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# Study Guide

# Mathematics

## Grade 10

Via Afrika Publishers

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Chapter 5

# Trigonometry

## Overview

Here, we will introduce you to trigonometry. Trigonometry is the field of mathematics in which we study the relationship between the sides and angles of triangles. The word trigonometry is derived from the Greek words *trigonon*, which means triangle, and *metron*, which means measurement. In ancient times, mathematicians, astronomers and surveyors in Egypt, Babylon, India and China used trigonometry for navigation, surveying and astronomy. For example, they were able to calculate:

- the height of mountains
- the distance and direction across the sea (navigation)
- the dimensions of large areas of land for construction
- astronomical distances, for example, between the Earth, the moon and the sun.

You will also learn how to sketch the graphs of each of the trigonometric ratios.

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Trigonometry

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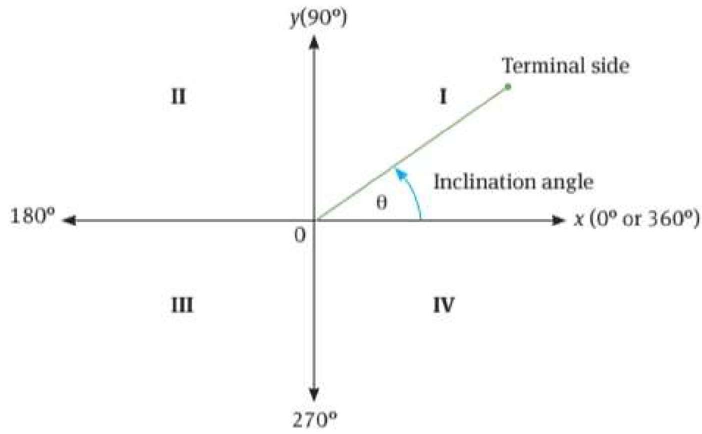
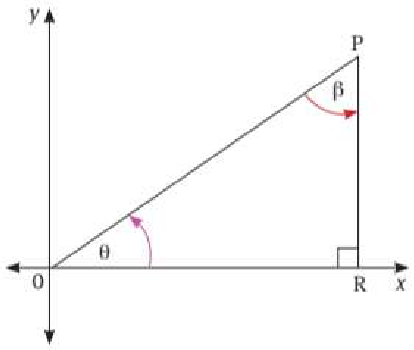
# Unit 1

# Right-angled triangles

## 1.1 Trigonometric ratios in right-angled triangles

Trigonometry is the measurement of triangles, specifically **right-angled triangles**. When we use the Cartesian plane in trigonometry, we call the positive $x$-axis the 0° line. If we move anti-clockwise by 360°, we have completed a full circle. The line that forms any angle ($\theta$) in this way is known as the **terminal side** and the angle ($\theta$) is known as the **inclination angle**.



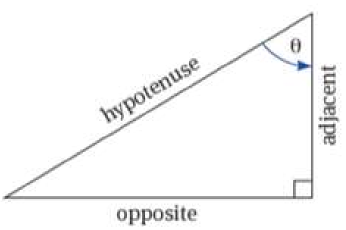



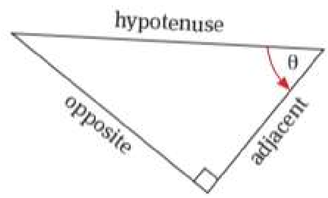

In a right-angled triangle, each side has a specific name:

- The *hypotenuse* (OP) is always opposite the 90° angle.
- The opposite side is opposite the angle. In this example, PR is opposite $\theta$ and OR is opposite $\beta$.
- The adjacent side forms the angle together with the hypotenuse. Therefore OR is adjacent to $\theta$, and PR is adjacent to $\beta$.

Adjacent is always next to the angle.

Opposite is opposite the angle.



Unit 2

# Definitions of the trigonometric ratios

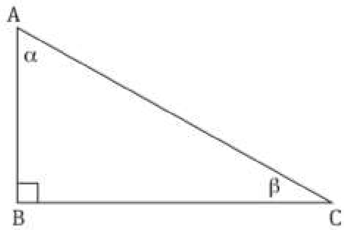
## 2.1 Definitions

In Trigonometry, the ratios between the sides are given certain names. In ΔABC with ∠BAC = α, we ca define the ratios as:

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}}$$




## 2.2 Reciprocals of the three trigonometric ratios

The **reciprocal ratios** are the inverse versions of each of the main ratios. The inverse of sin θ is cosec θ, the inverse of cos θ is sec θ, and the inverse of tan θ is cot θ. The reciprocal ratios are defined as follows:

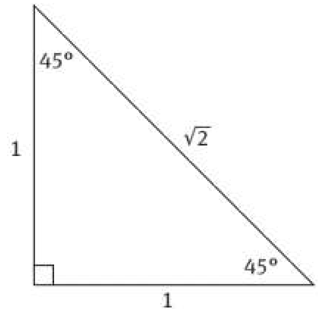
| Trigonometric ratio | Reciprocal ratio | Conclusion |
|---|---|---|
| $\sin \theta = \frac{o}{h}$ | $\text{cosec}\,\theta = \frac{h}{o}$ | $\text{cosec}\,\theta = \frac{1}{\sin \theta}$ |
| $\cos \theta = \frac{a}{h}$ | $\sec \theta = \frac{h}{a}$ | $\sec \theta = \frac{1}{\cos \theta}$ |
| $\tan \theta = \frac{o}{a}$ | $\cot \theta = \frac{a}{o}$ | $\cot \theta = \frac{1}{\tan \theta}$ |

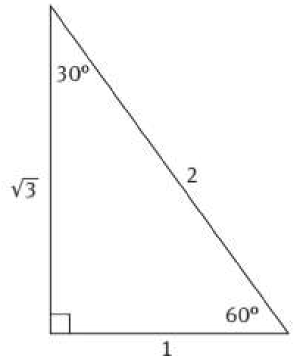
Unit 3

# Special triangles

The ratios of certain angles are well known, and we can use them to solve problems. These angles are $0°$, $30°$, $45°$, $60°$ and $90°$.

We can show these angles (except for $0°$) in two triangles, as shown below.




The ratios formed by each of these special angles is shown below.

| | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| **sin θ** | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| **cos θ** | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| **tan θ** | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | undefined |

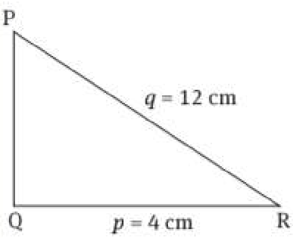
# Unit 4

# Using your calculator

When all you know is the size of an angle, you can use a calculator to work out the value of a ratio. Your scientific calculator has keys for the trigonometric ratios. Make sure your calculator is in DEG mode when working with degrees.


**Note:** $\sin^{-1}$ on your calculator is not the same as $\frac{1}{\sin}$. In trigonometry, $\frac{1}{\sin}$ refers to the reciprocal *ratio* for sin, namely cosec. The $\sin^{-1}$ key on the calculator is the reciprocal *function* of sin, which is what you use to calculate the value of an angle when you know the value of the ratio.

The following example shows how to use your calculator to work out the value of a ratio for a particular angle.

| Example | Answer | Calculator |
|---|---|---|
| In $\Delta PQR$; $\angle Q$ is $90°$, $q = 12$ cm and $p = 4$ cm. Find the values of $\angle P$ and $\angle R$.   | $\sin P = \frac{p}{q}$<br>$\sin P = \frac{4}{12}$<br>$\angle P = \sin^{-1}\frac{1}{3}$<br>$\angle P = 19{,}47°$<br>$\angle R = 90° - \angle P$<br>$= 70{,}53°$ | sin ( 1 ÷ 3 ) = |
| **Note:** We name the sides of a right-angled triangle according to the angle names. So for example, the side opposite angle Q is $q$, and the side opposite angle P is $p$. | | |

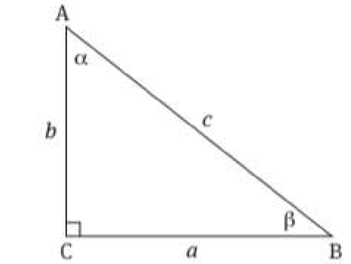
Unit 5

# Solving trigonometric equations

## 5.1 Solve diagrams in two dimensions

Use the following tools to solve right-angled triangles:

- Theorem of Pythagoras: $a^2 + b^2 = c^2$
- Sine: $\sin A = \frac{a}{c}$, $\sin B = \frac{b}{c}$
- Cosine: $\cos A = \frac{b}{c}$, $\cos B = \frac{a}{c}$
- Tangent: $\tan A = \frac{a}{b}$, $\tan B = \frac{b}{a}$



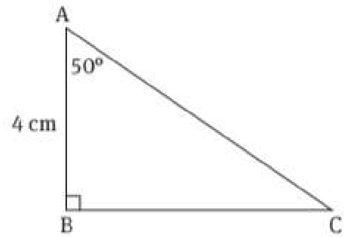

**Remember**: In triangle ABC, $a$ is opposite ∠A, $b$ is opposite ∠B, and c opposite ∠C.

You can calculate an unknown side or angle when you know the sizes of:

- an angle and a side
- two sides and we need the angle.

### Example

1 Using the diagram alongside, calculate the length of BC if AB is adjacent to ∠A and BC is the opposite side.




$\tan 50° = \frac{BC}{AB}$
BC = AB.tan 50°
BC = 4.tan 50°
BC = 4,77 cm (to 2 decimal numbers)

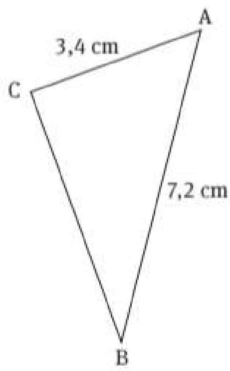
2 Calculate the length of AC, given that AB is adjacent to ∠A and AC is the hypotenuse.

$\cos 50° = \frac{AB}{AC}$
AC.cos 50° = AB
AC = cos 50° = A cos 50°
AC = 6,22 cm (to 2 decimal numbers)

We can also use calculate an angle if we know the value of a ratio.

## Example

Using the diagram alongside, calculate the size of $\angle B$ if $\angle C = 90°$.




AC is the opposite side to $\angle B$ and AB is the hypotenuse:

$\sin B = \frac{AC}{AB}$

$\sin B = \frac{3,4}{7,2}$

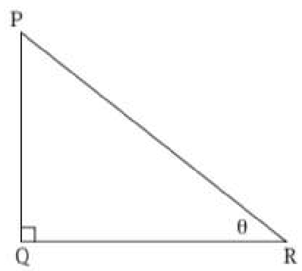
$\angle B = \sin^{-1}\left(\frac{3,4}{7,2}\right)$

Therefore, $\angle B = 28,18°$ (to two decimal places)

Let's work through another example.

## Example

In the triangle alongside, $\tan \theta = \frac{5}{12}$.




Calculate the following using the triangle:

1 $\angle\theta$
2 $\cos \theta$
3 The length of the hypotenuse

1 $\theta = \tan^{-1} = 22,62°$
2 $\cos \theta = \cos 22,62° = 0,92$
3 The tan ratio is $\frac{\text{opposite}}{\text{adjacent}}$. Therefore, we know that the side opposite $\theta$ (PR) is 5 units long, and the adjacent side is 12 units long (QR). Now we can use the theorem of Pythagoras to calculate the length of the hypotenuse, PR:

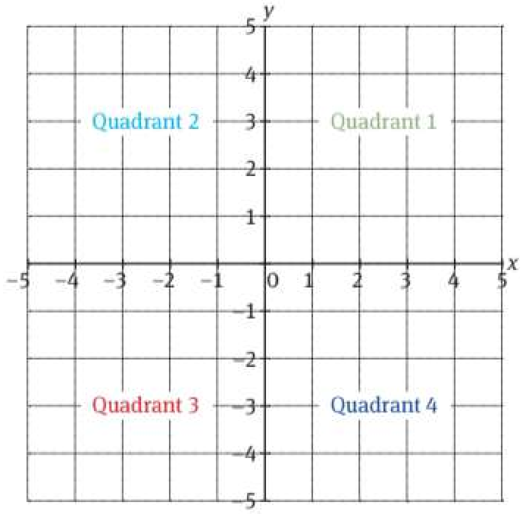
$PR^2 = 5^2 + 12^2 = 25 + 144 = 169$
$PR = \sqrt{169}$
$PR = 13$

Unit 6

# Extending the ratios to $0° \leq \theta \leq 360°$

## 6.1 Trigonometry ratios in four quadrants

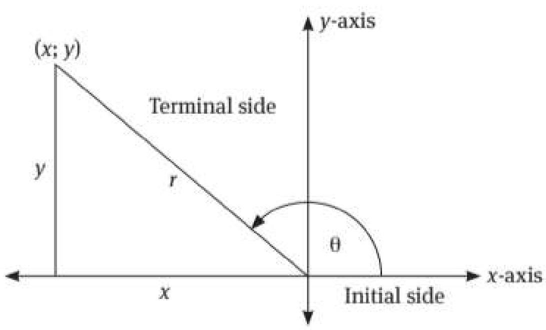
By convention we label the four quadrants of a Cartesian plane as follows:




In trigonometry, we tend to think of an angle as created by a rotating radius (arm). The beginning position of the "arm" is called the **initial side** (usually, the initial side coincides with the positive $x$-axis; angles with such an initial side are said to be in *standard position*). When the "arm" rotates and ends up in one of the quadrants, the side ("arm") is now called the **terminal side**. The measure of the angle is a number which describes the amount of rotation. If the rotation of the "arm" was clockwise, the angle measure is a negative number; if the rotation was anti-clockwise, the angle measure is positive.

### Angles greater than 90°

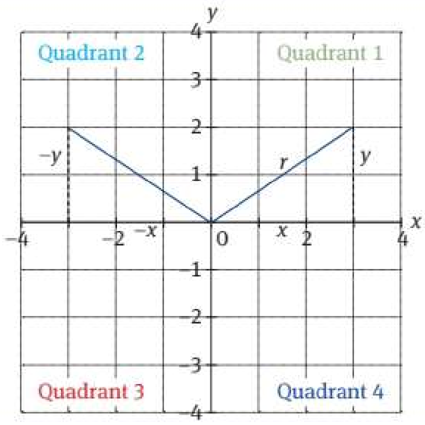
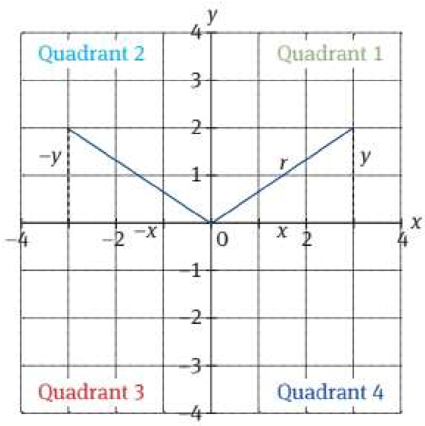
We **define** the trigonometric functions for angles greater than 90° in the following way:

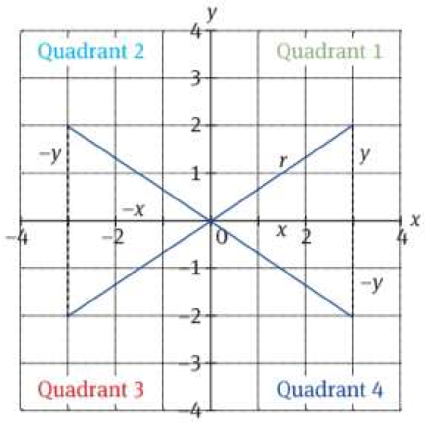
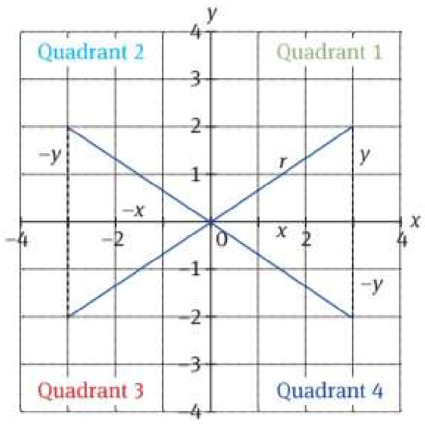


The trigonometric definitions do not rely on the lengths of the sides of the triangle, but only on the angle. When negative or obtuse angles are used in trigonometric functions, they will sometimes produce negative values. The **CAST diagram** will help you to remember the signs of trigonometric functions for different angles.

The functions will be negative in all quadrants except those that indicate that the function is positive. For example, when the angle is between 0° and 90°, the side *r* is in the first quadrant. All functions will be positive in this quadrant. When the angle is between 90° and 180°, the terminal side is in the second quadrant. This means that only the sine function is positive. All other functions will be negative. When the angle is between 180° and 270°, the terminal side is in the third quadrant. This means that only the tan function is positive. All other functions will be negative. When the angle is between 270° and 360°, the terminal side is in the fourth quadrant. This means that only the cosine function is positive. All other functions will be negative.

| | |
|---|---|
|  | **Quadrant 1**<br>sine of an angle = $\frac{y}{r}$<br>cosine of an angle = $\frac{x}{r}$<br>tan of an angle = $\frac{y}{x}$<br>All positive, because $x$, $y$ and $r > 0$ |
|  | **Quadrant 2**<br>sine of an angle = $\frac{y}{r}$<br>cosine of an angle = $\frac{-x}{r}$<br>tan of an angle = $\frac{y}{-x}$<br>Because $x$ is negative in the second quadrant, all the functions containing $x$ will be negative, i.e. cos and tan.<br>Only the sine function is positive, because $y$ and $r > 0$. |

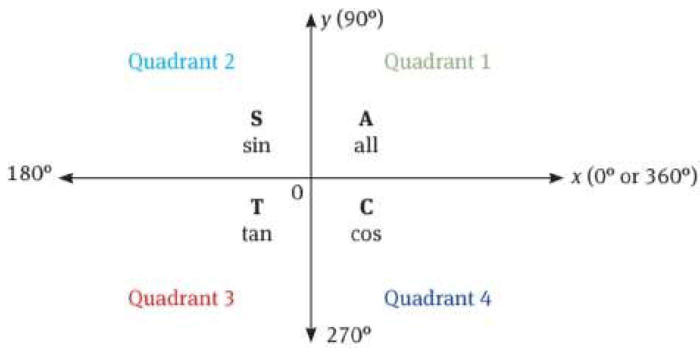
| | |
|---|---|
|   | **Quadrant 3**<br>sine of an angle = $\frac{-y}{r}$<br>cosine of an angle = $\frac{-x}{r}$<br>tan of an angle = $\frac{-y}{-x} = \frac{y}{x}$<br>Only the **tan** function is positive in the third quadrant (neg ÷ neg = pos). |
|   | **Quadrant 4**<br>sine of an angle = $\frac{-y}{r}$<br>cosine of an angle = $\frac{x}{r}$<br>tan of an angle = $\frac{-y}{x}$<br>In quadrant 4 only the cos function is positive, because $x$ and $r > 0$. |

### The CAST rule

There is a simple rule by which you can remember all of these results. Notice in the first quadrant, **A**ll the functions are positive; in the second quadrant, only the **S**ine is positive; in the third, only the **T**angent is positive; and in the fourth quadrant, only the **C**osine is positive.

This is called the **CAST Rule** and tells you which function is **positive** in each quadrant. And the C, A, S and T stand for **cosine**, **all**, **sine** and **tangent**.

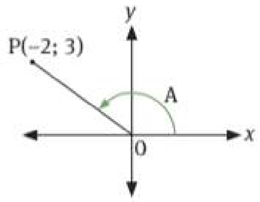
| | |
|---|---|
| **S** | **A** |
| **T** | **C** |



Unit 6

## Examples using the CAST rule

1. Without using a calculator, use the figure to calculate the following:
   1.1 OP
   1.2 tan A
   1.3 cos A
   1.4 $\sin^2 A - \cos^2 A$




*Notice that the point P is in the SECOND quadrant, only sine is positive in the second quadrant.*

| | |
|---|---|
| S | A |
| T | C |

1.1 Use Pythagoras' theorem to calculate the length of OP:

$OP^2 = 3^3 + (-2)^2$

$\therefore OP = \sqrt{9+4} = \sqrt{13}$

1.2 $\tan A = \frac{y}{x} = \frac{3}{-2} = -\frac{3}{2}$

(tan is negative in the 2nd quadrant!)

1.3 $\cos A = \frac{x}{r} = \frac{-2}{\sqrt{13}}$

(cos is also negative in the 2nd quadrant!)

1.4 $\sin A = \frac{y}{r} = \frac{3}{\sqrt{13}}$

therefore, $\sin^2 A - \cos^2 A$

$= \frac{4}{13} + \frac{9}{13}$

$= \frac{13}{13}$

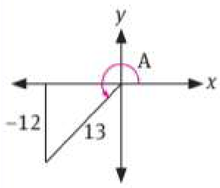
$= 1$

2. If $\sin A = -\frac{12}{13}$ and $\hat{A} \in [90°; 270°]$ determine, without the use of a calculator, the values of
   2.1 cos A
   2.2 tan A

*Notice that the angle A could be in the second or third quadrants, but sine is negative in the third quadrant. So, angle A has to be in the THIRD quadrant.*

| | |
|---|---|
| S | A |
| T | C |

*It is useful to make a drawing:*




*The x-value = −5 (Pythagoras)*

2.1 $\cos A = \frac{x}{r} = \frac{-5}{13}$ (cos is negative in the 3rd quadrant!)

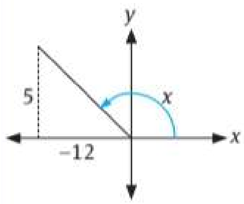
2.2 $\tan A = \frac{y}{x} = \frac{-12}{-5} = \frac{12}{5}$ (tan is positive in the 3rd quadrant!)

3. If $\tan x = -\frac{5}{12}$, $0° < x < 180°$, use a sketch (no calculator) to determine the value of $3 \sin x - 2 \cos x$.

*Notice that the tan function is negative and the given domain covers the first two quadrants. Tan is negative in the second quadrant.*

| | |
|---|---|
| S | A |
| T | C |

The terminal side: 13 *(Pythagoras)*




*Sine is positive in the 2nd quadrant and cosine is negative.*

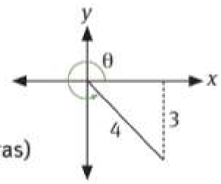
$3 \sin x - 2 \cos x$

$= 3\left(\frac{5}{13}\right) - 2\left(\frac{-12}{13}\right)$

$= \frac{15}{13} + \frac{24}{13}$

$= \frac{39}{13}$

$= 3$

| | |
|---|---|
| 4. If $4 \sin \theta + 3 = 0$ and $0 > 270°$ determine, without using a calculator:<br>4.1 $\tan \theta$<br>4.2 $\cos^2\theta$<br><br>*Notice that angle is greater than 270°. So, the operating quadrant must be the fourth quadrant and the sine function is also negative in the fourth quadrant.*<br><br>**S** \| **A**<br>**T** \| **C** | $4 \sin \theta + 3 = 0$<br>$\therefore \sin \theta = -\frac{3}{4}$<br><br>$x = \sqrt{4^2 - (-3)^2}$<br>$= \sqrt{16 - 9}$ (Pythagoras)<br>$= \sqrt{7}$<br><br>4.1 $\tan \theta = -\frac{3}{\sqrt{7}}$ (tan is negative in the 4th quadrant)<br>4.2 $\cos 20 = \left(\frac{\sqrt{7}}{4}\right)^2 = \frac{7}{16}$ (cos is positive in the 4th quadrant) |