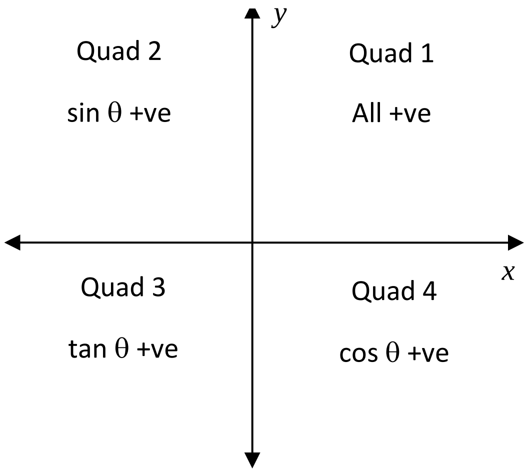
GRADE 11 & 12
TRIGONOMETRY NOTES

DEFINITIONS

The Cartesian plane is divided into four quadrants.

The value of *r* is always positive, while the values of *x* and *y* depend on the quadrant in which they are positioned.

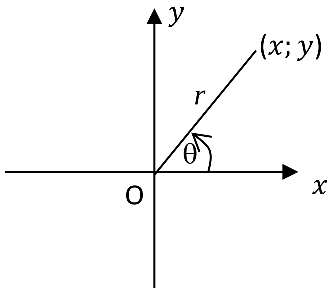


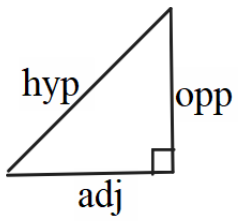
N.B. THE RATIO NAMES CANNOT BE WRITTEN WITHOUT AN ANGLE.

sine θ = $\frac{y}{r} = \frac{\text{opp}}{\text{hyp}}$

cosine θ = $\frac{x}{r} = \frac{\text{adj}}{\text{hyp}}$

tangent θ = $\frac{y}{x} = \frac{\text{opp}}{\text{adj}}$





We do not need to use cosecant θ, secant θ and cotangent θ in grade 11 and 12.

N.B. ALWAYS DRAW A SKETCH WHEN WORKING WITH DEFINITIONS.

Example:

If 13.sin θ = 5 and θ ∈ (90°; 270°) find, without the use of a calculator:

a. tan θ

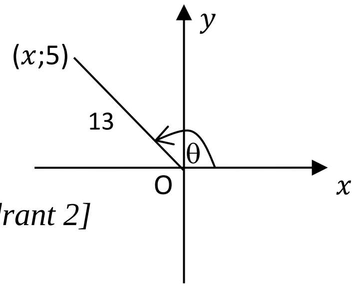
b. sin θ + cos θ

Solution:

$\sin\theta = \frac{5}{13}$ *Must start with the ratio on its own so need to rearrange.*

sin θ is positive ∴ in quadrants 1 and 2, but θ ∈ (90°; 270°) ∴ only quadrant 2.

$\sin\theta = \frac{5}{13} = \frac{y}{r}$



∴ by Pythagoras Thm $x^2 = 169 - 25$

$x^2 = 144$

$x = \pm 12$

∴ $x = -12$ *[x negative in quadrant 2]*

a. $\tan\theta = \frac{5}{-12}$

b. $\sin\theta + \cos\theta = \frac{5}{13} + \frac{-12}{13}$

$= -\frac{7}{13}$

Example:

Given 5.cos A = -4 and A > 180°.

Determine, with the aid of a sketch and WITHOUT the use of a calculator, the value of tan A.

5.cos A = -4

$\cos A = \frac{-4}{5}$ *Isolate the ratio*

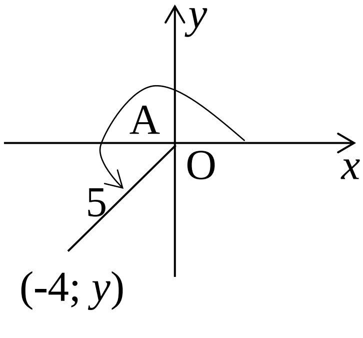
cos A negative in quad 2 and 3 but A > 180° so only quad 3

By Pythagoras Thm $y^2 = 5^2 - (-4)^2$ **Be careful of signs**

$y^2 = 9$

$y = \pm 3$

∴ $y = -3$ In quad 3



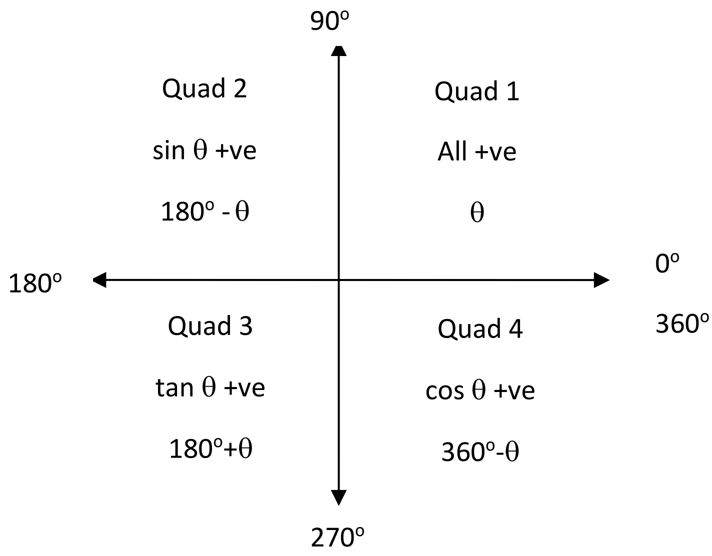
∴ $\tan A = \frac{-3}{-4}$

$= \frac{3}{4}$

REDUCTION FORMULAE

N.B. The **SIGN** of the function value is determined from the **ORIGINAL FUNCTION** using the CAST diagram.

N.B. When determining the values of functions of ($180^o \pm \theta$) or ($360^0 - \theta$) the function **NEVER** changes, but the sign may (i.e. when changing to an acute angle the name does not change).



Example: *Determine the quadrant, then the sign of the ratio and then the angle*

a. $\cos 130^o = \cos(180^o - 50^o)$
$\cos 130^o = -\cos 50^o$ *[130° is in Quad 2 ∴ cos θ is negative]*

b. $\sin(180^o + \theta) = -\sin\theta$ *[(180°+ θ) is in Quad 3 ∴ sin θ is negative]*

Example:

Simplify: $\dfrac{\sin(180°+\theta).\sin(180°-\theta).\cos\theta}{\cos(180°+\theta).\sin(360°-\theta)}$

$$\frac{\sin(180°+\theta).\sin(180°-\theta).\cos\theta}{\cos(180°+\theta).\sin(360°-\theta)}$$

$$= \frac{-\sin\theta.\sin\theta.\cos\theta}{-\cos\theta.(-\sin\theta)}$$

$= -\sin\theta$

Determine the quadrant, then the sign of the ratio and then the angle

e.g ($360^o - \theta$) *is in quad 4, sin θ is - in quad 4 ∴ will be -sin θ*
and ($180^o - \theta$) *is in quad 2, sin θ is + in quad 2 ∴ will be sin θ*

When multiplying a negative number put it in brackets to avoid confusing with subtraction.

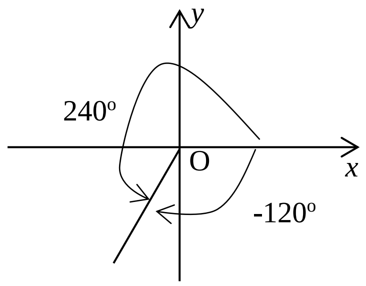
The sign is VERY important.

NEGATIVE ANGLES

A negative angle is formed when the initial arm is rotated in a clockwise direction.

Example:

cos (-120^o)



Convert to the equivalent positive angle and then calculate as usual.

cos (-120^o) = cos ($360^o - 120^o$)
cos (-120^o) = cos 240^o *240^o is in Quad 3 $\therefore$ cos 60^o will be negative*
cos (-120^o) = cos ($180^o + 60^o$)
cos (-120^o) = -cos 60^o

SPECIAL ANGLES

These are angles for which the function values are determined without the use of a calculator, namely 0^o, 30^o, 45^o, 60^o, 90^o, 180^o, 270^o and 360^o. Where necessary they are left in surd form. The ratios must never be learnt off by heart! Rather you must learn how to work them out.

N.B. When using a calculator to square a function that is negative you MUST use BRACKETS.

Example:

Without the use of a calculator evaluate:

a. $3.\sin 30^o.\tan 45^o.\cos^2 30^o$

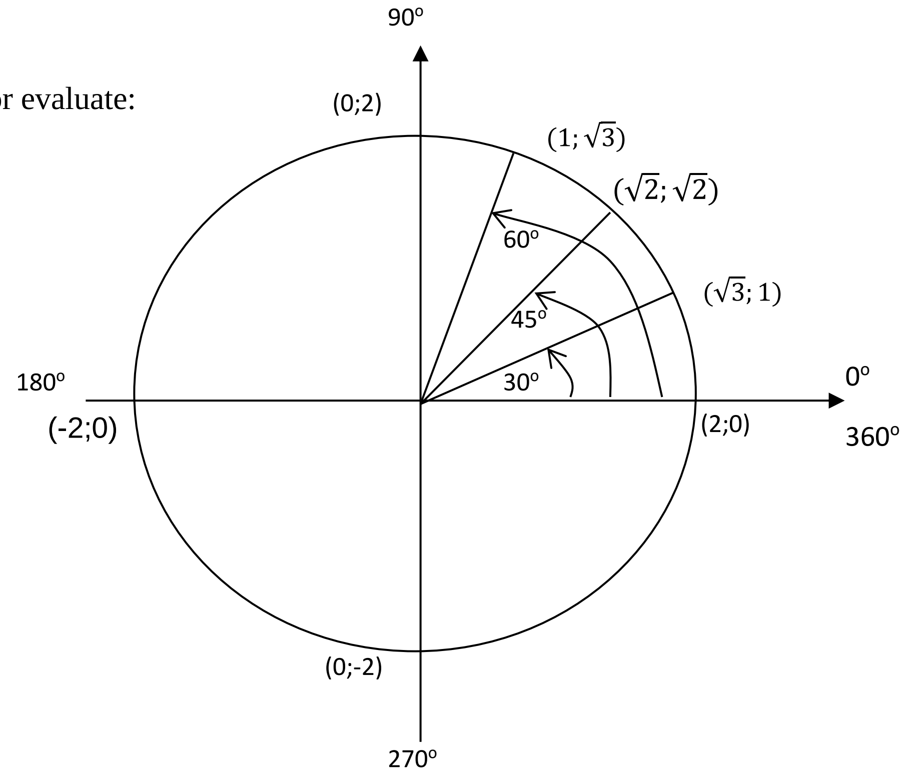
Solution:

$3.\sin 30^o.\tan 45^o.\cos^2 30^o$

$= \frac{3}{1}.\frac{1}{2}.\frac{\sqrt{2}}{\sqrt{2}}.\left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{3}{2}.\frac{3}{4}$

$= \frac{9}{8}$



b. $\tan 315^o - \cos^2 240^o$

Solution: *Always change to positive acute angles before simplifying*

$\tan 315^o - \cos^2 240^o$

$= -\tan 45^o - (-\cos 60^o)^2$

$= -1-\left(-\frac{1}{2}\right)^2$

$= -1-\frac{1}{4}$

$= -\frac{5}{4}$

CO-RATIOS

$\sin(90^o - \theta) = \cos\theta$ $\quad\quad$ $\cos(90^o - \theta) = \sin\theta$

$\sin(90^o + \theta) = \cos\theta$ $\quad\quad$ $\cos(90^o + \theta) = -\sin\theta$

N.B. Always change to an acute angle before using co-ratios.

NB We do not ever use 270^o.

When simplifying ratios you need to ask
1. What quadrant?
2. What sign?
3. Does the name change?

Examples:

1. Write the following as a ratio of 30^o:

a. $\cos 60^o$ $\quad\quad$ b. $\sin 240^o$

Solution:

a. $\cos 60^o$
$= \cos(90^o - 30^o)$
$= \sin 30^o$

b. $\sin 240^o$
$= -\sin(180^o + 60^o)$
$= -\sin 60^o$
$= -\sin(90^o - 30^o)$
$= -\cos 30^o$

2. Simplify: $\frac{2.\sin(90°-\alpha)+\cos(180°-\alpha)}{-\sin(90°-\alpha)-2.\cos(360°-\alpha)}$

$$\frac{2.\sin(90°-\alpha)+\cos(180°-\alpha)}{-\sin(90°-\alpha)-2.\cos(360°-\alpha)}$$

$$= \frac{2.\cos\alpha+(-\cos\alpha)}{-\cos\alpha-2.\cos\alpha}$$

$$= \frac{\cos\alpha}{-3\cos\alpha}$$

$$= -\frac{1}{3}$$

3. Simplify: $\dfrac{\tan 210°.\sin 240°.\sin 170°}{\cos 100°.\sin 225°.\cos 135°}$

$$\frac{\tan 210°.\sin 240°.\sin 170°}{\cos 100°.\sin 225°.\cos 135°}$$

$$= \frac{\tan 30°.(-\sin 60°).\sin 10°}{(-\cos 80°).(-\sin 45°).(-\cos 45°)}$$

$$= \frac{\frac{\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{2}\right)\sin 10°}{(-\sin 10°)\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)}$$

Could have changed sin 10^o to cos 80^o.

Must show the conversion.

$$= +\frac{\frac{3}{6}}{\frac{2}{4}}$$

$$= \left(\frac{3}{6}\right)\left(\frac{4}{2}\right)$$

$= 1$

4. Find cos (90^o - α) if tan α = $\frac{4}{3}$ and α ∈ (0^o; 90^o).

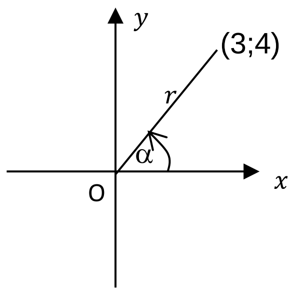
Solution:

tan α = $\frac{4}{3}$ = $\frac{y}{x}$ *[α is in Quad 1 ∴ x and y positive]*

By Pythagoras Thm, $r^2 = 16 + 9$

$r^2 = 25$

$r = 5$



cos (90^o- α) = sin α

$= \frac{4}{5}$

WRITING TERMS OF A GIVEN ANGLE

Given: $\cos 20^\circ = t$

Write the following in terms of t:
$\cos 160^\circ$
$= \cos(180^\circ - 20^\circ)$
$= -\cos 20^\circ$
$= -t$

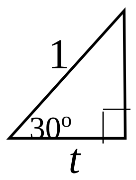
$\cos 200^\circ$
$= \cos(180^\circ + 20^\circ)$
$= -\cos 20^\circ$
$= -t$

$\cos(-340^\circ)$
$= \cos 20^\circ$
$= t$

NB this is a very common type of question - need to use Pythgoras Thm to calculate the third side

$\sin 70^\circ$
$= \cos 20^\circ$
$= \sqrt{1-t^2}$

$y^2 = 1^2 - t^2$ Pythagoras Thm
$y = \sqrt{1-t^2}$ **NB can't square root over + or - sign**



IDENTITIES

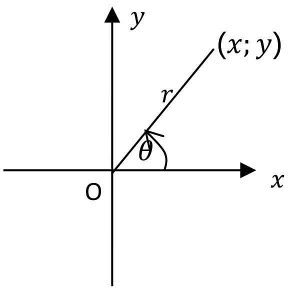
1. Quotient Identity

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \qquad \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

2. Pythagorean Identity

$\sin^2\theta + \cos^2\theta = 1$

PROOF:
LHS $= \sin^2\theta + \cos^2\theta$
LHS $= \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$
LHS $= \frac{x^2}{r^2} + \frac{y^2}{r^2}$
LHS $= \frac{x^2+y^2}{r^2}$
LHS $= \frac{r^2}{r^2}$ [by Pythagoras Thm]
LHS $= 1$
LHS $=$ RHS



Example:

1. Write sin θ. tan θ + cos θ in terms of sin θ and/or cos θ, in its simplest form.

Solution:

$\sin\theta.\tan\theta + \cos\theta$

$= \dfrac{\sin\theta.\sin\theta}{\cos\theta} + \cos\theta$

$= \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta}$

2. Simplify: $\dfrac{\tan\theta.\cos(360° - \theta).\sin(180° - \theta)}{\sin(180° + \theta)}$

$\dfrac{\tan\theta.\cos(360° - \theta).\sin(180° - \theta)}{\sin(180° + \theta)}$

$= \dfrac{\sin\theta.\cos\ \theta.\sin\theta}{\cos\theta(-\sin\theta)}$

$= -\sin\theta$

3. Simplify to one trig ratio:

a. $\tan^2 x - \tan^2 x.\sin^2 x$

$= \tan^2 x(1 - \sin^2 x)$

$= \dfrac{\sin^2 x.\cos^2 x}{\cos^2 x}$

$= \sin^2 x$

b. $(\cos x + \sin x)^2$

$= \cos^2 x + 2\cos x.\sin x + \sin^2 x$

$= 1 + 2\cos x.\sin x$

c. $(1 - \cos A)(1 + \cos A)$

$= 1 - \cos^2 A$

$= \sin^2 A$

d. $(\sin A - 1)(\sin A + 1)$

$= \sin^2 A - 1$

$= -1(-\sin^2 A + 1)$

$= -\cos^2 A$

PROVING IDENTITIES

To prove an identity you need to transform one side to the exact form of the other side or transform both sides to the same expression.

Hints:

1. If in doubt change everything to sin θ and cos θ.
2. Start with the more "complicated" side and try and write it like the other side.
3. You can use any of the fundamental identities.
4. Avoid using surds as they carry the ambiguous ± sign.
5. Do not use the definitions - it makes life too complicated and have to have a sketch.
6. Sometimes it helps to change 1 back to $\sin^2\theta + \cos^2\theta$.

N.B. ALWAYS WORK WITH THE LHS AND THE RHS SEPARATELY.

Example:

Prove that $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta}$

Solution:

LHS $= \dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$

LHS $= \dfrac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$

LHS $= \dfrac{\sin^2\theta + 1 + 2.\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$

LHS $= \dfrac{1+1+2.\cos\theta}{\sin\theta(1+\cos\theta)}$

LHS $= \dfrac{2+2.\cos\theta}{\sin\theta(1+\cos\theta)}$

LHS $= \dfrac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$

LHS $= \dfrac{2}{\sin\theta}$

LHS = RHS

USE OF THE CALCULATOR

N.B. It is important to remember BODMAS when using the calculator.

N.B. ALWAYS USE A POSITIVE RATIO IN THE CALCULATOR.

N.B. NEVER INVERT A DEGREE.

To find the ratio of a given angle.

Given sin θ / cos θ / tan θ use sin/cos/tan key.

Example: $\sin 50^{o} = 0{,}766$

To find the angle given the ratio.

Given sin θ / cos θ / tan θ you use the $\sin^{-1}/\cos^{-1}/\tan^{-1}$ key.

Example:
If $\sin\theta = 0{,}5$, then one solution for θ is 30^{o}.