7. Finance, Growth and Decay

A. Simple and Compound Interest

Formula for simple interest:

Growth $A = P(1 + i.n)$

A is the final value of the money invested or borrowed, or the total Accumulated value, i.e. P + the interest accrued.
P is the principal or initial amount invested or borrowed, also known as the capital.
i is the interest rate at which growth takes place over a year, represented as a decimal.
n is the number of compounds that the money has been invested or borrowed – i.e. number of years × no of times interest is given per year.

This formula is for calculating the growth of an investment or loan. By replacing the plus sign with a minus sign, we change the formula to simple interest decay/depreciation:

Decay $A = P(1 - i.n)$ See Straight-line decay

Compound Interest formula:

Growth $A = P(1 + i)^n$

A is the final value of the money invested or borrowed, or the total accumulated value, i.e. P + the interest accrued.
P is the principal or initial amount invested or borrowed, also known as the capital.
i is the interest rate at which growth takes place over a year, represented as a decimal.
n is the number of compounds that the money has been invested or borrowed.
$(i.e. number\ of\ years \times no\ of\ times\ interest\ is\ given\ per\ year)$

This formula is for calculating the growth of an investment or loan. By replacing the plus sign with a minus sign, we change the formula to compound interest decay:

Decay $A = P(1-i)^n$ See Reducing-balance decay.

Example: Simple and Compound interest

John has R10 000 that he wants to save for a period of 10 years. Which of these three options would yield the most money in his savings account?

Option A: Simple interest at 8,5% per annum (p.a.)

Option B: 8,5% p.a. interest compounded annually

Option C: 8,5% p.a. interest compounded quarterly

Answer: *For all three options:*

P = R10 000

$$i = \frac{8,5}{100} = 0,085$$

n = 10

Option A

$$\begin{aligned} A &= P(1 + i.n) && \text{write the formula for simple interest.} \\ &= 10\,000(1 + 0,085 \times 10) && \text{substitute values for } P, i \text{ and } n. \\ &= R\ 18500.00 && \text{find the answer on the calculator} \end{aligned}$$

Option B

$$\begin{aligned} A &= P(1 + i)^n && \text{write the formula for compound interest} \\ &= 10\,000(1 + 0,085)^{10} && \text{substitute values for } P, i \text{ and } n \\ &= R22\ 609,83 && \text{find the answer using your calculator} \end{aligned}$$

Option C

If the interest paid is 8,5% interest p.a. this is the interest paid over 1 full year. Therefore if interest is compounded quarterly, then four equal interest amounts of $\frac{0{,}085}{4}$ *or 0,02125 will be paid every three months. Also, over each year interest is paid four times (every 3 months), therefore interest will be paid 40 times* $(10 \ \times 4)$.

$A = P(1+i)^n$ write the formula for compound interest

$= 10\ 000\left(1+\frac{0{,}085}{4}\right)^{10\times 4}$ substitute values for *P, i* and *n*. Remember to adjust by dividing *i* by 4 and to multiply *n* by 4 as the interest is compounded four times per year(quarterly).

$= \ R23\ 189{,}04$ find the answer on your Calculator.

As you can see option C yields the most money.

Answer for Option C

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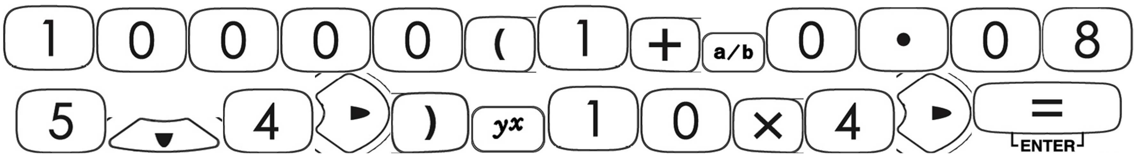

B. The use of Logarithms to calculate the value of n in the formulae

$$A = P(1+i)^n \ \ and \ \ A = P(1-i)^n$$

A very important part of Grade 12 Financial mathematics is to calculate *n*, the time period of the investment or loan.

In grade 10 and 11 we calculate by trial-and-error but in Grade 12 we can now use Logarithms to calculate n.

Example

Andrew is a financially savvy young boy. He wants to start saving money for a deposit on a car. To buy his favourite car he would need a deposit of R20 000. To date he has R5 000 in his savings account. His dad promised that he would double any amount that he saves up. The bank that his father suggests offers an interest rate of 9,8% p.a. compounded annually.

As his dad will meet him half way, he will only have to save R10 000. How long will he have to save in order to double the money in his savings account?

$A = P(1+i)^n$ Always write the formula for compound interest.

$10\ 000 = 5000(1+0{,}098)^n$ substitute A, P and i

$\frac{10\ 000}{5000} = 1{,}098^n$ divide A by P

$2 = 1{,}098^n$ simplify RHS

$\log_{1{,}098} 2 = n$ Log Definition

$\therefore \mathrm{n} = 7{,}4$

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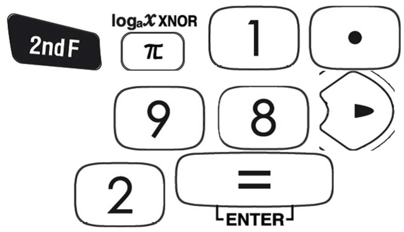


It will thus take him 7,4 years to save R10 000.

The term n is calculated by the formula:

$$n = \log_{(1+i)} \frac{A}{P}$$

Nominal and effective interest rates:

Interest rates cannot be compared unless their compounding periods are the same. By "converting" nominal rates into effective interest rates, we can compare two interest rates properly.

In many cases, depending on local regulations, interest rates as quoted by lenders and advertisements are based on nominal, not effective interest rates. This may give the lender or investor the wrong impression about the actual effective interest rate.

To convert from nominal to effective interest rate, we use the formula:

$$i_{eff} = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1 \qquad \text{or} \qquad i_{eff} = \left(1 + \frac{i_{nom}}{m}\right)^m - 1$$

i is the effective interest rate per annum

$i^{(m)}$ is the nominal interest rate per annum compounded m times in one year

m is the number of times that interest has been compounded in one year.

NB! Both m values must be the same.

It is important to take note that the effective interest produces the same rand value as nominal interest however, effective interest is the actual rate received over a full year whereas nominal interest rate is the adjusted rate over the compound period.

Example:

You have a choice of three banks at which you are going to invest a sum of money:

Bank A: 12% compounded semi-annually

Bank B: 11.9% compounded quaterly and

Bank C: 12.1% compounded annually

Use effective interest rates to determine which option would be best.

Answer:

Bank A:
$$i_{eff} = \left(1 + \frac{i_{nom}}{m}\right)^{m} - 1$$
$$\therefore i_{eff} = \left(1 + \frac{0.12}{2}\right)^{2} - 1$$
$$\therefore i_{eff} = 0.1236$$
$\therefore$ Effective interest rate for Bank A: 12.36%

Bank B:
$$i_{eff} = \left(1 + \frac{0.119}{4}\right)^{4} - 1$$
$$\therefore i_{eff} = 0.12441648$$
$\therefore$ Effective interest rate for Bank B: 12.44%

Bank C: Effective interest rate will only differ from nominal interest rate if it is compounded more than once a year, therefore, for Bank C, the effective interest rate equals the Nominal interest rate which equals 12.1%.

Therefore the best option would be Bank B.

Activity 7.1

The following are nominal rate statements:

Nominal Rate (r)   Time Period (t)   Compounding Period (CP)

1) 12% interest per year, compounded monthly.
2) 12% interest per year, compounded quarterly.
3) 3% interest per quarter, compounded monthly.

What are the corresponding effective annual interest rates?

C. Annuities

An annuity is a fixed amount paid into an account at regular intervals that accrues interest. There are two types of annuities:

Savings fund
- Retirement annuity
- Education policy
- monthly savings

Future-value annuity:

A fixed sum of money paid at regular intervals into a savings or investment account with the idea of growing capital. Compound interest is paid on money deposited into the account. In future-value annuities you mostly save money. These fixed payments are usually paid at the start of each month or time period so that the last payment will have a chance to accrue interest.

If this is not the case, then one should be very careful when calculating the number of payments made and which of them will be allowed to accrue interest.

Future-value annuities are for capital accumulated by equal regular payments!

The formula for a future value annuity is: $F_v = \frac{x((1+i)^n - 1)}{i}$

Where:
$i -$ interest rate represented as a decimal.
$n -$ number of times the annuity will be paid.
Example: If there are monthly payments for 10 years, then $n = 10\ (no\ of\ years) \times 12 (no\ of\ months) = 120$
$x -$ fixed payment that is invested in the annuity at regular intervals.

We use x if the last payment will not have a chance to accrue any interest.

If the last payment will accrue interest, then we use replace x in the F_v equation with $x(1+i)$.

Thus the equation will be: $F_v = \frac{x(1+i)[(1+i)^n - 1]}{i}$
This is a very important as most textbooks do not differentiate between x and $x(1+i)$. It will never be stated whether or not the last payment will accrue interest or not. You have to read each question carefully and decide whether or not the last payment will accrue interest.

The future value annuity formulas are derived from the sum of a geometric sequence formula. This is why some teachers prefer to solve annuity problems by means of the geometric series formula:

$$S_n = \frac{a(r^n - 1)}{r - 1};\ r \neq 1$$

Where:
$S_n = F_v$
a = the fixed payment x, if the last payment does not accrue interest.

or

a = $a(1+r)$ if the last payment has one more period to accrue interest.
r = interest rate represented as a decimal.
n = number of years $\times$ number of times payments that are made per year.

Using these two methods to solve Future-value annuities

R 2000 is paid into an investment fund at the beginning of each year for 4 consecutive years. The interest rate is 12,5% p.a. compounded annually. Use the compound interest formula to calculate the total amount of money at the end of the four year investment.

Method 1

Calculate the total amount by calculating the growth of each deposit one at a time.

$$A = 4th\ deposit + 3rd\ deposit + 2nd\ deposit + last\ deposit$$

Each deposit accrues interest individually (12,5% compounded annually).

The first R2000 deposited will accumulate interest for 4 years:
$A_1 = 2000(1 + 0{,}125)^4$

$\therefore A_1 = 2000(1{,}125)^4$

Notice that a does not equal A_1 but A_4

The second deposit of R2000 will accumulate interest for 3 years:
$A_2 = 2000(1{,}125)^3$

The third deposit of R2000 will accumulate interest for 2 years:
$A_3 = 2000(1{,}125)^2$

The fourth deposit of R2000 will only accumulate interest for 1 year:
$A_4 = 2000(1{,}125)^1$

To calculate the total amount saved we calculate A. The sum of $A_4 + A_3 + A_2 + A_1$ is equal to (total accumulated value):
$A = A_4 + A_3 + A_2 + A_1$

$A = 2000(1{,}125) + 2000(1{,}125)^2 + 2000(1{,}125)^3 + 2000(1{,}125)^4$
$A = R10832{,}52$

The sum of $A_1 + A_2 + A_3 + A_4$ forms a Geometric sequence

This brings us to:

Method 2

Therefore we can also calculate ***A*** using the as the sum of the geometric sequence formula:

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

$$A = A_1 + A_2 + A_3 + A_4$$
$$= 2000(1{,}125) + 2000(1{,}125)^2 + 2000(1{,}125)^3 + 2000(1{,}125)^4$$

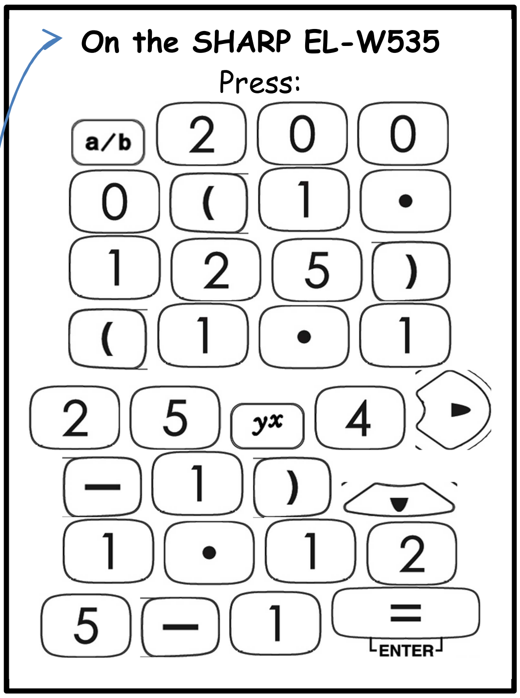
$$A = S_n = f_v$$

$a = first\ term = 2000(1{,}125)^1$
as the last deposit will accrue interest.

$$r = \frac{T_2}{T_1} = \frac{2000(1{,}125)^2}{2000(1{,}125)} = 1{,}125$$
according to sequences and series theory.

Thus:

$$A = S_n = \frac{2000(1{,}125)(1{,}125^4 - 1)}{1{,}125 - 1}$$
$A =$ R10 832,52



Example:

Samuel realizes that his daughter is quite clever. He will have to start putting money away for her to go to University. He decides to pay R200 per month into a savings account for the next 10 years. The interest rate of the savings account is 8% p.a. compounded monthly. Let us see what impact this has on his total amount saved at the end of the 10 years.

The interest rate is 8% p.a. compounded monthly, therefore each month they receive $\frac{0,08}{12}$ in interest.
After carefully reading the question, you will realize that there will be $10 \times 12 = 120$ payments.

Payment 1 will accumulate interest on a monthly basis for 10 years. That is $10 \times 12 = 120$ times.

Payment 2 will accumulate interest on a monthly basis for 9 years and 11 months = 119 times.
Payment 3 will accumulate interest on a monthly basis for 9 years and 10 months = 118 times.
And so forth…
Surely at the end of the ten years, if payments are made at the start of each month, then payment 120 will accumulate interest for only one month.

Once again this will form a geometric sequence, so we can use the sum of a geometric sequence formula or the future-value annuity formula.

Since the final payment will accrue interest we use $a = 200\left(1 + \frac{0,08}{12}\right)$

Sum of Geometric Sequence formula

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

Future value annuity formula

$$F_v = \frac{x(1+i)[(1+i)^n - 1]}{i}$$

Where: $\boldsymbol{A} = S_n = F_v$

$$i = 8\% = \frac{0.08}{12}$$

$$a = 200\left(1 + \frac{0,08}{12}\right)^1$$

$$r = \left(1 + \frac{0,08}{12}\right)^n$$

$$n = 12 \times 10$$

Thus:

$$A = S_n = \frac{200\left(1+\frac{0,08}{12}\right)^1\left[\left(1+\frac{0,08}{12}\right)^{10\times 12}-1\right]}{\left(1+\frac{0,08}{12}\right)-1} \quad F_v = \frac{200\left(1+\frac{0.08}{12}\right)[\left(1+\frac{0.08}{12}\right)^{10\times 12}-1]}{\frac{0,08}{12}}$$

$A =$ R36 833,14 $\qquad F_v = R36\,833{,}14$

Therefore if he pays R200 per month for 10 years, at 8% compound interest compounded monthly, he will have saved R36 833,14 for his daughter's studies.

<u>Activity 7.2</u>

1. Calculate the total amount of money saved if Adam pays an amount of R15 000 out of his yearly bonus into a savings account. He makes the payments at the beginning of each year for a period of 10 years. The interest rate is 11% p.a.

2. You want to save money to buy a house in 8-years-time. You decide to save R800 at the end of each month in a savings account. The interest on the savings account is 7,8% compounded monthly. How much money will you have saved by the end of the 8 years?

Present-value annuity:

Repayments:
- Bond
- Car payments

Money is paid by regular equal payments in order to pay back a loan. Interest is charged on the outstanding balance.

Instalments on paying back loans are usually paid at the end of each month.

The formula for a present-value annuity is:

$$P_v = \frac{x(1-(1+i)^{-n})}{i}$$

Where:

$i -$ interest rate as a decimal.

$n -$ number of loan repayments made..
Example: If there are monthly payments for 10 years, then $n = 10 \times 12 = 120$

$x -$ the value of the loan repayment.
We use x if the last payment pays off the loan.

Or

If the last fixed payment is made a term before the end of the loan, then we replace x by $x(1+i)$. Thus the equation will be:

$$P_v = \frac{x(1+i)[1-(1+i)^{-n}]}{i}$$

A question that you might be asked is to deduct the Present-value annuity from the future- value annuity formula. To do this we:

$P_v(1+i)^n = \frac{x[(1+i)^n - 1]}{i}$ set the Compound interest formula equal to the future-value formula but let $P = P_v$.

$P_v = \frac{x[(1+i)^n - 1](1+i)^{-n}}{i}$ divide both sides by $(1+i)^n$, thus times RHS by $(1+i)^{-n}$.

$P_v = \frac{x[(1+i)^n(1+i)^{-n}-1(1+i)^{-n}]}{i}$ distribute $(1+i)^{-n}$ into the bracket.

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$ simplify, as $(1+i)^n(1+i)^{-n} = 1$

You now have the present-value annuity formula!

<u>Example</u>

Andrew takes out a personal loan of R50 000 to do some repairs to his house. Interest is charged at 12,5% p.a. compounded monthly. The loan is paid in 36 equal payments. He will start paying back the loan one month after he receives the loan. Determine:

a) His monthly instalments on the loan.
b) The amount due after the first year of paying off the loan.

Answer:

a) He is paying back a loan so we are using the present-value annuity formula.

$P_v = 50\,000$
$x = ?$
$i = \frac{0{,}125}{12}$
$n = 36$

$\therefore\ P_v = \frac{x[1-(1+i)^{-n}]}{i}$ Always first write the equation, you will use this equation as the last payment will not accrue interest.

$50000 = \frac{x\left[1-\left(1+\frac{0{,}125}{12}\right)^{-36}\right]}{\frac{0{,}125}{12}}$ Substitute all the values and simplify the R.H.S.

$\frac{50000}{29{,}8921} = \frac{x \times 29{,}8921}{29{,}8921}$ Divide both sides by 29,8921

$\therefore x = 1672{,}68$
His monthly instalments will thus be R1672,68.

b) After 1 year he has made 12 payments of R1672,68, $\therefore n = 24$ as he has 24 payments left.

$P_v = ?$

$x = 1672{,}68$

$i = \frac{0{,}125}{12}$

$n = 24$

$\therefore\ P_v = \frac{x[1-(1+i)^{-n}]}{i}$ Always first write the relevant equation.

$P_v = \frac{1672{,}68\left[1-\left(1+\frac{0{,}125}{12}\right)^{-24}\right]}{\frac{0{,}125}{12}}$ Substitute all the values.

$P_v = 35\ 357{,}75$ Calculate the present value of the loan.

He will still have to pay R35 357,75 if he continues paying back the loan.

Deferred payments

It can be arranged or advertised that repayments only start 3 months after the loan starts. Then you would have to add three months interest on the loan amount to the principal loan.

This is never a good option as starting repayments later than the standard one month, effectively increases the loan amount.

Activity 7.3

1. Michelle buys a motorbike. The total loan is R75 000. She decides to pay the motorbike off over a 24 month period from the current date but her first payment will be two months later. RBC credit services offer her an 11,5% p.a. interest rate compounded monthly.

   Determine:
   a) Her monthly instalments on the loan.
   b) What is her outstanding balance after 10 months of payments (therefore 12 months after she received the loan)?

D. Depreciation

Straight-line depreciation: $A = P(1 - i.n)$

A is the future value of the asset after depreciation.
P is the principal or initial amount value of the asset.
i is the rate of depreciation per annum as a decimal.
n is the number of years.

Reducing-balance depreciation: $A = P(1 - i)^n$

A is the future value of the money after depreciation.
P is the principal or initial value of the asset.
i is the interest rate per annum as a decimal.
n is the number of years that the money is invested or borrowed.

Example:

The value of a car after 6 years is R120 000. The car loses value at 11.75% p.a. interest rate annually on a reducing-balance basis. The owner of the car did not have cash to pay the R4034.65 administrative fee when he bought the car, so he decided to add this to the loan amount of the car. What was the original value of the car excluding the administrative fee?

$A = 120000$
$P = ?$
$i = 0{,}1175$
$n = 6$

$A = P(1-i)^n$ Always first write the equation.

$120\ 000 = P(1 - 0{,}1175)^6$ Substitute the values into the equation and simplify both sides.

$\frac{120\ 000}{0{,}4723\ldots} = \frac{P \times 0{,}4723\ldots}{0{,}4723\ldots}$ Divide both sides by 0,4723…
$P = 254\ 034{,}65$ Calculate the principle amount(P)
$P = R254\ 034{,}65 - R4\ 034{,}65.$ Subtract the administrative fee.

$P = R250\ 000{,}00$

The original value of the car was R250 000.

Sinking funds

Equipment does not last forever. Companies regularly need to replace old equipment with new equipment to run their business effectively. Old equipment can be sold as scrap or as second hand goods.

Buying new equipment can be very costly. Companies have to plan ahead in order to buy this new equipment. They can do this by “saving” money in a fund so that they can make sure that they will have the finances available in due time to buy such new equipment. This is called a sinking fund.

Old equipment has a book value. This book value is the current value of an asset after depreciation has been taken into account. If an asset is at the end of its useful life and cannot be sold as second hand equipment, then we say that the asset has a scrap value.

Example

A school buys a bus for R850 000. The bus depreciates at 11% p.a. on reducing balance basis. Its' useful life is 15 years. The inflation rate of a new bus will be 5% p.a. The old bus will be sold as a second hand bus in 10 years- time.

A sinking fund (savings account) will be set up by the school. Monthly deposits earning 11.5 % p.a. compounded monthly will be made. The first payment will be made a month after the bus was purchased.

a) Calculate the second-hand value of the bus in 10 years.
b) What will the same bus cost in 10 years?
c) What is the amount required to save in the sinking fund in 10 years?
d) Calculate the fixed equal monthly payments into sinking fund.

Answers:

a) *Second hand value* (Use **compound decay** formula and depreciate the bus's value for 10 years).

$$A = P(1-i)^n$$
$$= 850\ 000(1-0{,}11)^{10}$$
$$= R265\ 044{,}62$$

b) *The Cost of the same bus in 10 years-time*(Use **compound interest** to inflate the bus's value for 10 years)

$$A = P(1+i)^n$$
$$= 850\ 000(1+0.05)^{10}$$
$$= R1\ 384\ 560.43$$

c) *Amount required to save in the sinking fund in 10 years:*

Sinking fund = Cost of new bus – second hand value

$$= 1\ 384\ 560{,}43 - 265\ 044{,}62$$
$$= R1\ 119\ 515{,}81$$

d) *Fixed equal monthly payments into sinking fund.*

Number of payments: $n = 10 \times 12 = 120$ equal monthly payments

$F_v = R1\ 19\ 515{,}81$

$i = \frac{0{,}115}{12}$

$F_v = \frac{x[(1+i)^n - 1]}{i}$ Always first write the equation.

$1\ 119\ 515{,}81 = \frac{x\left[\left(1+\frac{0{,}115}{12}\right)^{10\times 12} - 1\right]}{\frac{0{,}115}{12}}$ We substitute al the values. We

will use this formula with x as the last payment is made at the end of the 10 year period and will not accrue any interest.

$\frac{1134035{,}11}{223{,}40} = \frac{x \times 223{,}40}{223{,}40}$ Divide both sides by 223,4

$\therefore x = R5\ 011{,}26$

They will have to pay $R5\ 011{,}26$ per month into the sinking fund in order to replace the bus in 10 years time.

Activity 7.4

1. A newspaper company buys a printing press for R 179 800 in 2003. They decide that the printing press should be replaced in 7 years-time.

   a) If the machine depreciates at a rate of 7,6% per year, what will the value of the printing press be in 7 years-time?
   b) If the value of the printing press appreciates by 5,98% per year, what will the value of a new printing press be in 7 years-time?

c) What is the total amount that the company should save in a sinking fund?

d) What will the monthly instalments be if the company uses this amount for their sinking-fund and the bank gives them an interest rate of 11,58% compounded monthly?

2. Your mom decides to buy a new car. She buys a car that costs R146 000 in total including all the administration costs. She decides that the next time she buys a car she would like to pay for it in cash.

   a) If her current car depreciates at a rate of 6,32%, how much will her car be worth in 4 years-time?

   b) If she buys the same type of car in 4 years-time and it appreciates at a rate of 7,13%, how much will the new car cost?

   c) What amount should your mom have to save in a sinking-fund in order to buy a new car?

   d) If the bank offers an interest rate of 9,34% compounded monthly, what monthly instalment should your mom pay every month?

3. A coffee shop buys a multi-functional coffee machine for R14 999. They also buy a fire-alarm system for R 12 000. They ask you to help them calculate how much they would need to pay every month in order to upgrade their coffee machine and fire-alarm system in 5 and a half years-time. They tell you that a coffee machine depreciates at a rate of 3,78% per annum and a fire-alarm system depreciates 11,75% per annum. A coffee machine's value appreciates at a rate of 6,4% per annum and a fire-alarm system's value appreciates at a rate of 7,9% per annum. If the bank gives an interest rate of 14,34% compounded quarterly what will the coffee shop's quarterly instalments be?

E. Loan Repayments

We now use the Present value formula:

$$Loan\ with\ (+)\ interest\ =\ repayments\ with\ interest$$

Loan repayments using the Present-Value formula:

Example:

Adam buys a car for R240 000,00. He takes a loan from his bank. The bank charges 11% interest p.a. compounded monthly. He will repay the loan over a period of 5 years. Repayments start at the end of the month the loan is drawn. What are his monthly repayments?

Answers:

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$

$$240\ 000 = \frac{x\left[1-\left(1+\frac{0.11}{12}\right)^{-5\times 12}\right]}{\frac{0.11}{12}}$$

$P = 240000$

$i = \dfrac{0,11}{12}$

$n = 5 \times 12 = 60$

$x = ?$

When using the present-value formula:

$$Loan\ +\ interest\ =\ Repayment\ with\ interest$$

$P(1+i)^n = \frac{x[(1+i)^n - 1]}{i}$ Set Compound interest formula = Future-value formula

$240000\left(1+\frac{0.11}{12}\right)^{5\times 12} = \frac{x\left[\left(1+\frac{0,11}{12}\right)^{5\times 12}-1\right]}{\frac{0,11}{12}}$ Substitute all the values.

$414939{,}78 = x \times 79{,}5180\ldots$ Calculate L. H. S. and simplify R. H. S.

$x = 5218{,}18$ Solve for x by dividing both sides by 79,5180…

His monthly repayments will be R5218,18.

Or: When using the Future-value formula:

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$ First write the formula

$240000 = \frac{x\left[1-\left(1+\frac{0.11}{12}\right)^{-5\times12}\right]}{\frac{0.11}{12}}$ Substitute all the values

$240000 = x \times 45{,}99303\ldots$ Calcualte the L.H.S. and Simplify the R.H.S.

$\frac{240000}{45{,}99303\ldots} = \frac{x\times45{,}99303\ldots}{45{,}99303\ldots}$ Isolate x by dividing by 45,99303…

$x = 5218{,}18$ His monthly repayments will be R5218,18.

Example:

A loan is taken out to buy a new sound system for a wedding venue. The loan is repaid with 3 equal payments of R4000,00. The first payment is made one year after he bought the set, and the second one year later and so forth. Interest is calculated at 7,5% p.a. What was the initial value of the loan?

<u>Use the Present-value formula.</u>

The repayments represent the present-value and include the interest needed to repay the loan.

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$ Always write the formula first

$P_v = \frac{4000[1-(1+0{,}075)^{-3}]}{0{,}075}$ Substitute your values.

$P_v = 10402{,}10$ Solve for P_v

The initial amount of the loan was $R10\ 402{,}10$.

Activity 7.5

1. A couple takes out a loan from the bank in order to pay for their wedding. They pay the bank R 3 560 every month for 3 years. The bank charges them 9,5% p.a. compounded monthly. What was the original value of their loan?

2. Mrs Busisiwe takes a loan from the bank in order to re-furnish her house. She decides that she needs R 50 000 in order to get the furniture that she wants. The bank agrees to give her the loan and charges her 15,7% interest compounded quarterly. If her repayment is R 2 500 every third month, how long will it take her to pay back the loan rounded off to the nearest year?

3. Joe and Kate decide to start their own business. They go to NBC Bank and ask them what their repayments would be if they took out a loan of R 100 000. The bank says that they can pay the loan back over 5 years and with at an interest rate of 12,3% compounded monthly. What amount will the NBC Bank staff tell Joe and Kate if they have to make monthly instalments?

4. Susan's mother decides to take out a loan in order to pay for Susan's first year at university. The fees for the first year of university are R23 500. The bank tells Susan's mom that they will charge her 3.67% interest compounded monthly and that she has to pay the loan back over 2 years. What will Susan's mother's monthly instalments be?

5. Judy decides to buy musical equipment for her school. The music shop says that Judy can pay off the loan over 2 years at R 350 per month. They say that they will charge her 5,8% interest p.a. compounded monthly. What was the original loan that Judy took out from the music shop?

6. George offers short term loans. He decides that for a loan of a certain value, he will charge a monthly instalment of R 2 380. He will ask people to pay the loan back over 4 years and charges these people 16,78% compounded monthly. What is the value of the loan that George will give?

F. Real Life Growth and Decay

Examples:

1. Given that a population increased from 120 000 to 214 000 in 10 years, at what annual (compound) rate was the population growing?

Answer:

$A = P(1+i)^n$	Always first write the equation
$214000 = 120000(1+i)^{10}$	Substitute $A, P\ and\ n$.
$\frac{214000}{120000} = (1+i)^{10}$	Divide both sides by P (120 000)
$\sqrt[10]{\frac{107}{60}} = \sqrt[10]{(1+i)^{10}}$	Take $\sqrt[10]{ANS}$ (tenth root) of both sides.
$1{,}059554\ldots = 1 + i$	Subtract 1 from each side.
$\therefore 0{,}059554\ldots = i$	for the rate, multiply i by 100
$(\therefore rate\ =\ i \times 100)$	

$\therefore rate = 5{,}96\%$

Activity 7.6

1. In order to buy a car, John takes out a loan of R250 000 from the bank. The bank charges an annual interest rate of 11%, compounded monthly. The instalments start a month after he received the money from the bank.

   a) Calculate his monthly instalments if he has to pay back the loan over a period of 5 years.

   b) Determine the outstanding balance of his loan after two years (immediately after the 24th instalment).

G. Micro Lenders

Micro lenders offer short term loans at very high interest rates and simple interest is used.

There is no advantage in paying back the full amount before the time as interest is calculated upfront using simple interest based on the full amount of the loan over the repayable period.

Do you need cash instantly? You can have cash in your hands within 24 hours. Up to R100 000 cash available to you right now!!!
Call us on 0860 505 505
You can fund that dream holiday that you have always wanted.

Use this handy installment table to choose the loan that suits your budget:

Finance amount:	2 years(7,2%)	5 years(7,5%)	7 years(7,5%)
30 000	1 430	688	545
45 000	2 145	1 032	817
60 000	2 860	1 375	1 090
75 000	3 575	1 719	1 362
100 000	4 767	2 292	1 816

Examples:
Peter wants to take a R45 000 loan to consolidate his debt. He decides to repay the loan over a period of 5 years. How much will he end up paying back for the loan?

Answers:
First locate the monthly instalment: 45 000 over 5 years = R1032 per month.

$1032 \times 5 \times 12 = 61\ 920$ 5 years of 12 months each.

He will thus pay back R61 920 for his R45 000 loan.

Activity 7.7

1) A loan of R120 000,00 is taken out and repaid by equal monthly instalments over a 4 year period. Interest is set at 10,5% p.a. compounded monthly. Calculate the monthly repayment.

2) At what annual percentage interest rate, compounded monthly, should a lump sum be invested in order for it to triple in 15 years?

3) Joe buys a machine for work to the value of R10 000. He borrows the money on 1 January 2011 from a financial institution that charges interest at a rate of 11,25% p.a. compounded monthly. Joe agrees to pay monthly instalments of R495,15. The agreement of the loan allows Joe to start paying these equal monthly instalments from 1 July 2011.

    a) Calculate the total amount owing to the financial institution on 1 July 2011.
    b) How many months will it take Joe to pay back the loan?
    c) What is the balance of the loan immediately after Joe has made the 15th payment? **Similar to Paper 1 November 2010**

Answers for Activities

Activity 7.1

Answers:

Note, only ever round off your FINAL answer!

1. i what we are trying to calculate
   $i^{(m)}$ 12% compounded once a year i.e. 0,12
   m 12 times per year.

$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$ We always first write the formula

$1 + i = \left(1 + \frac{0{,}12}{12}\right)^{12}$ Now we substitute the values.

$1 + i = 1{,}12682 \ldots$ Calculate the RHS on the calculator

$i = 1{,}12682 \ldots - 1$ Subtract one from both sides.

$i = 0{,}12682 \ldots$

Now write the effective interest rate is a percentage:

$\therefore i \times 100$

$= 0{,}12682 \ldots \times 100$ Multiply i by 100.

$= 12{,}682 \ldots \%$ Round your final answer.

2) i = 0,125508….. thus the effective interest rate is 12,5508…%

3) i = 0,1268….. thus the rate is 12.68…%

Activity 7.2

1. Savings - F_v

The last payment will accrue interest!

$x = 15\ 000$

$i = 0{,}11$

$n = 10$

$F_v = \frac{x(1+i)[(1+i)^n - 1]}{i}$ First write the equation.

$F_v = \frac{15000(1+0{,}11)[(1+0{,}11)^{10}-1]}{0{,}11}$ Then substitute all the values

$F_v = R278\ 421{,}45$ Calculate the Future value of the annuity.

2. Savings - F_v

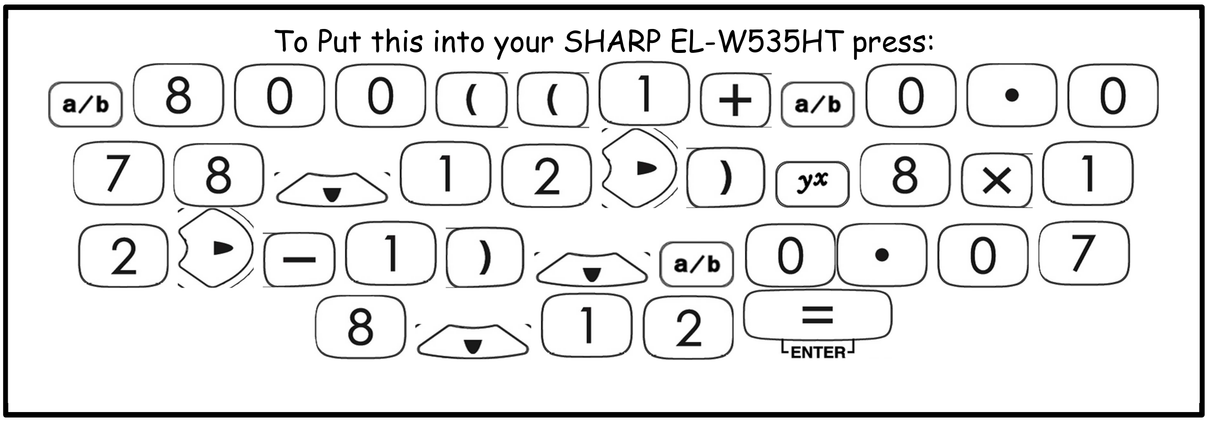
The last payment will not accrue interest!

$x = 800$
$i = 0{,}078$
$n = 8$

$$F_v = \frac{x[(1+i)^n-1]}{i}$$

$$F_v = \frac{800[\left(1+\frac{0{,}078}{12}\right)^{8\times12}-1]}{\frac{0{,}078}{12}}$$

$$F_v = R106\ 167.85$$



Activity 7.3

1. Her principal amount is now higher as she only starts paying off the loan two months after the normal starting time. Therefore we add two months' interest to the principal value to calculate the new principal value.

   a) The easiest way to start with this problem is to write down the formula to the solution: $P_v = \frac{x[1-(1+i)^{-n}]}{i}$

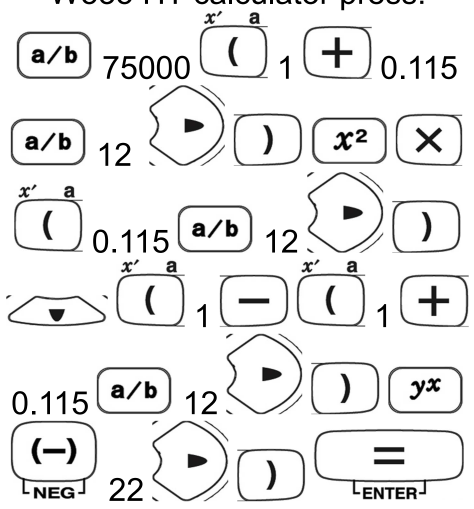
   Now Remember that your principle amount will be larger, therefore you need to calculate the larger principle amount:

$$75\ 000\left(1+\frac{0.115}{12}\right)^2 = \frac{x\left[1-\left(1+\frac{0.115}{12}\right)^{-22}\right]}{\frac{0.115}{12}}$$

   Now get x alone in the equation and use your calculator to calculate the repayment:

$$\frac{75\ 000\left(1+\frac{0.115}{12}\right)^2 \times \left(\frac{0.115}{12}\right)}{\left[1-\left(1+\frac{0.115}{12}\right)^{-22}\right]} = x$$

To put this into the SHARP EL-W535 HT calculator press:



   Therefore the repayment will be: R3 870.46

   b) As this is a deferred payment, only 22 payments are made. Therefore after ten payments, $n = 22 - 10 = 12$

$$P_v = \frac{3870.46\left[1-\left(1+\frac{0,115}{12}\right)^{-12}\right]}{\frac{0,115}{12}}$$

$P_v = 43\ 677{,}28$ Her outstanding balance is R43 677.28.

NB! Should she wish to settle the loan, this will not be the settlement amount as the outstanding balance includes interest charged on the outstanding balance over another 12 months and the settlement amount does not.

Take note that both the basic annuity formulas <u>only</u> hold when payment commences one period from the present and ends after periods.

<u>Activity 7.4</u>

1.

a) A = ?
P = 179 800
$i = 7.6\% \, p.a.$
n = 7 years

$A = P(1-i)^n$
$\therefore A = 179\,800\,(1-0.076)^7$
$\therefore A = 103\,393.56$

b) A = ?
P = 179 800
$i = 5.98\% \, p.a.$
n = 7 years

$A = P(1+i)^n$
$\therefore A = 179\,800\,(1+0.0598)^7$
$\therefore A = 269\,995.853$

c) Total Amount = 269 995, 853 – 103 393, 5593
= 166 602.2937

d) F_v = 166 602.2937
$i = 11.58\% \div 12$
$x = ?$
$n = 7 \times 12$

$F_v = \frac{x[(1+i)^n - 1]}{i}$

$$\therefore 166\,602.2937 = \frac{x\left[\left(1+\frac{0.1158}{12}\right)^{7\times12} - 1\right]}{\frac{0.1158}{12}}$$

$$\therefore 1\,607.712099 = x[1.240533116]$$

$$\therefore x = 1\,295,98$$

2. a) P = 146 000 $A = P(1-i)^n$

A = ? $\therefore A = 146\ 000\ (1-0.0632)^4$

$i = 6.32\%\ p.a.$ $\therefore A = \ 112\ 445.0609$

n = 4 years

b) P = 146 000 $A = P(1+i)^n$

A = ? $\therefore A = 146\ 000\ (1+0.0713)^4$

$i = 7.13\%\ p.a.$ $\therefore A = \ 192\ 307.9664$

n = 4 years

c) Total Amount = 192 302, 9664 – 112 445, 0609

= 79 862, 90553

d) F_v = 79 862, 90553 $F_v = \frac{x[(1+i)^n - 1]}{i}$

$i = 9.34\ \%\ \div 12$

$x = ?$

$n = 4\ \times 12$

$$\therefore 79\ 862.90553 = \frac{x\left[\left(1+\frac{0.0934}{12}\right)^{4\times12} - 1\right]}{\frac{0.0934}{12}}$$

$$\therefore 621.5996147 = x[0.450855777]$$

$$\therefore x = 1\ 378.71$$

3. First work out how much each item depreciates:

Therefore for the coffee machine:

P = 14 999 $A = P(1-i)^n$

A = ? $\therefore\ A = 14\ 999(1-0.0378)^{5\frac{1}{2}}$

$i = 3.78\%\ p.a.$ $\therefore A = 12\ 134.49604$

$n = 5\frac{1}{2}$ years

Therefore for the fire alarm system:

P = 12 000 $A = P(1-i)^n$

A = ? $\therefore A = 12\ 000(1-0.1175)^{5\frac{1}{2}}$

$i = 11.75\%\ p.a.$ $\therefore A = 6\ 034.095417$

$n = 5\frac{1}{2}$ years

Then work out how much each item appreciates:
Therefore for the coffee machine:

P = 14 999 $A = P(1+i)^n$

A = ? $\therefore A = 14\,999(1+0.064)^{5\frac{1}{2}}$

$i = 6.4\%\ p.a.$ $\therefore A = 21\,097.99859$

$n = 5\frac{1}{2}$ years

Therefore for the fire alarm system:

P = 12 000 $A = P(1+i)^n$

A = ? $\therefore A = 12000(1+0.079)^{5\frac{1}{2}}$

$i = 7.9\%\ p.a.$ $\therefore A = 18\,230.52568$

$n = 5\frac{1}{2}\ years$

Next work out how much they need for each item:
Coffee Machine = 21 097, 99859 – 12 134, 49604
= 8 963, 50255

Fire Alarm System = 18 230, 52568 – 6 034, 095417
= 12 196, 43026

Add those two totals together to get F_v:
F_v = 8 963, 50255 + 12 196, 43026
= 21 159, 93281

Now find the monthly instalment:

F_v = 21 159, 93281 $F_v = \frac{x[(1+i)^n-1]}{i}$

$i = 14.34\% \div 4$

$x = ?$

$n = 5\frac{1}{2} \times 4$ $21\,159.93281 = \frac{x\left[\left(1+\frac{0.1434}{4}\right)^{5\frac{1}{2}\times 4}-1\right]}{\frac{0.1434}{4}}$

$$758.5835913 = x[1.17035686]$$

$$\therefore x = 648.16$$

Activity 7.5

1. P_v = ?

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$i = 9.5\% \div 12$

$\therefore P_v = \frac{3560\left[1-\left(1+\frac{0.095}{12}\right)^{-3 \times 12}\right]}{\frac{0.095}{12}}$

$x = 3\,560$

$n = 3 \times 12$

$\therefore P_v = 111\,135.566$

2. P_v = 50 000

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$i = 15.7\% \div 4$

$\therefore 50\,000 = \frac{2500\left[1-\left(1+\frac{0.157}{4}\right)^{-n\times 4}\right]}{\frac{0.157}{4}}$

$x = 2\,500$

$\therefore 1\,962.5 = 2500\left[1-\left(\frac{4\,157}{4\,000}\right)^{-4n}\right]$

$n = ? \times 4$

$\therefore \frac{157}{200} = 1 - \left(\frac{4\,157}{4\,000}\right)^{-4n}$

$\therefore -\frac{43}{200} = -\left(\frac{4\,157}{4\,000}\right)^{-4n}$

$\therefore \frac{43}{200} = \left(\frac{4\,157}{4\,000}\right)^{-4n}$

$\therefore -4n = \log_{\left(\frac{4\,157}{4\,000}\right)}\left(\frac{43}{200}\right)$

$\therefore -4n = -39.9259$

$\therefore n = 9.9815$

$\therefore n \approx 10 \text{ years}$

3. P_v = 100 000

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$i = 12.3\% \div 12$

$100\,000 = \frac{x\left[1-\left(1+\frac{0.123}{12}\right)^{-5\times 12}\right]}{\frac{0.123}{12}}$

$x = ?$

$1025 = x[0.457663975]$

$n = 5 \times 12$

$\therefore x = 2\,239.63$

4. P_v = 23 500

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$i = 3.67\% \div 12$

$23\,500 = \frac{x\left[1-\left(1+\frac{0.0367}{12}\right)^{-2\times 12}\right]}{\frac{0.0367}{12}}$

$x = ?$

$71.87083333 = x[0.070666844]$

$n = 2 \times 12$ $\therefore x = 1\,017.04$

5. P_v = ?

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$i = 5.8\% \div 12$

$\therefore P_v = \frac{350\left[1-\left(1+\frac{0.058}{12}\right)^{-2\times12}\right]}{\frac{0.058}{12}}$

$x = 350$

$n = 2 \times 12$

$\therefore P_v = 7\,913.08$

6. P_v = ?

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$i = 16.78\% \div 12$

$\therefore P_v = \frac{2\,380\left[1-\left(1+\frac{0.1678}{12}\right)^{-4\times12}\right]}{\frac{0.1678}{12}}$

$x = 2\,380$

$n = 4 \times 12$

$\therefore P_v = 82\,807.53$

Activity 7.6

1.

a) Calculate his monthly installments if he has to pay back the loan over a period of 5 years.

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$ Always first write the equation

$250000 = \frac{x\left[1-\left(1+\frac{0,11}{12}\right)^{-5\times12}\right]}{\frac{0,11}{12}}$ Substitute all the values

$250000 = x \times 45{,}9930\ldots$ Simplify the R.H.S.

$\frac{250000}{45{,}9930} = \frac{x\times45{,}9930\ldots}{45{,}9930\ldots}$ Divide both sides by 45,9930…

$\therefore x = 5\,435{,}61$

His monthly instalments are R5 435,61

b) $P_v = \frac{x[1-(1+i)^{-n}]}{i}$ Always first write the equation.

$P_v = \frac{5435{,}61\left[1-\left(1+\frac{0{,}11}{12}\right)^{-36}\right]}{\frac{0{,}11}{12}}$ Substitute all the values.

$P_v = 166\ 030{,}02$

The outstanding balance on the loan is R166 030,02 after 24 months.

Activity 7.7

1. A loan of R120 000,00 is taken out and repaid by equal monthly instalments over a 4 year period. Interest is set at 10,5% p.a. compounded monthly. Calculate the monthly repayment.

$P = R120\ 000{,}00$
$n = 4$
$i = 10{,}5\%$ p.a. compounded monthly.
$x = ?$

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$

$120\ 000 = \frac{x\left(1-\left(1+\frac{0{,}105}{12}\right)^{-4\times12}\right)}{\frac{0{,}105}{12}}$

$x = R3\ 072{,}41$
He pays R3 072,41 per month on this loan.

2. At what annual percentage interest rate, compounded monthly, should a lump sum be invested in order for it to triple in 15 years?

$Lump\ sum = x$
$F_v = 3x$
$i = \frac{?}{12}$ $A = P(1+i)^n$

$n = 15 \times 12$ $3x = x(1+i)^{15\times12}$
$3 = (1+i)^{15\times12}$

$$\sqrt[15\times12]{3} = \sqrt[15\times12]{\left(1+\frac{i}{12}\right)^{15\times12}}$$

$$1{,}006122\ldots = 1+\frac{i}{12}$$

$$\therefore \frac{i}{12} = 0{,}006122\ldots$$

$$\therefore i = 0.0735\ldots$$

$$\therefore rate = 7{,}4\ \%$$

Rounded off to one decimal place.

3) Joe buys a machine for work to the value of R10 000. He borrows the money on 1 January 2011 from a financial institution that charges interest at a rate of11,25% p.a. compounded monthly. Joe agrees to pay monthly installments of R494,15. The agreement of the loan allows Joe to start paying these equal monthly instalments from 1 July 2011.

a) Calculate the total amount owing to the financial institution on 1 July 2011.

$$A = P(1+i)^n$$

$$A = 10000\left(1+\frac{0{,}1125}{12}\right)^6$$

$A =$R10 575,85

He will owe R10575,85 by 1 July 2011.

b) How many months will it take Joe to pay back the loan?

$$P_v = \frac{x[1-(1+i)^{-n}]}{i}$$ Always first write down the equation.

$$10575{,}85 = \frac{494{,}15\left[1-\left(1+\frac{0{,}1125}{12}\right)^{-n}\right]}{\frac{0{,}1125}{12}}$$ Substitute all the values and simplify R.H.S.

$$\frac{10575{,}85}{52709{,}3333\ldots} = \frac{52709{,}3333\ldots\times\left[1-\left(1+\frac{0{,}1125}{12}\right)^{-n}\right]}{52709{,}3333\ldots}$$ Divide both sides by 52709,3333…

$0,2006447\ldots = 1 - \left(1 + \frac{0,1125}{12}\right)^{-n}$

$\left(1 + \frac{0,1125}{12}\right)^{-n} = 0,7993552\ldots$ isolate $\left(1 + \frac{0,1125}{12}\right)^{-n}$ on one side.

$\log_{\left(1+\frac{0,1125}{12}\right)} 0,7993552 = -n$ change to log form and plug into calculator

$-n = -23,999787\ldots$ Simplify R.H.S. and divide

$\therefore n = 23,999787\ldots$ both sides by -1.

$\therefore n = 24\ months$

It will take him 24 months, but if we include the first 6 months it will take him 30 months.

c) $n = 24 - 15 = 9$

$P_v = \frac{x[1-(1+i)^{-n}]}{i}$ Always first write the equation.

$P_v = \frac{494,15\left[1-\left(1+\frac{0,1125}{12}\right)^{-9}\right]}{\frac{0,1125}{12}}$ substitute values.

It should be noted that there are many methods to solve this problem but your answers should all be the same.

$P_v = 4245,8498\ldots$ calculate P_v.

The balance is R4245,85 after the 15th payment.