Chapter 3

Number Patterns

Number Patterns

Types of Number Patterns	
1. Arithmetic (linear) sequence	$T_n = a + (n-1)d$ a : 1st term (T_1) d : constant difference (common difference)
2. Geometric (exponential) sequence	$T_n = a.r^{n-1}$ a : 1st term (T_1) r : common ratio

3. Quadratic Sequence $T_n = an^2 + bn + c$

$a+b+c = T_1$; T_2 ; T_3 ; T_4 ; ... sequence

$3a+b = \boxed{T_2 - T_1}$; $\boxed{T_3 - T_2}$; $\boxed{T_4 - T_3}$;... 1st differences

$2a = \boxed{}$ $\boxed{}$ 2nd differences

NB!

n : number of terms

T_n : term "*n*" (the *n*th term)

T_4 : term 4

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① Arithmetic (linear) sequence

$$\boxed{\begin{array}{l} * \ T_n = a + (n -)d \\ \bullet \ d = T_2 - T_1 \\ \bullet \ T_2 - T_1 = T_3 - T_2 \\ \bullet \ a \, ; \, a+d \, ; \, a+2d \, ; \, \ldots \end{array}}$$

$$\begin{array}{ccccc} & T_1 & T_2 & T_3 & T_4 \\ \text{E.g.} & 1 \, ; & 3 \, ; & 5 \, ; & 7 \, ; \ldots \end{array}$$

$$+2 \quad +2 \quad +2$$

Constant difference: $\underline{+2}$

- $a = 1$
- $d = +2$

$* \ T_n = a + (n-1)d$

$T_n = 1 + (n-1)(2)$

$T_n = 1 + 2n - 2$

$T_n = 2n - 1$

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2. Geometric (exponential) sequence

* $T_n = a \cdot r^{n-1}$
- $r = \dfrac{T_2}{T_1}$
- $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$
- $a \,;\, ar \,;\, ar^2 \,;\, \ldots$

E.g. $\underset{T_1}{3} \,;\, \underset{T_2}{6} \,;\, \underset{T_3}{12} \,;\, \underset{T_4}{24} \,;\, \ldots$

$\times 2 \quad \times 2 \quad \times 2$

Common ratio: $\times \underline{2}$

- $a = 3$
- $r = 2$

$* \; T_n = a \cdot r^{n-1}$

$T_n = 3 \cdot (2)^{n-1}$

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③ Quadratic sequence

* $T_n = an^2 + bn + c$

- 1st difference form an arithmetic sequence

- 2nd differences remain constant

E.g. $2\ ;\ 4\ ;\ 8\ ;\ 14\ ;\ \ldots$

$+2 \quad +4 \quad +6$

$+2 \quad +2$ ⇒ 2nd differences are constant

$\therefore$ quadratic sequence

$a + b + c = 2\ ;\ 4\ ;\ 8\ ;\ 14$

$3a + b = +2 \quad +4 \quad +6$ 1st differences

$2a = +2 \quad +2$ 2nd differences

- $2a = 2$

$\therefore a = 1$

- $3a + b = 2$

$3(1) + b = 2$

$3 + b = 2$

$\therefore b = -1$

- $a + b + c = 2$

$1 + (-1) + c = 2$

$\therefore c = 2$

$\boxed{T_n = an^2 + bn + c}$

$\therefore T_n = 1n^2 - 1n + 2$

$T_n = n^2 - n + 2$

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Mixed sequences

* consists of a combination of 2 sequences

Eg. $\frac{1}{2}; \frac{3}{4}; \frac{5}{6}; \frac{7}{8}; \ldots$

- top
  and
- bottom

separate sequences

Find T_n :

T_n of top section:

- $a = 1$
- $d = 2$

$\therefore T_n = 1 + (n-1)(2)$

$T_n = 1 + 2n - 2$

$T_n = 2n - 1$

or

$1 ; 3 ; 4 ; 6 ; 7 ; 12 ; 10 ;$

- $1 ; 4 ; 7 ; 10 ; \ldots$
  and
- $3 ; 6 ; 12 ; \ldots$

Calculate T_{55}:

$1 ; 4 ; 7 ; 10 ; \ldots$ is $T_1 ; T_3 ; T_5 ; \ldots$ of the original sequence.

$3 ; 6 ; 12 ; \ldots$ is $T_2 ; T_4 ; T_6 ; \ldots$ of the original sequence.

T_n of the bottom section:

- $a = 2$
- $d = 2$

$\therefore T_n = 2 + (n-1)(2)$

$T_n = 2 + 2n - 2$

$T_n = 2n$

Therefore :

T_n of pattern :

$T_n = \frac{2n-1}{2n}$

In which of the two sequences will T_{55} be?

The term T_{55} has an uneven number.

Therefore we need to find term $(55 + 1) \div 2$ of this sequence :

$1 ; 4 ; 7 ; 10 ; \ldots$

- $a = 1$
- $d = 3$
- $n = 28$

$\left.\begin{array}{l} a = 1 \\ d = 3 \end{array}\right\}$ Arithmetic

$\boxed{(55+1) \div 2 = 28}$

$T_n = a + (n-1)d$

$T_n = 1 + (n-1)(3)$

$T_n = 1 + 3n - 3$

$T_n = 3n - 2$

T_{28} of "pink" sequence:

$$T_{28} = 3(28) - 2$$
$$= 82$$

$\therefore T_{55}$ of mixed sequence :

$$= 82$$

Example 1

Consider the following sequence: $8 ; 5 ; 2 ; \ldots$

(a) Find the nth term.

(b) Find the 25th term.

(c) Which term is equal to -109?

Solutions:

(a) $\underbrace{8 ; 5}_{-3} \underbrace{; 2}_{-3} ; \ldots$

$-3 \quad -3 \quad \Rightarrow$ Constant difference

$\therefore$ Arithmetic sequence

- $a = 8$
- $d = -3$

$\therefore T_n = a + (n-1)d$

$T_n = 8 + (n-1)(-3)$

$T_n = 8 - 3n + 3$

$T_n = -3n + 11$

(b) $T_n = -3n + 11$ with $n = 25$

$T_{25} = -3(25) + 11$

$T_{25} = -75 + 11 = -64$

(c) $T_n = -3n + 11$ with $T_n = -109$

$-109 = -3n + 11$

$3n = 11 + 109$

$3n = 120$

$n = 40$

Example 2

Consider the number pattern: $-2\,;\,3\,;\,10\,;\,19\,;\ldots$

(a) Find the general term for this sequence.

(b) Find the value of term 50.

(c) Which term will have a value of 138?

Solutions:

(a) $-2\,;\,3\,;\,10\,;\,19\,;\ldots$

$+5 \quad +7 \quad +9$ ⇒ 1st differences not constant

$+2 \quad +2$ ⇒ 2nd differences are constant

∴ quadratic sequence

$a+b+c = -2\,;\,3\,;\,10\,;\,19\,;\ldots$

$3a+b = +5 \quad +7 \quad +9$

$2a = +2 \quad +2$

• $2a = +2$

$\therefore a = 1$

• $3a + b = +5$

$3(1) + b = 5$

$3 + b = 5$

$b = 2$

• $a+b+c = -2$

$1+2+c = -2$

$c = -5$

$$\boxed{T_n = an^2 + bn + c}$$

$\therefore T_n = 1n^2 + 2n - 5$

$\therefore T_n = n^2 + 2n - 5$

(b) Find the value of term 50.

$T_n = n^2 = 2n - 5$ with $n = 50$

$T_n = (50)^2 + 2(50) - 5$

$= 2595$

(c) Which term will have a value of 138?

$T_n = n^2 + 2n - 5$ with $T_n = 138$

$138 = n^2 + 2n - 5$

$0 = n^2 + 2n - 143$

$n = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- $a = +1$
- $b = +2$
- $c = -143$

$n = \dfrac{-(2) \pm \sqrt{(2)^2 - 4(1)(-143)}}{2(1)}$

$n = -13$ or $n = 11$

n/a

(because term number is always positive)

$\therefore$ term 11

Example 3

Consider the sequence :

$20 ; 28 ; 34 ; 38 ; 40 ; \ldots$

(a) Find the general formula for the 1^{st} differences.

(b) Find the general formula for the sequence.

Solutions:

(a) $20 ; 28 ; 34 ; 38 ; 40 ; \ldots$

1st differences: $+8 \quad +6 \quad +4 \quad +2$

Sequence formed by 1st differences:

$\underbrace{8 ; 6}_{-2} \, ; \underbrace{\, ; 4}_{-2} \, ; \underbrace{\, ; 2}_{-2} \, ; \ldots$

$\left.\begin{array}{l} \bullet\ a = 8 \\ \bullet\ d = -2 \end{array}\right\}$ Arithmetic sequence

$T_n = a + (n-1)d$

$T_n = 8 + (n-1)(-2)$

$T_n = 8 - 2n + 2$

$T_n = -2n + 10$

(b) Find the general formula for the sequence.

$$\underbrace{a+b+c}_{} = 20 \; ; \; 28 \; ; \; 34 \; ; \; 38 \; ; \; 40 \; ; \; \ldots$$

$$3a + b = +8 \quad +6 \quad +4 \quad +2$$

$$2a = -2 \quad -2 \quad -2$$

• $2a = -2$

$\therefore a = -1$

• $3a + b = 8$

$3(-1) + b = 8$

$-3 + b = 8$

$b = 11$

• $a + b + c = 20$

$-1 + 11 + c = 20$

$10 + c = 20$

$c = 10$

$\therefore T_n = -n^2 + 11n + 10$

Example 4

Consider the number pattern:

$-3 ; -5 ; -5 ; -3 ; \ldots$

(a) Find the general term for this sequence

(b) Between which two terms of the quadratic sequence will the first difference be 72?

Solutions:

(a) $T_1 \quad T_2 \quad T_3 \quad T_4$

$-3 ; -5 ; -5 ; -3 ; \ldots$

$-2 \quad 0 \quad +2$ ⇒ 1st differences not constant

$+2 \quad +2$ ⇒ 2nd differences are constant

∴ quadratic sequence

$T_2 - T_1 = -5-(-3)$

$a+b+c = -3 ; -5 ; -5 ; -3 ; \ldots$

$3a + b = -2 \quad 0 \quad +2$

$2a = +2 \quad +2$

- $2a = +2$

$\therefore a = 1$

- $3a+b = -2$

$3(1)+b = -2$

$3+b = -2$

$b = 5$

- $a+b+c = -3$

$1+(-5)+c = -3$

$c = 1$

$\boxed{T_n = an^2 + bn + c}$

$\therefore T_n = 1n^2 - 5n + 1$

$\therefore T_n = n^2 - 5n + 1$

(b) Between which two terms of the quadratic sequence will the first difference be 72?

$$\begin{array}{ccccccc} T_1 & & T_2 & & T_3 & & T_4 \\ -3 & ; & -5 & ; & -5 & ; & -3 \ ; \ \ldots \end{array} \Rightarrow \text{Quadratic sequence}$$

$$\begin{array}{ccccc} T_1 & & T_2 & & T_3 \\ -2 & & 0 & & +2 \\ & +2 & & +2 & \end{array} \Rightarrow \text{Sequence formed by 1st differences}$$

Sequence formed by 1st differences: $-2 \quad 0 \quad +2 \ ; \ \ldots$

$+2 \quad +2 \Rightarrow$ constant

$\therefore$ Arithmetic sequence

- $a = -2$
- $d = 2$

$$T_n = a + (n-1)d$$
$$= -2 + (n-1)(2)$$
$$= -2 + 2n - 2$$

$$T_n = 2n - 4$$

$$T_n = 2n - 4 \text{ with } T_n = 72$$

$$72 = 2n - 4$$

$$76 = 2n$$

$$38 = n$$

$\therefore T_{38}$ of sequence formed by 1st differences

$\therefore$ Between T_{38} and T_{39} of quadratic sequence

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Example 5

A quadratic number pattern has a constant second difference of 4 and $T_4 = T_{14} = 18$. Find the general term for this sequence.

Solution: $\boxed{T_n = an^2 + bn + c}$

- $2a$ = constant 2nd difference

$2a = 4$

$\therefore a = 2$

- $T_n = 2n^2 + bn + c$ with $T_4 = 18$

$T_4 = 2(4)^2 + b(4) + c$

$18 = 32 + 4b + c$

$\boxed{-14 - 4b = c}$ (1)

- $T_n = 2n^2 + bn + c$ with $T_{14} = 18$

$T_{14} = 2(14)^2 + b(14) + c$

$18 = 392 + 14b + c$

$\boxed{-374 = 14b + c}$ (2)

Substitute (1) into (2) : $-374 = 14b + (-14 - 4b)$

$-374 = 14b - 14 - 4b$

$-360 = 10b$

$-36 = b$

Now substitute $b = -36$ into (1):

$$c = -14 - 4b$$

$$= -14 - 4(-36)$$

$$= -14 + 144$$

$$= 130$$

$$\therefore T_n = 2n^2 - 36n + 130$$

Example 6

Find the values of x and y if

$$6\,;x\,;16\,;27\,;y\,;\ldots$$

is a quadratic sequence.

Solution:

Quadratic $\Rightarrow$ 2nd differences constant

$6\;;\;x\;;\;16\;;\;27\;;\;y\;;\ldots$

1st: $x-6$; $16-x$; $27-16 = 11$; $y-27$

2nd: $\boxed{16-x-(x-6)}$ $\boxed{11-(16-x)}$ $\boxed{y-27-(11)}$

2nd differences remain constant:

Therefore we can equate the 2nd differences.

- $\boxed{16-x-(x-6) = 11-(16-x)}$

$$16-x-x+6 = 11-16+x$$
$$22-2x = -5+x$$
$$27 = 3x$$
$$9 = x$$

- $\boxed{11-(16-x) = y-27-(11)}$

$$11-16+x = y-38$$
$$33+x = y$$

Now substitute $x = 9$:

$$\therefore\ 33+9 = y$$
$$42 = y$$