CAPS

Mind the Gap!

Mathematics
Study Guide

Grade 12

basic education

Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

3 Unit

Number patterns, sequences and series

3.1 Number patterns

A list of numbers in order is called a number pattern or number sequence.

We need at least three numbers in the list to work out if the numbers form a pattern. If we only have two numbers, we cannot be sure what the pattern is.

For example, if we have the list 2; 4; ... many different number patterns are possible:

The pattern could be	2; 4; 6; ...	add 2 to each number to get the next number
OR	2; 4; 8; ...	multiply each number by 2 to get the next number
OR	2; 4; 2; 4; ...	repeat the pattern

A single number in a pattern or sequence is called a **term**.

Term 1 is written as T_1, term 2 is written as T_2 and so on. The number of the term shows its position in the sequence.

T_{10} is the 10^{th} term in the sequence.
T_n is the nth term in a sequence.

e.g. 1

1. Look at the number pattern 3; 8; 13; …

T_1		T_2		T_3		T_4		T_5
3		8		13		...		
	+5		+5		+5		+5	

If we keep adding **5** to each term we get the next term:
$T_4 = 13 + 5 = 18$; $T_5 = 23$; $T_6 = 28$, etc.

2. Look at the number pattern 5; 15; 45; …

T_1		T_2		T_3		T_4		T_5
5		15		45		...		
	× 3		× 3		× 3		× 3	

In this pattern, each term is multiplied by 3 to get the next term.
So $T_4 = 45 \times 3 = 135$; $T_5 = 405$; $T_6 = 1\ 215$, and so on.

3. Look at the sequence: 1; 4; 9; …
$T_1 = 1^2$; $T_2 = 2^2$; $T_3 = 3^2$
These are all perfect square numbers. Each number is the number of the term squared.
So $T_4 = (4)^2 = 16$; $T_5 = (5)^2 = 25$; $T_6 = (6)^2 = 36$, and so on.

It is important to learn to recognise square numbers.

3.2 Arithmetic sequences

Arithmetic sequence is a sequence where the common difference (d) between consecutive terms is constant.

$T_2 - T_1 = T_3 - T_2 = T_n - T_{n-1} = d$ (common difference)

e.g. 2 Given the sequence: 5; 9; 13; 17 ; . . .

a) Determine the common difference

b) Determine the next two terms

Solution

$d = 9 - 5 = 13 - 9 = 4$

$T_5 = 17 + 4 = 21$ and $T_6 = 21 + 4 = 25$

If we use a for the first term T_1, d for the common difference, then the general term T_n for an arithmetic sequence is: $\mathbf{T_n = a + (n - 1)d}$

e.g. 3 Given the sequence 4; 10; 16; . . .

a) Determine a formula for the nth term of the sequence.

b) Calculate the 50^{th} term.

c) Which term of the sequence is equal to 310

Solutions

a) $a = 4$ and $d = 10 - 4 = 16 - 10 = 6$

$$\begin{aligned} T_n &= a + (n - 1)d \\ &= 4 + (n - 1)6 \\ &= 4 + 6n - 6 \\ &= 6n - 2 \end{aligned}$$

b)
$$\begin{aligned} T_{50} &= 6 \times 50 - 2 \\ &= 300 - 2 \\ &= 298 \end{aligned}$$

c)
$$\begin{aligned} 6n - 2 &= 310 \\ 6n &= 312 \\ n &= 52 \end{aligned}$$

Activity 1

1. Given the sequence 6; 13; 20; …

a) Determine a formula for the nth term of the sequence.

b) Calculate the 21st term of this sequence.

c) Determine which term of this sequence is 97. (5)

2. Consider this number pattern: 8; 5; 2; …

a) Calculate the 15^{th} term.

b) Determine which term of this sequence is –289. (4)

3. a) Given the arithmetic sequence $1 - p; 2p - 3; p + 5; \ldots$ determine the value of p.

b) Determine the values of the first three terms of the sequence. (5)

[14]

Solutions

1. a) It is an arithmetic sequence because there is a common difference.

$a = 6; d = 7 \quad T_n = a + (n - 1)d$ ✓

$T_n = 6 + (n - 1)(7)$

$T_n = 7n - 1$ ✓

b) $T_{21} = 7(21) - 1 = 147 - 1 = 146$ ✓

c) $97 = 7n - 1$ ✓

$\therefore 98 = 7n$

$\therefore 14 = n$ ✓

$\therefore 97$ is the 14th term of the sequence. (5)

2. a) It is an arithmetic sequence: $a = 8; d = 5 - 8 = 2 - 5 = -3$

$T_n = a + (n - 1)d$

$\therefore T_{15} = 8 + (15 - 1)(-3)$ ✓

$T_{15} = 8 + 14(-3)$

$T_{15} = 8 - 42 = -34$ ✓

b) $T_n = a + (n - 1)d$

$-289 = 8 + (n - 1)(-3)$ ✓

$\therefore -289 = 8 - 3n + 3$

$\therefore -300 = -3n$

$\therefore 100 = n$ ✓ $\quad \therefore -289$ will be the 100^{th} term (4)

3. a) Since this is an arithmetic sequence, you can assume that there is a common difference between the terms.

$d = T_2 - T_1 = T_3 - T_2$

$\therefore (2p - 3) - (1 - p) = (p + 5) - (2p - 3)$ ✓

$3p - 4 = -p + 8$ ✓

$4p = 12$

$p = 3$ ✓

b) $p = 3$

$T_1 = 1 - p = 1 - 3 = -2$

$T_2 = 2p - 3 = 2(3) - 3 = 3$ ✓

$T_3 = p + 5 = 3 + 5 = 8$ ✓

So the first three terms of the sequence are –2; 3; 8 (5)

[14]

3.3 Quadratic sequences

At least four numbers are needed to determine whether the sequence is quadratic or not.

Consider this number pattern:

There is no common difference between the numbers.

The differences are 6; 10; 14; 18.

Now we can see if there is a second common difference.

	T_1		T_2		T_3		T_4		T_5
	6		12		22		36		54
First difference		$(T_2 - T_1) = 6$		$(T_3 - T_2) = 10$		$(T_4 - T_3) = 14$		$(T_5 - T_4) = 18$	
Second difference			$10 - 6 = 4$		$14 - 6 = 4$		$18 - 4 = 4$		

In this sequence, there is a second common difference of 4.

The next term will be: $T_6 = 54 + (18 + 4) = 76$

A pattern with a common second difference is called a **quadratic number sequence.**

The general formula for any term of a quadratic sequence is: $T_n = an^2 + bn + c$

	$T_1 = a + b + c$		$T_2 = 4a + 2b + c$		$T_3 = 9a + 3b + c$		$T_4 = 16a + 4b + c$
1[st] difference		$3a + b$		$5a + b$		$7a + b$	
2[nd] difference			$2a$		$2a$		

If	$T_n = an^2 + bn + c$
then	$2a$ is the second difference
	$3a + b$ is $T_2 - T_1$
	$a + b + c$ is the first term

e.g. 4 Look at the number sequence 12; 20; 32; 48; . . .

2^{nd} common difference is 4
So $2a = 4$ $\qquad \therefore a = 2$

2. $T_2 - T_1 = 8$
So $3a + b = 8$ $\qquad \therefore 3(2) + b = 8$
$\qquad \therefore b = 2$

3. *1^{st} term is 12*
So $a + b + c = 12$ $\quad \therefore 2 + 2 + c = 12$
$\qquad \therefore c = 8$

$\therefore T_n = 2n^2 + 2n + 8$

$\therefore T_5 = 2(5)^2 + 2(5) + 8 = 68$
$\therefore T_6 = 2(6)^2 + 2(6) + 8 = 92$

3. 12		20		32		48
	$T_2 -$		$T_3 -$		$T_4 -$	
	2. =8		=12		=16	
		1. 4		4		

Activity 2

1. Consider the number pattern: 3; 13; 31; 57; 91; …
   - **a)** Determine the general term for this pattern.
   - **b)** Calculate the 7th term of this pattern.
   - **c)** Which term is equal to 241? (9)

2. Find term 6 of this pattern and then find the rule in the form $T_n = an^2 + bn + c$
   -1 ; 3; 9; 17; 27 … (4)

[13]

Solutions

1. a) It helps to make a diagram:

	T_1		T_2		T_3		T_4		T_5
	3		13		31		57		91
First difference		$(T_2 - T_1)$ $= 10$		$(T_3 - T_2)$ $= 18$		$(T_4 - T_3)$ $= 26$		$(T_5 - T_4)$ $= 34$	
Second difference			$18 - 10$ $= 8$		$26 - 18$ $= 8$		$34 - 26$ $= 8$		

$\therefore$ it is a quadratic sequence.

$2a = 8 \therefore a = 4$ ✓

$3a + b = 10 \quad \therefore 3(4) + b = 10$

$b = -2$ ✓

$a + b + c = 3 \therefore 4 + (-2) + c = 3$

$c = 1$ ✓

$\therefore T_n = 4n^2 - 2n + 1$ ✓

b) $T_7 = 4(7)^2 - 2(7) + 1$ ✓
$= 4(49) - 14 + 1$
$= 183$ ✓

c) $241 = 4n^2 - 2n + 1$

$0 = 4n^2 - 2n + 1 - 241$ ✓ make the equation = 0 to solve

$0 = 4n^2 - 2n - 240$

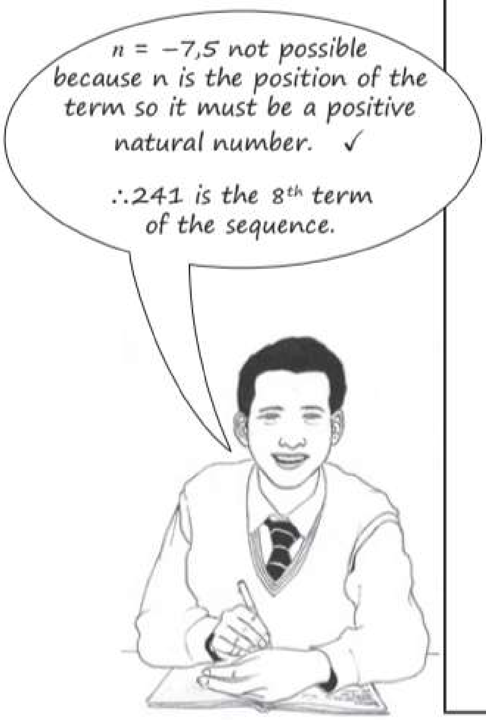
$0 = 2n^2 - n - 120$ divide through by 2

$0 = (2n + 15)(n - 8)$ ✓

factorise

$\therefore 2n + 15 = 0$ OR $n - 8 = 0$

$\therefore n = -7{,}5$ OR $n = 8$ ✓ (9)



2.

T_1	T_2	T_3	T_4	T_5	
-1	3	9	17	27	...

$$\begin{array}{ccccccc} & 4 & & 6 & & 8 & & 10 \\ & & 2 & & 2 & & 2 & & \checkmark \end{array}$$

$\therefore T_6 = 27 + (10 + 2) = 39$ ✓use the pattern of the numbers

$2a = 2 \therefore a = 1$

$3a + b = 4$

$3(1) + b = 4 \therefore b = 1$

$a + b + c = -1$

$1 + 1 + c = -1 \therefore c = -3$

$T_n = n^2 + n - 3$ ✓✓ (4)

[13]

3.4 Geometric sequences

When there is a **common ratio (r)** between consecutive terms, we can say this is a **geometric sequence.**

If the first term (T_1) is a, the common ratio is r, and the general term is T_n, then:

$$r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_n}{T_{n-1}} \text{ and}$$
$$T_n = ar^{n-1}$$

Look at the sequence 5; 15; 45; 135; 405; …

$\frac{15}{5} = 3$ $\frac{45}{15} = 3$ and $\frac{135}{45} = 3$ and so the common ratio is 3.

Therefore the sequence is geometric. To get the next term you multiply the preceding term by the common ratio.

e.g. 5

Given the sequence $1; \frac{2}{3}; \frac{4}{9}; ...$

a) Determine the next two terms

b) Which term of the sequence is equal to $\frac{32}{243}$?

Solutions

The common ratio is $\frac{2}{3}$ because $\frac{2}{3} \div 1 = \frac{2}{3} = \frac{4}{9} \div \frac{2}{3}$

a) $T_4 = ar^3 = 1\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ and $T_5 = 1\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

b) $a = 1$; $r = \frac{2}{3}$ and $T_n = ar^{n-1} = \frac{32}{243}$

$\therefore T_n = (1)\left(\frac{2}{3}\right)^{n-1} = \frac{2^5}{3^5} = \left(\frac{2}{3}\right)^5$

$\therefore \left(\frac{2}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^5$

$\therefore n - 1 = 5$

$n = 6$

e.g. 6

In a geometric sequence, the fifth term is 80 and the common ratio is –2. Determine the first three terms of the sequence.

$T_5 = 80$ and $r = -2$

$T_5 = ar^4 = a(-2)^4 = 80$

$16a = 80$

$a = 5$

$\therefore T_1 = 5; T_2 = 5(-2)^1 = -10; T_3 = 5(-2)^2 = 20$

Activity 3

a) Determine the 10th term of the sequence: 3; 6; 12; . . . (2)

b) Determine the number of terms in the sequence: 2; 4; 8; . . .; 1024 (2)

c) If 5; x; 45 are the first three terms of a geometric sequence, determine the value of x. (2)

d) Determine the geometric sequence whose 8th term is 9 and whose 10th term is 25. (3)

[9]

Solutions

a) $a = 3;\ r = \frac{6}{3} = \frac{12}{6} = 2$

$T_n = ar^{n-1}$

$T_{10} = 3(2)^{10-1} = 3(2)^9 = 3 \times 512 = 1536$ ✓✓ (2)

b) $a = 2;\ r = \frac{4}{2} = \frac{8}{4} = 2$

$ar^{n-1} = 1024$

$2(2)^{n-1} = 2^{10} = 2^n = 2^{10}$ ✓

$\therefore n = 10$ ✓ (2)

c) $\frac{x}{5} = \frac{45}{x}$ ✓

$x = \pm\sqrt{225} = \pm 15$ ✓ (2)

d) $ar^7 = 9$

$ar^9 = 25$

$\frac{ar^9}{ar^7} = \frac{25}{9}$

$\therefore r^2 = \frac{25}{9}$

$r = \frac{5}{3}$ ✓

$a = \frac{9}{\left(\frac{5}{3}\right)^7} = 9 \times \left(\frac{3}{5}\right)^7$ ✓

The sequence is: $9\left(\frac{3}{5}\right)^7;\ 9\left(\frac{3}{5}\right)^6;\ 9\left(\frac{3}{5}\right)^5;\ 9\left(\frac{3}{5}\right)^4;\ 9\left(\frac{3}{5}\right)^3$ ✓ (3)

[9]

3.5 Arithmetic and geometric series

When we add the terms of a sequence together, we form a series.

We use the symbol S_n to show the sum of the first n terms of a series.

So $\quad S_n = T_1 + T_2 + T_3 + T_4 + \ldots + T_n$

3.5.1 Arithmetic series

The formula is $S_n = \frac{n}{2}[2a + (n - 1)d]$	where S*n* is the sum of *n* terms, *a* is the first term, *n* is the number of terms and *d* is the common difference.

Proof

The general term of an arithmetic series is $T_n = a + (n - 1)d$

So $S_n = T_1 + T_2 + T_3 + T_4 + \ldots + T_n$

$S_n = a + [a + d] + a + 2d + \ldots + [a + (n - 2)d] + [a + (n - 1)d] \ldots$ equation 1

If we write the series in reverse we get:

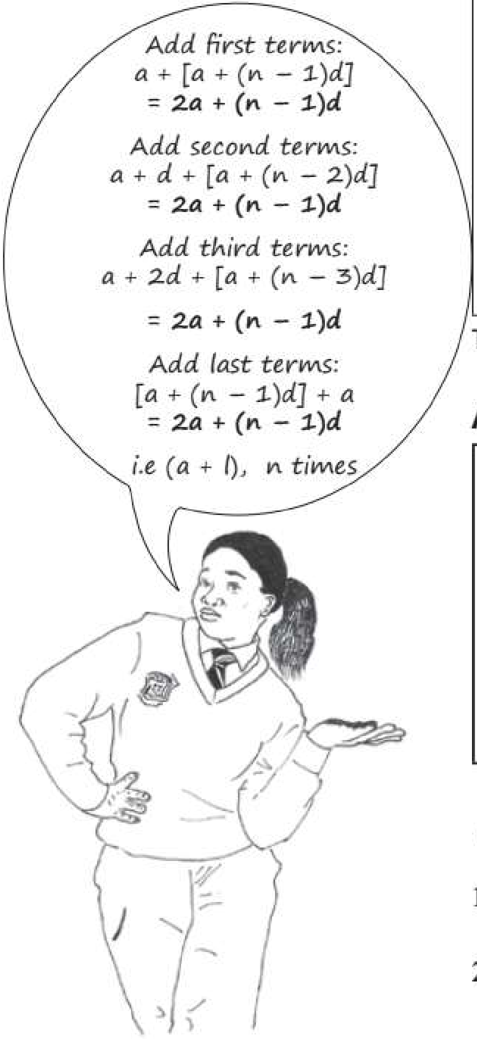
$S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \ldots + [a + d] + a \ldots$ equation 2

We can add equation 1 and equation 2.

So $2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d] + [2a + (n - 1)d]$

$2S_n = n[2a + (n - 1)d]$

$S_n = \frac{n}{2}[2a + (n - 1)d]$



This formula is provided on the **information sheet** in the final exam.

Alternative Proof

Or $S_n = a + [a + d] + [a + 2d] + \ldots + [l - d] + l \ldots$ equation 1

In reverse $S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \ldots + [a + d] + a$

$S_n = l + [l - d] + [l - 2d] + \ldots + [a + d] + a \ldots$ equation 2

Adding equation 1 and equation 2

$2S_n = [a + l] + [a + l] + \ldots + [a + l] \qquad n$ times

$2S_n = n[a + l]$

$\therefore S_n = \frac{n}{2}[a + l]$

e.g. 7

1. Determine the sum of the first 20 terms of the series:
   $3 + 7 + 11 + 15 + \ldots$
2. The sum of the series $5 + 3 + 1 + \ldots$ is -216, determine the number of terms in the series

Solutions

1. $a = 3, n = 20, d = 4$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2(3) + (19)4]$

$S_{20} = 10(6 + 76)$

$S_{20} = 820$

The sum of the first 20 terms is 820

2. $a = 5 \quad d = -2 \quad S_n = -216 \quad S_n = \frac{n}{2}[2a + (n-1)d] \quad n = ?$

Substitute into the formula:

$-216 = \frac{n}{2}[2(5) + (n-1)(-2)]$

$-216 = \frac{n}{2}[10 + -2n + 2]$

$-216 = \frac{n}{2}[12 - 2n]$

$-432 = 12n - 2n^2$

$-432 = -2n^2 + 12n$ .... Make equation = 0

$2n^2 - 12n - 432 = 0$ .... Divide through by 2 (common factor)

$n^2 - 6n - 216 = 0$ .... Factorise trinomial

$(n - 18)(n + 12) = 0$

$\therefore n - 18 = 0$ or $n + 12 = 0$

$n = 18$ or $n = -12$

$n > 0 \quad \therefore n = 18$

$\therefore$18 terms of the series add up to –216.

Activity 4

1. Determine the sum of the series: 19 + 22 + 25 + . . . + 121 (3)
2. The sum of the series 22 + 28 + 34 + . . . is 1870. Determine the number of terms. (2)
3. Given the arithmetic sequence -3; 1; 5; …,393
   - **a)** Determine a formula for the nth term of the sequence.
   - **b)** Write down the 4^{th}, 5^{th}, 6^{th} and 7^{th} terms of the sequence.
   - **c)** Write down the remainders when each of the first seven terms of the sequence is divided by 3.
   - **d)** Calculate the sum of the terms in the arithmetic sequence that are divisible by 3. (10)
4. The sum of n terms is given by $S_n = \frac{n}{2}(1 + n)$. Determine T_5. (3)
5. $3x + 1$; $2x$; $3x - 7$ are the first three terms of an arithmetic sequence. Calculate the value of x. (3)
6. The first and second terms of an arithmetic sequence are 10 and 6 respectively.
   - **a)** Calculate the 11^{th} term of the sequence.
   - **b)** The sum of the first n terms of this sequence is –560. Calculate n. (6)

[27]

Solutions

1. $a = 19$ and $d = 3$

$T_n = 3n + 16 = 121$

$3n = 105$

$n = 35$ ✓

$Sn = \frac{n}{2}(a + l)$

$S_{35} = \frac{35}{2}(19 + 121) = \frac{35}{2}(140) = 35 \times 70 = 2450$ ✓ ✓ (3)

2. $a = 22$ and $d = 6$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$\frac{n}{2}[2 \times 22 + (n - 1)6] = 1870$ ✓

$19n + 3n^2 = 1870$

$3n^2 + 19n - 1870 = 0$

$(3n + 85)(n - 22) = 0$ ✓

$\therefore n = 22$

n cannot be a negative because it is the number of terms (2)

3. **a)** $T_n = -3 + (n - 1)4$ ✓

$4n - 7 = T_n$ ✓

b) $T_4 = 5 + 4 = 9$; $T_5 = 9 + 4 = 13$; ✓ $T_6 = 13 + 4 = 17$ *and*
$T_7 = 17 + 4 = 21$ ✓

c) 0; 1; 2; 0; 1; 2; 0 ✓✓

d) $T_n = -3 + 12(n - 1)$ ✓

$393 = 12n - 15$

$12n = 393 + 15 = 408$ ✓

$n = 34$

$S_{34} = \frac{34}{2} \times (-3 + 393)$

$= 17 \times 390$ ✓✓

$= 6630$ (10)

4. $S_5 = \frac{5}{2}(1 + 5) = 15$ ✓

$S_4 = \frac{4}{2}(1 + 4) = 10$ ✓

$T_5 = 15 - 10 = 5$ ✓ (3)

5. $T_2 - T_1 = T_3 - T_2$

$2x - (3x + 1) = (3x - 7) - 2x$ ✓

$2x - 3x - 1 = 3x - 7 - 2x$

$-2x + 6 = 0$ ✓

$2x = 6$

$x = 3$ ✓ (3)

6. a) $T_n = a + (n-1)d$
$T_{11} = 10 + (11-1)(-4)$ ✓
$= -30$ ✓

b) $S_n = \frac{n}{2}[2a + (n-1)d]$
$-560 = \frac{n}{2}[2(10) + (n-1)(-4)]$ ✓
$-1120 = -4n^2 + 24n$
$4n^2 - 24n - 1120 = 0$
$n^2 - 6n - 280 = 0$ ✓
$(n-20)(n+14) = 0$ ✓
$n = 20$ or $n = -14$
$n = 20$ only ✓ because number of terms cannot be a negative number (6)

[27]

3.5.2 Geometric series

The formula is
$S_n = \frac{a(r^n - 1)}{r - 1}$ for $r > 1$ or $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r < 1$
where a is the first term
r is the common ratio
n is the number of terms
S_n is the sum of the terms

Proof:

The general term of a geometric series is $T_n = ar^n - 1$

So $S_n = T_1 + T_2 + T_3 + T_4 + \dots + T_n$
$S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$ multiply each term by r

$$\begin{array}{l} S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \\ \hline rS_n - S_n = -a + 0 + 0 + \dots + 0 + 0 + ar^n \end{array}$$
write down the series again with like terms under each other

$\therefore rS_n - S_n = ar^n - a$ subtract each bottom term from each top term
$S_n(r-1) = a(r^n - 1)$ S_n and a are common factors
So $S_n = \frac{a(r^n - 1)}{r - 1}$ Divide through by $(r-1)$
We can also use for $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r < 1$

e.g. 8

Evaluate: 25 + 50 + 100 + … to 6 terms

Solution

We need to check if this is an arithmetic series or a geometric series first.

You should see that there is a common ratio of 2 because $\frac{50}{2} = 2$ and $\frac{100}{50} = 2$

$r = 2$

$\therefore$ It is a geometric series and $a = 25$, $n = 6$, $r = 2$

$S_n = \frac{a(1 - r^n)}{1 - r}$

$S_6 = \frac{25(1 - 2^6)}{1 - 2}$ $2^6 = 64$

$S_6 = \frac{25(1 - 64)}{-1}$

$S_6 = \frac{25(-63)}{-1}$

$= 1\,575$

So the sum of the first 6 terms of the series is 1 575.

Activity 5

1. Determine 3 + 6 + 12 + 24 + . . . to 10 terms (2)
2. If 2 + 6 + 18 + . . . = 728, determine the value of n. (3)

[5]

Solutions

1. $a = 3$ *and* $r = \frac{6}{3} = \frac{12}{6} = 2$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{10} = \frac{3(2^{10} - 1)}{2 - 1} = 3(1024 - 1) = 3069$ ✓ ✓ (2)

2. $a = 2$ and $r = \frac{6}{2} = \frac{18}{6} = 3$

$S_n = \frac{2(3^n - 1)}{3 - 1} = 728$ ✓

$\frac{2(3^n - 1)}{2} = 728$

$3^n - 1 = 728$

$3^n = 729 = 3^6$ ✓

$\therefore n = 6$ ✓ (3)

[5]

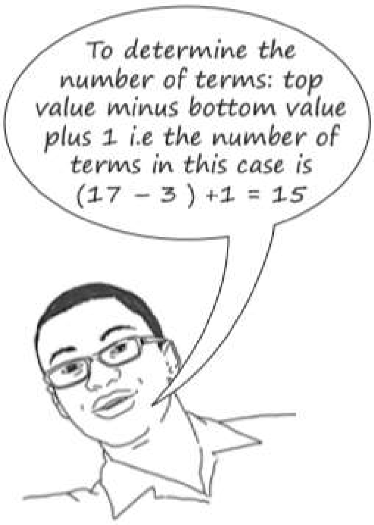
3.5.3 Sigma notation

Here is another useful way of **representing** a series.
The sum of a series can be written in **sigma notation**.
The symbol sigma is a Greek letter that stands for 'the sum of'.

$\sum$ is the symbol for 'the sum of'

$\sum_{k=1}^{n} T_k$ means 'the sum of the terms Tk from k = 1 to k = n.

In other words, $\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + T_4 + \ldots + T_n$



e.g. 9

$\sum_{k=3}^{17} p^k = p^3 + p^4 + p^5 + \ldots + p^{17}$

Activity 6

1. Evaluate: $\sum_{n=4}^{70} (2n - 4)$ (3)
2. What is the value of m for which $\sum_{k=1}^{m} 5(3)^{k-1} = 65$? (4)
3. Consider the sequence: $\frac{1}{2}; 4; \frac{1}{4}; 7; \frac{1}{8}; 10; \ldots$
   a) If the pattern continues in the same way, write down the next TWO terms in the sequence.
   b) Calculate the sum of the first 50 terms of the sequence. (5)

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hint
Look for two different sequences in the pattern and separate them

Solutions

1. The question asks you to find the sum of the terms from $n = 4$ to $n = 70$ if the n^{th} term is $2n - 4$.
   $a = T_1 = 2(4) - 4 = 4$ Find the first term a
   $T_2 = 2(5) - 4 = 6$
   $T_3 = 2(6) - 4 = 8$
   So the sequence is 4; 6; 8; … and this is an arithmetic series. ✓

   To check d, calculate $T_2 - T_1$
   $d = T_2 - T_1 = 6 - 4 = 2$
   $n = (70 - 4) + 1 = 67$ ✓ There are 67 terms

Now we can substitute these values into the formula to find the sum of 67 terms.

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_{67} = \frac{67}{2}[2(4) + (67 - 1)2]$

$S_{67} = 33.5\,[8 + 132] = 4690$

So $\sum_{n=4}^{70} (2n - 4) = 4690$ ✓ (3)

2. This is a geometric series because $5(3)^{k-1}$ has the form ar^{k-1}, $T_1 = 5(3)^{1-1} = 5$;
$T_2 = 5(3)^{2-1} = 15$; $T_3 = 5(3)^{3-1} = 45$
$a = 5$; $r = 3$; $n = m$ and $S_m = 65$

$S_n = \frac{a(r^n - 1)}{r - 1}$ ✓ … substitute

$65 = \frac{5(3^m - 1)}{3 - 1}$ ✓

$65 = \frac{5(3^m - 1)}{2}$ … multiply through by 2

$130 = 5.3^m - 5$ … add like terms

$135 = 5.3^m$ ✓ … divide through by 5

$27 = 3^m$ … write 27 as a power of 3

$3^3 = 3^m$ … bases are the same, so the powers are equal

$\therefore m = 3$ ✓ (4)

3. a) T_1, T_3 and T_5 form a sequence with a common ratio of $\frac{1}{2}$, so T_7 is $\frac{1}{16}$. ✓
T_2, T_4 and T_6 form a sequence with a common difference of 3, so T_8 is 13.

b) S_{50} = 25 terms of 1st sequence + 25 terms of 2nd sequence

$S_{50} = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \text{ to 25 terms}) + (4 + 7 + 10 + 13 + \ldots \text{ to 25 terms})$ ✓

$S_{50} = \dfrac{\frac{1}{2}\left[\left(\frac{1}{2}\right)^{25} - 1\right]}{\frac{1}{2} - 1} + \frac{25}{2}[2(4) + 24(3)]$ ✓

$S_{50} = 0{,}99999997 + 1\,000$ ✓

$S_{50} \approx 1\,001{,}00$ ✓ (5)

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3.5.4 Infinite geometric series

An infinite series is one in which there is no last term, i.e. the series goes on without ending.

e.g. 10

$6 + 3 + \frac{3}{2} + \frac{3}{4} + \ldots$

$S_\infty = \sum_{k=1}^{\infty} 2(3)^{k-1} = 2 + 6 + 18 + 54 + \ldots$ the sum from term1 to infinity of $2(3)^{k-1}$

$T_1 = 2(3)^0 = 2$
$T_2 = 2(3)^1 = 6$
$T_3 = 2(3)^2 = 18$
$T_4 = 2(3)^3 = 54$ …

The terms of this series are all positive numbers and the sum will get bigger and bigger without any end. This is called a **divergent** series.

e.g. 11

Look at this infinite series:

$S_\infty = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$S_2 = 1 + \frac{1}{2} = 1\frac{1}{2} = 1.5$

$S_3 = 1\frac{1}{2} + \frac{1}{4} = 1\frac{3}{4} = 1.75$

$S_4 = 1\frac{3}{4} + \frac{1}{8} = 1\frac{7}{8} = 1.675$

$S_5 = 1\frac{7}{8} + \frac{1}{16} = 1\frac{15}{16} = \ldots..$

This series will converge to 2. It is therefore called a convergent series and we can write the sum to infinity equals 2: $S_\infty = 2$

You can identify a convergent infinite series by looking at the value r

An infinite series is convergent if $-1 < r < 1,\ r \neq 0$

The formula for the sum of a convergent infinite series: $S_\infty = \frac{a}{1-r}$ where a is the first term, r is the common ratio
This formula is provided on the information sheet in the final exam.

e.g. 12

1. Look again at the example where $S_\infty = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$a = 1$ and $r = \frac{1}{2}$ $\quad 0 < r < 1$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{1}{1-\frac{1}{2}} = 1 \div \frac{1}{2}$

$S_\infty = 1 \times 2 = 2$

2. For which value(s) of x will $8x^2 + 4x^3 + 2x^4 + \ldots$ be convergent?

For convergent geometric series, $-1 < r < 1$

$r = T_2 \div T_1$

$= 4x^3 \div 8x^2$

$= \frac{x}{2}$

$\therefore -1 < \frac{x}{2} < 1$ multiply through by 2

$-2 < x < 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. x \neq 0$

Activity 7

1. Calculate S_∞ if $\sum_{p=1}^{\infty} 8(4)^{1-p}$ (3)
2. Given the series: $3(2x-3)^2 + 3(2x-3)^3 + 3(2x-3)^4 + \ldots$ for which values of x will the series converge? (4)
3. Find the value of m if: $\sum_{k=1}^{m} 3(2)^{k-1} = 93$ (4)
4. For which values of x will $\sum_{k=1}^{\infty} (4x-1)^k$ exists. (3)

[14]

Solutions

1. $T_1 = 8(4)^{1-1} = 8 = a$ ✓

To find r, find the common ratio using T_1 and T_2, T_2 and T_3.

$T_2 = 8(4)^{1-2} = 8(4)^{-1} = 8 \times \frac{1}{4} = 2$

$T_3 = 8(4)^{1-3} = 8(4)^{-2} = 8 \times \frac{1}{16} = \frac{1}{2}$

$T_2 \div T_1 = \frac{2}{8} = \frac{1}{4}$ and $T_3 \div T_2 = \frac{\frac{1}{2}}{2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

so $r = \frac{1}{4}$ and $a = 8$ ✓

$\therefore S_\infty = \frac{a}{1-r} = \frac{8}{1-\frac{1}{4}} = \frac{8}{\frac{3}{4}}$ ✓ When dividing by a fraction, you can multiply by the inverse

$= 8 \times \frac{4}{3} = \frac{32}{3}$

$\therefore S_\infty = \frac{32}{3}$ or $10\frac{2}{3}$ (3)

2. This is a geometric series with $r = 2x - 3$ ✓

To converge $-1 < r < 1$ ✓

$-1 < 2x - 3 < 1$ Add 3 to both sides

$2 < 2x < 4$ Divide by 2 on both sides

$1 < x < 2$ ✓ $x \neq \frac{3}{2}$ ✓ (4)

The series will converge for $1 < x < 2$

3. $a = 3$; $r = 2$; $S_m = 93$

$S_n = \frac{a(1-r^n)}{1-r}$ ✓

$93 = \frac{3(1-2^m)}{1-2}$ ✓

$93 = \frac{3(1-2^m)}{-1}$

$-93 = 3(1-2^m)$

$-31 = 1 - 2^m$

$2^m = 32$ ✓

$2^m = 2^5$

$\therefore m = 5$ ✓ (4)

4. $r = 4x - 1$ ✓

$-1 < r < 1$

$-1 < 4x - 1 < 1$; $x \neq \frac{1}{4}$ ✓

$0 < 4x < 2$

$0 < x < \frac{1}{2}$ ✓ (3)

[14]