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Study Guide

Mathematics

Grade 10

Via Afrika Publishers

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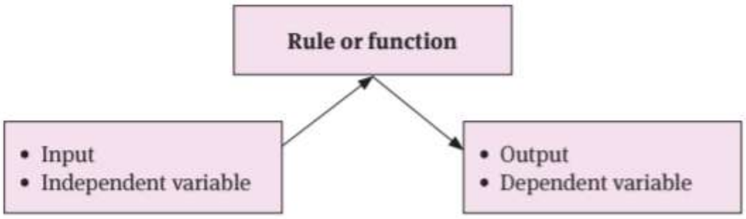
Unit 1

What is a function?

1.1 Dependent and independent variables

A function is like a machine:

- it has an input and an output
- there is a relationship between input and output values
- the output depends on the input, so we say that the output is a dependent variable and the input is the independent variable.
- "$f(x) = ...$" is the classic way of writing a function.



1.2 Relationship between variables

- A function relates inputs to outputs
- A function is a special type of relation in which:
  - the output value is dependent on the input value
  - any input produces only one output (not this or that)
  - an input and its matching output are together called an ordered pair
  - a function can also be seen as a set of ordered pairs

For example: $y = 2x + 3$ is a function.

Input	Relationship	Output
0	×2 + 3	3
1	×2 + 3	5
2	×2 + 3	7
3	×2 + 3	9
...	...	...

First, it is useful to give a function a **name**. The most common name is "f", but you can have other names, such as "g". But f is used most often in mathematics.

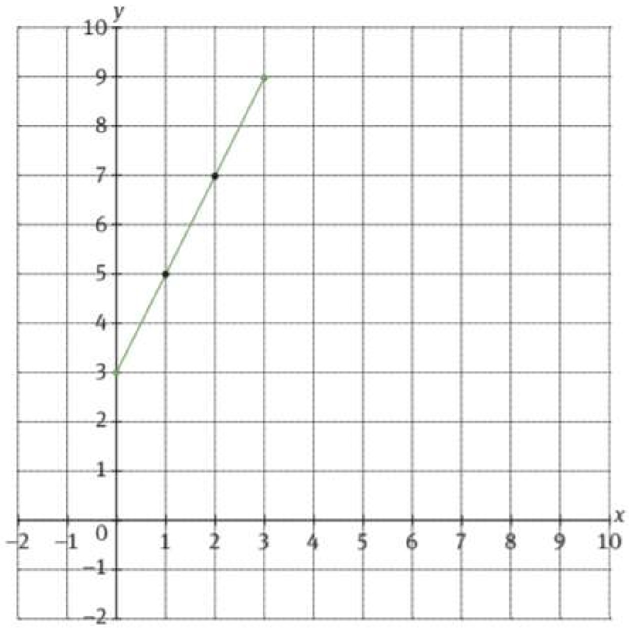
$$\underbrace{f}_{\text{function name}}(\underbrace{x}_{\text{input}}) = \underbrace{2x + 3}_{\text{what to output}}$$

We read the function statement as: *"f of x equals two x plus three"*.

What goes into the function is put inside brackets ( ) after the name of the function. So, $f(x)$ shows you the function is called "f", and "$2x + 3$" tells you that function "f" takes "x", *multiplies it by two and adds three.*

The answer is the output. The output value **uniquely** depends on the input value. An input of 3 gives an output of 9: $f(3) = 9$.

The input and output values can be plotted as coordinates on a graph, in the form (input; output) or $(x; f(x))$: (0; 3), (1; 5), (2; 7), (3; 9), etc.



So, when plotted in the Cartesian plane, the function $y = 2x + 3$ forms a straight line of the kind $y = ax + q$, also written as $f(x) = ax + q$.

Example

A function f is defined by $f(x) = 2 + x - x^2$. What is the value of $f(-3)$?

$f(-3) = 2 + (-3) -(-3)^2 = 2 - 3 - 9 = -10$

Unit 2

Graphs of functions

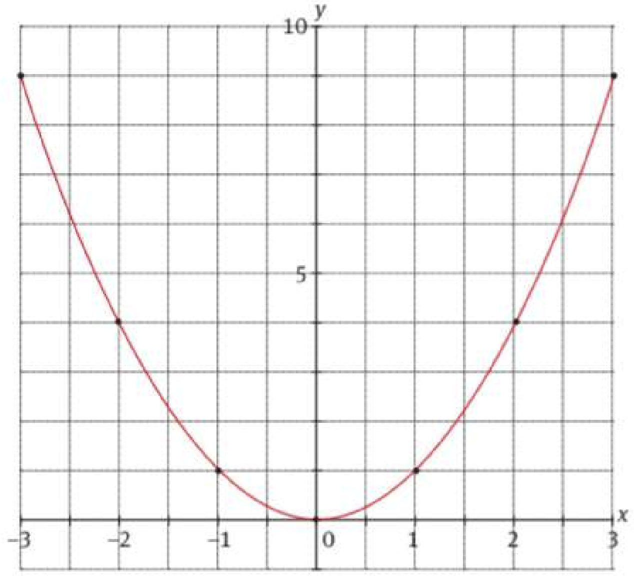
At first, it may be helpful to generate a table of values using numbers from the domain (x-values).

Your x-values should include examples of negative numbers and fractions.

2.1 The graph of the parabola, $y = x^2$

- The graph forms a parabola
- Axis of symmetry: $y = 0$ (y-axis)
- Turning point at (0; 0)
- Intercept with x-axis at $x = 0$ and intercept with y-axis at $y = 0$

x **(domain)**	-2	-1	0	1	2
y **(range)**	4	1	0	1	4

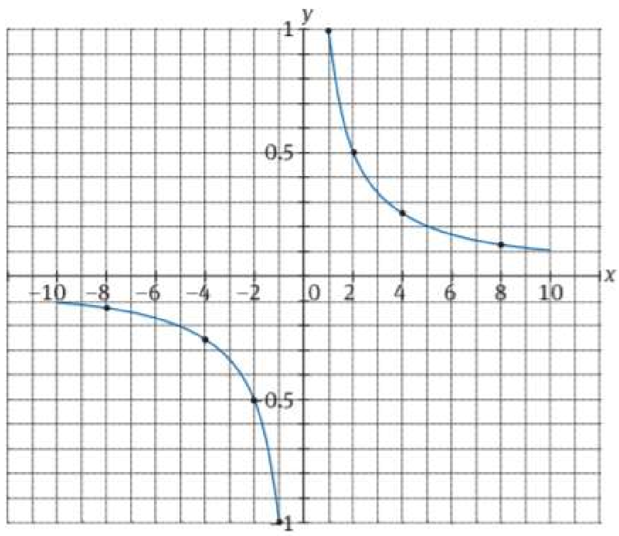


Equation 1: $y = x^2$

2.2 The graph of $y = \frac{1}{x}$

- The graph forms a hyperbola
- Axis of symmetry: $y = x$
- Intercept with y-axis at $y = 1$

x (domain)	-2	-1	0	1	2
y (range)	$-0,5$	-1	undefined	1	0,5

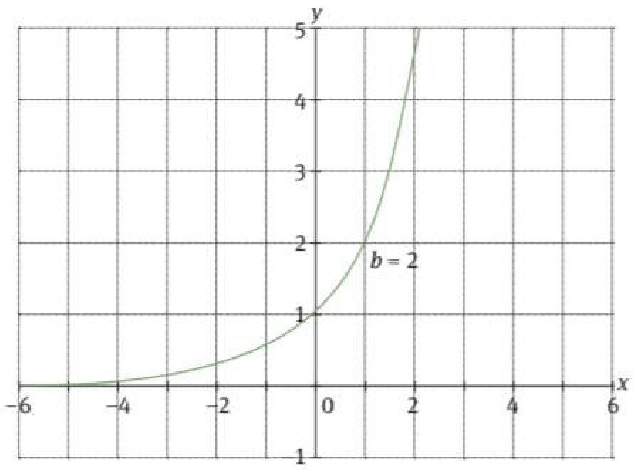


Equation 1: $y = \frac{1}{x}$

2.3 The graph of $y = b^x$, $b > 0$ and $b \neq 1$

- The graph forms an exponential graph
- Axis of symmetry: $y = x$
- Intercept with y-axis at $y = 1$

x (domain)	-2	-1	0	1	2
y (range)	0,25	0,5	1	2	4



Equation 1: $y = 2^x$

Unit 3

The graph of $y = ax^2 + q$

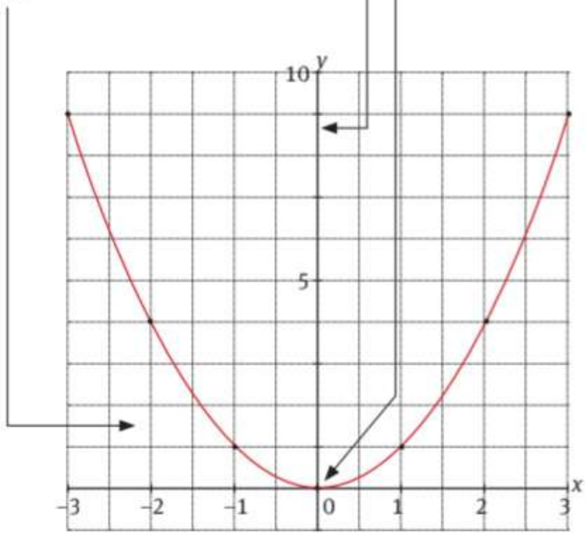
3.1 The effect of a

- The effect of a is a vertical stretch and/or a reflection about the x-axis.
- The value of a gives an indication of the steepness (wide or narrow) of the two "arms".
- For $a > 0$:
  - The parabola will have a minimum value.
  - The "arms" point upwards.
  - Range is $y \geq q, y \in R$

- For $a < 0$:
  - The parabola will have a maximum value.
  - The "arms" point downwards.
  - Range is $y \leq q, y \in R$

3.2 The effect of q

- The y-intercept (where the graph cuts the y-axis) is equal to the value of q or $y = q$ (this is also the maximum or minimum value).
- The effect of q is a vertical shift.
- The turning point of the graph is at $(0; q)$.
- The axis of symmetry is the y-axis ($x = 0$).
- Domain is $x \in R$.

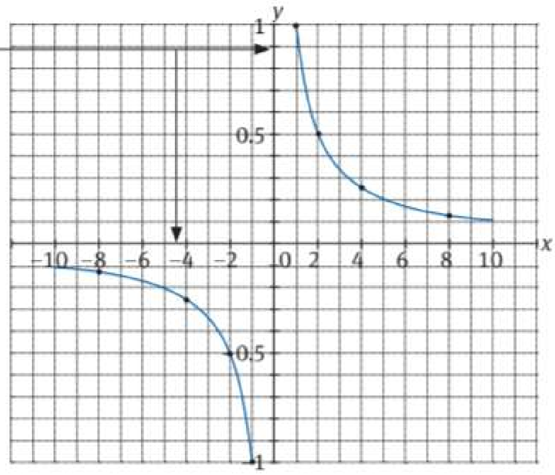


Unit 4

The graph of $y = \frac{a}{x} + x$

4.1 The effect of *a*

- Graphs that represent the equation $y = \frac{a}{x} + q$ form a hyperbola.
- We call the x- and y-axes the asymptotes.
  - The line $x = 0$ (y-axis) is called a vertical asymptote of the function if y approaches infinity (positive or negative) as x approaches 0.
  - The line $y = 0$ (parallel to x-axis) is called a horizontal asymptote of the function.
- For $a > 0$, the graph lies in the 1st and 4th quadrants
- For $a < 0$, the graph lies in the 2nd and 3rd quadrants.



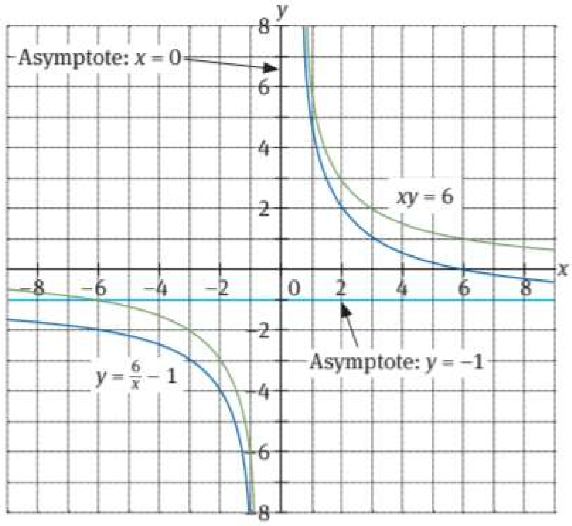
4.2 The effect of *q*

- For $q = 0$, the graph will never intercept the x- or y-axis because division by 0 is undefined.
- For $q \neq 0$, the line $y = q$ is the horizontal asymptote and $x = 0$ the vertical asymptote.
- Domain: all real x-values, except 0
- Range: all real y-values, except $y = q$

Example

Sketch the graph of $y = \frac{6}{x} - 1$
Domain: $x \in \mathrm{R}, x \neq 0$
Range: $y \in \mathrm{R}, y \neq 1$
x-intercept (let $y = 0$): $x = 6$
y-intercept: none
Asymptotes:

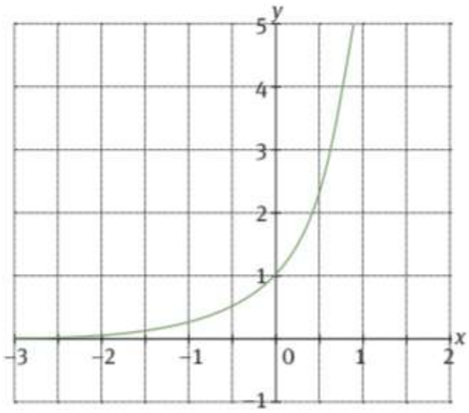
- The graph is only decreasing, for $x \in \mathrm{R}, x \neq 0$
- The graph shifts down one unit: new asymptote, $y = -1$
- Therefore, the asymptotes are: y-axis and $y = -1$


Asymptote: $x = 0$
$xy = 6$
Asymptote: $y = -1$
$y = \frac{6}{x} - 1$
y
x
8
6
4
2
0
−2
−4
−6
−8
2
4
6
8
−8
−6
−4
−2

Unit 5

The graph of $y = ab^x + q$, $b > 0$ and $b \neq 1$

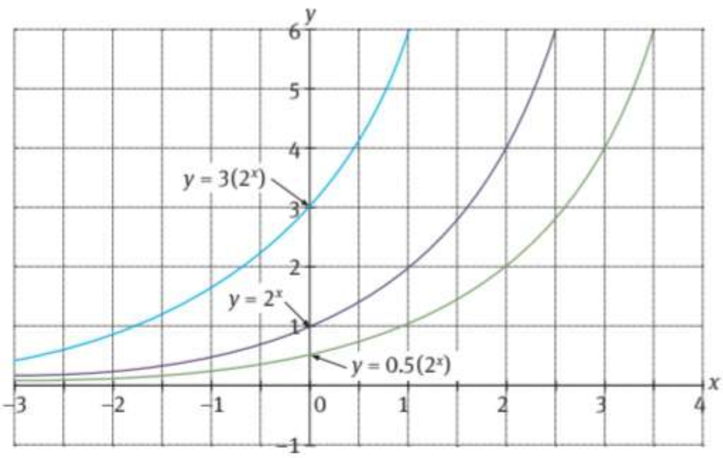
5.1 Possible values of b



- Graphs that represent the equation $y = ab^x + q$ form an exponential graph in two quadrants of the Cartesian plane.
- When $b > 1$ the y-values increase as x increases and the function is called an increasing function.
- The graph will incline to the right.
- When $0 < b < 1$ the y values will decrease and the function is a decreasing function.
- The graph will decline to the right.
- The y-intercept (where the graph cuts the y-axis) is equal to the value of a for $q = 0$ and is therefore $a + q$ for any other value.
- There are no x-intercepts.
- The line $y = q$ is the asymptote for the graph.
- Domain: $x \in R$
- Range: $y > 0, x \in R$

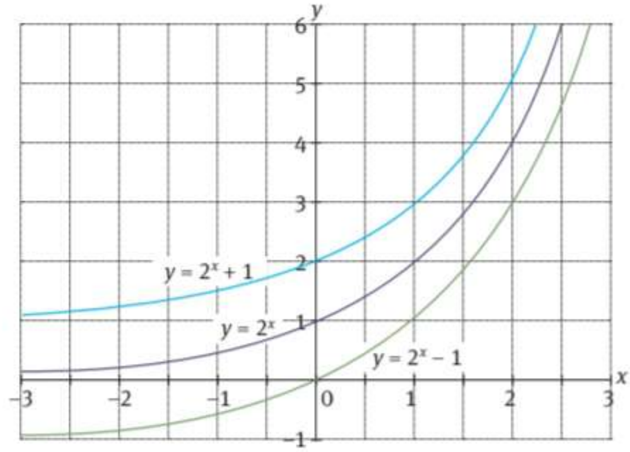
5.2 The effect of a

- a affects the steepness of the graph
- y-intercept: $y = a$
- If $a < 0$, then the graph reflects about the y-axis



5.3 The effect of q

- The value of q shifts the graph up or down (vertical shift)
- y-intercept: $1 + q$



Unit 6

Sketching graphs

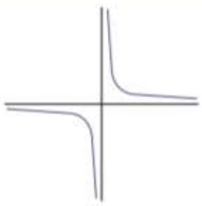
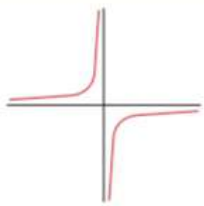
6.1 The parabola

To plot graph of any quadratic function (parabola), we need answers to these questions:

- What is the sign of a (the coefficient of x^2 in quadratic function)?
- Does the graph of quadratic function intersect the x-axis? And if it does, at what point does it intersect?
- Does the graph of quadratic function intersect the y-axis?
- What is the maximum or minimum value of function?

For $a > 0$: The parabola will have a minimum value. The "arms" point upwards. 	For $a < 0$: The parabola will have a maximum value. The "arms" point downwards. 
x-intercept(s): Let $y = 0$ and solve for x	y-intercept(s): Let $x = 0$ and solve for y
Because $y = ax^2 + q$ has no term in x, the axis of symmetry is the y-axis.	The turning point of the parabola is $(0; q)$

6.2 The hyperbola

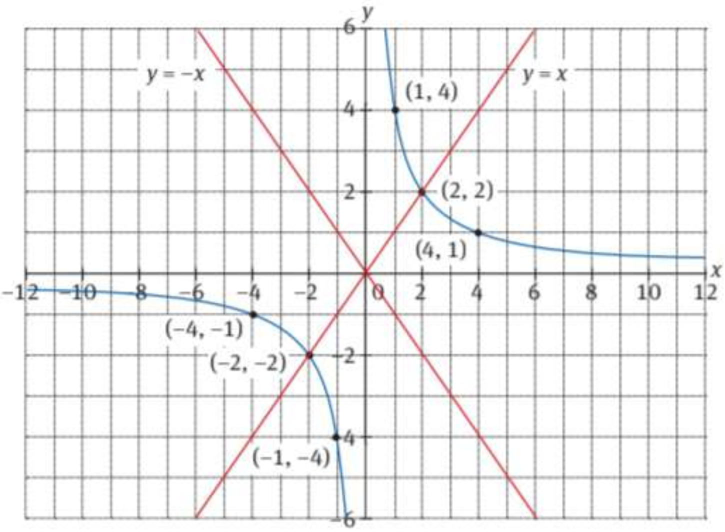
$a > 0$ 	$a < 0$ 
x-intercepts: None, except when $q \neq 0$. Then put $y = 0$ and solve for x	y-intercepts: None, the y-axis is an asymptote
Asymptotes: The y-axis ($x = 0$) The 2nd asymptote is the line $y = q$	
The closest points to the axes: $(\sqrt{a}; \sqrt{a} + q)$ and $(-\sqrt{a}; -\sqrt{a} + q)$	Domain: $x \in R; x \neq 0$ Range: $y \in R; y \neq q$

Example

Given $y = \frac{4}{x}$, sketch the graph of the function.

- Show the coordinates where $x = 1$, $x = -1$, $x = 4$ and $x = -4$ clearly on your graph. To do so, work out the y-values and plot the points: $(1; 4)$, $(-1; -4)$, $(4; 1)$, $(-4; -1)$

- Draw two axes of symmetry on your sketch and give their equations. ($y = x$ and $y = -x$)
- Find the point(s) of intersection of the hyperbola and the axes of symmetry. $(\sqrt{a}; \sqrt{a} + q)$ and $(-\sqrt{a}; -\sqrt{a} + q) = (-2; -2)$ and $(2; 2)$
- The graph will be in quadrants 1 and 3, because $a > 0$
- The domain is $x \in R, x \neq 0$
- The range is $y \in R, y \neq q$



6.3 The exponential function

Use the characteristics of the graph to determine the general shape.

a	b	Characteristic
$a < 0$	$0 < b < 1$ (a positive fraction)	decreasing
$a > 0$	$0 < b < 1$ (a positive fraction)	Increasing
$a < 0$	$b > 1$	Decreasing
$a > 0$	$b > 1$	Increasing

- If $a < 0$, the graph is below the asymptote and if $a > 0$, the graph is above the asymptote.
- asymptote: $y = q$
- y-intercept: $y = a + q$
- Domain: $x \in R$
- Range if $a < 0$: $y < q, y \in R$
- Range if $a > 0$: $y > q, y \in R$

Example

Given: $f(x) = 3$, sketch the graph of f. Show any intercepts with the axes.

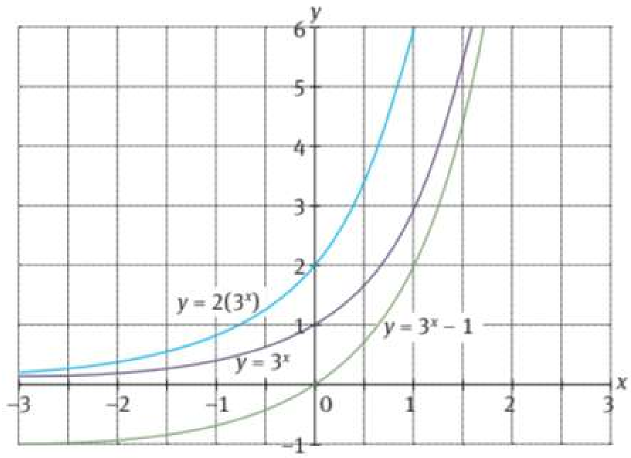
General form: $y = ab^x + q$

$a > 0$ (=1)	$b > 1$ (=3)	Increasing

Therefore:

- Asymptote: $y = 0$ ($q = 0$)
- y-intercept: $y = 1$ ($a = 1$)
- Domain: $x \in R$
- Range if $a > 0$: $y > q$, $y \in R$

On the same system of axes, sketch the graph of $g(x) = 2.3^x$. Clearly show any intercepts with the axes.



Unit 7

Finding the equations of graphs

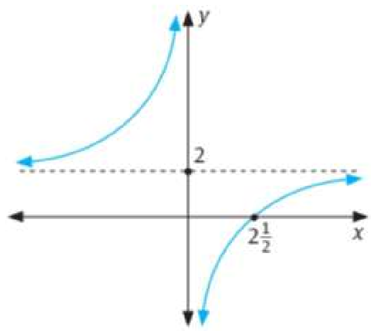
To find the equation of a graph, use the general form of the equation and the characteristics of the graph to write down what you know. Then, substitute point(s) from the graph into the equation to find the values of any unknown variables.

Example

The graph alongside is the graph of $y = \frac{a}{x} + q$.

1 Find the equation of the graph.

2 Give a reason why $y = 2$ is an asymptote.



1 The asymptote is at $y = 2$, so we know $q = 2$.
We can then use the point (2; 0) in the general equation to find a:

$0 = \frac{a}{\frac{5}{2}} + 2$

$\frac{a}{\frac{5}{2}} = -2$

$a = -2 \times \frac{5}{2}$

$a = -5$

Therefore, $y = \frac{-5}{x} + 2$

The range of the graph is $y \in \mathbb{R}; y \neq 2$

2 If we make x the subject of the equation, we have:

$y = \frac{-5}{x} + 2$

$xy = -5 + 2x$

$x(y - 2) = -5$

$x = \frac{-5}{y - 2}$

Therefore, we cannot have $y = 2$, because division by 0 is undefined.