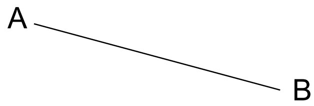
Analytical Geometry

Term	Explanation
distance	Length (in units) from one point to another. Found by using the distance formula using two points given.
gradient	How steep a line is. Found by using the gradient formula using two points given.
mid-point	The co-ordinate that represents the middle of a line segment. Found by using the mid-point formula using two points given.
parallel	Lines that have the same gradient are parallel to each other. Parallel = same gradient.
perpendicular	Two line segments meeting at a right angle.
x-intercept	The point at which a graph cuts the x –axis.
y-intercept	The point at which a graph cuts the y –axis.
point of intersection	The co-ordinate where two graphs intersect each other.
diagonal	The line segment joining opposite corners of a quadrilateral.
rectangle	A 4-sided shape (quadrilateral) where both pairs of opposite sides are equal in length and all 4 angles are 90^0.
square	A 4-sided shape (quadrilateral) where all 4 sides are equal in length and all 4 angles are 90^0.
kite	A 4-sided shape (quadrilateral) where the adjacent sides (those next to each other) are equal in length. The diagonals are perpendicular to each other.
rhombus	A 4-sided shape (quadrilateral) is a parallelogram with 4 equal sides.
parallelogram	A 4-sided shape (quadrilateral) that has 2 pairs of parallel sides.
equilateral triangle	A triangle with 3 equal sides and 3 equal angles.
isosceles triangle	A triangle with 2 equal sides and 2 equal angles.
collinear	Points that lie on the same line.
origin	The point where the x and y axis meet on a Cartesian plane.

line segment	All points between two given points. 
perimeter	The distance around the outside of a shape (the length of the outline of the shape)
angle of inclination	The angle between a line and the horizontal line (most often the x –axis). It can be any measurement from 0^0 to 180^0. It is always measured from the horizontal line in an anti-clockwise direction. If the line has a positive gradient, the angle of inclination will be less than 90^0. If the line has a negative gradient, the angle of inclination will be between 90^0 and 180^0.
circle	A curve where all points are the same distance from a given fixed point (the centre).
circumference	The distance around the circle (the perimeter of the circle).
equidistant	Exactly the same distance.
radius	The distance from the centre point of a circle to the circumference.
concentric circles	Circles of different sizes that have a common centre point. (A smaller one would lie inside a larger one).
tangent	A line which touches a circle at one point only.
secant	A line which intersects the circle at 2 points.

Revision of Grade 10 and 11 work

You should already know the following from Grade 10 and 11:

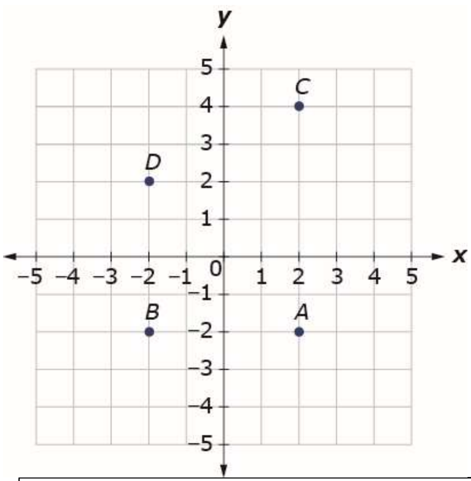
- Distance between two points
- Midpoint of a line segment
- Gradient of a line segment
- Equation of a straight line
- Angle of inclination

The formulae for all of the above are supplied on the formula sheet.

Basic example to demonstrate how to work with all of the above.

Plot the following points on a Cartesian Plane.

$A(2\,;\,-2)$ $\quad B\,(-2\,;\,-2)$ $\quad C(2\,;4)$ $\quad D(-2\,;2)$



Find:

a) The length of AD, correct to 2 decimal places

b) The gradient of AB and BD

c) The midpoint of BC

d) The equation of line BC

e) The angle of inclination of BC and AD.

Solutions:	Notes
$a)\quad A\,(2\,;-2)\quad D\,(-2\,;\,2)$ $x_1;y_1\qquad x_2;y_2$ $AD=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $AD=\sqrt{(-2-2)^2+(2-(-2))^2}$ $AD=\sqrt{(-4)^2+(2+2)^2}$ $AD=\sqrt{(-4)^2+(4)^2}$ $AD=\sqrt{16+16}$ $AD=\sqrt{32}=5{,}66$ units	Ensure you label the points accordingly in order to avoid careless errors. Check that the answer 'looks' reasonable. Does the distance look like it is about 5 or 6 units long?
b) $mAB=0$ $mBD=undefined$	You could have found these answers by using the formula, $m=\frac{y_2-y_1}{x_2-x_1}$ and should have come to the same answers. However, note that you should know that a horizontal line has a gradient of zero and a vertical line has an undefined gradient.
c) $B(-2;-2)$ and $C(2;4)$ $x_1\,;\,y_1\qquad x_2\,;\,y_2$ $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$ $=\left(\frac{-2+2}{2};\frac{-2+4}{2}\right)$ $=\left(\frac{0}{2};\frac{2}{2}\right)=(0;1)$	Always ensure you label the points accordingly in order to avoid careless errors. Check that the answer 'looks' reasonable. Plot the point and look if it is halfway between B and C.

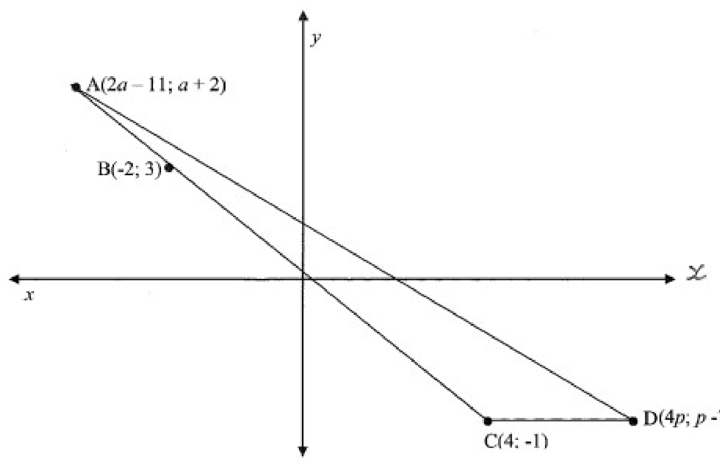
d) $y - y_1 = m(x - x_1)$ $B(-2;-2)$ and $C(2;4)$ $x_1\,;\,y_1 \qquad x_2\,;\,y_2$ $m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{4-(-2)}{2-(-2)}$ $m = \frac{4+2}{2+2}$ $m = \frac{6}{4}$ $m = \frac{3}{2}$ $y - y_1 = m(x - x_1)$ $m = \frac{3}{2} \quad C(2;4)$ $y - 4 = \frac{3}{2}(x-2)$ $y - 4 = \frac{3}{2}x - 3$ $y = \frac{3}{2}x - 3 + 4$ $y = \frac{3}{2}x + 1$	In order to use the formula, $y - y_1 = m(x - x_1)$, you need a point and the gradient. This is an important skill for Calculus as well. We currently have two points. These will be used to find the gradient then any one of the points can be used to find the equation of the line. Keep checking your answer. The gradient is positive – does that look correct? Is the line sloping upwards? Does a y –intercept of 1 look correct? Use the diagram to check.
e) BC: $\tan\theta = m$ $mBC = \frac{3}{2}$ (proved in (d)) $\therefore \tan\theta = \frac{3}{2}$ $\therefore \theta = 56{,}3^o$ AD: $\tan\theta = m$ $A\,(2\,;-2) \quad D\,(-2\,;\,2)$ $x_1; y_1 \qquad x_2; y_2$	The angle of inclination is always linked directly to the gradient. The formula for inclination is given. When the gradient is negative, you need to expect an obtuse angle as the angle of inclination.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{2-(-2)}{-2-2}$ $m = \frac{4}{-4} = -1$ $\tan\theta = -1$ RA: 45^o $\therefore 180^o - 45^o = 135^o$ $\therefore \theta = 135^o$	Note this on the Cartesian plane by joining A to D and marking the angle of inclination by starting on the x –axis and rotating in an anti-clockwise direction to meet up with the line.

It is important that you find the above 5 skills relatively easy. These are the basics to almost all Analytical geometry questions. However, as there is often more to questions in this section, below are a further 2 fully worked examples from Grade 11 past papers in order to show how these skills are used in combination with other knowledge and skills.

Example 1

The points $A(2a-11\,;a+2), C(4\,;\,-1)$ and $D(4p\,;p-7)$ are the vertices of ΔACD with $B(-2\,;3)$ on AC.



(EC Nov 2015)

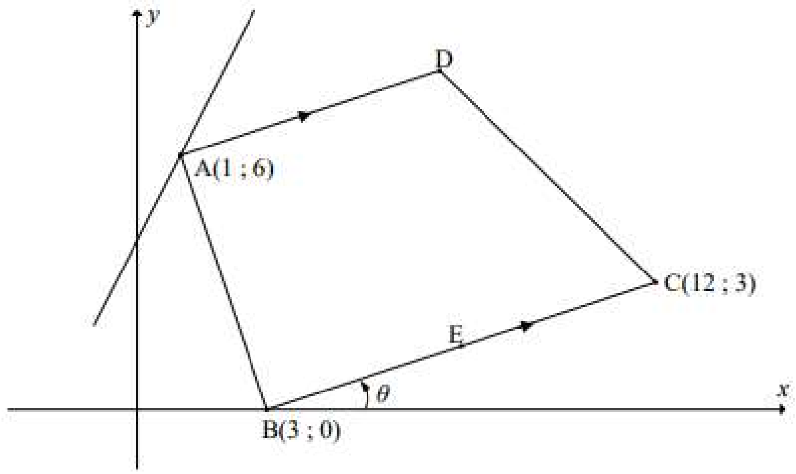
Question	Notes:
a) If points A, B and C are collinear, find the value of a.	Collinear means – they are all in a line. If they are all in a line, then they must share the same gradient. $\therefore mAB = mBC = mAC$

b) Determine the equation of line AC.	Whenever you read 'find the equation of a line', you need to remember that all you require is the gradient and a point. The gradient was found in the previous question and you have 2 points. Either can be used.
c) Hence, determine the co-ordinates of midpoint M of AB.	This should be straightforward
d) Determine the value of p, if CD is parallel to the x −axis.	If a line is parallel to the x −axis, then the y −co-ordinates must be equal. Equate the y −co-ordinates and solve.
Solutions	
a) $mBC = \frac{y_2-y_1}{x_2-x_1}$ $(-2\,;3)$ $(4\,;-1)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $mBC = \frac{-1-3}{4-(-2)}$ $mBC = \frac{-4}{6}$ $mBC = -\frac{2}{3}$ $mAB = mBC$ $A(2a-11\,;a+2)$ $B(-2\,;3)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $mAB = \frac{3-(a+2)}{-2-(2a-11)}$	b) $m = -\frac{2}{3}$ $C(4\,;-1)$ $y - y_1 = m(x - x_1)$ $y-(-1) = -\frac{2}{3}(x-4)$ $y+1 = -\frac{2}{3}x + \frac{8}{3}$ $y = -\frac{2}{3}x + \frac{8}{3} - 1$ $y = -\frac{2}{3}x + \frac{5}{3}$

$-\frac{2}{3} = \frac{3-a-2}{-2-2a+11}$ $-\frac{2}{3} = \frac{1-a}{-2a+9}$ $-2(-2a+9) = 3(1-a)$ $4a-18 = 3-3a$ $4a+3a = 3+18$ $7a = 21$ $a = 3$ Remember to check if your answer looks reasonable – this makes $A(2(3)-11\,;3+2)$ which equals $A(-5\,;5)$. You should check if this looks feasible on the diagram	c) $A(-5;5)$ $\quad B(-2;3)$ $x_1\,;\,y_1$ $\quad x_2\,;\,y_2$ $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$ $=\left(\frac{-5-2}{2};\frac{5+3}{2}\right)$ $=\left(\frac{-7}{2};\frac{8}{2}\right)$ $=\left(\frac{-7}{2};4\right)$ Check again that this looks correct on the diagram.
d) $-1 = p-7$ $6 = p$	

Example 2

$A(1\,;6), B(3\,;0), C(12\,;3)$ and D are the vertices of a trapezium with $AD//BC$. E is the midpoint of BC. The angle of inclination of the straight line BC is θ, as shown in the diagram.



(DBE Exemplar 2013)

Question	Notes
a) Calculate the co-ordinates of E.	This should be straightforward
b) Determine the gradient of line BC.	This should be straightforward. Remember that as the line is sloping upwards, you should expect the gradient to be positive.
c) Calculate the magnitude of θ.	This should be straightforward. Remember that as the gradient is positive, you should be expecting an acute angle.
d) Prove that AD is perpendicular to AB.	What aspect of this section would you link to 'perpendicular'? (gradient). You therefore need to find the gradient of both lines. What is required to make lines perpendicular? (the product of their gradients should be -1).
e) A straight line passing though vertex A does not pass through any of the sides of the trapezium. This line makes an angle of 45^o with side AD of the trapezium. Determine the equation of this straight line.	This is a level 3/4 question and some will find it very difficult. If you are asked to find the equation of a line, you should remind yourself that you need a point and the gradient to do that. You already have a point (A is given), therefore the focus should be on finding the gradient. Remember that AD//BC.

	(And that information is never given in a question if it will not be useful). Therefore, if a horizontal line is drawn through A, the angle of inclination from that line to AD will be equal to the angle of inclination of BC (found in (c)). The inclination of line AD can now be found by adding 45^o to the answer from (c) - $18{,}43^o$. Inclination can be used to find the gradient. Once a gradient and a point are available, the formula can be used to find the equation.
Solutions:	
a) $B(3;0)$ $C(12;3)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$ $=\left(\frac{3+12}{2};\frac{0+3}{2}\right)$ $=\left(\frac{15}{2};\frac{3}{2}\right)$ Check again that this looks correct on the diagram.	b) $B(3\,;0)$ $C(12\,;3)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $mBC=\frac{3-0}{12-3}$ $mBC=\frac{3}{9}$ $mBC=\frac{1}{3}$
c) $tan\,\theta=m$ $tan\,\theta=\frac{1}{3}$ $\therefore\ \theta=18{,}43^o$	d) $mAD=\frac{1}{3}$ (AD//BC) $A(1\,;6)$ $B(3\,;0)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $mAB=\frac{0-6}{3-1}$ $mAB=\frac{-6}{2}$ $mAB=-3$ $mAB\times mAD=\frac{1}{3}\times -3=\ -1$ $\therefore AB\perp AD$

e)

$mAD = \frac{1}{3}$ (proved above)

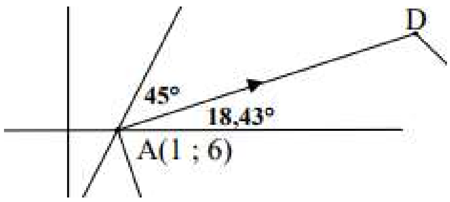
$\therefore < of\ inclination = 18{,}43^{o}$

$\therefore$ inclination of new line = $18{,}43^{o} + 45^{o} = 63{,}43^{o}$

$$tan\ 63{,}43^{o} = m$$

$$tan\ 63{,}43^{o} = 2$$

This diagram may explain further:



$m = 2$ $\qquad A(1\,;6)$

$$y - y_1 = m(x - x_1)$$

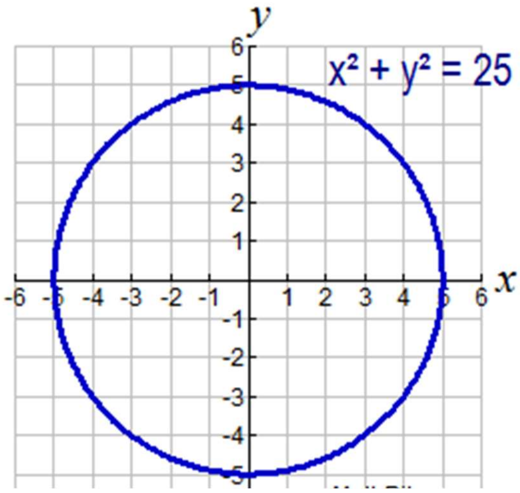
$$y - 6 = 2(x - 1)$$

$$y = 2x - 2 + 6$$

$$y = 2x + 4$$

Circles centred at the origin

This circle has its centre at the origin. The radius is 5 ($\sqrt{25}$)



It is NOT a function (because for every x –value there are 2 possible y –values)

$$x^2 + y^2 = r^2$$

The $'r'$ represents the radius. This is the general form of a circle that has it's centre at the origin.

Application of circle graphs and equations

Example 1 a) State the centre and radius of the circle $x^2 + y^2 = 30$ b) Find the equation of a circle centred at the origin that passes through the point $(2\,;\,-6)$	
Solution	Notes
a) Centre $(0\,;0)$ Radius: $\sqrt{30}$	You need to recognize that the centre will be at the origin when the equation is written in this format. The radius will always be the square root of the constant.
b) $x^2 + y^2 = r^2$ $(2\,;\,-6)$ $2^2 + (-6)^2 = r^2$ $4 + 36 = r^2$ $40 = r^2$ $\therefore\ x^2 + y^2 = 40$	It is good practice to draw a rough sketch. Substitute the x –value and y –value of the coordinate to find the value of r^2.

Example 2 Determine the value of $'a'$ if $(2;a)$ lies on the circle $x^2+y^2=20$	
Solution	Notes
$x^2+y^2=20$ $2^2+a^2=20$ $4+a^2=20$ $a^2=16$ $a=4 \quad or\ a=-4$	If the point given lies on the circle then by substituting the coordinates of the point, you can solve for a.

Circles not centred at the origin

Understanding the transformations of functions and relations in general is important. Here is a refresher of functions previously covered:

Equation:	Asymptotes, turning points, horizontal and vertical shifts are all important. The one most useful for the circle though is the horizontal shift. So those shifts have been made bold.
$y=(x\mathbf{-2})^2+5$	This is a parabola (quadratic function). **There has been a horizontal shift, 2 units to the right** and a vertical shift 5 units up. The coordinate of the turning point is $(2;5)$
$y=\frac{2}{x\mathbf{+3}}-4$	This is a hyperbola. **There has been a horizontal shift, 3 units to the left** and a vertical shift 4 units down. The asymptotes are: $x=-3$ & $y=-4)$
$y=2.3^{x\mathbf{+1}}-2$	This is an exponential function. **There has been a horizontal shift, 1 unit to the left** and a vertical shift of 2 down. The asymptote is $y=-2$

Even though all of this is not part of Analytical geometry it is important to discuss for 2 reasons:

(1) The functions from Grade 11 are not covered in detail again this year but are assessed.

(2) It is a reminder that there is often a link between two (or more) topics and that skills you learn in one topic are often required in another.

$$(x-a)^2+(y-b)^2=r^2$$

This is the standard form of a circle.

$'r'$ still represents the radius

The $'a'$ and $'b'$ represent shifts in the graph, which means that the centre will have shifted form the origin. Therefore $(a\,;b)$ also represents the centre.

In the same way the horizontal shift is found by using the ‘opposite’ sign (‘plus’ means shift left and ‘minus’ means shift right), so finding the centre works the same.

This equation is no different from the one already learned.

$$(x-0)^2+(y-0)^2=r^2$$

is the same as:

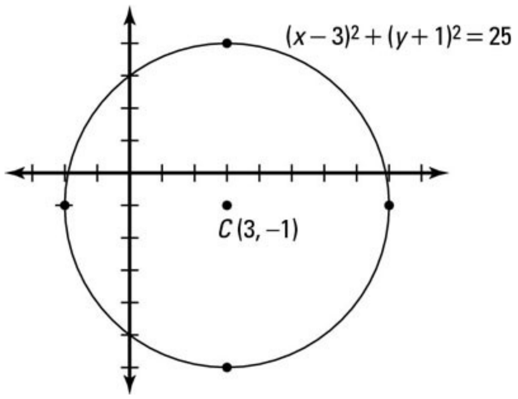
$$x^2+y^2=r^2$$

Example:

$$(x-3)^2+(y+1)^2=25$$

- centre is: $(3\,;\,-1)$ - remember the change in signs
- radius is: 5 - remember to square root the constant

Sketch



You can count the units from the centre to find another 4 points on the circle 9north, south, east and west from the centre).

Use algebra to remove brackets and collect like terms for this equation:

$$(x-3)^2+(y+1)^2=25$$

$$x^2-6x+9+y^2+2y+1=25$$

$$x^2-6x+y^2+2y+10=25$$

If you subtract 10 on both sides to make one constant:

$$x^2-6x+y^2+2y=15$$

Or subtract 25 from both sides

$$x^2-6x+y^2+2y-15=0$$

Both are acceptable to note the following:

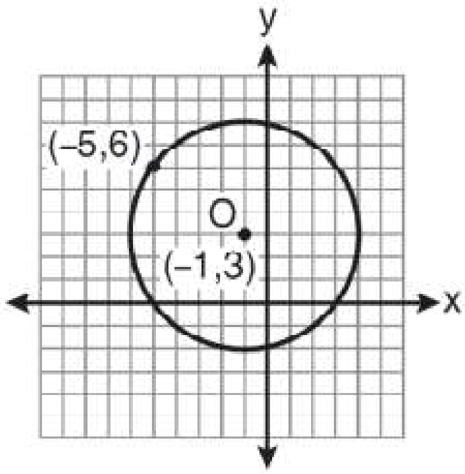
You could be given the equation of the circle in any format that you have seen so far. If it is in either of the last formats, there is an algebraic manipulation that you need to be confident in – completing the square.

By completing the square, it can be changed into the format where the centre and the radius are easier to see.

Below is an example:

Example: $x^2 - 6x + y^2 + 2y - 15 = 0$

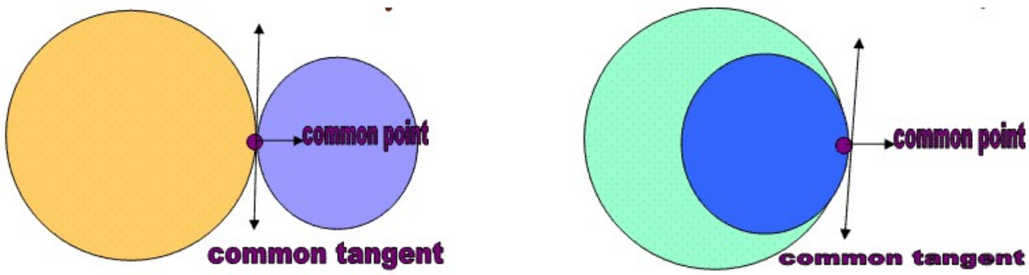
Notes	Solution
Ensure the constant is on the RHS	$x^2 - 6x + y^2 + 2y = 15$
Re-write the expression, ready to form a perfect square trinomial with both the x and y –values.	$x^2 - 6x\_\_\_\_ + y^2 + 2y\_\_\_\_ = 15$
Take the coefficient of x (in this case -6), and y (in this case 2) halve them and square them $\left(\frac{1}{2} \times -6\right)^2 = 9$ $\left(\frac{1}{2} \times 2\right)^2 = 1$ Add these to form the perfect square trinomials. In order to keep the balance, remember to add these numbers to BOTH sides.	$x^2 - 6x + \mathbf{9} + y^2 + 2y + \mathbf{1} = 15 + \mathbf{9} + \mathbf{1}$
Factorise the perfect square trinomials that have been formed and simplify the right-hand side. The format given has now been changed into the standard format where the centre and radius are easy to see.	$(x-3)^2 + (y+1)^2 = 25$ Centre: $(3; -1)$ Radius: 5

Example 1 Determine the coordinates of the centre and the length of the radius for the following circle: $$x^2+2x+y^2-10y+16=0$$	
$x^2+2x+y^2-10y=-16$ $x^2+2x+1+y^2-10y+25=-16+1+25$ $(x+1)^2+(y-5)^2=10$ The centre is $(-1\,;5)$ and the radius is $\sqrt{10}$	The steps above need to be followed to get the equation into standard form.
Example 2 Find the equation of the circle shown in the diagram:  	
$(x+1)^2+(y-3)^2=r^2$ $(-5\,;6)$ $(-5+1)^2+(6-3)^2=r^2$ $(-4)^2+(3)^2=r^2$ $16+9=r^2$ $25=r^2$ $(x+1)^2+(y-3)^2=25$	As the centre is already given, write the correct format and fill the centre in. Substitute the other point for x and y to find r^2. Substitute back into the first equation that had the centre.

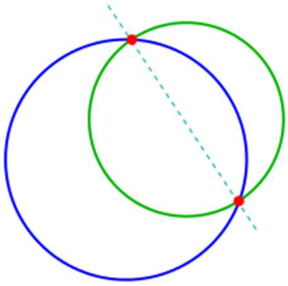
Example 3 a) Determine the equation of the circle passing through the points $(2\,;\,-5)$ and $(4\,;\,-1)$ which form the diameter of the circle b) Find the diameter in simplest surd form.	
a) $(2;-5)$ $(4;-1)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$ $=\left(\frac{2+4}{2};\frac{-5-1}{2}\right)$ $=\left(\frac{6}{2};\frac{-6}{2}\right)$ $=(3;\,-3)$	The centre must be the midpoint of the diameter.
$(x-3)^2+(y+3)^2=r^2$ $(2\,;-5)$ $(2-3)^2+(-5+3)^2=r^2$ $(-1)^2+(-2)^2=r^2$ $1+4=r^2$ $5=r^2$ $(x-3)^2+(y+3)^2=5$	Once the centre has been found, the rest is similar to the example above. Note that essentially you are using the distance formula in this step.
b) Radius = $\sqrt{5}$ ∴ Diameter = $2\sqrt{5}$	As the radius is known, the diameter must be twice as long.

The following relationship with circles needs to be considered.

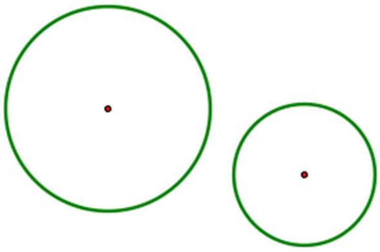
Circles that intersect at one point only can either touch externally or internally.



Circles that intersect at two points:

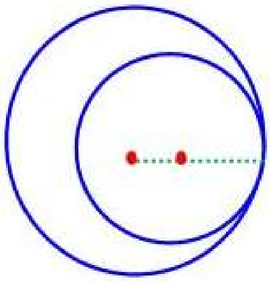
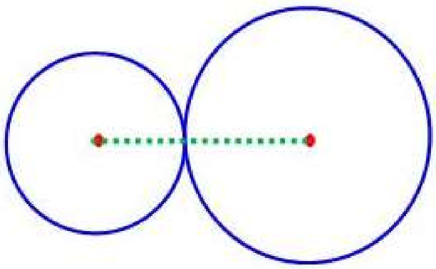
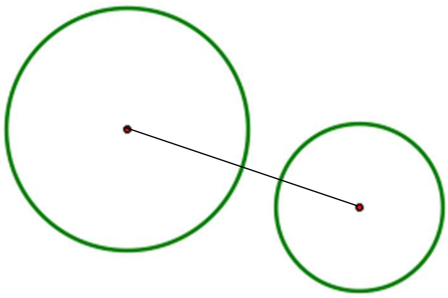
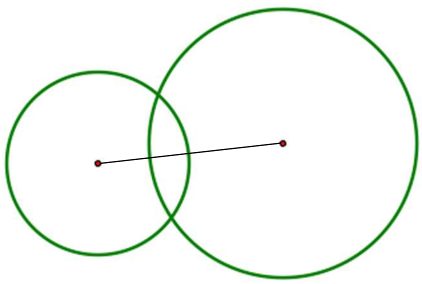


Circles that do not intersect:



The relationship between the radii and the distance between the two centres of each of the circles in the above diagrams is important to understand.

Use R_1 for the radius of the larger circle and r_2 for the radius of the smaller circle.

	Distance from centre to centre $= R_1 - r_2$
	Distance from centre to centre $= R_1 + r_2$
	Distance from centre to centre $> R_1 + r_2$ (the distance from centre to centre is larger than the two radii added)
	Distance from centre to centre $< R_1 + r_2$ (the distance from centre to centre is smaller than the two radii added)

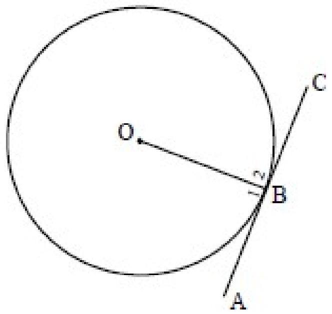
These aspects of circles are often important in understanding how to answer a question.

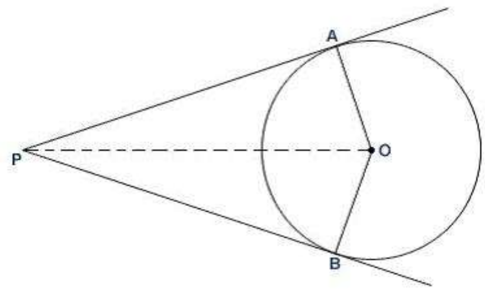
Equations of tangents to circles

This topic combines knowledge of Grade 11 Euclidean geometry as well as that of functions and finding equations of straight lines.
Analytical Geometry always brings in knowledge of these other two topics. Topics rarely stand alone in mathematics and many skills are required from previous knowledge, no matter what topic is currently being covered.

A tangent is a straight line that touches a curve at only one point.

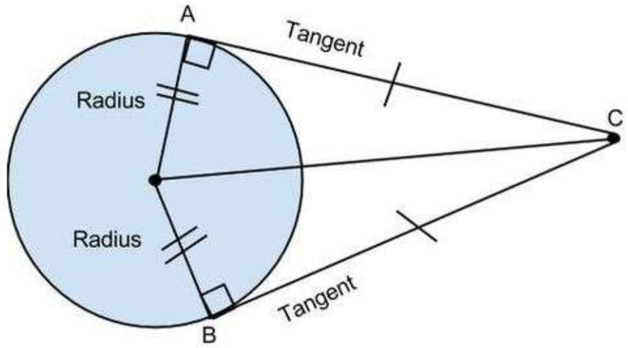
Below are two theorems from Grade 11 relating to tangents:
(You learned the tan-chord theorem too but that isn't really used in Analytical geometry)





Note the following:
A tangent is always perpendicular to a radius ($B_1 = B_2 = 90^o$ in the above sketch).
Fill the right angles in.
Tangents from the same point are equal in length. ($AP = BP$ in the above sketch).
Mark these equal.

Refer back to the second diagram.
Below, the right angles have been filled in. This is a key point and is used often in this topic.
The theorem of Pythagoras is often required to find a length within the right-angled triangle formed.



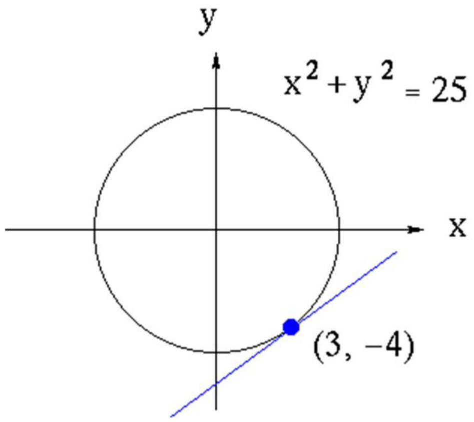
What is happening at 'C' on the diagram? (the point of intersection of the two tangents).
Remember how important it is to know other aspects of mathematics in order to excel within a particular topic.

What is covered in this aspect of Analytical Geometry is finding the equation of a tangent to a circle.

What is always needed to find the equation of a straight line? (a point and the gradient!)

Example 1

Find the equation of the following tangent:



Solution

$(3\,;-4)$ $\quad$ $(0\,;0)$

$x_1\,;\,y_1$ $\quad$ $x_2\,;\,y_2$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$m=\frac{0-(-4)}{0-3}$$

$$m=\frac{4}{-3}$$

$$m=-\frac{4}{3}$$

$\therefore \perp$ gradient = $\frac{3}{4}$

$m=\frac{3}{4}$ $\quad$ $(3\,;-4)$

$$y-y_1=m(x-x_1)$$

$$y-(-4)=\frac{3}{4}(x-3)$$

$$y+4=\frac{3}{4}x-\frac{9}{4}$$

$$y=\frac{3}{4}x-\frac{9}{4}-4$$

$$y=\frac{3}{4}x-\frac{25}{4}$$

Notes

Whenever you are asked to find the equation of a line you need a point and the gradient. In this case a point is given so your focus should be on finding the gradient. To find gradient we need two points.

Notice that with the form of the circle equation we know that the origin is the centre. This can be used to find the gradient of the radius.

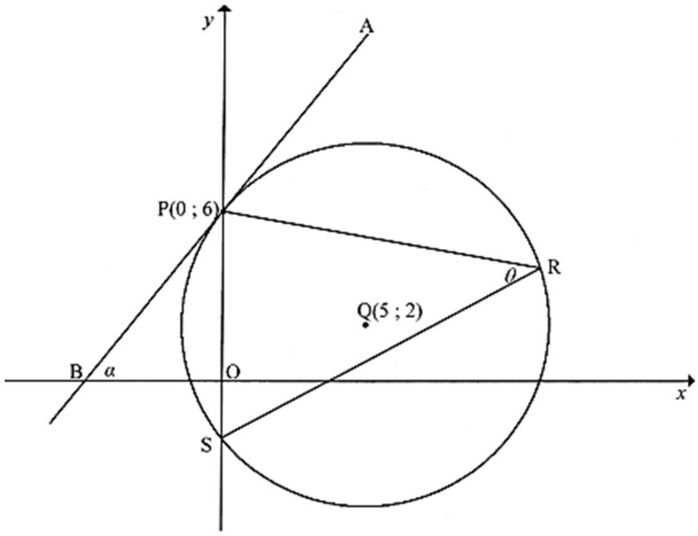
Your knowledge of the theorem rad⊥tan and perpendicular gradients will then give you the gradient required to use the formula.

Check your answer again –

does $-\frac{25}{4}$ $(-6{,}25)$ look like a reasonable y –intercept?

Example 2

In the diagram below, $Q(5\,;2)$ is the centre of a circle that intersects the $y-axis$ at $P(0\,;6)$ and S. The tangent APB at P intersects the x –axis at B and makes the angle α with the positive x –axis. R is a point on the circle and $P\hat{R}S = \theta$



a) Determine the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$.

b) Calculate the co-ordinate of S.

c) Determine the equation of the tangent APB in the form $y = mx + c$.

d) Calculate the size of α.

e) Calculate, with reasons, the size of θ.

f) Calculate the area of ΔPQS.

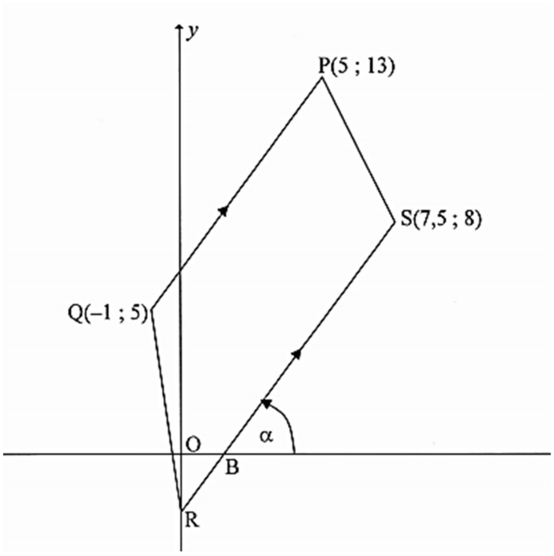
Solutions	Notes
a) $(x-a)^2 + (y-b)^2 = r^2$ $Q(5\,;2)$ $(x-5)^2 + (y-2)^2 = r^2$ $P(0\,;6)$ $(0-5)^2 + (6-2)^2 = r^2$ $(-5)^2 + (4)^2 = r^2$ $25 + 16 = r^2$ $41 = r^2$ $\therefore (x-5)^2 + (y-2)^2 = 41$	When the centre and a point is available, substitute the centre for a and b then substitute the other known point to find r^2.

b) $(x-5)^2+(y-2)^2=41$ $(0-5)^2+(y-2)^2=41$ $25+y^2-4y+4=41$ $y^2-4y-12=0$ $(y-6)(y+2)=0$ $y=6 \quad or \quad y=-2$ $\therefore S(0\,;\,-2)$	What 'happens' at S? S is a y –intercept (make $x=0$) Note that there are 2 y –intercepts but as one is given, and it should be easy to see that the answer being looked for is negative.
c) $P(5\,;2)$ $\quad Q(0\,;6)$ $x_1\,;\,y_1$ $\quad x_2\,;\,y_2$ $mPQ=\dfrac{y_2-y_1}{x_2-x_1}$ $mPQ=\dfrac{6-2}{0-5}$ $mPQ=-\dfrac{4}{5}$ ∴ ⊥ gradient = $\frac{5}{4}$ $m=\frac{5}{4}$ $\quad (0\,;6)$ $y-y_1=m(x-x_1)$ $y-6=\dfrac{5}{4}(x-0)$ $y=\dfrac{5}{4}x+6$	What is needed to find the equation of a tangent? (gradient and a point). Gradient needs to be found as in the example above.
d) $tan\ \alpha=m$ $tan\ \alpha=\dfrac{5}{4}$ $\therefore\ \alpha=51{,}34^o$	What does α represent? (angle of inclination). How do we find that? ($tan\ \alpha=m$)
e) $B\hat{P}S=\theta$ (tan-chord) $\alpha=51{,}34^o$ $\therefore B\hat{P}S=38{,}66^o$ (<'s of Δ)	The tan-chord theorem is useful here. $B\hat{P}S=\theta$ Note that $B\hat{P}S$ is in a right-angled triangle and that the size of α is already known.
f) PS = 8 units and ⊥ht = 5 units $\therefore Area\ \Delta\ PQS=\dfrac{1}{2}(8)(5)$ $=20units^2$	The length of PS is known. Therefore, the perpendicular height from PS to Q is required. As this is from the y –axis where $x=0$, the height to the centre is simple to count.

Past Paper examples

Example

In the diagram below, points $P(5\,;13), Q(-1\,;5)$ and $S(7{,}5\,;8)$ are given. SR//PQ where R is the y −intercept of SR. The x −intercept of SR is B. QR is joined.



Determine:

a) The gradient of PQ.
b) Calculate the length of PQ.
c) Determine the equation of the line RS in the form $ax + by + c = 0$.
d) Determine the x −co-ordinate of B.
e) Calculate the size of $O\hat{R}B$.
f) Prove that QBSP is a parallelogram.

DBE Feb/Mar 2015

Solution	Notes
a) $P(5\,;13)$ $\quad Q(-1\,;5)$ $\quad x_1\,;\,y_1 \quad x_2\,;\,y_2$ $mPQ = \frac{y_2 - y_1}{x_2 - x_1}$ $mPQ = \frac{5 - 13}{-1 - 5}$ $mPQ = \frac{-8}{-6}$ $mPQ = \frac{4}{3}$	(a) and (b) should be two straightforward questions to find gradient and distance.

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b) $P(5\,;13)$ $Q(-1\,;5)$

$x_1\,;\,y_1$ $x_2\,;\,y_2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$PQ = \sqrt{(-1-5)^2 + (5-13)^2}$

$PQ = \sqrt{(-6)^2 + (-8)^2}$

$PQ = \sqrt{36 + 64}$

$PQ = \sqrt{100}$

$PQ = 10$

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c) $mPQ = \frac{4}{3}$

$\therefore mRS = \frac{4}{3}$

$m = \frac{4}{3}$ $(7{,}5\,;8)$

$y - y_1 = m(x - x_1)$

$y - 8 = \frac{4}{3}(x - 7{,}5)$

$y - 8 = \frac{4}{3}x - 10$

$y = \frac{4}{3}x - 2$

Remember the form the equation needs to be in. To remove fractions, find LCD and multiply each term.

$3y = 4x - 6$

$-4x + 3y + 6 = 0$

$4x - 3y - 6 = 0$

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Remember that in order to find the equation of a straight line you need gradient and a point.

Always take note of ALL information given – as the parallel lines are important.

Although there is not enough information on RS to find gradient, there is on PQ and parallel lines have equal gradients.

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d) $4x - 3y - 6 = 0$

$y = 0$

$4x - 3(0) - 6 = 0$

$4x = 6$

$x = \frac{6}{4} = \frac{3}{2}$

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Using the equation from (c), make $y = 0$ to find x –intercept.

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e) $tan\ \alpha = m$ $tan\ \alpha = \frac{4}{3}$ $\therefore \alpha = 53{,}13^o$ $\therefore O\hat{B}R = 53{,}13^o$ (vert opp <'s) $\therefore\ O\hat{R}B = 36{,}87^o$ (<'s of Δ)	When asked to find the size of an angle that is not an angle of inclination: you will need to find an angle of inclination that is useful then use some Grade 8 geometry from there. In this case, first find α then work in ΔORB which is right-angled.
f) Option 1: $B\left(\frac{3}{2}\,;0\right)$ $S(7{,}5\,;8)$ $x_1\,;\,y_1$ $x_2\,;\,y_2$ $BS = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $BS = \sqrt{\left(7{,}5-\frac{3}{2}\right)^2+(8-0)^2}$ $BS = \sqrt{(6)^2+(8)^2}$ $BS = \sqrt{36+64}$ $BS = \sqrt{100}$ $BS = 10$ $\therefore BS = PQ$ And $BS//PQ$ (given) $\therefore QBSP$ is a parm (one pair opp sides = and //)	Remember that there are 5 ways to prove that a quadrilateral is a parallelogram. Make sure you know them well. As you list all 5 think which are impossible (opposite angles equal – this would be a large amount of work and quite difficult) and which seem more likely. There are quite a few that could be found with one or two calculations. In this case we could use: One pair of oppos sides equal and parallel (would need to find distance of BS) Both pairs of oppos sides parallel (one pair is already given – would have to find gradients of other pair, QB and PS) Diagonals bisect (would need to calculate 2 midpoints) Both pairs of oppos sides equal (one side has already been calculated previously – would have to find length of other 3 sides)