CAPS

Mind the Gap!

Mathematics

Study Guide

Grade 12

basic education

Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

Unit 10

Trigonometry

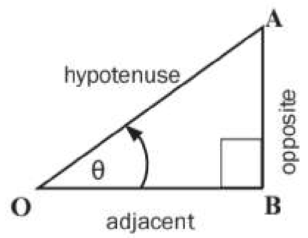
10.1 Revise: Trig ratios

Trigonometry is the study of the relationship between the sides and angles of triangles.

The word trigonometry means 'measurement of triangles'.

The trigonometric ratios

Using θ as the **reference angle** in $\triangle ABO$



- The side opposite the 90° is the hypotenuse side, therefore side AO is the hypotenuse side.
- The side opposite θ is the opposite side, therefore AB is the opposite side.
- The side adjacent to θ is called the adjacent side, therefore OB is the adjacent side.

We work with the ratios of the sides of the triangle:

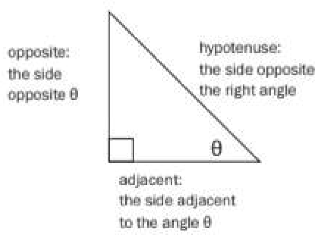
- The ratio $\frac{\text{opposite}}{\text{hypotenuse}}$ is called **sine** θ (abbreviated to **sin θ**)
- The ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ is called **cosine** θ (abbreviated to **cos θ**)
- The ratio $\frac{\text{opposite}}{\text{adjacent}}$ is called **tangent** θ (abbreviated to **tan θ**)

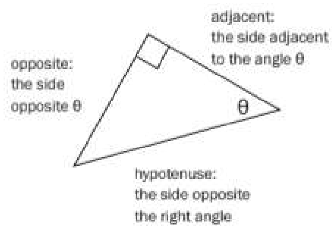
Therefore

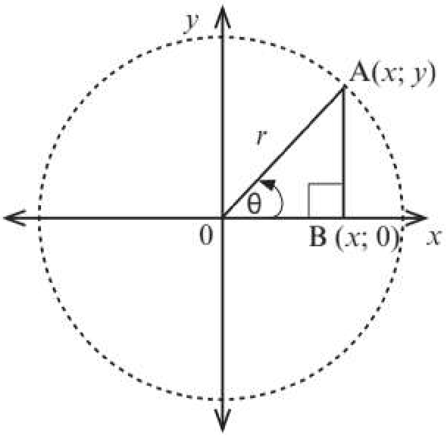
$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = AB/AO$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = OB/AO$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = AB/OB$$







We can also place the same triangle on the Cartesian plane in **standard position**, with a vertex at the origin and one side on the x-axis like this:

- On the Cartesian plane, A is the point $(x; y)$.
- The angle $A\hat{O}B$ or θ is positive (we rotate in an anti-clockwise direction)
- The length of OB is x units and the length of AB is y units.
- We can find the length of AO, using the Theorem of Pythagoras.

In ΔABO, $AO^2 = AB^2 + OB^2$

$AO^2 = x^2 + y^2$

$r^2 = x^2 + y^2$

NB

The Theorem of Pythagoras

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

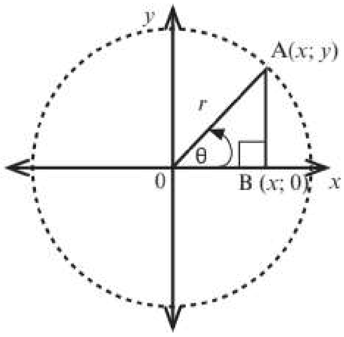
hint

NOTE:

Look at the circle centre O. AO is also a **radius** of this circle.

Now we can name the trigonometric ratios in terms of x, y and r.

- The ratio $\frac{y}{r}$ is called **sin θ**
- The ratio $\frac{x}{r}$ is called **cos θ**
- The ratio $\frac{y}{x}$ is called **tan θ**



NB

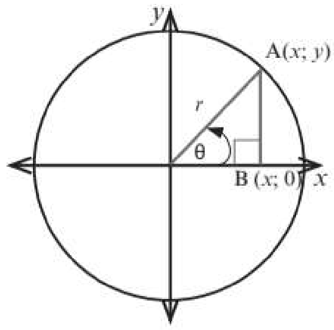
Learn these ratios:

$\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$ $\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\tan\theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$

hint

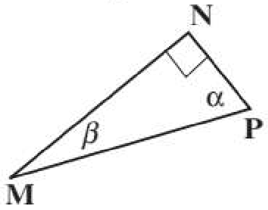
Remember the word SOHCAHTOA:

SOH	CAH	TOA
$\sin\theta = \frac{O}{H}$	$\cos\theta = \frac{A}{H}$	$\tan\theta = \frac{O}{A}$



Activity 1

1. ΔMNP is a right-angled triangle. Write down the trig ratios for:
   a) $\sin \alpha$ b) $\sin \beta$ (4)
   c) $\tan \beta$ d) $\cos \alpha$ (3)
2. If MP = 13 and NP = 5, calculate $\cos \beta$.



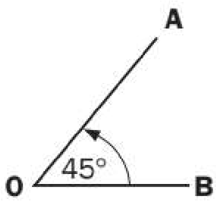
[7]

Solutions

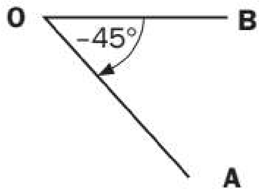
1. a) $\sin \alpha = \frac{MN}{MP}$ ✓ (1) b) $\sin \beta = \frac{NP}{MP}$ ✓ (1)
   c) $\tan \beta = \frac{NP}{MN}$ ✓ (1) d) $\cos \alpha = \frac{NP}{MP}$ ✓ (1) (4)

2. MP = 13 and NP = 5, so we can find MP,
   $MP^2 = MN^2 + NP^2$ ……….Pythagoras ✓
   $13^2 = MN^2 + 5^2$
   $169 = MN^2 + 25$
   $MN^2 = 169 - 25$
   $MN^2 = 144$ ✓
   $\therefore MN = 12$
   $\cos \beta = \frac{MN}{MP} = \frac{12}{13}$ ✓ (3)

[7]



Angles measured in an anti-clockwise direction from the x-axis are positive



Angles measured in an anti-clockwise direction from the x-axis are negative.
$\therefore$ angle is negative

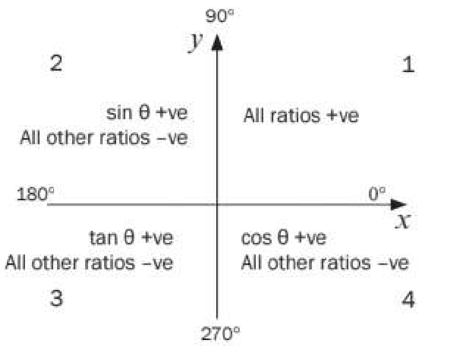
10.2 Trig ratios in all the quadrants of the Cartesian plane

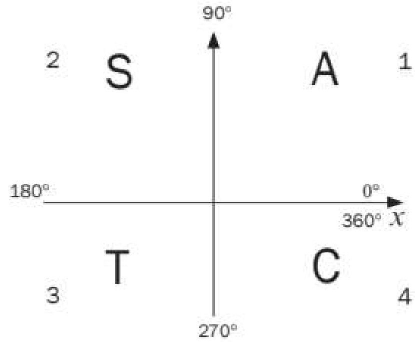
The Cartesian plane has four quadrants (quarters). We label them 1, 2, 3 and 4 starting from the quadrant with positive x- and y-values. We can calculate trig ratios for any angle size in the Cartesian plane.

hint

CAST

For trig ratios that are positive in the 4 quadrants.





- In the first quadrant x, y and r are positive. Therefore, all the trig functions are positive.
- In the second quadrant, y and r are positive, therefore sin θ is positive. In the second quadrant, x is negative, therefore cos θ and tan θ are negative.
- In the third quadrant, x and y are negative, therefore tan θ is positive. In the third quadrant, r is positive, therefore cos θ and sin θ are negative
- In the fourth quadrant, x and r are positive, therefore cos θ is positive. In the fourth quadrant, y is negative, therefore sin θ and tan θ are negative.

Activity 2

1. If sin θ is negative and cos θ is positive, then which statement is true?
   A. $0° < \theta < 90°$ **B.** $90° < \theta < 180°$
   C. $180° < \theta < 270°$ **D.** $270° < \theta < 360°$ (1)
2. If $\tan \theta < 0$ and $\cos \theta < 0$, then which statement is true?
   A. $0° < \theta < 90°$ **B.** $90° < \theta < 180°$
   C. $180° < \theta < 270°$ **D.** $270° < \theta < 360°$ (1)
3. Will the following trig ratios be positive or negative?
   a) sin 315°
   b) cos (–215°)
   c) tan 215°
   d) cos 390° (4)

[6]

Solutions

1. Sin θ is negative in 3rd and 4th quadrants; cos θ is positive in 1st and 4th quadrants.
   So θ is in the 4th quadrant. **D. $270° < \theta < 360°$** ✓ (1)
2. $\tan \theta < 0$ in 2nd and 4th quadrants; $\cos \theta < 0$ in 2nd and 3rd quadrants.
   So θ is in the 2nd quadrant. **B. $90° < \theta < 180°$** ✓ (1)
3. **a)** sin 315° is in 4th quadrant so it is negative. ✓ (1)
   b) cos (–215°) is in 2nd quadrant so it is negative. ✓ (1)
   c) tan 215° is in 3rd quadrant, so it is positive. ✓ (1)
   d) cos 390° is the same as cos 30° in the 1st quadrant, so it is positive. ✓ (1)

[6]

10.3 Solving triangles with trig

For some trigonometry problems, it is helpful to draw a diagram showing the angle involved and the x, y and r values.

e.g. 1

If $\tan\theta = -\sqrt{3}$ and $180° < \theta < 360°$, determine, using a diagram, the value of:

a) $\sin\theta$ **b)** $3\cos\theta$

Solutions

a) $\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1}$.

$180° < \theta < 360°$ and $\tan\theta$ is negative in the 4th quadrant

By Pythagoras, $r^2 = x^2 + y^2$

$r^2 = (1)^2 + (-\sqrt{3})^2$

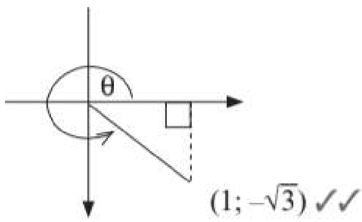
$r^2 = 1 + 3 = 4$

$r = 2$ ✓

$\therefore \sin\theta = \frac{y}{r} = \frac{-\sqrt{3}}{2}$ ✓(4)

b) $3\cos\theta$

$= 3(\frac{x}{r}) = 3(\frac{1}{2})$ ✓ $= \frac{3}{2} = 1{,}5$ ✓ (2)



[6]

Activity 3

If $\cos\beta = \frac{p}{\sqrt{5}}$ where $p < 0$ and $\beta \in [180°; 360°]$, determine, using a diagram, an expression in terms of p for:

a) $\tan\beta$ **b)** $2\cos^2\beta - 1$

[6]

Solutions

a) $\cos\beta = \frac{p}{\sqrt{5}} = \frac{x}{r}$; so $x = p$ and $r = \sqrt{5}$

By Pythagoras, $y^2 = r^2 - x^2$

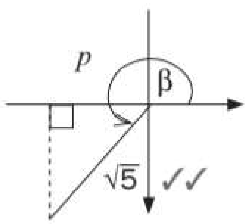
$\therefore y^2 = (\sqrt{5})^2 - p^2$

$= 5 - p^2$

$\therefore y = \pm\sqrt{5 - p^2}$.

$\therefore y = -\sqrt{5 - p^2}$ ✓ since β is in quadrant 3, y is negative

$\therefore \tan\beta = \frac{-\sqrt{5 - p^2}}{p}$ ✓ (4)



b) $2\cos^2\beta - 1 = 2\left(\frac{p}{\sqrt{5}}\right)^2 - 1$ ✓

$= \frac{2p^2}{5} - 1$ ✓ (2)

[6]

10.4 Using a calculator to find trig ratios

The scientific calculator calculates trigonometric ratios as decimal fractions.

e.g. 2

1. $\sin 58° = 0{,}8480480962....$ [Press: sin 58 =]
2. $\cos 222° = -0{,}7431448255...$ [Press: cos 222 =]
3. Calculate (correct to 2 decimal places):
   $\cos 238° \tan 132° = 0{,}5885349 \ldots \approx 0{,}59$ (to 2 decimal places)
   [Press: cos 238 × tan 132 =]
4. $\frac{\sin^2 327}{5+\tan 37} = 0{,}05155 \ldots \approx 0{,}052$ [NOTE: $\sin^2 327° = (\sin 327°)^2$]
5. $\sin 30° = \frac{1}{2}$

10.5 The trig ratios of special angles

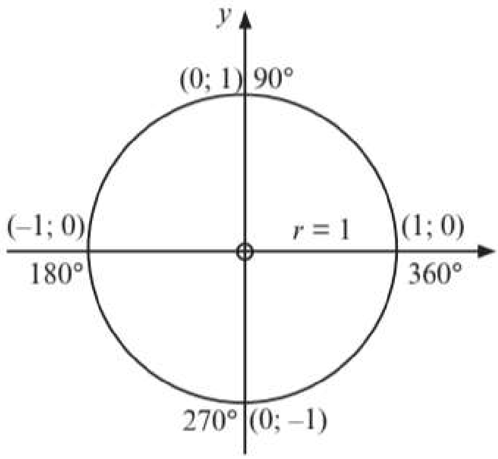
1. Special trig ratios using the unit circle

Consider a circle on the Cartesian plane that has a radius of **one unit.**

We can find the trig ratios for 0° (or 360°), 90°, 180° and 270° using the unit circle.

Label the $(x; y)$ coordinates on each axis.

Label the angles on each axis.



From the unit circle:

- At 0 *or* 360°: $x = 1$, $y = 0$ *and* $r = 1$
- At 90°: $x = 0$, $y = 1$ *and* $r = 1$
- At 180°: $x = -1$, $y = 0$ *and* $r = 1$
- At 27°: $x = 0$, $y = -1$ *and* $r = 1$

$\sin 0° = \frac{0}{1} = 0$ $\qquad$ $\sin 90° = \frac{1}{1} = 1$

$\cos 0° = \frac{1}{1} = 1$ $\qquad$ $\cos 90° = \frac{0}{1} = 0$

$\tan 0° = \frac{0}{1} = 0$ $\qquad$ $\tan 90° = \frac{1}{0}$ is undefined

$\sin 180° = \frac{0}{1} = 0$ $\qquad$ $\sin 270° = \frac{-1}{1} = -1$

$\cos 180° = \frac{-1}{1} = -1$ $\qquad$ $\cos 270° = \frac{0}{1} = 0$

$\tan 180° = \frac{0}{-1} = 0$ $\qquad$ $\tan 270° = \frac{-1}{0}$ is undefined

NB

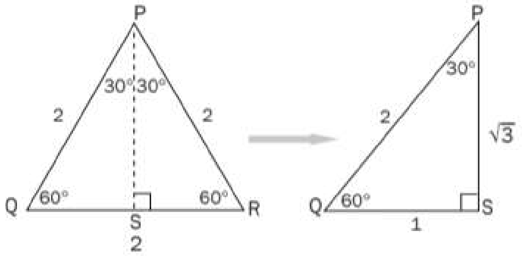
Summary

θ	0°	90°	180°	270°	360°
sin θ	0	1	0	–1	0
cos θ	1	0	–1	0	1
tan θ	0	undefined	0	undefined	0

2. Special trig ratios using an equilateral triangle

We use an equilateral triangle that has sides of **2 units** to find the trig ratios for the special angles **30°** and **60°**. The perpendicular bisector of one side creates two triangles. The angles of an equilateral triangle are equal, so angles P, Q and R are each 60°.

P is bisected, so $Q\hat{P}S = R\hat{P}S = 30°$.



By Pythagoras,

$PR^2 = PS^2 + RS^2$

$2^2 = PS^2 + 1^2$

$PS^2 = 4 - 1 = 3$

$\therefore PS = \sqrt{3}$

Now we can use ΔPQS to work out trig ratios of 30° and 60°.

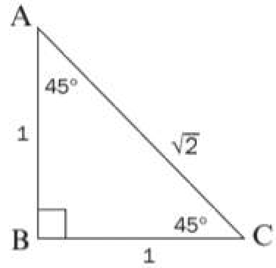
$\sin 60° = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\qquad$ $\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

3. Special trig ratios using a right-angled isosceles triangle

Use a right-angled isosceles triangle with sides of **one unit** to work out the trig ratios for **45°**. The angles opposite the equal sides are equal, so they are each 45° (sum of angles in Δ).



By Pythagoras,

$AC^2 = AB^2 + BC^2$

$AC^2 = 1^2 + 1^2$

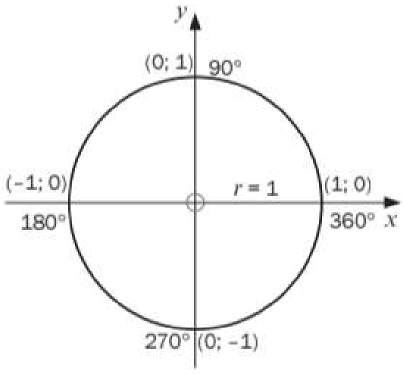
$AC^2 = 1 + 1 = 2$

$\therefore AC = \sqrt{2}$

The hypotenuse will be $\sqrt{2}$ units.

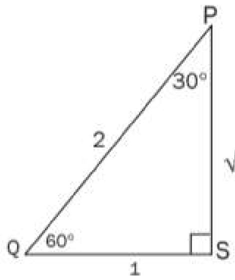
$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\qquad$ $\tan 45° = \frac{1}{1} = 1$

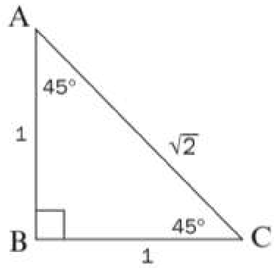
Summary of special angles



You should **memorise** the special angles, as you will use them often. You will be asked exam questions where you are not allowed to use a calculator, and must show that you have used the special angles.

If you just remember these three diagrams, you can work out all of the special angles.





If you find it difficult to remember the diagrams, then learn this summary of the special angles.

θ	30°	45°	60°
sin θ	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan θ	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

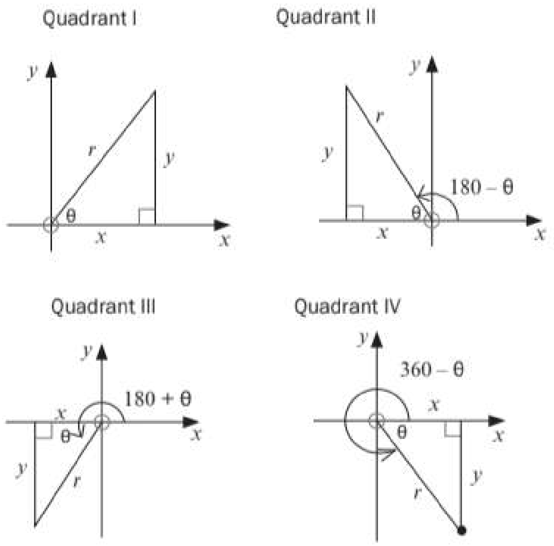
You can also use a scientific calculator to find these special angle ratios.

10.6 Using reduction formulae

Look at the angles here. If $\theta < 90°$, it is in the first quadrant, therefore θ is an acute angle.

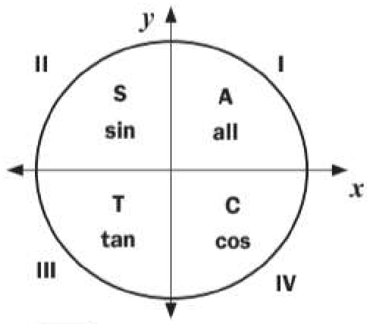
Therefore
- angle $(180° - \theta)$ in quadrant II
- angle $(180° + \theta)$ in the quadrant III
- angle $(360° - \theta)$ in quadrant IV.

You can work out which trig ratios will be positive and which will be negative, according to the quadrants they are in.



a) **Reduction formulae**

Quadrant II: 180° – θ	Quadrant III: 180°+ θ	Quadrant IV: 360° – θ
$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = \cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = \tan\theta$	$\tan(360° - \theta) = -\tan\theta$



b) **Angles greater than 360°**
We can add or subtract 360° (or multiples of 360°) and will always end up with an angle in the first revolution. For example, 390° can be written as (30° + 360°), so 390° has the same terminal arm as 30°.

c) **Negative angles:**
- $(-\theta)$ lies in quadrant IV and is the same as $360° - \theta$.

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

- $(\theta - 180)$lies in the third quadrant

$\sin(\theta - 180) = -\sin\theta$ $\cos(\theta - 180) = -\cos\theta$ $\tan(\theta - 180) = \tan\theta$

- $(-\theta - 180)$lies in the second quadrant

$\sin(-\theta - 180) = \sin\theta$ $\cos(-\theta - 180) = -\cos\theta$ $\tan(-\theta - 180) = -\tan\theta$

- $(\theta - 360)$ lies in the first quadrant

$\sin(\theta - 360) = \sin\theta$ $\cos(\theta - 360) = -\cos\theta$ $\tan(\theta - 360) = -\tan\theta$

hint

When you divide, you sometimes need to round off to the closest numbers that are easier to divide.

hint

When you divide, you sometimes need to round off to the closest numbers that are easier to divide.

e.g. 3

$\sin(360°+\theta) = \sin\theta \qquad \cos(360°+\theta) = \cos\theta \qquad \tan(360°+\theta) = \tan\theta$

Activity 4

Without using a calculator, determine the value of:

1. $\cos 150°$ **2.** $\sin(-45°)$ **3.** $\tan 480°$

[7]

Solutions

1. $\cos 150°$ — rewrite as (180 – ?)
$= \cos(180° - 30°)$ — quadrant II, $\cos\theta$ negative
$= -\cos 30°$ ✓ — special ratios
$= -\frac{\sqrt{3}}{2}$ ✓ (2)

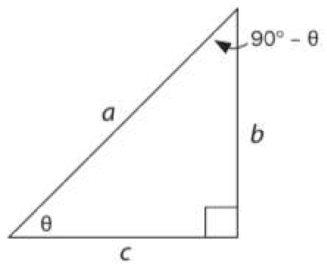
2. $\sin(-45°)$ — $\sin(-\theta) = -\sin\theta$; quadrant IV, $\sin\theta$ negative
$= -\sin 45°$ ✓ — special ratios
$= -\frac{1}{\sqrt{2}}$ ✓ (2)

3. $\tan 480°$ — write as an angle in the first rotation of 360°
$= \tan(480° - 360°)$
$= \tan 120°$ ✓ — quadrant II, rewrite as (180 – ?)
$= \tan(180° - 60°)$ — $\tan\theta$ negative
$= -\tan 60°$ ✓ — special ratios
$= -\sqrt{3}$ ✓ (3)

[7]

d) Co-functions

In this right-angled triangle, the sides are a, b and c and $\hat{B} = \theta$.
A = 90° and angles of a triangle are supplementary
$\therefore \hat{C} = (90° - \theta)$.
Look at the sine and cosine ratios for the triangle:



$\sin\theta = \frac{b}{a}$ and $\cos(90° - \theta) = \frac{b}{a}$
$\therefore \cos(90° - \theta) = \sin\theta$
$\cos\theta = \frac{c}{a}$ and $\sin(90° - \theta) = \frac{c}{a}$
$\therefore \sin(90° - \theta) = \cos\theta$

Trig ratios of angles that add up to 90°, are called **co-functions.**

NB

$\sin(90^\circ - \theta) = \cos\theta$	(quadrant I)
$\sin(90^\circ + \theta) = \cos\theta$	(sin θ positive in quadrant II)
$\sin(\theta - 90^\circ) = \sin[-(90^\circ - \theta)]$	(common factor of –1)
$= -\sin(90^\circ - \theta)$	(sin θ negative in quadrant IV)
$= -\cos\theta$	
$\cos(90^\circ - \theta) = \sin\theta$	(quadrant I)
$\cos(90^\circ + \theta) = -\sin\theta$	(cos θ negative in quadrant II)
$\cos(\theta - 90^\circ) = \cos[-(90^\circ - \theta)]$	(common factor of –1)
$= +\cos(90^\circ - \theta)$	(cos θ positive in quadrant IV)
$= +\sin\theta$	

Activity 5

Write the trig ratios as the trig ratios of their co-functions:

1. sin 50° **2.** cos 70° **3.** sin 100° **4.** cos 140°

[4]

Solutions

1. $\sin 50^\circ = \sin(90^\circ - 40^\circ) = \cos 40^\circ$ ✓
2. $\cos 70^\circ = \cos(90^\circ - 20^\circ) = \sin 20^\circ$ ✓
3. $\sin 100^\circ = \sin(90^\circ + 10) = \cos 10^\circ$ ✓
4. $\cos 140^\circ = \cos(90^\circ + 50^\circ) = -\sin 50^\circ$ ✓

[4]

Summary

Any angle (obtuse or reflex) can be reduced to an acute angle by using:

- Convert negative angles to positive angles
- Reduce angles greater than 360°
- Use reduction formulae
- Use co-functions

Activity 6

Simplify without using a calculator:

1. $\dfrac{\sin(180° + x).\cos 330°.\tan 150°}{\sin x}$ (4)

2. $\dfrac{\cos 750°.\tan 315°.\cos(-\theta)}{\cos(360° - \theta).\sin 300°.\sin(180° - \theta)}$ (8)

3. $\dfrac{\tan 480°.\sin 300°.\cos 14°.\sin(-135°)}{\sin 104°.\cos 225°}$ (9)

4. $\dfrac{\cos 260°.\cos 170°}{\sin 10°.\sin 190°.\cos 350°}$ (7)

[28]

Solutions

1. $\dfrac{\sin(180° + x).\cos 330°.\tan 150°}{\sin x}$ reduction formulae in numerator

$= \dfrac{\overset{\checkmark}{(-\sin x)}\overset{\checkmark}{(+\cos 30°)}\overset{\checkmark}{(-\tan 30°)}}{\sin x}$ (use brackets to separate ratios)

$= \dfrac{+\sin x.\frac{\sqrt{3}}{2}\checkmark.\frac{\sqrt{3}}{3}}{\sin x}$ special angles

$= \dfrac{\sqrt{3}}{2}.\dfrac{\sqrt{3}}{3}$

$= \dfrac{3}{6} = \dfrac{1}{2}$ (4)

2. $\dfrac{\cos 750°.\tan 315°.\cos(-\theta)}{\cos(360° - \theta).\sin 300°.\sin(180° - \theta)}$ use reduction formulae

$= \dfrac{\cos 30°\checkmark.(-\tan 45°)\checkmark.\cos\theta\checkmark}{\cos\theta\checkmark.(-\sin 60°)\checkmark.\sin\theta\checkmark}$ use special angles

$= \dfrac{\frac{\sqrt{3}}{2}.(-1)\cos\theta}{\cos\theta.\left(-\frac{\sqrt{3}}{2}\right)\sin\theta}\checkmark$

$= \dfrac{-1}{-\sin\theta} = \dfrac{1}{\sin\theta}\checkmark$ (8)

3. $\dfrac{\tan 480°.\sin 300°.\cos 14°.\sin(-135°)}{\sin 104°.\cos 225°}$

$= \dfrac{\tan 120°.(-\sin 60)\checkmark.\cos 14°.\sin 225°}{\sin 76°\checkmark.(-\cos 45°)\checkmark}$

$= \dfrac{\cos(180° + 80°).\cos(180° - 10°)}{\sin 10°.\sin(180° + 10°).\cos(360° - 10°)}$

$= \dfrac{(-\tan 60°)\checkmark.(-\sin 60°).\sin 76°\checkmark.(-\sin 45°)\checkmark}{\sin 76°.(-\cos 45°)}$

$= \dfrac{(-\sqrt{3}).\left(\frac{-\sqrt{3}}{2}\right).\sin 76.\left(\frac{-\sqrt{2}}{2}\right)}{\sin 76°.\left(\frac{-\sqrt{2}}{2}\right)}\checkmark\checkmark$

$= \dfrac{3}{2}\checkmark$ (9)

4. $\dfrac{\cos 260°.\cos 170°}{\sin 10°.\sin 190°.\cos 350°}$

$= \dfrac{-\cos 80°\checkmark.(-\cos 10°)}{\sin 10°.(-\sin 10°)\checkmark.\cos 10°\checkmark}$

$= \dfrac{(-\sqrt{3}).\left(\frac{-\sqrt{3}}{2}\right).\sin 76.\left(\frac{-\sqrt{2}}{2}\right)}{\sin 76°.\left(\frac{-\sqrt{2}}{2}\right)}\checkmark\checkmark$

$= \dfrac{-\sin 10°\checkmark.(-\cos 10°)}{\sin 10°.(-\sin 10°).\cos 10°}$

$= \dfrac{-1}{\sin 10°}\checkmark$ (7)

[28]

10.7 Trigonometric identities

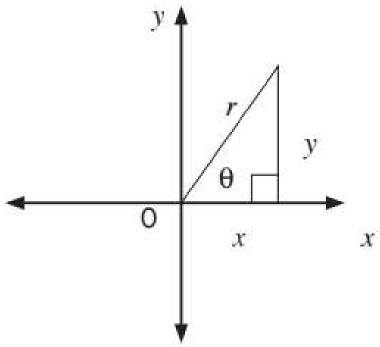
1. $\tan\theta = \frac{\sin\theta}{\cos\theta}$; $(\cos\theta \neq \theta)$
(the quotient identity)

2. $\sin^2\theta + \cos^2\theta = 1$ → $\sin^2\theta = 1 - \cos^2\theta$; → $\cos^2\theta = 1 - \sin^2\theta$
(the square idedtity)

NB

PROOF OF IDENTITIES

Proof of the identities are examinable with the RHS and break it down into its x, y and r values.



Proof: $\frac{\sin\theta}{\cos\theta}$

$= \frac{y}{r} \div \frac{x}{r}$

$= \frac{y}{r} \times \frac{r}{x}$

$= \frac{y}{x} = \tan\theta$

Proof: $\sin^2\theta + \cos^2\theta$

$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$

$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$ Use LCD r^2

$= \frac{x^2 + y^2}{r^2}$ $x^2 + y^2 = r^2$ (Pythagoras)

$= \frac{r^2}{r^2} = 1$

We can use the identities and the reduction formulae to help us simplify trig expressions.

Activity 7

Simplify the following expressions.

1. $\dfrac{\cos(180°-x)\sin(x-90°)-1}{\tan^2(540°+x)\sin(90°+x)\cos(-x)}$ (8)

2. $[\sin(-\theta)+\cos(360°+\theta)][\cos(\theta-90°)+\cos(180°+\theta)]$ (3)

3. $\cos^2\theta\,(1+\tan^2\theta)$ (3)

4. $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}$ (3)

[17]

Solutions

1. $\dfrac{\cos(180°-x)\sin(x-90°)-1}{\tan^2(540°+x)\sin(90°+x)\cos(-x)}$ – use reduction formulae and co-functions

$= \dfrac{(-\cos x)✓\,(-\cos x)✓-1}{\tan^2(540°-360°+x)\cos x✓.\cos x✓}$ – multiply out numerator and denominator reduction of angle > 360°

$= \dfrac{\cos^2 x-1}{\tan^2(180°+x).\cos^2 x}$ – use trig identity format for $\cos^2 x - 1$ reduction formula

$= \dfrac{-(1-\cos^2 x)}{\tan^2 x✓.\cos^2 x}$ – use trig identities for $1-\cos^2 x$ and for $\tan x$

$= \dfrac{-\sin^2 x✓}{\frac{\sin^2 x}{\cos^2 x}✓.\frac{\cos^2 x}{1}}$ – simplify

$= \dfrac{-\sin^2 x}{\sin^2 x} = -1$ ✓ (8)

2. $[\sin(-\theta)+\cos(360°+\theta)][\cos(\theta-90°)+\cos(180°+\theta)]$ – reduce to angle < 90°

$=[-\sin\theta+\cos\theta][\cos(-(90°-\theta))+(-\cos\theta)]$ – simplify; use co-functions

$=(-\sin\theta+\cos\theta)(\sin\theta-\cos\theta)$ – multiply out using FOIL

$=-\sin^2\theta+\sin\theta\cos\theta+\cos\theta\sin\theta-\cos^2\theta$ ✓✓

$=-(\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta$ – use trig identity

$=-1+2\sin\theta\cos\theta$ – use double angle identity

$=-1+\sin 2\theta$ ✓ (3)

3. $\cos^2\theta\,(1+\tan^2\theta)$ – multiply out the bracket

$=\cos^2\theta+\cos^2\theta.\tan^2\theta$ ✓ – use trig identity for $\tan\theta$

$=\cos^2\theta+\dfrac{\cos^2\theta}{1}.\dfrac{\sin^2\theta}{\cos^2\theta}$ – simplify

$=\cos^2\theta+\sin^2\theta$ ✓ $=1$ ✓ – use trig identity $\sin^2\theta+\cos^2\theta=1$ (3)

4. $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}$ – use trig identity $\sin^2\theta+\cos^2\theta=1$

$=\dfrac{\sin^2\theta✓}{\cos^2\theta✓}$ – use trig identity for $\tan\theta$

$=\tan^2\theta$ ✓ (3)

[17]

10.8 More trig identities

You need to be able to use all the information you have about trig ratios and ways to simplify them in order to solve more complicated trig identities.

Activity 8

Prove the following identities:

1. $\sin x \cdot \tan x + \cos x = \frac{1}{\cos x}$ (4)

2. $(\sin x + \tan x)\left(\frac{\sin x}{1 + \cos x}\right) = \sin x.\ \tan x$ (7)

3. $\frac{1}{\cos x} = \frac{\cos x}{1 + \sin x} + \tan x$ (6)

4. $\frac{1}{\tan x} + \tan x = \frac{\tan x}{\sin^2 x}$ (5)

[22]

hint

Hints for solving trig identities:

- Choose either the **left-hand side** or the **right-hand side** and simplify it to look like the other side.
- If both sides look difficult, you can try to simplify on both sides until you reach a point where both sides are the same.
- It is usually helpful to write $\tan \theta$ as $\frac{\sin \theta}{\cos\theta}$.
- Sometimes you need to simplify $\frac{\sin \theta}{\cos \theta}$ to $\tan \theta$.
- If you have $\sin^2 x$ or $\cos^2 x$ with +1 or –1, use the squares identities ($\sin^2\theta + \cos^2\theta = 1$).
- Find a common denominator when fractions are added or subtracted.
- Factorise if necessary – specify with examples i.e. common factor, DOPS, Trinomial, sum/diff of two cubes

Solutions

1. LHS: $\sin x.\ \tan x + \cos x$

$= \sin x\ .\ \frac{\sin x}{\cos x} + \cos x\ ✓ + \cos x$

$= \frac{\sin^2 x}{\cos x} + \frac{\cos x}{1}$

$= \frac{\sin^2 x + \cos^2 x\ ✓}{\cos x\ ✓} = \frac{1}{\cos x}\ ✓ = \text{RHS (4)}$

$\therefore \sin x.\ \tan x + \cos x = \frac{1}{\cos x}$ (4)

2. LHS: $(\sin x + \tan x)\left(\frac{\sin x}{1+ \cos x}\right)$

$= \left(\sin x + \frac{\sin x}{\cos x}\ ✓\right)\left(\frac{\sin x}{1+ \cos x}\right)$

$= \left(\frac{\sin x \cos x + \sin x\ ✓}{\cos x\ ✓}\right)\left(\frac{\sin x}{1 + \cos x}\right)$

$= \left(\frac{\sin x\ (\cos x + 1)\ ✓}{\cos x}\right)\left(\frac{\sin x}{1 + \cos x}\right)$

$= \frac{\sin^2 x}{\cos x}\ ✓\ (7)$

$\therefore$ LHS = RHS (7)

RHS: $\sin x.\ \tan x$

$= \sin x.\ \frac{\sin x}{\cos x}\ ✓$

$= \frac{\sin^2 x}{\cos x}\ ✓$

3. RHS: $\frac{\cos x}{1 + \sin x} + \tan x$

$= \frac{\cos x}{1 + \sin x} + \frac{\sin x}{\cos x}\ ✓$

$= \frac{\cos^2 x + \sin x\ (1 + \sin x)\ ✓}{\cos x\ (1 + \sin x)\ ✓}$

$= \frac{\cos^2 x + \sin x + \sin^2 x\ ✓}{\cos x\ (1 + \sin x)}$ trig identity: $\cos^2 x + \sin^2 x = 1$

$= \frac{1 + \sin x\ ✓}{\cos x\ (1 + \sin x)}$

$= \frac{1}{\cos x}\ ✓ = \text{LHS}$

$\therefore \frac{1}{\cos x} = \frac{\cos x}{1 + \sin x} + \tan x$ (6)

4. $\frac{1}{\tan x} + \tan x = \frac{\tan x}{\sin^2 x}$

LHS: $\frac{1}{\tan x} + \tan x$ RHS: $\frac{\tan x}{\sin^2 x}$

$= \frac{1}{\frac{\sin x}{\cos x}} + \frac{\sin x}{\cos x}$ ✓ $= \frac{\sin x}{\cos x}$ ✓ $. \frac{1}{\sin^2 x}$

$= \frac{\cos x}{\sin x}$ ✓ $+ \frac{\sin x}{\cos x}$ $= \frac{1}{\sin x.\cos x}$

$= \frac{\cos^2 x + \sin^2 x \checkmark}{\sin x.\cos x \checkmark}$

$= \frac{1}{\sin x.\cos x}$

$\therefore$ LHS = RHS (5)

[22]

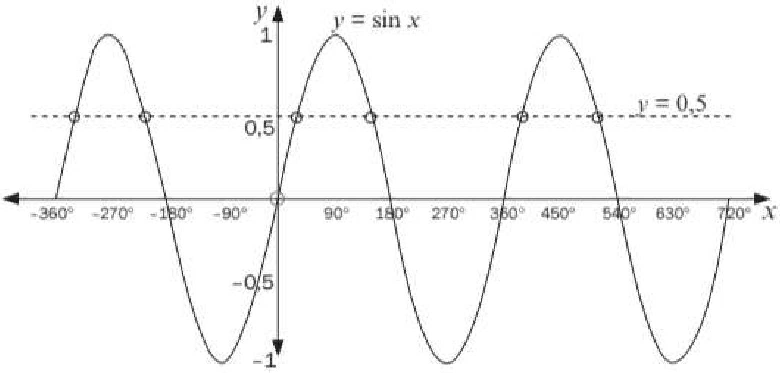
10.9 Solving trigonometric equations

To solve a trig equation where the angle is unknown, you need to find all the possible values of the angle.

For example, if $\sin \theta = \frac{1}{2}$, we know that θ could be 30°. However, there are other values for θ in the other quadrants. Have a look at the graph for sin

$\theta = \frac{1}{2}$, $\theta \in [-360°; 720°]$.

There are six values for θ between −360° and 720°.



If 30° is our reference angle in quadrant I.

In quadrant II: $\sin (180° - 30°) = \sin 30° = \frac{1}{2}$ So θ is 150°

In quadrant III and IV, the sine ratio is negative, so there is no solution for θ.

The angle could be greater than 360°.

In quadrant I: $\sin (360° + 30°) = \sin 30° = \frac{1}{2}$ So θ is 390°

In quadrant II: $\sin (540° - 30°) = \sin ((540° - 360°) - 30°)$

$= \sin (180° - 30°) = \sin 30 = \frac{1}{2}$ So θ is 510°

You can also work out that $\theta = -210°$ or $\theta = -330°$

You do not need to draw a graph to solve these equations.

NB

A method to find the general solution of trig equations:

1. Isolate the trig function on one side of the equation.
2. Determine the reference angle: put the **positive** number for the angle in the calculator and press the trig key and the inverse key:

   shift sin / shift cos /shift tan.

   Use special angles if the question does not allow you to use a calculator.
3. For sin x and cos x, place the reference angle in the two possible quadrants where they are positive or negative (according to the question). The period of the sine and cosine graphs is 360°, so add k 360° to each solution. Always put $k \in \mathbb{Z}$.
4. For tan x, place the reference angle in one correct quadrant where it is positive or negative (according to the question). The period of the tan graph is 180°, so add k 180°. Always put $k \in \mathbb{Z}$.
5. If x must be solved for a given interval:
   a) Find the general solution
   b) Substitute k with –1; 0; 1; 2, etc. to find the solutions in the correct interval.

e.g. 4

1. Solve for x: $\sin x = 0{,}7$ [On your calculator, press: $\sin^{-1} 0{,}7 =$]
The calculator answer is 44,42……°
We call this the **reference angle**, as it is not the only solution to the equation.
$\sin x$ is positive, so angle x must be in quadrant I or quadrant II in the first revolution.
In quadrant I: $x = 44{,}42\ldots\ldots°$
AND
In quadrant II: $x = 180° - 44{,}42\ldots° = 135{,}57\ldots\ldots°$
The period of the sin graph is 360°, so the other points of intersection occur 360° to the right or left of these solutions.
We add k revolutions to the two angles in the first revolution.
k is an integer (…–1; 0; 1; …). We call this **the general solution** of the equation.
So we can say the solution to **sin x = 0,7** is
$x = 44{,}42° + k360°$ or $x = 135{,}57° + k360°$; $k \in \mathbb{Z}$.
(Correct to two decimal place)

2. Solve for x: $\sin x = -0{,}7$
This time, place the reference angle in quadrants III and IV ($\sin x$ is negative)
$x = 180° + 44{,}42\ldots\ldots° + k360°$ or $x = 360° - 44{,}42\ldots\ldots° + k360°$ $k \in \mathbb{Z}$
$x = 224{,}42° + k360°$ or $x = 315{,}57° + k360$; $k \in \mathbb{Z}$
(Correct to two decimal place)

3. Solve for x: $\cos x = -0{,}7$ Reference angle $= 134{,}427\ldots°$
$\cos x$ is negative in quadrants II and III.
$x = 360° - 134{,}43° = 225{,}57°$
$x = 134{,}43° + k360°$ or $x = 225{,}57° + k360°$; $k \in \mathbb{Z}$
(Correct to two decimal place)

4. Solve for x: $\cos x = 0{,}7$ Reference angle $= 45{,}57\ldots\ldots°$
This time, place the reference angle in quadrants I and IV where $\cos x$ is positive:
$x = 45{,}57\ldots° + k360°$ or $x = 360° - 45{,}57\ldots\ldots° + k360°$
$x = 45{,}57° + k360°$ or $x = 314{,}43° + k360°$; $k \in \mathbb{Z}$.
(Correct to two decimal place)

5. Solve for x: $\tan x = 0{,}7$
$\tan x$ is positive in quadrants I and III.
Reference angle = 34,99° (correct to 2 dec places)
$x = 34{,}99\ldots°$ or $180° + 34{,}99\ldots\ldots° = 214{,}99\ldots\ldots°$
Now the period of the tan graph is 180°, so the other points of intersection occur 180° to the right or left of the solutions.
$x = 34{,}99° + k180°$; $k \in \mathbb{Z}$
(Correct to two decimal place)

hint

You do not need to write down the solution of 215°.
This solution is already there because
34,99° + (1)180° = 215°

6. Solve for x: $\tan x = -0{,}7$
$\tan x$ is negative in quadrants II and IV.
The reference angle is $-34{,}99\ldots\ldots°$
$180° - 34{,}99\ldots\ldots° = 145{,}01\ldots\ldots°$
$x = 145{,}01° + k180°$; $k \in \mathbb{Z}$.

Activity 9

1. If $\cos 20° = p$, determine the following ratios in terms of p:
   a) $\cos 380°$
   b) $\sin 110°$
   c) $\sin 200°$ (6)
2. Determine the general solution for x in the following equations:
   a) $5 \sin x = \cos 320°$ (correct to 2 decimal places)
   b) $3 \tan x + \sqrt{3} = 0$ (without using a calculator)
   c) $\frac{\tan x - 1}{2} = -3$ (correct to one decimal place) (10)
3. Determine x for $x \in [-180°; 180°]$ if $2 + \cos(2x - 10°) = 2{,}537$ (6)

[22]

Solutions

1. $\cos 20° = \frac{p}{1}$ so $x = p$ and $r = 1$

   By Pythagoras, $y^2 = r^2 - x^2$

   $y^2 = 1^2 - p^2 = 1 - p^2$

   $y = \sqrt{1 - p^2}$ first quadrant, so y is positive

   a) $\cos 380° = \cos(360° + 20°) = \cos 20°$ ✓ $= p$ ✓ (2)

   b) $\sin 110°$ reduction formula
   $= \sin(180° - 70°)$
   $= \sin 70°$ ✓ co-function
   $= \sin(90° - 20°)$
   $= \cos 20°$ ✓ $= p$ ✓ (3)

   c) $\sin 200° = \sin(180° + 20°)$
   $= -\sin 20°$ ✓
   $= \frac{-\sqrt{1-p^2}}{1} = -\sqrt{1 - p^2}$ (1) (6)

2. a) $5 \sin x = \cos 320°$ ✓
   $5 \sin x = 0{,}766044$
   $\sin x = 0{,}15320...$ ✓
   Ref angle $= 8{,}81°$
   $x = 8{,}81° + k360°$ OR $x = 180° - 8{,}81° + k360°$ ✓
   $x = 171{,}19° + k360°$ ✓ $k \in \mathbb{Z}$ (4)

   Calculator keys:
   cos 320 =
   ÷ 5 =
   SHIFT sin ANS =

   b) $3 \tan x + \sqrt{3} = 0$
   $3 \tan x = -\sqrt{3}$
   $\tan x = \frac{-\sqrt{3}}{3}$ ✓ [special angle: tan 30° $\tan 30° = \frac{\sqrt{3}}{3}$]

   Ref angle $= 30°$
   $x = 180° - 30° + k180°$ ✓
   $x = 150° + k180°$ ✓ $k \in \mathbb{Z}$ (3)

c) $\frac{\tan x - 1}{2} = -3$ multiply both sides by 2

$\tan x - 1 = -6$

$\tan x = -5$ ✓ reference angle is 78,69…°

$\therefore x = 180° - 78,69…° + k180°$ ✓

$x = 101,31° + k180°;\ k \in \mathbb{Z}$ ✓ (3) (10)

3. $2 + \cos(2x - 10°) = 2,537$

$\cos(2x - 10°) = 0,537$

Ref angle = 57,52…..°

$2x - 10° = 57,52…..° + k360°$ or $2x - 10° = 360° - 57,52° + k360°$

[solve equations]

$2x = 67,52….° + k360°$ or $2x = 312,48…° + k360°$ ✓

[divide all terms on both sides by 2]

$x = 33,76° + k180°$ or $x = 156,24° + k180°$ ✓ $k \in \mathbb{Z}$

$x \in [-180°; 180°]$

So for $k = -1$: $x = 33,76° - 180° = -146,24°$ or $x = 156,24° - 180° = -23,76°$ ✓

For $k = 0$: $x = 33,76°$ or $x = 156,24°$ ✓

(For $k = 1$, x will be > 180°, so it is too big)

Solution: $x \in \{-146,24°;\ -28,76°;\ 33,76°;\ 156,24°\}$ ✓✓ (6)

[22]

10.10 More solving trig equations using identities

NB

- $a \sin \theta = b \cos \theta$: single sin and cos function with the same angle

1) Divide by the cos function
2) Change sin $\frac{\theta}{\cos \theta}$ to tan θ

e.g. 5

Solve for x (give general solution) and round off your answer to 2 decimal places.

1. $3 \sin x = 4 \cos x$

2. $4 \cos^2 x + 4 \sin x \cos x + 1 = 0$

[6]

Solutions

1. $3 \sin x = 4 \cos x$ Divide both sides by $\cos x$ to create $\tan x$ on LHS

$\frac{3 \sin x}{\cos x} = \frac{4 \cos x}{\cos x}$ ✓ Trig identity for $\tan x$

$3 \tan x = 4$

$\tan x = \frac{3}{4}$ ✓

Ref angle $= 53{,}13°$

$x = 53{,}13° + k180°\ k \in \mathbb{Z}$ ✓ (3)

2. $4 \cos^2 x + 4\sin x \cos x + 1 = 0$ use $1 = \sin^2 x + \cos^2 x$

$4 \cos^2 x + 4\sin x \cos x + (\sin^2 x + \cos^2 x)$ ✓ $= 0$

$5 \cos^2 x + 4\sin x \cos x + \sin^2 x = 0$

$(5 \cos x + \sin x)(\cos x + \sin x) = 0$

$5 \cos x + \sin x = 0$ or $\cos x + \sin x = 0$

$\frac{5 \cos x}{\cos x} = \frac{-\sin x}{\cos x}$ or $\frac{\cos x}{\cos x} = \frac{-\sin x}{\cos x}$

$5 = -\tan x \therefore \tan x = -5$ | $1 = -\tan x \therefore \tan x = -1$

Reference angle $= 78{,}69°$ | Reference angle $= -45°$

$x = 180° - 78{,}69° + k180°$ or $\therefore x = 180° - 45° + k180°$

$\therefore x = 101{,}3° + k180°$ ✓ | $\therefore x = 135° + k180°$ ✓ $k \in \mathbb{Z}$ (3)

[6]

NB

- $a \sin \theta = b \cos \beta$: single sin and cos function with the different angles
  1. Use co-functions to get the same function i.e. change the sin function to a cos function or the cos function to a sin function.
  2. If $\sin\theta = \sin\beta$, we equate the angles then $\theta = \beta$ and $\theta = 180° - \beta$.

     If $\cos\theta = \cos\beta$ we equate the angles then $\theta = \beta$ and $\theta = 360° - \beta$

e.g. 6

Solve for x (give general solution) and round off your answer to 2 decimal places.

$\sin(x + 20°) = \cos 3x$ **[7]**

Solution

$\sin(x + 20°) = \cos 3x$	Use co-functions
$\sin(x + 20°) = \sin(90° - 3x)$ ✓	Choose one angle to be the reference angle
Ref angle $= (90° - 3x)$	
$x + 20° = 90° - 3x + k360°$ ✓	or $x + 20° = 180° - (90° - 3x) + k360$ ✓
$4x = 70° + k360°$ ✓	$x + 20° = 180° - 90° + 3x + k360°$
$x = 17{,}5° + k90°$ ✓	$-2x = 70° + k360°$ ✓
	$x = -35° - k180°$ ✓ $k \in \mathbb{Z}$ **[7]**

e.g. 7

1. $\sin^2 A - \sin A \cos A = 0$
2. $\cos^2 A - 2\cos A - 3 = 0$
3. $\cos^2 x + 3\sin x = -3$ **[16]**

NB

Trigonometric equations that leads to quadratic equations

Solutions

1. $\sin^2 A - \sin A \cos A = 0$

   $\sin A(\sin A - \cos A) = 0$......✓.......factorise by means of a HCF

   $\therefore \sin A = 0$ or $\sin A - \cos A = 0$ ✓

   $\therefore \sin A = 0$ or $\sin A = \cos A$

   $\therefore A = 0° + 360°n$ ✓ or $\tan A = 1$ ✓

   $\therefore A = 45° + 180°n$.......$n \in \mathbb{Z}$ ✓ (5)

2. $\cos^2 A - 2\cos A - 3 = 0$

   $(\cos A + 1)(\cos A - 3) = 0$ ✓

   $\therefore \cos A + 1 = 0$✓ or $\cos A - 3 = 0$

   $\therefore \cos A = -1$ ✓ or $\cos A = 3$ ✓

   $\therefore A = -180° + 360°n$ .....$n \in \mathbb{Z}$

   if $\cos A = 3$....no solution ✓ (5)

3. $\cos^2 x + 3\sin x = -3$ Use $\cos^2 x = 1 - \sin^2 x$ to make a quadratic equation in $\sin x$

   $1 - \sin^2 x + 3\sin x + 3 = 0$

   $-\sin^2 x + 3\sin x + 4 = 0$

   $\sin^2 x - 3\sin x - 4 = 0$ ✓

   $(\sin x - 4)(\sin x + 1) = 0$ ✓

   $\sin x - 4 = 0$ or $\sin x + 1 = 0$

   $\sin x = 4$ ✓ $\sin x = -1$ ✓

   No solution ✓ ref angle $= -90°$

   $(-1 \le \sin x \le 1)$ $x = -90° + k360°$ or $x = 360° - 90° + k360°$

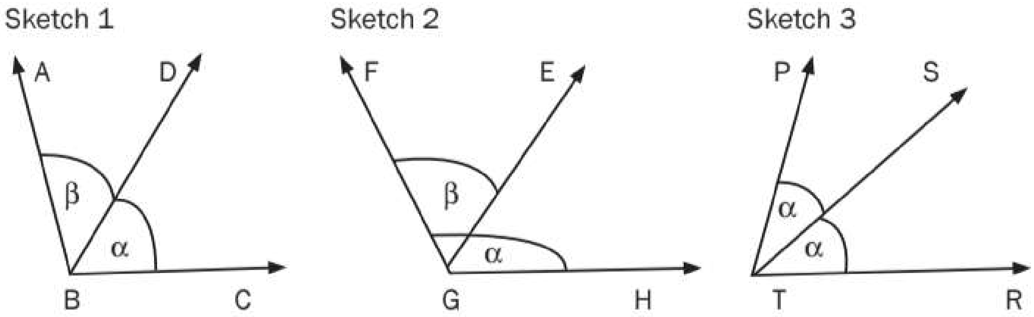
   $x = 270° + k360°$ ✓ (6)

[16]

10.11 Compound and double angle identities

$\sin(20° + 30°) \neq \sin 20° + \sin 30°$

When two angles are added or subtracted to form a new angle, then a **compound** or a **double** angle is formed.



Sketch 1 : The compound angle $A\hat{B}C$ is equal to the sum of α and β
eg. **75°** = 45° + 30°

Sketch 2 : The compound angle $E\hat{G}H$ is equal to the difference between α and β
eg. **15°** = 60° – 45° or 15° = 45° – 30°

Sketch 3 : The double angle $P\hat{T}R$ is equal to the sum of α and α
eg. **45°** = 22,5° + 22,5°

NOTE:
$\sin(\alpha + \beta) \neq \sin\alpha + \sin\beta$ **and** $\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$

Using the same methods as we did to establish the reduction formulae, we can also establish the compound angle identities.

Given any angles α and β, we can find the values of the sine and cosine ratios of the angles $\alpha + \beta$, $\alpha - \beta$ and 2α.

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= 2\cos^2\alpha - 1$
$= 1 - 2\sin^2\alpha$

These formulae are provided on the **information sheet** in the final exam.

You should **learn** these formulae, as you will use them often.

Accept:

$$\cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta,$$

and derive the other compound angle identities

NB

These are examinable, learn them well.

Proof:

$$\begin{aligned}\cos(\alpha + \beta) = \cos[\alpha - (-\beta)] &= \cos\alpha.\cos(-\beta) + \sin\alpha.\sin(-\beta)\\ &= \cos\alpha.\cos\beta + \sin\alpha.(-\sin\beta)\\ &= \cos\alpha.\cos\beta - \sin\alpha.\sin\beta\end{aligned}$$

Proof:

$$\begin{aligned}\sin(\alpha + \beta) = \cos[90^\circ - (\alpha + \beta)] &= \cos[90^\circ - \alpha - \beta] = \cos[(90^\circ - \alpha) - \beta]\\ &= \cos(90^\circ - \alpha).\cos(\beta) + \sin(90^\circ - \alpha).\sin(\beta)\\ &= \sin\alpha.\cos\beta + \cos\alpha.\sin\beta\end{aligned}$$

Proof:

$$\begin{aligned}\sin(\alpha - \beta) = \cos[90^\circ - (\alpha - \beta)] &= \cos[90^\circ - \alpha + \beta] = \cos[(90^\circ + \beta) - \alpha]\\ &= \cos(90^\circ + \beta).\cos\alpha + \sin(90^\circ + \beta).\sin\alpha\\ &= -\sin\beta.\cos\alpha + \cos\beta.\sin\alpha\\ &= \sin\alpha.\cos\beta - \cos\alpha.\sin\beta\end{aligned}$$

e.g. 8

Simplify without the use of a calculator:

1. $\cos 70° \cos 10° + \cos 20° \cos 80°$
2. $2 \sin 15° \cos 15°$
3. $\sin 15°$ **[10]**

Solutions

1. $\cos 70° \cos 10° + \cos 20° \cos 80°$

$= \cos 70° \cos 10° + \sin 70° \sin 10°$

$= \cos(70° - 10°)$ ✓

$= \cos 60°$ ✓

$= \frac{1}{2}$ ✓ (3)

2. $2 \sin 15° \cos 15°$

$= \sin 2(15°)$ ✓

$= \sin 30°$ ✓

$= \frac{1}{2}$ ✓ (3)

3. $\sin 15°$

$= \sin(45° - 30°)$ *n*

$= \sin 45° . \cos 30° - \cos 45° . \sin 30°$ *n*

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$ ✓✓

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$ ✓ $\times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}(\sqrt{3}-1)}{4}$ ✓ (4)

[10]

Activity 10

Do NOT use a calculator to answer this question. Show ALL calculations. Prove that:

1. $\cos 75° = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$ (5)
2. Prove that $\cos(90° - 2x).\tan(180° + x) + \sin^2(360° - x) = 3\sin^2 x$ (7)
3. Prove that $(\tan x - 1)(\sin 2x - 2\cos^2 x) = 2(1 - 2\sin x \cos x)$ (7)

[19]

Solutions

1. LHS= $\cos 75° = \cos(45° + 30°)$ ✓

$= \cos 45°.\cos 30° - \sin 45°.\sin 30°$ ✓

$= \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}.\frac{1}{2}$ ✓✓

$= \frac{\sqrt{2}.\sqrt{3}}{4} - \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{2}(\sqrt{3}-1}{4}$ ✓ = RHS (5)

2. LHS = $\cos(90° - 2x).\tan(180° + x) + \sin^2(360° - x)$ co-functions and reductions

$= \sin 2x$✓. $\tan x$✓ + $\sin^2 x$✓ double angle for sin $2x$

trig identity for tan x

$= 2\sin x.\cos x$✓. $\frac{\sin x}{\cos x}$ ✓+ $\sin^2 x$ simplify

$= 2\sin^2 x + \sin^2 x$✓

$= 3\sin^2 x$✓ = RHS (7)

3. There are several ways to prove this. Here is one solution.

LHS = $(\tan x - 1)(\sin 2x - 2\cos^2 x)$

$= \left(\frac{\sin x}{\cos x}\right.$ ✓ $\left. - 1\right)(2\sin x.\cos x$✓ $- 2\cos^2 x)$ double angle identity for sin $2x$

$= 2\sin^2 x - 2\sin x.\cos x - 2\sin x.\cos x + 2\cos^2 x$✓✓ multiply out

$= 2\sin^2 x - 4\sin x\cos x + 2\cos^2 x$✓

$= 2(\sin^2 x - 2\sin x.\cos x + \cos^2 x)$✓ trig identity $\sin^2 x + \cos^2 x = 1$

$= 2(1 - 2\sin x.\cos x)$✓ = RHS (7)

[19]

Activity 11

Determine the general solution for x in the following:

a) $\sin 2x.\cos 10° - \cos 2x.\sin 10° = \cos 3x$ (8)

b) $\cos^2 x = 3\sin 2x$ (11)

c) $2\sin x = \sin(x + 30°)$ (5)

[24]

Solutions

a) $\sin 2x.\cos 10° - \cos 2x.\sin 10° = \cos 3x$ use compound angle identity

$\therefore \sin(2x - 10°)✓ = \cos 3x$ use co-functions

$\therefore \sin(2x - 10°) = \sin(90° - 3x)✓$

$\therefore 2x - 10° = 90° - 3x + k360°✓$ or $2x - 10° = 180° - (90° - 3x) + k360°✓\ k \in \mathbb{Z}$

$\therefore 5x = 100° + k360°✓$ $\quad 2x - 10° = 90° + 3x + k360°$

$\therefore x = 20° + k72°✓$ $\quad -x = 100 + k360°✓$

$x = -100 - k360\ ✓\ k \in \mathbb{Z}$ (8)

b) $\cos^2 x = 3\sin 2x$ use double angles for $\sin 2x$

$\cos^2 x = 3(2\sin x.\cos x)✓$ make LHS = 0

$\cos^2 x - 3(2\sin x.\cos x) = 0$ multiply out

$\cos^2 x - 6\sin x.\cos x = 0✓$ common factor

$\cos x(\cos x - 6\sin x)✓ = 0$

$\therefore \cos x = 0✓$ or $\cos x - 6\sin x = 0✓$

$\cos x = 0$ or $\frac{\cos x}{\cos x} = \frac{6\sin x}{\cos x}$

$\cos x = 0$ or $1 = 6\tan x$

$\cos x = 0$ or $\tan x = \frac{1}{6}✓$

Reference angle = 90° or reference angle = 9,46°

$\therefore x = 90° + k360°✓$ or $x = 360° - 90° + k360°$ or $x = 9{,}46° + k180°\ ✓k \in \mathbb{Z}$

$x = 270° + k360°✓$ or $x = 180° + 9{,}46° + k360°✓k \in \mathbb{Z}$

$= 189{,}46° + k360°\ ✓\ k \in \mathbb{Z}$ (11)

c) $2\sin x = \sin(x + 30°)$

$2\sin x = \sin x.\cos 30° + \cos x.\sin 30°✓$

$2\sin x = \sin x.\frac{\sqrt{3}}{2} + \cos x.\frac{1}{2}✓$ multiply by 2

$4\sin x = \sqrt{3}\sin x + \cos x$ divide by $\cos x$

$4\tan x = \sqrt{3}\tan x + 1$

$4\tan x - \sqrt{3}\tan x = 1✓$

$\tan x = \frac{1}{4 - \sqrt{3}}✓$

$x = 23{,}79° + k180°;\ k \in \mathbb{Z}\ ✓$ (5)

[24]

10.12 Determining x for which the identity is undefined

NB

- $\frac{\text{any number}}{0}$ is undefined. Therefore if the denominator of an identity $= 0$, then the identity is undefined
- $y = \tan x$ is undefined for certain value of x. Therefore if a tan function is in an identity then the identity is undefined where the tan function is undefined.

e.g. 9

For which values of x is this identity undefined? $\frac{1}{\tan x} + \tan x = \frac{\tan x}{\sin^2 x}$

[4]

Solution

$\frac{1}{\tan x} + \tan x = \frac{\tan x}{\sin^2 x}$ is undefined if $\tan x = 0$ or if $\sin^2 x = 0$ or if $\tan x$ is undefined

[division by 0 is undefined]

if $\tan x = 0$	OR	if $\sin^2 x = 0$	OR	$\tan x$ is undefined
$x = 0° + k180°$✓	OR	$\sin x = 0$ ✓✓		$x = 90° + k180°$ ✓ (4)
$x = 0° + k360°$	OR	$x = 180° + k360°$		

So the identity is undefined for $x = 0° + k360°$ or $x = 180° + k360°$ or $x = 90° + k180°$

All these solutions are the same as $x = 0° + k90°$ for $k \in \mathbb{Z}$.

[4]

11 Unit

Trigonometry: Sine, cosine and area rules

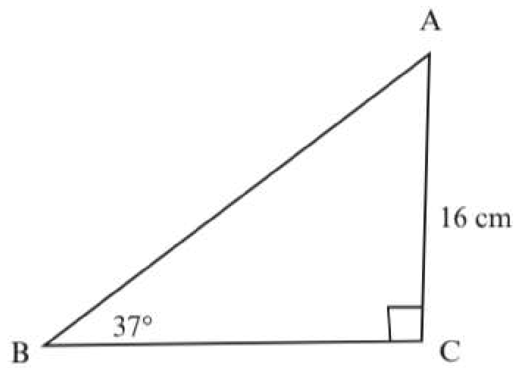
We use these three rules to find the lengths of sides, sizes of angles and the area of any kind of triangle. To 'solve a triangle' means you must calculate the unknown sides and angles.

11.1 Right-angled triangles

You can use the trig ratios to find angles and lengths of a right-angled triangle.

Activity 1

In triangle ABC, $\hat{B} = 37°$ and AC = 16 cm. $\hat{C} = 90°$. Calculate the length of AB and BC (correct to one decimal place). (3)



[3]

Solution

To calculate the length of AB, use 37° as the reference angle, then AC = 16 cm is the opposite side and AB is the hypotenuse. Use the sine ratio.

$\sin 37° = \frac{\text{opp}}{\text{hyp}} = \frac{16}{\text{AB}}$

AB $\sin 37° = 16$

AB $= \frac{16}{\sin 37°} = 26{,}6$ cm ✓

To find the length of BC, you can use

$\cos 37° = \frac{\text{adj}}{\text{hyp}} = \text{BC}/26.6$

$26{,}6 \cos 37° = \text{BC}$ ✓

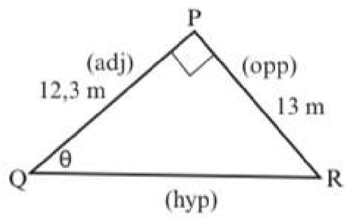
BC = 21,2 cm (to one decimal place) ✓

You can also use Pythagoras' theorem:

$\text{AB}^2 = \text{AC}^2 + \text{BC}^2$ [3]

Activity 2

In triangle PQR, PQ = 12,3 m and PR = 13 m. Calculate the size of $\hat{Q}$. (2)



[2]

Solution

Use PQ and PR.

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{13}{12.3} \checkmark$$

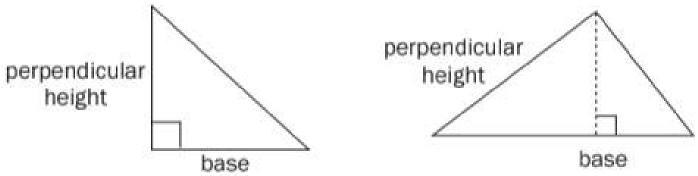
$$\theta = \tan^{-1}\left(\frac{13}{12.3}\right) = 46{,}58° \checkmark$$ **[2]**

11.2 Area rule

Area of a right-angled triangle:

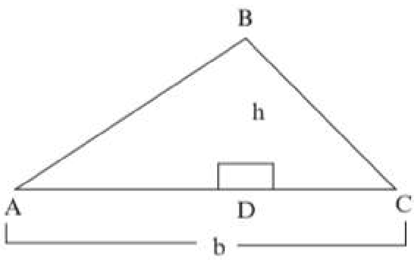
Area $\Delta = \frac{1}{2}$ base $\times$ perpendicular height

Area $\Delta = \frac{1}{2} bh$



Proof of Area rule [STUDY FOR EXAM PURPOSE]

If $\hat{A}$ is acute



Area of ΔABC $= \frac{1}{2} bh$...............(1)

But sin A $= \frac{h}{c} \therefore h = c$ sin A

Substituting into (1)

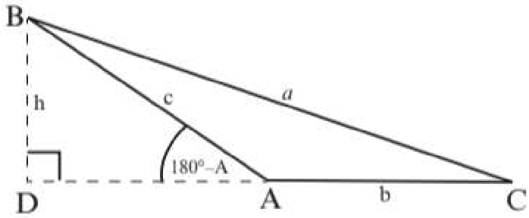
Area of ΔABC $= \frac{1}{2} bc$ sin A

Similarly it can be shown that

Area of ΔABC $= \frac{1}{2} ab$ sin C

$= \frac{1}{2} ac$ sin B

If $\hat{A}$ is obtuse



Area of ΔABC $= \frac{1}{2} bh$...............(1)

But sin (180° – A) $= \frac{h}{c} \therefore h = c$ sin A

Substituting into (1)

Area of ΔABC $= \frac{1}{2} bc$ sin A

Similarly it can be shown that

Area of ΔABC $= \frac{1}{2} ab$ sin C

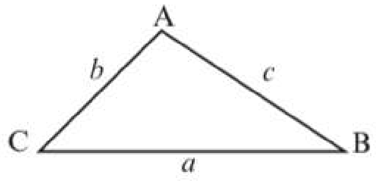
$= \frac{1}{2} ac$ sin B

NB

If a base or height is unknown, you can use trig ratios to work them out. If the perpendicular height is not given and cannot be worked out, then we need a different area formula.

There is a formula that works to find the area of *any* triangle, even if we do not know the perpendicular height.

The area of any ΔABC is half the product of two sides and sine of the included angle.



So if you choose to use angle A, then

Area ΔABC = $\frac{1}{2}$ ***bc*** **sin A**

If you choose to use angle B, then

Area ΔABC = $\frac{1}{2}$ ***ac*** **sin B**

If you choose to use angle C, then

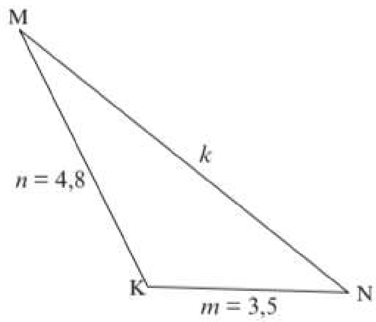
Area ΔABC = $\frac{1}{2}$ ***ab*** **sin C**

Learn one form of the formula – you can work out the others from that.

To find the area of any triangle, you need to know the lengths of two sides and the size of the angle between the two sides.

e.g. 1

Calculate the area of ΔMNK with $m = 3{,}5$ cm; $n = 4{,}8$ cm and $\hat{K} = 112°$.



Choose the version of the formula that uses the sides m and n and the angle K because these are known values.

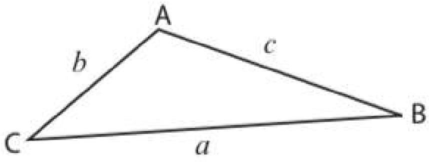
$$\begin{aligned} \text{Area } \Delta\text{ MNK} &= \tfrac{1}{2}mn\sin K \\ &= \tfrac{1}{2}(3{,}5)(4{,}8)\sin 112° \\ &= 8{,}4 \sin 112° \\ &= 7{,}788\text{ cm}^2 \text{ (correct to 3 decimal places)} \end{aligned}$$

11.3 Sine rule

If you have enough information about the sides and angles of any triangle, you can use the sine rule to find the other sides and angles.

Sine rule

The ratio of sine of the angle divided by the side opposite that angle is the same for all three pairs of sides and angles.



So ...

In any triangle ABC:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

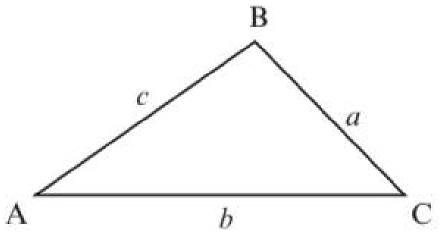
We can also use the ratios with the sides in the numerator:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

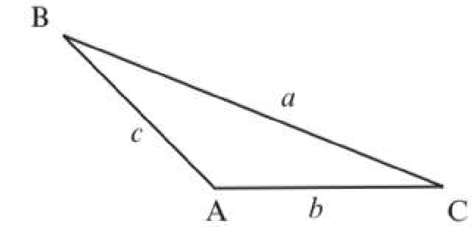
The formula will be provided on the information sheet

Proof of Sine rule [STUDY FOR EXAM PURPOSE]

If $\hat{A}$ is acute



If $\hat{A}$ is obtuse



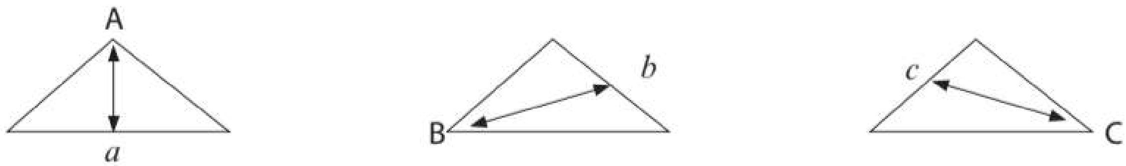
Using Area rule for ΔABC:

$$\frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$$

Dividing each by $\frac{1}{2}abc$ results: $\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}$

NB

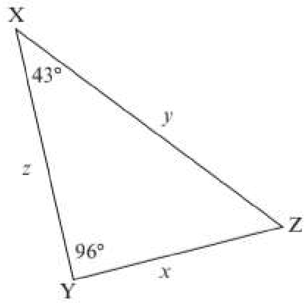
To use the **sine rule** you need to know at least one side and its matching opposite angle and one more side or angle.



The sine rule can be used to solve many problems if the right information about the triangle is given.

Activity 3

Solve ΔXYZ in which $z = 7{,}3\ m$, $\hat{X} = 43°$ and $\hat{Y} = 96°$. Give your solutions correct to 3 decimal places. (4)



[4]

Solution

The angle opposite the known side is not given, but you can work it out.

$\hat{Z} = 180° - (43° + 96°)$ (sum angles of Δ)

$\hat{Z} = 41°$ ✓

To find y: $\frac{y}{\sin 96°} = \frac{7{,}3}{\sin 41°}$ ✓

$y = \frac{7{,}3 \sin 96°}{\sin 41°}$

$y = 11{,}066$ m ✓

Using the sine rule again to find x:

$\frac{x}{\sin 43°} = \frac{7{,}3}{\sin 41°}$

$x = \frac{7{,}3 \sin 43°}{\sin 41°}$

$x = 7{,}589$ m ✓

[4]

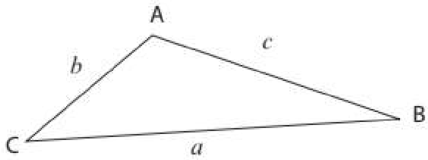
11.4 Cosine rule

You apply the **cosine rule** If you are given the values of:

- two sides and the included angle OR
- three sides of a triangle,

Cosine rule:

In any triangle ABC:



If you choose to use angle A, then

$$a^2 = b^2 + c^2 - 2bc \cos A$$

If you choose to use angle B, then

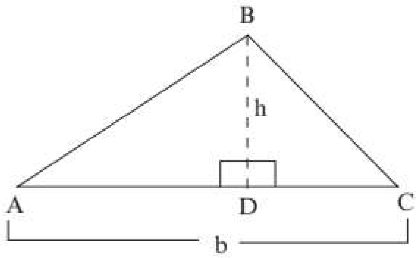
$$b^2 = a^2 + c^2 - 2ac \cos B$$

If you choose to use angle C, then

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof of Cosine rule [STUDY FOR EXAM PURPOSE]

If $\hat{A}$ is acute



In ΔBDC: $a^2 = BD^2 + CD^2$ (Pythagoras Theorem)

$= BD^2 + (b - AD)^2$

$= BD^2 + b^2 - 2bAD + AD^2$

But $BD^2 + AD^2 = c^2$ (Pythagoras Theorem)

Thus $a^2 = b^2 + c^2 - 2bAD$ ....(1)

In Δ ABD: $\cos A = \frac{AD}{c} \therefore AD = c \cos A$ ....(2)

Substituting (2) into (1)

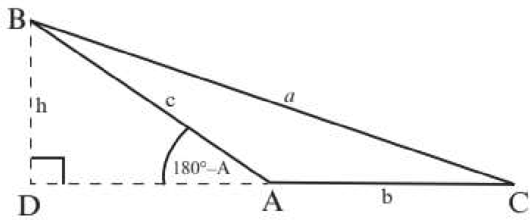
$\therefore a^2 = b^2 + c^2 - 2bc \cos A$

Similarly it can be shown that:

$b^2 = a^2 + c^2 - 2ac \cos B$ and

$c^2 = a^2 + b^2 - 2ab \cos C$

If $\hat{A}$ is obtuse



In ΔBDC: $a^2 = BD^2 + CD^2$ (Pythagoras Theorem)

$= BD^2 + (b + AD)^2$

$= BD^2 + b^2 + 2bAD + AD^2$

But $BD^2 + AD^2 = c^2$ (Pythagoras Theorem)

Thus $a^2 = b^2 + c^2 + 2bAD$ ....(1)

InΔABD:.

$\cos(180° - A) = \frac{AD}{c} \therefore AD = -c \cos A$..........(2)

Substituting (2) into (1)

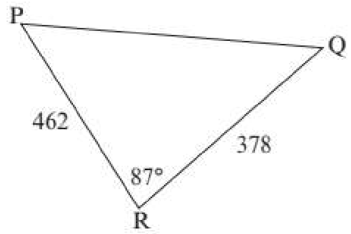
$\therefore a^2 = b^2 + c^2 - 2bc \cos A$

Similarly it can be shown that:

$b^2 = a^2 + c^2 - 2ac \cos B$ and

$c^2 = a^2 + b^2 - 2ab \cos C$

e.g. 2



1. Solve ΔPQR if $q = 462$ mm, $p = 378$ mm and $\hat{R} = 87°$.

Using the cosine rule
(two sides and the included angle are given so you can find the side opposite the given angle)
$PQ^2 = p^2 + q^2 - 2pq \cos R$
$PQ^2 = (378)^2 + (462)^2 - 2(378)(462).\cos 87°$
$PQ^2 = 338\ 048{,}5159$
$PQ = 581{,}42$ mm [take square root]

Using the sine rule:

$\frac{378}{\sin P} = \frac{581{,}42}{\sin 87°}$

$\frac{\sin P}{378} = \frac{\sin 87°}{581{,}42}$ (it is easier to have $\hat{P}$ in the numerator)

$\sin P = \frac{378 \times \sin 87°}{581{,}42}$

$\sin P = 0{,}649$

$\hat{P} = \sin^{-1}(0{,}649) = 40{,}48°$

$\therefore \hat{Q} = 180° - (87° + 40{,}48°) = 52{,}52°$ [sum angles of Δ]

2. Determine the biggest angle in ΔABC if $a = 7$ cm; $b = 9$ cm and $c = 15$ cm.
You are given three sides, so use the cosine rule.
The biggest angle will be $\hat{C}$ (opposite the longest side).

$$\left.\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ 2ab\cos C &= a^2 + b^2 - c^2 \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\}$$ rearrange the formula to get cos C on its own

$\cos C = \frac{7^2 + 9^2 - 15^2}{2(7)(9)}$

$\cos C = -0{,}753968\ldots$ cos θ is negative in quad 2, so $\hat{C}$ is obtuse.
reference angle is $41{,}06°$

$\hat{C} = 180° - 41{,}064\ldots° = 138{,}94°$ (correct to two decimal places)

11.5 Problems in two and three dimensions

Activity 4

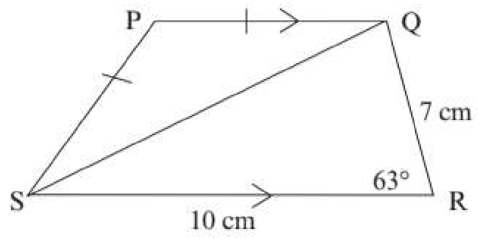
1. PQRS is a trapezium with PQ // SR, PQ = PS, SR = 10 cm, QR = 7 cm, $\hat{R} = 63°$.

Calculate:

a) SQ (2)

b) PS (6)

c) area of quadrilateral PQRS. (correct to 2 decimal places) (5)



[13]

NB

When solving triangles, start with the triangle which has most information (i.e. triangle with three sides or two sides and an angle or two angles and a side given)

Solutions

a) In Δ QSR, you know two sides and the included angle, so use the cosine rule.

$SQ^2 = 7^2 + 10^2 - 2(7)(10)\cos 63°$ ✓

$SQ^2 = 85,44\ldots$ find the square root

$SQ = 9,24$ cm ✓ (2)

b) In Δ PQS, you know that PQ = PS and you worked out that SQ = 9,24 cm.

Think about the question first

If you can find $\hat{P}$ then you can use the sine rule to find PS.

To find $\hat{P}$, you need to first find $P\hat{Q}S$ or $P\hat{S}Q$.

$P\hat{Q}S = P\hat{S}Q$ (alternate angles, PQ // SR)

Now can you work out a value for $Q\hat{S}R$?

In ΔQSR, you know three sides and $\hat{R}$.

So it is easiest to use the sine rule to find $Q\hat{S}R$.

$\frac{\sin Q\hat{S}R}{7} = \frac{\sin 63°}{9,24}$ ✓

$\sin Q\hat{S}R = \frac{7 \sin 63°}{9,24} = 0.675004$

$\therefore Q\hat{S}R = 42,45°$ ✓

$P\hat{Q}S = Q\hat{S}R = 42,45°$ (alternate angles, PQ // SR)

$P\hat{Q}S = P\hat{S}Q = 42,45°$ (base angles of isosceles Δ)

$\therefore \hat{P} = (180° - (42,45° + 42,45°)$ ✓

$= 95,1°$ ✓ (sum angles in Δ)

Now we can find PS using the sine rule and $\hat{P}$.

In ΔPQS $\frac{PS}{\sin 42,45°} = \frac{9,24}{\sin 95,1}$ ✓

$PS = \frac{9,24 \sin 42,45°}{\sin 95,1}$

$PS = 6,26$ cm ✓ (6)

c) To find the areas of PQRS, find the area of the two triangles and add them together.

To find the area of ΔPQS, use $\hat{P} = 95{,}1°$ and PS = PQ = 6,26 cm.

Area $\Delta PQS = \frac{1}{2}qs \sin P$ ✓

Area $\Delta PQS = \frac{1}{2}(6{,}26)(6{,}26)\sin 95{,}1°$

Area ΔPQS = 19,52 m² ✓

To find the area of ΔRQS, use $\hat{R} = 63°$, QR = 7 cm and SR = 10 cm.

Area $\Delta RQS = \frac{1}{2}(7)(10)\sin 63$ ✓

Area ΔRQS = 31,19 m² ✓

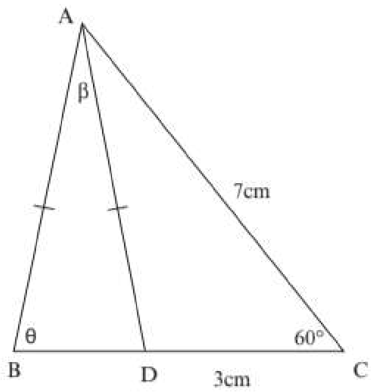
∴ Area PQRS = 19,52 + 31,19 = 50,71 m² ✓ (5)

[13]

Activity 5

In the diagram alongside, AC = 7 cm, DC = 3 cm, AB = AD, $D\hat{C}A = 60°$, $D\hat{A}B = \beta$ and $A\hat{B}D = \theta$.

Show that $BD = \frac{\sqrt{37}\sin\beta}{\sin\theta}$



[3]

Solution

$AD^2 = AC^2 + CD^2 - 2AC.CD\cos 60° = (7)^2 + (3)^2 - 2 \times 7 \times 3 \times 0{,}5$ ✓

$AD^2 = 58 - 21$

$AD^2 = 37$

$AD = \sqrt{37}$ P

Applying sine rule:

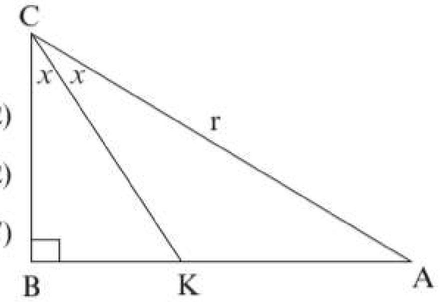
$\frac{BD}{\sin\beta} = \frac{AD}{\sin\theta} \Rightarrow BD = \frac{AD\sin\beta}{\sin\theta}$ but $AD = \sqrt{37}$ ✓

$\therefore BD = \frac{\sqrt{37}\sin\beta}{\sin\theta}$ ✓

[3]

Activity 6

1. In the diagram alongside, ABC is a right angled triangle. KC is the bisector of $A\hat{C}B$. AC = r units and $B\hat{C}K = x$

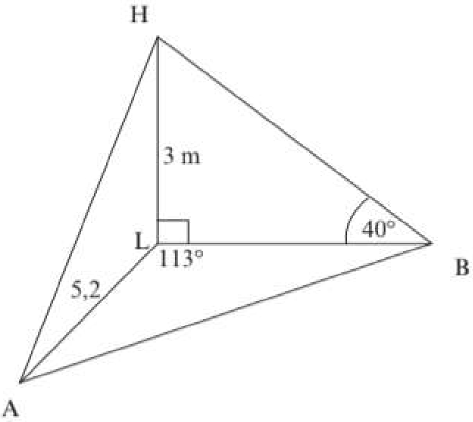


   1.1 Write down AB in terms of x (2)

   1.2 Give the size of $A\hat{K}C$ in terms of x (2)

   1.3 If it is given that $\frac{AK}{AB} = \frac{2}{3}$, calculate the value of x (7)

2. A, B and L are points in the same horizontal plane, HL is a vertical pole of length 3 metres, AL = 5,2 m, the angle $A\hat{L}B = 113°$ and the angle of elevation of H from B is 40°.

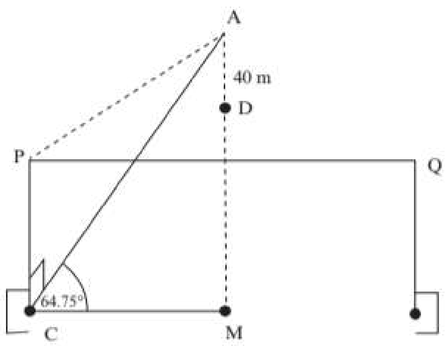


   2.1 Calculate the length of LB. (3)

   2.2 Hence, or otherwise, calculate the length of AB. (3)

   2.3 Determine the area of ΔABL. (3)

3. The angle of elevation from a point C on the ground, at the centre of the goalpost, to the highest point A of the arc, directly above the centre of the Moses Mabhida soccer stadium, is 64,75°. The soccer pitch is 100 metres long and 64 metres wide as prescribed by FIFA for world cup stadiums. Also AC ⊥ PC.
In the figure below PQ = 100 metres and PC = 32 metres



   3.1 Determine AC (2)

   3.2 Calculate $P\hat{A}C$ (2)

   3.3 A camera is placed at D, 40 m directly below point A, calculate the distance from D to C (4)

[28]

Solutions

1.1 $\sin 2x = \frac{AB}{r} \therefore AB = r\sin 2x$ ✓✓ (2)

1.2 $A\hat{K}C = 90° + x$ [ext. angle of ΔCBK] ✓✓ (2)

1.3 $\frac{AK}{\sin x} = \frac{r}{\sin(90° + x)} \therefore AK = \frac{r\sin x}{\cos x}$ ✓✓

$\frac{AK}{AB} = \frac{\frac{r\sin x}{\cos x}}{r\sin 2x} = \frac{r\sin x}{r\cos x.2\cos x\sin x} = \frac{1}{2\cos^2 x} = \frac{2}{3}$ ✓✓

$\therefore \cos^2 x = \frac{3}{4}$ ✓

$\cos x = \frac{\sqrt{3}}{2}$ ✓

Hence $x = 30^0$ ✓ (7)

2.1 In ΔHLB, $\tan 40° = \frac{3}{LB}$ ✓

[ΔHLB is right-angled, so use a trig ratio]

$LB = \frac{3}{\tan 40^0}$ ✓

$LB = 3{,}5752\ldots \approx 3{,}58$ metres ✓ (3)

2.2 In ΔABL,

[ΔABL not right-angled. You have two sides and included angle, so use the Cosine Rule]

$AB^2 = AL^2 + BL^2 - 2(AL)(BL).\cos L$ ✓

$AB^2 = (5{,}2)^2 + (3{,}58)^2 - 2(5{,}2)(3{,}58).\cos 113°$ ✓

$AB^2 = 54{,}40410\ldots\ m^2$

$AB = 7{,}38$ m ✓ (3)

2.3 Area $\Delta ABL = \frac{1}{2} AL \times BL \times \sin A\hat{L}B$ ✓

$= \frac{1}{2}(5{,}2) \times (3{,}58) \times \sin 113°$ ✓

$= 8{,}56805\ldots$ ✓

$\approx 8{,}57\ m^2$ (3)

3.1 $\cos 64{,}750° = \frac{CM}{AC} \therefore AC = \frac{CM}{\cos 64{,}75^0} = \frac{50m}{0{,}426569} = 117{,}21$ ✓✓ (2)

3.2 $\tan P\hat{A}C = \frac{PC}{AC}$

$P\hat{A}C = \tan^{-1}\left(\frac{32}{AC}\right)$ ✓

$= 15{,}27^0$ ✓ (2)

3.3 $DC^2 = AC^2 + AD^2 - 2AC.AD\cos(90^0 - 64{,}75^0)$ ✓

$DC^2 = (117{,}21)^2 + (40)^2 - 2(117{,}21).40\cos(25{,}25^0)$ ✓

$= 6857{,}289$ ✓

$DC = 82{,}81$ m ✓ (4)

[28]