Stochastic Processes 1

Probability Examples c-8

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Introduction

This is the eighth book of examples from the *Theory of Probability.* The topic *Stochastic Processes* is so huge that I have chosen to split the material into two books. In the present first book we shall deal with examples of *Random Walk* and Markov chains, where the latter topic is very large. In the next book we give examples of *Poisson processes, birth and death processes, queueing theory* and other types of stochastic processes.

The prerequisites for the topics can e.g. be found in the *Ventus: Calculus 2* series and the *Ventus: Complex Function Theory* series, and all the previous *Ventus: Probability c1-c7.*

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
27th October 2014

1 Stochastic processes; theoretical background

1.1 General about stochastic processes

A *stochastic process* is a family $\{X(t) \mid t \in T\}$ of random variables $X(t)$, all defined on the same sample space Ω, where the domain T of the parameter is a subset of $\mathbb{R}$ (usually $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $[0,+\infty[$ or $\mathbb{R}$ itself), and where the parameter $t \in T$ is interpreted as the time.

We note that we for every fixed ω in the sample space Ω in this way define a so-called *sample function* $T(\cdot,\omega) : T \to \mathbb{R}$ on the domain T of the parameter.

In the description of such a stochastic process we must know the *distribution function of the stochastic process*, i.e.

$$P\left\{X\left(t_1\right) \le x_1 \,\wedge\, X\left(t_2\right) \le x_2 \,\wedge\, \cdots \,\wedge\, X\left(t_n\right) \le x_n\right\}$$

for every $t_1, \dots, t_n \in T$, and every $x_1, \dots, x_n \in \mathbb{R}$, for every $n \in \mathbb{N}$.

This is of course not always possible, so one tries instead to find less complicated expressions connected with the stochastic process, like e.g. means, which to some extent can be used to characterize the distribution.

A very important special case occurs when the random variables $X(t)$ are all *discrete* of values in $\mathbb{N}_0$. If in this case $X(t) = k$, then we say that the *process at time t is at state E_k*. This can now be further specialized.

A *Markov process* is a discrete stochastic process of values in $\mathbb{N}_0$, for which also

$$P\{X(t_{n+1}) = k_{n+1} \mid X(t_n) = k_n \wedge \cdots \wedge X(t_1) = k_1\} = P\{X(t_{n+1}) = k_{n+1} \mid X(t_n) = k_n\}$$

for any $k_1, \dots, k_{n+1}$ in the range, for any $t_1 < t_2 < \cdots < t_{n+1}$ from T, and for any $n \in \mathbb{N}$.

We say that when a Markov process is going to be described at time t_{n+1}, then we have just as much information, if we know the process at time t_n, as if we even know the process at the times $t_1, \dots, t_n$, provided that these times are all smaller than t_{n+1}. One may coin this in the following way: *If the present is given, then the future is independent of the past.*

1.2 Random walk

Consider a sequence (X_k) of mutually independent identically distributed random variables, where the distribution is given by

$$P\{X_k = 1\} = p \qquad \text{and} \qquad P\{X_k = -1\} = q, \qquad\qquad p,\, q > 0 \text{ and } p + q = 1 \text{and} k \in \mathbb{N}.$$

We define another sequence of random variables (S_n) by

$$S_0 = 0 \qquad \text{and} \qquad S_n = S_0 + \sum_{k=1}^{n} X_k, \qquad \text{for } n \in \mathbb{N}.$$

In this special construction the new sequence $(S_n)_{n=0}^{+\infty}$ is called a *random walk.* In the special case of $p = q = \dfrac{1}{2}$, we call it a *symmetric random walk.*

An *outcome* of $X_1, X_2, \dots, X_n$ is a sequence $x_1, x_2, \dots, x_n$, where each x_k is either 1 or -1.

A random walk may be interpreted in several ways, of which we give the following two:

1) A person walks on a road, where he per time unit with probability p takes one step to the right and with probability q takes one step to the left. At time 0 the person is at state E_0. His position at time n is given by the random variable S_n. If in particular, $p = q = \frac{1}{2}$, this process is also called the *"drunkard's walk"*.

2) Two persons, Peter and Paul, are playing a series of games. In one particular game, Peter wins with probability p, and Paul wins with probability q. After each game the winner receives 1 $ from the loser. We assume at time 0 that they both have won 0 $. Then the random variable S_n describes Peter's gain (positive or negative) after n games, i.e. at time n.

We mention

Theorem 1.1 (The ballot theorem). *At an election a candidate A obtains in total a votes, while another candidate B obtains b votes, where $b < a$. The probability that A is leading during the whole of the counting is equal to $\dfrac{a-b}{a+b}$.*

Let Peter and Paul be the two gamblers mentioned above. Assuming that Peter to time 0 has 0 $, then the probability of Peter at some (later) time having the sum of 1 $ is given by

$$\alpha = \min\left\{1\,,\, \frac{p}{q}\right\},$$

hence the probability of Peter at some (later) time having the sum of N \$, where $N > 0$, is given by

$$\alpha^N = \min\left\{1\,,\,\left(\frac{p}{q}\right)^N\right\}.$$

The corresponding probability that Paul at some time has the sum of 1 \$ is

$$\beta = \min\left\{1\,,\,\frac{q}{p}\right\},$$

and the probability that he at some later time has a positive sum of N \$ is

$$\beta^N = \min\left\{1\,,\,\left(\frac{q}{p}\right)^N\right\}.$$

Based on this analysis we introduce

$$p_n := P\{\text{return to the initial position at time } n\}, \qquad n \in \mathbb{N},$$

$$f_n := P\{\text{the first return to the initial position at time } n\}, \qquad n \in \mathbb{N},$$

$$f := P\{\text{return to the initial position at some later time}\} = \sum_{n=1}^{+\infty} f_n.$$

Notice that $p_n = f_n = 0$, if n is an odd number.

We shall now demonstrate how the corresponding generating functions profitably can be applied in such situation. Thus we put

$$P(s) = \sum_{n=0}^{+\infty} p_n\, s^n \qquad \text{and} \qquad F(s) = \sum_{n=0}^{+\infty} f_n\, s^n,$$

where we have put $p_0 = 1$ and $f_0 = 0$. It is easily seen that the relationship between these two generating functions is

$$F(s) = 1 - \frac{1}{P(s)}.$$

Then by the binomial series

$$P(s) = \frac{1}{\sqrt{1-4pqs^2}},$$

so we conclude that

$$F(s) = \sum_{k=1}^{+\infty} \frac{1}{2k-1}\binom{2k}{k}(pq)^k s^{2k},$$

which by the definition of $F(s)$ implies that

$$f_{2k} = \frac{1}{2k-1}\binom{2k}{k}(pq)^k = \frac{p_{2k}}{2k-1}.$$

Furthermore,

$$f = \lim_{s \to 1-} F(s) = 1 - \sqrt{1-4pq} = 1 - |1-2p| = \begin{cases} 2p, & \text{for } p < \dfrac{1}{2}, \\[2mm] 1, & \text{for } p = \dfrac{1}{2}, \\[2mm] 2q, & \text{for } p > \dfrac{1}{2}. \end{cases}$$

In the *symmetric case*, where $p = \dfrac{1}{2}$, we define a random variable T by

$$T = n, \qquad \text{if the first return occurs at time } n.$$

Then it follows from the above that T has the distribution

$$P\{T = 2k\} = f_{2k} \qquad \text{and} \qquad P\{T = 2k-1\} = 0, \qquad\qquad \text{for } k \in \mathbb{N}.$$

The generating function is

$$F(s) = 1 - \sqrt{1-s^2},$$

hence

$$E\{T\} = \lim_{s \to 1-} F(s) = +\infty,$$

which we formulate as *the expected time of return to the initial position is* $+\infty$.

1.3 The ruin problem

The initial position is almost the same as earlier. The two gamblers, Peter and Paul, play a series of games, where Peter has the probability p of winning 1 \$ from Paul, while the probability is q that he loses 1 \$ to Paul. At the beginning Peter owns k \$, and Paul owns $N-k$ \$, where $0 < k < N$. The games continue, until one of them is ruined. The task here is to find the probability that Peter is ruined.

Let a_k be the probability that Peter is ruined, if he at the beginning has k \$, where we allow that $k = 0, 1, \dots, N$. If $k = 0$, then $a_0 = 1$, and if $k = N$, then $a_N = 0$. Then consider $0 < k < N$, in which case

$$a_k = p\,a_{k+1} + q\,a_{k-1}.$$

We rewrite this as the *homogeneous, linear difference equation of second order*,

$$p\,a_{k+1} - a_k + q\,a_{k-1} = 0, \qquad k = 1, 2, \dots, N-1.$$

Concerning the solution of such difference equations, the reader is referred to e.g. the *Ventus: Calculus 3*series. We have two possibilities:

1) If $p \neq \frac{1}{2}$, then the probability for Peter being ruined, if he starts with k \$, is given by

$$a_k = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}, \qquad \text{for } k = 0,\, 1,\, 2,\, \dots,\, N.$$

2) If instead $p = q = \frac{1}{2}$, then

$$a_k = \frac{N-k}{N}, \qquad \text{for } k = 1,\, 2,\, \dots,\, N.$$

We now change the problem to finding the expected number of games μ_k, which must be played before one of the two gamblers is ruined, when Peter starts with the sum of k \$. In this case,

$$\mu_k = p\,\mu_{k+1} + q\,\mu_{k-1} + 1, \qquad \text{for } k = 1,\, 2,\, \dots,\, N-1.$$

We rewrite this equation as an *inhomogeneous linear difference equation of second order*. Given the boundary conditions above, its solution is

1) For $p \neq \frac{1}{2}$ we get

$$\mu_k = \frac{k}{q-p} - \frac{N}{q-p} \cdot \frac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N}, \qquad \text{for } k = 0, 1, 2, \dots, N.$$

2) For $p = q = \frac{1}{2}$ we get instead

$$\mu_k = k(N-k), \qquad \text{for } k = 0, 1, 2, \dots, N.$$

In the special case, where we consider a *symmetric random walk*, i.e.. $p = \frac{1}{2}$, we sum up the results:

Let (X_k) be a sequence of mutually independent identically distributed random variables of distribution given by

$$P\{X_k = 1\} = P\{X_k = -1\} = \frac{1}{2}, \qquad \text{for } n \in \mathbb{N}.$$

In this case, the random variables $S_0 = 0$ and $S_{2n} = S_0 + \sum_{k=1}^{2n} X_k$ have the distribution given by

$$p_{2n,2r} := P\{S_{2n} = 2r\} = \binom{2n}{n+r} 2^{-2n}, \qquad \text{for } r = -n, -n+1, \dots, n, \qquad n \in \mathbb{N}.$$

In particular,

$$u_{2n} := p_{2n,0} = \binom{2n}{n} e^{-2n} \quad \left(\sim \frac{1}{\sqrt{\pi n}} \text{ for large } n\right).$$

Then we define a random variable T by

$$T = n, \qquad \text{if the first return to } E_0 \text{ occurs to time } n.$$

This random variable has the values 2, 4, 6, ..., with the probabilities

$$f_{2n} := P\{T = 2n\} = \frac{u_{2n}}{2n-1} = \frac{1}{2n-1}\binom{2n}{n} 2^{-2n}, \qquad \text{for } n \in \mathbb{N},$$

where

$$E\{T\} = +\infty.$$

For every $N \in \mathbb{Z}$ there is the probability 1 for the process reaching state E_N to some later time.

Finally, if the process is at state E_k, $0 < k < N$ to time 0, then the process reaches state E_0 before E_N with the probability $1 - \frac{k}{N}$, and it reaches state E_N before E_0 with the probability $\frac{k}{N}$. The expected time for the process to reach either E_0 or E_N from E_k is $k(N-k)$.

Theorem 1.2 (The Arcus sinus law for the latest visit). *The probability that the process up to time $2n$ the last time is in state E_0 to time $2k$, is given by*

$$\alpha_{2k,2n} = u_{2k} \cdot u_{2n-2k},$$

where we have put $u_{2n} = P\{S_{2n} = 0\}$.

The distribution which has the probability $\alpha_{2k,2n}$ at the point $2k$, where $0 \le k \le n$, is also called the *discrete Arcus sinus distribution of order n*. The reason for this name is the following: If β and γ are given numbers, where $0 < \beta < \gamma < 1$, then

$$P\{\text{last visit of } E_0 \text{ is between } 2\beta n \text{ and } 2\gamma n\} = \sum_{\beta n \le k \le \gamma n} u_{2k} \cdot u_{2n-2k}$$

$$\sim \sum_{\beta \le \frac{k}{n} \le \gamma} \frac{1}{n} \cdot \frac{1}{\pi \sqrt{\frac{k}{n}\left(1 - \frac{k}{n}\right)}} \sim \int_\beta^\gamma \frac{1}{\pi\sqrt{x(1-x)}}\, dx = \frac{2}{\pi} \operatorname{Arcsin} \sqrt{\gamma} - \frac{2}{\pi} \operatorname{Arcsin} \sqrt{\beta},$$

where we recognize the sum as a mean sum of the integral of Arcsin. This implies that

$$P\{\text{last visit of } E_0 \text{ before } 2nx\} \sim \frac{2}{\pi} \operatorname{Arcsin} \sqrt{x}, \qquad \text{for } x \in]0\,,\,1[.$$

One interpretation of this result is that if Peter and Paul play many games, then there is a large probability that one of them is almost all the time on the winning side, and the other one is almost all the time on the losing side.

1.4 Markov chains

A *Markov chain* is a (discrete) stochastic process $\{X(t) \mid t \in \mathbb{N}_0\}$, which has a *finite* number of states, e.g. denoted by $E_1, E_2, \dots, E_m$, and such that for any $1 \le k_0, k_1, \dots, k_n \le m$ and every $n \in \mathbb{N}$,

$$P\{X(n) = k_n \mid X(n-1) = k_{n-1} \wedge \cdots \wedge X(0) = k_0\} = P\{X(n) = k_n \mid X(n-1) = k_{n-1}\}.$$

If furthermore the Markov chain satisfies the condition that the conditional probabilities

$$p_{ij} := P\{X(n) = j \mid X(n-1) = i\}$$

do not depend on n, we call the process a *stationary Markov chain*.

We shall in the following only consider *stationary Markov chains*, and we just write *Markov chains*, tacitly assuming that they are stationary.

A Markov chain models the situation, where a particle moves between the m states $E_1, E_2, \dots, E_m$, where each move happens at discrete times $t \in \mathbb{N}$. Then p_{ij} represents the probability that the particle in one step moves from state E_i to state E_j. In particular, p_{ii} is the probability that the particle stays at state E_i.

We call the p_{ij} the *transition probabilities*. They are usually lined up in a *stochastic matrix:*

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}.$$

In this matrix the element p_{ij} in the i-th row and the j-th column represents the probability for the transition from state E_i to state E_j.

For every stochastic matrix we obviously have

$$p_{ij} \geq 0 \qquad \text{for every } i \text{ and } j,$$

$$\sum_j p_{ij} = 1 \qquad \text{for every } i, \text{ thus all sums of the rows are 1.}$$

The *probabilities of state* $p_i^{(n)}$ are defined by

$$p_i^{(n)} := P\{X(n) = i\}, \qquad \text{for } i = 1, 2 \dots, m \text{ og } n \in \mathbb{N}_0.$$

The corresponding *vector of state* is

$$\mathbf{p}^{(n)} := \left(p_1^{(n)}, p_2^{(n)}, \dots, p_m^{(n)}\right), \qquad \text{for } n \in \mathbb{N}_0.$$

In particular, the *initial distribution* is given by

$$\mathbf{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, \dots, p_m^{(0)}\right).$$

Then by ordinary matrix computation (note the order of the matrices),

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)}\,\mathbf{P}, \qquad \text{hence by iteration} \qquad \mathbf{p}^{(n)} = \mathbf{p}^{(0)}\,\mathbf{P}^n,$$

proving that the probabilities of state at time $t = n$ is only determined by the initial condition $\mathbf{p}^{(0)}$ and the stochastic matrix $\mathbf{P}$, iterated n timers. The elements $p_{ij}^{(n)}$ in $\mathbf{P}^n$ are called the transition probabilities at step n, and they are given by

$$p_{ij}^{(n)} = P\{X(k+n) = j \mid X(k) = i\}.$$

We define a *probability vector* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ as a vector, for which

$$\alpha_i \geq 0, \quad \text{for } i = 1, 2, \dots, m, \qquad \text{and} \qquad \sum_{i=1}^m \alpha_i = 1.$$

A probability vector α is called *invariant* with respect to the stochastic matrix $\mathbf{P}$, or a *stationary distribution* of the Markov chain, if

$$\alpha\,\mathbf{P} = \alpha.$$

The latter name is due to the fact that if $X(k)$ has its distribution given by α, then every later $X(n+k)$, $n \in \mathbb{N}$ has also its distribution given by α.

In order to ease the computations in practice we introduce the following new concepts:

1) We say that a Markov chain is *irreducible*, if we to any pair of indices (i,j) can find an $n = n_{ij} \in \mathbb{N}$, such that that $p_{ij}^{n} > 0$. This means that the *state* E_j *can be reached from state* E_i, *no matter the choice of* (i,j). (However, all the $n_{ij} \in \mathbb{N}$ do not have to be identical).

2) If we even can choose $n \in \mathbb{N}$ *independently* of the pair of indices (i,j), we say that the Markov chain is *regular*.

Remark 1.1 Notice that *stochastic regularity* has nothing to do with the concept of a *regular matrix* known from Linear Algebra. We must not confuse the two definitions. $\Diamond$

Theorem 1.3 *Let* $\mathbf{P}$ *be an* $m \times m$ regular *stochastic matrix.*

1) The sequence $(\mathbf{P}^n)$ *converges towards a stochastic* limit matrix $\mathbf{G}$ *for* $n \to +\infty$.

2) Every row in $\mathbf{G}$ *has the same probability vector*

$$\mathbf{g} = (g_1, g_2, \dots, g_m), \qquad \textit{where } g_i > 0 \textit{ for every } i = 1, 2, \dots, m.$$

3) If $\mathbf{p}$ *is any probability vector, then*

$$\mathbf{p}\,\mathbf{P}^n \to \mathbf{g} \qquad \textit{for } n \to +\infty.$$

4) The regular matrix $\mathbf{P}$ *has precisely one invariant probability vector,* $\mathbf{g}$.

The theorem shows that for a *regular* stochastic matrix $\mathbf{P}$ the *limit distribution is uniquely determined by the invariant probability vector.*

It may occur for an *irreducible* Markov chain that $\mathbf{P}^n$ diverges for $n \to +\infty$. We have instead

Theorem 1.4 *Let* $\mathbf{P}$ *be an* $m \times m$ irreducible *stochastic matrix.*

1) The sequence $\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{P}^i\right)$ *converges towards a stochastic* limit matrix $\mathbf{G}$ *for* $n \to +\infty$.

2) Every row in $\mathbf{G}$ *is the same probability vector*

$$\mathbf{g} = (g_1, g_2, \dots, g_m), \qquad \textit{where } g_i > 0 \textit{ for ethvert } i = 1, 2, \dots, m.$$

3) Given any probability vector $\mathbf{p}$*, then*

$$\mathbf{p}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{P}^i\right) \to \mathbf{g} \qquad \textit{for } n \to +\infty.$$

4) The irreducible matrix $\mathbf{P}$ *has precisely one invariant probability vector, namely* $\mathbf{g}$.

Given a Markov chain of the m states $E_1, E_2, \dots, E_m$ with the corresponding stochastic matrix $\mathbf{P}$. A subset $\mathcal{C}$ of the states $E_1, E_2, \dots, E_m$ is called *closed*, if no state outside $\mathcal{C}$ can be reached from any state in $\mathcal{C}$. This can also be expressed in the following way: A subset $\mathcal{C}$ of the m states is *closed*, if for every $E_i \in \mathcal{C}$ and every $E_j \notin \mathcal{C}$ we have $p_{ij} = 0$.

If a closed set only contains one state, $\mathcal{C} = \{E_i\}$, we call E_i an *absorbing* state. This is equivalent to $p_{ii} = 1$, so we can immediately find the absorbing states from the numbers 1 in the diagonal of the stochastic matrix $\mathbf{P}$.

The importance of a closed set is described by the following theorem, which is fairly easy to apply in practice.

Theorem 1.5 *A Markov chain is irreducible, if and only if it does not contain any proper closed subset of states.*

A necessary condition of irreducibility of a Markov chain is given in the following theorem:

Theorem 1.6 *Assume that a Markov chain of the m states $E_1, E_2, \dots, E_m$ is irreducible. Then to every pair of indices (i,j), $1 \le i, j \le m$ there exists an $n = n_{ij}$, such that $1 \le n \le m$ and $p_{ij}^{(n)} > 0$.*

Concerning the proof of regularity we may use the following method, if the matrix $\mathbf{P}$ is not too complicated: Compute successively the matrices $\mathbf{P}^i$, until one at last (hopefully) reaches a number $i = n$, where all elements of $\mathbf{P}^n$ are different from zero. Since we already know that all elements are ≥ 0, we can ease the computations by just writing $*$ for the elements of the matrices, which are $\neq 0$. We do not have to compute their exact values. But we must be very careful with the zeros. This method is of course somewhat laborious, and one may often apply the following theorem instead.

Theorem 1.7 *If a stochastic matrix $\mathbf{P}$ is irreducible, and there exists a positive element in the diagonal $p_{ii} > 0$, then $\mathbf{P}$ is regular.*

It is usually easy to prove that $\mathbf{P}$ is irreducible. The difficult part is to prove that it is also regular. We give here another result:

We introduce for every $m \times m$ irreducible stochastic matrix $\mathbf{P}$ the following numbers

$$d_i := \text{ largest common divisor of all } n \in \mathbb{N}, \text{ for which } p_{ii}^{(n)} > 0.$$

It can be proved that $d_1 = d_2 = \cdots = d_m := d$, so we only have to find one single of the d_i-erne. (For one's own convenience, choose always that d_i, which gives the easiest computations).

Theorem 1.8 *An irreducible Markov chain is regular, if and only if $d = 1$.*
If $d > 1$, then the Markov chain is periodic of period d.

One may consider a *random walk* on the set $\{1, 2, 3, \dots, N\}$ as a *Markov chain* of the transition probabilities

$$p_{i,i-1} = q \quad \text{and} \quad p_{i,i+1} = p, \qquad \text{for } i = 2, 3, \dots, N-1, \qquad \text{where } p, q > 0 \text{ og } p + q = 1.$$

1) If $p_{11} = p_{NN} = 1$, then we have a *random walk of two absorbing barriers.*

2) If $p_{12} = p_{N,N-1} = 1$, then we have a *random walk of two reflecting barriers.* In this case the corresponding Markov chain is irreducible.

2 Random walk

Example 2.1 *Consider a ruin problem of total capital N \$, where $p < \frac{1}{2}$.*
In every game the loss/gain is only 50 cents. Is this game more advantageous for Peter than if the stake was 1 \$ (i.e. a smaller probability that Peter is ruined)?

We have $2N+1$ states $E_0, E_1, \dots, E_{2N}$, where state E_i means that A has $\frac{i}{2}$ \$. If A initially has k \$, then $a_{2N} = 0$ and $a_0 = 1$.
We get for the values in between,

$$a_k = p\, a_{k+1} + q\, a_{k-1}, \qquad 0 < k < 2N,$$

which we rewrite as

$$p\,(a_{k+1} - a_k) = q\,(a_k - a_{k-1})\,.$$

Hence by recursion,

$$a_k - a_{k-1} = \frac{q}{p}\,(a_{k-1} - a_{k-2}) = \cdots = \left(\frac{q}{p}\right)^{k-1}(a_1 - a_0)\,,$$

and we get

$$\begin{aligned} a_k &= (a_k - a_{k-1}) + (a_{k-1} - a_{k-2}) + \cdots + (a_1 - a_0) + a_0 \\ &= \left\{\left(\frac{q}{p}\right)^{k-1} + \left(\frac{q}{p}\right)^{k-2} + \cdots + \left(\frac{q}{p}\right)^{1} + 1\right\}(a_1 - a_0) + a_0 \\ &= \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \frac{q}{p}}\,(a_1 - a_0) + a_0 = \frac{\left(\frac{q}{p}\right)^k - 1}{\frac{q}{p} - 1}\,(a_1 - a_0) + a_0. \end{aligned}$$

Now, $a_0 = 1$, so we get for $k = 2N$ that

$$0 = a_{2N} = \frac{\left(\frac{q}{p}\right)^{2N} - 1}{\frac{q}{p} - 1}\,(a_1 - 1) + 1 = a_1\,\frac{\left(\frac{q}{p}\right)^{2N} - 1}{\frac{q}{p} - 1} + 1 - \frac{\left(\frac{q}{p}\right)^{2N} - 1}{\frac{q}{p} - 1},$$

hence by a rearrangement,

$$a_1 = \frac{\frac{q}{p} - 1}{\left(\frac{q}{p}\right)^{2N} - 1}\left\{\frac{\left(\frac{q}{p}\right)^{2N} - 1}{\frac{q}{p} - 1} - 1\right\} = \frac{q}{p}\,\frac{\left(\frac{q}{p}\right)^{2N-1} - 1}{\left(\frac{q}{p}\right)^{2N} - 1}.$$

Then by insertion,

$$a_{2k} = \frac{\left(\frac{q}{p}\right)^{2k} - 1}{\frac{q}{p} - 1}\left\{-\frac{\frac{q}{p} - 1}{\left(\frac{q}{p}\right)^{2N} - 1}\right\} + 1 = \frac{\left(\frac{q}{p}\right)^{2N} - \left(\frac{q}{p}\right)^{2k}}{\left(\frac{q}{p}\right)^{2N} - 1} = \frac{\left(\frac{q}{p}\right)^{N} - \left(\frac{q}{p}\right)^{k}}{\left(\frac{q}{p}\right)^{N} - 1}\cdot\frac{\left(\frac{q}{p}\right)^{N} + \left(\frac{q}{p}\right)^{k}}{\left(\frac{q}{p}\right)^{N} + 1}.$$

These expressions should be compared with

$$\tilde{a}_k = \frac{\left(\dfrac{q}{p}\right)^N - \left(\dfrac{q}{p}\right)^k}{\left(\dfrac{q}{p}\right)^N - 1}$$

from the ruin problem with 1 \$ at stake in each game. Notice the indices $\tilde{a}_k$ and a_{2k}, because 1 \$ $= 2 \cdot 50$ cents. It clearly follows from $q > \frac{1}{2} > p$, that

$$a_{2k} > \tilde{a}_k.$$

Since the a indicate the probability that Peter is ruined, if follows that there is larger probability that he is ruined if the stake is 50 cents than if the stake is 1 \$.

Example 2.2 *Peter and Paul play in total N games. In each game Peter has the probability $\frac{1}{2}$ for winning (in which case he receives 1 \$ from Paul) and probability $\frac{1}{2}$ for losing (in which case he delivers 1 \$ to Paul). The games are mutually independent of each other. Find the probability that Peter's total gain never after the start of the games is 0 \$.*

The probability that Peter's gain is never 0 \$, is

$$1 - P\{\text{return to the initial position at some time}\} = 1 - \sum_{n=1}^{N} f_n,$$

where

$$f_n = P\{\text{first return is to time } n\}.$$

The parity assures that $f_{2k-1} = 0$, because we can only return to the initial position after an *even* number of steps. It follows from $p = q = \frac{1}{2}$ that

$$f_{2k} = \frac{1}{2k-1}\binom{2k}{k}(p\,q)^k = \frac{1}{2k-1}\binom{2k}{k}\left(\frac{1}{4}\right)^k.$$

By insertion we get the probability

$$1 - \sum_{n=1}^{N} f_n = 1 - \sum_{k=1}^{\left[\frac{N}{2}\right]} \frac{1}{2k-1}\binom{2k}{k}\left(\frac{1}{4}\right)^k = \sum_{k=\left[\frac{N}{2}\right]+1}^{\infty} \frac{1}{2k-1}\binom{2k}{k}\left(\frac{1}{4}\right)^k.$$

Alternatively, we may use the following considerations which somewhat simplify the task.

1) If $N = 2n$ is even, then the wanted probability is

$$P\{S_1 \neq 0 \wedge S_2 \neq 0 \wedge \cdots \wedge S_{2n} \neq 0\}.$$

This expression is equal to u_{2n}, which again is equal to

$$u_{2n} = \binom{2n}{n} 2^{-2n} \qquad \left(\sim \frac{1}{\sqrt{\pi n}}\right).$$

2) If $N = 2n+1$ is odd, then S_{2n+1} is always $\neq 0$. Hence, the probability is

$$\begin{aligned} &P\{S_1 \neq 0 \wedge S_2 \neq 0 \wedge \cdots \wedge S_{2n} \neq 0 \wedge S_{2n+1} \neq 0\} \\ &\quad = P\{S_1 \neq 0 \wedge S_2 \neq 0 \wedge \cdots \wedge S_{2n} \neq 0\} = \binom{2n}{n} 2^{-2n} \qquad \left(\sim \frac{1}{\sqrt{\pi n}}\right) \end{aligned}$$

according to the first question.

3 Markov chains

Example 3.1 *Let* $\mathbf{P}$ *be a stochastic matrix for a Markov chain of the states* $E_1, E_2, \dots, E_m$.

1) Prove that

$$p_{ij}^{(n_1+n_2+n_3)} \geq p_{ik}^{(n_1)} \cdot p_{kk}^{(n_2)} \cdot p_{kj}^{(n_3)},$$

for

$$1 \leq i,\, j,\, k \leq m, \qquad n_1,\, n_2,\, n_3 \in \mathbb{N}_0.$$

2) Prove that if the Markov chain is irreducible and $p_{ii} > 0$ *for some* i*, then the Markov chain is regular.*

1) Since $\mathbf{P}^{n_1+n_2+n_3} = \mathbf{P}^{n_1}\mathbf{P}^{n_2}\mathbf{P}^{n_3}$, and since all matrix elements are ≥ 0, it follows that

$$p_{ij}^{(n_1+n_2+n_3)} = \sum_{k=1}^{m}\sum_{\ell=1}^{m} p_{ik}^{(n_1)} p_{k\ell}^{(n_2)} p_{\ell j}^{(n_3)} \geq p_{ik}^{(n_1)} \cdot p_{kk}^{(n_2)} \cdot p_{kj}^{(n_3)}.$$

2) Assume that the Markov chain is irreducible and that there is an i, such that $p_{ii} > 0$.
Since $p_{ii}^{(n)} \geq (p_{ii})^n < 0$, we must have $p_{ii}^{(n)} > 0$ for all $n \in \mathbb{N}_0$.
Now, $\mathbf{P}$ is irreducible, so to every j there exists an n_1, such that

$$p_{ji}^{(n_1)} > 0, \qquad \text{[index “}i\text{” as above]},$$

and to every k there exists an n_2, such that also

$$p_{ik}^{(n_2)} > 0.$$

Then follow this procedure on all pairs of indices (j, k).
If we choose N_1 as the largest of the possible n_1, and N_2 as the largest of the possible n_2, then it follows from **1.** that

$$p_{jk}^{(N_1+N_2)} \geq p_{ji}^{(n_1)} p_{ik}^{(n_2)} p_{ii}^{(N_1-n_1+N_2-n-2)} > 0,$$

where $n_1 = n_1(j)$ and $n_2 = n_2(k)$ depend on j and k, respectively.
Hence all elements of $\mathbf{P}^{\mathbf{N_1}+\mathbf{N_2}}$ are > 0, so the stochastic matrix $\mathbf{P}$ is regular.

Example 3.2 *Let* $\mathbf{P}$ *be an irreducible stochastic matrix. We introduce for every* i *the number* d_i *by*

$$d_i = \text{ largest common divisor of all } n, \text{ for which } p_{ii}^{(n)} > 0.$$

1) Prove that d_i *does not depend on* i*. We denote the common value by* d*.*

2) Prove that if $\mathbf{P}$ *is regular, then* $d = 1$*.*

1) If we use Example 3.1 with $j = i$, we get

$$p_{ii}^{(n_1+n_2+n_2)} \geq p_{ik}^{(n_1)} p_{kk}^{(n_2)} p_{ki}^{(n_3)},$$

and analogously

$$p_{kk}^{(n_1+n_2+n_3)} \geq p_{ik}^{(n_1)} p_{ii}^{(n_2)} p_{ki}^{(n_3)}.$$

Using that $\mathbf{P}$ is irreducible, we can find n_1 and n_3, such that $p_{ik}^{(n_1)} > 0$ and $p_{ki}^{(n_3)} > 0$.
Let n_1 and n_3 be as small as possible. Then

$$p_{ii}^{(n_1+n_3)} \geq p_{ik}^{(n_1)} > 0 \quad \text{and} \quad p_{kk}^{(n_1+n_3)} \geq p_{ik}^{(n_1)} p_{ki}^{(n_3)} > 0.$$

Hence, $d_i | n_1 + n_3$ and $d_k | n_1 + n_3$, where "$a\,|b$" means that a is a divisor in b.
By choosing $n_2 = m\,d_k$, such that $p_{kk}^{(n_2)} > 0$, we also get

$$p_{ii}^{(n_1+n_2+n_3)} \geq p_{ik}^{(n_1)} p_{kk}^{(n_2)} p_{ki}^{(n_3)} > 0, \qquad \text{thus } d_i | n_1 + n_2 + n_3.$$

We conclude that $d_i | n_2 = m \cdot d_k$.
If $n_2 = n\,d_i$ is chosen, such that $p_{kk}^{(n_2)} > 0$, then analogously

$$d_k | n_1 + n_2 + n_3, \qquad \text{thus } d_k | n \cdot d_i.$$

It follows that d_1 is divisor in all numbers $n_2 = m \cdot d_k$, for which $p_{kk}^{(n_2)} A0$. Since d_k is the largest common divisor, we must have $d_i | d_k$.
Analogously, $d_k | d_i$, hence $d_k = d_i$.
Since i and k are chosen arbitrarily, we have proved **1.**.

2) If $\mathbf{P}$ is regular, there exists an $n \in \mathbb{N}$, such that all $p_{ij}^{(n)} > 0$. Then also $p_{ij}^{(n+m)} > 0$, $m \in \mathbb{N}_0$, and the largest common divisor is clearly 1, hence $d = 1$.
The proof that conversely $d = 1$ implies that $\mathbf{P}$ is regular is given in Example 3.3.

Example 3.3 . *Let* $\mathbf{P}$ *be a stochastic matrix of an irreducible Markov chain* $E_1, E_2, \dots, E_n$*, and assume that* $d = 1$ *(cf. Example 3.2).*
Prove that there exists an $N \in \mathbb{N}$*, such that we for all* $n \geq N$ *and all* i *and* j *have* $p_{ij}^{(n)} > 0$ *(which means that the Markov chain is regular).*
HINT: *One may in the proof use without separate proof the following result from Number Theory: Let* $a_1, a_2, \dots, a_k \in \mathbb{N}$ *have the largest common divisor 1. Then there exists an* $N \in \mathbb{N}$*, such that for all* $n \geq N$ *there are integers* $c_1(n), c_2(n), \dots, c_k(n) \in \mathbb{N}$*, such that*

$$n = \sum_{j=1}^{k} c_j(n)\, a_j.$$

Since $\mathbf{P}$ is irreducible, we can to every pair of indices (i, j) find $n_{ij} \in \mathbb{N}$, such that $p_{ij}^{(n_{ij})} > 0$.
Since $d = 1$, we have for every index "i" a finite sequence $a_{i1}, a_{i2}, \dots a_{in_i} \in \mathbb{N}$, such that the largest common divisor is 1 and $p_{ii}^{(a_{ij})} > 0$.
Then by the result from Number Theory mentioned in the hint there exists an N_i, such that one to every $n \geq N_i$ can find $c_{i1}(n), \dots, c_{in_i}(n) \in \mathbb{N}$, such that

$$n = \sum_{j=1}^{n_i} c_{ij}(n)\, a_{ij}.$$

Then let $N \geq \max\{n_{ij}\} + \max\{N_i\}$. If $n \geq N$, then

$$p_{ij}^{(n)} \geq p_{ij}^{(n_{ij})} p_{jj}^{(n-p_{ij})}.$$

Since $n - n_{ij} \geq N_i$, it follows that

$$n - n_{ij} = \sum_{k=1}^{n_j} \tilde{c}_{jk}(n)\, a_{jk},$$

and we conclude that

$$p_{ij}^{(n)} \geq p_{ij}^{(n_{ij})} p_{jj}^{(n-n_{ij})} \geq p_{ij}^{(n_{ij})} \prod_{k=1} n_j \left(p_{jj}^{(a_{jk})}\right)^{\tilde{c}_{jk}} > 0.$$

This is true for every pair of indices (i, j), thus we conclude that $\mathbf{P}$ is regular.

Example 3.4 *Let $\mathbf{P}$ be an $m \times m$ stochastic matrix.*

1) Prove that if $\mathbf{P}$ is regular, then $\mathbf{P}^2$ is also regular.

2) Assuming instead that $\mathbf{P}$ is irreducible, can one conclude that $\mathbf{P}^2$ is also irreducible?

1) If $\mathbf{P}$ is regular, then there is an $N \in \mathbb{N}$, such that $p_{ij}^{(n)} > 0$ for all $n \geq N$ and all i, j. In particular, $p_{ij}^{(2N)} > 0$ for all (i, j). Now, $p_{ij}^{(2N)}$ are the matrix elements of $\mathbf{P}^{2N} = \left(\mathbf{P}^2\right)^N$, thus $\mathbf{P}^2$ is also regular.

2) The answer is "no"! In fact, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is irreducible, while

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is not.

Example 3.5 *Let $\mathbf{P}$ be an $m \times m$ stochastic matrix. Assume that $\mathbf{P}$ is irreducible, and that there is an i, such that*

$$p_{ii}^{(3)} > 0 \qquad \textit{and} \qquad p_{ii}^{(5)} > 0.$$

Prove that $\mathbf{P}$ is regular.

The result follows immediately from Example 3.3, because the largest common divisor for 3 and 5 is $d_i = 1$.

Notice that

$$p_{ii}^{(8)} \geq p_{ii}^{(3)} \cdot p_{ii}^{(5)} > 0, \qquad p_{ii}^{(9)} = \left(p_{ii}^{(3)}\right)^3 > 0, \qquad p_{ii}^{(10)} = \left(p_{ii}^{(5)}\right)^2 > 0, \qquad p_{ii}^{(11)} = p_{ii}^{(5)} \left(p_{ii}^{(3)}\right)^2 > 0,$$

hence the succeeding $p_{ii}^{(n)} > 0$, $n \geq 12$, because one just multiply this sequence successively by $\mathbf{P}^3$.

Example 3.6 *Let* $\mathbf{P}$ *be an* $m \times m$ *irreducible matrix. Prove for every pair* (i, j) *(where* $1 \leq i,\, j \leq m$*) that there exists an* n*, depending on* i *and* j*, such that* $1 \leq n \leq m$ *and* $p_{ij}^{(n)} > 0$*.*

When $\mathbf{P}$ is irreducible, we can get from every state E_i to any other state. When we sketch the graph, we see that it must contain a cycle,

$$E_{i_1} \quad \to \quad E_{i_2} \quad \to \quad \cdots \quad \to \quad E_{i_m} \quad \to \quad E_{i_1},$$

where $(i_1, i_2, \dots, i_m)$ is a permutation of $(1, 2, \dots, m)$. It follows that we can get from every E_i to any other E_j in n steps, where $1 \leq n \leq m$. This means that the corresponding matrix $\mathbf{P}^n$ has $p_{ij}^{(n)} > 0$.

Example 3.7 *Let* $\mathbf{P}$ *be a stochastic matrix for an irreducible Markov chain of the states* $E_1, E_2, \dots, E_m$*. Given that* $\mathbf{P}$ *has the invariant probability vector*

$$\mathbf{g} = \left(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}\right).$$

Prove that $\mathbf{P}$ *is double stochastic.*

The condition $\mathbf{g}\,\mathbf{P} = \mathbf{g}$ is written

$$\frac{1}{m}\sum_{j=1}^{m} p_{ij} = \frac{1}{m}, \qquad \text{thus} \qquad \sum_{j=1}^{m} p_{ij} = 1.$$

This proves that the sum of every column is 1, thus the matrix is double stochastic.

Example 3.8 *Given a regular Markov chain of the states* $E_1, E_2, \dots, E_m$ *and with the stochastic matrix* $\mathbf{P}$*. Then*

$$\lim_{n\to\infty} p_{ij}^{(n)} = g_i,$$

where $\mathbf{g} = (g_1, g_2, \dots, g_m)$ *is the uniquely determined invariant probability vector of* $\mathbf{P}$*. Prove that there exist a positive constant* K *and a constant* $a \in]0,1[$*, such that*

$$\left|p_{ij}^{(n)} - g_j\right| \leq K\,a^n \qquad \text{for } i,\, j = 1, 2, \dots, m \text{ and } n \in \mathbb{N}.$$

If $\mathbf{P}$ is regular, then there is an n_0, such that $p_{ij}^{(n)} > 0$ for all $n \geq n_0$ and all $i, j = 1, 2, \dots, m$. Let ϱ_j be the column vector which has 1 in its row number j and 0 otherwise. Then

$$\mathbf{P}^n \varrho_j = \left(p_{1j}^{(n)}, \dots, p_{mj}^{(n)}\right)^T.$$

If

$$0 < \varepsilon := \min_{i,\,j} p_{ij}^{(n_0)} \qquad \left[\text{clearly } < \frac{1}{2}\right]$$

and

$$m_n^j = \min_i p_{ij}^{(n)} \quad \text{and} \quad M_n^j = \max_i p_{ij}^{(n)},$$

then

$$M_n^j - m_n^j \leq (1 - 2\,\varepsilon)^n = a^n \qquad \text{for all } j,$$

and

$$m_n^j \leq \left\{ \begin{array}{c} q_j \\ \\ p_{ij}^{(n)} \end{array} \right\} M_n^j \qquad \text{for } n \geq n_0.$$

Thus

$$\left| p_{ij}^{(n)} - g_j \right| \leq M_n^j - m_n^j \leq a^n \qquad \text{for all } i,\ j \text{ and all } n \geq n_0.$$

Now, $\left| p_{ij}^{(n)} - g_j \right| \leq 1$ for all $n \in \mathbb{N}$. We therefore get the general inequality if we put

$$K = (1 - 2\,\varepsilon)^{-n_0} = \left(\frac{1}{a} \right)^{n_0}.$$

Notice that since $0 < \varepsilon < \dfrac{1}{2}$, we have $a = 1 - 2\,\varepsilon \in]0,1[$.

Example 3.9 *Given a stochastic matrix by*

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

Prove that $\mathbf{P}$ *is irreducible, but not regular.*
Find the limit matrix $\mathbf{G}$.
Compute $\mathbf{P}^2$ *and find all invariant probability vectors for* $\mathbf{P}^2$.

We conclude from the matrix that

$$E_1 \leftrightarrow E_3 \leftrightarrow E_2,$$

thus $\mathbf{P}$ is irreducible. We conclude from

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{P}^3 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} = \mathbf{P},$$

that

$$\mathbf{P}^{2n+1} = \mathbf{P} \qquad \text{and} \qquad \mathbf{P}^{2n} = \mathbf{P}^2.$$

Since every $\mathbf{P}^n$ contains zeros, we conclude that $\mathbf{P}$ is not regular.

The solution of $\mathbf{g}\,\mathbf{P} = \mathbf{g}$, $g_1 + g_2 + g_3 = 1$, is the solution of

$$\frac{1}{3}\,g_3 = g_1, \qquad \frac{2}{3}\,g_3 = g_2, \qquad g_1 + g_2 = g_3, \qquad g_1 + g_2 + g_3 = 1,$$

thus

$$\mathbf{g} = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right),$$

and the limit matrix is

$$\mathbf{G} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \dfrac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

Alternatively,

$$\begin{aligned} \mathbf{G} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{P}^i = \lim_{n\to} \frac{1}{2n} \left\{ \sum_{i=1}^{n} \mathbf{P}^{2i-1} + \sum_{i=1}^{n} \mathbf{P}^n \right\} = \frac{1}{2}\,\mathbf{P} + \frac{1}{2}\,\mathbf{P}^2 \\ &= \frac{1}{2} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 1 \\ \frac{1}{3} & \frac{2}{3} & 1 \\ \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Clearly, $\mathbf{P}^2$ is not irreducible ($E_3 \leftrightarrow E_3$).

The probability vectors for $\mathbf{P}^2$ are the solutions of

$$\mathbf{g}\,\mathbf{P}^2 = \mathbf{g}, \qquad g_1 + g_2 + g_3 = 1, \quad g_i \geq 0,$$

thus

$$g_1 = \frac{1}{3}\,g_1 + \frac{1}{3}\,g_2, \qquad g_2 = \frac{2}{3}\,g_1 + \frac{2}{3}\,g_2, \qquad g_3 = g_3, \qquad g_1 + g_2 + g_3 = 1,$$

and hence

$$\mathbf{g} = (x, 2x, 1-3x), \qquad x \in \left[0, \frac{1}{3}\right].$$

Remark 3.1 We see that in $\mathbf{P}^2$ we have the closed set $\{E_1, E_2\}$, and $\{E_3\}$ is another closed set. $\Diamond$

Example 3.10 *A Markov chain of three states E_1, E_2 and E_3 has the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \dfrac{2}{3} & 0 \end{pmatrix}.$$

Prove that the state E_1 is absorbing, and prove that the only invariant probability vector for $\mathbf{P}$ is $\mathbf{g} = (1,0,0)$. Prove for any probability vector $\mathbf{p}$ that $\mathbf{p}\,\mathbf{P}^n \to \mathbf{g}$ for $n \to \infty$.

Obviously E_1 is an absorbing state.
We get the equations for the probability vectors from $\mathbf{g} = \mathbf{g}\,\mathbf{P}$, i.e.

$$g_1 = g_1 + \frac{1}{3}\,g_3, \qquad g_2 = \frac{1}{2}\,g_2 + \frac{2}{3}\,g_3, \qquad g_3 = \frac{1}{2}\,g_2.$$

It follows from the former equation that $g_3 = 0$, which by insertion into the latter one implies that $g_2 = 0$, so $\mathbf{g} = (1,0,0)$ is the only invariant probability vector.

Let $\mathbf{p}$ be any probability vector. We put

$$\mathbf{p}\,\mathbf{P}^n = \left(p_1^{(n)}, p_2^{(n)}, p_3^{(n)}\right).$$

Then

$$\left(p_1^{(n+1)}, p_2^{(n+1)}, p_3^{(n+1)}\right) = \left(p_1^{(n)}, p_2^{(n)}, p_3^{(n)}\right)\mathbf{P},$$

implies that

$$p_1^{(n+1)} = p_1^{(n)} + \frac{1}{3}\,p_3^{(n)}, \qquad p_2^{(n+1)} = \frac{1}{2}\,p_2^{(n)} + \frac{2}{3}\,p_3^{(n)}, \qquad p_3^{(n+1)} = \frac{1}{2}\,p_2^{(n)}.$$

In particular, $\left(p_1^{(n)}\right)$ is (weakly) increasing, and since the sequence is bounded ≤ 1, we get

$$\lim_{n\to\infty} p_1^{(n)} = p_1 \leq 1.$$

By taking the limit of the first coordinate it follows that $\left(p_3^{(n)}\right)$ is also convergent, and that

$$\lim_{n\to\infty} p_3^{(n)} = \lim_{n\to\infty} p_1^{(n+1)} - \lim_{n\to} p_1^{(n)} = p_1 - p_1 = 0.$$

Finally we get from the third coordinate that $\left(p_2^{(n)}\right)$ is convergent,

$$\lim_{n\to\infty} p_2^{(n)} = 2 \lim_{n\to\infty} p_3^{(n+1)} = 0.$$

Then $p_1 = 1$, and we have proved that

$$\mathbf{p}\,\mathbf{P}^n \to (1,0,0) = \mathbf{g} \qquad \text{for } n \to \infty.$$

Example 3.11 *1) Find for every $a \in [0,1]$ the invariant probability vector(s) (g_1, g_2, g_3) of the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & a & 1-a \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}.$$

2) Prove for every $a \in]0,1[$ that

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & a & 0 & 1-a \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

is a regular stochastic matrix, and find the invariant probability vector (g_1, g_2, g_3, g_4) of $\mathbf{Q}$.

1) We write the equation $\mathbf{g}\,\mathbf{P} = \mathbf{g}$ as the following system of equations,

$$\begin{cases} g_1 & = & & & & & \frac{1}{3}\,g_3, \\ g_2 & = & g_1 & + & a\,g_2 & & \\ g_3 & = & & & (1-a)g_2 & + & \frac{2}{3}\,g_3. \end{cases}$$

Thus $g_3 = 3\,g_1$ and $g_1 = 1-a$, so

$$1 + g_1 + g_2 + g_3 = g_2\,\{1 + 1 - a + 3 - 3a\} = g_2(5-4a).$$

If $a \in [0,1]$, then

$$g_2 = \frac{1}{5-4a} \in \left[\frac{1}{5}\,,\,1\right] \qquad \text{for } a \in [0,1],$$

and the probability vector is

$$\mathbf{g} = \left(\frac{1-a}{5-4a}\,,\,\frac{1}{5-4a}\,,\,\frac{3-3a}{5-4a}\right).$$

2) When $0 < a < 1$, it follows from

$$\mathbf{Q}^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & a & 0 & 1-a \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & a & 0 & 1-a \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{a}{2} & \frac{1}{4} & \frac{1}{2}(1-a) \\ \frac{1}{6} & \frac{2}{3}\,a & \frac{1}{6} & \frac{2}{3}(1-a) \\ \frac{a}{3} & \frac{3}{4}(1-a) & \frac{2}{3}\,a & \frac{1}{4}(1-a) \\ \frac{1}{4} & \frac{3}{16} & \frac{1}{2} & \frac{1}{6} \end{pmatrix},$$

that all elements of $\mathbf{Q}^2$ are > 0, hence $\mathbf{Q}$ is regular.
Then $\mathbf{g} = \mathbf{g}\,\mathbf{Q}$ implies that

$$\begin{array}{rcrcrcrcr} g_1 & = & \frac{1}{2}\,g_1 & + & \frac{1}{3}\,g_2 & & & & \\ g_2 & = & & & & & a\,g_3 & + & \frac{3}{4}\,g_4, \\ g_3 & = & \frac{1}{2}\,g_1 & + & \frac{2}{3}\,g_2 & & & & \\ g_4 & = & & & & & (1-a)g_3 & + & \frac{1}{4}\,g_4. \end{array}$$

We get from the first equation that $\frac{1}{2}\,g_1 = \frac{1}{3}\,g_2$, hence $g_2 = \frac{3}{2}\,g_1$.
By adding the first and the third equation we get $g_1 + g_3 = g_1 + g_2$, so $g_2 = g_3$.
Finally, we conclude from the fourth equation that $\frac{3}{4}\,g_4 = (1-a)g_3$, thus

$$g_4 = \frac{4}{3}(1-a)g_3 = \frac{4}{3}(1-a)\frac{3}{2}\,g_1 = 2(1-a)g_1,$$

and

$$1 = g_1 + g_2 + g_3 + g_4 = g_1\left\{1 + \frac{3}{2} + \frac{3}{2} + 2(1-a)\right\} = g_1\{4 + 2(1-a)\} = g_1(6-2a),$$

hence

$$g_1 = \frac{1}{6-2a},$$

and the invariant probability vector is

$$\mathbf{g} = \left(\frac{1}{6-2a}\, , \, \frac{3}{12-4a}\, , \, \frac{3}{12-4a}\, , \, \frac{1-a}{3-a} \right).$$

Example 3.12 *Given a Markov chain of four states E_1, E_2, E_3 and E_4 and with the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. *Find for $\mathbf{P}$ its invariant probability vector(s).*

2. *Prove for a randomly chosen initial distribution*

$$\mathbf{p}^{(0)} = (\alpha_0, \beta_0, \gamma_0, \delta_0)$$

for the distribution $\mathbf{p}^{(n)} = (\alpha_n, \beta_n, \gamma_n, \delta_n)$ that

$$\alpha_n + \beta_n + \gamma_n = \left(\frac{3}{4}\right)^n (\alpha_0 + \beta_0 + \gamma_0)\, .$$

3. *Let $\mathbf{p}^{(0)} = (1,0,0,0)$. Find $\mathbf{p}^{(1)}$, $\mathbf{p}^{(2)}$ and $\mathbf{p}^{(3)}$.*

Given a sequence of random variables $(Y_n)_{n=0}^{\infty}$ by the following: The possible values of Y_n are 1, 2, 3, 4, and the corresponding probabilities are α_n, β_n, γ_n and δ_n, resp. (as introduced above).

Prove for any initial distribution $\mathbf{p}^{(0)} = (\alpha_0, \beta_0, \gamma_0, \delta_0)$ that the sequence (Y_n) converges in probability towards a random variable Y. Find the distribution of Y.

1) The last coordinate of the matrix equation $\mathbf{g} = \mathbf{g}\,\mathbf{P}$ with $\mathbf{g} = (\alpha, \beta, \gamma, \delta)$ is given by

$$\delta = \frac{1}{4}(\alpha+\beta+\gamma) + \delta, \qquad \text{thus} \qquad \alpha+\beta+\gamma = 0.$$

Now α, β, $\gamma \ge 0$, so $\alpha = \beta = \gamma = 0$, and hence $\delta = 1$. The only invariant probability vector is $(0,0,0,1)$.

2) Consider again the last coordinate,

$$\delta_n = \frac{1}{4}\left(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}\right) + \delta_{n-1}.$$

We have in general $\delta = 1 - (\alpha+\beta+\gamma)$, so

$$1 - (\alpha + \beta_n + \gamma_n) = \frac{1}{4}\left(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}\right) + 1 - \left(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}\right),$$

thus

$$\alpha_n + \beta_n + \gamma_n = \frac{3}{4}\left(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}\right),$$

and hence by recursion,

$$\alpha_n + \beta_n + \gamma_n = \left(\frac{3}{4}\right)^n \left(\alpha_0 + \beta_0 + \gamma_0\right).$$

3) In general,

$$\begin{aligned}
\alpha_n &= \frac{1}{4}\,\alpha_{n-1}, \\
\beta_n &= \frac{1}{4}\,\alpha_{n-1} + \frac{1}{2}\,\beta_{n-1}, \\
\gamma_n &= \frac{1}{4}\,\alpha_{n-1} + \frac{1}{4}\,\beta_{n-1} + \frac{3}{4}\,\gamma_{n-1}, \\
\delta_n &= \frac{1}{4}\left(\alpha_{n-1} + \beta_{n-1} + \gamma_{n-1}\right) + \delta_{n-1}.
\end{aligned}$$

Put

$$\mathbf{p}^{(0)} = (\alpha_0, \beta_0, \gamma_0) = (1, 0, 0, 0),$$

then

$$\begin{aligned}
\mathbf{p}^{(1)} &= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \\
\mathbf{p}^{(2)} &= \left(\frac{1}{16}, \frac{1}{16} + \frac{1}{8}, \frac{1}{16} + \frac{1}{16} + \frac{3}{16}, \frac{3}{16} + \frac{1}{4}\right) = \left(\frac{1}{16}, \frac{3}{16}, \frac{5}{6}, \frac{7}{16}\right), \\
\mathbf{p}^{(3)} &= \left(\frac{1}{64}, \frac{1}{64} + \frac{3}{32}, \frac{1}{64} + \frac{3}{64} + \frac{15}{64}, \frac{9}{64} + \frac{7}{16}\right) = \left(\frac{1}{64}, \frac{7}{64}, \frac{19}{64}, \frac{37}{64}\right).
\end{aligned}$$

4) A qualified guess is that $Y_n \to Y$ in probability, where

$$P\{Y = 4\} = 1 \qquad \text{and} \qquad P\{Y = j\} = 0 \quad \text{for } j = 1,\, 2,\, 3.$$

We shall prove that

$$P\left\{|Y_n - Y| \geq \varepsilon\right\} \to 0 \quad \text{for } n \to \infty \quad \text{for every fixed } \varepsilon > 0.$$

If $\varepsilon \in]0,1[$, then

$$\begin{aligned}
P\left\{|Y_n - Y| \geq \varepsilon\right\} &= 1 - P\left\{|Y_n - Y| < \varepsilon\right\} = 1 - P\left\{Y_n - Y = 0\right\} \\
&= 1 - \delta_n = \alpha_n + \beta_n + \gamma_n = \left(\frac{3}{4}\right)^n \left(\alpha_0 + \beta_0 + \gamma_0\right) \\
&= \left(\frac{3}{4}\right)^n \left(1 - \delta_0\right) \to 0 \qquad \text{for } n \to \infty,
\end{aligned}$$

and the claim is proved.

Example 3.13 *Five balls, 2 white ones and 3 black ones, are distributed in two boxed A and B, such that A contains 2, and B contains 3 balls. At time n (where $n = 0, 1, 2, \ldots$) we choose at random from each of the two boxes one ball and let the two chosen balls change boxes. In this way we get a Markov chain with 3 states: E_0, E_1 and E_2, according to whether A contains 0, 1 or 2 black balls.*

1. *Find the corresponding stochastic matrix* $\mathbf{P}$.

2. *Prove that it is regular, and its invariant probability vector.*

We let in the following $\mathbf{p}^{(n)} = (\alpha_n, \beta_n, \gamma_n)$ denote the distribution immediately befor the interchange at time $t = n$.

3. *Given the initial distribution $\mathbf{p}^{(0)} = (1, 0, 0)$, find the probabilities of state,*

$$\mathbf{p}^{(3)} = (\alpha_3, \beta_3, \gamma_3) \qquad \textit{and} \qquad \mathbf{p}^{(4)} = (\alpha_4, \beta_4, \gamma_4)$$

and prove that

$$\sqrt{(\alpha_3 - \alpha_4)^2 + (\beta_3 - \beta_4)^2 + (\gamma_3 - \gamma_4)^2} < 0,07.$$

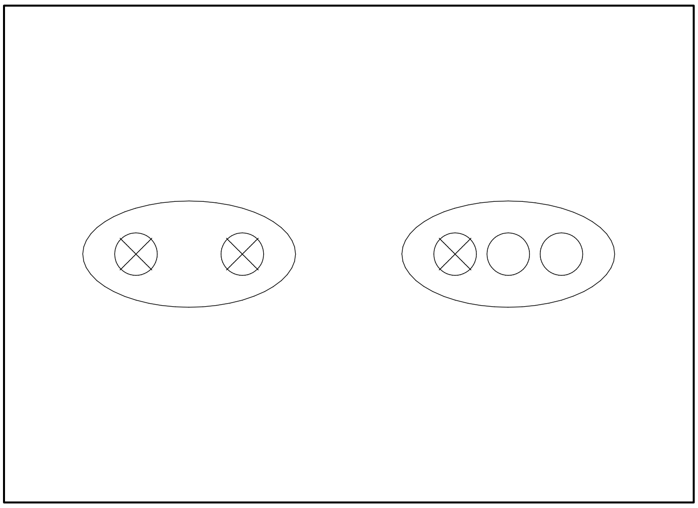


Figure 1: The two boxes with two black balls in A to the left and 1 black ball in B to the right.

1) Since $p_{ij} = P\{X(n) = j \mid X(n-1) = i\}$, the stochastic matrix is with i as the row number and j as the column number,

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} \cdot \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \cdot \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

2) All elements of

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \\ \frac{1}{9} & \frac{5}{9} & \frac{1}{3} \end{pmatrix}$$

are > 0, thus $\mathbf{P}$ is regular.

We imply from $\mathbf{g} = \mathbf{g}\,\mathbf{P}$, i.e.

$$g_1 = \frac{1}{6}\,g_2, \qquad g_2 = g_1 + \frac{1}{2}\,g_2 + \frac{2}{3}\,g_3, \qquad g_3 = \frac{1}{3}\,g_2 + \frac{1}{3}\,g_3,$$

that

(1) $g_2 = 6\,g_1, \qquad g_2 = 2\,g_1 + \frac{4}{3}\,g_3, \qquad 2\,g_3 = g_2,$

hence $g_3 = \frac{1}{2}\,g_2 = 3\,g_1$, and thus by insertion,

$$g_1 + g_2 + g_3 = g_1 + 6\,g_1 + 3\,g_1 = 10\,g_1 = 1.$$

The probability vector is

$$\mathbf{g} = \left(\frac{1}{10}, \frac{6}{10}, \frac{3}{10}\right).$$

3) From (1) follows

$$\alpha_n = \frac{1}{6}\,\beta_{n-1}, \qquad \beta_n = \alpha_{n-1} + \frac{1}{2}\,\beta_{n-1} + \frac{2}{3}\,\gamma_{n-1}, \qquad \gamma_n = \frac{1}{3}\,\beta_{n-1} + \frac{1}{3}\,\gamma_{n-1}.$$

Put $\mathbf{p}^{(0)} = (\alpha_0, \beta_0, \gamma_0) = (1,0,0)$. Then

$$\begin{aligned}
\mathbf{p}^{(1)} &= (\alpha_1, \beta_1, \gamma_1) = (0,1,0), \\
\mathbf{p}^{(2)} &= (\alpha_2, \beta_2, \gamma_2) = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right), \\
\mathbf{p}^{(3)} &= (\alpha_3, \beta_3, \gamma_3) = \left(\frac{1}{12}, \frac{23}{36}, \frac{5}{18}\right), \\
\mathbf{p}^{(4)} &= (\alpha_4, \beta_4, \gamma_4) = \left(\frac{23}{216}, \frac{127}{216}, \frac{11}{36}\right).
\end{aligned}$$

Then by insertion,

$$\begin{aligned}
&\sqrt{(\alpha_3-\alpha_4)^2 + (\beta_3-\beta_4)^2 + (\gamma_3-\gamma_4)} \\
&\quad = \sqrt{\left(\frac{1}{12} - \frac{23}{216}\right)^2 + \left(\frac{23}{36} - \frac{127}{216}\right)^2 + \left(\frac{5}{18} - \frac{11}{36}\right)^2} \\
&\quad = \sqrt{\left(\frac{18-23}{216}\right)^2 + \left(\frac{138-127}{216}\right)^2 + 6^2\left(\frac{10-11}{216}\right)^2} \\
&\quad = \frac{1}{216}\sqrt{5^2+11^2+6^2} = \frac{1}{216}\sqrt{25+121+36} \\
&\quad = \frac{\sqrt{182}}{216} < \frac{14}{216} = \frac{7}{108} < 0.07.
\end{aligned}$$

Example 3.14 *Consider a Markov chain of the states E_0, E_1, …, E_m and transition probabilities*

$$\begin{aligned}
p_{i,i+1} &= 1 - \frac{i}{m}, \qquad i = 0, 1, 2, \dots, m-1; \\
p_{i,i-1} &= \frac{i}{m}, \qquad i = 1, 2, \dots, m; \\
p_{ij} &= 0 \qquad \textit{otherwise.}
\end{aligned}$$

Prove that the Markov chain is irreducible, and find its invariant probability vector.
(This Markov chain is called Ehrenfest's model: *There are in total m balls in two boxes A and B; at time n we choose at random one ball and move it to the other box, where E_i denotes the state that there are i balls in the box A).*

The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{m} & 0 & 1-\frac{1}{m} & 0 & \cdots & 0 & 0 \\
0 & \frac{2}{m} & 0 & 1-\frac{2}{m} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{m} \\
0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.$$

We get from the second oblique diagonal the diagram

$$E_0 \;\to\; E_1 \;\to\; \cdots \;\to\; E_{m-1} \;\to\; E_m,$$

and similarly from the first oblique diagonal,

$$E_m \;\to\; E_{n-1} \;\to\; \cdots \;\to\; E_1 \;\to\; E_0.$$

Hence $\mathbf{P}$ is irreducible.

The first and last coordinate of $\mathbf{g} = \mathbf{g}\,\mathbf{P}$ give

$$g_0 = \frac{1}{m}\,g_1 \qquad \text{and} \qquad g_m = \frac{1}{m}\,g_{m-1}.$$

For the coordinates in between, i.e. for $i = 1, \dots, m-1$, we get

$$g_i = g_{i-1}p_{i-1,i} + g_{i+1}p_{i+1,i} = \left(1 - \frac{1-i}{m}\right) g_{i-1} + \frac{i+1}{m}\,g_{i+1}.$$

Hence, $g_1 = m\,g_0$. If $i = 1$, then

$$g_1 = g_0 + \frac{2}{m}\,g_2, \qquad \text{thus} \qquad g_2 = \frac{m(m-1)}{2}\,g_0 = \binom{m}{2}\,g_0.$$

A qualified guess is that

$$g_i = \binom{m}{i}\,g_0.$$

This is obviously true for $i = 0$ and $i = 1$. Then a check gives

$$\begin{aligned}
&\left(1 - \frac{i-1}{m}\right)\binom{m}{i-1} + \frac{i+1}{m}\binom{m}{i+1} \\
&\qquad = \frac{m+1-i}{m} \cdot \frac{m!}{(m+1-i)!} \cdot \frac{1}{(i-1)!} + \frac{i+1}{m} \cdot \frac{m!}{(i+1)!(m-1-i)!} \\
&\qquad = \frac{m!}{(m-i)!i!}\left\{\frac{i}{m} + \frac{m-i}{m}\right\} = \binom{m}{i},
\end{aligned}$$

and we have tested the claim. The rest then follows from that the solution is unique and from the fact that $g_i = \binom{m}{i} g_0$ solves the problem. Therefore,

$$g_i = \binom{m}{i}\,g_0, \qquad i = 0, 1, 2, \dots, m.$$

We conclude that

$$\sum_{i=0}^{m} g_i = g_0 \sum_{i=0}^{m} \binom{m}{i} 1^i 1^{m-i} = g_0 \cdot 2^m = 1,$$

thus $g_0 = 2^{-m}$, and

$$g_i = \frac{1}{2^m}\binom{m}{i}, \qquad i = 0, 1, 2, \dots, m,$$

corresponding to the probability vector

$$\mathbf{g} = \frac{1}{2^m}\left(1, \binom{m}{1}, \binom{m}{2}, \cdots, \binom{m}{m-1}, 1\right).$$

Example 3.15 (Continuation of Example 3.14).
Let $Y(n) = 2X(n) - m$, and let $e_n = E\{Y(n)\}$, $n \in \mathbb{N}_0$.
Find e_{n+1} expressed by e_n and m.
Find e_n, assuming that the process at time $t = 0$ is in state E_j.

If we put $p_i^{(n)} = P\{X(n) = i\}$, then (cf. Example 3.14),

$$p_0^{(n+1)} = p_1^{(n)} p_{1,0} = \frac{1}{m}\, p_1^{(n)} \qquad \text{and} \qquad p_m(n+1) = p_{m-1}^{(n)} p_{m-1,m} = \frac{1}{m}\, p_{m-1}^{(n)},$$

and

$$p_i^{(n+1)} = \left(1 - \frac{i-1}{m}\right) p_{i-1}^{(n)} + \frac{i+1}{m}\, p_{i+1}^{(n)}, \qquad i = 1, \dots, m-1.$$

Furthermore,

$$e_n = E\{Y(n)\} = 2E\{X(n)\} - m = 2\sum_{i=1}^{m} i\, p_i^{(n)} - m.$$

Hence

$$\begin{aligned}
e_{n+1} &= E\{Y(n+1)\} = 2E\{X(n+1)\} - m = 2\sum_{i=1}^{m} i\, p_i^{(n+1)} - m = 2\sum_{i=1}^{m-1} i\, p_i^{(n+1)} + 2m\, p_m^{(n+1)} - m \\
&= 2\sum_{i=1}^{m-1} i\left\{1 - \frac{i-1}{m}\right\} p_{i-1}^{(n)} + 2\sum_{i=1}^{m-1} i \cdot \frac{i+1}{m}\, p_{i+1}^{(n)} + 2m\, p_m^{(n+1)} - m \\
&= 2\sum_{i=0}^{m-2} (i+1)\left\{1 - \frac{i}{m}\right\} p_i^{(n)} + 2\sum_{i=2}^{m} (i-1) \cdot \frac{i}{m}\, p_i^{(n)} + 2m\, p_m^{(n+1)} - m \\
&= 2\sum_{i=0}^{m-2} (i+1) p_i^{(n=} - 2\sum_{i=0}^{m-2} \frac{i}{m}\,(i+1) p_i^{(n)} + 2\sum_{i=0}^{m} \frac{i}{m}\,(i-1)\, p_i^{(n)} + 2m\, p_m^{(n+1)} - m \\
&= 2\sum_{i=0}^{m} (i+1) p_i^{(n)} - 2m\, p_{m-1}^{(n)} - 2(m+1) p_m^{(n)} - 2\sum_{i=0}^{m} \frac{i}{m}\,(i+1)\, p_i^{(n)} + 2\,\frac{m-1}{m} \cdot m\, p_{m-1}^{(n)} \\
&\qquad + 2 \cdot \frac{m}{m}(m+1) P_m^{(n)} + 2\sum_{i=0}^{m} \frac{i}{m}\,(i-1) p_i^{(n)} + 2m\, p_m^{(n+1)} - m,
\end{aligned}$$

and thus

$$\begin{aligned}
e_{n+1} &= 2\sum_{i=0}^{m} i\,p_i^{(n)} - m + 2\sum_{i=0}^{m} p_i^{(n)} - 2\sum_{i=0}^{m} \frac{i}{m}\,\{i+1-i+1\}p_i^{(n)} \\
&\qquad +2(m-1)\,p_{m-1}^{(n)} - 2m\,p_{m-1}^{(n)} + 2(m+1)p_m^{(n)} - 2(m+1)p_m^{(n)} + 2m\,p_m^{(n+1)} \\
&= e_n + 2 - \frac{4}{m}\sum_{i=0}^{m} i\,p_i^{(n)} - 2p_{m-1}^{(n)} + 2m\cdot\frac{1}{m}\cdot p_{m-1}^{(n)} \\
&= e_n + 2 - \frac{2}{m}\left\{2\sum_{i=0}^{m} i\,p_i^{(n)} - m\right\} - \frac{2}{m}\cdot m = e_n + 2 - \frac{2}{m}\cdot e_n - 2 = \left(1-\frac{2}{m}\right)e_n.
\end{aligned}$$

Then by recursion,

$$e_n \left(1-\frac{2}{m}\right)^n e_0,$$

where

$$\begin{aligned}
e_0 &= 1\left\{(j-1)\cdot\frac{j}{m} + (j+1)\cdot\left(1-\frac{j}{m}\right)\right\} - m = 2\left\{\frac{j}{m}\,(j-1-j-1)+j+1\right\} - m \\
&= 2j + 2 - \frac{4j}{m} - m = 2j\left(1-\frac{2}{m}\right) - m\left(1-\frac{2}{m}\right) = (2j-m)\left(1-\frac{2}{m}\right),
\end{aligned}$$

hence

$$e_n = \left(1-\frac{2}{m}\right)^{n+1}(2j-m).$$

Example 3.16 *Consider a Markov chain of the states E_0, E_1, …, E_m and transition probabilities*

$$\begin{aligned} p_{i,i+1} &= p, && i = 2, 3, \dots, m-1; \\ p_{i,i-1} &= q, && i = 2, 3, \dots, m-1; \\ p_{1,2} &= 1, && p_{m,m-1} = 1; \\ p_{ij} &= 0 && \textit{otherwise}. \end{aligned}$$

Her er $p > 0$, $q > 0$, $p + q = 1$.

1) Prove that the Markov chain is irreducible, and find its invariant probability vector.

2) Is the given Markov chain regular?

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The first oblique diagonal gives

$$E_m \;\rightarrow\; E_{m-1} \;\rightarrow\; \cdots \;\rightarrow\; E_1 \;\rightarrow\; E_0,$$

and the last oblique diagonal gives the diagram

$$E_0 \;\rightarrow\; E_1 \;\rightarrow\; \cdots \;\rightarrow\; E_{m-1} \;\rightarrow\; E_m.$$

It is therefore possible to come from every state to any other by using these diagrams. Hence, the Markov chain is irreducible.

The equations of the invariant probability vector are

$$g_0 = q\,g_1, \qquad g_1 = g_0 + q\,g_2, \qquad g_{m-1} = p\,g_{m-2} + g_m, \qquad g_m = p\,g_{m-1},$$

and for the coordinates in between,

$$g_j = p\,g_{j-1} + q\,g_{j+1}, \qquad j = 2, 3, 4, \dots, m-2.$$

It follows from the first equation that $g_1 = \dfrac{1}{q}\,g_0$.
Similarly, from the second equation,

$$q\,g_2 = g_1 - g_0 = \frac{1}{q}\,g_0 - g_0 = \frac{1-q}{q}\,g_0 = \frac{p}{q}\,g_0,$$

thus

$$g_2 = \frac{p}{q^2}\,g_0.$$

A qualified guess is

(2) $g_j = \frac{1}{q}\left(\frac{p}{q}\right)^{j-1} g_0, \qquad \text{for } j \geq 1 \text{ and } j \leq m-1.$

It follows from the above that this formula is true for $j = 1$ and $j = 2$.
Then we check for $j = 2, 3, 4, \dots, m-2$, i.e.

$$p\, g_{j-1} + q\, g_{j+1} = \frac{p}{q} \cdot \left(\frac{p}{q}\right)^{j-2} g_0 + \frac{q}{q}\left(\frac{p}{q}\right)^{j} g_0 = \frac{1}{q}\left(\frac{p}{q}\right)^{j-1}\left\{q + q \cdot \frac{p}{q}\right\} = g_j.$$

If $j = m-1$, then

$$g_{m-1} = p\, g_{m-2} + g_m = p\, g_{m-2} + p\, g_{m-1}, \qquad \text{i.e. } g_{m-1} = \frac{p}{q}\, g_{m-2},$$

proving that (2) also holds for $j = m-1$. Finally,

$$g_m = p\, g_{m-1} = \frac{p}{q}\left(\frac{p}{q}\right)^{m-2} g_0 = \left(\frac{p}{q}\right)^{m-1} g_0.$$

Summing up we get

(3) $\mathbf{g} = g_0\left(1, \frac{1}{q}, \frac{p}{q^2}, \dots, \frac{1}{q}\left(\frac{p}{q}\right)^{m-2}, \left(\frac{p}{q}\right)^{m-1}\right).$

If $p \neq q$, i.e. $p \neq \frac{1}{2}$, then we get the condition

$$\begin{aligned}
1 &= \sum_{j=0}^{n} g_j = \left\{1 + \frac{1}{q}\left\{1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^{m-2}\right\} + \left(\frac{p}{q}\right)^{m-1}\right\} g_0 \\
&= \left\{1 + \frac{1}{q} \cdot \frac{1 - \left(\frac{p}{q}\right)^{m-1}}{1 - \frac{p}{q}} + \left(\frac{p}{q}\right)^{m-1}\right\} g_0 = \left\{1 + \frac{1 - \left(\frac{p}{q}\right)^{m-1}}{q-p} + \left(\frac{p}{q}\right)^{m-1}\right\} g_0 \\
&= \frac{1}{q-p}\left\{q - p + 1 + (q-p-1)\left(\frac{p}{q}\right)^{m-1}\right\} g_0 = \frac{1}{q-p}\left\{2q - 2p\left(\frac{p}{q}\right)^{m-1}\right\} g_0 \\
&= \frac{2q}{q-p}\left\{1 - \left(\frac{p}{q}\right)^{m}\right\} g_0,
\end{aligned}$$

hence

$$g_0 = \frac{q-p}{2q\left\{1 - \left(\frac{p}{q}\right)^m\right\}},$$

which is inserted into (3).
When $p = q = \frac{1}{2}$, formula (3) is reduced to

$$\mathbf{g} = g_0(1, 2, 2, \cdots, 2, 1), \qquad \text{where } 1 = g_0\{1 + 2(m-1) + 1\} = 2m\, g_0,$$

so $g_0 = \dfrac{1}{2m}$, and

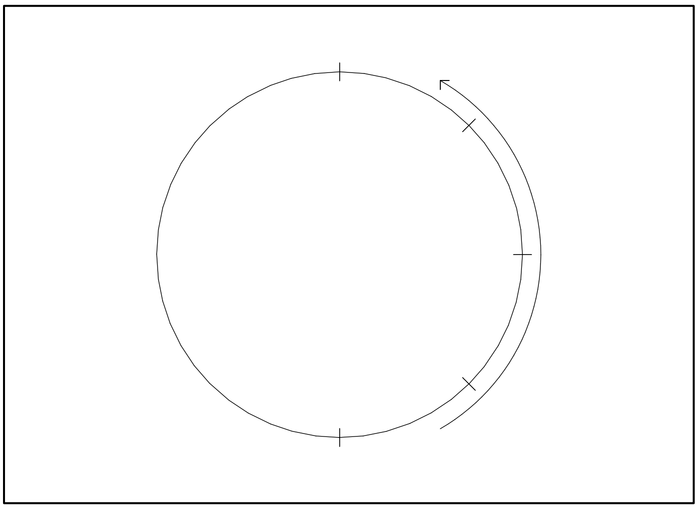
$$\mathbf{g} = \left(\frac{1}{2m}, \frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m}, \frac{1}{2m}\right).$$

2) The Markov chain is not regular. In fact, all $\mathbf{P}^n$ contain zeros. This is easily seen by an example. Let $m = 3$, and let $\star$ denote any number > 0. Then

$$\mathbf{P} = \begin{pmatrix} 0 & \star & 0 & 0 \\ \star & 0 & \star & 0 \\ 0 & \star & 0 & \star \\ 0 & 0 & \star & 0 \end{pmatrix}, \qquad \mathbf{P}^2 = \begin{pmatrix} \star & 0 & \star & 0 \\ 0 & \star & 0 & \star \\ \star & 0 & \star & 0 \\ 0 & \star & 0 & \star \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} 0 & \star & 0 & \star \\ \star & 0 & \star & 0 \\ 0 & \star & 0 & \star \\ \star & 0 & \star & 0 \end{pmatrix},$$

and we see that $\mathbf{P}^2$ has zeros at all places where $i + j$ is odd, and $\mathbf{P}^3$ has zeros at all places, where $i + i$ is even.
This pattern is repeated, so $\mathbf{P}^{2n}$ has the same structure as $\mathbf{P}^2$, and $\mathbf{P}^{2n+1}$ has the same structure as $\mathbf{P}^3$, concerning zeros.

Example 3.17 *A circle is divided into m arcs E_1, E_2, ..., E_m, where $m \geq 3$.*



A particle moves in the following way between the states E_1, E_2, ... , E_m:
There is every minute the probability $p \in]0,1[$ that it moves from a state to the neighbouring state in the positive sense of the plane, and the probability $q = 1 - p$ that it moves to the neighbouring state in the negative sense of the plane, i.e-

$$\begin{array}{rcll} p_{i,i+1} & = & p, & i = 1, 2, \dots, m-1; \\ p_{i,i-1} & = & q, & i = 2, 3, \dots, m; \\ p_{m,1} & = & p; & p_{1,m} = q; \\ p_{ij} & = & 0 & \textit{otherwise.} \end{array}$$

1) Find the stochastic matrix.

2) Prove that the Markov chain is irreducible.

3) Prove that the Markov chain is double stochastic, and find the invariant probability vector.

4) Prove that if the particle at time $t = 0$ is in state E_i, then there is a positive probability that it is in the same state to all of the times $t = 2, 4, 6, 8, \dots$.

5) Prove that if the particle at time $t = 0$ is in state E_i, then there is a positive probability that the particle is in the same state at time $t = m$.

6) Find the values of m, for which the Markov chain is regular.

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & p & 0 & 0 & \cdots & 0 & 0 & q \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ p & 0 & 0 & 0 & \cdots & 0 & q & 0 \end{pmatrix}.$$

2) It follows from

$$E_1 \quad \to \quad E_2 \quad \to \quad \cdots \quad \to \quad E_m \quad \to \quad E_1,$$

that $\mathbf{P}$ is irreducible.

3) The sum of each column is $p + q = 1$, hence the Markov chain is double stochastic. Since $\mathbf{P}$ is irreducible,

$$\left(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}\right)$$

is the only invariant probability vector.

4) This is obvious from the parity. (The same number of steps forward as backward, hence in total an even number).

5) This is also obvious, because the probability is $\geq p^m + q^m$, because p^m is the probability of m steps forward, and q^m is the probability of m steps backward.

6) It follows from **4.** and **5.** that if m is odd, then there is a positive probability to be in any given state, when $t > 3m$ is even, thus the Markov chain is regular, when m is odd.

If m is even, then the Markov chain is not regular, because the difference of the indices of the possible states must always be an even number. We shall therefore have zeros in every matrix $\mathbf{P}^n$.

Example 3.18 *Given a Markov chain of the states E_0, E_1, E_2, $\dots$, E_m (where $m \geq 2$) and the transition probabilities*

$$\begin{aligned} p_{0,j} &= a_i, & i &= 1, 2, \cdots, m, \\ p_{i,i-1} &= 1, & i &= 1, 2, \cdots, m, \\ p_{ij} &= 0, & & \textit{otherwise}, \end{aligned}$$

where

$$a_i \geq 0, \quad i = 1, 2, \dots, m-1, \qquad a_m > 0, \qquad \sum_{i=1}^{m} a_i = 1.$$

1) *Prove that the Markov chain is irreducible.*

2) *Assume that the process at time $t = 0$ is at state E_0; let T_1 denote the random variable, which indicates the time of the first return to E_0.*
   Find the distribution of T_1.

3) *Compute the mean of T_1.*

4) *Let T_1, T_2, $\dots$, T_k denote the times of the first, second, $\dots$, k-th return to E_0. Prove for every $\varepsilon > 0$ that*

$$P\left\{\left|\frac{T_k}{k} - \mu\right| > \varepsilon\right\} \to 0 \qquad \textit{for } k \to \infty.$$

HINT: *Apply that*

$$T_k = T_1 + (T_2 - T_1) + \cdots + (T_k - T_{k-1}).$$

1) The corresponding stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{m-1} & a_m \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

From this we immediately get the diagram (cf. the oblique diagonal in the matrix consisting of only ones)

$$E_m \;\to\; E_{m-1} \;\to\; \cdots \;\to\; E_1 \;\to\; E_0.$$

Now $a_m > 0$, hence also

$$E_0 \;\to\; E_m,$$

and we conclude that we can get from any state to any other state, so $\mathbf{P}$ is irreducible.

2) The probability is a_j for getting from E_0 to E_j. We shall use j steps in order to move from E_j back to E_0, so

$$P\{T_1 = j+1\} = a_j, \qquad j = 1, 2, \dots, m.$$

3) The mean is

$$\mu = E\{T_1\} = \sum_{j=1}^{m} (j+1) a_j = 1 + \sum_{j=1}^{m} j\, a_j.$$

4) Clearly,

$$T_k = T_1 + (T_2 - T_1) + \cdots + (T_k - T_{k-1}).$$

Furthermore, $X_j = T_j - T_{j-1}$ has the same distribution as T_1, and T_1 and the X_j are mutually independent.
Using $V\{T_1\} = \sigma^2 < \infty$, it follows from Chebyshev's inequality that

$$P\left\{\left|\frac{T_k}{k} - \mu\right| \geq \varepsilon\right\} \leq \frac{\sigma^2}{k\,\varepsilon^2} \to 0 \qquad \text{for } k \to \infty.$$

Example 3.19 *Given a Markov chain of the states E_0, E_1, ..., E_m (where $m \geq 2$) and transition probabilities*

$$\begin{array}{ll} p_{i,i+1} = p, & i = 0, 1, 2, \dots, m-1, \\ p_{i,0} = q, & i = 0, 1, 2, \dots, m-1, \\ p_{m,0} = 1, & \\ p_{i,j} = 0 & \textit{otherwise}, \end{array}$$

(where $p > 0$, $q > 0$, $p + q = 1$).
The above can be considered as a model of the following situation:
A person participates in a series of games. He has in each game the probability p of winning and probability q of losing; if he loses in a game, he starts from the beginning in the next game. He does the same after m won games in a row. The state E_i corresponds for $i = 1, 2, \dots, m$ to the situation that he has won the latest i games.

1) Find the stochastic matrix.

2) Prove that the Markov chain is irreducible.

3) Prove that the Markov chain is regular.

4) Find the stationary distribution.

5) Assume that the process at time $t = 0$ is in state E_0; let T_1 denote the random variable, which indicagtes the time of the first return to E_0. Find

$$P\{T_k = k+1\}, \qquad k = 0, 1, 2, \dots, m.$$

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ q & 0 & 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ q & 0 & 0 & 0 & \cdots & 0 & p \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

2) The Markov chain is irreducible, because we have the transitions

$$E_0 \;\to\; E_1 \;\to\; E_2 \;\to\; \cdots \;\to\; E_m \;\to\; E_0.$$

3) Since the Markov chain is irreducible, and $p_{00} = q > 0$, it is also regular.

4) The matrix equation $\mathbf{g}\,\mathbf{P} = \mathbf{g}$, where $\mathbf{g} = (g_0, g_1, \dots, g_m)$, is written

$$\begin{array}{rcl} g_0 & = & (g_0 + \cdots + g_{m-1})\,q + g_m, \\ g_1 & = & g_0 \cdot p, \\ g_2 & = & g_1 \cdot p, \\ \vdots & & \vdots \\ g_m & = & g_{m-1} \cdot p, \end{array}$$

hence

$$g_k = g_0 \cdot p^k, \qquad k = 0, 1, \dots, m.$$

It follows from

$$1 = \sum_{k=0}^{m} g_k = g_0 \sum_{k=0}^{m} p^k = g_0 \cdot \frac{1-p^{m+1}}{1-p} = g_0 \cdot \frac{1-p^{m+1}}{q},$$

that

$$g_k = q \cdot \frac{p^k}{1-p^{m+1}}, \qquad k = 0, 1, \dots, m.$$

5) If $k \le m-1$, then

$$\begin{aligned} &P\{T_1 = k+1\} \\ &\quad = P\{k \text{ games are won successively, and the } (k+1)\text{-th game is lost}\} \\ &\quad = p^k q. \end{aligned}$$

If $k = m$, then

$$P\{T_1 = m+1\} = P\{m \text{ games are won successively}\} = p^m.$$

Example 3.20 *Given a Markov chain of states E_1, E_2, E_3, E_4 and E_5 and of its stochastic matrix* $\mathbf{P}$ *given by*

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

1. *Prove that the Markov chain is irreducible, and find its invariant probability vector.*

A Markov chain of the states E_1, E_2, E_3, E_4 and E_5 has its stochastic matrix $\mathbf{Q}$ *given by*

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{5} & 0 & 0 & \frac{4}{5} \\ \frac{4}{5} & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \end{pmatrix}.$$

2. *Find all probability vectors which are invariant with respect to* $\mathbf{Q}$.

3. *Assume that an initial distribution is given by*

$$\mathbf{q}^{(0)} = (1,0,0,0,0).$$

Prove that $\lim_{n\to\infty} \mathbf{q}^{(n)}$ *exists and find the limit vector.*

1) The two oblique diagonals next to the main diagonal give

$$E_5 \quad \to \quad E_4 \quad \to \quad E_3 \quad \to \quad E_2 \quad \to \quad E_1$$

and

$$E_1 \quad \to \quad E_2 \quad \to \quad E_3 \quad \to \quad E_4 \quad \to \quad E_5,$$

so we can get from any state to any other one, hence $\mathbf{P}$ is irreducible.

The equations of the invariant probability vector are

$$\begin{aligned} p_1 &= \frac{1}{3}\,p_2, \\ p_2 &= p_1 + \frac{1}{3}\,p_3, \\ p_3 &= \frac{2}{3}\,p_2 + \frac{1}{3}\,p_4, \\ p_4 &= \frac{2}{3}\,p_3 + p_5, \\ p_5 &= \frac{2}{3}\,p_4, \end{aligned}$$

thus

$$\begin{aligned}
p_2 &= 3p_1, \\
p_3 &= 3p_2 - 3p_1 = 9p_1 - 3p_2 = 6p_1, \\
p_4 &= 2p_3 = 12p_1, \\
p_5 &= 8p_1.
\end{aligned}$$

We get

$$1 = p_1 + p_2 + p_3 + p_4 + p_5 = p_1\{1+3+6+12+8\} = 30p_1,$$

thus

$$p_1 = \frac{1}{30},$$

and hence

$$\mathbf{p} = \left(\frac{1}{30}, \frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{4}{15}\right).$$

2) The equations of the probability vectors are

$$\begin{aligned}
q_1 &= \frac{4}{5}\,q_2 + \frac{1}{5}\,q_5, \\
q_2 &= \frac{1}{5}\,q_1 + \frac{4}{5}\,q_5, \\
q_3 &= q_3, \\
q_4 &= q_4, \\
q_5 &= \frac{4}{5}\,q_1 + \frac{1}{5}\,q_2,
\end{aligned}$$

from which it is seen that q_3 and q_4 can be chosen arbitrarily, if only

$$q_3 \geq 0, \qquad q_4 \geq 0 \quad \text{and} \quad q_3 + q_4 \leq 1.$$

The remaining equations are

$$\left\{\begin{aligned} 5q_1 - 4q_2 &= q_5, \\ -q_1 + 5q_2 &= 4q_5. \end{aligned}\right. \qquad \text{thus} \qquad q_1 = q_2 = q_5.$$

The invariant probability vectors of $\mathbf{Q}$ are given by

$$\mathbf{q} = (x,x,y,z,x), \qquad x,\ y,\ z \geq 0 \text{ and } 3x+y+z=1.$$

3) We have for every n that $q_3^{(n)} = q_4^{(n)} = 0$, so it suffices to consider

$$\mathbf{Q}^{(1)} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{4}{5} \\ \frac{4}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} & 0 \end{pmatrix}.$$

Now $\mathbf{Q}^{(1)}\mathbf{Q}^{(1)}$ contains only positive elements, so $\mathbf{Q}^{(1)}$ is regular.
The equations of the corresponding simplified invariant probability vector are

$$\begin{aligned} q_1 &= \frac{4}{5}\,q_2 + \frac{1}{5}\,q_5, \\ q_2 &= \frac{1}{5}\,q_1 + \frac{4}{5}\,q_5, \\ q_5 &= \frac{4}{5}\,q_1 + \frac{1}{5}\,q_2. \end{aligned}$$

There are the same equations as in **2.**, and since $q_1 + q_2 + q_5 = 1$, we have $q_1 = q_2 = q_5 = \frac{1}{3}$. It follows that

$$\lim_{n\to\infty} \mathbf{q}^{(n)} = \left(\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right).$$

Example 3.21 *Given a Markov chain of the 5 states E_1, E_2, E_3, E_4 and E_5 and the transition probabilities*

$$p_{i,i+1} = \frac{i}{i+2}, \quad p_{i,1} = \frac{2}{i+2}, \qquad i = 1,\, 2,\, 3,\, 4;$$

$$p_{5,1} = 1; \qquad\qquad p_{i,j} = 0 \quad \textit{otherwise.}$$

1) Find the stochastic matrix.

2) Prove the Markov chain is irreducible.

3) Prove that the Markov chain is regular.

4) Find the stationary distribution (the invariant probability vector).

5) Assume that the process at time $t = 0$ is in state E_1. Denote by T the random variable, which indicates the time of the first return to E_1.
Find $P\{T = k\}$, $k = 1$, 2, 3, 4, 5, and compute the mean and variance of T.

1) The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2) It follows from the diagram

$$E_1 \;\to\; E_2 \;\to\; E_3 \;\to\; E_4 \;\to\; E_5 \;\to\; E_1,$$

that $\mathbf{P}$ is irreducible.

3) Since $p_{1,1} = \frac{2}{3} > 0$, and $\mathbf{P}$ is irreducible, we conclude that $\mathbf{P}$ is regular.

4) The equations of the invariant probability vector are

$$p_1 = \frac{2}{3}\,p_1 + \frac{1}{2}\,p_2 + \frac{2}{5}\,p_3 + \frac{1}{3}\,p_4 + p_5,$$

$$p_2 = \frac{1}{3}\,p_1, \qquad p_3 = \frac{1}{2}\,p_2, \qquad p_4 = \frac{3}{5}\,p_3, \qquad p_5 = \frac{2}{3}\,p_4.$$

We get successively,

$$\begin{aligned} p_4 &= \frac{3}{2}\,p_5, \\ p_3 &= \frac{5}{3}\,p_4 = \frac{5}{2}\,p_5, \\ p_2 &= 2p_3 = 5p_5, \\ p_1 &= 3p_2 = 15p_5, \end{aligned}$$

so

$$\begin{aligned} 1 &= p_1 + p_2 + p_3 + p_4 + p_5 \\ &= p_5\left(1 + \frac{3}{2} + \frac{5}{2} + 5 + 15\right) = 25p_5. \end{aligned}$$

Thus

$$p_5 = \frac{1}{25}, \quad p_4 = \frac{3}{50}, \quad p_3 = \frac{1}{10}, \quad p_2 = \frac{1}{5} \quad \text{and} \quad p_1 = \frac{3}{5},$$

and hence

$$\mathbf{p} = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{10}, \frac{3}{50}, \frac{1}{25} \right).$$

5) We immediately get,

$$P\{T = 1\} = \frac{2}{3} \qquad \text{and} \qquad P\{T > 1\} = \frac{1}{3}.$$

We are in the latter case in state E_2, so

$$P\{T = 2\} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \qquad \text{and} \qquad P\{T > 2\} = \frac{1}{6}.$$

In the latter case we are in state E_3, so

$$P\{T = 3\} = \frac{1}{6} \cdot \frac{2}{15} = \frac{1}{15} \qquad \text{and} \qquad P\{T > 3\} = \frac{1}{6} \cdot \frac{3}{5} = \frac{1}{10}.$$

In the latter case we are in state E_4, so

$$P\{T = 4\} = \frac{1}{3} \cdot \frac{1}{10} = \frac{1}{30} \qquad \text{and} \qquad P\{Y = 5\} = \frac{2}{3} \cdot \frac{1}{10} = \frac{1}{15}.$$

Summing up,

$$P\{T = 1\} = \frac{2}{3}, \qquad P\{T = 2\} = \frac{1}{6}, \qquad P\{T = 3\} = \frac{1}{15},$$

$$P\{T = 4\} = \frac{1}{30}, \qquad P\{T = 5\} = \frac{1}{15}.$$

The mean is

$$E\{T\} = \frac{2}{3} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{15} + 4 \cdot \frac{1}{30} + 5 \cdot \frac{1}{15} = 1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}.$$

Then we get

$$\begin{aligned} E\left\{T^2\right\} &= \frac{2}{3} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{15} + 16 \cdot \frac{1}{30} + 25 \cdot \frac{1}{15} \\ &= \frac{2}{3} + \frac{2}{3} + \frac{3}{5} + \frac{8}{15} + \frac{5}{3} = 3 + \frac{17}{15} = \frac{62}{15}, \end{aligned}$$

hence

$$V\{T\} = E\left\{T^2\right\} - (E\{T\})^2 = \frac{62}{15} - \frac{25}{9} = \frac{1}{45}(186 - 125) = \frac{61}{45}.$$

Example 3.22 *Consider a Markov chain of the m states $E_1, E_2, \dots, E_m$ (where $m \geq 3$) and the transition probabilities*

$$p_{i,i+1} = \frac{i}{i+i+2}, \qquad p_{i,1} = \frac{1}{i+2}, \qquad i = 1, 2, \dots, m-1,$$

$$p_{m,1} = 1, \qquad\qquad p_{i,j} = 0 \text{ otherwise.}$$

1) *Find the stochastic matrix.*

2) *Prove that the Markov chain is irreducible.*

3) *Prove that the Markov chain is regular.*

4) *Find the stationary distribution.*

5) *Assume that the process at time $t = 0$ is in state E_1. Let T denote the random variable, which indicates the time of the first return to E_1. find*

$$P\{T = k\}, \qquad k = 1, 2, \dots, m.$$

6) *Find the mean $E\{T\}$.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{4} & 0 & \frac{2}{4} & 0 & \cdots & 0 & 0 \\ \frac{2}{5} & 0 & 0 & \frac{3}{5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{2}{m+1} & 0 & 0 & 0 & \cdots & 0 & \frac{m-1}{m+1} \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

2) We get immediately the diagram

$$E_1 \quad\rightarrow\quad E_2 \quad\rightarrow\quad E_3 \quad\rightarrow\quad \cdots \quad\rightarrow\quad E_m.$$

Since also $E_m \rightarrow E_1$, we conclude that the Markov chain is irreducible.

3) We have proved that $\mathbf{P}$ is irreducible, and since $p_{1,1} = \frac{2}{3} > 0$, it follows that $\mathbf{P}$ is regular.

4) The equations of the stationary distribution are

$$\begin{aligned} g_1 &= \frac{2}{3}\,g_1 + \frac{2}{4}\,g_2 + \frac{2}{5}\,g_3 + \cdots + \frac{2}{m+1}\,g_{m-1} + g_m, \\ g_2 &= \frac{1}{3}\,g_1, \\ g_3 &= \frac{2}{4}\,g_2, \\ \vdots \quad & \quad \vdots \\ g_{m-1} &= \frac{m-2}{m}\,g_{m-2}, \\ g_m &= \frac{m-1}{m+1}\,g_{m-1}, \end{aligned}$$

thus

$$\begin{aligned}
g_1 &= \frac{3}{2}\, g_2, \\
g_2 &= \frac{4}{2}\, g_3, \\
\vdots \quad & \quad \vdots \\
g_{m-2} &= \frac{m}{m-2}\, g_{m-1}, \\
g_{m-1} &= \frac{m+1}{m-1}\, g_m.
\end{aligned}$$

Since

$$g_j = \frac{j+2}{j}\, g_{j+1},$$

it follows by recursion for $j \leq m-2$ that

$$g_j = \frac{j+2}{j}\, g_{j+1} = \frac{j+2}{j} \cdot \frac{j+3}{j+1} \cdot \frac{j+4}{j+2} \cdots \frac{m-1}{m-3} \cdot \frac{m}{m-2} \cdot \frac{m+1}{m-1}\, g_m = \frac{m(m+1)}{j(j+1)}\, g_m.$$

A check shows that this is also true for $j = m-1$ and $j = m$, so in general,

$$g_j = \frac{m(m+1)}{j(j+1)}\, g_m, \qquad j = 1, 2, \dots, m.$$

Then

$$1 = \sum_{j=1}^{m} g_j = m(m+1) g_m \sum_{j=1}^{m} \frac{1}{j(j+1)} = m(m+1) \left\{ 1 - \frac{1}{m+1} \right\} g_m = m^2 g_m,$$

i.e.

$$g_m = \frac{1}{m^2}, \qquad \text{and} \qquad g_j = \frac{m+1}{m} \cdot \frac{1}{j(j+1)}, \qquad j = 1, \dots, m.$$

5) Clearly,

$$P\{T = 1\} = \frac{2}{3},$$

so by inspecting the matrix we get

$$P\{T = 2\} = \frac{1}{3} \cdot \frac{2}{4} = \frac{1}{6},$$

and

$$P\{T = 3\} = \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} = \frac{1}{15}.$$

In order to compute $P\{T = k\}$ we must in step number $m-1$ be on the oblique diagonal, so

$$P\{T = k\} = \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdots \frac{k-1}{k+1} \cdot \frac{2}{k+2} = \frac{1 \cdot 2 \cdot 2}{k(k+1)(k+2)} = \frac{4}{k(k+1)(k+2)}.$$

A small check shows that this result is correct for $k = 1, 2, 3$.
Finally,

$$P\{T = n\} = \frac{1}{3} \cdot \frac{2}{4} \cdots \frac{m-1}{m+1} = \frac{2}{m(m+1)}.$$

6) The mean is

$$\begin{aligned} E\{T\} &= \sum_{k=1}^{m} k \cdot P\{T = k\} = \sum_{k=1}^{m-1} \frac{4}{(k+1)(k+2)} + \frac{2}{m+1} \\ &= 4 \sum_{k=1}^{m-1} \left\{ \frac{1}{k+1} - \frac{1}{k+2} \right\} + \frac{2}{m+1} = 4 \left\{ \frac{1}{2} - \frac{1}{m+1} \right\} + \frac{2}{m+1} \\ &= 2 \left\{ 1 - \frac{1}{m+1} \right\} = \frac{2m}{m+1}. \end{aligned}$$

Example 3.23 *Given a Markov chain of the m states $E_1, E_2, \dots, E_m$ (where $m \geq 3$) and the transition probabilities*

$$\begin{array}{lcll} p_{i,i} & = & 1-2p, & i = 1, 2, \dots, m, \\ p_{i,i-1} & = & p_{i,i+1} = p, & i = 2, \dots, m-1, \\ p_{1,2} & = & p_{m,m-1} = 2p, & \\ p_{i,j} & = & 0 & \text{otherwise}, \end{array} \qquad \text{where } p \in \left]0, \frac{1}{2}\right].$$

1) Find then stochastic matrix.

2) Prove that the Markov chain is irreducible.

3) Find the invariant probability vector.

4) Prove that the Markov chain is regular for $p \in \left]0, \frac{1}{2}\right[$, and not regular for $p = \frac{1}{2}$.

5) Compute $p_{1,1}^{(2)}$.

6) In the case of $m = 5$ and $p = \frac{1}{2}$ one shall compute

$$\lim_{n\to\infty} p_{1,1}^{(2n-1)} \qquad \text{and} \qquad \lim_{n\to\infty} p_{1,1}^{(2n)}.$$

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 1-2p & 2p & 0 & 0 & \cdots & 0 & 0 & 0 \\ p & 1-2p & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & p & 1-2p & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p & 1-2p & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2p & 1-2p \end{pmatrix}.$$

2) It follows from the diagram

$$E_1 \;\to\; E_2 \;\to\; \cdots \;\to\; E_m \;\to\; E_{m-1} \;\to\; \cdots \;\to\; E_1,$$

that $\mathbf{P}$ is irreducible.

3) The equations are

$$\begin{aligned} p_1 &= (1-2p)p_1 + p \cdot p_2, \\ p_2 &= 2p \cdot p_1 + (1-2p)p_2 + p \cdot p_3, \\ p_i &= p \cdot p_{i-1} + (1-2p)p_i + p \cdot p_{i+1}, \qquad i = 3, 4, \dots, m-2, \\ p_{m-1} &= p \cdot p_{m-2} + (1-2p)p_{m-1} + 2p \cdot p_m, \\ p_m &= p \cdot p_{m-1} + (1-2p)p_m. \end{aligned}$$

They are reduced to

$$\begin{aligned} p_2 &= 2p_1, \\ p_3 &= 2p_2 - 2p_1, \\ p_{i+1} &= 2p_i - p_{i-1}, \qquad i = 3,\,4,\,\dots\,,\, m-2, \\ p_{m-2} &= 2p_{m-1} - p_m, \\ p_{m-1} &= 2p_m. \end{aligned}$$

Hence

$$p_2 = 2p_1, \qquad p_3 = 2p_2 - 2p_1 = 4p_1 - 2p_1 = 2p_1,$$

and

$$p_{i+1} - p_i = p_i - p_{i-1} = \cdots = p_3 - p_2 = 2p_1 - 2p_1 = 0,$$

thus

$$p_{m-1} = p_{m-2} = \cdots = p_3 = p_2.$$

Finally, $p_{m-1} = 2p_m$. Summing up we get

$$p_m = p_1 \qquad \text{and} \qquad p_2 = \cdots = p_{m-1} = 2p_1,$$

whence

$$1 = \sum_{n=1}^{m} p_n = p_1\{1 + 2(m-2) + 1\} = 2(m-1)p_1,$$

i.e.

$$p_1 = \frac{1}{2(m-1)},$$

and the invariant probability vector is

$$\mathbf{p} = \left(\frac{1}{2(m-1)}, \frac{1}{m-1}, \frac{1}{m-1}, \cdots, \frac{1}{m-1}, \frac{1}{2(m-1)} \right).$$

4) If $p \neq \dfrac{1}{2}$, then all $p_{i,i} \neq 0$. Since $\mathbf{P}$ according to **2.** is irreducible, $\mathbf{P}$ is regular.

If $p = \dfrac{1}{2}$, then

$$\mathbf{p} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

It follows that if $p_{i,j}^{(2)} \neq 0$ then $i-j$ is an even integer, and if $p_{i,j}^{(3)} \neq 0$, then $i-j$ is an odd integer, etc.. It therefore follows that $\mathbf{P}^n$ will always contain zeros, and we conclude that $\mathbf{P}$ is not regular for $p = \frac{1}{2}$.

5) The element $p_{1,1}^{(2)}$ of $\mathbf{P}^2$ is

$$p_{1,1}^{(2)} = (1-2p)(1-2p) + 2p \cdot p = 1 - 4p + 6p^2.$$

6) If we put $m = 5$ and $p = \dfrac{1}{2}$, then we get the stochastic matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $i = j = 1$, clearly $i - j = 0$ is an even integer, so it follows immediately from **4.** that $p_{1,1}^{(2n-1)} = 0$ for all n, and hence

$$\lim_{n\to\infty} p_{1,1}^{(2n-1)} = 0.$$

In the computation of $p_{1,1}^{(2)}$ we consider instead $q_{1,1}^{(n)}$ i $\mathbf{Q}^n = \left(\mathbf{P}^2\right)^n$, where

$$\mathbf{Q} = \mathbf{P}^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0\frac{1}{4} & \\ 0 & \frac{1}{4} & 0\frac{3}{4} & 0 & \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

By **3.** the invariant probability vector is given by

$$\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right),$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{P} \to \mathbf{G}, \qquad \text{for } n \to \infty,$$

where each row in $\mathbf{G}$ is $\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$. From $p_{1,1}^{(2i-1)} = 0$ follows that

$$\frac{1}{2n}\sum_{i=1}^{2n} p_{1,1}^{(i)} = \frac{1}{2n}\sum_{j=1}^{n} p_{1,1}^{(2j)} \to g_1 = \frac{1}{8} \qquad \text{for } n \to \infty.$$

If $\lim_{n\to\infty} p_{1,1}^{(2n)}$ exists, then this will imply that

$$\frac{1}{2}\lim_{n\to\infty} p_{1,1}^{(2n)} = \frac{1}{8}, \quad \text{hence} \quad \lim_{n\to\infty} p_{1,1}^{(2n)} = \frac{1}{4}.$$

Now, *since*

$$q_{1,1}^{(n+1)} = p_{1,1}^{(2n+2)} = \frac{1}{2}\, p_{1,1}^{(2n)} + \frac{1}{4}\, p_{1,3}^{(2n)} < p_{1,1}^{(2n)} = q_{1,1}^{(n)},$$

the sequence $\left(p_{1,1}^{(n)}\right)$ is decreasing and bounded from below, hence convergent. We therefore conclude that

$$\lim_{n\to\infty} p_{1,1}^{(2n)} = \frac{1}{4}.$$

Example 3.24 *A Markov chain has its stochastic matrix* $\mathbf{Q}$ *given by*

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

1. *Prove that the Markov chain is regular, and find the invariant probability vector.*

Another Markov chain with 5 states has its stochastic matrix $\mathbf{P}$ *given by*

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & 0 & \frac{3}{8} & \frac{1}{8} \end{pmatrix}.$$

2. *Prove that this Markov chain is not irreducible.*

3. *Prove for any initial distribution*

$$\mathbf{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, p_4^{(0)}, p_5^{(0)}\right)$$

that

$$p_4^{(n)} + p_5^{(n)} \le \frac{1}{2}\left\{p_4^{(n-1)} + p_5^{(n-1)}\right\}, \qquad n \in \mathbb{N},$$

and then prove that

$$p_4^{(n)} + p_5^{(n)} \le \left(\frac{1}{2}\right)^n, \qquad n \in \mathbb{N}.$$

4. *Prove that the limit* $\lim_{n\to\infty} \mathbf{p}^{(n)}$ *exists and find the limit vector.*

1) It follows from

$$\mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \qquad \text{that} \qquad \mathbf{Q}^2 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{9} & \frac{11}{18} & \frac{1}{6} \\ \frac{2}{9} & \frac{1}{6} & \frac{11}{18} \end{pmatrix}.$$

All elements of $\mathbf{Q}^2$ are positive, thus $\mathbf{Q}$ is regular.

The invariant probability vector $\mathbf{g} = (g_1, g_2, g_3)$ satisfies

$$i) \quad g_1,\, g_2,\, g_3 \ge 0, \qquad ii) \quad g_1 + g_2 + g_3 = 1, \qquad iii) \quad \mathbf{g}\,\mathbf{Q} = \mathbf{g}.$$

The latter condition iii) is written

$$\left\{\begin{array}{rclcrcrl} g_1 & = & & & \frac{1}{3}\,g_2 & + & \frac{1}{3}\,g_3, & \qquad (1) \\ g_2 & = & \frac{1}{2}\,g_1 & & & + & \frac{2}{3}\,g_3, & \qquad (2) \\ g_3 & = & \frac{1}{2}\,g_1 & + & \frac{2}{3}\,g_2. & & & \qquad (3) \end{array}\right.$$

When we insert (1) into (2), we get

$$g_2 = \left(\frac{1}{6}\,g_2 + \frac{1}{6}\,g_3\right) + \frac{2}{3}\,g_3, \qquad \text{thus} \qquad g_2 = g_3.$$

Then it follows from (1) that $g_1 = \frac{2}{3}\,g_2$. Furthermore, ii) implies that

$$\frac{2}{3}\,g_2 + g_2 + g_2 = \frac{8}{3}\,g_2 = 1,$$

thus

$$g_2 = \frac{3}{8} = g_3 \qquad \text{and} \qquad g_1 = \frac{2}{3} \cdot \frac{3}{8} = \frac{2}{8},$$

so

$$\mathbf{g} = \left(\frac{2}{8}, \frac{3}{8}, \frac{3}{8}\right) = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right).$$

2) We notice that $\mathbf{Q}$ is included in $\mathbf{P}$ as the upper (3×3) sub-matrix

$$\mathbf{P} = \left(\begin{array}{ccccccc} 0 & \frac{1}{2} & \frac{1}{2} & | & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & | & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & | & 0 & 0 \\ - & - & - & - & & \\ \frac{1}{3} & \frac{1}{3} & 0 & & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & 0 & & \frac{3}{8} & \frac{1}{8} \end{array}\right),$$

hence $\{E_1, E_2, E_3\}$ is a proper closed subset. Then $\mathbf{P}$ is not irreducible.

3) It follows from $\mathbf{p}^{(n)} = \mathbf{p}^{(n-1)}\,\mathbf{P}$ that

$$\begin{aligned} p_4^{(n)} &= \frac{1}{6}\,p_4^{(n-1)} + \frac{3}{8}\,p_5^{(n-1)}, \\ p_5^{(n)} &= \frac{1}{6}\,p_4^{(n-1)} + \frac{1}{8}\,p_5^{(n-1)}, \end{aligned}$$

hence by addition,

$$p_4^{(n)} + p_5^{(n)} = \frac{1}{3}\,p_4^{(n-1)} + \frac{1}{2}\,p_5^{(n-1)} \le \frac{1}{2}\left\{p_4^{(n-1)} + p_5^{(n-1)}\right\}.$$

When this inequality is iterated, we get

$$p_4^{(n)} + p_5^{(n)} \le \frac{1}{2}\left\{p_4^{(n-1)} + p_5^{(n-1)}\right\} \le \cdots \le \left(\frac{1}{2}\right)^n \left\{p_4^{(0)} + p_5^{(0)}\right\} \le \left(\frac{1}{2}\right)^n .$$

4) It follows from **3.** that

$$p_4^{(n)} \to 0 \quad \text{and} \quad p_5^{(n)} \to 0 \qquad \text{for } n \to \infty,$$

so we end in the closed subset $\{E_1, E_2, E_3\}$. Inside this closed subset the behaviour is governed by the stochastic matrix $\mathbf{Q}$, the probability vector of which was found in **1.**. Hence it also follows for $\mathbf{P}$ that

$$\mathbf{p}^{(n)} \to \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, 0, 0\right) \qquad \text{for } n \to \infty.$$

Example 3.25 *Given a Markov chain of 5 states E_1, E_2, E_3, E_4 and E_5 and transition probabilities*

$$\begin{aligned} p_{11} &= p_{55} = 1, \\ p_{23} &= p_{34} = p_{45} = \frac{2}{3}, \\ p_{21} &= p_{32} = p_{43} = \frac{1}{3}, \\ p_{ij} &= 0 \qquad \textit{otherwise.} \end{aligned}$$

1) Find the stochastic matrix $\mathbf{P}$.

2) Find all invariant probability vectors of $\mathbf{P}$.

3) Compute the matrix $\mathbf{P}^2$.

4) Prove for any initial distribution

$$\mathbf{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, p_4^{(0)}, p_5^{(0)} \right)$$

that

$$p_2^{(n+2)} + p_3^{(n+2)} + p_4^{(n+2)} \le \frac{2}{3} \left\{ p_2^{(n)} + p_3^{(n)} + p_4^{(n)} \right\}, \qquad n \in \mathbb{N}_0,$$

and then prove that

$$\lim_{n\to\infty} p_2^{(n)} = \lim_{n\to\infty} p_4^{(n)} = \lim_{n\to\infty} p_4^{(n)} = 0.$$

5) Let the initial distribution be given by

$$\mathbf{q}^{(0)} = (0,1,0,0,0).$$

Find $\lim_{n\to\infty} \mathbf{q}^{(n)}$.

6) We assume that the process at time $t = 0$ is in state E_2. Let T denote the random variable, which indicates the time when the process for the first time gets to either the state E_1 or the state E_5. Find the mean $E\{T\}$.

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.2 We notice that the situation can be interpreted as a random walk on the set $\{1,2,3,4,5\}$ with two absorbing barriers. $\diamond$

2) The equations of the invariant probability vectors are

$$\begin{aligned}
g_1 &= g_1 + \frac{1}{3}\,g_2,\\
g_2 &= \frac{1}{3}\,g_3,\\
g_3 &= \frac{2}{3}\,g_2 + \frac{1}{3}\,g_4,\\
g_4 &= \frac{2}{3}\,g_3\\
g_5 &= \frac{2}{3}\,g_4 + g_5,
\end{aligned}$$

from which it is immediately seen that

$$g_2 = 0, \qquad g_3 = 0, \qquad g_4 = 0,$$

so the only constraint on $g_1 \geq 0$ and $g_5 \geq 0$ is that $g_1 + g_5 = 1$. Putting $g_1 = x \in [0,1]$, it follows that all invariant probability vectors are given by

$$\mathbf{g}_x = (x, 0, 0, 0, 1-x), \qquad x \in [0,1].$$

Remark 3.3 We note that we have a single infinity of invariant probability vectors. $\Diamond$

3) By a simple computation,

$$\mathbf{P}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{9} & 0 & \frac{4}{9} & 0 \\ \frac{1}{9} & 0 & \frac{4}{9} & 0 & \frac{4}{9} \\ 0 & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4) It follows from $\mathbf{p}^{(n+2)} = \mathbf{p}^{(n)}\,\mathbf{P}^2$ that

$$\mathbf{p}^{(n+2)} = \begin{pmatrix} p_1^{(n)} + \frac{1}{3}\,p_2^{(n)} + \frac{1}{9}\,p_3^{(n)} \\ \frac{2}{9}\,p_2^{(n)} + \frac{1}{9}\,p_4^{(n)} \\ \frac{4}{9}\,p_3^{(n)} \\ \frac{4}{9}\,p_2^{(n)} + \frac{2}{9}\,p_4^{(n)} \\ \frac{4}{9}\,p_3^{(n)} + \frac{2}{3}\,p_4^{(n)} + p_5^{(n)} \end{pmatrix},$$

hence

$$\begin{aligned}
p_2^{(n+2)} + p_3^{(n+2)} + p_4^{(n+2)} &= \left(\frac{2}{9}\,p_2^{(n)} + \frac{1}{9}\,p_4^{(n)}\right) + \frac{4}{9}\,p_3^{(n)} + \left(\frac{4}{9}\,p_2^{(n)} + \frac{2}{9}\,p_4^{(n)}\right)\\
&= \left(\frac{2}{9} + \frac{4}{9}\right) p_2^{(n)} + \frac{4}{9}\,p_3^{(n)} + \left(\frac{1}{9} + \frac{2}{9}\right) p_4^{(n)}\\
&= \frac{2}{3}\,p_2^{(n)} + \frac{4}{9}\,p_3^{(n)} + \frac{1}{3}\,p_4^{(n)} \leq \frac{2}{3}\left\{p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right\}.
\end{aligned}$$

Since $p_i^{(n)} \geq 0$, this implies that

$$0 \leq p_2^{(2n)} + p_3^{(2n)} + p_4^{(2n)} \leq \left(\frac{2}{3}\right)^n \left\{p_2^{(0)} + p_3^{(0)} + p_4^{(0)}\right\} \to 0 \qquad \text{for } n \to \infty,$$

and

$$0 \leq p_2^{(2n+1)} + p_3^{(2n+1)} + p_4^{(2n+1)} \leq \left(\frac{2}{3}\right)^n \left\{p_2^{(1)} + p_3^{(1)} + p_4^{(1)}\right\} \to 0 \qquad \text{for } n \to \infty,$$

thus

$$\lim_{n\to\infty} \left\{p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right\} = 0.$$

Now, $p_i^{(n)} \geq 0$, so we conclude that

$$\lim_{n\to\infty} p_2^{(n)} = \lim_{n\to\infty} p_3^{(n)} = \lim_{n\to\infty} p_4^{(n)} = 0.$$

5) Let us rename the states to $E_0^\star$, $E_1^\star$, $E_2^\star$, $E_3^\star$, $E_4^\star$. Let us first compute the probability of starting at $E_1^\star$ and then ending again in $E_1^\star$. The parameters are here

$$N = 4, \qquad k = 1, \qquad q = \frac{1}{3}, \qquad p = \frac{2}{3},$$

so this probability is given by some known formula,

$$\frac{\left(\frac{1}{2}\right)^1 - \left(\frac{1}{2}\right)^4}{1 - \left(\frac{1}{4}\right)^4} = \frac{7}{15}.$$

It follows from the first results of the example that the structure of the limit vector is $(x, 0, 0, 0, 1 - x)$. Hence

$$\mathbf{q}^{(n)} \to \left(\frac{7}{15}, 0, 0, 0, \frac{8}{15}\right).$$

6) Here it is again advantageous to use the theory of the ruin problem. We get

$$\mu = E\{T\} = \frac{1}{-\frac{1}{3}} - \frac{4}{-\frac{1}{3}} \cdot \frac{1 - \left(\frac{1}{2}\right)^1}{1 - \left(\frac{1}{2}\right)^4} = -3 + 12 \cdot \frac{\frac{1}{2}}{\frac{15}{16}} = -3 + \frac{32}{5} = \frac{17}{5}$$

so the mean is $\frac{17}{5}$.

Remark 3.4 The questions 5 and 6 can of course also be solved without using the theory of the ruin problem. $\Diamond$

Example 3.26 *Given a Markov chain of the states E_1, E_2, ..., E_m, where $m \geq 3$, and of the transition probabilities*

$$\begin{array}{ll} p_{i,i} = 1-2p, & i = 2, 3, \dots, m; \\ p_{i,i+1} = p, & i = 1, 2, \dots, m-1; \\ p_{i,i-1} = p, & i = 2, 3, \dots, m-1; \\ p_{1,1} = 1-p; & \\ p_{m,m-1} = 2p; & \\ p_{i,j} = 0 & \textit{otherwise}, \end{array}$$

(here p is a number in the interval $\left]0, \frac{1}{2}\right]$).

1. *Find the stochastic matrix.*

2. *Prove that the Markov chain is irreducible.*

3. *Prove that the Markov chain is regular.*

4. *Find the invariant probability vector of* $\mathbf{P}$.

If the process to time $t = 0$ is in state E_k, then the process will with probability 1 på reach either state E_1 or E_m at some time.
Let a_k denote the probability of getting to E_1 before E_m, when we start at E_k, for $k = 1, 2, \dots, m$. In particular, $a_1 = 1$ and $a_m = 0$.

5. *Prove that*

$$(4) \quad a_k = p\, a_{k+1} + (1-2p)a_k + p\, a_{k-1}, \qquad k = 2, 3, \dots, m-1.$$

6. *Apply (4) to find the probabilities a_k, $k = 2, 3, \dots, m-1$.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 1-p & p & 0 & 0 & \cdots & 0 & 0 & 0 \\ p & 1-2p & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & p & 1-2p & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-2p & p & 0 \\ 0 & 0 & 0 & 0 & \cdots & p & 1-2p & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2p & 1-2p \end{pmatrix}.$$

2) Since $p \neq 0$, it follows by the two oblique diagonals that we have the following transitions

$$E_1 \;\rightarrow\; E_2 \;\rightarrow\; \cdots \;\rightarrow\; E_{m-1} \;\rightarrow\; E_m \;\rightarrow\; E_{m-1} \;\cdots\; \rightarrow\; E_2 \;\rightarrow\; E_1,$$

proving that $\mathbf{P}$ is irreducible.

3) It follows from $\mathbf{P}$ being irreducible, and $p_{1,1} = 1-p > 0$ for $p \in \left]0, \frac{1}{2}\right]$, that $\mathbf{P}$ is regular.

4) The equations of the invariant probability vector are

$$\begin{aligned}
g_1 &= (1-p)g_1 + p\,g_2,\\
g_j &= p\,g_{j-1} + (1-2p)g_j + p\,g_{j+1}, \qquad \text{for } j = 2,\,3,\,\dots,\,m-2,\\
g_{m-1} &= p\,g_{m-2} + (1-2p)g_{m-1} + 2p\,g_m,\\
g_m &= p\,g_{m-1} + (1-2p)g_m,
\end{aligned}$$

thus

$$\begin{aligned}
g_2 &= g_1,\\
2g_j &= g_{j-1} + g_{j+1}, \qquad \text{for } j = 2,\,3,\,\dots,\,m-2,\\
2g_{m-1} &= g_{m-2} + 2g_m,\\
2g_m &= g_{m-1}.
\end{aligned}$$

We get by a backwards recursion that

$$2g_m g_{m-1} = g_{m-2} = \cdots = g_2 = g_1,$$

so

$$1 = \sum_{k=1}^{m} g_k = 2(m-1)g_m + g_m = (2m-1)g_m,$$

and the invariant probability vector is

$$\mathbf{g} = \left(\frac{2}{2m-1}, \frac{2}{2m-1}, \cdots, \frac{2}{2m-1}, \frac{1}{2m-1} \right).$$

5) If the process starts at state E_k, $k = 2, 3, \ldots, m-1$, we end after 1 step with probability p in state E_{k-1}, with probability $1-2p$ in E_k, and with probability p in E_{k+1}. If a_k is defined as above, this gives precisely (4), so

$$a_k = p\, a_{k+1} + (1-2p)a_k + p\, a_{k-1}, \qquad k = 2, 3, \ldots, m-1.$$

6) A reduction of (4) gives

$$2a_k = a_{k+1} + a_{k-1}, \qquad k = 2, 3, \ldots, m-1,$$

or more convenient

$$a_{k-1} - a_k = a_k - a_{k+1}, \qquad k = 2, 3, \ldots, m-1.$$

Hence

$$1 - a_2 = a_1 - a_2 = a_2 - a_3 = \cdots = a_{m-1} - a_m = a_{m-1} - 0 = a_{m-1},$$

so

$$1 = a_2 + a_{m-1}.$$

On the other hand,

$$a_2 = (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_{m-1} - a_m) = (m-2)a_{m-1},$$

hence by insertion

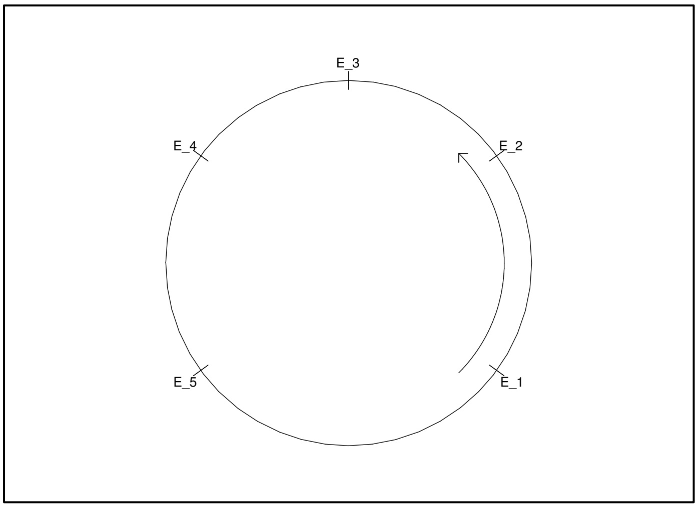
$$1 = a_2 + a_{m-1} = (m-1)a_{m-1}, \qquad \text{thus} \qquad a_{m-1} = \frac{1}{m-1}.$$

In general,

$$\begin{aligned} a_k &= (a_k - a_{k+1}) + (a_{k+1} - a_{k+2}) + \cdots + (a_{m-1} - a_m) \\ &= (m-k)a_{m-1} = \frac{m-k}{m-1} \qquad \text{for } k = 2, \ldots, m-1. \end{aligned}$$

A simple check shows that this is also true for $k = 1$ and $k = m$, so

$$a_k = \frac{m-k}{m-1}, \qquad k = 1, 2, \ldots, m.$$



Example 3.27 *A circle is divided into 5 arcs E_1, E_2, E_3, E_4 and E_5.*
A particle moves in the following way between the 5 states:
At every time unit there is the probability $p \in]0,1[$ of walking two steps forward in the positive direction, and the probability $q = 1 - p$ of walking one step backwards, so the transition probabilities are

$$\begin{aligned} p_{1,3} &= p_{2,4} = p_{3,5} = p_{4,1} = p_{5,2} = p, \\ p_{1,5} &= p_{2,1} = p_{3,2} = p_{4,3} = p_{5,4} = q, \\ p_{i,j} &= 0 \qquad \textit{otherwise.} \end{aligned}$$

1) Find the stochastic matrix $\mathbf{P}$ of the Markov chain.

2) Prove that the Markov chain is irreducible.

3) Find the invariant probability vector.

4) Assume that the particle at $t = 0$ is in state E_1.
Find the probability that the particle is in state E_1 for $t = 3$, and find the probability that the particle is in state E_1 for $t = 4$.

5) Check if the Markov chain is regular.

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & p & 0 & q \\ q & 0 & 0 & p & 0 \\ 0 & q & 0 & 0 & p \\ p & 0 & q & 0 & 0 \\ 0 & p & 0 & q & 0 \end{pmatrix}.$$

2) The oblique diagonal in the stochastic matrix below the main diagonal gives the transitions

$$E_5 \;\rightarrow\; E_4 \;\rightarrow\; E_3 \;\rightarrow\; E_2 \;\rightarrow\; E_1.$$

Since also $E_1 \;\rightarrow\; E_5$, we conclude that $\mathbf{P}$ is irreducible.

3) Since the matrix is double stochastic, the invariant probability vector is

$$\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right).$$

4) Let the particle start at E_1. Then we have the tree

$$\begin{array}{ccccccccc}
E_1 & \stackrel{p}{\to} & E_3 & \stackrel{p}{\to} & E_5 & \stackrel{p}{\to} & E_2 & \stackrel{p}{\to} & E_4 \\
q\downarrow & & q\downarrow & & q\downarrow & & q\downarrow & & \\
E_5 & \stackrel{p}{\to} & E_2 & \stackrel{p}{\to} & E_4 & \stackrel{p}{\to} & E_1 & & \\
q\downarrow & & q\downarrow & & q\downarrow & & & & \\
E_4 & \stackrel{p}{\to} & E_1 & \stackrel{p}{\to} & E_3 & & & & \\
q\downarrow & & q\downarrow & & & & & & \\
E_3 & \stackrel{p}{\to} & E_5 & & & & & & \\
q\downarrow & & & & & & & & \\
E_2 & & & & & & & &
\end{array}$$

We see that we can back to E_1 in three steps by

$$\begin{array}{ll}
E_1 \stackrel{p}{\to} E_3 \stackrel{q}{\to} E_2 \stackrel{q}{\to} E_1, & \text{probability } p\cdot q^2, \\
E_1 \stackrel{q}{\to} E_5 \stackrel{p}{\to} E_2 \stackrel{q}{\to} E_1, & \text{probability } p\cdot q^2, \\
E_1 \stackrel{q}{\to} E_5 \stackrel{q}{\to} E_4 \stackrel{p}{\to} E_1, & \text{probability } p\cdot q^2,
\end{array}$$

so

$$P\{T=3\}=3pq^2.$$

Analogously we reach E_1 in four steps along four paths, all of probability p^3q, so

$$P\{T=4\}=4p^3q.$$

5) It follows from the tree that we from E_1 with positive probability can reach any other state in 4 steps. It follows by the symmetry that this also holds for any other initial state E_k, so $\mathbf{P}^4$ has only positive elements, so we have that $\mathbf{P}$ is regular.

Example 3.28 *Given a Markov chain with 5 states E_0, E_1, E_2, E_3 and E_4 and the transition probabilities*

$$\begin{aligned} p_{0,2} &= p_{4,2}=1, \\ p_{1,2} &= p_{2,3}=p_{3,4}=\frac{1}{4}, \\ p_{1,0} &= p_{2,1}=p_{3,2}=\frac{3}{4}, \\ p_{i,j} &= 0 \qquad \text{otherwise}. \end{aligned}$$

This can be considered as a model of the following situation:
Two gamblers, Peter and Poul, participate in a series of games, in each of which Peter wins with the probability $\frac{1}{4}$ and loses with the probability $\frac{3}{4}$; if Peter wins, he receives 1 \$ from Paul, and if he loses, he gives 1 \$ to Paul.
The two gamblers have in total 4 \$, and the state E_i corresponds to that Peter has i \$ (and Paul has $4-i$ \$). To avoid that the game stops, because one of the gamblers has 0 \$, they agree that in that case the gambler with 0 \$ then receives 2 \$ from the gambler with 4 \$.

1) *Find the stochastic matrix.*

2) *Prove that the Markov chain is irreducible.*

3) *Find the invariant probability vector.*

4) *Compute $p_{22}^{(2)}$ and $p_{22}^{(3)}$.*

5) *Check if the Markov chain is regular.*

6) *At time $t=0$ the process is in state E_0. Find the probability that the process returns to E_0, before it arrives at E_4.*

1) The stochastic matrix is

$$\mathbf{P}=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

2) An analysis of $\mathbf{P}$ provides us with the diagram

$$\begin{array}{ccccccccc} E_0 & \leftarrow & E_1 & \leftrightarrow & E_2 & \leftrightarrow & E_3 & \rightarrow & E_4 \\ \downarrow & & & & \| & & & & \downarrow \\ E_2 & & = & & E_2 & & = & & E_2 \end{array}$$

It follows from this diagram that we can come from any state E_i to any other state E_j, so the Markov chain is irreducible.

3) The equations of the invariant probability vector are

$$\begin{aligned}
g_0 &= \frac{3}{4}\,g_1, \\
g_1 &= \frac{3}{4}\,g_2, \\
g_2 &= g_0 + \frac{1}{4}\,g_1 + \frac{3}{4}\,g_3 + g_4, \\
g_3 &= \frac{1}{4}\,g_2, \\
g_4 &= \frac{1}{4}\,g_3,
\end{aligned}$$

thus

$$g_3 = 4g_4, \qquad g_2 = 16g_4, \qquad g_1 = 12g_4, \qquad g_0 = 0g_4,$$

and

$$1 = g_0 + g_1 + g_2 + g_3 + g_4 = (9 + 12 + 16 + 4 + 1)g_4 = 42g_4,$$

hence

$$\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = \left(\frac{3}{14}, \frac{2}{7}, \frac{8}{21}, \frac{2}{21}, \frac{1}{42}\right).$$

4) It follows from

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0\frac{1}{4} & 0 & \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0\frac{1}{4} & 0 & \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{3}{16} & \frac{3}{4} & \frac{1}{16} & 0 \\ \frac{9}{16} & 0 & \frac{3}{8} & 0 & \frac{1}{16} \\ 0 & \frac{9}{16} & \frac{1}{4} & \frac{3}{16} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix},$$

that $p_{22}^{(2)} = \frac{3}{8}$. (Notice that the indices start at 0). Hence

$$p_{22}^{(3)} = \left(0, \frac{3}{4}, 0, \frac{1}{4}, 0\right) \cdot \left(0, \frac{3}{4}, \frac{3}{8}, \frac{1}{4}, 0\right) = \frac{9}{16} + \frac{1}{16} = \frac{5}{8}.$$

5) From $\mathbf{P}^2$ we get

$$E_0 \;\rightarrow\; E_1 \;\rightarrow\; E_2 \qquad \text{and} \qquad E_2 \;\rightarrow\; E_0,$$

thus

$$E_0 \;\leftrightarrow\; E_1 \;\leftrightarrow\; E_2.$$

In this case we get the diagram

$$\begin{array}{ccccc}
 & & E_0 & & \\
 & \swarrow & \updownarrow & & \\
E_3 & \rightarrow & E_1 & \leftarrow & E_4, \\
 & & \updownarrow & \nearrow & \\
 & & E_2 & &
\end{array}$$

and $\mathbf{P}^2$ is seen to be irreducible.
Since $p_{22}^{(2)} > 0$, it follows that $\mathbf{P}^2$, and hence also $\mathbf{P}$, is regular.

6) When $t = 0$ we are in state E_0.
When $t = 1$ we are with probability 1 in state E_2
If $t \geq 1$ we denote by a_k the probability that we can get from E_k to E_0 before E_4. Since $t > 0$, it follows that $a_0 = 1$ and $a_4 = 0$, and

$$\begin{aligned}
a_2 &= \frac{3}{4}\,a_1 + \frac{1}{4}\,a_3, \\
a_1 &= \frac{3}{4}\,a_0 + \frac{1}{4}\,a_2 = \frac{3}{4} + \frac{1}{4}\,a_2, \\
a_3 &= \frac{3}{4}\,a_2 + \frac{1}{4}\,a_4 = \frac{3}{4}\,a_2.
\end{aligned}$$

When the latter two equations are inserted into the first one, we get

$$a_2 = \frac{3}{4}\left(\frac{3}{4}+\frac{1}{4}\,a_2\right)+\frac{1}{4}\cdot\frac{3}{4}\,a_2 = \frac{9}{16}+\frac{3}{16}\,a_2+\frac{3}{16}\,a_2,$$

thus

$$16a_2 = 9+6a_2 \qquad \text{or} \qquad a_2 = \frac{9}{10}.$$

The wanted probability is $a_2 = \frac{9}{10}$, because we might as well start from E_2 as from E_0.

Example 3.29 *Given a Markov chain of 5 states E_1, E_2, E_3, E_4 and E_5 and the transition probabilities*

$$\begin{aligned} p_{1,2} &= p_{1,3} = p_{1,4} = p_{1,5} = \frac{1}{4}, \\ p_{2,3} &= p_{2,4} = p_{2,5} = \frac{1}{3}, \\ p_{3,4} &= p_{3,5} = \frac{1}{2}, \qquad p_{4,5} = 1, \\ p_{5,1} &= 1, \qquad p_{i,j} = 0 \quad \textit{otherwise.} \end{aligned}$$

1) *Find the stochastic matrix.*

2) *Prove that the Markov chain is irreducible.*

3) *Find the invariant probability vector.*

4) *Check if the Markov chain is regular.*

5) *To time $t = 0$ the process is in state E_1. Denote by T_1 the random variable, which indicates the time of the first return to E_1.*
Compute

$$P\{T_1 = k\}, \qquad k = 2,\,3,\,4,\,5,$$

and find the mean of T_1.

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2) It follows from the diagram

$$E_5 \;\rightarrow\; E_1 \;\rightarrow\; E_2 \;\rightarrow\; E_3 \;\rightarrow\; E_4 \;\rightarrow\; E_5,$$

that the Markov chain is irreducible.

3) The equations of the invariant probability vector are

$$\begin{aligned} g_1 &= g_5, \\ g_2 &= \frac{1}{4}\,g_1, \\ g_3 &= \frac{1}{4}\,g_1 + \frac{1}{3}\,g_2 = \frac{1}{3}\,g_1, \\ g_4 &= \frac{1}{4}\,g_1 + \frac{1}{3}\,g_2 + \frac{1}{2}\,g_3 = \frac{1}{6}\,g_1, \end{aligned}$$

where

$$1 = g_1 + g_2 + g_3 + g_4 + g_5 = g_1\left(1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right) = \frac{37}{12}\,g_1,$$

thus $g_1 = \dfrac{12}{37}$ and

$$\mathbf{g} = \frac{1}{37}\,(12, 3, 4, 6, 12).$$

4) It follows from the computation

$$\mathbf{P}^2 = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{12} & \frac{5}{24} & \frac{11}{24} \\ \frac{1}{3} & 0 & 0 & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

that we have in $\mathbf{P}^2$,

$$\begin{array}{ccccc} E_2 & & & & E_3 \\ & \searrow & & \nearrow & \\ E_3 & \rightarrow & E_1 & \rightarrow & E_4 \\ & \nearrow & & \searrow & \\ E_4 & & & & E_5, \end{array}$$

i.e.

$$E_1 \;\leftrightarrow\; E_3 \;\leftrightarrow\; E_4 \qquad \text{and} \qquad E_2 \;\rightarrow\; E_1 \;\rightarrow\; E_5 \;\rightarrow\; E_2,$$

and we can get from any state E_i to any other state E_j, so $\mathbf{P}^2$ is irreducible.
Now, $p_{1,1}^{(2)} = \dfrac{1}{4} > 0$, so $\mathbf{P}^2$ is regular, which implies that also $\mathbf{P}$ is regular, because there is an n, such that $\mathbf{P}^{2n}$ has only positive elements.

5) In this case we have the tree

$$
\begin{array}{ccccccccccc}
 & & & & E_5 & \stackrel{1}{\to} & \mathbf{E_1} & & & & \\
 & & & \frac{1}{3}\nearrow & & & & & & & \\
 & & E_2 & \stackrel{\frac{1}{3}}{\to} & E_3 & \stackrel{\frac{1}{2}}{\to} & E_4 & \stackrel{1}{\to} & E_5 & \stackrel{1}{\to} & \mathbf{E_1} \\
 & \frac{1}{4}\nearrow & & \frac{1}{3}\searrow & & \frac{1}{2}\searrow & & & & & \\
E_1 & \stackrel{\frac{1}{4}}{\to} & E_3 & \stackrel{\frac{1}{2}}{\to} & E_4 & \stackrel{1}{\to} & E_5 & \stackrel{1}{\to} & \mathbf{E_1} & & \\
 & \stackrel{\frac{1}{4}}{\searrow} & & \frac{1}{2}\searrow & & & & & & & \\
 & & E_4 & \stackrel{1}{\to} & E_5 & \stackrel{1}{\to} & \mathbf{E_1} & & & & \\
 & \frac{1}{4} & & & & & & & & & \\
 & & E_5 & \stackrel{1}{\to} & \mathbf{E_1}. & & & & & &
\end{array}
$$

We conclude from an analysis of this tree that $P\{T_1 = 1\} = 0$, and

$$
\begin{aligned}
P\{T_1 = 2\} &= \frac{1}{4} \cdot 1 = \frac{1}{4}, \\
P\{T_1 = 3\} &= \frac{1}{4}\left(\frac{1}{3} + \frac{1}{2} + 1\right) \cdot 1 = \frac{11}{24}, \\
P\{T_1 = 4\} &= \frac{1}{4}\left(\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 + \frac{1}{2} \cdot 1\right) \cdot 1 = \frac{1}{4}\left(\frac{1}{6} + \frac{1}{3} + \frac{1}{2}\right) = \frac{1}{4}, \\
P\{T_1 = 5\} &= \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{24}.
\end{aligned}
$$

The mean is

$$E\{T_1\} = 2 \cdot \frac{1}{4} + 3 \cdot \frac{11}{24} + 4 \cdot \frac{1}{4} + 5 \cdot \frac{1}{24} = \frac{1}{24}(12+33+24+5) = \frac{74}{24} = \frac{37}{12}.$$

Example 3.30 *Given a Markov chain of 4 states and the transition probabilities*

$$\begin{aligned} p_{11} &= 1-a, \qquad p_{12} = a, \\ p_{23} &= p_{34} = p_{21} = p_{32} = \frac{1}{2}, \\ p_{43} &= 1, \qquad p_{ij} = 0 \quad \textit{otherwise.} \end{aligned}$$

(Here a is a constant in the interval $[0,1]$).

1. *Find the stochastic matrix* $\mathbf{P}$*.*

2. *Prove that the Markov chain is irreducible for $a \in]0,1]$, though not irreducible for $a = 0$.*

Find for $a \in [0,1]$ the invariant probability vector.

Find all values of a for which the Markov chain is regular.

Assume in the following that $a = 0$.

5. *Prove that $p_{i1}^{(3)} \geq \frac{1}{4}$, $i = 1, 2, 3, 4$.*

6. *Let*

$$\mathbf{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, p_4^{(0)}\right)$$

be any initial distribution.
Prove that

$$p_2^{(n+3)} + p_3^{(n+3)} + p_4^{(n+3)} \leq \frac{3}{4}\left(p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right), \qquad \textit{for alle } n \in \mathbb{N},$$

and then find

$$\lim_{n\to\infty} \mathbf{p}^{(0)}\,\mathbf{P}^n.$$

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 1-a & a & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0\frac{1}{2} & \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

2) If $a > 0$, then

$$E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1,$$

so the Markov chain is irreducible.
If $a = 0$, then E_1 is an absorbing state, so the Markov chain is not irreducible for $a = 0$.

3) The equations of the invariant probability vector are

$$\begin{array}{ll} g_1 = (1-a)g_1 + \frac{1}{2}\,g_2, & \text{i.e. } g_2 = 2ag_1 \\ g_2 = a\,g_1 + \frac{1}{2}\,g_3, & \text{i.e. } g_3 = 2ag_1, \\ g_3 = \frac{1}{2}\,g_2 + g_4, & \\ g_4 = \frac{1}{2}\,g_3, & \text{i.e. } g_4 = 4ag_1. \end{array}$$

It follows from

$$1 = g_1 + g_2 + g_3 + g_4 = g(1 + 2a + 2a + 4a) = (1 + 8a)g_1,$$

that $g_1 = \dfrac{1}{1+8a}$, and

$$\mathbf{g} = \frac{1}{1+8a}\,(1, 2a, 2a, 4a).$$

4) Since the Markov chain is irreducible, we must at least require that $a \in]0,1]$.
If $a < 1$, then $p_{11} = 1 - a > 0$, so the Markov chain is regular for $a \in]0,1[$.
If $a = 1$, then

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and it is obvious that $\mathbf{P}^n$ contains zeros for every $n \in \mathbb{N}$. Hence the Markov chain is not regular for $a = 1$.
Summing up, the Markov chain is regular, if and only if $a \in]0,1[$.

5) If $a = 0$, then

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{P}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0\frac{1}{4} & \end{pmatrix}, \qquad \mathbf{P}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{8} & 0 & \frac{3}{8} & 0 \\ \frac{1}{4} & \frac{3}{8} & 0 & \frac{3}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{pmatrix}.$$

(It is not necessary to compute the full matrix $\mathbf{P}^3$; however, the alternative proof is just as long as the above).
It follows that $p_{i1}^{(3)} \geq \frac{1}{4}$ for $i = 1, 2, 3, 4$.

6) By **5.**,

$$p_2^{(n+3)} + p_3^{(n+3)} + p_4^{(n+3)} = 1 - p_1^{(n+3)} \leq \left(1 - \frac{1}{4}\right)\left(p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right).$$

Hence by recursion,

$$0 \leq p_2^{(n+3p)} + p_3^{(n+3p)} + p_4^{(n+3p)} \leq \left(\frac{3}{4}\right)^p \left(p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right) \to 0 \qquad \text{for } p \to \infty,$$

so

$$0 \leq \left\{ \begin{array}{l} \lim_{n\to\infty} p_2^{(n)} \\ \lim_{n\to\infty} p_3^{(n)} \\ \lim_{n\to\infty} p_4^{(n)} \end{array} \right\} \leq \lim_{n\to\infty} \left(p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right) = 0,$$

and

$$\lim_{n\to\infty} p_1^{(n)} = 1 - \lim_{n\to\infty}\left(p_2^{(n)} + p_3^{(n)} + p_4^{(n)}\right) = 1 - 0 = 1.$$

Therefore,

$$\lim_{n\to\infty} \mathbf{p}^{(0)}\,\mathbf{P}^n = (1,0,0,0).$$

Example 3.31 *Given a Markov chain of 5 states E_0, E_1, E_2, E_3 and E_4, and transition probabilities*

$$\begin{aligned} p_{0,1} &= p_{0,1} = p_{0,3} = p_{0,4} = \frac{1}{4}, \\ p_{1,1} &= p_{2,2} = p_{3,3} = p_{4,4} = \frac{3}{4}, \\ p_{1,0} &= p_{2,1} = p_{3,2} = p_{4,3} = \frac{1}{4}, \\ p_{i,j} &= 0 \qquad \textit{otherwise.} \end{aligned}$$

1. *Find the stochastic matrix.*

2. *Prove that the Markov chain is irreducible.*

3. *Prove that the Markov chain is regular.*

4. *Find the invariant probability vector.*

For $t = 0$ the process is at state E_1. We denote by T_1 the stochastic variable, which indicates the time, when process for the first time is in state E_0.

5. *Find $P\{T_1 = k\}$, $k \in \mathbb{N}$, and the mean of T_1 (i.e. the expected time of getting from E_1 to E_0).*

6. *Find for $i = 2$, 3, 4, the expected time for getting from E_i to E_0.*

When $t = 0$, the process is in state E_0. Let T denote the time of the first return to E_0.

7. *Find the mean of T.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

2) It follows from the diagram

$$E_0 \;\to\; E_4 \;\to\; E_3 \;\to\; E_2 \;\to\; E_1 \;\to\; E_0,$$

that the Markov chain is irreducible.

3) Since e.g. $p_{2,2} = \frac{3}{4} > 0$, and the Markov chain is irreducible, it is also regular.

4) The equations of the invariant probability vector are

$$\begin{array}{ll}
g_0 = \frac{1}{4}\, g_1, & \text{thus } g_1 = 4g_0, \\
g_1 = \frac{1}{4}\, g_0 + \frac{3}{4}\, g_1 + \frac{1}{4}\, g_2, & \text{thus } g_2 = 4g_1 - g_0 - 3g_1 = g_1 - g_0 = 3g_0, \\
g_2 = \frac{1}{4}\, g_0 + \frac{3}{4}\, g_2 + \frac{1}{4}\, g_3, & \text{thus } g_3 = 4g_2 - g_0 - 3g_2 = g_2 - g_0 = 2g_0, \\
g_3 = \frac{1}{4}\, g_0 + \frac{3}{4}\, g_3 + \frac{1}{4}\, g_4, & \\
g_4 = \frac{1}{4}\, g_0 + \frac{3}{4}\, g_4, & \text{thus } g_4 = g_0,
\end{array}$$

so

$$1 = g_0 + g_1 + g_2 + g_3 + g_4 = g_0(1+4+3+2+1) = 11g_0,$$

and

$$\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = \frac{1}{11}\,(1,4,3,2,1).$$

5) It follows from the matrix that

$$P\{T_1 = 1\} = \frac{1}{4} \quad \text{and} \quad P\{T_1 = 2\} = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^1,$$

and in general,

$$P\{T_1 = k\} = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \quad \text{with} \quad E\{T_1\} = \frac{1}{4} \cdot \frac{1}{\left(1 - \frac{3}{4}\right)^2} = 4.$$

6) Let $\tilde{T}_1$ denote the random variable, which gives the time when the process is at state E_{i-1} for the first time when we start at E_i. Then

$$P\left\{\tilde{T}_i = k\right\} = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \quad \text{with} \quad E\left\{\tilde{T}_i\right\} = 4.$$

Let T_i denote the time, when the process for the first time is in state E_0, when we start at E_i. Then

$$T_i = \tilde{T}_i + \tilde{T}_{i-1} + \cdots + \tilde{T}_1,$$

hence

$$E\{T_i\} = 4i, \qquad i = 1, 2, 3, 4.$$

7) In the first step we get to one of the states E_1, E_2, E_3, E_4, each of the probability $\frac{1}{4}$, hence

$$E\{T\} = 1 + \frac{1}{4} \cdot 4\{1+2+3+4\} = 11.$$

Example 3.32 *Given a Markov chain of 7 states E_0, E_1, E_2, E_3, E_4, E_5, E_6, and the transition probabilities*

$$\begin{array}{rcll}
p_{0,i} & = & \frac{1}{6}, & i = 1,\,2,\,3,\,4,\,5,\,6, \\
p_{i,0} & = & r, & i = 1,\,2,\,3,\,4,\,5,\,6, \\
p_{i,i+1} & = & p, & i = 1,\,2,\,3,\,4,\,5, \\
p_{i,i-1} & = & p, & i = 2,\,3,\,4,\,5,\,6, \\
p_{1,6} = p_{6,1} & = & p, & p_{i,j} = 0 \quad \textit{otherwise},
\end{array}$$

where $p \geq 0$, $r > 0$ and $2p + r = 1$.

1. *Find the stochastic matrix $\mathbf{P}$.*

2. *Prove that the Markov chain is irreducible.*

3. *Prove that the Markov chain is regular, if $p \in \left]0, \frac{1}{2}\right[$, but not regular for $p = 0$.*

4. *Find the value of r, for which the invariant probability vector $\mathbf{g}$ is given by*

$$\mathbf{g} = \left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right).$$

At time $t = 0$ the process is in state E_0. Let T_0 denote the time of the first return to E_0.

5. *Find for every value of $p \in \left[0, \frac{1}{2}\right[4$,*

$$P\{T_0 = k\}, \qquad k = 2,\,3,\,4,\,\dots .$$

6. *Find the mean of T_0.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
r & 0 & p & 0 & 0 & 0 & p \\
r & p & 0 & p & 0 & 0 & 0 \\
r & 0 & p & 0 & p & 0 & 0 \\
r & 0 & 0 & p & 0 & p & 0 \\
r & 0 & 0 & 0 & p & 0 & p \\
r & p & 0 & 0 & 0 & p & 0
\end{pmatrix}.$$

2) It follows from the first row that $E_0 \;\to\; E_i$ for all $i = 1, \dots, 6$. It follows from the first column that $E_i \;\to\; E_0$ for all $i = 1, \dots, 6$. Thus $E_0 \;\leftrightarrow\; E_i$ for all $i = 1, \dots 6$, and we can via E_0 always get from any E_i to any other E_j, and the Markov chain is irreducible.

3) If $0 < p < \frac{1}{2}$, then clearly $p_{ij}^{(2)} > 0$ for all (i, j), and the Markov chain is regular.
If $p = 0$, then $p_{ij}^{(2)} = 0$ for i, $j = 1, \dots, 6$, and hence also for $p_{ij}^{(n)}$, which means that every $\mathbf{P}^n$ contains zeros, and the Markov chain is not regular for $p = 0$.

4) If the vector

$$\mathbf{g} = \left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$$

satisfies $\mathbf{g} = \mathbf{g}\,\mathbf{P}$, then we get from the first coordinate that

$$\frac{1}{7} = \frac{1}{7} \cdot 6r,$$

so $r = \frac{1}{6}$ is the only possibility. In this case,

$$p = \frac{1}{2}(1 - r) = \frac{5}{12},$$

and it follows that the matrix is double stochastic for this value, and the given vector is indeed an invariant probability vector.

5) Due to the extreme symmetry we may introduce the new state

$$E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6.$$

Then we have a new stochastic matrix for E_0 and E alone,

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ r & 2p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 2p & 2p \end{pmatrix}.$$

It follows that

$$P\{T_0 = 2\} = 1 - 2p \quad \text{and} \quad P\{T_0 = 3\} = (1 - 2p) \cdot (2p)^1,$$

and in general,

$$P\{T_0 = k\} = (1 - 2p) \cdot (2p)^{k-2}, \qquad k \geq 2.$$

6) The mean is

$$\begin{aligned} E\{T_0\} &= (1 - 2p) \sum_{k=2}^{\infty} k\,(2p)^{k-2} = (1 - 2p) \sum_{k=1}^{\infty} (k+1)\,(2p)^{k-1} \\ &= (1 - 2p) \left\{ \frac{1}{(1 - 2p)^2} + \frac{1}{1 - 2p} \right\} = \frac{1}{1 - 2p} + 1 = 2\,\frac{1 - p}{1 - 2p}. \end{aligned}$$

Example 3.33 *Given a Markov chain of 5 states E_0, E_1, E_2, E_3 and E_4 and transition probabilities*

$$\begin{aligned} p_{0,1} &= 1, \\ p_{1,2} &= p_{2,3} = p_{3,4} = \frac{2}{3}, \\ p_{1,0} &= p_{2,1} = p_{3,2} = \frac{1}{3}, \\ p_{4,2} &= p_{4,3} = \frac{1}{2}, \\ p_{i,j} &= 0 \qquad \textit{otherwise}. \end{aligned}$$

1) Find the stochastic matrix $\mathbf{P}$.

2) Prove that the Markov chain is irreducible.

3) Check if the Markov chain is regular.

4) Find the invariant probability vector.

5) Find $p_{2,2}^{(2)}$.

6) At time $t = 0$ the process is in state E_2. Find for every $n \in \mathbb{N}$ the probability that the process is in state E_0 to time $2n$ without *in the meantime having been in any of the states E_0 or E_4.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

2) We obtain from the oblique diagonals the transitions

$$E_0 \;\to\; E_1 \;\to\; E_2 \;\to\; E_3 \;\to\; E_4 \;\to\; E_3 \;\to\; E_2 \;\to\; E_1 \;\to\; E_0,$$

so we conclude that the Markov chain is irreducible.

3) We get by a computition,

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{5}{9} & 0 & \frac{4}{9} & 0 \\ \frac{1}{9} & 0 & \frac{4}{9} & 0 & \frac{4}{9} \\ 0 & \frac{1}{9} & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

The elements of the diagonal of $\mathbf{P}^2$ are > 0, so we shall only check if $\mathbf{P}^2$ is irreducible. We have the transitions

$$E_0 \;\leftrightarrow\; E_2, \qquad E_1 \;\leftrightarrow\; E_3 \quad \text{and} \quad E_2 \;\leftrightarrow\; E_4,$$

so we can get from any "even" state to any other "even" state, and from any "odd" state to any other "odd" state. Since also

$$E_4 \quad \leftrightarrow \quad E_1 \qquad \text{and} \qquad E_3 \quad \leftrightarrow \quad E_2,$$

we have also a connection in both directions between "even" and "odd" states, so $\mathbf{P}^2$ is irreducible, and hence also regular, because $p_{1,1}^{(2)} > 0$. Then also $\mathbf{P}$ is regular.

4) The equations of the invariant probability vector are

$$\begin{array}{ll}
g_0 = \frac{1}{3}\,g_1, & \text{thus } g_1 = 3g_0, \\
g_1 = g_0 + \frac{1}{3}\,g_2, & \text{thus } g_2 = 3g_1 - 3g_0 = 6g_0, \\
g_2 = \frac{2}{3}\,g_1 + \frac{1}{3}\,g_3 + \frac{1}{2}\,g_4, & \text{thus } 6g_0 = 2g_0 + \frac{1}{3}\,g_3 + \frac{1}{2}\,g_4, \\
g_3 = \frac{2}{3}\,g_2 + \frac{1}{2}\,g_4, & \text{thus } g_3 = 4g_0 + \frac{1}{2}\,g_4, \\
g_4 = \frac{2}{3}\,g_3, &
\end{array}$$

so in particular,

$$4g_0 = \frac{1}{3}\,g_0 + \frac{1}{2}\,g_4 = \frac{1}{3}\,g_3 + \frac{1}{3}\,g_3 = \frac{2}{3}\,g_3,$$

and

$$g_3 = 6g_0 \qquad \text{and} \qquad g_4 = \frac{2}{3}\,g_3 = 4g_0.$$

It follows from

$$1 = g_0 + g_1 + g_2 + g_3 + g_4 = g_0(1+3+6+6+4) = 20g_0$$

that $g_0 = \frac{1}{20}$ and

$$\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = \left(\frac{1}{20}, \frac{3}{20}, \frac{3}{10}, \frac{3}{10}, \frac{1}{5}\right).$$

5) According to the computation in **3.** we have $p_{22}^{(2)} = \frac{4}{9}$.

6) Starting at E_2 we get in the first step either to E_1 or to E_3, thus neither to E_0 nor to E_4. It follows from the matrix of $\mathbf{P}^2$ that

$$\begin{aligned} &P\{E_0 \text{ after 2 steps}\} = \tfrac{1}{9}, \\ &P\{E_2 \text{ after 2 steps}\} = \tfrac{4}{9}, \\ &P\{E_4 \text{ after 2 steps}\} = \tfrac{4}{9}. \end{aligned}$$

Then the process can be iterated,

$$P\{E_0 \text{ after } 2n \text{ steps without passing } E_0 \text{ or } E_4\} = \frac{1}{9}\left(\frac{4}{9}\right)^{n-1}.$$

Example 3.34 *Given a Markov chain of 2 states E_1 and E_2 and of the stochastic matrix*

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

1. *Prove that* $\mathbf{Q}$ *is regular, and find the invariant probability vector.*

Another Markov chain with 6 states, E_1, E_2, E_3, E_4, E_5 and E_6, has the stochastic matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. *Prove that* $\mathbf{P}$ *is not irreducible, and find all closed subsets.*

3. *Find all invariant probability vectors of* $\mathbf{P}$.

4. *Prove for any initial distribution*

$$\mathbf{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, p_4^{(0)}, p_5^{(0)}, p_6^{(0)}\right)$$

that

$$p_3^{(n+2)} + p_4^{(n+2)} + p_5^{(n+2)} \le \frac{2}{3}\left(p_3^{(n)} + p_4^{(n)} + p_5^{(n)}\right), \qquad n \ni \mathbb{N}_0,$$

and then prove that

$$\lim_{n\to\infty} p_3^{(n)} = \lim_{n\to\infty} p_4^{(n)} = \lim_{n\to\infty} p_5^{(n)} = 0.$$

5. *Let the initial distribution be given by*

$$\mathbf{q}^{(0)} = (0,1,0,0,0,0).$$

Find $\lim_{n\to\infty} \mathbf{q}^{(n)}$.

Then let the initial distribution be given by

$$\mathbf{r}^{(0)} = (0,0,0,1,0,0).$$

Prove that $\lim_{n\to\infty} \mathbf{r}^{(n)}$.

1) All elements of $\mathbf{Q}$ are > 0, so $\mathbf{Q}$ is regular.
The equation of the invariant probability vector is

$$g_1 = \frac{1}{4}\,g_1 + \frac{1}{2}\,g_2 \qquad \text{thus} \qquad g_2 = \frac{3}{2}\,g_1,$$

so the probability vector is

$$\mathbf{g} = \left(\frac{2}{5}, \frac{3}{5}\right).$$

2) Obviously, $\{E_1, E_2\}$ and $\{E_6\}$ are closed subsets. Since we from E_5 can get to both E_4 and E_6, from E_4 can reach both E_3 and E_5, and from E_3 get to both E_2 and E_4, we have – with the exception of the union $\{E_1, E_2, E_6\}$ – no other possibility of closed subsets. Since we have non-trivial closed subsets, the Markov chain is not irreducible.

3) The equations of the invariant probability vectors are

$$\begin{array}{ll} g_1 = \frac{1}{4}\, g_1 + \frac{1}{2}\, g_2, & g_2 = \frac{3}{4}\, g_1 + \frac{1}{2}\, g_2 + \frac{1}{3}\, g_3, \\ g_3 = \frac{1}{3}\, g_4, & g_4 = \frac{2}{3}\, g_3 + \frac{1}{3}\, g_5, \\ g_5 = \frac{2}{3}\, g_4, & g_6 = \frac{1}{2}\, g_5 + g_6. \end{array}$$

When we solve these equations backwards, we obtain successively,

$$g_5 = 0, \qquad g_4 = 0, \quad \text{and} \quad g_3 = 0.$$

The closed system $\{E_1, E_2\}$ corresponds to the matrix $\mathbf{Q}$, so the invariant probability vectors are

$$\mathbf{g} = \left(\frac{2}{5}\, x, \frac{3}{5}\, x, 0, 0, 0, 1 - x \right), \qquad 0 \le x \le 1.$$

4) It follows from

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{16} & \frac{9}{16} & 0 & 0 & 0 & 0 \\ \frac{3}{8} & \frac{5}{8} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{9} & 0 & \frac{4}{9} & 0 \\ 0 & \frac{1}{9} & 0 & \frac{4}{9} & 0 & \frac{4}{9} \\ 0 & 0 & \frac{1}{9} & 0 & \frac{2}{9} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

that

$$\begin{aligned} p_3^{(n+2)} + p_4^{(n+2)} + p_5^{(n+2)} &= \left(\frac{2}{9} + \frac{4}{9} \right) p_3^{(n)} + \frac{4}{9}\, p_4^{(n)} + \left(\frac{1}{9} + \frac{2}{9} \right) p_5^{(n)} \\ &\le \frac{2}{3} \left\{ p_3^{(n)} + p_4^{(n)} + p_5^{(n)} \right\}. \end{aligned}$$

This estimate gives by recursion,

$$0 \le p_3^{(n+2p)} + p_4^{(n+2p)} + p_5^{(n+2p)} \le \left(\frac{2}{3} \right)^p \left\{ p_3^{(n)} + p_4^{(n)} + p_5^{(n)} \right\} \to 0 \qquad \text{for } p \to \infty.$$

Since all $p_i^{(n)} \ge 0$, we get

$$\lim_{n\to\infty} p_3^{(n)} = \lim_{n\to\infty} p_4^{(n)} = \lim_{n\to\infty} p_5^{(n)} = 0.$$

5) If $\mathbf{q}^{(0)} = (0, 1, 0, 0, 0, 0)$, we are in the closed set $\{E_1, E_2\}$, which actually is described by the matrix $\mathbf{Q}$ given in **1.**. Then

$$\lim_{n\to\infty} \mathbf{q}^{(n)} = \left(\frac{2}{5}, \frac{3}{5}, 0, 0, 0, 0 \right).$$

6) If $\mathbf{r}^{(0)} = (0,0,0,1,0,0)$, then we start in E_4. Hence by **4.**,

$$\mathbf{r}^{(2)} = \left(0, \frac{1}{9}, 0, \frac{4}{9}, 0, \frac{4}{9}\right),$$

so $\frac{1}{9}$ reaches the closed set $\{E_1, E_2\}$, and $\frac{4}{9}$ reaches the closed set $\{E_6\}$.
The rest, $\frac{4}{9}$, lies again in E_4, and the process is repeated.
In total, $\frac{5}{9}$ disappears in each step, where $\frac{1}{5}$ goes to $\{E_1, E_2\}$, and $\frac{4}{5}$ goes to E_6. Hence we conclude that

$$\lim_{n\to\infty} \mathbf{r}^{(n)} = \left(\frac{2}{25}, \frac{3}{25}, 0, 0, 0, \frac{4}{5}\right).$$

Alternatively we may apply the *theory of the ruin problem*. We first re-numerate the states to

$$F_0 = \{E_1, E_2\}, \qquad F_1 = E_3, \qquad F_2 = E_4, \qquad F_3 = E_5, \quad \text{and} \quad F_4 = E_6.$$

Then we get the diagram

$$F_0 \longleftarrow F_1 \longleftrightarrow F_2 \longleftrightarrow F_3 \longrightarrow F_4.$$

Starting at $E_4(= F_2)$, the parameters of the ruin problem in order to reach $F_0 = \{E_1, E_2\}$ before $F_4 = E_6$ is given by

$$k = 2, \qquad N = 4, \qquad q = \frac{1}{3}, \qquad p = \frac{2}{3},$$

hence the probability is

$$a_2 = \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^4}{1 - \left(\frac{1}{4}\right)^4} = \frac{\frac{1}{4} - \frac{1}{16}}{1 - \frac{1}{16}} = \frac{3}{15} = \frac{1}{5}.$$

Once one has arrived to $F_0 = \{E_1, E_2\}$, one stays there forever, an we approach the stationary distribution of (1). This gives

$$\mathbf{r}^{(n)} \to \left(\frac{1}{5} \cdot \frac{2}{5}, \frac{1}{5} \cdot \frac{3}{5}, 0, 0, 0, \frac{4}{5}\right) = \left(\frac{2}{25}, \frac{3}{25}, 0, 0, 0, \frac{4}{5}\right).$$

Example 3.35 *Given a Markov chain of 5 states E_1, E_2, E_3, E_4 and E_5, and transition probabilities*

$$\begin{array}{ll} p_{1,1} = a, & p_{1,2} = 1-a, \\ p_{3,1} = p_{3,5} = \frac{1}{2}, & p_{2,3} = p_{4,5} = p_{5,4} = 1, \\ p_{i,j} = 0 \qquad \textit{otherwise.} & \end{array}$$

(Here a is a constant in the interval $[0,1]$).

1) Find the stochastic matrix $\mathbf{P}$.

2) Prove that the Markov chain is irreducible for $a < 1$, and not irreducible for $a = 1$.

3) Find for every a them invariant probability vector.

4) Find the values of a, for which the Markov chain is regular.

5) To time $t = 0$ the process is in state E_5. Denote by T the time when the process for the first time is in state E_1.
Find the distribution of the random variable T.

6) Find the mean of T.

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} a & 1-a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

2) For $a < 1$ we get the diagrams

$$E_1 \quad \to \quad E_2 \quad \to \quad E_3 \quad \to \quad E_1,$$

and

$$E_3 \quad \to \quad E_5 \quad \to \quad E_4 \quad \to \quad E_3,$$

so we can get from any state E_i to any other state E_j, thus the Markov chain is irreducible.
If $a = 1$, then E_1 is absorbing, and the Markov chain is not irreducible.

3) The equations of the invariant probability vectors are

$$\begin{array}{lll}
g_1 = a\, g_1 + \frac{1}{2}\, g_3, & \text{thus} & g_3 = 2(1-a)g_1, \\
g_2 = (1-a)g_1, & \text{thus} & g_2 = (1-a)g_1, \\
g_3 = g_2 + g_4, & & \\
g_4 = g_5, & & \\
g_5 = \frac{1}{2}\, g_3, & \text{thus} & g_4 = g_5 = (1-a)g_1.
\end{array}$$

We now get

$$1 = g_1 + g_2 + g_3 + g_4 + g_5 = g_1\ \{1 + 1 - a + 2 - 2a + 1 - a + 1 - a\} = g_1(6 - 5a).$$

Since $6 - 5a > 0$ for $a \in [0,1]$, we get $g_1 = \dfrac{1}{6-5a}$, and

$$\mathbf{g} = \frac{1}{6-5a}\,(1, 1-a, 2(2/1-a), 1-a, 1-a).$$

4) Now, $\mathbf{P}$ is irreducible for $a \in [0,1[$, and $p_{1,1} = a > 0$ for $a \in]0,1[$, hence the Markov chain is (at least) regular for $a \in]0,1[$. When $a = 0$, then

$$\mathbf{P}^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{P}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

It follows that E_3 is an absorbing state for $\mathbf{P}^3$, hence the Markov chain corresponding to $\mathbf{P}3$ is not irreducible. In particular, $\mathbf{P}$ is not regular.

5) We derive from the diagram

$$\begin{array}{lllllll}
 & & & & & & E_1, \\
 & & & & & \tfrac{1}{2} \nearrow & \\
E_5 & \stackrel{1}{\to} & E_4 & \stackrel{1}{\to} & E_3 & & \\
 & & & & & \tfrac{1}{2} \searrow & \\
 & & & & & & E_5,
\end{array}$$

that

$$P\{T=1\}=P\{T=2\}=0 \qquad \text{and} \qquad P\{T=3\}=\frac{1}{2},$$

and the process is repeated from E_5. Hence

$$P\{T=3k\}=\left(\frac{1}{2}\right)^k \qquad \text{and} \qquad P\{T=j\}=0 \quad \text{for } j \neq 3k.$$

6) The mean is

$$P\{T\}=\sum_{k=1}^{\infty} 3k\left(\frac{1}{2}\right)^k=\frac{3}{2}\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^{k-1}=\frac{3}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2}=6.$$

Example 3.36 *Given a Markov chain of 5 states E_0, E_1, E_2, E_3 and E_4, and transition probabilities*

$$\begin{array}{ll}
p_{0,i}=\frac{1}{4}, & i=1,\,2,\,3,\,4; \\
p_{i,i-1}=\frac{1}{i}, & i=1,\,2,\,3,\,4; \\
p_{i,i}=\frac{i-1}{i}, & i=2,\,3,\,4; \\
p_{i,j}=0 & \textit{ellers}.
\end{array}$$

1. *Find the stochastic matrix* $\mathbf{P}$.

2. *Prove that the Markov chain is irreducible.*

3. *Check if the Markov chain is regular.*

Find the invariant probability vector.

At time $t=0$ the process is in state E_i, where i is one of the numbers 1, 2, 3, 4. Let T_i denote the random variable, which indicates the time, when the process for the first time is in state E_{i-1}.

5. *Find for $i=1$, 2, 3, 4, the probabilities $P\{T_i=k\}$, $k \in \mathbb{N}$, and find the mean of T_i (i.e. the expected time for getting from E_i to E_{i-1}).*

6. *Find for $i=1$, 2, 3, 4, the expected time for getting from E_i to E_0.*

Let the process at time $t=0$ be in state E_0. Denote by T the time of the the first return to E_0.

7. *Find the mean of T.*

1) The stochastic matrix is

$$\mathbf{P}=\begin{pmatrix}
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{3}{4}
\end{pmatrix}.$$

2) It follows from the diagram

$$E_4 \quad \to \quad E_3 \quad \to \quad E_2 \quad \to \quad E_1 \quad \to \quad E_0 \quad \to \quad E_4,$$

that the Markov chain is irreducible.

3) Since e.g. $p_{2,2} = \dfrac{1}{2} > 0$, and the Markov chain is irreducible, it is also regular.

4) The equations of the invariant probability vectors are

$$\begin{array}{lll}
g_0 = g_1, & \text{thus} & g_1 = g_0, \\
g_1 = \frac{1}{4}\,g_0 + \frac{1}{2}\,g_2, & \text{thus} & g_2 = \frac{3}{2}\,g_0, \\
g_2 = \frac{1}{4}\,g_0 + \frac{1}{2}\,g_2 + \frac{1}{3}\,g_3, & \text{thus} & g_3 = 3\left(\frac{1}{2}\,g_2 - \frac{1}{4}\,g_0\right) = \frac{3}{2}\,g_0, \\
g_3 = \frac{1}{4}\,g_0 + \frac{2}{3}\,g_3 + \frac{1}{4}\,g_4, & & \\
g_4 = \frac{1}{4}\,g_0 + \frac{3}{4}\,g_4, & \text{thus} & g_4 = g_0.
\end{array}$$

Hence

$$1 = g_0 + g_1 + g_2 + g_3 + g_4 = g_0\left(1 + 1 + \frac{3}{2} + \frac{3}{2} + 1\right) = 6\,g_0,$$

from which $g_0 = \dfrac{1}{6}$, and

$$\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}\right).$$

5) Clearly,

$$P\{T_1 = 1\} = 1, \quad P\{T_1 = k\} = 0 \quad \text{for } k \geq 2, \quad \text{and} \quad E\{T_1\} = 1.$$

We get for T_2,

$$P\{T_2 = k\} = \left(\frac{1}{2}\right)^k, \quad k \in \mathbb{N}, \qquad \text{and} \qquad E\{T_2\} = 2.$$

We get for T_3,

$$P\{T_3 = k\} = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}, \quad k \in \mathbb{N}, \qquad \text{and} \qquad E\{T_3\} = 3.$$

We get for T_4,

$$P\{T_4 = k\} = \frac{1}{4}\left(\frac{3}{4}\right)^{k-1}, \quad k \in \mathbb{N}, \qquad \text{and} \qquad E\{T_4\} = 4.$$

6) Let $\tilde{T}_i$ denote the time when the process of initial state E_i for the first time is in E_0. Then

$$\begin{aligned}
E\left\{\tilde{T}_1\right\} &= E\{T_1\} = 1, \\
E\left\{\tilde{T}_2\right\} &= E\{T_2\} + E\{T_1\} = 2 + 1 = 3, \\
E\left\{\tilde{T}_3\right\} &= E\{T_3\} + E\left\{\tilde{T}_2\right\} = 3 + 3 = 6, \\
E\left\{\tilde{T}_4\right\} &= E\{T_4\} + E\left\{\tilde{T}_3\right\} = 4 + 6 = 10.
\end{aligned}$$

7) In the first step we get to one of the states E_1, E_2, E_3, E_4, each of probability $\frac{1}{4}$. Then we shall move back to E_0, so

$$\begin{aligned} E\{T\} &= 1+\frac{1}{4}\left(E\left\{\tilde{T}_1\right\}+E\left\{\tilde{T}_2\right\}+E\left\{\tilde{T}_3\right\}+E\left\{\tilde{T}_4\right\}\right) \\ &= 1+\frac{1}{4}(1+3+6+10)=1+\frac{1}{4}\cdot 20=6. \end{aligned}$$

Example 3.37 *Given a Markov chain of 4 states E_1, E_2, E_3 and E_4, and transition probabilities*

$$\begin{array}{ll} p_{1,1} = a, & p_{1,2} = 1-a, \\ p_{2,3} = p_{3,2} = \frac{2}{3}, & p_{2,1} = p_{3,4} = \frac{1}{3}, \\ p_{4,3} = 1, & p_{i,j} = 0 \text{ otherwise.} \end{array}$$

(Here a is a constant in the interval $[0,1]$).

1) *Find the stochastic matrix* $\mathbf{P}$.

2) *Find the values of a, for which the Markov chain is irreducible.*

3) *Find the values of a, for which the Markov chain is regular.*

4) *Find for every a the invariant probability vector.*

5) *Find for every a the limit* $\lim_{n\to\infty} p_{22}^{(2n)}$.

6) *Put $a = 1$, and assume that the process at time $t = 0$ is in state E_2. Find the probability that the process at any later time reaches state E_4.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} a & 1-a & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

2) When $a \in [0,1[$, we have the diagram

$$E_1 \;\rightarrow\; E_2 \;\rightarrow\; E_3 \;\rightarrow\; E_4 \;\rightarrow\; E_3 \;\rightarrow\; E_2 \;\rightarrow\; E_1,$$

and we conclude that the Markov chain is irreducible.
If $a = 1$, then E_1 is absorbing, and the Markov chain is not irreducible for $a = 1$.

3) When we check the possible regularity we shall only consider $a \in [0,1[$.
If $a \in]0,1[$, then $p_{1,1} = a > 0$, and the Markov chain is regular for $a \in]0,1[$.
If $a = 0$, then every second element of $\mathbf{P}^n$ is 0 for every $n \in \mathbb{N}$, thus the Markov chain is not regular for $a = 0$.

4) The equations of the invariante probability vector are

$$\begin{array}{lll} g_1 = a\,g_1 + \frac{1}{3}\,g_2, & \text{thus} & g_2 = 3(1-a)g_1, \\ g_2 = (1-a)g_1 + \frac{2}{3}\,g_3, & \text{thus} & g_3 = \frac{3}{2}\,(3-3a-1+a)g_1 = 3(1-a)g_1, \\ g_3 = \frac{2}{3}\,g_2 + g_4, & \text{thus} & g_4 = (1-a)g_1. \end{array}$$

It follows from

$$1 = g_1 + g_2 + g_3 + g_4 = g_1\,\{1 + 3(1-a) + 3(1-a) + (1-a)\} = (8-7a)g_1$$

and $7a \le 7 < 8$ that $g_1 = \dfrac{1}{8-7a} \le 1$, hence

$$\mathbf{g} = \frac{1}{8-7a}\,(1, 3(1-a), 3(1-a), 1-a).$$

5) If $a \in]0,1[$, the Markov chain is regular, so $\mathbf{P}^n$ konverges, hence $\mathbf{P}^{(2n)}$ also converges towards $\mathbf{G}$, where each row of $\mathbf{g}$ is the invariant probability vector found in **4.**. Hence

$$\lim_{n\to\infty} p_{2,2}^{(2n)} = g_2 = \frac{3(1-a)}{8-7a}.$$

If $a = 0$, the Markov chain is irreducible, but not regular.
Then compute

$$\mathbf{P}^2 = \begin{pmatrix} a & 1-a & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & 1-a & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{7}{9} & 0 & \frac{2}{9} \\ \frac{4}{5} & 0 & \frac{5}{9} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

It follows that $\{E_2, E_4\}$ is a closed system. The corresponding stochastic sub-matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{7}{9} & \frac{2}{9} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

is regular, and the equations of the invariant probability vector are

$$\begin{aligned} g_2 &= \tfrac{7}{9}\, g_2 + \tfrac{2}{3}\, g_4, \\ g_4 &= \tfrac{2}{9}\, g_2 + \tfrac{1}{3}\, g_4, \end{aligned} \qquad \text{thus} \qquad g_2 = 3g_4,$$

hence

$$(g_2, g_4) = \left(\frac{3}{4}, \frac{1}{4}\right), \quad \text{and} \quad \lim_{n\to\infty} p_{2,2}^{(2n)} = \frac{3}{4},$$

(and not $\frac{3}{8}$, which we get by inserting $a = 0$ into the formula of **4.**.
This result is in agreement with the theoretical result, because $p_{2,2}^{(2n+1)} = 0$, so

$$\frac{1}{2n}\sum_{i=1}^{2n} \mathbf{P} = \frac{1}{2n}\sum_{i=1}^{n} \mathbf{P}^{2i} + \frac{1}{2n}\sum_{i=1}^{n} \mathbf{P}^{2i-1} \to \mathbf{G} \quad \text{for } n \to \infty.$$

If $a = 1$, then

$$\mathbf{P}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \dfrac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \dfrac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{4}{9} & 0 & \frac{2}{9} \\ \frac{2}{9} & 0 & \frac{7}{9} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Thus

$$g_2^{(n+2)} + g_3^{(n+2)} + g_4^{(n+2)} \le \frac{7}{9}\left(g_2^{(n)} + g_3^{(n)} + g_4^{(n)}\right),$$

and hence $g_2^{(2n)} \to 0$ for $n \to \infty$, and in particular, $p_{2,2}^{(2n)} \to 0$.

6) Let $a = 1$. If the proces at time $t = 0$ starts in E_2, we get $P\{T = 1\} = 0$.
If $t = 2$, then $P\{T = 2\} = \dfrac{2}{9}$, while $\dfrac{4}{9}$ of the "mass" lies in E_2, and $\dfrac{1}{3}$ is "lost" in the absorbing state E_1. Thus

$$P\{T = 2k+1\} = 0 \qquad \text{for } k \in \mathbb{N}_0,$$

and

$$P\{T = 2k\} = \frac{2}{9} \cdot \left(\frac{4}{9}\right)^{k-1} \qquad \text{for } k \in \mathbb{N}.$$

Finally, by a summation, the wanted probability is

$$\sum_{k=1}^{\infty} \frac{2}{9} \cdot \left(\frac{4}{9}\right)^{k-1} = \frac{\frac{2}{9}}{1 - \frac{4}{9}} = \frac{2}{5}.$$

Example 3.38 *Given a Markov chain of 5 states E_0, E_1, E_2, E_3 and E_4, and transition probabilities*

$$\begin{array}{ll}
p_{i,i} = \frac{4}{5}, & i = 1,\, 2,\, 3,\, 4, \\
p_{i,i-1} = \frac{1}{5}, & i = 1,\, 2,\, 3,\, 4, \\
p_{0,2} = p_{0,4} = a, & p_{0,0} = 1 - 2a, \\
p_{i,1} = 0 & \textit{otherwise.}
\end{array}$$

Here a is a constant in the interval $\left[0, \frac{1}{2}\right]$.

1. *Find the stochastic matrix* $\mathbf{P}$.

2. *Find the values of a, for which the Markov chain is irreducible.*

3. *Find for every a the invariant probability vector.*

Ay time $t = 0$ the process is in state E_i, where i is one of the numbers 1, 2, 3, 4.
Let T_i denote the random variable, which indicates the time when the process for the first time is in state E_0.

4. *Find $P\{T_2 = k\}$, $k = 2$, 3, 4, …, and the mean of T_2 (i.e. the expected time for getting from E_2 to E_0).*

5. *Find the mean of T_4.*

Now put $a = \frac{1}{2}$, and assume that the process to time $t = 0$ is in state E_0. Let T denote the time of the first return to E_0.

6. *Find the mean of T.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 1-2a & 0 & a & 0 & a \\ \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \end{pmatrix}.$$

2) When $a = 0$, we see that E_0 is absorbing, and the Markov chain is not irreducible.
For $0 < a \le \frac{1}{2}$ we get the diagram

$$E_0 \;\rightarrow\; E_4 \;\rightarrow\; E_3 \;\rightarrow\; E_2 \;\rightarrow\; E_1 \;\rightarrow\; E_0,$$

proving that the Markov chain is irreducible, and even regular, because e.g. $p_{1,1} = \frac{4}{5} > 0$.

3) The equations of the invariant probability vector are

$$\begin{array}{lll} g_0 = (1-2a)g_0 + \frac{1}{5}\,g_1, & \text{thus} & g_1 = 10a\,g_0, \\ g_1 = \frac{4}{5}\,g_1 + \frac{1}{5}\,g_2, & \text{thus} & g_2 = g_1 = 10a\,g_0, \\ g_2 = a\,g_0 + \frac{4}{5}\,g_2 + \frac{1}{5}\,g_3, & \text{thus} & g_3 = 10a\,g_0 - 5a\,g_0 = 5a\,g_0, \\ g_3 = \frac{4}{5}\,g_3 + \frac{1}{5}\,g_4, & \text{thus} & g_4 = g_3 = 5a\,g_0, \\ g_4 = a\,g_0 + \frac{4}{5}\,g_4, & \text{thus} & g_4 = 5a\,g_0. \end{array}$$

It follows that

$$1 = g_0 + g_1 + g_2 + g_3 + g_4 = g_0\,(1 + 10a + 10a + 5a + 5a) = (1+30a)\,g_0,$$

thus $g_0 = \dfrac{1}{1+30a}$, and therefore

$$\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = \frac{1}{1+30a}\,(1, 10a, 10a, 5a, 5a).$$

4) Let $\tilde{T}_i$ denote the random variable, which indicates the first time, when the process is in state E_{i-1}, when we start in E_i. Then

$$P\left\{\tilde{T}_i = k\right\} = \frac{1}{5}\cdot\left(\frac{4}{5}\right)^{k-1}, \quad k \in \mathbb{N}, \qquad \text{med } E\left\{\tilde{T}_i\right\} = 5.$$

It follows from $T_2 = \tilde{T}_2 + \tilde{T}_1$ for $k \geq 2$ that

$$\begin{aligned} P\left\{T_2 = k\right\} &= \sum_{j=1}^{k-1} P\left\{\tilde{T}_2 = j\right\} \cdot P\left\{\tilde{T}_1 = k-j\right\} = \sum_{j=1}^{k-1} \frac{1}{5}\left(\frac{4}{5}\right)^{j-1} \cdot \frac{1}{5}\left(\frac{4}{5}\right)^{k-j-1} \\ &= \frac{1}{25}\,(k-1)\left(\frac{4}{5}\right)^{k-2}, \qquad k \geq 2. \end{aligned}$$

The mean is

$$E\left\{T_2\right\} = E\left\{\tilde{T}_2\right\} + E\left\{\tilde{T}_1\right\} = 5 + 5 = 10.$$

Alternatively,

$$E\left\{T_2\right\} = \frac{1}{25}\sum_{k=2}^{\infty} k(k-1)\left(\frac{4}{5}\right)^{k-2} = \frac{1}{25} \cdot \frac{2!}{\left(1-\frac{4}{5}\right)^3} = 10.$$

5) The mean of T_4 is

$$E\left\{T_4\right\} = E\left\{\tilde{T}_4\right\} + E\left\{\tilde{T}_3\right\} + E\left\{\tilde{T}_2\right\} + E\left\{\tilde{T}_1\right\} = 4 \cdot 5 = 20.$$

6) Let $a = \frac{1}{2}$, and assume that we at time $t = 0$ are in state E_0. Then we are to time $t = 1$ either in E_2 or in E_4, each of probability $\frac{1}{2}$. This gives

$$E\{T\} = 1 + \frac{1}{2}\,E\left\{T_2\right\} + \frac{1}{2}\,E\left\{T_4\right\} = 1 + \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 20 = 16.$$

Example 3.39 *Given a Markov chain of the states $E_1, E_2, \dots, E_m$. We assume that $\{E_1, E_2, \dots, E_r\}$ is a closed subset C, and that we from any other of the states $E_{r+1}, E_{r+2}, \dots, E_m$, have positive probability eventually of reaching the closed subset. Thus, the stochastic matrix looks like*

$$\mathbf{P} = \begin{pmatrix} \mathbf{S} & | & \mathbf{0} \\ - & + & - \\ \mathbf{R} & | & \mathbf{Q} \end{pmatrix},$$

where $\mathbf{S}$ is an $r \times r$ stochastic matrix, $\mathbf{Q}$ is an $(m-r)\times(m-r)$-matrix, $\mathbf{0}$ is an $r \times (m-r)$-matrix consisting of zeros, and $\mathbf{R}$ is an $(m-r)\times r$-matrix.

1. *Prove for every pair (i,j) with $r+1 \le i,\ j \le m$ that $p_{ij}^{(n)} \to 0$ for $n \to \infty$, and conclude that $\mathbf{Q}^n \to \mathbf{0}$ for $n \to \infty$.*

2. *Prove that there are constants $b > 0$, $c \in]0,1[$, such that $p_{ij}^{(n)} \le b\,c^n$ for every pair (i,j) with $r+1 \le i,\ j \le m$ and every $n \in \mathbb{N}$, and conclude that for every (i,j) as above, the infinite series $\sum_{n=0}^{\infty} p_{ij}^{(n)}$ is convergent.*

3. *Prove that the matrix $\mathbf{I} - \mathbf{Q}$ has the reciprocal matrix*

$$\mathbf{N} = \sum_{k=0}^{\infty} \mathbf{Q}^k.$$

We define for every $j \in \{r+1, \dots, m\}$ a random variable X_j by

$X_j = k$, *if the process is in state E_j in total k times.*

For $i \in \{r+1, \dots, m\}$ we let $E_i\{X_j\}$ denote the expected number of times the process is in state E_j, if the processen at time $t = 0$ is in state E_i.

4. *Prove that*

$$E_i\{X_j\} = \sum_{n=0}^{\infty} p_{ij}^{(n)}.$$

5. *Prove that $E_i\{X_j\}$ can be found as the (i,j)-th element of the matrix $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$.*

We denote for $i \in \{r+1, \dots, m\}$ and $j \in \{1, 2, \dots, r\}$ by b_{ij} the probability that the process by starting in E_i reachers state E_j before any of the states in C.

6. *Prove that b_{ij} is equal to the (i,j)-th element of the matrix $\mathbf{B} = \mathbf{N}\,\mathbf{R}$.*

1) For every fixed $i \in \{r+1, \dots, m\}$ there exists an n_i, such that the i-th row in $\mathbf{P}^{n_i}$ contains elements. Then

$$\sum_{j=r+1}^{m} p_{i,j}^{(m+n_i)} \le \alpha_i \sum_{j=r+1}^{m} p_{i,j}^{(n)} \le \alpha_i, \qquad \text{where } 0 \le \alpha_i < 1,$$

hence

$$\sum_{j=r+1}^{m} p_{i,j}^{(n+s\,n_i)} \le \alpha_i^s \to 0 \qquad \text{for } s \to \infty.$$

We conclude that $p_{i,j}^{(n)} \to 0$ for $n \to \infty$ and $i, j = r+1, \dots, m$. This implies precisely that $\mathbf{Q}^n \to \mathbf{0}$ for $n \to \infty$.

2) It follows from **1.** that

$$p_{i,j}^{(n+sn_i)} = \alpha_i^s = \left(\sqrt[n_i]{\alpha_i}\right)^{sn_i} = \alpha_i^{-n/n_i}\left(\sqrt[n_i]{\alpha_i}\right)^{n+sn_i} \leq \frac{1}{\alpha_i}\left(\sqrt[n_i]{\alpha_i}\right)^{n+sn_i},$$

(with a trivial modification for $\alpha_i = 0$).
If we choose $b_i = \dfrac{1}{\alpha_i}$ and $c_i = \sqrt[n_i]{\alpha_i}$, then we get the estimate

$$p_{i,j}^{(n)} \leq b_i \cdot c_i^n.$$

Then choose $b = \max_i b_i > 0$ and $c = \max_i c_i < 1$, and we get the inequality

$$\sum_{n=0}^{\infty} p_{i,j}^{(n)} \leq b \sum_{n=0}^{\infty} c^n = \frac{b}{1-c} < \infty.$$

3) If we put $\mathbf{N}^{(n)} = \sum_{k=0}^{n} \mathbf{Q}^n$, then

$$\mathbf{N}^{(n)}(\mathbf{I} - \mathbf{Q}) = (\mathbf{O} - \mathbf{Q})\mathbf{N}^{(n)} = \mathbf{I} - \mathbf{Q}^{(n)} \to \mathbf{I} \qquad \text{for } n \to \infty,$$

hence

$$(\mathbf{I} - \mathbf{Q})^{-1} = \mathbf{N} = \sum_{k=0}^{\infty} \mathbf{Q}^k.$$

4) Since $p_{i,j}^{(n)}$ is the probability that we are in state E_j after n steps, when we start in E_i, then the expected number of times, the process is in state E_j, is the sum of all these probabilities, thus

$$E_i\{X_j\} = \sum_{n=0}^{\infty} p_{i,j}^{(n)}.$$

5) The claim follows from that

$$E_i\{X_j\} = \sum_{n=0}^{\infty} p_{i,j}^{(n)}$$

is the (i,j)-th element of the matrix

$$\sum_{n=0}^{\infty} \mathbf{Q}^n = \mathbf{N} = \mathbf{I} - \mathbf{Q}.$$

6) We can only reach a state E_j, $j \in \{1, 2, \dots, r\}$, through the matrix $\mathbf{R}$, i.e. through one of the possibilities

$$\mathbf{Q}^0\mathbf{R},\ \mathbf{Q}^1\mathbf{R},\ \dots,\ \mathbf{Q}^n\mathbf{R},\ \dots.$$

An addition of these gives precisely $\mathbf{B} = \mathbf{N}\,\mathbf{R}$, where the (i,j)-th element b_{ij} is the probability that we end in $E_j \in C$, without (by the construction) being in any state earlier from C.

Example 3.40 *Given an irreducible Markov chain $E_1, E_2, \dots, E_m$ with the stochastic matrix $\mathbf{P}$ and invariant probability vector*

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n),$$

where $\alpha_j \neq 0$ for every j.

1. *Prove that if the process to time $t = 0$ is in state E_i, then for every $j \in \{1, 2, \dots, m\}$, the process reaches eventually with probability 1 the state E_j.*

We define for every j an random variable T_j by putting $T_j = n$, if the process is in state E_j for the first time after the time 0 to the time n.
Denote for every i by m_{ij} the mean of T_j, if the process to time $t = 0$ is in state E_i.

2. *Prove that m_{ij} is finite for every (i, j).*

3. *Prove by a convenient splitting of what happens in the first step,*

$$(5) \quad m_{ij} = \sum_k p_{ik} m_{kj} - p_{ij} m_{jj} + 1 \qquad \textit{for every } (i, j).$$

4. *Prove that the* mean of the time of return m_{ii} *is given by*

$$m_{ii} = \frac{1}{\alpha_i}, \qquad i = 1, 2, \dots, m.$$

HINT: *Multiply the i-th equation of (5) by α_i, and then sum over i.*

1) This follows from the fact that the Markov chain is irreducible, so E_i is transferred into E_k after some steps.

2) When the Markov chain is irreducible, it follows by considering the graph that there exists a transition diagram of M transitions, by which one comes from any E_i back to E_i through all the other states in at most M steps. Hence there exists an $a \in]0, 1[$, such that

$$m_{ij} \leq \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

3) If we start in E_i, then after the first step the one on the right hand side of (5) is in state E_k of probability $p_{i,k}$, hence

$$\sum_{k=1}^{m} p_{ik} m_{kj} + 1.$$

However, if we end in the state E_j, we count $p_{ij} m_{jj}$ too much, so

$$m_{ij} = \sum_k p_{ik} m_{kj} - p_{ij} m_{jj} + 1 \qquad \text{for every } (i, j).$$

4) By using the hint and that α is invariant, we get

$$\begin{aligned}\sum_i \alpha_i m_{ij} &= \sum_k \left(\sum_i \alpha_i p_{ik}\right) m_{kj} - \left(\sum_i \alpha_i p_{ij}\right) m_{jj} + \sum_i \alpha_i \\ &= \sum_k \alpha_k m_{kj} - \alpha_j m_{jj} + 1.\end{aligned}$$

The two sums are equal, hence by a rearrangement,

$$m_{jj} = \frac{1}{\alpha_j}.$$

Example 3.41 *Given a Markov chain of the states 0, 1, 2, ..., and transition probabilities*

$$p_{i,i+1} = p, \qquad p_{i,0} = q, \quad i \in \mathbb{N}_0, \qquad p_{ij} = 0 \quad \textit{otherwise},$$

(where $p > 0$, $q > 0$, $p + q = 1$).
Prove that the Markov chain is regular, and find its stationary distribution.

The corresponding stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ q & 0 & 0 & p & 0 & \cdots \\ q & 0 & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We have trivially the transitions

$$E_i \;\to\; E_0 \;\to\; E_1 \;\to\; E_2 \;\to\; \cdots, \qquad i \in \mathbb{N},$$

so we can get from any state E_i to any other state E_j. This shows that the Markov chain is irreducible. From $p_{0,0} = q > 0$ follows that $d = 1$, and the Markov chain is regular.

A possible stationary distribution $\mathbf{g}$ must fulfil the equations

$$g_0 = q \sum_{j=0}^{\infty} g_j \qquad \text{(convergent series)},$$

and

$$g_n = p\, g_{n-1}, \qquad n \in \mathbb{N}_0.$$

When we divide the recursion formula by $p^n > 0$, we get

$$\frac{1}{p^n}\, g_n = \frac{1}{p^{n-1}}\, g_{n-1} = \cdots = g_0,$$

hence

$$g_n = p^n g_0, \qquad n \in \mathbb{N}_0,$$

is the only possibility. We see by insertion that the series $\sum_{j=0}^{\infty} g_j$ is in fact convergent and

$$1 = \sum_{j=0}^{\infty} g_j = g_0 \sum_{j=0}^{\infty} p^n = g_0 \cdot \frac{1}{1-p} = g_0 \cdot \frac{1}{q},$$

from which we conclude that $g_0 = q$. Therefore, the Markov chain has a stationary distribution, which is given by the coordinates

$$g_n = q\, p^n, \qquad n \in \mathbb{N}_0.$$

Example 3.42 *A Markov chain has the countably many states E_1, E_2, E_3, $\dots$, and transition probabilities*

$$p_{i,i+1} = \frac{i}{i+1}, \quad p_{i,1} = \frac{2}{i+2}, \quad i \in \mathbb{N}, \qquad p_{ij} = 0 \quad \textit{otherwise.}$$

1) Prove that the Markov chain is regular.

2) Prove that there exists a stationary distribution, and then find it.

3) Assume that the process at time $t = 0$ is in state E_1. Let T denote the random variable, which indicates the time of the first return to E_1. Find $P\{T = k\}$, $k \in \mathbb{N}$.

4) Find the mean $E\{T\}$.

5) Prove that T does not have a variance.

1) The infinite stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{2}{4} & 0 & \frac{2}{4} & 0 & \cdots \\ \frac{2}{5} & 0 & 0 & \frac{3}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We conclude from

$$E_1 \ \rightarrow \ E_2 \ \rightarrow \ E_3 \ \rightarrow \ \cdots \ \rightarrow \ E_n \ \rightarrow \ \cdots,$$

and

$$E_n \ \rightarrow \ E_1 \qquad \text{for alle } n,$$

that the Markov chain is irreducible.
Since $d_1 = 1$, the Markov chain is regular.

2) The equations of a possible invariant probability vector are

$$g_{i+1} = \frac{i}{i+2}\, g_i, \qquad i \in \mathbb{N}.$$

Then by recursion,

$$g_i = \frac{2}{(i+1)i}\, g_1, \qquad i \in \mathbb{N}.$$

Using that the sectional series is telescopic we conclude from

$$\sum_{i=1}^{\infty} \frac{2}{(i+1)i} = 2\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right) = 2,$$

that the stationary distribution exists and that its coordinates are given by

$$g_i = \frac{1}{(i+1)i}, \qquad i \in \mathbb{N}.$$

3) It follows that

$$
\begin{aligned}
P\{T=1\} &= \frac{2}{3}, \\
P\{T=2\} &= \left(1-\frac{2}{3}\right)\cdot\frac{2}{4}=\frac{1}{6}, \\
P\{T=3\} &= \left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\cdot\frac{2}{5}=\frac{1}{15}, \\
P\{T=4\} &= =\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdot\frac{2}{6}=\frac{1}{30}.
\end{aligned}
$$

We conclude from this pattern that

$$
\begin{aligned}
P\{T=k\} &= \prod_{i=1}^{k-1}\left(1-\frac{2}{i+2}\right)\cdot\frac{2}{k+2}=\prod_{i=1}^{k-1}\frac{i}{i+2}\cdot\frac{2}{k+2} \\
&= \frac{2(k-1)!}{(k+2)!}\cdot 2=\frac{4}{(k+2)(k+1)k}.
\end{aligned}
$$

4) The mean is

$$E\{T\}=4\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+1)}=4\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\frac{4}{2}=2.$$

5) Now

$$k^2 \cdot \frac{4}{(k+2)(k+1)k} \sim \frac{4}{k},$$

and the series $\sum \frac{4}{k} = \infty$ is divergent, hence the variance does not exist.

Example 3.43 *Given a Markov chain of the states E_1, E_2, E_3, E_4 and E_5 and with the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 1-a \end{pmatrix},$$

where a is a constant in the interval $[0,1]$.

1) Find the values of a, for which the Markov chain is irreducible.

2) Find the values of a, for which the Markov chain is regular.

3) Find for every a the invariant probability vector.

4) At time $t = 0$ the process is in state E_1. Let T denote the time when the process for the first time is in state E_5. Find the distribution of the random variable T.

5) Find the mean and variance of T.

6) Assume that $a = 0$. Prove that all the matrices $\mathbf{P}^n$ for $n \geq 4$ are equal to the same matrix $\mathbf{Q}$, and find $\mathbf{Q}$.

1) If $a = 0$, then E_5 is absorbing, and the Markov chain is not irreducible.
If $a \in]0,1]$, then we get the transitions

$$E_5 \quad \rightarrow \quad E_1 \quad \rightarrow \quad E_2 \quad \rightarrow \quad E_3 \quad \rightarrow \quad E_4 \quad \rightarrow \quad E_5,$$

proving that the Markov chain is irreducible for $a \in]0,1]$.

2) If $a \in]0,1[$, then the diagonal element $p_{55} = 1 - a > 0$, hence the Markov chain is regular.
On the other hand, if $a = 1$, then

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 1-a \end{pmatrix}, \qquad \mathbf{P}^2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{P}^4 = \begin{pmatrix} \frac{5}{8} & \frac{3}{32} & \frac{1}{32} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{3}{8} & \frac{3}{8} & \frac{3}{16} & \frac{1}{16} \\ 0 & 0 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & 0 & 0 & \frac{1}{4} & \frac{5}{8} \end{pmatrix}.$$

Since $\mathbf{P}^4$ has the transitions

$$E_5 \;\rightarrow\; E_1 \;\rightarrow\; E_2 \;\rightarrow\; E_3 \;\rightarrow\; E_4 \;\rightarrow\; E_5,$$

the corresponding Markov chain is irreducible.
Furthermore, all elements of the diagonal are > 0, so the Markov chain corresponding to the matrix $\mathbf{P}^4$ is regular. This implies that the original Markov chain is also regular.

3) The equations of the invariant probability vector are

$$\begin{aligned} g_1 &= a g_5, \\ g_2 &= \frac{3}{4}\, g_1, \\ g_3 &= \frac{1}{4}\, g_1 + \frac{2}{3}\, g_2, \\ g_4 &= \frac{1}{3}\, g_2 + \frac{1}{2}\, g_2, \\ g_5 &= \frac{1}{2}\, g_3 + g_4 + (1-a) g_5, \end{aligned}$$

thus

$$\begin{aligned} g_1 &= a g_5, \\ g_2 &= \frac{3}{4}\, a\, g_5, \\ g_3 &= \frac{1}{4}\, a\, g_5 + \frac{1}{2}\, a\, g_5 = \frac{3}{4}\, a\, g_5, \\ g_4 &= \frac{1}{4}\, a\, g_5 + \frac{3}{8}\, a\, g_5 = \frac{5}{8}\, a\, g_5. \end{aligned}$$

A check gives

$$a\, g_5 = \frac{3}{8}\, a\, g_5 + \frac{5}{8}\, a\, g_5 = a\, g_5,$$

so it is OK.
Furthermore,

$$1 = g_1 + g_2 + g_3 + g_4 + g_5 = g_5 \left(a + \frac{3}{4}\, a + \frac{3}{4}\, a + \frac{5}{8}\, a + 1 \right) = \left(1 + \frac{25}{8}\, a \right) g_5,$$

from which $g_5 = \dfrac{8}{8 + 25a}$, and thus

$$\mathbf{g} = \frac{1}{8 + 25a}\, (8a, 6a, 6a, 5a, 8).$$

4) Here a consideration of the graph is the easiest method:

$$\begin{array}{ccccccccc} E_1 & \xrightarrow{\frac{3}{4}} & E_2 & \xrightarrow{\frac{2}{3}} & E_3 & \xrightarrow{\frac{1}{2}} & E_4 & \xrightarrow{1} & E_5 \\ & \searrow \frac{1}{4} & & \searrow \frac{1}{3} & & \searrow \frac{1}{2} & & & \\ & & E_3 & \xrightarrow{\frac{1}{2}} & E_4 & \xrightarrow{1} & E_5 & & \\ & & & \searrow \frac{1}{2} & & & & & \\ & & & & E_5 & & & & \\ & & & & & & & & \\ t=0 & & t=1 & & t=2 & & t=3 & & t=4 \end{array}$$

It follows that

$$P\{T=1\}=0 \qquad \text{and} \qquad P\{T=2\}=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}.$$

To time $t=3$ we get the paths

$$E_1 \;\rightarrow\; E_2 \;\rightarrow\; E_3 \;\rightarrow\; E_5, \qquad \text{sandsynligheld:} \quad \frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{4},$$

$$E_1 \;\rightarrow\; E_2 \;\rightarrow\; E_4 \;\rightarrow\; E_5, \qquad \text{sandsynligheid:} \quad \frac{3}{4}\cdot\frac{1}{3}\cdot 1=\frac{1}{4},$$

$$E_1 \;\rightarrow\; E_3 \;\rightarrow\; E_4 \;\rightarrow\; E_5, \qquad \text{sandsynligheid:} \quad \frac{1}{4}\cdot\frac{1}{2}\cdot 1=\frac{1}{8},$$

hence

$$P\{T=3\}=\frac{1}{4}+\frac{1}{4}+\frac{1}{8}=\frac{5}{8}.$$

Finally,

$$P\{T=4\}=\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{4},$$

hence, summing up,

$$P\{T=2\}=\frac{1}{8}, \qquad P\{T=3\}=\frac{5}{8}, \qquad P\{T=4\}=\frac{1}{4}.$$

5) The mean is

$$E\{T\}=2\cdot\frac{1}{8}+3\cdot\frac{5}{8}+4\cdot\frac{1}{4}=\frac{25}{8}.$$

Furthermore,

$$E\left\{T^2\right\}=4\cdot\frac{1}{8}+9\cdot\frac{5}{8}+16\cdot\frac{1}{4}=\frac{81}{8},$$

so

$$V\{T\}=\frac{81}{8}-\frac{625}{64}=\frac{648-625}{64}=\frac{23}{64}.$$

6) If $a=0$, then

$$\mathbf{P}=\begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^2=\begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^4=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then it is obvious that $\mathbf{P}^5=\mathbf{P}\,\mathbf{P}^4=\mathbf{P}^4$, because the sums of the rows of $\mathbf{P}$ are 1. We conclude that

$$\mathbf{P}^n=\mathbf{P}^4, \qquad \text{for } n\geq 4.$$

Example 3.44 *Given a Markov chain of 5 states E_1, E_2, E_3, E_4 and E_5, and transition probabilities*

$$\begin{aligned} p_{1,1} &= 1-4a, \qquad\qquad p_{1,2}=p_{1,3}=p_{1,4}=p_{1,5}=a, \\ p_{2,1} &= p_{2,2}=p_{3,2}=p_{3,3}=p_{4,3}=p_{4,4}=p_{5,4}=p_{5,5}=\frac{1}{2}, \\ P_{i,j} &= 0 \qquad \textit{otherwise}. \end{aligned}$$

Here a is a constant in the interval $\left[0,\frac{1}{4}\right]$.

1. *Find the stochastic matrix* $\mathbf{P}$.

2. *Find the values of a, for which the Markov chain is irreducible.*

3. *Find the values of a, for which the Markov chain is regular.*

4. *Find for every a the invariant probability vector.*

At time $t=0$ the process is in state E_2. Let T_2 denote the random variable, which indicates the time, when the process for the first time is in state E_1.

5. *Find $P\{T_2=k\}$, $k\in\mathbb{N}$, and compute the mean of T_2.*

Then put $a=\frac{1}{4}$ and assume that the process at time $t=0$ is in state E_1, and let T denote the time of its first return to E_1.

6. *Find the mean of T.*

1) The stochastic matrix is

$$\mathbf{P}=\begin{pmatrix} 1-4a & a & a & a & a \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

2) If $a=0$, then E_1 is clearly an absorbing state, so $\mathbf{P}$ is not irreducible for $a=0$.
When $0<a\le\frac{1}{4}$ we notice the oblique diagonal below the main diagonal. All elements of this diagonal are $\frac{1}{2}$, so we have always the flow

$$E_5 \;\rightarrow\; E_4 \;\rightarrow\; E_3 \;\rightarrow\; E_2 \;\rightarrow\; E_1.$$

Now $a>0$ implies that also $E_1 \;\rightarrow\; E_5$, so we get e.g.

$$E_1 \;\rightarrow\; E_5 \;\rightarrow\; E_4 \;\rightarrow\; E_3 \;\rightarrow\; E_2 \;\rightarrow\; E_1,$$

proving that $\mathbf{P}$ is irreducible for $0<a\le\frac{1}{4}$.

3) If $a\in\left]0,\frac{1}{4}\right]$, then $\mathbf{P}$ is irreducible. Since there exist positive elements in the diagonal (e.g. $p_{2,2}=\frac{1}{2}$), it follows that $\mathbf{P}$ is regular for every $a\in\left]0,\frac{1}{4}\right]$.

4) The system of equations $\mathbf{g}\,\mathbf{P} = \mathbf{g}$ is written

$$\begin{array}{rcccccccccl}
(1-4a)g_1 & = & \frac{1}{2}\,g_2 & & & & & & & = & g_1, \\
a\,g_1 & = & \frac{1}{2}\,g_2 & = & \frac{1}{2}\,g_3 & & & & & = & g_2, \\
a\,g_1 & & & + & \frac{1}{2}\,g_3 & + & \frac{1}{2}\,g_4 & & & = & g_3, \\
a\,g_1 & & & & & + & \frac{1}{2}\,g_4 & = & \frac{1}{2}\,g_5 & = & g_4, \\
a\,g_1 & & & & & & & + & \frac{1}{2}\,g_5 & = & g_5,
\end{array}$$

from which clearly

$$g_2 = 8a\,g_1 \qquad \text{and} \qquad g_5 = 2a\,g_1.$$

Then by insertion of these values,

$$g_3 = 6a\,g_1 \qquad \text{and} \qquad g_4 = 4a\,g_1.$$

Finally,

$$1 = \sum_{i=1}^{5} g_i = g_1(1+8a+6a+4a+2a) = (20a+1)g_1,$$

so

$$\mathbf{g} = \frac{1}{20a+1}\,(1, 8a, 6a, 4a, 2a).$$

In particular, $\mathbf{g} = (1,0,0,0,0)$ for $a = 0$.

5) We note that T_2 is geometrically distributed, so

$$P\{T_2 = k\} = \left(\frac{1}{2}\right)^k, \qquad k \in \mathbb{N},$$

and we have $E\{T_2\} = 2$.

6) Starting at E_1 we reach in the first step one of the states E_2, E_3, E_4 or E_5, all of probability $\frac{1}{4}$. From these states it takes in average 2, 4, 6 or 8 steps to get back to E_1. Consequently

$$E\{T\} = 1 + \frac{1}{4}\{2+4+6+8\} = 6.$$

Example 3.45 *Given a Markov chain of 5 states E_0, E_1, E_2, E_3 and E_4, and transition probabilities*

$$\begin{aligned} p_{0,1} &= p_{4,3} = 1, \qquad\qquad p_{3,2} = p_{2,1} = p_{1,0} = \frac{2}{3}, \\ p_{1,2} &= p_{3,4} = \frac{1}{3}, \qquad p_{2,3} = \frac{a}{3}, \qquad p_{2,4} = \frac{1-a}{3}, \\ p_{i,j} &= 0 \qquad\qquad \textit{otherwise.} \end{aligned}$$

Here a is a constant in the interval $[0,1]$

1. *Find the stochastic matrix* $\mathbf{P}$.

2. *Prove that the Markov chain is irreducible for every $a \in [0,1]$.*

3. *Find for $a \in [0,1]$ the invariant probability vector.*

4. *Prove that the Markov chain is regular for $a \in [0,1[$, but not for $a = 1$.*

We assume in the following that the process at time $t = 0$ is in state E_2.

5. *Find for $a = 1$ the probability that the process gets to the state E_4 before the state E_0.*
HINT: *One may apply results concerning the ruin problem.*

6. *Find for $a = 0$ the probability that the process gets to state E_4 before to state E_0-*

7. *Find for every $a \in [0,1]$ the probability that the process reaches state E_4 before state E_0.*

1) The stochastic matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{a}{3} & \frac{1-a}{3} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad a \in [0,1].$$

2) When $a \in]0,1]$, then we have the transitions

$$E_0 \longleftrightarrow E_1 \longleftrightarrow E_2 \longleftrightarrow E_3 \longleftrightarrow E_4,$$

and when $a \in [0,1[$, then we have the transitions

$$\begin{array}{ccc} E_0 \longleftrightarrow E_1 \longleftrightarrow E_2 & \longleftarrow & E_3, \\ & \searrow & \updownarrow \\ & & E_4, \end{array}$$

and it follows that the chain is irreducible.
One might e.g. split into the three cases

$$\begin{array}{lccc} a = 0: & E_0 \longleftrightarrow E_1 \longleftrightarrow E_2 & \longleftarrow & E_3, \\ & & \searrow & \updownarrow \\ & & & E_4, \\ a = 1: & E_0 \longleftrightarrow E_1 \longleftrightarrow E_2 & \longleftrightarrow & E_3, \\ & & & \updownarrow \\ & & & E_4, \\ 0 < a < 1: & E_0 \longleftrightarrow E_1 \longleftrightarrow E_2 & \longleftrightarrow & E_3, \\ & & \searrow & \updownarrow \\ & & & E_4, \end{array}$$

from which one also derives the irreducibility.

3) The system of equations $\mathbf{g}\,\mathbf{P} = \mathbf{g}$ is written

$$\left\{ \begin{array}{ccccccccc} & \frac{2}{3}\,g_1 & & & & & & = & g_0, \\ g_0 & & + & \frac{2}{3}\,g_2 & & & & = & g_1, \\ & \frac{1}{3}\,g_1 & & & + & \frac{2}{3}\,g_3 & & = & g_2, \\ & & & \frac{a}{3}\,g_2 & & & +g_4 & = & g_3, \\ & & & \frac{1-a}{3}\,g_3 & + & \frac{1}{3}\,g_3 & & = & g_4, \end{array} \right.$$

thus

$$\begin{aligned} g_1 &= \frac{3}{2}\,g_0, \\ g_2 &= \frac{3}{2}\,(g_1 - g_0) = \frac{3}{4}\,g_0, \\ g_3 &= \frac{3}{2}\left(g_2 - \frac{1}{3}\,g_1\right) = \frac{3}{8}\,g_0, \\ g_4 &= g_3 - \frac{a}{3}\,g_2 = \left(\frac{3}{8} - \frac{a}{4}\right) g_0. \end{aligned}$$

Recalling that

$$1 = \sum_{i=0}^{4} g_i = g_0 \left\{ 1 - \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{8} - \frac{a}{4} \right\} = g_0 \left\{ 4 - \frac{a}{4} \right\} = \frac{16-a}{4}\,g_0,$$

we conclude that

$$g_0 = \frac{4}{16-a},$$

and hence

$$\mathbf{g} = \frac{4}{16-a}\left(1, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{8} - \frac{a}{4}\right) = \frac{1}{32-2a}\,(8, 12, 6, 3, 3-2a).$$

4) If $a = 1$, then we have a random walk $\{E_0, E_1, E_2, E_3, E_4\}$. It follows from the diagram

$$E_0 \longleftrightarrow E_1 \longleftrightarrow E_2 \longleftrightarrow E_3 \longleftrightarrow E_4,$$

that it is only possible to get from E_0 back to E_0 through an *even* number of steps. Hence the Markov chain is periodic of period 2, and it is *not regular*.
If $a < 1$, then it was mentioned previously that we have the diagram

$$\begin{array}{ccccccc} E_0 & \longleftrightarrow & E_1 & \longleftrightarrow & E_2 & \longleftrightarrow & E_3, \\ & & & & & \searrow & \updownarrow \\ & & & & & & E_4. \end{array}$$

The chain $E_4 \longrightarrow E_3 \longrightarrow E_4$ shows that $p_{44}^{(2)} > 0$, and thus $p_{44}^{(2n)} > 0$.
The chain $E_4 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_4$ shows that $p_{44}^{(3)} > 0$. By a composition with the first chain it follows that $p_{44}^{(2n+1)} > 0$, thus $p_{44}^{(n)} > 0$ for $n > 2$.
Since $\mathbf{P}$ is irreducible, we conclude that $\mathbf{P}$ is regular.
Alternatively we compute $\mathbf{P}^n$, and it is easily seen that all elements of $\mathbf{P}^6$ are > 0, and the claim is proved-

5) We now return to the diagram for $a = 1$, i.e.

$$E_0 \longleftrightarrow E_1 \longleftrightarrow \mathbf{E_2} \longleftrightarrow E_3 \longleftrightarrow E_4,$$

where $\mathbf{E_2}$ is the initial state.
This can be interpreted as a ruin problem, where we shall get to E_4 before E_0.
The interpretation gives

$$N = 4, \qquad k = 2, \qquad p = \frac{1}{3}, \qquad q = \frac{2}{3},$$

so we are in case b) with $\frac{q}{p} = 2$, hence the probability of reaching E_0 before E_4 is

$$a_2 = \frac{2^2 - 2^4}{1 - 2^4} = \frac{12}{15} = \frac{4}{5}.$$

Thus the wanted probability (of the complementary event) is

$$b_2 = 1 - a_2 = \frac{1}{5}.$$

Alternatively. Let b_k denote the probability that we by starting in E_k reaches E_4 before E_0.
When we split the investigation according to what happens after one step, we get

$$b_k = \frac{1}{3}\,b_{k+1} + \frac{2}{3}\,b_{k-1}, \qquad k = 1, 2, 3, \quad \text{and} \quad b_0 = 0, \quad b_4 = 1,$$

thus

$$\frac{1}{3}\,b_{k+1} - b_k + \frac{2}{3}\,b_{k-1} = 0.$$

The complete solution is

$$b_k = c_1 \cdot 1 + c_2 \cdot 2^k.$$

We get

$$\begin{array}{ll} k = 0: \qquad & c_1 + c_2 = b_0 = 0, \\ k = 4: \qquad & c_1 + 16c_2 = b_4 = 1, \end{array}$$

hence

$$\left\{ \begin{array}{rcl} c_1 & = & -\dfrac{1}{15}, \\[2mm] c_2 & = & \dfrac{1}{15}, \end{array} \right.$$

and whence

$$b_k = \frac{1}{15}\left(2^k - 1\right), \qquad \text{and in particular} \qquad b_2 = \frac{3}{15} = \frac{1}{5}.$$

ALTERNATIVELY. We can reach E_4 without passing E_0 by either

go directly to E_4 (in two steps), probability $\frac{1}{9}$,

or

be back at E_2 after two steps,

$$p_{22}^{(2)} = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9},$$

and then go directly to E_4,

or

be back at E_2 after $2 \cdot 2$ steps etc..

Summing up we get the probability

$$\frac{1}{9} \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n = \frac{1}{9} \cdot \frac{1}{1 - \frac{4}{9}} = \frac{1}{9} \cdot \frac{9}{5} = \frac{1}{5}.$$

6) If $a = 0$, then we have the diagram

$$E_0 \longleftrightarrow E_1 \longleftrightarrow \mathbf{E_2} \longleftrightarrow E_4,$$

which again may be interpreted as a ruin problem. The probability of reaching E_0 before E_4 is for

$$N = 3, \qquad k = 2, \qquad p = \frac{1}{3}, \qquad q = \frac{2}{3}$$

given by

$$a_2 = \frac{2^2 - 2^3}{1 - 2^3} = \frac{4}{7}.$$

The wanted probability (of the complementary event) is

$$b_2 = 1 - \frac{4}{7} = \frac{3}{7}.$$

ALTERNATIVELY this question can also be solved by the two alternatives described in **5.**.

7) THE GENERAL CASE. Let c_k denote the probability when starting in E_k to reach E_4 before E_0. We get by the usual splitting

$$\begin{aligned} c_0 &= 0, \\ c_4 &= 1, \\ c_1 &= \frac{1}{3}\, c_2, \\ c_2 &= \frac{2}{3}\, c_1 + \frac{1-a}{3} + \frac{a}{3}\, c_3, \\ c_3 &= \frac{1}{3} + \frac{2}{3}\, c_2. \end{aligned}$$

When we insert the expressions of c_1 and c_3 into the equations of c_2, we get

$$c_2 = \frac{2}{3} \cdot \frac{1}{3}\, c_2 + \frac{1-a}{3} + \frac{a}{3}\left(\frac{1}{3} + \frac{2}{3}\, c_2\right) = \frac{2}{9}\, c_2 + \frac{2}{9}\, a\, c_2 + \frac{1}{3} - \frac{a}{3} + \frac{a}{9},$$

which is reduced to

$$c_2 \left\{\frac{7}{9} - \frac{2}{9}\, a\right\} = \frac{1}{3} - \frac{2a}{9} = \frac{3-2a}{9},$$

hence

$$c_2 = \frac{3-2a}{7-2a}.$$

CHECK. If $a = 0$, then $c_2 = \dfrac{3}{7}$, cf. **6.**, and if $a = 1$, then $c_2 = \dfrac{1}{5}$, cf. **5**.
ALTERNATIVELY we split according to when we last time were in state E_2,

$$p_{22}^{(2)} = \frac{2}{3} \cdot \frac{1}{3} + \frac{a}{3} \cdot \frac{2}{3} = \frac{2}{9}\,(1+a).$$

The probability of going from E_2 to E_4 in one or two steps is

$$\frac{a}{3}\cdot\frac{1}{3}+\frac{1-a}{3}=\frac{1}{3}-\frac{2}{9}\,a=\frac{3-2a}{9}.$$

The wanted probability is

$$\frac{3-2a}{9}\sum_{n=0}^{\infty}\left\{\frac{2}{9}\,(1+a)\right\}^n=\frac{3-2a}{9}\cdot\frac{1}{1-\frac{2}{9}-\frac{2a}{9}}=\frac{3-2a}{7-2a}.$$

Example 3.46 *A Markov chain of the states E_1, E_2, E_3 and E_4 has the stochastic matrix*

$$\mathbf{P}=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ a & 0 & 0 & 1-a \end{pmatrix},$$

where a is a constant in the interval $[0,1]$.

1. *Find the values of a, for which the Markov chain is irreducible, and the values of a, for which it is regular.*

2. *Find for every a the invariant probability vector.*

A particle moves between the states E_1, E_2, E_3 and E_4 of the given transition probabilities. At time $t=0$ the particle is in state E_1. Let T denote the random variable, which indicates the time, when the particle for the first time is in state E_4.

3. *Find $P\{T=2\}$.*
HINT: *Split the investigation according to whether the particle is passing through state E_2 or state E_3.*

4. *Find $P\{T=n\}$ for $n=2,\,3,\,4,\,\dots$.*

5. *Find the mean of T.*

6. *Explain why we have in the case $a=0$ that*

$$\mathbf{P}^n\rightarrow\begin{pmatrix} 0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\qquad \textit{for } n\rightarrow\infty.$$

1) If $a=0$, then E_4 is absorbing, and the Markov chain is neither irreducible nor regular for $a=0$.
If $a\in\,]0,1]$, then we have the transitions

$$E_1\longrightarrow E_2\longrightarrow E_3\longrightarrow E_4\longrightarrow E_1,$$

and the Markov chain is irreducibel.
Since $p_{3,3}=\frac{1}{2}>0$, it is also regular for $a\in\,]0,1]$.

2) The equations of the invariant probability vector are

$$
\begin{array}{ll}
g_1 = a\, g_4, & \text{thus } g_1 = a\, g_4, \\
g_2 = \frac{1}{2}\, g_1, & \text{thus } g_2 = \frac{1}{2}\, a\, g_4, \\
g_3 = \frac{1}{2}\, g_1 + \frac{1}{2}\, g_2 + \frac{1}{2}\, g_3, & \text{thus } g_3 = g_1 + g_2 = \frac{3}{2}\, a\, g_4,
\end{array}
$$

hence

$$1 = g_1 + g_2 + g_3 + g_4 = g_4 \left(a + \frac{1}{2}\, a + \frac{3}{2}\, a + 1 \right))(1 - 3a)g_4.$$

The invariant probability vector is

$$\mathbf{g} = \frac{1}{1+3a} \left(a, \frac{a}{2}, \frac{3a}{2}, 1 \right).$$

3) We derive from the matrix the tree

$$
\begin{array}{ccccc}
E_1 & \rightarrow & E_2 & \rightarrow & E_3 \\
 & \searrow & & \searrow & \\
 & & R_3 & \rightarrow & E_4 \\
 & & & \searrow & \\
 & & & & E_3
\end{array}
$$

where all arrows have the weight $\frac{1}{2}$, thus

$$P\{T=2\} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

4) We have at step $n=2$ only the possibilities E_3 and E_4. Hence

$$P\{T=n\} = \left(\frac{1}{2} \right)^{n-1} \qquad \text{for } n \geq 2.$$

5) The mean is

$$E\{T\} = \sum_{n=2}^{\infty} n \left(\frac{1}{2} \right)^{n-1} = \sum_{n=1}^{\infty} n \left(\frac{1}{2} \right)^{n-1} - 1 = \frac{1}{\left(1 - \frac{1}{2} \right)^2} - 1 = 3.$$

6) If $a=0$, then

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

thus

$$
\left\{
\begin{array}{l}
p_{i,1}^{(n+1)} = 0, \\[2ex]
p_{i,2}^{(n+1)} = \dfrac{1}{2}\, p_{2,i}^{(n)} = 0, \\[2ex]
p_{i,3}^{(n+1)} = \dfrac{1}{2} \left\{ p_{3,1}^{(n)} + p_{3,2}^{(n)} + p_{3,3}^{(n)} \right\} = \dfrac{1}{2}\, p_{3,3}^{(n)},
\end{array}
\right.
$$

and hence $p_{1,j}^{(n)} = p_{2,j}^{(n)} = 0$, and

$$p_{3,1}^{(n+1)} + p_{3,2}^{(n+1)} + p_{3,3}^{(n+1)} = \frac{1}{2}\left\{p_{3,1}^{(n)} + p_{3,2}^{(n)} + p_{3,3}^{(n)}\right\}.$$

This shows that

$$p_{i,j}^{(n)} \to 0 \qquad \text{for } n \to \infty, \quad \text{if} i = 1,\, 2,\, 3,\, 4 \text{ and } j = 1,\, 2,\, 3,$$

and we get

$$p_{i,4}^{(n)} \to 1 \qquad \text{for } n \to \infty,$$

and the claim is proved.

Example 3.47 *A Markov chain of the states E_1, E_2, E_3, E_4 and E_5 has the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 1-a \end{pmatrix},$$

where a is a constant in the interval $[0,1]$.

1. *Find the values of a, for which the Markov chain is irreducible.*

2. *Find the values of a, for which the Markov chain is regular.*

3. *Find for every a the invariant probability vector.*

Assume in the following that $a = \frac{1}{2}$.
At time $t = 0$ the process is in state E_1. Let T denote the random variable, which indicates the time, when the particle for the first time is in state E_5, and let U denote the random variable, which indicates the time, when the particle for the first time returns to the state E_1.

4. *Find $P\{T = k\}$ for $k = 2$, 3, 4, and the mean of T.*

5. *Find $P\{U = 3\}$ and $P\{Y = 4\}$.*

6. *Find $P\{U = k\}$ for $k = 5$, 6, ...*
HINT: *Split into the cases $T = 2$, $T = 3$ or $T = 4$.*

1) When $a = 0$, then E_5 is absorbing, and the Markov chain is not irreducible.
If $a \in]0,1]$, then we have the transitions

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5 \longrightarrow E_1,$$

and the Markov chain is irreducible for $a \in]0,1]$.

2) If $a \in]0,1[$, then the element of the diagonal $p_{5,5} = 1 - a > 0$, and since the Markov chain is irreducible for $a \in]0,1[$, it is also regular for $a \in]0,1[$.
If $a = 1$, then

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{P}^2 = \begin{pmatrix} 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}, \qquad \mathbf{P}^3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{12} & \frac{5}{12} \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

The Markov chain corresponding to $\mathbf{P}^3$ has the transitions

$$E_1 \longrightarrow E_5 \longrightarrow E_4 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1,$$

so it is irreducible. Since the element of the diagonal $p_{1,1}^{(3)} = \dfrac{1}{2} > 0$, it is also regular. Hence the original Markov chain is also regular for $a = 1$, so the Markov chain is regular for $a \in]0,1]$.

3) The equations of the invariant probability vector are

$$\begin{array}{ll} g_1 = a\, g_5, & \text{thus } g_1 = a\, g_5, \\ g_2 = \frac{1}{3}\, g_1, & \text{thus } g_2 = \frac{1}{3}\, a\, g_5, \\ g_3 = \frac{1}{3}\, g_1 + \frac{1}{2}\, g_2, & \text{thus } g_3 = \frac{3}{2}\, g_2 = \frac{1}{2}\, a\, g_5, \\ g_4 = \frac{1}{3}\, g_1 + \frac{1}{2}\, g_2 + \frac{1}{2}\, g_3, & \text{thus } g_4 = \frac{3}{2}\, g_3 = \frac{3}{4}\, a\, g_5. \end{array}$$

Hence

$$1 = g_1 + g_2 + g_3 + g_4 + g_5 = g_5 \left(a + \frac{1}{3}\,a + \frac{1}{2}\,a + \frac{3}{4}\,a + 1 \right) = \left(1 + \frac{31}{12}\,a \right) g_5,$$

and the invariant probability vector is

$$\mathbf{g} = \frac{12}{12+31a} \left(a, \frac{a}{3}, \frac{a}{2}, \frac{3}{4}\,a, 1 \right).$$

4) We have the tree

$$\begin{array}{ccccccccc}
t=0 & & t=1 & & t=2 & & t=3 & & t=4 \\
E_1 & \xrightarrow{\frac{1}{3}} & E_2 & \xrightarrow{\frac{1}{2}} & E_3 & \xrightarrow{\frac{1}{2}} & E_4 & \xrightarrow{1} & \mathbf{E}_5 \\
\| & \searrow \frac{1}{3} & & \searrow \frac{1}{2} & & \searrow \frac{1}{2} & & \nearrow \frac{1}{2} & \\
E_1 & & E_3 & \xrightarrow{\frac{1}{2}} & E_4 & \xrightarrow{1} & \mathbf{E}_5 & & \\
 & \searrow \frac{1}{3} & & \searrow \frac{1}{2} & & \nearrow \frac{1}{2} & & & \\
 & & E_4 & \xrightarrow{1} & \mathbf{E}_5 & & & &
\end{array}$$

When we compute $P\{T=2\}$ we have the paths

$$\begin{array}{ll}
E_1 \longrightarrow E_3 \longrightarrow E_5 & \quad \text{probability: } \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}, \\
E_1 \longrightarrow E_4 \longrightarrow E_5 & \quad \text{probability: } \frac{1}{3} \cdot 1 = \frac{1}{3},
\end{array}$$

thus

$$P\{T=2\} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

When we compute $P\{T=3\}$ we have the paths

$$\begin{array}{ll}
E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_5, & \quad \text{probability: } \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}, \\
E_1 \longrightarrow E_2 \longrightarrow E_4 \longrightarrow E_5, & \quad \text{probability: } \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}, \\
E_1 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5, & \quad \text{probability: } \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6},
\end{array}$$

hence

$$P\{T=3\} = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}.$$

When we compute $P\{T=4\}$ we shall only consider the path

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5, \qquad \text{probability: } \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

5) It is only possible to reach E_1 via E_5. Since $a = \frac{1}{2}$, we have

$$\begin{aligned}
P\{U=3\} &= P\{T=2\} \cdot P\left\{E_5 \to E_1\right\} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \\
P\{U=4\} &= P\{T=2\} \cdot P\left\{E_5 \to E_5 \to E_1\right\} + P\{T=3\} \cdot P\left\{E_5 \to E_1\right\} \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{5}{12} \cdot \frac{1}{2} = \frac{3+5}{24} = \frac{1}{3}.
\end{aligned}$$

6) When $k \geq 5$, we shall find how much "mass", which is collected in total in E_5 at $t = 4$. This mass of probability is

$$P\{T=2\} \cdot \frac{1}{2} \cdot \frac{1}{2} + P\{T=3\} \cdot \frac{1}{2} + P\{T=4\} = \frac{1}{8} + \frac{5}{24} + \frac{1}{12} = \frac{10}{24} = \frac{5}{12}.$$

In every one of the following steps, half of it remains in E_5, and the other half is transferred to E_1, so

$$P\{U=k\} = \frac{5}{12} \cdot \left(\frac{1}{2}\right)^{k-4} \qquad \text{for } k \geq 5.$$

Example 3.48 *A Markov chain of 2 states E_1 and E_2 has the stochastic matrix*

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}.$$

1. *Prove that* $\mathbf{Q}$ *is regular, and find the invariant probability vector.*

Another Markov chain of 4 states E_1, E_2, E_3 and E_4 has the stochastic matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} & \frac{2}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}.$$

2. *Prove that* $\mathbf{P}$ *is not irreducible, and find its closed subsets.*

3. *Prove for every initial distribution*

$$\mathbf{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, p_4^{(0)}\right)$$

that

$$p_3^{(n)} + p_4^{(n)} = \frac{4}{5}\left\{p_3^{(n-1)} + p_4^{(n-1)}\right\}, \qquad n \in \mathbb{N},$$

and then prove that

$$\lim_{n\to\infty} p_3^{(n)} = \lim_{n\to\infty} p_4^{(n)} = 0.$$

4. *Show that* $\lim_{n\to\infty} \mathbf{p}^{(n)}$ *exists and find the limit vector.*

5. *At time $t = 0$ the process is in state E_3. Find for every $n \in \mathbb{N}$ the probability that the process for $t = n$ for the first time is in state E_1* without *previously having been in state E_2.*

1) All elements of $\mathbf{Q}$ are > 0, so the Markov chain is regular. The equations of the invariant probability vector are

$$g_1 = \tfrac{1}{5}\, g_1 + \tfrac{3}{5}\, g_2, \qquad \text{thus } g_2 = \tfrac{4}{3}\, g_1,$$

$$1 = g_1 + g_2 = \left(1 + \tfrac{4}{3}\right) g_1 = \tfrac{7}{3}\, g_1,$$

so

$$\mathbf{g} = \left(\frac{3}{7}, \frac{4}{7}\right).$$

2) Clearly, $\{E_1, E_2\}$ is a closed subset, and there is no other proper closed subset. However, since there exists a proper closed subset, the Markov chain is not irreducible.

3) It follows immediately that

$$p_3^{(n)} = \frac{2}{5}\,p_3^{(n-1)} + \frac{2}{5}\,p_4^{(n-1)},$$

and

$$p_4^{(n)} = \frac{2}{5}\,p_3^{(n-1)} + \frac{2}{5}\,p_4^{(n-1)} = p_3^{(n)},$$

hence

$$p_3^{(n)} + p_4^{(n)} = \frac{4}{5}\left\{p_3^{(n-1)} + p_4^{(n-1)}\right\}, \qquad n \in \mathbb{N}.$$

Then by iteration,

$$p_3^{(n)} + p_4^{(n)} = \left(\frac{4}{5}\right)^n \left\{p_3^{(0)} + p_4^{(0)}\right\} \to 0 \qquad \text{for } n \to \infty,$$

so

$$0 = \lim_{n\to\infty}\left\{p_3^{(n)} + p_4^{(n)}\right\} = 2\lim_{n\to\infty} p_3^{(n)} = 2\lim_{n\to\infty} p_4^{(n)}.$$

4) It follows from **3.** that the latter two coordinates tend towards 0 for $n \to \infty$. The former two coordinates are governed by the matrix $\mathbf{Q}$, so we conclude from **1.** that

$$\lim_{n\to\infty} \mathbf{p}^{(n)} = \left(\frac{3}{7}, \frac{4}{7}, 0, 0\right).$$

5) The two states E_3 and E_4 occur in every step of the same weight, thus $\frac{1}{2}\cdot\frac{1}{5} = \frac{1}{10}$ of the total weight goes to E_1, and $\frac{1}{2}\cdot\frac{1}{5} = \frac{1}{10}$ of the total weight goes to E_2. Then we have the diagram

$$\begin{array}{ccccccc}
 & & & & E_2 & & \\
 & & & \frac{1}{10}\nearrow & & & \nearrow \\
E_3 & \xrightarrow{\frac{4}{5}} & \{E_3, E_4\} & \xrightarrow{\frac{4}{5}} & \{E_3, E_4\} & & \to, \\
 & \frac{1}{5}\searrow & & \frac{1}{10}\searrow & & & \searrow \\
 & & \mathbf{E}_1 & & \mathbf{E}_1 & &
\end{array}$$

hence

$$P\{T=1\} = \frac{1}{5}, \quad P\{T=2\} = \frac{4}{5}\cdot\frac{1}{10}, \quad \dots, \quad P\{T=n\} = \frac{1}{10}\left(\frac{4}{5}\right)^{n-1},$$

which can also be written

$$P\{T=n\} = \frac{2^{2n-3}}{5^n} = \frac{1}{8}\left(\frac{4}{5}\right)^n \qquad \text{for } n \geq 2,$$

together with

$$P\{T=1\} = \frac{1}{5} \qquad \text{for } n = 1.$$

Example 3.49 *A Markov chain of the states E_1, E_2, E_3, E_4 and E_5 has the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 1-a \end{pmatrix},$$

where a is a constant in the interval $[0,1]$.

1. *Find the values of a, for which the Markov chain is irreducible.*

2. *Find the values of a, for which the Markov chain is regular.*

3. *Find for every a the invariant probability vector.*

The process is at time $t = 0$ in the state E_1. Let T denote the random variable, which indicates the time when the process for the first time is in state E_5.

4. *Find $P\{T = k\}$ for $k = 1$, 2, 3, 4, and then the mean of T.*

Assume that $a > 0$ and that the process at time $t = 0$ is in state E_1. Let U denote the random variable, which indicates the time when the process for the first time returns to E_1.

5. *Find the mean of U.*

1) If $a = 0$, then E_5 is absorbing, and the Markov chain is not irreducible.
If $a \in]0,1]$, then we have the transitions

$$E_5 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5,$$

and the Markov chain is irreducible for $a \in]0,1]$.

2) We have proved for $a \in]0,1[$ that the Markov chain is irreducible, and since the element of the diagonal $p_{5,5} = 1 - a > 0$, we conclude that the Markov chain is regular for every $a \in]0,1[$.
If $a = 1$, and we let $\star$ denote elements > 0, then

$$\mathbf{P}^2 = \begin{pmatrix} 0 & \star & \star & 0 & \star \\ 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \star & \star & 0 & \star \\ 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \star & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ \star & 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 & 0 \\ 0 & \star & \star & 0 & \star \end{pmatrix}.$$

The Markov chain corresponding to $\mathbf{P}^2$ has the transitions

$$E_1 \longrightarrow E_3 \longrightarrow E_5 \longrightarrow E_2 \longrightarrow E_4 \longrightarrow E_1,$$

from which follows that it is irreducible. Furthermore, $\mathbf{P}^2$ has elements in its diagonal which are > 0, hence it is also regular. This implies again that the original Markov chain is regular for $a \in]0,1]$.

3) The equations of the invariant probability vector are

$$
\begin{aligned}
g_1 &= a\,g_5, && \text{thus } g_1 = a\,g_5, \\
g_2 &= \frac{1}{3}\,g_1, && \text{thus } g_2 = \frac{1}{3}\,a\,g_5 \\
g_3 &= \frac{1}{3}\,g_1 + g_2, && \text{thus } g_3 = \frac{2}{3}\,a\,g_5, \\
g_4 &= \frac{1}{2}\,g_3, && \text{thus } g_4 = \frac{1}{3}\,a\,g_5,
\end{aligned}
$$

where the latter equation is used as a check:

$$g_5 = \frac{1}{3}\,g_1 + \frac{1}{2}\,g_3 + g_4 + (1-a)g_5.$$

We have furthermore the condition

$$1 = g_1 + g_2 + g_3 + g_4 + g_5 = g_5\left(a + \frac{1}{3}\,a + \frac{2}{3}\,a + \frac{1}{3}\,a + 1\right) = g_5\left(\frac{7}{3}\,a + 1\right),$$

hence $g_5 = \dfrac{3}{7a+3}$, and

$$\mathbf{g} = \frac{1}{7a+3}\,(3a, a, 2a, a, 3).$$

4) If to $t = 0$ we start in E_1, then we have the tree

$$
\begin{array}{ccccccccc}
E_1 & \xrightarrow{\frac{1}{3}} & E_2 & \xrightarrow{1} & E_3 & \xrightarrow{\frac{1}{2}} & E_4 & \xrightarrow{1} & \mathbf{E}_5 \\
\| & \searrow \frac{1}{3} & & & & \searrow \frac{1}{2} & & & \\
E_1 & & E_3 & \xrightarrow{\frac{1}{2}} & E_4 & \xrightarrow{1} & \mathbf{E}_5 & & \\
 & \searrow \frac{1}{3} & & \searrow \frac{1}{2} & & & & & \\
 & & \mathbf{E}_5 & & \mathbf{E}_5 & & & & \\
 & & t=1 & & t=2 & & t=3 & & t=4
\end{array}
$$

From this we infer that

$$
\begin{aligned}
P\{T=1\} &= \frac{1}{3}, \\
P\{T=2\} &= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}, \\
P\{T=3\} &= \frac{1}{3} \cdot \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 \cdot \frac{1}{2} = \frac{1}{3}, \\
P\{T=4\} &= \frac{1}{3} \cdot 1 \cdot \frac{1}{2} = \frac{1}{6}.
\end{aligned}
$$

The mean of T is

$$E\{T\} = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{6} = \frac{1}{3} + \frac{1}{3} + 1 + \frac{2}{3} = \frac{7}{3}.$$

5) We first notice that we can only reach E_1 via E_5.
If the process is in E_5, then we have the probability a of in the next step to be in E_1, and probability $1-a$ of to remain in E_5. This gives a geometric distribution $\mathrm{Pas}(1,a)$ of mean $\frac{1}{a}$. Then finally,

$$E\{U\} = E\{T\} + \frac{1}{a} = \frac{7}{3} + \frac{1}{a}.$$

Example 3.50 *A Markov chain of states E_1, E_2, E_3 and E_4 has the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & a & 0 & 1-a \end{pmatrix},$$

where a is a constant in the interval $[0,1]$.

1. *Find the values of a, for which the Markov chain is irreducible.*

2. *Find the values of a, for which the Markov chain is regular.*

3. *Find for every a the invariant probability vector.*

At $t = 0$ the process is in state E_2. Let T denote the random variable, which indicates the time, when the process for the first time is in state E_4.

4. *Find the probabilities $P\{T = 1\}$ and $P\{T = 2\}$.*

5. *Prove that for every $k \in \mathbb{N}_0$,*

$$P\{T = 2k+1\} = P\{T = 2k+2\},$$

and find these probabilities.

6. *Find the mean of the random variable T.*

1) By analyzing the stochastic matrix we obtain the diagram

$$\begin{array}{ccccccc} E_1 & \longleftrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & E_4 \ \circlearrowright 1-a \\ & & \downarrow & & & & \downarrow a \\ & & E_4 & & & & E_2 \end{array}$$

If $a = 0$, then the Markov chain is absorbing with E_4 as absorbing state.
If $a > 0$, then clearly the Markov chain is irreducible.

2) The Markov chain is not irreducible, and therefore not regular either for $a = 0$.
If $a > 0$, then $p_{22}^{(2)} > 0$ and $p_{22}^{(3)} > 0$, and the Markov chain is regular.
Alternatively one may prove that all elements of $\mathbf{P}^5$ are > 0.
Alternatively we have for $0 < a < 1$ that $p_{44} > 0$, and we shall only investigate the case $a = 1$ separately.

3) Ligningerne $\mathbf{g}\,\mathbf{P} = \mathbf{g}$ for sandsynlighedsvektoren skrives

$$\frac{1}{2}\,g_2 = g_1, \qquad \text{thus } g_2 = 2g_1,$$

$$g_1 + a\,g_4 = g_2, \qquad \text{thus } a\,g_4 = g_1,$$

$$\frac{1}{4}\,g_2 = g_3,$$
$$g_3 = \frac{1}{2}\,g_1,$$

$$\frac{1}{4}\,g_2 + g_3 + (1-a)g_4 = g_4.$$

If $a = 0$, then $g_1 = 0$, hence

$$\mathbf{g} = (0, 0, 0, 1).$$

If $a \neq 0$, then $g_4 = \frac{1}{a}\, g_1$, hence

$$1 = \sum_{i=1}^{4} g_i = g_1 \left\{1 + 2 + \frac{1}{2} + \frac{1}{a}\right\} = \frac{7a+2}{2a}\, g_1,$$

from which

$$g_1 = \frac{2a}{7a+2},$$

and

$$\mathbf{g} = \frac{1}{7a+2}\,(2a, 4a, a, 2).$$

4) The event $\{T = 1\}$ can only occur by the transition $E_2 \longrightarrow E_4$, so

$$P\{T = 1\} = \frac{1}{4}.$$

The event $\{T = 2\}$ can only occur by the process

$$E_2 \longrightarrow E_3 \longrightarrow E_4,$$

thus

$$P\{T = 2\} = \frac{1}{4}.$$

5) We can only obtain E_4 to time $2k+1$ by repeating $E_2 \longrightarrow E_1 \longrightarrow E_2$ in total k times, follows by $E_2 \longrightarrow E_4$.
Analogously for $2k+2$, with the modicication that we at last replace $E_2 \longrightarrow E_4$ by $E_2 \longrightarrow E_3 \longrightarrow E_4$. It follows (cf. **4.**) that

$$P\{T = 2k+1\} = \frac{1}{4} \cdot \left(\frac{1}{2}\right)^k = P\{T = 2k+2\}, \qquad k \in \mathbb{N}.$$

When we compare with **4.**, we see that this is also true for $k = 0$.

6) Using the results of **5.** it follows by straightforward computations that

$$
\begin{aligned}
E\{T\} &= \sum_{n=1}^{\infty} n\, P\{T_n\} = \sum_{k=0}^{\infty} ((2k+1)P\{T=2k+1\} + (2k+2)P\{T=2k+2\}) \\
&= \sum_{k=0}^{\infty} \{(2k+1)+(2k+2)\} \cdot \frac{1}{4}\left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} \frac{1}{4}(4k+3)\left(\frac{1}{2}\right)^k \\
&= \sum_{k=0}^{\infty} \left\{(k+1) - \frac{1}{4}\right\}\left(\frac{1}{2}\right)^k = \sum_{\ell=1}^{\infty} \ell \cdot \left(\frac{1}{2}\right)^{\ell-1} - \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \\
&= \left[\frac{d}{dx}\sum_{\ell=0}^{\infty} x^{\ell}\right]_{x=\frac{1}{2}} - \frac{1}{4}\cdot 2 = \left[\frac{d}{dx}\left(\frac{1}{1-x}\right)\right]_{x=\frac{1}{2}} - \frac{1}{2} \\
&= \frac{1}{\left(1-\frac{1}{2}\right)^2} - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}.
\end{aligned}
$$

Example 3.51 *A Markov chain of states E_1, E_2, E_3, E_4 and E_5 has the stochastic matrix*

$$\mathbf{P} = \begin{pmatrix} a & 0 & 0 & 0 & 1-a \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

where a is a constant in the interval $[0,1]$.

1. *Find the values of a, for which the Markov chain is irreducible.*

2. *Find the values of a, for which the Markov chain is regular.*

3. *Find for every a the invariant probability vector.*

At time $t = 0$ the process is in state E_5. Let T denote the random variable, which indicates the time, when the process for the first time is in state E_1

4. *Find $P\{T = k\}$ for $k = 2$, 3, 4, and then the mean and variance of T.*

Then assume that $a < 1$, and that the process at $t = 0$ is in state E_5.
Let U denote the random variable, which indicates the time when the process for the first time returns to E_5.

5. *Find the mean of U.*

1) If $a = 1$, then E_1 is absorbing, and the Markov chain is not irreducible.
If $a \in [0,1[$, then we have the transitions

$$E_1 \longrightarrow E_5 \longrightarrow E_4 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1,$$

and the Markov chain is irreducible for $a \in [0,1[$.

2) If $a \in [0,1[$, then e.g.

$$E_1 \longrightarrow E_5 \longrightarrow E_4 \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1, \qquad 5 \text{ steps},$$

and

$$E_1 \longrightarrow E_5 \longrightarrow E_4 \longrightarrow E_3 \longrightarrow E_1, \qquad 4 \text{ steps}.$$

The largest common divisor for 4 and 5 is 1, so the Markov chain is regular.

ALTERNATIVELY, let $\star$ denote that $p_{i,j} > 0$, and let A denote that a is a factor (so $A = 0$ for $a = 0$, and $A = \star$ for $a \in]0,1[$). Then successively,

$$\mathbf{P} = \begin{pmatrix} A & 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 0 \\ 0 & \star & 0 & \star & 0 \end{pmatrix}, \qquad \mathbf{P}^2 = \begin{pmatrix} A & \star & 0 & \star & A \\ A & 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ \star & \star & 0 & 0 & 0 \\ \star & 0 & \star & 0 & 0 \end{pmatrix},$$

$$\mathbf{P}^4 = \begin{pmatrix} \star & \star & A & A & \star \\ \star & A & \star & A & A \\ \star & \star & \star & \star & A \\ A & \star & 0 & \star & \star \\ \star & \star & 0 & \star & \star \end{pmatrix}, \qquad \mathbf{P}^8 = \begin{pmatrix} \star & \star & \star & \star & \star \\ \star & \star & A & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \end{pmatrix},$$

and we see that all elements of e.g. $\mathbf{P}^{16}$ are > 0, so the Markov chain is regular for $a \in [0,1[$.

3) The equations of the invariant probability vector are

$$\begin{aligned} p_1 &= a\,p_1 + p_2 + \frac{1}{2}\,p_3, \\ p_2 &= \frac{1}{2}\,p_3 + \frac{1}{2}\,p_5, \\ p_3 &= p_4, \\ p_4 &= \frac{1}{2}\,p_5, \\ p_5 &= (1-a)p_1. \end{aligned}$$

Then we get, expressed by p_1,

$$\begin{aligned} p_5 &= (1-a)p_1, \\ p_3 &= p_4 = \frac{1}{2}\,p_5 = \frac{1}{2}\,(1-a)p_1, \\ p_2 &= \frac{1}{2}\,p_3 + \frac{1}{2}\,p_5 = \frac{3}{4}\,(1-a)p_1. \end{aligned}$$

thus

$$1 = p_1+p_2+p_3+p_4+p_5 = p_1 + \left(\frac{3}{4}+\frac{1}{2}+\frac{1}{2}+1\right)(1-a)p_1 = p_1\left\{1+\frac{11}{4}\,(1-a)\right\},$$

hence

$$p_1 = \frac{4}{15-11a}, \qquad p_2 = \frac{3(1-a)}{15-11a}, \qquad p_3 = p_4 = \frac{2(1-a)}{15-11a}, \qquad p_5 = \frac{4(1-a)}{15-11a},$$

and the invariant probability vector is

$$\mathbf{p} = \frac{1}{15-11a}\,(4, 3(1-a), 2(1-a), 2(1-a), 4(1-a)).$$

4) We have here the tree

$$\begin{array}{ccccccccc} E_5 & \xrightarrow{\frac{1}{2}} & E_2 & \xrightarrow{1} & E_1 & & E_2 & \xrightarrow{1} & E_1, \\ & \tfrac{1}{2}\searrow & & & & \tfrac{1}{2}\nearrow & & & \\ & & E_4 & \xrightarrow{1} & E_3 & \xrightarrow{\frac{1}{2}} & E_1 & & \end{array}$$

from which we conclude that

$$\begin{aligned} P\{T=2\} &= \frac{1}{2}\cdot 1 = \frac{1}{2}, \\ P\{T=3\} &= \frac{1}{2}\cdot 1\cdot\frac{1}{2} = \frac{1}{4}, \\ P\{T=4\} &= \frac{1}{2}\cdot 1\cdot\frac{1}{2}\cdot 1 = \frac{1}{4}. \end{aligned}$$

Notice that the sum is 1, so there is no other possibility.
The mean is

$$E\{T\} = 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{4} = \frac{11}{4}.$$

It follows from

$$E\left\{T^2\right\} = 4\cdot\frac{1}{2} + 9\cdot\frac{1}{4} + 16\cdot\frac{1}{4} = \frac{33}{4},$$

that the variance is

$$V\{T\} = \frac{33}{4} - \left(\frac{11}{4}\right)^2 = \frac{132-121}{16} = \frac{11}{16}.$$

5) We can only reach E_5 via E_1, and since

$$P\left\{E_1 \longrightarrow E_5 \text{ i } k \text{ steps}\right\} = P\left\{E_1 \xrightarrow{a} E_1 \xrightarrow{a} E_1 \xrightarrow{a} \cdots \xrightarrow{a} E_1 \xrightarrow{1-a} E_5\right\} = a^{k-1}(1-a),$$

we get

$$\begin{aligned} E\{U\} &= E\{T\} + \sum_{k=1}^{\infty} k\, a^{k-1}(1-a) = \frac{11}{4} + (1-a)\cdot\frac{1}{(1-a)^2} \\ &= \frac{11}{4} + \frac{1}{1-a} = \frac{15-11a}{4(1-a)} = \frac{1}{p_5}. \end{aligned}$$

Example 3.52 *Given a Markov chain of 5 states E_1, E_2, E_3, E_4 and E_5 and transition probabilities*

$$p_{1,1} = p_{2,2} = p_{3,3} = \frac{2}{3}, \qquad p_{1,2} = p_{2,3} = \frac{1}{3},$$

$$p_{3,4} = \frac{a}{3}, \qquad p_{3,5} = \frac{1-a}{3}, \qquad p_{4,5} = p_{5,1} = 1,$$

and $p_{i,j} = 0$ oherwise.
Here a is a constant in the interval $[0,1]$.

1. *Find the stochastic matrix $\mathbf{P}$.*

2. *Find the values of a, for which the Markov chain is irreducible.*

3. *Find the values of a, for which the Markov chain is regular.*

4. *Find for every a the invariant probability vector.*

We assume that the process at time $t = 0$ is in state E_1. Let T denote the random variable, which indicates the time, when the process for the first time is in state E_2.

5. *Find the probabilities $P\{T = k\}$, $k \in \mathbb{N}$, and then the mean $E\{T\}$.*

Then assume instead that the process at time $t = 0$ is in state E_3. Let U denote the random variable, which indicates the time, when the process for the first time is in state E_5.

6. *Find the mean $E\{U\}$.*

1) If $a \in [0,1]$, then

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{a}{3} & \frac{1-a}{3} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2) We shall consider three cases:

$$(a) \quad a = 0, \qquad (b) \quad a = 1, \qquad (c) \quad 0 < a < 1.$$

a) If $a = 0$, then we get the matrix

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We notice that the fourth column is the zero column, so we can never get to E_4 from any E_i, hence the Markov chain is not irreducible for $a = 0$.

b) If $a = 1$, then we get the matrix

$$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have in particular the diagram

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5 \longrightarrow E_1,$$

and the Markov chain is irreducible for $a = 1$.

c) If $0 < a < 1$, then we have in particular the diagram

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5 \longrightarrow E_1,$$

and the Markov chain is irreducible for $0 < a < 1$.

Summing up, the Markov chain is irreducible for $0 < a \le 1$.

3) Since the Markov chain is irreducible for $0 < a \le 1$, and $p_{1,1} = \dfrac{2}{3} > 0$, it follows that the Markov chain is also regular for $0 < a \le 1$.

4) The equation of the invariant probability vector $\mathbf{g}$ is

$$\mathbf{g}\,\mathbf{P} = \mathbf{g},$$

which we expand as

$$\begin{cases} p_1 = \frac{2}{3}\,p_1 + p_5, \\ p_2 = \frac{1}{3}\,p_1 + \frac{2}{3}\,p_2, \\ p_3 = \frac{1}{3}\,p_2 + \frac{2}{3}\,p_3, \\ p_4 = \frac{a}{3}\,p_3, \\ p_5 = \frac{1-a}{3}\,p_3 + p_4, \end{cases} \qquad \text{thus} \qquad \begin{cases} p_1 = 3p_5, \\ p_2 = p_1, \\ p_3 = p_2, \\ p_4 = \frac{a}{3}\,p_3, \\ p_5 = \frac{1-a}{3}\,p_3 + p_4. \end{cases}$$

When $p_1, \dots, p_4$ are expressed by p_5, then

$$p_1 = p_2 = p_3 = 3p_5 \qquad \text{and} \qquad p_4 = \frac{a}{3}\,p_3 = a\,p_5,$$

thus

$$\mathbf{g} = p_5\,(3, 3, 3, a, 1)$$

where we have the condition

$$1 = p_1 + p_2 + p_3 + p_4 + p_5 = p_5\,(3+3+3+a+1) = (10+a)p_5,$$

so

$$p_5 = \frac{1}{10+a},$$

and

$$\mathbf{g} = \frac{1}{10+a}\,(3,3,3,a,1).$$

5) We start for $t=0$ at E_1, corresponding to the diagram

$$\begin{array}{ccccccccccc}
E_1 & \xrightarrow{\frac{2}{3}} & E_1 & \xrightarrow{\frac{2}{3}} & E_1 & \xrightarrow{\frac{2}{3}} & E_1 & \xrightarrow{\frac{2}{3}} & E_1 & \longrightarrow & \cdots \\
 & \searrow \frac{1}{3} & & \searrow \frac{1}{3} & & \searrow \frac{1}{3} & & \searrow \frac{1}{3} & & \searrow & \\
 & & E_2 & & E_2 & & E_2 & & E_2 & & \cdots \\
 & & & & & & & & & & \\
t=0 & & t=1 & & t=2 & & t=3 & & t=4 & & \cdots
\end{array}$$

It follows that

$$P\{T=k\} = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}, \qquad k \in \mathbb{N},$$

thus T is geometrically distributed, $T \in \operatorname{Pas}\left(1,\frac{1}{3}\right)$, $p=\frac{1}{3}$.

The mean is

$$E\{T\} = \frac{1}{p} = 3.$$

Alternatively,

$$E\{T\} = \sum_{k=1}^{\infty} k\,P\{T=k\} = \frac{1}{3}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k-1} = \frac{1}{3}\cdot\frac{1}{\left(1-\frac{2}{3}\right)^2} = 3.$$

6) Here we get the diagram

$$\begin{array}{cccccccccc}
 & & \mathbf{E}_5 & & E_4 & \xrightarrow{1} & \mathbf{E}_5 & & E_4 & \longrightarrow \\
 & \frac{1-a}{3}\nearrow & & \frac{a}{3}\nearrow & & \frac{1-a}{3}\nearrow & & \frac{a}{3}\nearrow & & \nearrow \\
E_3 & \xrightarrow{\frac{2}{3}} & E_3 & \xrightarrow{\frac{2}{3}} & E_3 & \xrightarrow{\frac{2}{3}} & E_3 & \xrightarrow{\frac{2}{3}} & E_3 & \longrightarrow \\
 & \searrow \frac{2}{3} & & \searrow \frac{1-a}{3} & & \searrow \frac{a}{3} & & \searrow \frac{1-a}{3} & & \searrow \\
 & & E_4 & \xrightarrow{1} & \mathbf{E}_5 & & E_4 & \xrightarrow{1} & \mathbf{E}_5 &
\end{array}$$

A simple counting gives

$$\begin{aligned}
P\{U=1\} &= \frac{1-a}{3}, \\
P\{U=2\} &= P\{E_3 \to E_3\}\cdot P\{E_3 \to E_5\} + P\{E_3 \to E_4\}\cdot P\{E_4 \to E_5\} \\
&= \frac{2}{3}\cdot\frac{1-a}{3} + \frac{a}{3}\cdot 1 = \frac{2-2a}{9} + \frac{a}{3} = \frac{2+a}{9}.
\end{aligned}$$

Now $P\{U=3\}$ is obtained by the paths

$$
\begin{array}{lll}
E_3 \xrightarrow{\frac{2}{3}} E_3 \xrightarrow{\frac{2}{3}} E_3 \xrightarrow{\frac{1-a}{3}} E_5, & \text{probability} & \dfrac{2}{3}\cdot\dfrac{2}{3}\cdot\dfrac{1-a}{3} = \dfrac{2}{3}\cdot\dfrac{2-2a}{9}, \\
E_3 \xrightarrow{\frac{2}{3}} E_3 \xrightarrow{\frac{a}{3}} E_4 \xrightarrow{1} E_5, & \text{probability} & \dfrac{2}{3}\cdot\dfrac{a}{3} = \dfrac{2}{3}\cdot\dfrac{3a}{9},
\end{array}
$$

so

$$P\{U=3\} = \frac{2}{3}\cdot\frac{2+a}{9} = \left(\frac{2}{3}\right)^1 \cdot P\{U=2\}.$$

Then repeat the pattern

$$P\{U=4\} = P\left\{E_3 \to E_3\right\}\cdot P\{U=3\} = \left(\frac{2}{3}\right)^2 P\{U=2\},$$

and in general

$$P\{U=k\} = \left(\frac{2}{3}\right)^{k-2} P\{U=2\} = \frac{2+a}{9}\left(\frac{2}{3}\right)^{k-2} \qquad \text{for } k \geq 2.$$

The mean is

$$
\begin{aligned}
E\{U\} &= \sum_{k=1}^{\infty} k\, P\{U=k\} = \frac{1-a}{3}\cdot 1 + \frac{2+a}{9}\sum_{k=2}^{\infty} k\left(\frac{2}{3}\right)^{k-2} \\
&= \frac{1-a}{3} + \frac{2+a}{9}\sum_{k=1}^{\infty}(k+1)\left(\frac{2}{3}\right)^{k-1} \\
&= \frac{1-a}{3} + \frac{2+a}{9}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} + \frac{2+a}{9}\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^{k+1} \\
&= \frac{1-a}{3} + \frac{2+a}{9}\cdot\frac{1}{1-\frac{2}{3}} + \frac{2+a}{9}\cdot\frac{1}{\left(1-\frac{2}{3}\right)^2} \\
&= \frac{1-a}{3} + \frac{2+a}{3} + (2+a) = 3+a.
\end{aligned}
$$

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