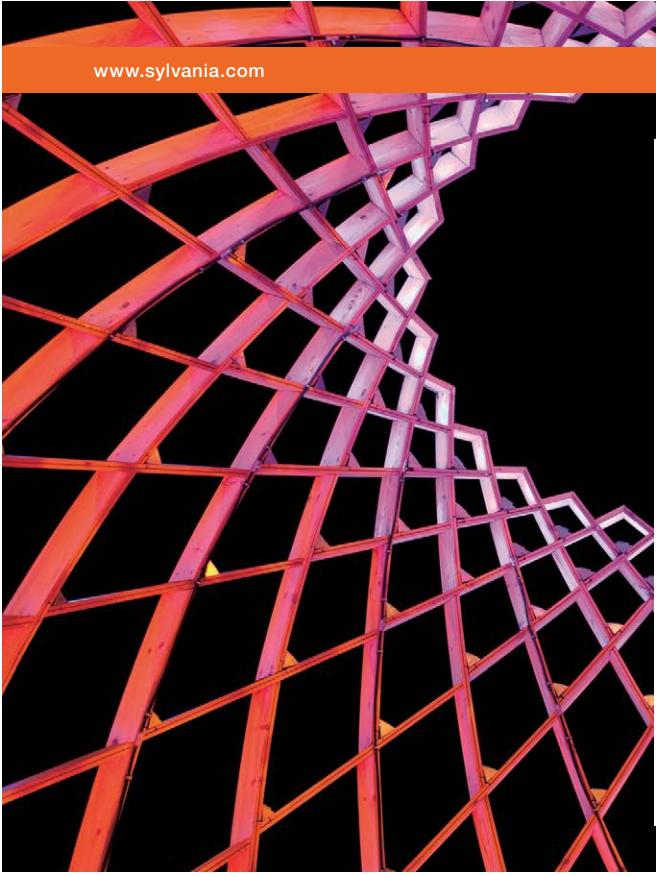


Figure 35.30: The surface \mathcal{F} of equation $z = 4 + xy$ for $x^2 + y^2 \leq 4$

hence the equation of the tangent is

$$(x, y, z) = (\sqrt{2} - \sqrt{2} u, \sqrt{2} + \sqrt{2} u, 6) = (\sqrt{2}(1 - u), \sqrt{2}(1 + u), 6), \quad u \in \mathbb{R}.$$

www.sylvania.com



We do not reinvent
the wheel we reinvent
light.

Fascinating lighting offers an infinite spectrum of possibilities: Innovative technologies and new markets provide both opportunities and challenges. An environment in which your expertise is in high demand. Enjoy the supportive working atmosphere within our global group and benefit from international career paths. Implement sustainable ideas in close cooperation with other specialists and contribute to influencing our future. Come and join us in reinventing light every day.

Light is OSRAM

OSRAM
SYLVANIA 

2) Since

$$4 + x(t)y(t) = 4 + 4 \cos t \cdot \sin t = 4 + 2 \sin 2t = z,$$

the curve \mathcal{K} lies on the surface \mathcal{F} .

3) It follows from Stokes's theorem that

$$\int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS,$$

where

$$\operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & y^2 + 2z \end{vmatrix} = (2y, -1, 0).$$

Since $z = 4 + xy$, $x^2 + y^2 \leq 4$, we get for the surface \mathcal{F} that

$$\frac{\partial z}{\partial x} = y \quad \text{and} \quad \frac{\partial z}{\partial y} = x,$$

hence

$$\mathbf{N}(x, y) = (-y, -x, 1).$$

ALTERNATIVELY, $(x, y, z) = (u, v, 4 + uv)$, thus

$$\mathbf{N}(u, v) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1) = (-y, -x, 1).$$

If we put $B = \{(x, y) \mid x^2 + y^2 \leq 4\}$, then

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS = \int_B \mathbf{N} \cdot \operatorname{rot} \mathbf{V} \, dx \, dy \\ &= \int_B (-2y^2 + x + 0) \, dx \, dy = -2 \int_0^{2\pi} \left\{ \int_0^2 (\varrho^2 \sin^2 \varphi + 0) \varrho \, d\varrho \right\} \, d\varphi \\ &= -2 \left[\frac{2\pi}{2} \right] \cdot \left[\frac{\varrho^4}{4} \right]_0^2 = -2\pi \cdot \frac{16}{4} = -8\pi. \end{aligned}$$

ALTERNATIVELY a direct computation gives

$$\begin{aligned} \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{2\pi} \mathbf{V} \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{2\pi} (2 \sin t, 2 \cos t, 4 \sin^2 t + 8 + 4 \sin 2t) \cdot (-\sin t, 2 \cos t, 4 \cos 2t) \, dt \\ &= \int_0^{2\pi} \{-4 \sin^2 t + 4 \cos^2 t + 16 \sin^2 t \cos 2t + 32 \cos 2t + 16 \sin 2t \cos 2t\} \, dt \\ &= 16 \int_0^{2\pi} \sin^2 t \cos 2t \, dt = 16 \int_0^{2\pi} \sin^2 t (2 \cos^2 t - 1) \, dt \\ &= 8 \int_0^{2\pi} \sin^2 2t \, dt - 16\pi = 8 \cdot \frac{2\pi}{2} - 16\pi = -8\pi. \end{aligned}$$

Example 35.15 Let α be a constant, and consider the vector field

$$\mathbf{V}(x, y, z) = (\alpha x^2 + xz + yz, \alpha y^2 - xz - yz, \alpha(x^2 - y^2 + z^2)), \quad (x, y, z) \in \mathbb{R}^3.$$

1. Find $\operatorname{div} \mathbf{V}$.

2. Show that \mathbf{V} is not a gradient field in \mathbb{R}^3 for any choice of α .

Let \mathcal{K} denote the circle given by $x^2 + y^2 = a^2$, $z = a$.

3. Find the circulation of \mathbf{V} along \mathcal{K} ; indicate the chosen orientation.

Let the domain $\Omega \subset \mathbb{R}^3$ be given by $x^2 + y^2 \leq a^2$, $y \geq 0$, $0 \leq z \leq a$.

4. Find the flux of \mathbf{V} through $\partial\Omega$.

A Divergence, circulation and flux.

D Compute $\operatorname{div} \mathbf{V}$. Check $\frac{\partial V_i}{\partial x_j}$ for some i and j . Find the circulation, e.g. by Stokes's theorem.
Finally, apply Gauß's theorem to find the flux.

I 1) The divergence is

$$\operatorname{div} \mathbf{V} = 2\alpha x + z + 2\alpha y - z + 2\alpha z = 2\alpha(x + y + z).$$

2) It follows from

$$\frac{\partial V_1}{\partial y} = z \quad \text{and} \quad \frac{\partial V_2}{\partial z} = -z,$$

that

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x} \quad \text{for } z \neq 0.$$

The surface $z = 0$ does not contain inner points, thus \mathbf{V} is not a gradient field for any value of α .

3) It follows by the definition of the circulation that

$$\begin{aligned} & \int_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds \\ &= \int_0^{2\pi} (\alpha a^2 \cos^2 t + a^2 \cos^2 t + a^2 \sin t, \alpha a^2 \sin^2 t - a^2 \cos t - a^2 \sin t, \alpha a^2 (\cos^2 t - \sin^2 t + a)) \cdot \\ & \quad \cdot (-\sin t, \cos t, 0) a \, dt \\ &= a^2 \int_0^{2\pi} \{-\alpha \cos^2 t \sin t - \cos t \sin t - \sin^2 t + \alpha \sin^2 t \cos t - \cos^2 t - \sin t \cos t + 0\} \, dt \\ &= a^3 \int_0^{2\pi} (-\sin^2 t - \cos^2 t) \, dt = -2\pi a^2. \end{aligned}$$

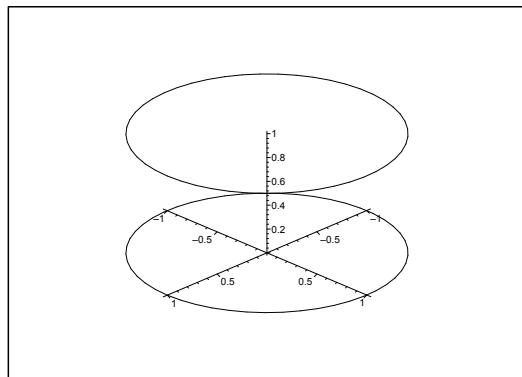


Figure 35.31: The curve \mathcal{K} and its projection onto the (X, Y) -plane for $a = 1$.

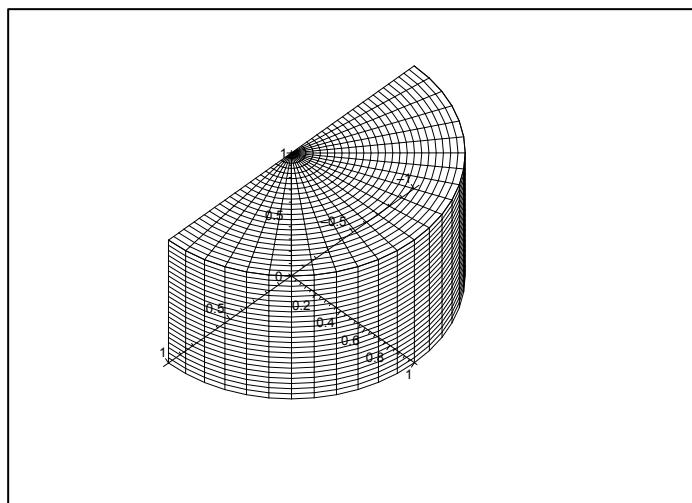


Figure 35.32: The body Ω for $a = 1$.

ALTERNATIVELY,

$$\operatorname{rot} \mathbf{V} \cdot \mathbf{e}_z = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz + yz & -xz - yz \end{vmatrix} = -z - z = -2z.$$

Choose \mathcal{F} as the disc $x^2 + y^2 \leq a^2$, $z = a$. Then we get by Stokes's theorem that

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \operatorname{rot} \mathbf{V} \cdot \mathbf{e}_z \, dS = \int_{\mathcal{F}} (-2a) \, dS = -2a \operatorname{area}(\mathcal{F}) = -2\pi a^3.$$

- 4) When we apply Gauß's theorem and 1), it follows that the flux is given by

$$\begin{aligned} \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, dS &= \int_{\Omega} \operatorname{div} \mathbf{V} \, d\Omega = 2\alpha \int_{\Omega} (x + y + z) \, d\Omega = 2\alpha \int_{\Omega} (y + z) \, d\Omega \\ &= 2\alpha a \int_{-a}^a \left\{ \int_0^{\sqrt{a^2-x^2}} y \, dy \right\} dx + 2\alpha \cdot \frac{\pi a^2}{2} \int_0^a z \, dz \\ &= \alpha a \cdot 2 \int_0^a (a^2 - x^2) \, dx + \alpha \pi a^2 \cdot \frac{a^2}{2} = 2\alpha a \left(a^3 - \frac{1}{3} a^3 \right) + \frac{1}{2} \alpha \pi a^4 \\ &= \frac{4}{3} \alpha a^4 + \frac{1}{2} \alpha \pi a^4 = \alpha a^4 \left(\frac{4}{3} + \frac{\pi}{2} \right). \end{aligned}$$



Deloitte.

Discover the truth at www.deloitte.ca/careers

© Deloitte & Touche LLP and affiliated entities.



Click on the ad to read more

Example 35.16 Consider the space curve \mathcal{K} given by the parametric description

$$(x, y, z) = \left(1 + \cos t, \sin t, 2 \sin \frac{t}{2}\right), \quad t \in [0, 2\pi].$$

1. Find a parametric description of the tangent of \mathcal{K} at the point corresponding to $t = \frac{\pi}{2}$.

2. Show that \mathcal{K} lies on a sphere of centrum at $(0, 0, 0)$, and find an equation of the sphere.

Furthermore, consider the surface \mathcal{F} given by the parametric description

$$(x, y, z) = \left(1 + \cos t, \sin t, 2u \sin \frac{t}{2}\right), \quad (t, u) \in [0, 2\pi] \times [0, 1],$$

and the vector field $\mathbf{V}(x, y, z) = (x, y, z)$, $(x, y, z) \in \mathbb{R}^3$.

3. Find the area of \mathcal{F} .

4. Find the circulation of the vector field along the curve \mathcal{K} .

A Space curve, surface area, circulation of a vector field.

D 1) First calculate $\mathbf{r}'(t)$.

2) Show that $x^2 + y^2 + z^2 = r^2 > 0$ and find $r > 0$.

3) Compute the surface area.

4) Apply Stokes's theorem. Alternatively the circulation is computed directly as a line integral.

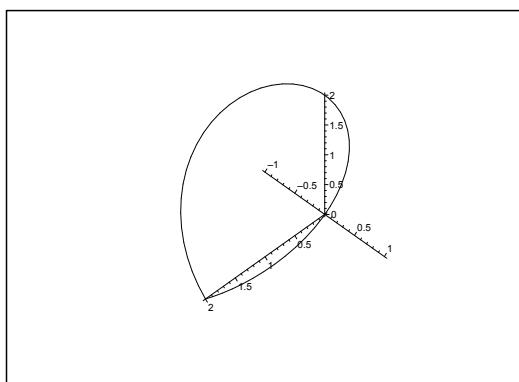


Figure 35.33: The curve \mathcal{K} .

I 1) We find

$$\mathbf{r}'(t) = \left(-\sin t, \cos t, \cos \frac{t}{2}\right), \quad \mathbf{r}'\left(\frac{\pi}{2}\right) = \left(-1, 0, \frac{1}{\sqrt{2}}\right).$$

Now, $\mathbf{r}\left(\frac{\pi}{2}\right) = (1, 1, \sqrt{2})$, hence a parametric description of the tangent is given by

$$(1, 1, \sqrt{2}) + u \left(-1, 0, \frac{1}{\sqrt{2}}\right), \quad u \in \mathbb{R}.$$

2) Since

$$\begin{aligned} x(t)^2 + y(t)^2 + z(t)^2 &= (1 + \cos t)^2 + \sin^2 t + 4 \sin^2 \frac{t}{2} \\ &= 1 + 2 \cos t + \cos^2 t + \sin^2 t + 2(1 - \cos t) \\ &= 1 + 1 + 2 = 4 = 2^2, \end{aligned}$$

it follows that \mathcal{K} lies on the sphere of centrum $(0, 0, 0)$ and radius 2.

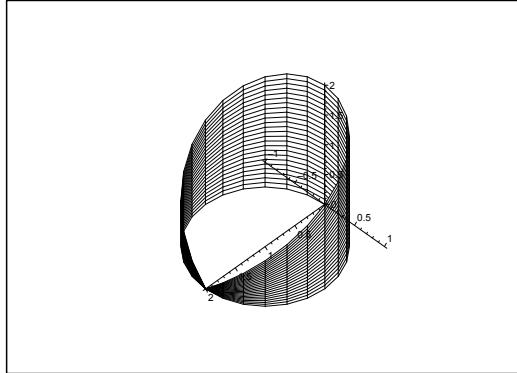


Figure 35.34: The surface \mathcal{F} .

3) It follows from

$$\frac{\partial \mathbf{r}}{\partial t} = \left(-\sin t, \cos t, u \cos \frac{t}{2} \right) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial u} = \left(0, 0, 2 \sin \frac{t}{2} \right)$$

that

$$\mathbf{N}(t, u) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\sin t & \cos t & u \cos \frac{t}{2} \\ 0 & 0 & 2 \sin \frac{t}{2} \end{vmatrix} = 2 \sin \frac{t}{2} (\cos t, \sin t, 0),$$

hence

$$\|\mathbf{N}(t, u)\| = \left| 2 \sin \frac{t}{2} \right| \cdot 1 = 2 \sin \frac{t}{2}, \quad t \in [0, 2\pi], \quad u \in [0, 1].$$

Hence

$$\text{area}(\mathcal{F}) = \int_0^1 \left\{ \int_0^{2\pi} 2 \sin \frac{t}{2} dt \right\} = \left[-4 \cos \frac{t}{2} \right]_0^{2\pi} = 4(1 + 1) = 8.$$

4) Since $\text{rot } \mathbf{V} = \mathbf{0}$, it follows by Stokes's theorem no matter how we choose the surface \mathcal{F}_1 with boundary curve \mathcal{K} that

$$\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} dt = \int_{\mathcal{F}_1} \mathbf{n} \cdot \text{rot } \mathbf{V} dS = 0.$$

ALTERNATIVELY we get by the definition that

$$\begin{aligned}\oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} dt &= \int_0^{2\pi} \mathbf{V} \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \mathbf{r}(t) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \frac{d}{dt} \left(\frac{1}{2} \|\mathbf{r}(t)\|^2 \right) dt = \frac{1}{2} (\|\mathbf{r}(2\pi)\|^2 - \|\mathbf{r}(0)\|^2) = 0,\end{aligned}$$

because the curve is closed.

ALTERNATIVELY it is possible though extremely tedious to insert the parametric description and then reduce.

WHY WAIT FOR PROGRESS?

DARE TO DISCOVER

Discovery means many different things at Schlumberger. But it's the spirit that unites every single one of us. It doesn't matter whether they join our business, engineering or technology teams, our trainees push boundaries, break new ground and deliver the exceptional. If that excites you, then we want to hear from you.

Dare to discover.
careers.slb.com/recentgraduates



Schlumberger

Example 35.17 Given the vector field

$$\mathbf{V}(x, y, z) = (y^3 - xz^2, -x^3 + yz^2, z^3), \quad (x, y, z) \in \mathbb{R}^3.$$

1. Find the divergence $\nabla \cdot \mathbf{V}$ and the rotation $\nabla \times \mathbf{V}$.

Let a be a positive constant, and let L denote the half spherical shell given by

$$z \geq 0, \quad a^2 \leq x^2 + y^2 + z^2 \leq 3a^2.$$

2. Find the flux of \mathbf{V} through ∂L .

Let C be the circle in the plane $z = a$ of centrum $(0, 0, a)$ and radius a .

3. Find the absolute value of the circulation $\oint_C \mathbf{V} \cdot \mathbf{t} ds$.

4. Check if there exists a vector field $\mathbf{W} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, such that

$$\mathbf{V} = \nabla \times \mathbf{W},$$

in the whole space.

5. Check if there exists a scalar field $F : \mathbb{R}^3 \rightarrow \mathbb{R}$, such that $\mathbf{V} = \nabla F$ in the whole space.

A Divergence, rotation, flux, circulation, vector potential, gradient field.

D Apply Gauß's theorem and Stokes's theorem, whenever it is possible.

I 1) We get by some very simple calculations that

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = -z^2 + z^2 + 3z^2 = 3z^2,$$

and

$$\nabla \times \mathbf{V} = \operatorname{rot} \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 - xz^2 & -x^3 + yz^2 & z^3 \end{vmatrix} = (-2yz, -2xz, -3zx^2 - 3y^2).$$

2) It follows from Gauß's theorem and the result of 1) that

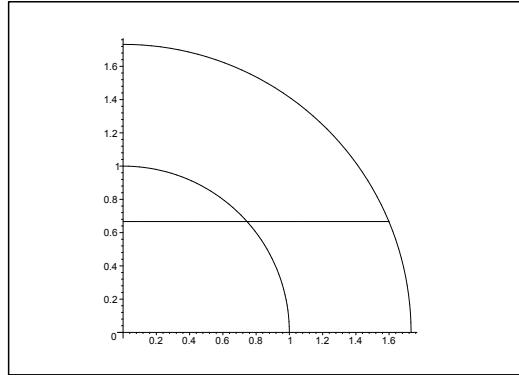
$$\int_{\partial L} \mathbf{V} \cdot \mathbf{n} dS = \int_L \operatorname{div} \mathbf{V} d\Omega = \int_L 3z^2 d\Omega.$$

At height $z \in [0, a]$ the body L is cut into an annulus of the area

$$\pi(3a^2 - z^2) - \pi(a^2 - z^2) = 2\pi a^2.$$

At height $z \in [a, \sqrt{3}a]$ the body L is cut into a circle of area

$$\pi(3a^2 - z^2).$$

Figure 35.35: The meridian cut for $a = 1$ with the cut at height z .

Hence by insertion,

$$\begin{aligned}
 \int_{\partial L} \mathbf{V} \cdot \mathbf{n} dS &= \int_L 3z^2 d\Omega = \int_0^a 3z^2 \cdot 2\pi a^2 dz + \int_a^{\sqrt{3}a} 3z^2 \pi(3a^2 - z^2) dz \\
 &= 6\pi a^2 \int_0^a z^2 dz + 9\pi a^2 \int_a^{\sqrt{3}a} z^2 dz - 3\pi \int_a^{\sqrt{3}a} z^4 dz \\
 &= 9\pi a^2 \int_0^{\sqrt{3}a} z^2 dz - 3\pi a^2 \int_0^a z^2 dz - 3\pi \int_a^{\sqrt{3}a} z^4 dz \\
 &= 3\pi a^2 \cdot 3\sqrt{3}a^3 - \pi a^2 \cdot a^3 - \frac{3}{5}\pi(9\sqrt{3} - 1)a \\
 &= \pi a^5 \left(9\sqrt{3} - 1 - \frac{27}{5}\sqrt{3} + \frac{3}{5} \right) = \frac{\pi}{5}(18\sqrt{3} - 2)a^5.
 \end{aligned}$$

3) Put $B = \{(x, y) \mid x^2 + y^2 \leq a^2\}$. Then we get by Stokes's theorem and the result of 1) that

$$\begin{aligned}
 \left| \oint_C \mathbf{V} \cdot \mathbf{t} ds \right| &= \left| \int_B \operatorname{rot} \mathbf{V} \cdot \mathbf{n} dx dy \right| = \left| \int_B (-2ya, -2xa, -3x^2 - 3y^2) \cdot (0, 0, 1) dx dy \right| \\
 &= 3 \int_B (x^2 + y^2) dx dy = 3 \cdot 2\pi \int_0^a \varrho^2 \cdot \varrho d\varrho = 6\pi \cdot \frac{a^4}{4} = \frac{3\pi a^4}{2}.
 \end{aligned}$$

4) Since $\operatorname{div} \mathbf{V} = 3z^2 \neq 0$ for $z \neq 0$, there exists no vector potential \mathbf{W} of \mathbf{V} in all of the space.

5) It follows from

$$\frac{\partial V_1}{\partial y} = 3y^2 \quad \text{and} \quad \frac{\partial V_2}{\partial x} = -3x^2,$$

that

$$\frac{\partial V_1}{\partial y} \neq \frac{\partial V_2}{\partial x} \quad \text{for } (x, y) \neq (0, 0),$$

and \mathbf{V} is not a gradient field, and there exists no integral F of \mathbf{V} .

Example 35.18 Given the vector fields

$$\mathbf{U}(x, y, z) = (z^2 + y \cos x, x^2 + \sin x, y^2), \quad \mathbf{V}(x, y, z) = (y, z, x),$$

in the space \mathbb{R}^3 .

1) Find the divergence and the rotation of both vector fields.

2) Find the flux of \mathbf{U} through the surface of the cube

$$\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}.$$

3) Let a be a positive constant. Find the circulation of \mathbf{V} along the circle in the (X, Z) -plane of centrum at $(a, 0, 2a)$ and radius a . Choose an orientation of the circle.

4) Find a vector potential for \mathbf{V} .

A Vector analysis.

D Apply Gauß's theorem and Stokes's theorem.

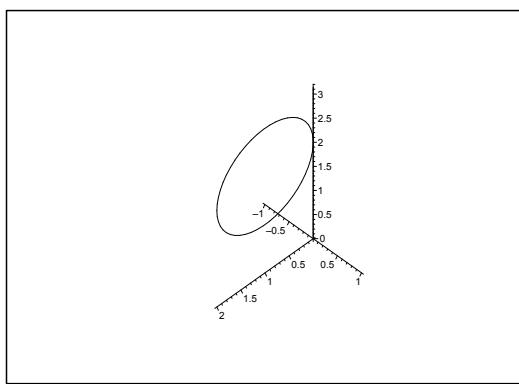


Figure 35.36: The circle of 3).

I 1) By simple calculations,

$$\operatorname{div} \mathbf{U} = -y \sin x, \quad \operatorname{div} \mathbf{V} = 0,$$

$$\begin{aligned} \operatorname{rot} \mathbf{U} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 + y \cos x & x^2 + \sin x & y^2 \end{vmatrix} = (2y, 2z, 2x) = 2\mathbf{V}, \\ \operatorname{rot} \mathbf{V} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1). \end{aligned}$$

In particular, $\mathbf{V} = \nabla \times \left(\frac{1}{2} \mathbf{U} \right)$, thus $\frac{1}{2} \mathbf{U}$ is a vector potential of \mathbf{V} (cf. 4)).

- 2) According to Gauß's theorem the flux of \mathbf{U} through ∂T is given by

$$\begin{aligned}\int_{\partial T} \mathbf{U} \cdot \mathbf{n} dS &= \int_T \operatorname{div} \mathbf{U} d\Omega = \int_0^1 \left\{ \int_0^1 \left\{ \int_0^1 (-y \sin x) dx \right\} dy \right\} dz \\ &= 1 \cdot \left[\frac{y^2}{2} \right]_0^1 \cdot [\cos x]_0^1 = -\frac{1}{2} (1 - \cos 1).\end{aligned}$$

- 3) According to Stokes's theorem,

$$\oint_{\partial A} \mathbf{t} \cdot \mathbf{V} ds = \int_A \mathbf{n} \cdot \operatorname{rot} \mathbf{V} dS = \int_A (0, -1, 0) \cdot (-1, -1, -1) dS = \operatorname{area}(A) = \pi a^2.$$

- 4) According to the result of 1), the field $\frac{1}{2} \mathbf{U}$ is a vector potential of \mathbf{V} .

SIMPLY CLEVER

ŠKODA



We will turn your CV into
an opportunity of a lifetime



Do you like cars? Would you like to be a part of a successful brand?
We will appreciate and reward both your enthusiasm and talent.
Send us your CV. You will be surprised where it can take you.

Send us your CV on
www.employerforlife.com



Click on the ad to read more

Example 35.19 Let $p \in \mathbb{R}$ and $b \in \mathbb{R}_+$ be constants. consider the circle \mathcal{K} given by $x^2 + y^2 = b^2$, $z = p$; the circle is given the orientation which forms a right hand turn with the Z-axis. Furthermore, consider the vector field

$$\mathbf{W}(x, y, z) = \left(\frac{yz}{\sqrt{x^2 + y^2}}, \frac{-xz}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} \right), \quad (x, y) \neq (0, 0).$$

Denote the circulation of \mathbf{W} along the oriented circle \mathcal{K} by $C(b, p)$.

1) Show that $C(b, p) = -2\pi pb$.

2) Let $\mathbf{V} = \text{rot } \mathbf{W}$. Show that

$$\mathbf{V}(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}} (x + y, y - x, -z), \quad (x, y) \neq (0, 0).$$

3) Show that \mathbf{W} is not a gradient field.

4) Show that \mathbf{W} has zero divergence.

5) Let \mathcal{O} be the surface of revolution which is given in semi-polar coordinates by

$$\varrho \in [a, 2a], \quad \varphi \in [0, 2\pi], \quad z = 2a - \frac{\varrho^2}{a},$$

where $a > 0$ is a positive constant.

Find the flux

$$\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} dS,$$

where we also shall choose an orientation of \mathcal{O} .

A Circulation, rotation, gradient field, divergence, flux.

D The circulation can be computed in various ways. The computation of the flux has also some variants.

I 1) We have two variants.

First variant. The definition of the circulation as a line integral.

We use the following parametric description of the circle \mathcal{K} ,

$$\mathbf{r}(t) = (x, y, z) = (b \cos t, b \sin t, p), \quad t \in [0, 2\pi].$$

Then

$$\mathbf{r}'(t) = b(-\sin t, \cos t, 0),$$

and the circulation is according to the definition given by

$$\begin{aligned} C(b, p) &= \int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t} ds = \int_0^{2\pi} \left(\frac{b \sin t \cdot p}{b}, -\frac{b \cos t \cdot p}{b}, b \right) \cdot b(-\sin t, \cos t, 0) dt \\ &= -pb \int_0^{2\pi} \{\sin^2 t + \cos^2 t + 0\} dt = -2\pi pb. \end{aligned}$$

Second variant. *Stokes's theorem.*

An application of Stokes's theorem gives

$$\int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \operatorname{rot} \mathbf{W} \cdot \mathbf{n} \, dS,$$

where \mathcal{F} is the disc at height $z = p$ and radius b , and where the unit normal vector is parallel to the Z -axis.

The unit normal is trivially $\mathbf{n} = (0, 0, 1)$. Then by 2),

$$\operatorname{rot} \mathbf{W} = \mathbf{V}.$$

When we apply the expression of \mathbf{V} , we obtain in polar coordinates

$$\int_{\mathcal{K}} \mathbf{W} \cdot \mathbf{t} \, ds = \int_{\mathcal{F}} \mathbf{V} \cdot \mathbf{n} \, dS = \int_{\mathcal{F}} \left(-\frac{p}{\sqrt{x^2 + y^2}} \right) \, dS = -p \int_0^{2\pi} \left\{ \int_0^b \frac{1}{\varrho} \cdot \varrho \, d\varrho \right\} \, d\varphi = -2\pi pb.$$

2) Let $(x, y) \neq (0, 0)$. Then

$$\begin{aligned} \mathbf{V} &= \operatorname{rot} \mathbf{W} = \nabla \times \mathbf{W} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{\sqrt{x^2 + y^2}} & \frac{-xz}{\sqrt{x^2 + y^2}} & \sqrt{x^2 + y^2} \end{vmatrix} \\ &= \begin{pmatrix} \frac{y}{\sqrt{x^2 + y^2}} - \left(-\frac{x}{\sqrt{x^2 + y^2}} \right) \\ \frac{y}{\sqrt{x^2 + y^2}} - \frac{x}{\sqrt{x^2 + y^2}} \\ -\frac{z}{\sqrt{x^2 + y^2}} + \frac{x^2 z}{(\sqrt{x^2 + y^2})^3} - \frac{z}{\sqrt{x^2 + y^2}} + \frac{y^2 z}{(\sqrt{x^2 + y^2})^3} \end{pmatrix} \\ &= \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} y + x \\ y - x \\ -2z + \frac{x^2 z + y^2 z}{x^2 + y^2} \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}} (x + y, y - x, -z). \end{aligned}$$

3) Suppose \mathbf{W} was a gradient field, ∇F . Then

$$\mathbf{V} = \nabla \times \mathbf{W} = \nabla \times \nabla F = \mathbf{0}.$$

But $\mathbf{V} \neq \mathbf{0}$, thus we conclude that \mathbf{W} is not a gradient field.

4) By just computing,

$$\begin{aligned} \operatorname{div} \mathbf{W} &= \frac{\partial}{\partial x} \left(\frac{yz}{\sqrt{x^2 + y^2}} \right) - \frac{\partial}{\partial y} \left(\frac{xz}{\sqrt{x^2 + y^2}} \right) + \frac{\partial}{\partial z} \left(\sqrt{x^2 + y^2} \right) \\ &= -\frac{xyz}{(x^2 + y^2)^{3/2}} - \left(-\frac{xyz}{(x^2 + y^2)^{3/2}} \right) + 0 = 0. \end{aligned}$$

ALTERNATIVELY it follows that if \mathbf{U} is defined by

$$3\mathbf{U} = \frac{z^2}{\sqrt{x^2 + y^2}} (-x, -y, 0) + \sqrt{x^2 + y^2} (-y, x, z), \quad (x, y) \neq (0, 0),$$

then

$$\mathbf{W} = \text{rot } \mathbf{U} = \nabla \times \mathbf{U},$$

and hence

$$\text{div } \mathbf{W} = \nabla \cdot (\nabla \times \mathbf{U}) = 0.$$

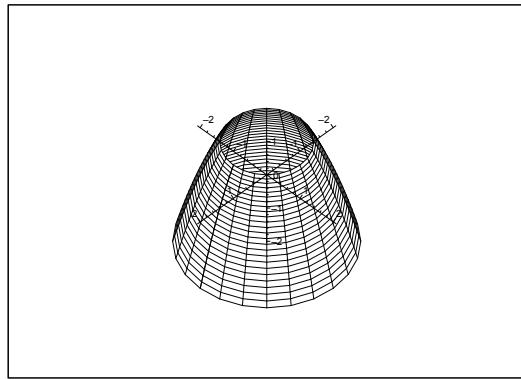


Figure 35.37: The surface \mathcal{O} . The upper boundary curve \mathcal{K}_1 is oriented as a left hand screw, while the lower boundary curve \mathcal{K}_2 is oriented as a right hand screw. Hence the normal vector field on \mathcal{O} is everywhere pointing away from the Z -axis.

- 5) Choose the orientation on \mathcal{O} as described in the caption of the figure. Then $\delta\mathcal{O} = \mathcal{K}_2 - \mathcal{K}_1$, where the minus sign in front of \mathcal{K}_1 means that this circle is run through in the opposite direction of the usual one, i.e. as a left handed screw.

There are two variants.

1. variant. *Stokes's theorem combined with 1).*

We get by Stokes's theorem,

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} dS &= \int_{\mathbf{0}} (\nabla \times \mathbf{W}) \cdot \mathbf{n} dS = \oint_{\delta\mathcal{O}} \mathbf{W} \cdot \mathbf{t} ds = - \int_{\mathcal{K}_1} \mathbf{W} \cdot \mathbf{t} ds + \int_{\mathcal{K}_2} \mathbf{W} \cdot \mathbf{t} ds \\ &= -C(a, a) + C(2a, -2a) = +2\pi a \cdot a + (+2\pi \cdot 2a \cdot 2a) = 10\pi a^2. \end{aligned}$$

Second variant. *Surface integral.*

The meridian curve has the equation

$$z = 2a - \frac{\varrho^2}{a},$$

so we conclude that the tangent vector is $\left(1, -\frac{2\varrho}{a}\right)$. Hence the normal vector $\mathbf{N} = (2\varrho, a)$, and thus the unit normal vector

$$\mathbf{n} = \frac{1}{\sqrt{a^2 + 4\varrho^2}} (2\varrho, a).$$

Then the outgoing unit normal vector field of the surface \mathcal{O} is

$$\mathbf{n}(\varrho, \varphi) = \frac{1}{\sqrt{a^2 + 4\varrho^2}} (2\varrho \cos \varphi, 2\varrho \sin \varphi, a).$$

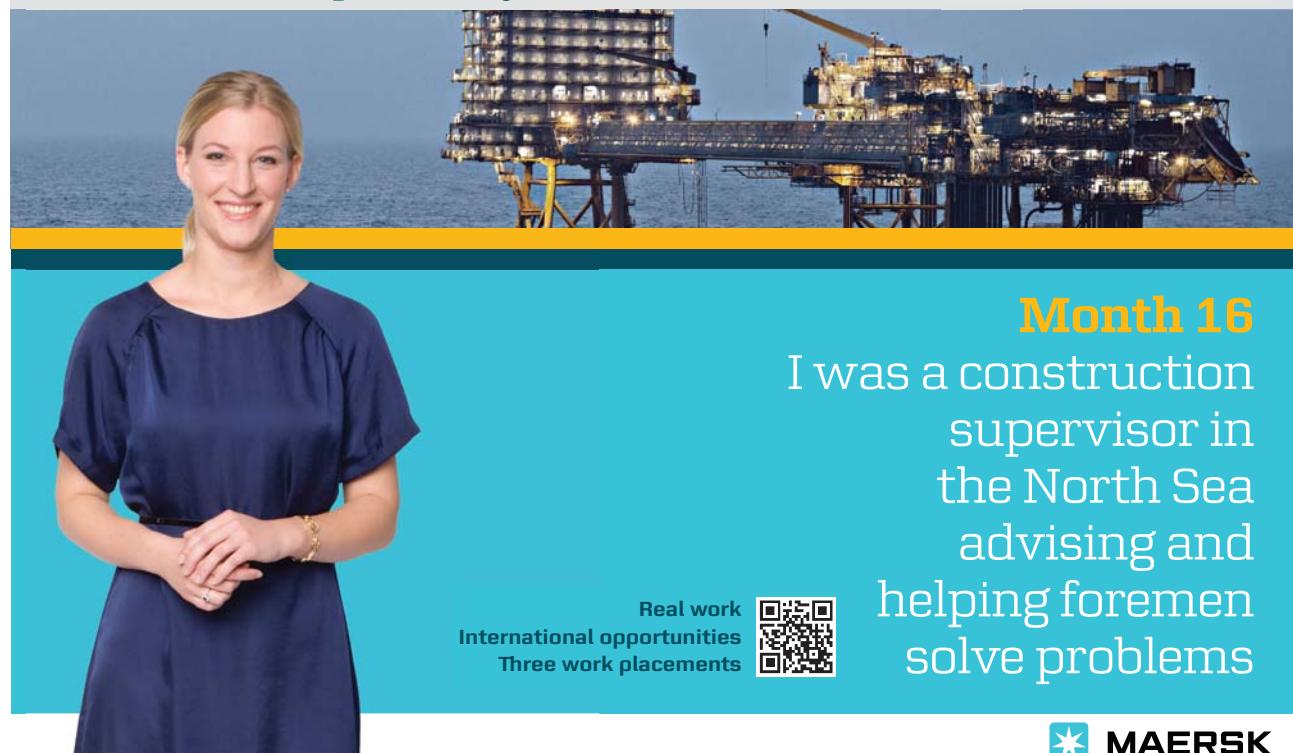
We have on \mathcal{O} ,

$$\begin{aligned} \mathbf{V} &= \frac{1}{\sqrt{x^2 + y^2}} (x + y, y - x, -z) \\ &= \frac{1}{\varrho} \left(\varrho \cos \varphi + \varrho \sin \varphi, \varrho \sin \varphi - \varrho \cos \varphi, -2a + \frac{\varrho^2}{a} \right) \\ &= \left(\cos \varphi + \sin \varphi, \sin \varphi - \cos \varphi, \frac{\varrho}{a} - 2 \frac{a}{\varrho} \right), \end{aligned}$$

hence the integrand over \mathcal{O} is written

$$\begin{aligned} f(x, y, z) &= \mathbf{V} \cdot \mathbf{n} = \frac{1}{\sqrt{a^2 + 4\varrho^2}} \left\{ 2\varrho (\cos^2 \varphi + \cos \varphi \sin \varphi + \sin^2 \varphi - \cos \varphi \sin \varphi) + \varrho - 2 \frac{a^2}{\varrho} \right\} \\ &= \frac{1}{\sqrt{a^2 + 4\varrho^2}} \left\{ 3\varrho - 2 \frac{a^2}{\varrho} \right\}. \end{aligned}$$

I joined MITAS because
I wanted **real responsibility**



Real work
International opportunities
Three work placements

Month 16

I was a construction supervisor in the North Sea advising and helping foremen solve problems

The Graduate Programme
for Engineers and Geoscientists
www.discovermitas.com



Finally, by inserting into a known formula of the surface integral over surfaces of revolution, we get

$$\begin{aligned}\int_{\mathcal{O}} \mathbf{V} \cdot \mathbf{n} dS &= \int_a^{2a} \left\{ \int_0^{2\pi} \frac{1}{\sqrt{a^2 + 4\rho^2}} \left\{ 3\rho - 2 \frac{a^2}{\rho} \right\} d\varphi \right\} \rho \sqrt{1 + \frac{4\rho^2}{a^2}} d\rho \\ &= \frac{2\pi}{a} \int_a^{2a} (3\rho^2 - 2a^2) d\rho = \frac{2\pi}{a} [\rho^3 - 2a^2\rho]_a^{2a} \\ &= \frac{2\pi}{a} \{8a^3 - 4a^3 - a^3 + 2a^3\} = 10\pi a^2.\end{aligned}$$

Example 35.20 Let \mathcal{F} be one eighth of a sphere given by

$$x^2 + y^2 + z^2 = a^2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0,$$

where a is a positive constant. Thus the boundary curve $\delta\mathcal{F}$ is composed of three circular arcs.

Also, consider the vector field

$$\mathbf{V}(x, y, z) = (ay + yz, -ax + zx, z^2 - 2xy), \quad (x, y, z) \in \mathbb{R}^3.$$

- 1) Find the rotation $\nabla \times \mathbf{V}$.
- 2) Show that \mathbf{V} is not a gradient field.
- 3) Find the circulation

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} ds,$$

where we choose an orientation of $\delta\mathcal{F}$.

A Rotation, circulation, Stokes's theorem.

D Sketch a figure. Apply Stokes's theorem.

I 1) The rotation is

$$\begin{aligned}\text{rot } \mathbf{V} &= \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay + yz & -ax + zx & z^2 - 2xy \end{vmatrix} \\ &= (-2x - x, y + 2y, -a + zaz) = (3x, 3y, -2a).\end{aligned}$$

- 2) It follows from $\text{rot } \mathbf{V} \neq \mathbf{0}$ that \mathbf{V} is not a gradient field.
- 3) Choose the orientation of $\delta\mathcal{F}$ as described on the figure. Then the unit normal vector field on \mathcal{F} is pointing outwards, i.e.

$$\mathbf{n} = \frac{1}{a} (x, y, z), \quad \text{for } (x, y, z) \in \mathcal{F}.$$

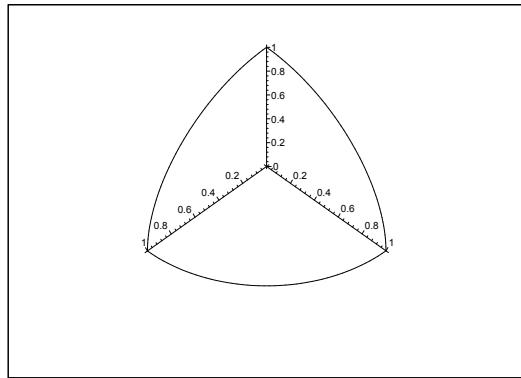


Figure 35.38: The surface \mathcal{F} and the boundary curve $\delta\mathcal{F}$ for $a = 1$. On the surface \mathcal{F} the unit normal vector field is always directed away from $(0, 0, 0)$, and the curve $\delta\mathcal{F}$ is oriented correspondingly, i.e. from the X -axis towards the y -axis, then towards the z -axis and finally back to the x -axis.

When we apply Stokes's theorem we conclude that the circulation along $\delta\mathcal{F}$ is

$$\begin{aligned}\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} \, dS = \int_{\mathcal{F}} \frac{1}{a} (x, y, z) \cdot (-3x, 3y, -2a) \, dS \\ &= \frac{1}{a} \int_{\mathcal{F}} (-3x^2 + 3y^2 - 2az) \, dS = -2 \int_{\mathcal{F}} z \, dS,\end{aligned}$$

where it follows by the symmetry that

$$\int_{\mathcal{F}} x^2 \, dS = \int_{\mathcal{F}} y^2 \, dS.$$

The following computations can be given in various variants.

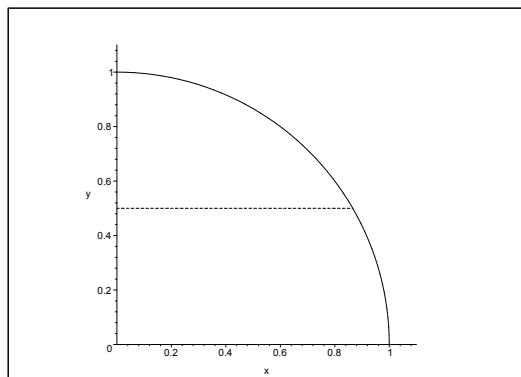


Figure 35.39: The meridian cut of \mathcal{F} for $a = 1$. We have at height z that $\varrho = \sqrt{a^2 - z^2}$.

First variant. If we first (i.e. innermost) at height z and denote the circle by ℓ , then

$$\begin{aligned}\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds &= -2 \int_{\mathcal{F}} z \, dS = -2 \int_{\ell} z \cdot \frac{\pi}{2} \cdot \sqrt{a^2 - z^2} \, ds \\ &= -\pi \int_{\ell} z \sqrt{a^2 - z^2} \, ds \quad \left(-\pi \int_{\ell} z \varrho \, ds \right).\end{aligned}$$

Using the parametric description

$$\varrho = a \cos \varphi, \quad z = a \sin \varphi,$$

of ℓ we get $ds = a \, d\varphi$, and the computations continue as follows,

$$\begin{aligned}\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds &= -\pi \int_0^{\frac{\pi}{2}} a \sin \varphi \cdot a \cos \varphi \cdot a \, d\varphi = -\pi a^3 \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi \, d\varphi \\ &= -\pi a^3 \left[\frac{\sin^2 \varphi}{2} \right]_0^{\frac{\pi}{2}} = -\frac{\pi}{2} a^3.\end{aligned}$$

Second variant. The surface \mathcal{F} is described in spherical coordinates by

$$\begin{cases} x = a \sin \theta \cos \varphi, \\ y = a \sin \theta \sin \varphi, \\ z = a \cos \theta, \end{cases} \quad \begin{cases} \theta \in \left[0, \frac{\pi}{2}\right], \\ \varphi \in \left[0, \frac{\pi}{2}\right], \end{cases} \quad \text{weight: } a^2 \sin \theta,$$

hence by insertion,

$$\begin{aligned}\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds &= - \int_{\mathcal{F}} 2z \, dS = -2 \int_0^{\frac{\pi}{2}} \left\{ \int_0^{\frac{\pi}{2}} a \cos \theta \cdot a^2 \sin \theta \, d\theta \right\} \, d\varphi \\ &= -2 \cdot \frac{\pi}{2} \cdot a^3 \left[\frac{\sin^2 \varphi}{2} \right]_0^{\frac{\pi}{2}} = -\frac{\pi}{2} a^3.\end{aligned}$$

Third variant. Direct computation of the line integrals without the use of Stokes's theorem.

First note that the boundary curve $\delta\mathcal{F}$ is composed of the subcurves:

Γ_1 : $(x, y, z) = (a \cos \varphi, a \sin \varphi, 0)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$, with the unit tangent vector $\mathbf{t} = (-\sin \varphi, \cos \varphi, 0)$, and the line element $ds = a \, d\varphi$.

Γ_2 : $(x, y, z) = (0, a \cos \varphi, a \sin \varphi)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$, with the unit tangent vector $\mathbf{t} = (0, -\sin \varphi, \cos \varphi)$, and the line element $ds = a \, d\varphi$,

Γ_3 : $(x, y, z) = (a \sin \varphi, 0, a \cos \varphi)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$, with the unit tangent vector $\mathbf{t} = (\cos \varphi, 0, -\sin \varphi)$ and the line element $ds = a \, d\varphi$.

We get by insertion,

$$\oint_{\delta\mathcal{F}} \mathbf{V} \cdot \mathbf{t} \, ds = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} (1t + yz, -ax + zx, z^2 - 2xy) \cdot \mathbf{t} \, ds.$$

The integrals are computed separately,

$$\begin{aligned}\int_{\Gamma_1} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} (a^2 \sin \varphi, -a^2 \cos \varphi, -2a^2 \sin \varphi \cos \varphi) \cdot (-\sin \varphi, \cos \varphi, 0) a \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} a^3 (-\sin^2 \varphi - \cos^2 \varphi) \, d\varphi = -\frac{\pi}{2} a^3, \\ \int_{\Gamma_2} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} (a^2 \cos \varphi + a^2 \sin \varphi \cos \varphi, 0, a^2 \sin^2 \varphi) \cdot (0, -\sin \varphi, \cos \varphi) a \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} a^3 \sin^2 \varphi \cos \varphi \, d\varphi = 3 \left[\frac{\sin^3 \varphi}{3} \right]_0^{\frac{\pi}{2}} = \frac{1}{3} a^3, \\ \int_{\Gamma_3} \mathbf{V} \cdot \mathbf{t} \, ds &= \int_0^{\frac{\pi}{2}} (0, -a^2 \sin \varphi + a^2 \sin \varphi \cos \varphi, a^2 \cos^2 \varphi) \cdot (\cos \varphi, 0, -\sin \varphi) a \, d\varphi \\ &= \int_0^{\frac{\pi}{2}} (-a^3 \cos^2 \varphi \sin \varphi) \, d\varphi = a^3 \left[\frac{\cos^3 \varphi}{3} \right]_0^{\frac{\pi}{2}} = -\frac{1}{3} a^3.\end{aligned}$$

Summarizing,

$$\oint_{\delta F} \mathbf{V} \cdot \mathbf{t} \, ds = -\frac{\pi}{2} a^3 + \frac{1}{3} a^3 - \frac{1}{3} a^3 = -\frac{\pi}{2} a^3.$$

IE business school

#1 EUROPEAN BUSINESS SCHOOL
FINANCIAL TIMES 2013

#gobeyond

MASTER IN MANAGEMENT

Because achieving your dreams is your greatest challenge. IE Business School's Master in Management taught in English, Spanish or bilingually, trains young high performance professionals at the beginning of their career through an innovative and stimulating program that will help them reach their full potential.

- Choose your area of specialization.
- Customize your master through the different options offered.
- Global Immersion Weeks in locations such as London, Silicon Valley or Shanghai.

Because you change, we change with you.

www.ie.edu/master-management | mim.admissions@ie.edu | [Facebook](#) [Twitter](#) [LinkedIn](#) [YouTube](#) [Instagram](#)



Click on the ad to read more

Example 35.21 Consider the vector field

$$\mathbf{V}(x, y, z) = (xz, yz + xz, 2xz - yz), \quad (x, y, z) \in \mathbb{R}^3.$$

1. Find the divergence $\nabla \cdot \mathbf{V}$ and the rotation $\nabla \times \mathbf{V}$.

Let A denote the half ball given by

$$x^2 + y^2 + z^2 \leq c^2, \quad z \geq 0,$$

where c is a positive constant, and let \mathbf{n} be the outwards unit normal vector of the surface ∂A .

2. Find the flux

$$\Phi = \int_{\partial A} \mathbf{V} \cdot \mathbf{n} dS.$$

3. The surface ∂A is the union of a disc \mathcal{F}_1 and a half sphere \mathcal{F}_2 . Find the fluxes

$$\Phi_1 = \int_{\mathcal{F}_1} \mathbf{V} \cdot \mathbf{n} dS \quad \text{and} \quad \Phi_2 = \int_{\mathcal{F}_2} \mathbf{V} \cdot \mathbf{n} dS.$$

Let \mathcal{K} denote a circle in the plane of equation $z = b$. We denote the centrum of the circle by (x_0, y_0, b) , and its radius is called a .

4. Choose an orientation of the circle \mathcal{K} . Then find the circulation

$$C = \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds.$$

A Divergence, rotation, flux, circulation.

D Follow the guidelines which give the simplest variant.

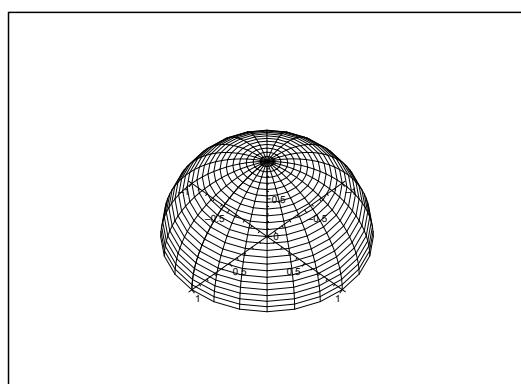


Figure 35.40: The half ball A for $c = 1$.

I 1) By just computing we get

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = z + z + 2x - y = 2x - y + 2z$$

and

$$\begin{aligned} \operatorname{rot} \mathbf{V} &= \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz + xz & 2xz - yz \end{vmatrix} \\ &= (-z - y - x, x - 2x, z) = (-x - y - z, -x, z). \end{aligned}$$

2) Then by Gauß's theorem,

$$\begin{aligned} \Phi &= \int_{\partial A} \mathbf{V} \cdot \mathbf{n} dS = \int_A \operatorname{div} \mathbf{V} d\Omega = \int_A (2x - y + 2z) d\Omega \\ &= 0 + 0 + 2 \int_A z d\Omega = 2 \cdot \frac{\pi}{4} c^4 = \frac{\pi}{2} c^4. \end{aligned}$$

3) Now $\mathbf{n} = (0, 0, -1)$ on \mathcal{F}_1 , where also $z = 0$. Hence

$$\Phi_1 = \int_{\mathcal{F}_1} (0, 0, 0) \cdot \mathbf{n} dS = 0.$$

Then apply the result of 2) and that $\Phi = \Phi_1 + \Phi_2$, to get

$$\Phi_2 = \Phi - \Phi_1 = \frac{\pi}{2} c^4.$$

4) Choose the orientation such that the projection onto the (X, Y) -plane has a positive orientation. Then the corresponding unit normal vector is $\mathbf{n} = (0, 0, 1)$.

By Stokes's theorem, the circulation along \mathcal{K} (which encircles the disc B) is given by

$$\begin{aligned} C &= \oint_{\mathcal{K}} \mathbf{V} \cdot \mathbf{t} ds = \int_B \mathbf{n} \cdot \nabla \times \mathbf{V} dS = \int_B (0, 0, 1) \cdot (-x - y - b, -x, -b) dS \\ &= -b \int_B dS = -b \cdot \operatorname{area}(B) = -b \cdot \pi \cdot a^2. \end{aligned}$$

36 Nabla calculus

From time to time we have previously in \mathbb{R}^3 used the notations

$$\mathbf{grad} = \nabla, \quad \mathbf{div} = \nabla \cdot \quad \text{and} \quad \mathbf{rot} = \mathbf{curl} = \nabla \times.$$

In this section we shall more systematically use nabla (∇) instead of **grad**, **div**, **rot** and **curl**. It turns up that this change of notation is very convenient, because the formulæ containing ∇ will be very similar to those already known from *Linear Algebra* in the three-dimensional Euclidean space $E_3 \sim \mathbb{R}^3$.

Remark 36.1 The gradient, divergence etc. were introduced in the middle of the 19th century. Clearly, one needed a shorthand for these new operations, and one chose to put delta, Δ , upside down to get ∇ . Concerning the name of this new operator one first tried to spell delta backwards, “atled”, but this suggestion was never successful. It probably resembled “Amled =Hamlet” too much. At that time, however, one had started to excavate the ruins of ancient Iraq, and on some of the clay tablets one also found pictures of harps. These harps had the same shape as ∇ , so this new symbol was afterwards called “nabla”, which is the Assyrian word for harp. One may still find a remnant of it in David’s 57th psalm, verse 9, where the word “nevel” is used. This is an ancient Hebrew word for a harp, which no longer is commonly understood. \diamond

36.1 The vectorial differential operator ∇

The vectorial differential operator ∇ is defined in all spaces \mathbb{R}^n in a rectilinear coordinate system by

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

In most applications, however, we restrict ourselves to $E_3 \sim \mathbb{R}^3$, where we write

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

We collect the definitions and rules, which we have met previously.

- 1) **Gradient of a scalar field** $f : \Omega \rightarrow \mathbb{R}$, where $\Omega \subseteq \mathbb{R}$, and $f \in C^1(\Omega)$,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

- 2) **Divergence of a vector field** $\mathbf{V} : \Omega \rightarrow \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, and $\mathbf{V} \in C^1(\Omega) \times \cdots \times C^1(\Omega)$,

$$\nabla \cdot \mathbf{V} = \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n}.$$

- 3) **Rotation of a vector field** $\mathbf{V} : \Omega \rightarrow \mathbb{R}$, $\Omega \subseteq \mathbb{R}^3$, and $\mathbf{V} \in C^1(\Omega)^3$. Note that rotation is only introduced in \mathbb{R}^3 .

$$\nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix},$$

The formal determinant is interpreted in the following way as if we were dealing with an ordinary determinant. The only difference is that we *always* calculate downwards and let the differential operators in the second line act on the functions in the third line. Note that if ∇ is replaced by a vector field \mathbf{U} , then we just get the usual determinant formula for $\mathbf{U} \times \mathbf{V}$.

- 4) **Differential of a scalar field** $f : \Omega \rightarrow \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$ and $f \in C^1(\Omega)$. We introduce the infinitesimal vector

$$d\mathbf{x} = (dx_1, \dots, dx_n).$$

Then

$$df = (d\mathbf{x} \cdot \nabla)f = \nabla f \cdot d\mathbf{x} = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.$$

- 5) **Differential of a vector field** $\mathbf{V} : \Omega \rightarrow \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{V} \in C^1(\Omega)^n$. This is similar to the case of a scalar field above, so

$$d\mathbf{V} = (d\mathbf{x} \cdot \nabla)\mathbf{V} = (\nabla V_1 \cdot d\mathbf{x}, \dots, \nabla V_n \cdot d\mathbf{x}) = (dV_1, \dots, dV_n).$$

It follows from these definitions that all these differential operators are linear.

**"I studied English for 16 years but...
...I finally learned to speak it in just six lessons"**

Jane, Chinese architect

ENGLISH OUT THERE

Click to hear me talking before and after my unique course download



Click on the ad to read more

We shall in the following let α and β denote scalars, $f, g \in C^1(\Omega)$ real functions, $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{U}, \mathbf{V} \in C^1/\Omega)^n$ vector fields. Finally, $d\mathbf{x} = (dx_1, \dots, dx_n)$ denotes the infinitesimal vector. Then the linearities goe as follows:

1) **Gradient**

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g.$$

2) **Divergence**

$$\nabla \cdot (\alpha \mathbf{U} + \beta \mathbf{V}) = \alpha \nabla \cdot \mathbf{U} + \beta \nabla \cdot \mathbf{V}.$$

3) **Rotation** (only in \mathbb{R}^3)

$$\nabla \times (\alpha \mathbf{U} + \beta \mathbf{V}) = \alpha \nabla \times \mathbf{U} + \beta \nabla \times \mathbf{V}.$$

4) **Differential of a scalar field**

$$d(\alpha f + \beta g) = \alpha df + \beta dg = \alpha \nabla f \cdot d\mathbf{x} + \beta \nabla g \cdot d\mathbf{x}.$$

5) **Differential of vector fields**

$$d(\alpha \mathbf{U} + \beta \mathbf{V}) = (d\mathbf{x} \cdot \nabla)(\alpha \mathbf{U} + \beta \mathbf{V}) = \alpha d\mathbf{U} + \beta d\mathbf{V}.$$

36.2 Differentiation of products

Let f, g be C^1 functions and let \mathbf{U}, \mathbf{V} be C^1 vector fields on the open set Ω . Then fg and $\mathbf{U} \cdot \mathbf{V}$ are C^1 functions, i.e. scalar fields, on Ω , so we can form the gradient and the differential of them. The rules are

1) **Gradient of fg .**

$$\nabla(fg) = g \nabla f + f \nabla g.$$

this is proved by considering each coordinate separately,

$$\frac{\partial}{\partial x_j}(fg) = g \frac{\partial f}{\partial x_j} + f \frac{\partial g}{\partial x_j}.$$

We note that if $g = f$, then the formula above reduces to

$$\nabla(f^2) = 2f \nabla f.$$

2) **Differential of fg .**

$$d(fg) = d\mathbf{x} \cdot \nabla(fg) = f d\mathbf{x} \cdot \nabla f + g d\mathbf{x} \cdot \nabla g = g df + f dg.$$

3) **Gradient of $\mathbf{U} \cdot \mathbf{V}$.** In general, this formula becomes messy, so it is not given here. However, we shall later return to the special case, when we are dealing with three dimensions.

Then we consider the product $f\mathbf{U}$, which is a vector field. We get only one possibility.

1) Divergence of $f\mathbf{U}$,

$$\operatorname{div}(f\mathbf{U}) = \nabla \cdot (f\mathbf{U}) = (\nabla f) \cdot \mathbf{U} + f \nabla \cdot \mathbf{U} = \mathbf{grad}f \cdot \mathbf{U} + f \operatorname{div} \mathbf{U}.$$

We leave the straightforward proof to the reader. Note here that the formula with nabla is easier to remember than the interpretation

$$\operatorname{div}(f\mathbf{U}) = \mathbf{grad}f \cdot \mathbf{U} + f \operatorname{div} \mathbf{U}.$$

In the remaining formulæ we assume that we consider the three dimensional case, so we can form the vector product. Then the possibilities of differentiation of a product, using these differential operators, are

$$\mathbf{rot}(f\mathbf{U}) = \nabla \times (f\mathbf{U}), \quad \operatorname{div}(\mathbf{U} \times \mathbf{V}) = \nabla \cdot (\mathbf{U} \times \mathbf{V}), \quad \mathbf{rot}(\mathbf{U} \times \mathbf{V}) = \nabla \times (\mathbf{U} \times \mathbf{V}),$$

and

$$\mathbf{grad}(\mathbf{U} \cdot \mathbf{V}) = \nabla(\mathbf{U} \cdot \mathbf{V}),$$

supplied with the differentials $d(f\mathbf{U})$, $d(\mathbf{U} \cdot \mathbf{V})$ and $d(\mathbf{U} \times \mathbf{V})$. Since the differentials follow from the rules $df = d\mathbf{x} \cdot \nabla f$ and $d\mathbf{U} = (d\mathbf{x} \cdot \nabla)\mathbf{U}$ above, we shall not further deal with the differentials.

The rules are

1) Rotation of $f\mathbf{U}$, $n = 3$.

$$\mathbf{rot}(f\mathbf{U}) = \nabla \times (f\mathbf{U}) = (\nabla f) \times \mathbf{U} + f \nabla \times \mathbf{U} = \mathbf{grad}f \times \mathbf{U} + f \mathbf{rot} \mathbf{U}.$$

We note that $\nabla \times$ behaves like a derivation.

2) Divergence of $\mathbf{U} \times \mathbf{V}$, $n = 3$.

$$\operatorname{div}(\mathbf{U} \times \mathbf{V}) = \nabla \cdot (\mathbf{U} \times \mathbf{V}) = (\nabla \times \mathbf{U}) \cdot \mathbf{V} - (\nabla \times \mathbf{V}) \cdot \mathbf{U} = \mathbf{rot} \mathbf{U} \cdot \mathbf{V} - \mathbf{rot} \mathbf{V} \cdot \mathbf{U}.$$

The structure is almost the same as for a derivation. The minus sign is caused by the fact that the vector product is anticommutative, $\mathbf{U} \times \mathbf{V} = -\mathbf{V} \times \mathbf{U}$. This follows from the proof, which again is left to the reader.

3) Rotation of $\mathbf{U} \times \mathbf{V}$, $n = 3$. The complicated formula below relies on the rule for the double vector product in \mathbb{R}^3 ,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \quad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3,$$

known from *Linear Algebra*. We quote again without proof the result

$$\mathbf{rot}(\mathbf{U} \times \mathbf{V}) = \nabla \times (\mathbf{U} \times \mathbf{V}) = (\mathbf{V} \cdot \nabla)\mathbf{U} - \mathbf{V}(\nabla \cdot \mathbf{V}) - (\mathbf{U} \cdot \nabla)\mathbf{V} + \mathbf{U}(\nabla \cdot \mathbf{V}).$$

4) Gradient of $\mathbf{U} \cdot \mathbf{V}$, $n = 3$.

$$\nabla(\mathbf{U} \cdot \mathbf{V}) = (\mathbf{V} \cdot \nabla)\mathbf{U} + \mathbf{V} \times (\nabla \times \mathbf{U}) + (\mathbf{U} \cdot \nabla)\mathbf{V} + \mathbf{U} \times (\nabla \times \mathbf{V}).$$

We shall not prove this rule either, although the proof is not too hard. We note that the expected terms are

$$(\mathbf{V} \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{V},$$

and then we have the two additional correction terms

$$\mathbf{V} \times (\nabla \times \mathbf{U}) + \mathbf{U} \times (\nabla \times \mathbf{V}) = \mathbf{V} \times \mathbf{rot}\mathbf{U} + \mathbf{U} \times \mathbf{rot}\mathbf{V},$$

which involve the rotations of \mathbf{U} and \mathbf{V} . Again we skip the proof and only mention that it uses the formula for $\nabla \times (\mathbf{U} \times \mathbf{V})$.

In the special case, where $\mathbf{V} = \mathbf{U}$, we get

$$\mathbf{grad}(\|\mathbf{U}\|^2) = \nabla(\|\mathbf{U}\|^2) = \nabla(\mathbf{U} \cdot \mathbf{U}) = 2\mathbf{U} \times (\nabla \mathbf{U}) = 2\mathbf{U} \times \mathbf{rot}\mathbf{U}.$$

It should be mentioned that if another variable is involved, e.g. t ($\neq x, y, z$), then we have the usual rules

$$\frac{\partial}{\partial t}(\mathbf{U} \cdot \mathbf{V}) = \frac{\partial \mathbf{U}}{\partial t} \cdot \frac{\partial \mathbf{V}}{\partial t}, \quad \text{and} \quad \frac{\partial}{\partial t}(\mathbf{U} \times \mathbf{V}) = \frac{\partial \mathbf{U}}{\partial t} \times \frac{\partial \mathbf{V}}{\partial t}.$$

36.3 Differentiation of second order

The three vectorial differential operators of first order,

$$\mathbf{grad} = \nabla, \quad (\text{from 1 dimension to 3}), \quad \mathbf{div} = \nabla \cdot, \quad (\text{from 3 dimensions to 1}),$$

$$\mathbf{rot} = \nabla \times, \quad (\text{from 3 dimensions to 3}),$$

can be combined to form differential operators of second order. Note that whenever $\mathbf{rot} = \nabla \times$ is involved, then we tacitly require that the dimension is 3. There are five possibilities:

- 1) **Rotation of gradient**, $n = 3$. From 1 via 3 to 3 dimensions,

$$\mathbf{rot}(\mathbf{grad}f) = \nabla \times \nabla f = \mathbf{0},$$

because $\nabla \times \nabla$ is the zero operator,

- 2) **Divergence of rotation**, $n = 3$. From 3 via 3 to 1 dimension,

$$\mathbf{div}(\mathbf{rot}\mathbf{U}) = \nabla \cdot (\nabla \times \mathbf{U}) = 0.$$

This follows straightforward by some tedious calculations.

- 3) **Divergence of a gradient**, all $n \in \mathbb{N}$. From 1 via n to 1 dimension.

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

This is the so-called *Laplace operator*, which is also denoted by Δ , thus

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We note that Δ can also act on a vector field,

$$\nabla^2 \mathbf{U} = \nabla^2(U_1 \mathbf{e}_1 + \cdots + U_n \mathbf{e}_n) = (\nabla^2 U_1) \mathbf{e}_1 + \cdots + (\nabla^2 U_n) \mathbf{e}_n.$$

We also mention the following formula for the Laplace operator acting on a product of two functions,

$$\nabla^2(fg) = f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f.$$

- 4) **Double rotation, $n = 3$** , From 3 via 3 to 3 dimensions.

$$\nabla \times (\nabla \times \mathbf{U}) = \nabla(\nabla \cdot \mathbf{U}) - \nabla \cdot \nabla \mathbf{U},$$

which can also be written

$$\mathbf{rot}(\mathbf{rot} \mathbf{U}) = \mathbf{grad}(\operatorname{div} \mathbf{U}) - \Delta \mathbf{U},$$

where

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the *Laplace operator*.

- 5) **Gradient of divergence, general** $n \in \mathbb{N}$. From n via 1 to n dimensions.

$$\nabla(\nabla \cdot \mathbf{U}) = \mathbf{grad}(\operatorname{div} \mathbf{U}) = \mathbf{grad} \left(\sum_{j=1}^n \frac{\partial U_j}{\partial x_j} \right),$$

which cannot be further simplified.

Excellent Economics and Business programmes at:


/
**university of
groningen**




**“The perfect start
of a successful,
international career.”**

CLICK HERE

to discover why both socially
and academically the University
of Groningen is one of the best
places for a student to be

www.rug.nl/feb/education



Click on the ad to read more

36.4 Nabla applied on \mathbf{x}

In the applications one often needs to consider, ∇ , $\nabla \cdot$, or $\nabla \times$, applied to expressions in \mathbf{x} . We list some of the formulæ below.

- 1) *Formulæ valid for all $n \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^n$,*

$$\operatorname{div} \mathbf{x} = \nabla \cdot \mathbf{x} = n,$$

$$\Delta \mathbf{x} = \nabla^2 \mathbf{x} = 0,$$

$$\Delta(\mathbf{x} \cdot \mathbf{x}) = \Delta(\|\mathbf{x}\|^2) = 2n,$$

$$(\mathbf{U} \cdot \nabla) \mathbf{x} = \mathbf{U},$$

$$\operatorname{grad}(\mathbf{a} \cdot \mathbf{x}) = \nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}, \quad \text{constant } \mathbf{a} \in \mathbb{R}^n.$$

- 2) *Formulæ valid for all $n \in \mathbb{N}$ and for $\mathbf{x} \neq \mathbf{0}$.*

If $f(\mathbf{x}) = F(\|\mathbf{x}\|)$ only depends on $\|\mathbf{x}\|$, then

$$\nabla f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} F(\|\mathbf{x}\|).$$

- 3) *Formulæ for all $n \geq 3$ and $\mathbf{x} \neq \mathbf{0}$.*

$$\nabla(\|\mathbf{x}\|^{2-n}) = 0 \quad \text{for } \mathbf{x} \neq \mathbf{0}.$$

- 4) *If $n = 3$, then*

$$\operatorname{rot} \mathbf{x} = \nabla \times \mathbf{x} = \mathbf{0}, \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^3,$$

and

$$\Delta\left(\frac{1}{\|\mathbf{x}\|}\right) = \nabla^2\left(\frac{1}{\|\mathbf{x}\|}\right) = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}.$$

- 5) *When $n = 2$, then*

$$\nabla^2(\ln \|\mathbf{x}\|) = 0.$$

As an application we consider a rotating body in space of angle velocity ω with respect to an axis through the origin. Then the velocity at the point \mathbf{x} is $\mathbf{v} = \omega \times \mathbf{x}$, and we find

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \nabla \cdot (\omega \times \mathbf{x}) = (\nabla \times \omega) \cdot \mathbf{x} - (\nabla \times \mathbf{x}) \cdot \omega = 0,$$

and

$$\operatorname{rot} \mathbf{v} = \nabla \times \mathbf{v} = \nabla \times (\omega \times \mathbf{x}) = (\mathbf{x} \cdot \nabla) \omega - \mathbf{x} (\nabla \cdot \omega) - (\omega \cdot \nabla) \mathbf{x} + \omega (\nabla \cdot \mathbf{x}) = -\omega + 3\omega = 2\omega.$$

36.5 The integral theorems

We shall here see what happens when we apply the nabla calculus in the cases of Gauß's and Stokes's theorems. This will give us an inspiration to derive new formulæ, which are also valid.

We first recall *Gauß's theorem*

$$\int_{\Omega} \operatorname{div} \mathbf{V} = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} dS.$$

When we replace the operator div with the symbol $\nabla \cdot$, we get

$$\int_{\Omega} \nabla \cdot \mathbf{V} d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} dS,$$

which shows the structure of the theorem. When we go from the left to the right, the body Ω is replaced by the surface $\partial\Omega$, and the operator $\nabla \cdot$ is replaced by an inner product with the unit normal vector field, $\mathbf{n} \cdot$.

This gives us the hint that we might try to replace $\nabla \cdot$ by the other differential operator $\operatorname{rot} = \nabla \times$, and then of course write $\mathbf{n} \times$ instead of $\mathbf{n} \cdot$. This suggests the formula

$$\int_{\Omega} \nabla \times \mathbf{V} d\Omega = \int_{\partial\Omega} \mathbf{n} \times \mathbf{V} dS.$$

It can be proved that this is indeed true, so we have obtained the following formula, written with rot instead of $\nabla \times$,

$$\int_{\operatorname{rot} \mathbf{V}} d\Omega = \int_{\partial\Omega} \mathbf{n} \times \mathbf{V} dS,$$

where \mathbf{n} as usual denotes the outgoing unit normal vector field on the surface $\partial\Omega$.

If we instead replace $\nabla \cdot$ by ∇ alone (and the vector field \mathbf{V} by a function f), then we see that we may expect a formula of the form

$$\int_{\Omega} \nabla f d\Omega = \int_{\partial\Omega} \mathbf{n} f dS, \quad \text{i.e.} \quad \int_{\Omega} \operatorname{grad} f d\Omega = \int_{\partial\Omega} \mathbf{n} f dS.$$

Again this formula can be proved to be correct. Note that the result here is a vector field.

We have above given three versions of *Gauß's theorem*.

Then let us turn to *Stokes's theorem*,

$$\int_{\mathcal{F}} \mathbf{n} \cdot \operatorname{rot} \mathbf{V} dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} ds,$$

which using the nabla notation is written

$$\int_{\mathcal{F}} \mathbf{n} \cdot \nabla \times \mathbf{V} dS = \int_{\mathcal{F}} (\mathbf{n} \cdot \nabla) \times \mathbf{V} dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} ds.$$

When we replace $\mathbf{t} \cdot \mathbf{V}$ by $\mathbf{t} \times \mathbf{V}$, we must apparently by analogy replace $(\mathbf{n} \cdot \nabla)$ by $(\mathbf{n} \times \nabla)$ to see what happens. In this way we see that we may expect the following formula, only derived by analogy,

$$\int_{\mathcal{F}} (\mathbf{n} \times \nabla) \times \mathbf{V} dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} ds.$$

Once we have got a hunch of the formula, it is not difficult to prove that it is indeed correct. As usual we shall not do it here.

Finally, let us see what we should expect, when the dot has disappeared on the right hand side of the original Stokes's theorem. Clearly, we now must consider a function f times the unit tangent vector field \mathbf{n} as the integrand, because we have not defined pointwise multiplication of two vector fields. So we derive that

$$\oint_{\delta\mathcal{F}} \mathbf{t} f \, ds$$

should be expected to occur on the right hand side.

By analogy, the integrand on the left hand side must contain \mathbf{n} and \times and ∇f . This is only possible in the following way,

$$\int_{\mathcal{F}} \mathbf{n} \times \nabla f \, dS = \oint_{\delta\mathcal{F}} \mathbf{t} f \, ds.$$

Again it can be proved that this formula indeed is valid.

Summarizing we have got three versions of *Gauß's theorem*,

1. $\int_{\Omega} \nabla \cdot \mathbf{V} \, d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} \, dS,$
2. $\int_{\Omega} \nabla \times \mathbf{V} \, d\Omega = \int_{\partial\Omega} \mathbf{n} \times \mathbf{V} \, dS,$
3. $\int_{\Omega} \nabla f \, d\Omega = \int_{\partial\Omega} \mathbf{n} f \, dS,$

where \mathbf{n} is the outgoing unit normal vector field of the surface $\partial\Omega$, and three versions of *Stokes's theorem*,

1. $\int_{\mathcal{F}} \mathbf{n} \cdot \nabla \times \mathbf{V} \, dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} \, ds,$
2. $\int_{\mathcal{F}} (\mathbf{n} \times \nabla) \times \mathbf{V} \, dS = \oint_{\delta\mathcal{F}} \mathbf{t} \times \mathbf{V} \, ds,$
3. $\int_{\mathcal{F}} \mathbf{n} \times \nabla f \, dS = \oint_{\delta\mathcal{F}} \mathbf{t} f \, ds,$

where \mathbf{t} is the unit tangent field of $\delta\mathcal{F}$, and \mathbf{n} is the unit normal vector field on \mathcal{F} , corresponding to the orientation of $\delta\mathcal{F}$.

As simple applications we first use the second variant of *Gauß's theorem* above with $\mathbf{V} = \nabla f$. We get in this case

$$\int_{\partial\Omega} \mathbf{n} \times \nabla f \, dS = \int_{\Omega} \nabla \times \nabla f \, d\Omega = \mathbf{0}.$$

Then we – also with $\mathbf{V} = \nabla f$ – apply the third version of Stokes's theorem. Since $\delta\mathcal{F} = \emptyset$, we get trivially

$$\int_{\partial\Omega} \mathbf{n} \times \nabla f \, dS = \oint_{\emptyset} \mathbf{t} f \, ds = \mathbf{0}.$$

Then we choose $\mathbf{V} = \mathbf{x}$ in the second version of Stokes's theorem,

$$\int_{\mathcal{F}} (\mathbf{n} \times \nabla) \times \mathbf{x} \, dS = \oint_{\delta\mathcal{F}} \mathbf{t} \times \mathbf{x} \, ds.$$

The integrand of the integrand of the left hand side is calculated,

$$(\mathbf{n} \times \nabla) \times \mathbf{x} = \nabla(\mathbf{x} \cdot \mathbf{n}) - \mathbf{n}(\nabla \cdot \mathbf{x}) = \mathbf{n} - 3\mathbf{n} = -2\mathbf{n},$$

and since $\mathbf{t} \times \mathbf{x} = -\mathbf{x} \times \mathbf{t}$, we conclude by insertion and reduction that

$$\int_{\mathcal{F}} \mathbf{n} \, dS = \frac{1}{2} \oint_{\delta\mathcal{F}} \mathbf{x} \times \mathbf{t} \, ds.$$

This expression is sometimes called the *vectorial area*, though this term may be misleading. In particular, every closed surface \mathcal{F} has the vectorial surface $\mathbf{0}$, because $\delta\mathcal{F} = \emptyset$.

American online LIGS University

is currently enrolling in the
Interactive Online BBA, MBA, MSc,
DBA and PhD programs:

- ▶ enroll by **September 30th, 2014** and
- ▶ save up to **16%** on the tuition!
- ▶ pay in 10 installments / 2 years
- ▶ Interactive Online education
- ▶ visit www.ligsuniversity.com to find out more!

Note: LIGS University is not accredited by any nationally recognized accrediting agency listed by the US Secretary of Education.
[More info here.](#)



Click on the ad to read more

36.6 Partial integration

The principle of partial integration is in its simplest form of functions in one real variable derived from the rule of differentiation,

$$\frac{d}{dx}(fg) = f \frac{dg}{dx} + g \frac{df}{dx},$$

from which by a rearrangement

$$f \frac{dg}{dx} = \frac{d}{dx}(fg) - g \frac{df}{dx}.$$

Hence by an integration,

$$\int f \frac{dg}{dx} dx = f(x)g(x) - \int g \frac{df}{dx} dx.$$

We can derive some similar rules for some of the differentiations introduced in Section 36.2 by using Gauß's and Stokes's theorem.

1) Gradient of a product of functions and Gauß's theorem

$$\nabla(fg) = g \nabla f + f \nabla g, \quad \text{or by a rearrangement} \quad f \nabla g = \nabla(fg) - g \nabla f,$$

from which by the third version of Gauß's theorem,

$$\int_{\Omega} f \nabla g d\Omega = \int_{\Omega} \nabla(fg) d\Omega - \int_{\Omega} f \nabla f d\Omega = \int_{\partial\Omega} \mathbf{n} fg dS - \int_{\Omega} g \nabla f d\Omega.$$

Note that we recover the third version of Gauß's theorem by choosing $f = 1$, so this formula may be considered as an extension of this third version.

Choosing $g = f$ we get by a rearrangement,

$$\int_{\Omega} f \nabla f d\Omega = \frac{1}{2} \int_{\partial\Omega} \mathbf{n} f^2 dS.$$

When furthermore $f = 1$, we just get the known result that $\int_{\partial\Omega} \mathbf{n} dS = \mathbf{0}$ for all closed surfaces $\partial\Omega$.

2) Divergence of a product of a function and a vector field and Gauß's theorem

$$\nabla \cdot (f\mathbf{U}) = (\nabla f) \cdot \mathbf{U} + f \nabla \cdot \mathbf{U}, \quad \text{or by a rearrangement} \quad f \nabla \cdot \mathbf{U} = \nabla \cdot (f\mathbf{U}) - (\nabla f) \cdot \mathbf{U},$$

so by an integration over the three dimensional body Ω followed by the first variant of Gauß's theorem,

$$\int_{\Omega} f \nabla \cdot \mathbf{U} d\Omega = \int_{\Omega} \nabla \cdot (f\mathbf{U}) d\Omega - \int_{\Omega} \nabla f \cdot \mathbf{U} d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot f\mathbf{U} dS - \int_{\Omega} \nabla f \cdot \mathbf{U} d\Omega.$$

When $f = 1$, this new formula degenerates to the first version of Gauß's theorem, so

$$\int_{\Omega} f \nabla \cdot \mathbf{U} d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot f\mathbf{U} dS - \int_{\Omega} \nabla f \cdot \mathbf{U} d\Omega,$$

may be considered as an extension of this first version of Gauß's theorem.

3) Gradient of a product of functions and Stokes's theorem

The formula

$$f \nabla g = \nabla(fg) - g \nabla f$$

is multiplied from the left by $\mathbf{n} \times$, so

$$f \mathbf{n} \times \nabla g = \mathbf{n} \times \nabla(fg) - g \mathbf{n} \times \nabla f.$$

Let \mathcal{F} be a surface of boundary $\delta\mathcal{F}$. Then we get by the third version of Stokes's theorem that

$$\int_{\mathcal{F}} f \mathbf{n} \times \nabla g \, dS = \int_{\mathcal{F}} \mathbf{n} \times \nabla(fg) \, dS - \int_{\mathcal{F}} g \mathbf{n} \times \nabla f \, dS = \oint_{\delta\mathcal{F}} f g \mathbf{t} \, ds - \int_{\mathcal{F}} g \mathbf{n} \times \nabla f \, dS,$$

which is an extension of the third version of Stokes's theorem. This is again obtained by putting $f = 1$.

4) Rotation of a product of a function and a vector field and Stokes's theorem,

$$f \nabla \times \mathbf{U} = \nabla \times (f \mathbf{U}) - \nabla f \times \mathbf{U}.$$

When this equation is multiplied from the left by $\mathbf{n} \cdot$, we get

$$\mathbf{n} \cdot f \nabla \times \mathbf{U} = \mathbf{n} \cdot \nabla \times (f \mathbf{U}) - \mathbf{n} \cdot \nabla f \times \mathbf{U}.$$

Let \mathcal{F} be a surface of boundary $\delta\mathcal{F}$. Then it follows from the first version of Stokes's theorem that

$$\int_{\mathcal{F}} \mathbf{n} \cdot f \nabla \times \mathbf{U} \, dS = \int_{\mathcal{F}} \mathbf{n} \cdot \nabla \times (f \mathbf{U}) \, dS - \int_{\mathcal{F}} \mathbf{n} \cdot \nabla f \times \mathbf{U} \, dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot f \mathbf{U} \, ds - \int_{\mathcal{F}} \mathbf{n} \cdot \nabla f \times \mathbf{U} \, dS,$$

which can be considered as an extension of the first version of Stokes's theorem, because this is recovered, when we put $f = 1$.

36.7 Overview of Nabla calculus

In complicated cases it may be useful to apply the abstract theory of *nabla calculus* instead of the enormous calculations with coordinates.

The theory may look a little confusing the first time one sees it, until one realizes that there are three different products in the vector analysis in \mathbb{R}^3 :

scalar multiplication:	no special notation	$\alpha \mathbf{V}$,	vector,
inner product (dot product):	dot	$\mathbf{U} \cdot \mathbf{V}$,	scalar,
vector product (cross product):	cross	$\mathbf{U} \times \mathbf{V}$,	vector.

These are transferred to a nabla notation,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

by the relations

no special notation $\mathbf{grad}f = \nabla f,$

dot $\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V},$

cross $\mathbf{rot} \mathbf{V} = \nabla \times \mathbf{V}$

When we apply the correspondence $\nabla \sim \mathbf{n}$ we get **Gauß's theorem** in three versions:

$$\int_{\partial\Omega} \mathbf{n} f dS = \int_{\Omega} \nabla f d\Omega = \int_{\Omega} \mathbf{grad} f d\Omega, \quad \text{no special notation}$$

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} dS = \int_{\Omega} \nabla \cdot \mathbf{V} d\Omega = \int_{\Omega} \operatorname{div} \mathbf{V} d\Omega, \quad \text{dot product, usual version}$$

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{V} dS = \int_{\Omega} \nabla \times \mathbf{V} d\Omega = \int_{\Omega} \mathbf{rot} \mathbf{V} d\Omega, \quad \text{cross product.}$$

When we apply the correspondence $(\mathbf{n}, \nabla, \times) \sim \mathbf{t}$, where \cdot and \times are always put in a meaningful connection, we obtain **Stokes's theorem** in three versions:

$$\oint_{\delta\mathcal{F}} \mathbf{t} f ds = \int_{\mathcal{F}} \mathbf{n} \times \nabla f dS = \int_{\mathcal{F}} \mathbf{n} \times \mathbf{grad} f dS, \quad \text{no special notation}$$

$$\oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{V} ds = \int_{\mathcal{F}} \mathbf{n} \cdot (\nabla \times \mathbf{V}) dS = \int_{\mathcal{F}} \mathbf{n} \cdot \mathbf{rot} \mathbf{V} dS, \quad \text{dot product, usual version}$$

$$\oint_{\delta\mathcal{F}} \mathbf{t} \times \mathbf{V} ds = \int_{\mathcal{F}} (\mathbf{n} \times \nabla) \times \mathbf{V} dS, \quad \text{cross product.}$$

..... Alcatel-Lucent 

www.alcatel-lucent.com/careers

What if you could build your future and create the future?

One generation's transformation is the next's status quo.
In the near future, people may soon think it's strange that devices ever had to be "plugged in." To obtain that status, there needs to be "The Shift".



Click on the ad to read more

36.8 Overview of partial integration in higher dimensions

The simplest case is described by

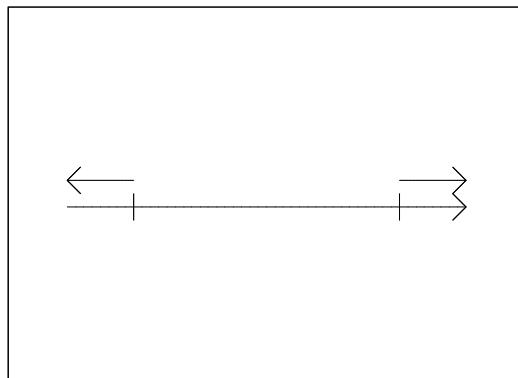


Figure 36.1: The interval $[a, b]$, where one goes from 1 dimension to 0 dimensions.

- 1) **Main theorem of the differential and integral calculus, 1 dimension:**

$$\int_a^b \frac{df}{dx} dx = [f(x)]_a^b = f(b) - f(a),$$

i.e. a 1-dimensional integral is transformed into a 0-dimensional “boundary integral”. Note that in the right end point, where the outer normal is equal to the direction of the axis, we use the sign +, and in the left end point, where the outer normal is opposite to the direction of the axis, we use the sign -.

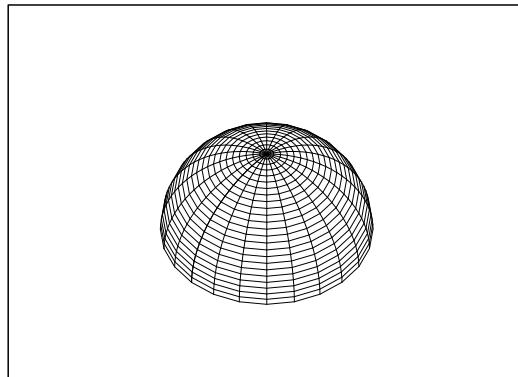


Figure 36.2: A surface \mathcal{F} with its limiting curve $\delta\mathcal{F}$.

- 2) **Stokes's theorem, 2 dimensions:**

$$\int_{\mathcal{F}} \mathbf{n} \cdot \text{rot} \mathbf{V} dS = \oint_{\delta\mathcal{F}} \mathbf{t} \cdot \mathbf{B} ds,$$

i.e. a 2-dimensional surface integral is transferred into a 1-dimensional “boundary integral”.

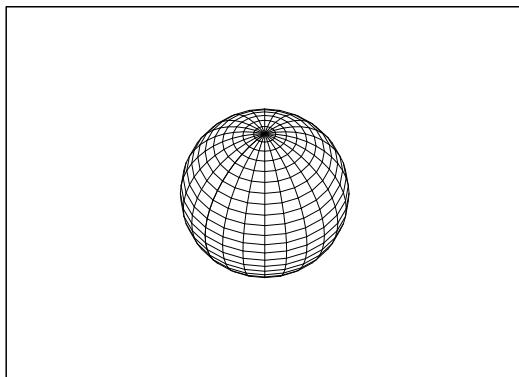


Figure 36.3: A domain in space Ω with surface $\partial\Omega$. The unit normal vector field is everywhere directed away from Ω .

3) Gauß's theorem, 3 dimensions:

$$\int_{\Omega} \operatorname{div} \mathbf{V} d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{V} dS,$$

i.e. a 3-dimensional space integral is transferred into a 2-dimensional “boundary integral”.

The more advanced applications use the nabla calculus, cf. the previous sections in this chapter. Here we have several special cases, because there in e.g. in \mathbb{R}^3 exist three types of product (scalar multiplication, dot and cross product) and three types of differentiation (gradient, divergence, rotation). The basic idea is that if the integrand is of the form $\Phi \otimes \mathcal{D}\Psi$, where \otimes stands for any of the possible products, and \mathcal{D} stands for any of the possible differentiations, then we get a new integrand (by a partial integration) of a simpler form $-\tilde{\mathcal{D}}\Phi \tilde{\otimes} \Psi$, where $(\tilde{\otimes}, \tilde{\mathcal{D}})$ does not have to be equal to (\otimes, \mathcal{D}) . The price for this is that we get an additional boundary integral.



**Join the best at
the Maastricht University
School of Business and
Economics!**

Top master's programmes

- 33rd place Financial Times worldwide ranking: MSc International Business
- 1st place: MSc International Business
- 1st place: MSc Financial Economics
- 2nd place: MSc Management of Learning
- 2nd place: MSc Economics
- 2nd place: MSc Econometrics and Operations Research
- 2nd place: MSc Global Supply Chain Management and Change

Sources: Keuzegids Master ranking 2013; Elsevier 'Beste Studies' ranking 2012; Financial Times Global Masters in Management ranking 2012

**Visit us and find out why we are the best!
Master's Open Day: 22 February 2014**

Maastricht
University is
the best specialist
university in the
Netherlands
(Elsevier)

www.mastersopenday.nl



Click on the ad to read more

36.9 Examples in nabla calculus

Example 36.1 Let \mathbf{V} denote a vector field, which is both divergence free and rotation free, and let \mathbf{e} be a fixed unit vector. We consider also the following fields,

$$F = -\mathbf{e} \cdot \mathbf{V}, \quad \mathbf{W} = \mathbf{V} \times \mathbf{e}, \quad \mathbf{U} = -\nabla F, \quad \mathbf{T} = \nabla \times \mathbf{W}.$$

1) show that

$$\nabla \times (\mathbf{V} \times \mathbf{x}) = \mathbf{V} + \nabla(\mathbf{V} \cdot \mathbf{x}).$$

2) Show that \mathbf{T} is the same vector field as \mathbf{U} , and that this field also is both divergence free and rotation free.

A Nabla calculus.

D Just exploit the assumptions,

$$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = 0 \quad \text{and} \quad \operatorname{rot} \mathbf{V} = \nabla \times \mathbf{V} = \mathbf{0},$$

and the rules of differentiation of products.

I 1) We shall use the following well-known rule of calculation

$$\nabla \times (\mathbf{V} \times \mathbf{W}) = (\mathbf{W} \cdot \nabla) \mathbf{V} - \mathbf{W}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla) \mathbf{W} + \mathbf{V}(\nabla \cdot \mathbf{W})$$

with $\mathbf{W} = \mathbf{x}$, thus

$$\begin{aligned} \nabla \times (\mathbf{V} \times \mathbf{x}) &= (\mathbf{x} \cdot \nabla) \mathbf{V} - \mathbf{x}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla) \mathbf{x} + \mathbf{V}(\nabla \cdot \mathbf{x}) \\ &= (\mathbf{x} \cdot \nabla) \mathbf{V} - \mathbf{0} - (\mathbf{V} \cdot \nabla) \mathbf{x} + 3\mathbf{V} \\ &= \mathbf{V} + (\mathbf{x} \cdot \nabla) \mathbf{V} - (\mathbf{V} \cdot \nabla) \mathbf{x} + 2(\mathbf{V} \cdot \nabla) \mathbf{x} \\ &= \mathbf{V} + (\mathbf{x} \cdot \nabla) \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{x} \\ &= \mathbf{V} + \nabla(\mathbf{V} \cdot \mathbf{x}), \end{aligned}$$

where we have used that

$$(\mathbf{V} \cdot \nabla) \mathbf{x} = \left\{ V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial}{\partial y} + V_3 \frac{\partial}{\partial z} \right\} (x, y, z) = (V_1, V_2, V_3) = \mathbf{V},$$

and that

$$\nabla(\mathbf{V} \cdot \mathbf{x}) = (\mathbf{V} \cdot \nabla) \mathbf{x} + (\mathbf{x} \cdot \nabla) \mathbf{V}.$$

2) Consider in particular \mathbf{T} and put $\mathbf{W} = \mathbf{x}$. Then

$$\begin{aligned} \mathbf{T} &= \nabla \times \mathbf{W} = \nabla \times (\mathbf{V} \times \mathbf{e}) \\ &= (\mathbf{e} \cdot \nabla) \mathbf{V} - \mathbf{e}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla) \mathbf{e} + \mathbf{V}(\nabla \cdot \mathbf{e}) \\ &= (\mathbf{e} \cdot \nabla) \mathbf{V} - \mathbf{0} - \mathbf{0} + \mathbf{0} = (\mathbf{e} \cdot \nabla) \mathbf{V} \\ &= (\mathbf{e} \cdot \nabla) \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{e} = \nabla(\mathbf{e} \cdot \mathbf{V}) = -\nabla F = \mathbf{U}, \end{aligned}$$

and the first claim is proved.

Since $\mathbf{T} = \mathbf{U} = -\nabla F$ is a gradient field, it is rotation free,

$$\nabla \times \mathbf{T} = -\nabla \times \nabla F = \mathbf{0}.$$

Since $\mathbf{T} = \mathbf{U} = \nabla \times \mathbf{W}$ is a rotation field, it is divergence free:

$$\nabla \cdot \mathbf{T} = \nabla \cdot \nabla \times \mathbf{W} = \mathbf{0}.$$

Example 36.2 Let f be a C^1 -function in r ($= \sqrt{x^2 + y^2 + z^2}$). We shall also (cf. the short hand notation in connection with the chain rule) consider f as a composed function $f(r(x, y, z))$, where $(x, y, z) \neq (0, 0, 0)$.

- 1) Express ∇f by the derivative f' and \mathbf{x} .
- 2) Then set up formulæ for $\nabla \times (\mathbf{x} f)$ and for $\nabla \cdot (\mathbf{x} f)$.
- 3) Find the integer n , for which $\nabla \cdot (r^n \mathbf{x}) = 0$.

A Nabla calculus.

D Just follow the guidelines.

I We shall of course always assume that $r \neq 0$. Then

$$\nabla r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = \frac{1}{r} \mathbf{x}.$$

- 1) We get by the chain rule,

$$\nabla f = \left(f'(r) \frac{\partial r}{\partial x}, f'(r) \frac{\partial r}{\partial y}, f'(r) \frac{\partial r}{\partial z} \right) = f'(r) \nabla r = \frac{f'(r)}{r} \mathbf{x}.$$

The advertisement features a woman teacher smiling and interacting with two young students (a boy and a girl) who are looking at a laptop screen. The background is yellow with orange swirling lines. In the top left corner is the logo for "e-learning for kids". In the bottom right corner, there is a green oval containing text about the organization's achievements.

About e-Learning for Kids Established in 2004, e-Learning for Kids is a global nonprofit foundation dedicated to fun and free learning on the Internet for children ages 5 - 12 with courses in math, science, language arts, computers, health and environmental skills. Since 2005, more than 15 million children in over 190 countries have benefitted from eLessons provided by EFK! An all-volunteer staff consists of education and e-learning experts and business professionals from around the world committed to making difference. eLearning for Kids is actively seeking funding, volunteers, sponsors and courseware developers; get involved! For more information, please visit www.e-learningforkids.org.

The number 1 MOOC for Primary Education
Free Digital Learning for Children 5-12
15 Million Children Reached



2) A direct computation gives

$$\begin{aligned}\nabla \times (\mathbf{x} f) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix} \\ &= \left(z \frac{f'(r)}{r} y - y \frac{f'(r)}{r} z, x \frac{f'(r)}{r} z - z \frac{f'(r)}{r} x, y \frac{f'(r)}{r} x - x \frac{f'(r)}{r} y \right) = (0, 0, 0).\end{aligned}$$

A variant is

$$\begin{aligned}\nabla \times (\mathbf{x} f) &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial f(r)}{\partial x} & \frac{\partial f(r)}{\partial y} & \frac{\partial f(r)}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \nabla f \times \mathbf{x} = \frac{f'(r)}{r} \mathbf{x} \times \mathbf{x} = \mathbf{0}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\nabla \cdot (\mathbf{x} f) &= \left(f(r) + x \frac{\partial f}{\partial x} \right) + \left(f(r) + y \frac{\partial f}{\partial y} \right) + \left(f(r) + z \frac{\partial f}{\partial z} \right) \\ &= 3f(r) + \mathbf{x} \cdot \nabla f = 3f(r) + \frac{f'(r)}{r} \mathbf{x} \cdot \mathbf{x} \\ &= 3f(r) + r f'(r).\end{aligned}$$

3) Choose $f(r) = r^n$. Then it follows from the above,

$$\nabla \cdot (r^n \mathbf{x}) = 3r^n + n r^{n-1} = (3+n)r^n.$$

When $r \neq 0$, this is equal to 0 for $n = -3$.

REMARK. In general, $\nabla \cdot (\mathbf{x} f(r)) = 0$ generates the differential equation

$$r f'(r) + 3f(r) = 0.$$

Then by separation of the variables,

$$\frac{f'(r)}{f(r)} \left[= \frac{\ln |f(r)|}{dr} \right] = -\frac{3}{r},$$

and the complete solution is obtained by an integration,

$$f(r) = C \cdot r^{-3}, \quad r \neq 0, \quad \text{where } C \in \mathbb{R}. \quad \diamond$$

Example 36.3 Let \mathbf{a} be a constant vector, and let f be a C^1 -function in one variable. We define

$$g(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}).$$

- 1) Express the gradient ∇g by the derivative f' .
(Use one of the special cases of the chain rule).
- 2) Let also \mathbf{V} be a gradient field, and let $k = 3$. Show that the vector $\nabla \times (g\mathbf{V})$ is perpendicular to both \mathbf{a} and \mathbf{V} .

A Nabla calculus.

D Just compute.

I 1) If $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{x} = (x_1, \dots, x_k)$, then

$$g(\mathbf{x}) = f(\mathbf{a} \cdot \mathbf{x}) = f \left(\sum_{j=1}^k a_j x_j \right),$$

hence

$$\frac{\partial g}{\partial x_j} = f'(\mathbf{a} \cdot \mathbf{x}) a_j,$$

hence

$$\nabla g = f'(\mathbf{a} \cdot \mathbf{x}) \mathbf{a}.$$

2) If \mathbf{V} is a gradient field, then there exists a function F , such that $\mathbf{V} = \nabla F$. Hence,

$$\begin{aligned} \nabla \times (g\mathbf{V}) &= (\nabla g) \times \mathbf{V} + g \nabla \times \mathbf{V} \\ &= f'(\mathbf{a} \cdot \mathbf{x}) \mathbf{a} \times \mathbf{V} + f(\mathbf{a} \cdot \mathbf{x}) \nabla \times (\nabla F) \\ &= f'(\mathbf{a} \cdot \mathbf{x}) \mathbf{a} \times \mathbf{V} + \mathbf{0} \\ &= f'(\mathbf{a} \cdot \mathbf{V}), \end{aligned}$$

which shows that $\nabla \times (g\mathbf{V})$ is perpendicular on both \mathbf{a} and \mathbf{V} .

Example 36.4 Show the formula

$$2(\nabla f) \cdot (\nabla \times (f \mathbf{V})) = (\nabla \times \mathbf{V}) \cdot \nabla(f^2).$$

A Nabla calculus.

D Just compute.

I We get straight away,

$$\begin{aligned} & 2(\nabla f) \cdot (\nabla \times (f \mathbf{V})) \\ &= 2 \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot \left(\frac{\partial}{\partial y}(fV_z) - \frac{\partial}{\partial z}(fV_y), \frac{\partial}{\partial z}(fV_x) - \frac{\partial}{\partial x}(fV_z), \frac{\partial}{\partial x}(fV_y) - \frac{\partial}{\partial y}(fV_x) \right) \\ &= 2 \frac{\partial f}{\partial x} \left\{ \frac{\partial f}{\partial y} V_z + f \frac{\partial V_z}{\partial y} - \frac{\partial f}{\partial z} V_y - f \frac{\partial V_y}{\partial z} \right\} \\ &\quad + 2 \frac{\partial f}{\partial y} \left\{ \frac{\partial f}{\partial z} V_x + f \frac{\partial V_x}{\partial z} - \frac{\partial f}{\partial x} V_z - f \frac{\partial V_z}{\partial x} \right\} \\ &\quad + 2 \frac{\partial f}{\partial z} \left\{ \frac{\partial f}{\partial x} V_y + f \frac{\partial V_y}{\partial x} - \frac{\partial f}{\partial y} V_x - f \frac{\partial V_x}{\partial y} \right\} \\ &= \frac{\partial(f^2)}{\partial x} \left\{ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right\} + \frac{\partial(f^2)}{\partial y} \left\{ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right\} + \frac{\partial(f^2)}{\partial z} \left\{ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right\} \\ &= \nabla(f^2) \cdot (\nabla \times \mathbf{V}). \end{aligned}$$

Example 36.5 Let \mathbf{V} be a C^1 vector field in the set $A \subseteq \mathbb{R}^3$. Show that if there exists a C^1 function $g : A \rightarrow \mathbb{R} \setminus \{0\}$, such that $g \mathbf{V}$ is a gradient field in A , then

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = 0$$

in the set A .

A Nabla calculus.

D Start by analyzing the assumption. Compute $\nabla \times \mathbf{V}$ by means of the rules of calculations.

I The assumption assures that there exists a C^2 function F , such that

$$g \mathbf{V} = \nabla F, \quad \text{i.e.} \quad \mathbf{V} = \frac{1}{g} \nabla F = h \nabla F,$$

where $h : A \rightarrow \mathbb{R} \setminus \{0\}$ is C^1 , because $g(\mathbf{x}) \neq 0$. Then

$$\begin{aligned} \nabla \times \mathbf{V} &= \nabla \times (h \nabla F) \\ &= (\nabla h) \times \nabla F + h \nabla \times \nabla F \\ &= (\nabla h) \times \nabla F, \quad [\text{the rotation of a gradient is } \mathbf{0}]. \end{aligned}$$

Now, ∇F is perpendicular to $(\nabla h) \times (\nabla F)$, hence

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = h \nabla F \cdot \{(\nabla h) \times \nabla F\} = 0.$$

Example 36.6 Let α be a constant. Find $\nabla(r^\alpha)$ and $\nabla^2(r^\alpha)$.

A Nabla calculus.

D Just compute.

I When $r \neq 0$, then

$$\nabla r = \frac{1}{r} (x, y, z),$$

hence by the chain rule,

$$\nabla(r^\alpha) = \alpha r^{\alpha-1} \nabla r = \alpha r^{\alpha-2} (x, y, z) = \alpha r^{\alpha-2} \mathbf{x}.$$

By taking the divergence we get

$$\begin{aligned}\nabla^2(r^\alpha) &= \nabla \cdot \nabla(r^\alpha) = \nabla \cdot \{\alpha r^{\alpha-2} (x, y, z)\} \\ &= \alpha(\alpha - 2) r^{\alpha-4} (x, y, z) \cdot (x, y, z) + 3\alpha r^{\alpha-2} \\ &= \alpha(\alpha - 2) r^{\alpha-4} \cdot r^2 + 3\alpha r^{\alpha-2} \\ &= \alpha(\alpha + 1) r^{\alpha-2}.\end{aligned}$$

BI NORWEGIAN BUSINESS SCHOOL

EFMD EQUIS ACCREDITED

Empowering People. Improving Business.

BI Norwegian Business School is one of Europe's largest business schools welcoming more than 20,000 students. Our programmes provide a stimulating and multi-cultural learning environment with an international outlook ultimately providing students with professional skills to meet the increasing needs of businesses.

BI offers four different two-year, full-time Master of Science (MSc) programmes that are taught entirely in English and have been designed to provide professional skills to meet the increasing need of businesses. The MSc programmes provide a stimulating and multi-cultural learning environment to give you the best platform to launch into your career.

- MSc in Business
- MSc in Financial Economics
- MSc in Strategic Marketing Management
- MSc in Leadership and Organisational Psychology

www.bi.edu/master

Click on the ad to read more

Example 36.7 Let \mathbf{e} be a constant unit vector. Show that

$$\mathbf{e} \cdot \{\nabla(\mathbf{V} \cdot \mathbf{e}) - \nabla \times (\mathbf{V} \times \mathbf{e})\} = \nabla \cdot \mathbf{V}.$$

A Nabla calculus.

D Just compute.

I We get by means of the rules of calculation,

$$\begin{aligned} & \mathbf{e} \cdot \{\nabla(\mathbf{V} \cdot \mathbf{e}) - \nabla \times (\mathbf{V} \times \mathbf{e})\} \\ &= \mathbf{e} \cdot \{[(\mathbf{e} \cdot \nabla)\mathbf{V} + \mathbf{e} \times (\nabla \times \mathbf{V}) + (\mathbf{V} \cdot \nabla)\mathbf{e} + \mathbf{V} \times (\nabla \times \mathbf{e})] - \nabla \times (\mathbf{V} \times \mathbf{e})\} \\ &= \mathbf{e} \cdot \{(\mathbf{e} \cdot \nabla)\mathbf{V} + [\mathbf{e} \times (\nabla \times \mathbf{V})] - \nabla \times (\mathbf{V} \times \mathbf{e})\} \\ &= \mathbf{e} \cdot \{(\mathbf{e} \cdot \nabla)\mathbf{V} - \nabla \times (\mathbf{V} \times \mathbf{e})\} + \mathbf{e} \cdot [\mathbf{e} \times (\nabla \times \mathbf{V})] \\ &= \mathbf{e} \cdot \{(\mathbf{e} \cdot \nabla)\mathbf{V} - [(\mathbf{e} \cdot \nabla)\mathbf{V} - \mathbf{e}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla)\mathbf{e} + \mathbf{V}(\nabla \cdot \mathbf{e})]\} + 0 \\ &= \mathbf{e} \cdot \mathbf{e}(\nabla \cdot \mathbf{V}) + 0 \\ &= \nabla \cdot \mathbf{V}. \end{aligned}$$

This formula can of course also be written in the form

$$\mathbf{e} \cdot \{\text{grad div } \mathbf{e}\} - \text{rot}(\mathbf{V} \times \mathbf{e}) = \text{div } \mathbf{V}.$$

Example 36.8 Let α be a constant vector. For $\mathbf{x} \neq \mathbf{0}$ we consider the fields

$$U(\mathbf{x}) = \frac{\mathbf{a} \cdot \mathbf{x}}{\|\mathbf{x}\|^3}, \quad \mathbf{W}(\mathbf{x}) = \frac{\mathbf{a} \times \mathbf{x}}{\|\mathbf{x}\|^3}.$$

Show that

$$\nabla \times \mathbf{W} = -\nabla U.$$

A Nabla calculus.

D Just compute by using the rule of calculation and the result of **Example 36.6**.

I Clearly, U and \mathbf{W} are C^∞ for $\mathbf{x} \neq \mathbf{0}$. Put $r = \|\mathbf{x}\|$. Then by **Example 36.6**,

$$\nabla(r^\alpha) = \alpha r^{\alpha-2} \mathbf{x} \quad \text{for } \mathbf{x} \neq \mathbf{0}.$$

Then we shall use the following result from Linear Algebra,

$$\mathbf{x} \times (\mathbf{a} \times \mathbf{x}) = (\mathbf{x} \cdot \mathbf{x})\mathbf{a} - (\mathbf{a} \cdot \mathbf{x})\mathbf{x} = r^2 \mathbf{a} - (\mathbf{a} \cdot \mathbf{x})\mathbf{x}.$$

Applying these preparations we get

$$\begin{aligned}
 \nabla \times \mathbf{W} &= \nabla \times \{r^{-3}(\mathbf{a} \times \mathbf{x})\} && \text{definition of } \mathbf{W} \\
 &= (\nabla r^{-3}) \times (\mathbf{a} \times \mathbf{x} + r^{-3} \nabla \times (\mathbf{a} \times \mathbf{x})) && \text{rule of calculation} \\
 &= -3r^{-5}\mathbf{x} \times (\mathbf{a} \times \mathbf{x}) + r^{-3} \nabla \times (\mathbf{a} \times \mathbf{x}) && \text{Example 36.6} \\
 &= -3r^{-5}\{r^2\mathbf{a} - (\mathbf{a} \cdot \mathbf{x})\mathbf{x}\} + r^{-3} \nabla \times (\mathbf{a} \times \mathbf{x}) && \text{Linear Algebra} \\
 &= -\frac{3}{r^3}\mathbf{a} + \frac{3\mathbf{a} \cdot \mathbf{x}}{r^5}\mathbf{x} + \frac{1}{r^3}\{0 - 0 - (\mathbf{a} \cdot \nabla)\mathbf{x} + \mathbf{a}(\nabla \cdot \mathbf{x})\} && \text{rule of computation} \\
 &= -\frac{3}{r^3}\mathbf{a} + \frac{3}{r^5}(\mathbf{a} \cdot \mathbf{x})\mathbf{x} + \frac{1}{r^3}(-\mathbf{a} + 3\mathbf{a}) && \text{computation} \\
 &= -\frac{1}{r^3}\mathbf{a} + \frac{3}{r^5}(\mathbf{a} \cdot \mathbf{x})\mathbf{x}, && \text{reduction,}
 \end{aligned}$$

and

$$\begin{aligned}
 \nabla U &= \nabla(r^{-3}(\mathbf{a} \cdot \mathbf{x})) && \text{definition of } U \\
 &= (\mathbf{a} \cdot \mathbf{x}) \nabla(r^{-3}) + \frac{1}{r^3} \nabla(\mathbf{a} \cdot \mathbf{x}) && \text{rule of calculation} \\
 &= (\mathbf{a} \cdot \mathbf{x}) \cdot \left\{ -\frac{3}{r^5}\mathbf{x} \right\} + \frac{1}{r^3}\mathbf{a} && \text{Example 36.6 and } \nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}.
 \end{aligned}$$

It follows by a comparison of these two expressions that

$$\nabla \times \mathbf{W} = -\nabla U.$$

This can also be written

$$\text{rot } \mathbf{W} = -\text{grad } U,$$

where U and \mathbf{W} are given above.

Example 36.9 Consider the composite vector function

$$\mathbf{V}(\mathbf{x}) = \mathbf{U}(w), \quad w = f(\mathbf{x}).$$

Find an expression for $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$.

A Nabla calculus.

D Just compute.

I In general,

$$\frac{\partial V_j}{\partial x_i} = \frac{\partial(U_j \circ f)}{\partial x_i} = \frac{dU_j}{dw} \cdot \frac{\partial f}{\partial x_i}, \quad w = f(\mathbf{x}).$$

When we change notation $(x_1, x_2, x_3) = (x, y, z)$, it follows that

$$\nabla \cdot \mathbf{V} = \sum_{i=1}^3 U'_i(w) \frac{\partial f}{\partial x_i} = \nabla f \cdot \mathbf{U}'(f(\mathbf{x})),$$

and

$$\begin{aligned} \nabla \times \mathbf{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \\ &= \begin{pmatrix} U'_z(w) \frac{\partial f}{\partial y} - U'_y(w) \frac{\partial f}{\partial z} \\ U'_x(w) \frac{\partial f}{\partial z} - U'_z(w) \frac{\partial f}{\partial x} \\ U'_y(w) \frac{\partial f}{\partial x} - U'_x(w) \frac{\partial f}{\partial y} \end{pmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ U'_x & U'_y & U'_z \end{vmatrix} = \nabla f \times \mathbf{U}'(f(\mathbf{x})). \end{aligned}$$

ALTERNATIVELY, a more sophisticated reasoning is the following,

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U_x \circ f & U_y \circ f & U_z \circ f \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ U'_x \circ f & U'_y \circ f & U'_z \circ f \end{vmatrix} = \nabla f \times \mathbf{U}'(f(\mathbf{x})).$$

Summarizing we obtain the results

$$\nabla \cdot (\mathbf{U} \circ f(\mathbf{x})) = \nabla f(\mathbf{x}) \cdot \mathbf{U} - (f(\mathbf{x})) \quad \text{and} \quad \nabla \times (\mathbf{U} \circ f(\mathbf{x})) = \nabla f(\mathbf{x}) \times \mathbf{U}'(f(\mathbf{x})).$$

Example 36.10 Given a C^1 vector field \mathbf{V} and a C^2 scalar field f with the following property:
The vector \mathbf{V} is at each point (x, y, z) perpendicular to the level surface of f through the point (x, y, z) .
Prove that $\mathbf{V} \cdot (\nabla \times \mathbf{V}) = 0$.

A Nabla calculus.

D Analyze the assumption. Then find a relation between f and \mathbf{V} . Finally, compute $\mathbf{V} \cdot (\nabla \times \mathbf{V})$.

I Since both ∇f and \mathbf{V} are perpendicular to the level surface, they are proportional at each point.
Hence, there exists a function g , such that (usually)

$$(36.1) \quad \mathbf{V}(x, y, z) = g(x, y, z) \nabla f(x, y, z).$$

When $\nabla f \neq \mathbf{0}$, then clearly g is os class C^1 . Thus, when $\nabla f \neq \mathbf{0}$, then

$$\nabla \times \mathbf{V} = \nabla \times (g \nabla f) = (\nabla g) \times (\nabla f) + g (\nabla \times \nabla f) = (\nabla g) \times (\nabla f) + \mathbf{0}.$$

Since ∇f is perpendicular to $\nabla g \times \nabla f$, we get

$$\mathbf{V} \cdot (\nabla \times \mathbf{V}) = g \nabla f \cdot \{\nabla g \times \nabla f\} = 0.$$

If $\nabla f = \mathbf{0}$, then (36.1) does not necessary hold. However, if (36.1) holds, the relation is trivial.

Now assume that (36.1) does not hold, i.e. $\mathbf{V}(x, y, z) \neq \mathbf{0}$ and $\nabla f(x, y, z) = \mathbf{0}$. We shall then use a continuity argument:

Need help with your dissertation?

Get in-depth feedback & advice from experts in your topic area. Find out what you can do to improve the quality of your dissertation!

Get Help Now



Go to www.helpmyassignment.co.uk for more info

Helpmyassignment

Click on the ad to read more

Since f has level surfaces, we must have $\nabla f \neq \mathbf{0}$ arbitrarily close to (x, y, z) , and it follows from the above that $\mathbf{V} \cdot (\nabla \times \mathbf{V}) = 0$ at these points. This relation is continuous, so it follows by a continuous extension that $\mathbf{V} \cdot \{\nabla \times \mathbf{V}\} = 0$ also is valid at points, where $\nabla f(x, y, z) = \mathbf{0}$.

Example 36.11 Show by means of Gauß's theorem that for any closed surface \mathcal{F} ,

$$\int_{\mathcal{F}} \mathbf{n} dS = \mathbf{0}.$$

A Gauß's theorem in its operator version.

D Insert the obvious into Gauß's theorem in its operator version.

I Let \mathcal{F} be the boundary of the domain Ω . Then by Gauß's theorem in its operator version,

$$\int_{\Omega} \nabla \square d\Omega = \int_{\partial\Omega} \mathbf{n} \square dS = \int_{\mathcal{F}} \mathbf{n} \square dS.$$

If we replace \square with 1, it follows that

$$\int_{\mathcal{F}} \mathbf{n} dS = \int_{\Omega} \nabla 1 d\Omega = \int_{\Omega} \mathbf{0} d\Omega = \mathbf{0}.$$

Example 36.12 Find the divergence of the vector field

$$\mathbf{V} = (\nabla f) \times (\nabla g).$$

[Cf. Example 38.17.]

A Nabla calculus.

D Just compute.

I The rotation of a gradient is $\mathbf{0}$, i.e. every gradient field is rotation free. Hence

$$\nabla \cdot (\nabla f \times \nabla g) = (\nabla \times \nabla f) \cdot \nabla g - (\nabla \times \nabla g) \cdot \nabla f = 0 - 0 = 0.$$

Example 36.13 Consider the vector field $\mathbf{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$\mathbf{V}(x, y, z) = f(x, y) \mathbf{e}_z,$$

which also satisfies

$$\nabla \times (\nabla \times \mathbf{V}) = \alpha \mathbf{V},$$

where α is a constant. Find a differential equation which has the function f as one of its solutions.

A Double rotation.

D Compute the left hand side.

I It follows from $\mathbf{V}(x, y, z) = f(x, y) \mathbf{e}_z$ that

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & f(x, y) \end{vmatrix} = (f'_y, -f'_x, 0)$$

and

$$\nabla \times (\nabla \times \mathbf{V}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f'_y & -f'_x & 0 \end{vmatrix} = (0, 0, -f''_{xx} - f''_{yy}) = (0, 0, \nabla^2 f),$$

and

$$\alpha \mathbf{V} = (0, 0, \alpha f(x, y)).$$

Then from $\nabla \times (\nabla \times \mathbf{V}) = \alpha \mathbf{V}$,

$$-\nabla^2 f = \alpha f \quad \text{or} \quad \nabla^2 f + \alpha f = \Delta f + \alpha f = 0.$$

Example 36.14 Let V denote the volume of a domain Ω in space with the outwards unit normal vector field \mathbf{n} , and let \mathbf{a} be a constant vector. Find

$$\frac{1}{V} \int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a}) dS.$$

A Nabla calculus.

D Apply a variant Gauß's theorem and use the nabla calculations.

I By a variant of Gauß's theorem,

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{V} dS = \int_{\Omega} \nabla \times \mathbf{V} d\Omega.$$

Put $\mathbf{V} = \mathbf{x} \times \mathbf{a}$. Then

$$\frac{1}{V} \int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a}) dS = \frac{1}{V} \int_{\Omega} \nabla \times (\mathbf{x} \times \mathbf{a}) d\Omega.$$

Then by a rule of calculation,

$$\begin{aligned} \nabla \times (\mathbf{x} \times \nabla a) &= (\mathbf{a} \cdot \nabla) \mathbf{x} - \mathbf{a} (\nabla \cdot \mathbf{x}) - (\mathbf{x} \cdot \nabla) \mathbf{a} + \mathbf{x} (\nabla \cdot \mathbf{a}) \\ &= (\mathbf{a} \cdot \nabla) \mathbf{x} - \mathbf{a} (\nabla \cdot \mathbf{x}) - \mathbf{0} + \mathbf{0} \\ &= \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z} \right) (x, y, z) - \mathbf{a} \cdot (1 + 1 + 1) \\ &= \mathbf{a} - 3\mathbf{a} = -2\mathbf{a}, \end{aligned}$$

which is a constant. Hence by insertion,

$$\frac{1}{V} \int_{\partial\Omega} \mathbf{n} \times (\mathbf{x} \times \mathbf{a}) dS = \frac{1}{V} \int_V (-2\mathbf{a} dS) = -2\mathbf{a}.$$

ADDITION. For completeness we here prove the variant of Gauß's theorem, which is applied above. First note that the usual version of Gauß's theorem can be written

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{W} dS = \int_{\Omega} \nabla \cdot \mathbf{W} d\Omega.$$

Choose $\mathbf{W} = \mathbf{V} \times \mathbf{b}$, where \mathbf{b} is any constant vector. Then

$$(36.2) \quad \int_{\partial\Omega} \mathbf{n} \cdot (\mathbf{V} \times \mathbf{b}) dS = \int_{\Omega} \nabla \cdot (\mathbf{V} \times \mathbf{b}) d\Omega.$$

The geometric interpretation of $\mathbf{n} \cdot (\mathbf{V} \times \mathbf{b})$ is that it is equal to the (signed) volume of the parallelepiped defined by the vectors \mathbf{n} , \mathbf{V} and \mathbf{b} . (This simple result is also known from Linear Algebra).

The same interpretation is true for $(\mathbf{n} \times \mathbf{V}) \cdot \mathbf{b}$ (with the same sign, because the sequence of the vectors is not changed), thus

$$\mathbf{n} \cdot (\mathbf{V} \times \mathbf{b}) = (\mathbf{n} \times \mathbf{V}) \cdot \mathbf{b}.$$

Since \mathbf{b} is constant, it follows by a rule of calculation,

$$\nabla \cdot (\mathbf{V} \times \mathbf{b}) = (\nabla \times \mathbf{V}) \cdot \mathbf{b} - (\nabla \times \mathbf{b} \cdot \mathbf{V}) = (\nabla \times \mathbf{V}) \cdot \mathbf{b}.$$

By inserting these two results into (36.2), we get

$$\int_{\partial\Omega} (\mathbf{n} \times \mathbf{V}) \cdot \mathbf{b} dS = \int_{\Omega} (\nabla \times \mathbf{V}) \cdot \mathbf{b} d\Omega.$$

Since \mathbf{b} is a constant vector, it follows by a rearrangement that

$$\left\{ \int_{\partial\Omega} \mathbf{n} \times \mathbf{V} dS - \int_{\Omega} \nabla \times \mathbf{V} d\Omega \right\} \cdot \mathbf{b} = 0.$$

Since $\mathbf{0}$ is the only vector, which is perpendicular to all vectors, the first factor must be $\mathbf{0}$, and we get by another rearrangement.

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{V} dS = \int_{\Omega} \nabla \times \mathbf{V} d\Omega,$$

and the variant of Gauß's theorem has been proved.

Example 36.15 Let \mathbf{V} , \mathbf{W} be vector fields in the space which also depend on time t and satisfy the equations

$$\nabla \times \mathbf{V} = \alpha \frac{\partial \mathbf{W}}{\partial t}, \quad \nabla \times \mathbf{W} = -\beta \frac{\partial \mathbf{V}}{\partial t},$$

where α and β are constants. Show that the vector field

$$\mathbf{U} = \beta \mathbf{V} \times \frac{\partial \mathbf{V}}{\partial t} + \alpha \mathbf{W} \times \frac{\partial \mathbf{W}}{\partial t}$$

and the scalar field

$$f = \beta \mathbf{V} \cdot \nabla \times \mathbf{V} + \alpha \mathbf{W} \cdot \nabla \times \mathbf{W}$$

satisfy the differential equation

$$\nabla \cdot \mathbf{U} + \frac{\partial f}{\partial t} = 0.$$

(An equation of this type is often called a continuity equation or a preservation theorem).

A Continuity equation.

D Nabla calculus.

I Since $\frac{\partial}{\partial t}$ is a differentiation with respect to a “parameter”, where can interchange $\frac{\partial}{\partial t}$ with any of the operators ∇ , $\nabla \cdot$ and $\nabla \times$. Hence

$$\begin{aligned} \frac{\partial f}{\partial t} &= \beta \frac{\partial \mathbf{V}}{\partial t} \cdot (\nabla \times \mathbf{V}) + \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \frac{\partial \mathbf{W}}{\partial t} \cdot (\nabla \times \mathbf{W}) + \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\ &= -(\nabla \times \mathbf{W}) \cdot (\nabla \times \mathbf{V}) + \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + (\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) + \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\ &= \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right). \end{aligned}$$

By using this rule of calculation we get similarly,

$$\begin{aligned} \nabla \cdot \mathbf{U} &= \beta \nabla \cdot \left(\mathbf{V} \times \frac{\partial \mathbf{V}}{\partial t} \right) + \alpha \nabla \cdot \left(\mathbf{W} \times \frac{\partial \mathbf{W}}{\partial t} \right) \\ &= \beta (\nabla \times \mathbf{V}) \cdot \frac{\partial \mathbf{V}}{\partial t} - \beta \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) \cdot \mathbf{V} + \alpha (\nabla \times \mathbf{W}) \cdot \frac{\partial \mathbf{W}}{\partial t} - \alpha \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \cdot \mathbf{W} \\ &= -(\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) + (\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{W}) - \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \\ &= -\beta \mathbf{V} \cdot \left(\nabla \times \frac{\partial \mathbf{V}}{\partial t} \right) - \alpha \mathbf{W} \cdot \left(\nabla \times \frac{\partial \mathbf{W}}{\partial t} \right) \end{aligned}$$

Finally, by adding these expressions we get

$$\nabla \cdot \mathbf{U} + \frac{\partial f}{\partial t} = 0.$$

37 Formulæ

Some of the following formulæ can be assumed to be known from high school. It is highly recommended that one *learns most of these formulæ in this appendix by heart*.

37.1 Squares etc.

The following simple formulæ occur very frequently in the most different situations.

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab, & a^2 + b^2 + 2ab &= (a+b)^2, \\ (a-b)^2 &= a^2 + b^2 - 2ab, & a^2 + b^2 - 2ab &= (a-b)^2, \\ (a+b)(a-b) &= a^2 - b^2, & a^2 - b^2 &= (a+b)(a-b), \\ (a+b)^2 &= (a-b)^2 + 4ab, & (a-b)^2 &= (a+b)^2 - 4ab. \end{aligned}$$

37.2 Powers etc.

Logarithm:

$$\begin{aligned} \ln |xy| &= \ln |x| + \ln |y|, & x, y \neq 0, \\ \ln \left| \frac{x}{y} \right| &= \ln |x| - \ln |y|, & x, y \neq 0, \\ \ln |x^r| &= r \ln |x|, & x \neq 0. \end{aligned}$$

Power function, fixed exponent:

$$(xy)^r = x^r \cdot y^r, x, y > 0 \quad (\text{extensions for some } r),$$

$$\left(\frac{x}{y} \right)^r = \frac{x^r}{y^r}, x, y > 0 \quad (\text{extensions for some } r).$$

Exponential, fixed base:

$$\begin{aligned} a^x \cdot a^y &= a^{x+y}, & a > 0 & \quad (\text{extensions for some } x, y), \\ (a^x)^y &= a^{xy}, & a > 0 & \quad (\text{extensions for some } x, y), \end{aligned}$$

$$a^{-x} = \frac{1}{a^x}, a > 0, \quad (\text{extensions for some } x),$$

$$\sqrt[n]{a} = a^{1/n}, a \geq 0, \quad n \in \mathbb{N}.$$

Square root:

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}.$$

Remark 37.1 It happens quite frequently that students make errors when they try to apply these rules. They must be mastered! In particular, as one of my friends once put it: “If you can master the square root, you can master everything in mathematics!” Notice that this innocent looking square root is one of the most difficult operations in Calculus. Do not forget the *absolute value!* ◇

37.3 Differentiation

Here are given the well-known rules of differentiation together with some rearrangements which sometimes may be easier to use:

$$\{f(x) \pm g(x)\}' = f'(x) \pm g'(x),$$

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x) = f(x)g(x) \left\{ \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \right\},$$

where the latter rearrangement presupposes that $f(x) \neq 0$ and $g(x) \neq 0$.

If $g(x) \neq 0$, we get the usual formula known from high school

$$\left\{ \frac{f(x)}{g(x)} \right\}' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

It is often more convenient to compute this expression in the following way:

$$\left\{ \frac{f(x)}{g(x)} \right\} = \frac{d}{dx} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f(x)}{g(x)} \left\{ \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)} \right\},$$

where the former expression often is *much easier* to use in practice than the usual formula from high school, and where the latter expression again presupposes that $f(x) \neq 0$ and $g(x) \neq 0$. Under these assumptions we see that the formulæ above can be written

$$\frac{\{f(x)g(x)\}'}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)},$$

$$\frac{\{f(x)/g(x)\}'}{f(x)/g(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}.$$

Since

$$\frac{d}{dx} \ln |f(x)| = \frac{f'(x)}{f(x)}, \quad f(x) \neq 0,$$

we also name these the *logarithmic derivatives*.

Finally, we mention the rule of **differentiation of a composite function**

$$\{f(\varphi(x))\}' = f'(\varphi(x)) \cdot \varphi'(x).$$

We first differentiate the function itself; then the insides. This rule is a 1-dimensional version of the so-called *Chain rule*.

37.4 Special derivatives.

Power like:

$$\frac{d}{dx} (x^\alpha) = \alpha \cdot x^{\alpha-1}, \quad \text{for } x > 0, \quad (\text{extensions for some } \alpha).$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \text{for } x \neq 0.$$

Exponential like:

$$\begin{aligned}\frac{d}{dx} \exp x &= \exp x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} (a^x) &= \ln a \cdot a^x, && \text{for } x \in \mathbb{R} \text{ and } a > 0.\end{aligned}$$

Trigonometric:

$$\begin{aligned}\frac{d}{dx} \sin x &= \cos x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \cos x &= -\sin x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \tan x &= 1 + \tan^2 x = \frac{1}{\cos^2 x}, && \text{for } x \neq \frac{\pi}{2} + p\pi, p \in \mathbb{Z}, \\ \frac{d}{dx} \cot x &= -(1 + \cot^2 x) = -\frac{1}{\sin^2 x}, && \text{for } x \neq p\pi, p \in \mathbb{Z}.\end{aligned}$$

Hyperbolic:

$$\begin{aligned}\frac{d}{dx} \sinh x &= \cosh x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \cosh x &= \sinh x, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \tanh x &= 1 - \tanh^2 x = \frac{1}{\cosh^2 x}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \coth x &= 1 - \coth^2 x = -\frac{1}{\sinh^2 x}, && \text{for } x \neq 0.\end{aligned}$$

Inverse trigonometric:

$$\begin{aligned}\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}}, && \text{for } x \in]-1, 1[, \\ \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}}, && \text{for } x \in]-1, 1[, \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \text{arccot } x &= \frac{1}{1+x^2}, && \text{for } x \in \mathbb{R}.\end{aligned}$$

Inverse hyperbolic:

$$\begin{aligned}\frac{d}{dx} \text{arsinh } x &= \frac{1}{\sqrt{x^2+1}}, && \text{for } x \in \mathbb{R}, \\ \frac{d}{dx} \text{arcosh } x &= \frac{1}{\sqrt{x^2-1}}, && \text{for } x \in]1, +\infty[, \\ \frac{d}{dx} \text{artanh } x &= \frac{1}{1-x^2}, && \text{for } |x| < 1, \\ \frac{d}{dx} \text{arcoth } x &= \frac{1}{1-x^2}, && \text{for } |x| > 1.\end{aligned}$$

Remark 37.2 The derivative of the trigonometric and the hyperbolic functions are to some extent exponential like. The derivatives of the inverse trigonometric and inverse hyperbolic functions are power like, because we include the logarithm in this class. ◇

37.5 Integration

The most obvious rules are dealing with linearity

$$\int \{f(x) + \lambda g(x)\} dx = \int f(x) dx + \lambda \int g(x) dx, \quad \text{where } \lambda \in \mathbb{R} \text{ is a constant,}$$

and with the fact that differentiation and integration are “inverses to each other”, i.e. modulo some arbitrary constant $c \in \mathbb{R}$, which often tacitly is missing,

$$\int f'(x) dx = f(x).$$

If we in the latter formula replace $f(x)$ by the product $f(x)g(x)$, we get by reading from the right to the left and then differentiating the product,

$$f(x)g(x) = \int \{f(x)g(x)\}' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Hence, by a rearrangement

The rule of partial integration:

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

The differentiation is moved from one factor of the integrand to the other one by changing the sign and adding the term $f(x)g(x)$.

Remark 37.3 This technique was earlier used a lot, but is almost forgotten these days. It must be revived, because MAPLE and pocket calculators apparently do not know it. It is possible to construct examples where these devices cannot give the exact solution, unless you first perform a partial integration yourself. ◇

Remark 37.4 This method can also be used when we estimate integrals which cannot be directly calculated, because the antiderivative is not contained in e.g. the catalogue of MAPLE. The idea is by a succession of partial integrations to make the new integrand smaller. ◇

Integration by substitution:

If the integrand has the special structure $f(\varphi(x)) \cdot \varphi'(x)$, then one can change the variable to $y = \varphi(x)$:

$$\int f(\varphi(x)) \cdot \varphi'(x) dx = " \int f(\varphi(x)) d\varphi(x) " = \int_{y=\varphi(x)} f(y) dy.$$

Integration by a monotonous substitution:

If $\varphi(y)$ is a *monotonous* function, which maps the y -interval *one-to-one* onto the x -interval, then

$$\int f(x) dx = \int_{y=\varphi^{-1}(x)} f(\varphi(y))\varphi'(y) dy.$$

Remark 37.5 This rule is usually used when we have some “ugly” term in the integrand $f(x)$. The idea is to put this ugly term equal to $y = \varphi^{-1}(x)$. When e.g. x occurs in $f(x)$ in the form \sqrt{x} , we put $y = \varphi^{-1}(x) = \sqrt{x}$, hence $x = \varphi(y) = y^2$ and $\varphi'(y) = 2y$. \diamond

37.6 Special antiderivatives

Power like:

$$\int \frac{1}{x} dx = \ln|x|, \quad \text{for } x \neq 0. \quad (\text{Do not forget the numerical value!})$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1}, \quad \text{for } \alpha \neq -1,$$

$$\int \frac{1}{1+x^2} dx = \arctan x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, \quad \text{for } x \neq \pm 1,$$

$$\int \frac{1}{1-x^2} dx = \operatorname{Artanh} x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{1-x^2} dx = \operatorname{Arcoth} x, \quad \text{for } |x| > 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\arccos x, \quad \text{for } |x| < 1,$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{Arsinh} x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1}, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{Arcosh} x, \quad \text{for } x > 1,$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln|x + \sqrt{x^2-1}|, \quad \text{for } x > 1 \text{ eller } x < -1.$$

There is an error in the programs of the pocket calculators TI-92 and TI-89. The *numerical signs* are missing. It is obvious that $\sqrt{x^2-1} < |x|$ so if $x < -1$, then $x + \sqrt{x^2-1} < 0$. Since you cannot take the logarithm of a negative number, these pocket calculators will give an error message.

Exponential like:

$$\int \exp x \, dx = \exp x, \quad \text{for } x \in \mathbb{R},$$

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x, \quad \text{for } x \in \mathbb{R}, \text{ and } a > 0, a \neq 1.$$

Trigonometric:

$$\int \sin x \, dx = -\cos x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cos x \, dx = \sin x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tan x \, dx = -\ln |\cos x|, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \cot x \, dx = \ln |\sin x|, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right), \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln \left(\frac{1 - \cos x}{1 + \cos x} \right), \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x, \quad \text{for } x \neq \frac{\pi}{2} + p\pi, \quad p \in \mathbb{Z},$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x, \quad \text{for } x \neq p\pi, \quad p \in \mathbb{Z}.$$

Hyperbolic:

$$\int \sinh x \, dx = \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \cosh x \, dx = \sinh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \tanh x \, dx = \ln \cosh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \coth x \, dx = \ln |\sinh x|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh x} \, dx = \operatorname{Arctan}(\sinh x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\cosh x} \, dx = 2 \operatorname{Arctan}(e^x), \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh x} \, dx = \frac{1}{2} \ln \left(\frac{\cosh x - 1}{\cosh x + 1} \right), \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\sinh x} dx = \ln \left| \frac{e^x - 1}{e^x + 1} \right|, \quad \text{for } x \neq 0,$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x, \quad \text{for } x \in \mathbb{R},$$

$$\int \frac{1}{\sinh^2 x} dx = -\coth x, \quad \text{for } x \neq 0.$$

37.7 Trigonometric formulæ

The trigonometric formulæ are closely connected with circular movements. Thus $(\cos u, \sin u)$ are the coordinates of a point P on the unit circle corresponding to the angle u , cf. figure A.1. This geometrical interpretation is used from time to time.

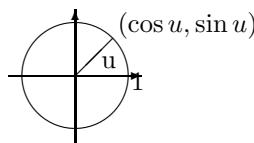


Figure 37.1: The unit circle and the trigonometric functions.

The fundamental trigonometric relation:

$$\cos^2 u + \sin^2 u = 1, \quad \text{for } u \in \mathbb{R}.$$

Using the previous geometric interpretation this means according to *Pythagoras's theorem*, that the point P with the coordinates $(\cos u, \sin u)$ always has distance 1 from the origo $(0, 0)$, i.e. it is lying on the boundary of the circle of centre $(0, 0)$ and radius $\sqrt{1} = 1$.

Connection to the complex exponential function:

The *complex exponential* is for imaginary arguments defined by

$$\exp(iu) := \cos u + i \sin u.$$

It can be checked that the usual functional equation for \exp is still valid for complex arguments. In other word: The definition above is extremely conveniently chosen.

By using the definition for $\exp(iu)$ and $\exp(-iu)$ it is easily seen that

$$\cos u = \frac{1}{2}(\exp(iu) + \exp(-iu)),$$

$$\sin u = \frac{1}{2i}(\exp(iu) - \exp(-iu)),$$

Moivre's formula: We get by expressing $\exp(inu)$ in two different ways:

$$\exp(inu) = \cos nu + i \sin nu = (\cos u + i \sin u)^n.$$

Example 37.1 If we e.g. put $n = 3$ into Moivre's formula, we obtain the following typical application,

$$\begin{aligned} \cos(3u) + i \sin(3u) &= (\cos u + i \sin u)^3 \\ &= \cos^3 u + 3i \cos^2 u \cdot \sin u + 3i^2 \cos u \cdot \sin^2 u + i^3 \sin^3 u \\ &= \{\cos^3 u - 3 \cos u \cdot \sin^2 u\} + i\{3 \cos^2 u \cdot \sin u - \sin^3 u\} \\ &= \{4 \cos^3 u - 3 \cos u\} + i\{3 \sin u - 4 \sin^3 u\} \end{aligned}$$

When this is split into the real- and imaginary parts we obtain

$$\cos 3u = 4 \cos^3 u - 3 \cos u, \quad \sin 3u = 3 \sin u - 4 \sin^3 u. \quad \diamond$$

Addition formulæ:

$$\begin{aligned} \sin(u+v) &= \sin u \cos v + \cos u \sin v, \\ \sin(u-v) &= \sin u \cos v - \cos u \sin v, \\ \cos(u+v) &= \cos u \cos v - \sin u \sin v, \\ \cos(u-v) &= \cos u \cos v + \sin u \sin v. \end{aligned}$$

Products of trigonometric functions to a sum:

$$\begin{aligned} \sin u \cos v &= \frac{1}{2} \sin(u+v) + \frac{1}{2} \sin(u-v), \\ \cos u \sin v &= \frac{1}{2} \sin(u+v) - \frac{1}{2} \sin(u-v), \\ \sin u \sin v &= \frac{1}{2} \cos(u-v) - \frac{1}{2} \cos(u+v), \\ \cos u \cos v &= \frac{1}{2} \cos(u-v) + \frac{1}{2} \cos(u+v). \end{aligned}$$

Sums of trigonometric functions to a product:

$$\begin{aligned} \sin u + \sin v &= 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right), \\ \sin u - \sin v &= 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right), \\ \cos u + \cos v &= 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right), \\ \cos u - \cos v &= -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right). \end{aligned}$$

Formulæ of halving and doubling the angle:

$$\sin 2u = 2 \sin u \cos u,$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2 \cos^2 u - 1 = 1 - 2 \sin^2 u,$$

$$\sin \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cos \frac{u}{2} = \pm \sqrt{\frac{1 + \cos u}{2}} \quad \text{followed by a discussion of the sign,}$$

37.8 Hyperbolic formulæ

These are very much like the trigonometric formulæ, and if one knows a little of Complex Function Theory it is realized that they are actually identical. The structure of this section is therefore the same as for the trigonometric formulæ. The reader should compare the two sections concerning similarities and differences.

The fundamental relation:

$$\cosh^2 x - \sinh^2 x = 1.$$

Definitions:

$$\cosh x = \frac{1}{2} (\exp(x) + \exp(-x)), \quad \sinh x = \frac{1}{2} (\exp(x) - \exp(-x)).$$

“Moivre’s formula”:

$$\exp(x) = \cosh x + \sinh x.$$

This is trivial and only rarely used. It has been included to show the analogy.

Addition formulæ:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$

$$\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y),$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$$

Formulæ of halving and doubling the argument:

$$\sinh(2x) = 2\sinh(x)\cosh(x),$$

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$$

$$\sinh\left(\frac{x}{2}\right) = \pm\sqrt{\frac{\cosh(x)-1}{2}} \quad \text{followed by a discussion of the sign,}$$

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}}.$$

Inverse hyperbolic functions:

$$\text{Arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R},$$

$$\text{Arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1,$$

$$\text{Artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad |x| < 1,$$

$$\text{Arcoth}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1.$$

37.9 Complex transformation formulæ

$$\begin{aligned}\cos(ix) &= \cosh(x), & \cosh(ix) &= \cos(x), \\ \sin(ix) &= i \sinh(x), & \sinh(ix) &= i \sin x.\end{aligned}$$

37.10 Taylor expansions

The generalized binomial coefficients are defined by

$$\binom{\alpha}{n} := \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{1 \cdot 2 \cdots n},$$

with n factors in the numerator and the denominator, supplied with

$$\binom{\alpha}{0} := 1.$$

The Taylor expansions for *standard functions* are divided into *power like* (the radius of convergency is finite, i.e. = 1 for the standard series) and *exponential like* (the radius of convergency is infinite).

Power like:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1,$$

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j, \quad n \in \mathbb{N}, x \in \mathbb{R},$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \alpha \in \mathbb{R} \setminus \mathbb{N}, |x| < 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

$$\text{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

Exponential like:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\exp(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n, \quad x \in \mathbb{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R},$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R},$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \quad x \in \mathbb{R}.$$

37.11 Magnitudes of functions

We often have to compare functions for $x \rightarrow 0+$, or for $x \rightarrow \infty$. The simplest type of functions are therefore arranged in an hierarchy:

- 1) logarithms,
- 2) power functions,
- 3) exponential functions,
- 4) faculty functions.

When $x \rightarrow \infty$, a function from a higher class will always dominate a function from a lower class. More precisely:

A) A *power function* dominates a *logarithm* for $x \rightarrow \infty$:

$$\frac{(\ln x)^\beta}{x^\alpha} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, \beta > 0.$$

B) An *exponential* dominates a *power function* for $x \rightarrow \infty$:

$$\frac{x^\alpha}{a^x} \rightarrow 0 \quad \text{for } x \rightarrow \infty, \quad \alpha, a > 1.$$

C) The *faculty function* dominates an *exponential* for $n \rightarrow \infty$:

$$\frac{a^n}{n!} \rightarrow 0, \quad n \rightarrow \infty, \quad n \in \mathbb{N}, \quad a > 0.$$

D) When $x \rightarrow 0+$ we also have that a *power function* dominates the *logarithm*:

$$x^\alpha \ln x \rightarrow 0-, \quad \text{for } x \rightarrow 0+, \quad \alpha > 0.$$

Index

- absolute value 162
- acceleration 490
- addition 22
- affinity factor 173
- Ampère-Laplace law 1671
- Ampère-Maxwell's law 1678
- Ampère's law 1491, 1498, 1677, 1678, 1833
- Ampère's law for the magnetic field 1674
- angle 19
- angular momentum 886
- angular set 84
- annulus 176, 243
- anticommutative product 26
- antiderivative 301, 847
- approximating polynomial 304, 322, 326, 336, 404, 488, 632, 662
- approximation in energy 734
- Archimedes's spiral 976, 1196
- Archimedes's theorem 1818
- area 887, 1227, 1229, 1543
- area element 1227
- area of a graph 1230
- asteroid 1215
- asymptote 51
- axial moment 1910
- axis of revolution 181
- axis of rotation 34, 886
- axis of symmetry 49, 50, 53
- barycentre 885, 1910
- basis 22
- bend 486
- bijective map 153
- body of revolution 43, 1582, 1601
- boundary 37–39
- boundary curve 182
- boundary curve of a surface 182
- boundary point 920
- boundary set 21
- bounded map 153
- bounded set 41
- branch 184
- branch of a curve 492
- Brownian motion 884
- cardioid 972, 973, 1199, 1705
- Cauchy-Schwarz's inequality 23, 24, 26
- centre of gravity 1108
- centre of mass 885
- centrum 66
- chain rule 305, 333, 352, 491, 503, 581, 1215, 1489, 1493, 1808
- change of parameter 174
- circle 49
- circular motion 19
- circulation 1487
- circulation theorem 1489, 1491
- circumference 86
- closed ball 38
- closed differential form 1492
- closed disc 86
- closed domain 176
- closed set 21
- closed surface 182, 184
- closure 39
- clothoid 1219
- colour code 890
- compact set 186, 580, 1813
- compact support 1813
- complex decomposition 69
- composite function 305
- conductivity of heat 1818
- cone 19, 35, 59, 251
- conic section 19, 47, 54, 239, 536
- conic sectional conic surface 59, 66
- connected set 175, 241
- conservation of electric charge 1548, 1817
- conservation of energy 1548, 1817
- conservation of mass 1548, 1816
- conservative force 1498, 1507
- conservative vector field 1489
- continuity equation 1548, 1569, 1767, 1817
- continuity 162, 186
- continuous curve 170, 483
- continuous extension 213
- continuous function 168
- continuous surfaces 177
- contraction 167
- convective term 492
- convex set 21, 22, 41, 89, 91, 175, 244
- coordinate function 157, 169
- coordinate space 19, 21

- Cornu's spiral 1219
 Coulomb field 1538, 1545, 1559, 1566, 1577
 Coulomb vector field 1585, 1670
 cross product 19, 163, 169, 1750
 cube 42, 82
 current density 1678, 1681
 current 1487, 1499
 curvature 1219
 curve 227
 curve length 1165
 curved space integral 1021
 cusp 486, 487, 489
 cycloid 233, 1215
 cylinder 34, 42, 43, 252
 cylinder of revolution 500
 cylindric coordinates 15, 21, 34, 147, 181, 182, 289, 477, 573, 841, 1009, 1157, 1347, 1479, 1651, 1801
 cylindric surface 180, 245, 247, 248, 499, 1230
 degree of trigonometric polynomial 67
 density 885
 density of charge 1548
 density of current 1548
 derivative 296
 derivative of inverse function 494
 Descartes'a leaf 974
 dielectric constant 1669, 1670
 difference quotient 295
 differentiability 295
 differentiable function 295
 differentiable vector function 303
 differential 295, 296, 325, 382, 1740, 1741
 differential curves 171
 differential equation 369, 370, 398
 differential form 848
 differential of order p 325
 differential of vector function 303
 diffusion equation 1818
 dimension 1016
 direction 334
 direction vector 172
 directional derivative 317, 334, 375
 directrix 53
 Dirichlet/Neumann problem 1901
 displacement field 1670
 distribution of current 886
 divergence 1535, 1540, 1542, 1739, 1741, 1742
 divergence free vector field 1543
 dodecahedron 83
 domain 153, 176
 domain of a function 189
 dot product 19, 350, 1750
 double cone 252
 double point 171
 double vector product 27
 eccentricity 51
 eccentricity of ellipse 49
 eigenvalue 1906
 elasticity 885, 1398
 electric field 1486, 1498, 1679
 electrical dipole moment 885
 electromagnetic field 1679
 electromagnetic potentials 1819
 electromotive force 1498
 electrostatic field 1669
 element of area 887
 elementary chain rule 305
 elementary fraction 69
 ellipse 48–50, 92, 113, 173, 199, 227
 ellipsoid 56, 66, 110, 197, 254, 430, 436, 501, 538, 1107
 ellipsoid of revolution 111
 ellipsoidal disc 79, 199
 ellipsoidal surface 180
 elliptic cylindric surface 60, 63, 66, 106
 elliptic paraboloid 60, 62, 66, 112, 247
 elliptic paraboloid of revolution 624
 energy 1498
 energy density 1548, 1818
 energy theorem 1921
 entropy 301
 Euclidean norm 162
 Euclidean space 19, 21, 22
 Euler's spiral 1219
 exact differential form 848
 exceptional point 594, 677, 920
 expansion point 327
 explicit given function 161
 extension map 153
 exterior 37–39
 exterior point 38
 extremum 580, 632
 Faraday-Henry law of electromagnetic induction 1676
 Fick's first law of diffusion 297

- Fick's law 1818
 field line 160
 final point 170
 fluid mechanics 491
 flux 1535, 1540, 1549
 focus 49, 51, 53
 force 1485
 Fourier's law 297, 1817
 function in several variables 154
 functional matrix 303
 fundamental theorem of vector analysis 1815

 Gaussian integral 938
 Gauß's law 1670
 Gauß's law for magnetism 1671
 Gauß's theorem 1499, 1535, 1540, 1549, 1580, 1718,
 1724, 1737, 1746, 1747, 1749, 1751, 1817,
 1818, 1889, 1890, 1913
 Gauß's theorem in \mathbb{R}^2 1543
 Gauß's theorem in \mathbb{R}^3 1543
 general chain rule 314
 general coordinates 1016
 general space integral 1020
 general Taylor's formula 325
 generalized spherical coordinates 21
 generating curve 499
 generator 66, 180
 geometrical analysis 1015
 global minimum 613
 gradient 295, 296, 298, 339, 847, 1739, 1741
 gradient field 631, 847, 1485, 1487, 1489, 1491,
 1916
 gradient integral theorem 1489, 1499
 graph 158, 179, 499, 1229
 Green's first identity 1890
 Green's second identity 1891, 1895
 Green's theorem in the plane 1661, 1669, 1909
 Green's third identity 1896
 Green's third identity in the plane 1898

 half-plane 41, 42
 half-strip 41, 42
 half disc 85
 harmonic function 426, 427, 1889
 heat conductivity 297
 heat equation 1818
 heat flow 297
 height 42
 helix 1169, 1235

 Helmholtz's theorem 1815
 homogeneous function 1908
 homogeneous polynomial 339, 372
 Hopf's maximum principle 1905
 hyperbola 48, 50, 51, 88, 195, 217, 241, 255, 1290
 hyperbolic cylindric surface 60, 63, 66, 105, 110
 hyperbolic paraboloid 60, 62, 66, 246, 534, 614,
 1445
 hyperboloid 232, 1291
 hyperboloid of revolution 104
 hyperboloid of revolution with two sheets 111
 hyperboloid with one sheet 56, 66, 104, 110, 247,
 255
 hyperboloid with two sheets 59, 66, 104, 110, 111,
 255, 527
 hysteresis 1669

 identity map 303
 implicit given function 21, 161
 implicit function theorem 492, 503
 improper integral 1411
 improper surface integral 1421
 increment 611
 induced electric field 1675
 induction field 1671
 infinitesimal vector 1740
 infinity, signed 162
 infinity, unspecified 162
 initial point 170
 injective map 153
 inner product 23, 29, 33, 163, 168, 1750
 inspection 861
 integral 847
 integral over cylindric surface 1230
 integral over surface of revolution 1232
 interior 37–40
 interior point 38
 intrinsic boundary 1227
 isolated point 39
 Jacobian 1353, 1355

 Kronecker symbol 23

 Laplace equation 1889
 Laplace force 1819
 Laplace operator 1743
 latitude 35
 length 23
 level curve 159, 166, 198, 492, 585, 600, 603

- level surface 198, 503
 limit 162, 219
 line integral 1018, 1163
 line segment 41
 Linear Algebra 627
 linear space 22
 local extremum 611
 logarithm 189
 longitude 35
 Lorentz condition 1824

 Maclaurin's trisectrix 973, 975
 magnetic circulation 1674
 magnetic dipole moment 886, 1821
 magnetic field 1491, 1498, 1679
 magnetic flux 1544, 1671, 1819
 magnetic force 1674
 magnetic induction 1671
 magnetic permeability of vacuum 1673
 magnostatic field 1671
 main theorems 185
 major semi-axis 49
 map 153
 MAPLE 55, 68, 74, 156, 171, 173, 341, 345, 350,
 352–354, 356, 357, 360, 361, 363, 364,
 366, 368, 374, 384–387, 391–393, 395–
 397, 401, 631, 899, 905–912, 914, 915,
 917, 919, 922–924, 926, 934, 935, 949,
 951, 954, 957–966, 968, 971–973, 975,
 1032–1034, 1036, 1037, 1039, 1040, 1042,
 1053, 1059, 1061, 1064, 1066–1068, 1070–
 1072, 1074, 1087, 1089, 1091, 1092, 1094,
 1095, 1102, 1199, 1200
 matrix product 303
 maximal domain 154, 157
 maximum 382, 579, 612, 1916
 maximum value 922
 maximum-minimum principle for harmonic functions 1895
 Maxwell relation 302
 Maxwell's equations 1544, 1669, 1670, 1679, 1819
 mean value theorem 321, 884, 1276, 1490
 mean value theorem for harmonic functions 1892
 measure theory 1015
 Mechanics 15, 147, 289, 477, 573, 841, 1009, 1157,
 1347, 1479, 1651, 1801, 1921
 meridian curve 181, 251, 499, 1232
 meridian half-plane 34, 35, 43, 181, 1055, 1057,
 1081

 method of indefinite integration 859
 method of inspection 861
 method of radial integration 862
 minimum 186, 178, 579, 612, 1916
 minimum value 922
 minor semi-axis 49
 mmf 1674
 Möbius strip 185, 497
 Moivre's formula 122, 264, 452, 548, 818, 984,
 1132, 1322, 1454, 1626, 1776, 1930
 monopole 1671
 multiple point 171

 nabla 296, 1739
 nabla calculus 1750
 nabla notation 1680
 natural equation 1215
 natural parametric description 1166, 1170
 negative definite matrix 627
 negative half-tangent 485
 neighbourhood 39
 neutral element 22
 Newton field 1538
 Newton-Raphson iteration formula 583
 Newton's second law 1921
 non-oriented surface 185
 norm 19, 23
 normal 1227
 normal derivative 1890
 normal plane 487
 normal vector 496, 1229

 octant 83
 Ohm's law 297
 open ball 38
 open domain 176
 open set 21, 39
 order of expansion 322
 order relation 579
 ordinary integral 1017
 orientation of a surface 182
 orientation 170, 172, 184, 185, 497
 oriented half line 172
 oriented line 172
 oriented line segment 172
 orthonormal system 23

 parabola 52, 53, 89–92, 195, 201, 229, 240, 241
 parabolic cylinder 613

- parabolic cylindric surface 64, 66
 paraboloid of revolution 207, 613, 1435
 parallelepiped 27, 42
 parameter curve 178, 496, 1227
 parameter domain 1227
 parameter of a parabola 53
 parametric description 170, 171, 178
 parfrac 71
 partial derivative 298
 partial derivative of second order 318
 partial derivatives of higher order 382
 partial differential equation 398, 402
 partial fraction 71
 Peano 483
 permeability 1671
 piecewise C^k -curve 484
 piecewise C^n -surface 495
 plane 179
 plane integral 21, 887
 point of contact 487
 point of expansion 304, 322
 point set 37
 Poisson's equation 1814, 1889, 1891, 1901
 polar coordinates 15, 19, 21, 30, 85, 88, 147, 163,
 172, 213, 219, 221, 289, 347, 388, 390,
 477, 573, 611, 646, 720, 740, 841, 936,
 1009, 1016, 1157, 1165, 1347, 1479, 1651,
 1801
 polar plane integral 1018
 polynomial 297
 positive definite matrix 627
 positive half-tangent 485
 positive orientation 173
 potential energy 1498
 pressure 1818
 primitive 1491
 primitive of gradient field 1493
 prism 42
 Probability Theory 15, 147, 289, 477, 573, 841,
 1009, 1157, 1347, 1479, 1651, 1801
 product set 41
 projection 23, 157
 proper maximum 612, 618, 627
 proper minimum 612, 613, 618, 627
 pseudo-sphere 1434
 Pythagoras's theorem 23, 25, 30, 121, 451, 547,
 817, 983, 1131, 1321, 1453, 1625, 1775,
 1929
 quadrant 41, 42, 84
 quadratic equation 47
 range 153
 rectangle 41, 87
 rectangular coordinate system 29
 rectangular coordinates 15, 21, 22, 147, 289, 477,
 573, 841, 1009, 1016, 1079, 1157, 1165,
 1347, 1479, 1651, 1801
 rectangular plane integral 1018
 rectangular space integral 1019
 rectilinear motion 19
 reduction of a surface integral 1229
 reduction of an integral over cylindric surface 1231
 reduction of surface integral over graph 1230
 reduction theorem of line integral 1164
 reduction theorem of plane integral 937
 reduction theorem of space integral 1021, 1056
 restriction map 153
 Riccati equation 369
 Riesz transformation 1275
 Rolle's theorem 321
 rotation 1739, 1741, 1742
 rotational body 1055
 rotational domain 1057
 rotational free vector field 1662
 rules of computation 296
 saddle point 612
 scalar field 1485
 scalar multiplication 22, 1750
 scalar potential 1807
 scalar product 169
 scalar quotient 169
 second differential 325
 semi-axis 49, 50
 semi-definite matrix 627
 semi-polar coordinates 15, 19, 21, 33, 147, 181,
 182, 289, 477, 573, 841, 1009, 1016, 1055,
 1086, 1157, 1231, 1347, 1479, 1651, 1801
 semi-polar space integral 1019
 separation of the variables 853
 signed curve length 1166
 signed infinity 162
 simply connected domain 849, 1492
 simply connected set 176, 243
 singular point 487, 489
 space filling curve 171
 space integral 21, 1015

- specific capacity of heat 1818
 sphere 35, 179
 spherical coordinates 15, 19, 21, 34, 147, 179, 181,
 289, 372, 477, 573, 782, 841, 1009, 1016,
 1078, 1080, 1081, 1157, 1232, 1347, 1479,
 1581, 1651, 1801
 spherical space integral 1020
 square 41
 star-shaped domain 1493, 1807
 star shaped set 21, 41, 89, 90, 175
 static electric field 1498
 stationary magnetic field 1821
 stationary motion 492
 stationary point 583, 920
 Statistics 15, 147, 289, 477, 573, 841, 1009, 1157,
 1347, 1479, 1651, 1801
 step line 172
 Stokes's theorem 1499, 1661, 1676, 1679, 1746,
 1747, 1750, 1751, 1811, 1819, 1820, 1913
 straight line (segment) 172
 strip 41, 42
 substantial derivative 491
 surface 159, 245
 surface area 1296
 surface integral 1018, 1227
 surface of revolution 110, 111, 181, 251, 499
 surjective map 153

 tangent 486
 tangent plane 495, 496
 tangent vector 178
 tangent vector field 1485
 tangential line integral 861, 1485, 1598, 1600, 1603
 Taylor expansion 336
 Taylor expansion of order 2, 323
 Taylor's formula 321, 325, 404, 616, 626, 732
 Taylor's formula in one dimension 322
 temperature 297
 temperature field 1817
 tetrahedron 93, 99, 197, 1052
 Thermodynamics 301, 504
 top point 49, 50, 53, 66
 topology 15, 19, 37, 147, 289, 477, 573, 841, 1009,
 1157, 1347, 1479, 1651, 1801
 torus 43, 182–184
 transformation formulæ 1353
 transformation of space integral 1355, 1357
 transformation theorem 1354
 trapeze 99

 triangle inequality 23, 24
 triple integral 1022, 1053

 uniform continuity 186
 unit circle 32
 unit disc 192
 unit normal vector 497
 unit tangent vector 486
 unit vector 23
 unspecified infinity 162

 vector 22
 vector field 158, 296, 1485
 vector function 21, 157, 189
 vector product 19, 26, 30, 163, 169, 1227, 1750
 vector space 21, 22
 vectorial area 1748
 vectorial element of area 1535
 vectorial potential 1809, 1810
 velocity 490
 volume 1015, 1543
 volumen element 1015

 weight function 1081, 1229, 1906
 work 1498

 zero point 22
 zero vector 22

 (r, s, t) -method 616, 619, 633, 634, 638, 645–647,
 652, 655
 C^k -curve 483
 C^n -functions 318
 1-1 map 153