CAPS

Mind the Gap!

Mathematics
Study Guide

Grade 12

basic education

Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

4 Unit

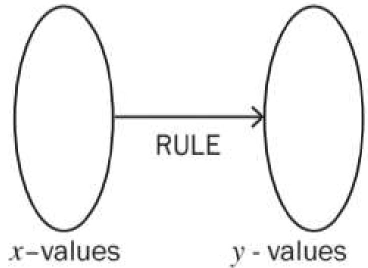
Functions

4.1 What is a function?

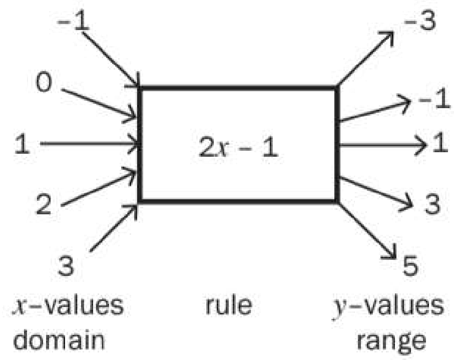
If you are given a set of x-values, you can work out the set of y-values or answers that came from *using a given rule* on each x-value.

So there is a **relationship** between the x-values and the y-values that is described by the rule.

The x-values are the input values and the y-values are the output values. In this flow diagram, the rule is $y = 2x - 1$



So for every x-value, we multiply it by 2 and subtract 1 to find the corresponding y-value.

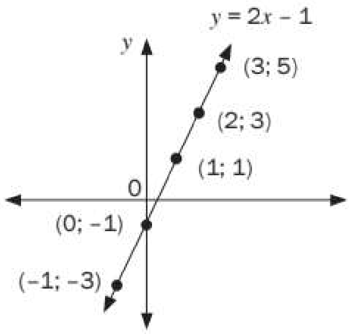


The input values or x-values are the elements of the **domain** of this set and the output values or y-values are the elements of the **range** of this set.

We can plot these values on the Cartesian plane.

If we extend the domain so that $x \in \mathbb{R}$, we get the graph for $y = 2x - 1$.

Look at the graph. For every x-value on this graph, there is only one y-value. If a rule or a formula produces only one y-value for each x-value, then we have a **function**.

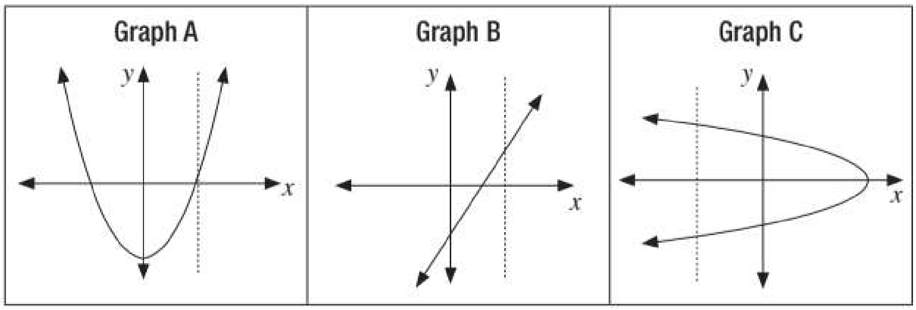


A **function** is a relationship between x and y, where for every x-value there is only one y-value.

One way to decide whether or not a graph represents a function is to use the **vertical line test.**

If any line drawn parallel to the y-axis cuts the graph only once, then the graph represents a function.

e.g. 1



Graph A and Graph B are functions.

Graph C is not a function because the vertical cuts the graph twice. So for an x-value on the graph, there are two y-values.

4.2 Function notation

We use function notation $f(x)$ to show that each y-value is a function of an x-value.

We can also use other letters too, such as $g(x)$, $h(x)$, etc.

So $y = 2x - 1$ can be written as $f(x) = 2x - 1$.

The value of $f(x)$ for any x-value can be worked out by substitution:

For example, at $x = -3$ we can find $f(-3) = 2(-3) - 1 = -7$

So the point $(-3; -7)$ lies on the graph of $f(x) = 2x - 1$

Activity 1

1. If $h(x) = \left(\frac{1}{2}\right)^x$ determine the value of $h(-4)$. (3)
2. If the function $g(x) = -x^2 - 3x$, find $g(x + h)$ (2)
3. If $f(x) = 4x + 1$, determine the value of:
   3.1 $f(x + a)$
   3.2 $f(x) + a$
   3.3 $af(x)$ (3)
4. If $g(x) = 2x^2$, determine the value of:
   4.1 $g(-x)$
   4.2 $-g(x)$ (2)

[10]

Solutions

1. $h(x) = \left(\frac{1}{2}\right)^x$

$\therefore h(-4) = \left(\frac{1}{2}\right)^{-4}$ ✓ $(2^{-1})^{-4} = 2^4 = 16$ ✓

So when $x = -4$, $y = 16$ and the point $(-4; 16)$ lies on the graph of the function ✓h. (3)

2. $g(x) = -x^2 - 3x$

$\therefore g(x + h) = -(x + h)^2 - 3(x + h)$ ✓ wherever there is an x, replace it with $(x + h)$

$= -(x^2 + 2xh + h^2) - 3x - 3h$

$= -x^2 - 2xh - h^2 - 3x - 3h$ ✓

This means that when $x = x + h$, $y = -x^2 - 2xh - h^2 - 3x - 3h$ (2)

3.1 $f(x) = 4x + 1$

$f(x + a) = 4(x + a) + 1$

$= 4x + 4a + 1$ ✓

3.2 $f(x) = 4x + 1$

$f(x) + a = 4x + 1 + a$

✓

3.3 $f(x) = 4x + 1$

$af(x) = a(4x + 1)$

$= 4ax + a$ ✓

(3)

4.1 $g(x) = 2x^2$

$g(-x) = 2(-x)^2$

$= 2x^2$ ✓

4.2 $g(x) = 2x^2$

$-g(x) = -2x^2$ ✓

(2)

[10]

hint

In each example, there is only one possible y-value for each x-value, so $f(x)$; $h(x)$ and $g(x)$ are functions.

4.3 The basic functions, formulas and graphs

Important terms to remember:

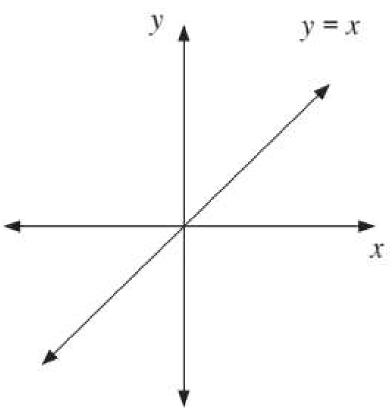
Domain:	the set of possible x-values
Range:	the set of possible y-values
Axis of symmetry:	an imaginary line that divides a graph into two mirror images of each other.
Maximum:	the highest possible y-value of a function.
Minimum:	the lowest possible y-value of a function.
Asymptote:	an imaginary line that a graph approaches but never touches.
Turning point:	The point at which a graph reaches its maximum or minimum value and changes direction.

4.3.1 The linear function (straight line)

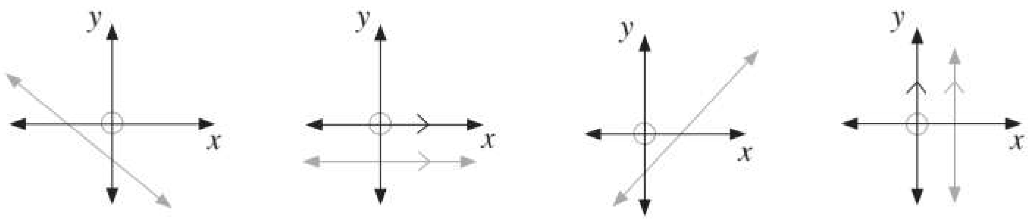
Linear functions have the form $f(x) = ax + q$ where a represents the **gradient** of a straight-line graph and q represents the y-intercept when $x = 0$.

The graph of y is a straight line with $a = 1$ and $q = 0$

Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}$



Also note the shape of the following linear functions



$a < 0$ $q < 0$	$a = 0$ $y = q$	$a > 0$ $q < 0$	a is undefined there is no q-value

SKETCHING THE LINEAR FUNCTION

To sketch the linear function using the dual intercept method.

- Determine the x-intercept (let $y = 0$)
- Determine the y-intercept (let $x = 0$)
- Plot these two points and draw a straight line through them.

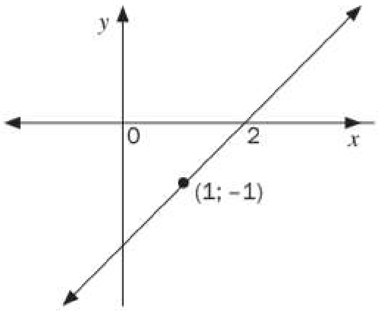
DETERMINING THE EQUATION OF A LINEAR FUNCTION

To determine the equation of the linear function follow the following steps:

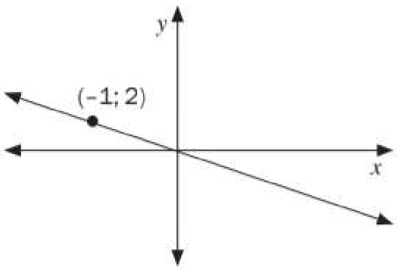
- Determine the gradient of the function.
- Substitute the value of the gradient into the general formula for the linear function.
- Solve for q.
- Write the equation in the form $f(x) = ax + q$

e.g. 2

1.



2.



Solutions

1.

$a = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-1 - 0}{1 - 2}$ ✓ ✓

$a = 1$

$\therefore y = 1x + c$

$0 = 1(2) + c$

$c = -2$ ✓

$\therefore f(x) = x - 2$

2.

$a = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2 - 0}{-1 - 0}$ ✓

$a = -2$

$\therefore y = -2x + c$

$0 = -2(0) + c$

$c = 0$ ✓

$\therefore f(x) = x - 2x$

[5]

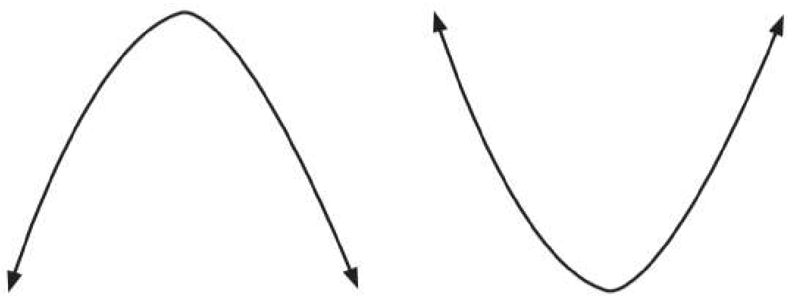
4.3.2 The quadratic functions (parabola)

A **quadratic function** is a **parabola** and can be represented with a general formula $y = ax^2 + bx + c$ or $y = a(x + p)^2 + q$

[PROPERTIES OF A PARABOLA]

1. **Shape**

$a < 0$ $\qquad$ $a > 0$



2. The graph has an axis of symmetry at $x = \frac{-b}{2a}$ or

3. The function has one turning point given by $\left(-\frac{b}{2a}; f\left(-\frac{b}{2a}\right)\right)$.

4. The function may have either a maximum or a minimum value but never both.

5. **Domain:** $x \in \mathbb{R}$

   Range: $y \geq f\left(-\frac{b}{2a}\right)$ or $y \leq f\left(-\frac{b}{2a}\right)$

SKETCHING THE QUADRATIC FUNCTION

To sketch any quadratic function, follow the following steps:

- Write down the y-intercept (let $x = 0$)
- To calculate the x-intercepts,
  - Write the equation in the form $ax^2 + bx + c = 0$
  - Factorise the left hand side of the equation.
  - Use the fact that if $(x - p)(x - q) = 0$, then $x = p$ or $x = q$, to calculate the x-intercepts.
- Determine the axis of symmetry.
- Substitute the x-value of the axis of symmetry into the original equation of the function to calculate the co-ordinates of the turning point.
- Plot the points and then draw the function using free hand.

e.g. 3

Sketch the graph of $f(x) = x^2 - 5x - 6$

1. ***y*-intercept**

   $f(0) = -6$

   Therefore the co-ordinates of the y-intercept are $(0; -6)$ ✓

2. ***x*-intercept**

   $x2 - 5x - 6 = 0$ ✓

   $(x - 6)(x + 1) = 0$ ✓

   $x = 6$ or $x = -1$ ✓

   $(6; 0)$ and $(-1; 0)$

3. **Axis of symmetry**

   $x = \frac{-b}{2a}$ ✓

   $= \frac{-(-5)}{2(1)}$ ✓

   $= \frac{5}{2}$ ✓

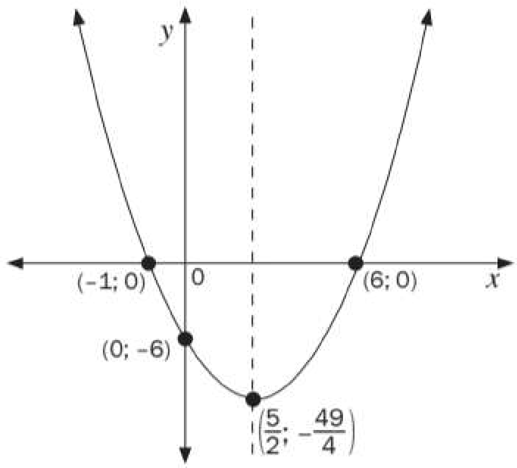
4. **Turning point**

   $f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) - 6$ ✓

   $= -12\frac{1}{4}$ ✓

   $\therefore TP\left(\frac{5}{2}; -12\frac{1}{4}\right)$ ✓

5. Sketch graph



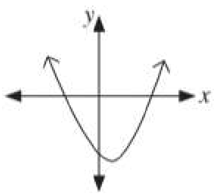
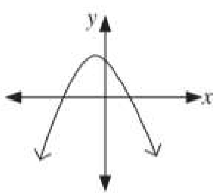
✓x-intercepts ✓y-intercept ✓shape ✓turning point

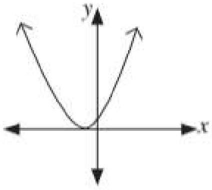
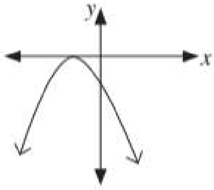
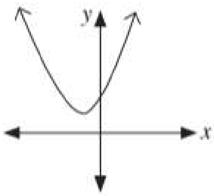
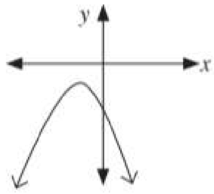
Determining the equation of a quadratic function

Given the x-intercept and one point	Given the turning point and one point
• Use the formula: $y = a(x - x_1)(x - x_2)$. • Substitute the values of the x-intercepts. • Substitute the given point which is not the x-intercept. • Solve for a. • Write the equation in the form $f(x) = ax^2 + bx + c$.	• Use the formula: $y = a(x + p)^2 + q$. • Substitute the co-ordinates of the turning point $(p; q)$. • Substitute the given point. • Solve for a. • Write the equation in the form $y = a(x + p)^2 + q$ or $f(x) = ax^2 + bx + c$ depending on the instruction in the question.

Given the co-ordinates of three points on the parabola
• Use the formula: $y = ax^2 + bx + c$. • One of the given point is the y-intercept, therefore c is given, so substitute its value. • Substitute the co-ordinates of the other two points into $y = ax^2 + bx + c$. • Solve the two equations simultaneously for a and b.

Nature of the roots and the quadratic function

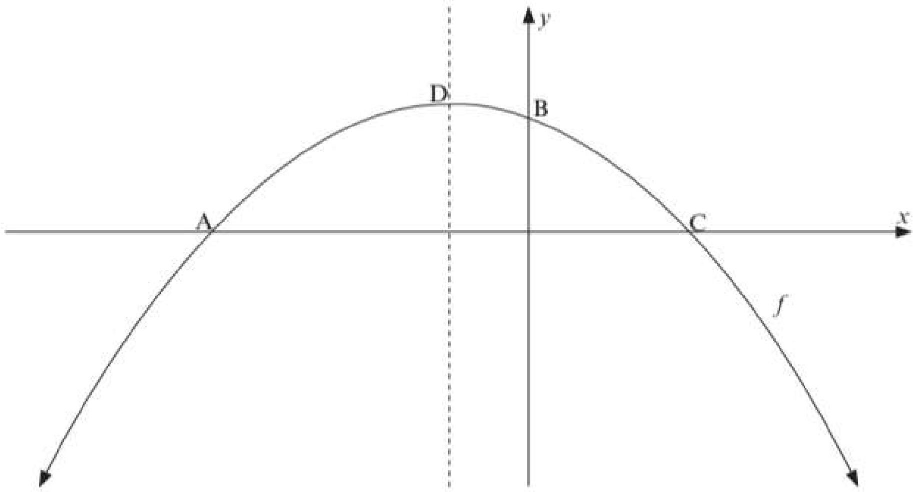
Nature of roots	Quadratic function
Real roots $\Delta > 0$	 or  NOTE: there are two x-intercepts.

Equal roots $\Delta = 0$	 or  NOTE: there is only one x-intercept.
Non-real roots $\Delta > 0$	 or  NOTE: there are no x-intercepts.

Activity 2

The sketch represents the graph of the parabola given by $f(x) = 2 - x - x^2$.

Points A, B and C are the intercepts on the axes and D is the turning point of the graph.



1.1 Determine the co-ordinates of A, B and C. (4)

1.2 Determine the co-ordinates of the turning point D. (3)

1.3 Write down the equation of the axes of symmetry of $f(x-5)$. (1)

1.4 Determine the values of x for which $-f(x) \geq 0$. (2)

[10]

Solutions

1.1 B(0; 2)

$2 - x - x^2 = 0$ ✓

$x^2 + x - 2 = 0$

$(x - 1)(x + 2) = 0$ ✓

$x = 1$ or $x = -2$ ✓

A(–2; 0) and C(1; 0) ✓ (4)

1.2 $x = \frac{-b}{2a}$

$= \frac{-(-1)}{2(-1)}$ ✓

$= -\frac{1}{2}$ ✓

$f\left(-\frac{1}{2}\right) = 2 - \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2$

$= \frac{9}{4} = 2\frac{1}{4}$

$D\left(-\frac{1}{2}; \frac{9}{4}\right)$ ✓ (3)

1.3 $x = \frac{9}{2}$ *or* $x = 4\frac{1}{2}$ ✓ (1)

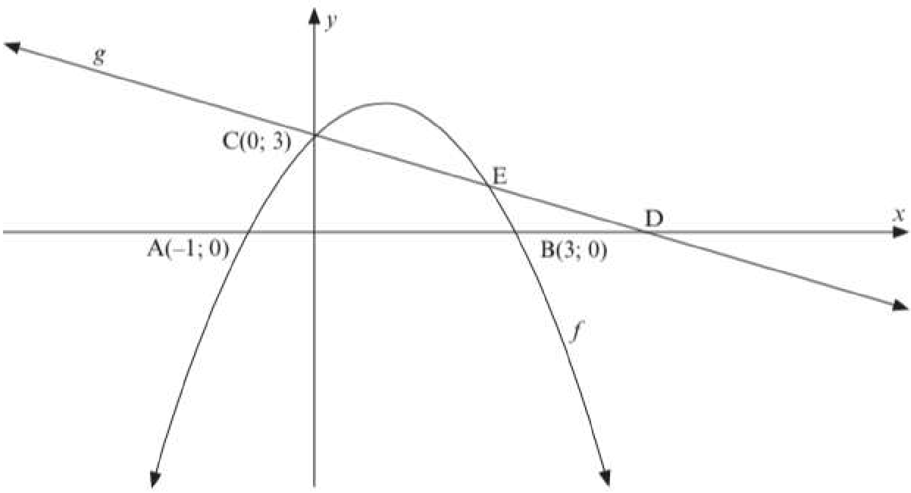
1.4 $x \leq -2$ ✓ *or* $x \geq 1$ ✓ (2)

[10]

Activity 3

The sketch represents the graph of the parabola given by $f(x) = ax^2 + bx + c$ and the straight line defined by $g(x) = mx + c$

Points A, B ,C and D are the intercepts on the axes. E is the point of intersection of the two graphs.



2.1 Write down the co-ordinates of point D if D is the image of B after B has been translated two units to the right. (1)

2.2 Determine the equation of g. (3)

2.3 Determine the equation of the function f in the form $f(x) = ax^2 + bx + c$. (4)

2.4 Determine the coordinates of E. (4)

2.5 Write down the values of x for which $f(x) \geq g(x)$. (2)

[14]

Solutions

2.1 $D(5; 0)$ ✓ (1)

2.2 $g(x) = mx + 3$

$0 = m(5) + 3$ or $m_g = \frac{3-0}{0-5} = -\frac{3}{5}$ ✓

$m = -\frac{3}{5}$ ✓

$g(x) = -\frac{3}{5}x + 3$ ✓ (3)

2.3 $f(x) = a(x + 1)(x - 3)$ ✓

$3 = a(0 + 1)(0 - 3)$ ✓

$a = 1$ ✓

$f(x) = -(x + 1)(x - 3)$

$f(x) = -x^2 + 2x + 3$ ✓ (4)

2.4 $-\frac{3}{5}x + 3 = -x^2 + 2x + 3$ ✓

$x^2 - \frac{13}{5}x = 0$

$x\left(x - \frac{13}{5}\right) = 0$ ✓

$x = 0$ *or* $x = \frac{13}{5} = 2{,}60$ ✓

$g\left(\frac{13}{5}\right) = -\frac{3}{5}\left(\frac{13}{5}\right) + 3$

$= \frac{36}{25}$

$= 1{,}44$ ✓

$\therefore E\left(\frac{13}{5}; \frac{36}{25}\right)$ *or* $E\left(2\frac{3}{5}; 1\frac{11}{25}\right)$ *or* $E\,(2{,}60; 1{,}44)$ (4)

2.5 $0 \leq x \leq \frac{13}{5}$ ✓✓ (2)

[14]

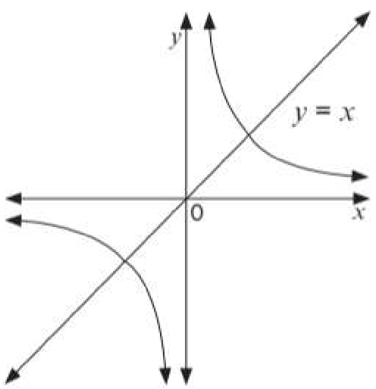
4.3.3 The hyperbolic function

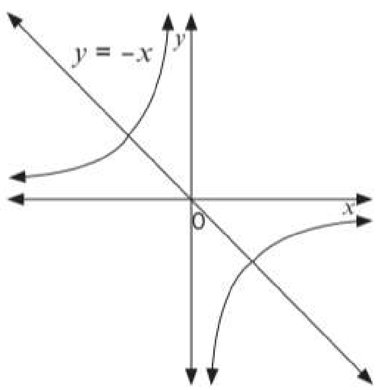
Hyperbola of the form $y = \frac{a}{x}$ or $xy = a$ where $a \neq 0$; $x \neq 0$; $y \neq 0$.

Properties

Shape

1. $a > 0$ $a < 0$





2. (i) Domain : $x \in \mathbb{R}; x \neq 0$ (i) Range: $y \in \mathbb{R}; y \neq 0$
3. The horizontal asymptote is the x-axis
4. The vertical asymptote is the y-axis
5. If $a < 0$, the graph lies in the 2nd and 4th quadrant
6. If $a > 0$, the graph lies in the 1st and 3rd quadrant
7. The lines of symmetry are: $y = x$ and $y = -x$.

SKETCHING THE HYPERBOLA OF THE FORM:

$y = \frac{a}{x}$ or $xy = a$

- The graph does not cut the x-axis and the y-axis (asymptotes)
- Use the table and consider both the negative and positive x-values
- a determine two quadrants where the graph will be drawn

Activity 4

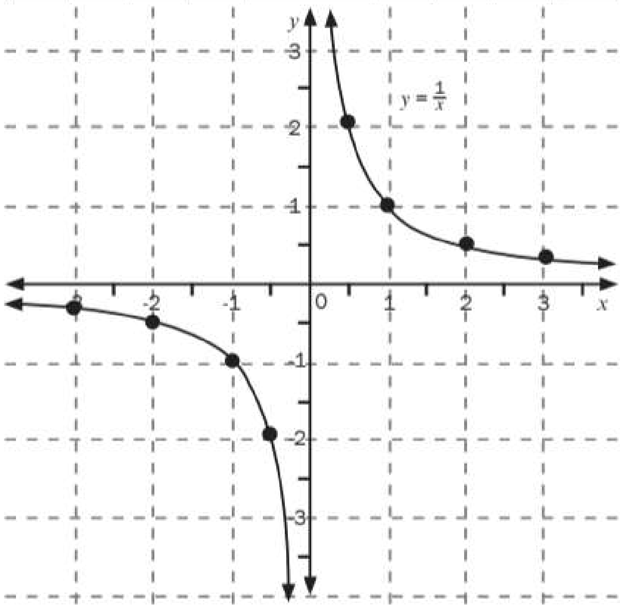
1. Sketch the graph of $y = \frac{1}{x}$ by plotting points. Describe the main features of the graph. (4)

Solution

$a = 1$

$a > 0$, the graph lies in the 1st and 3rd quadrant

-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	undefined	2	1	$\frac{1}{2}$	$\frac{1}{3}$



- Domain: $x \in \mathbb{R}; x \neq 0$ ✓
- Range: $y \in \mathbb{R}; y \neq 0$ ✓
- Asymptotes: $x = 0$ and $y = 0$ ✓
- Lines of symmetry $y = x$ and $y = -x$ ✓ (4)

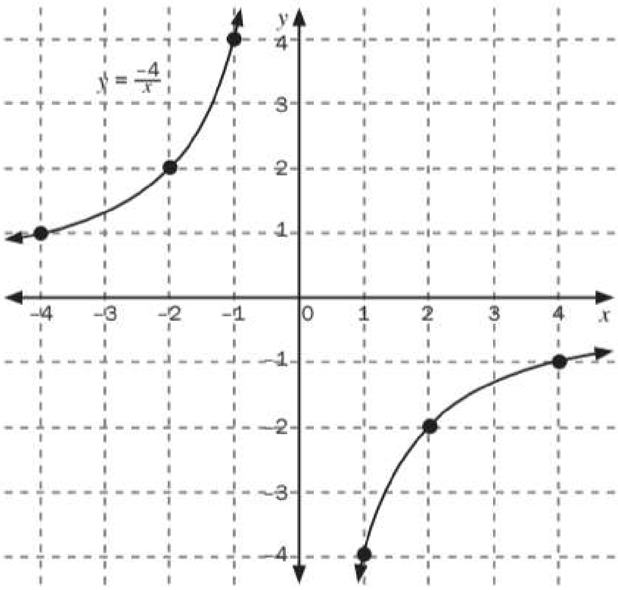
2. Sketch the graph of $y = \frac{-4}{x}$ by plotting the points. Describe the main features of the graphs. (4)

Solution

$a = -4$

$a < 0$, the graph lies in the 2nd and 4th quadrant

-4	-2	-1	0	1	2	4
1	2	4	undefined	-4	-2	-1



- Domain: $x \in \mathbb{R}; x \neq 0$ ✓
- Range: $y \in \mathbb{R}; y \neq 0$ ✓
- Asymptotes: $x = 0$ and $y = 0$ ✓
- Lines of symmetry $y = x$ and $y = -x$ ✓ (4)

[8]

4.3.4 The hyperbola

Hyperbola of the form $y = \frac{a}{x} + q$ is the translation of the graph of $y = \frac{a}{x}$ vertically by q units.

The Horizontal asymptote (x–axis) will also shift q units vertically (up or down).

Activity 5

1. Consider the function $y = \frac{1}{x} - 2$

   1.1 Determine :

   a) the equations of the asymptotes

   b) the coordinates of the x–intercepts

   1.2 Sketch the graph

   1.3 Write down:

   a) the domain and range

   b) the lines of symmetry $y = x + c$ and $y = -x + c$

(10)

Solutions

1.1

a) The horizontal asymptote is $y = -2$ since the graph moved 2 units down and the vertical asymptote is $x = 0$ ✓ denominator cannot equal to zero.

b) For x – intercepts let $y = 0$

$0 = \frac{1}{x} - 2$ ✓

$0 = 1 - 2x$ (✓multiplying by LCD which is x)

$2x = 1$ ✓

$x = \frac{1}{2}$ ✓

$\left(\frac{1}{2}; 0\right)$

2. Consider the function $f(x) = \frac{-4}{x} + 1$

   2.1 Determine:

   a) the equations of the asymptotes

   b) the coordinates of the x–intercepts

   2.2 Sketch the graph

   2.3 Write down the domain and range

   2.4 If the graph of f is reflected by the line having the equation $y = -x + c$, the new graph coincides with the graph of $f(x)$. Determine the value of c.

(9)

Solutions

2.1

a) The horizontal asymptote is $y = 1$ ✓ since the graph moved 1 units up and the vertical asymptote is $x = 0$ denominator cannot equal to zero.

b) For x–intercepts let $y = 0$

$0 = \frac{-4}{x} + 1$ ✓

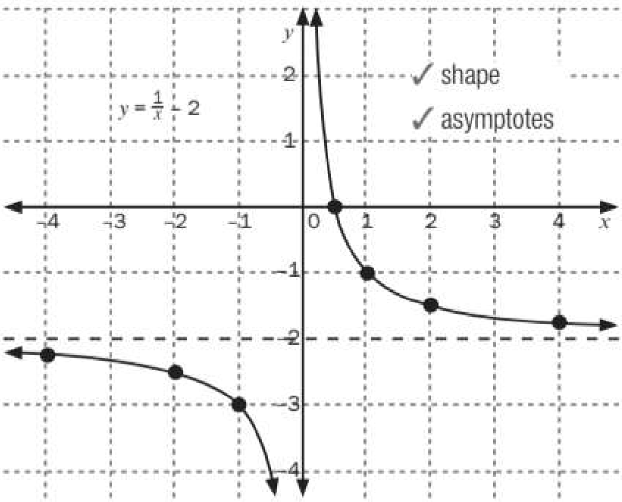
$0 = -4 + x$ (✓multiplying by LCD which is x

$x = 4$ ✓

$(4; 0)$

1.2

x	-4	-2	-1	0	1	2	4
y	$-2\frac{1}{4}$	$-2\frac{1}{2}$	-3	undefined	-1	$-1\frac{1}{2}$	$-1\frac{3}{4}$



1.3

a) Domain: $x \in \mathbb{R}; x \neq 0$ ✓

Range: $y \in \mathbb{R}; y \neq 2$ ✓

b) $y = x$ and $y = -x$

translation 2 units down therefore

$y = x - 2$ and $y = -x - 2$ ✓

$\therefore c = -2$

Or substitute (0; 2) point of intersection of the two asymptotes in $y = x + c$ or $y = -x + c$

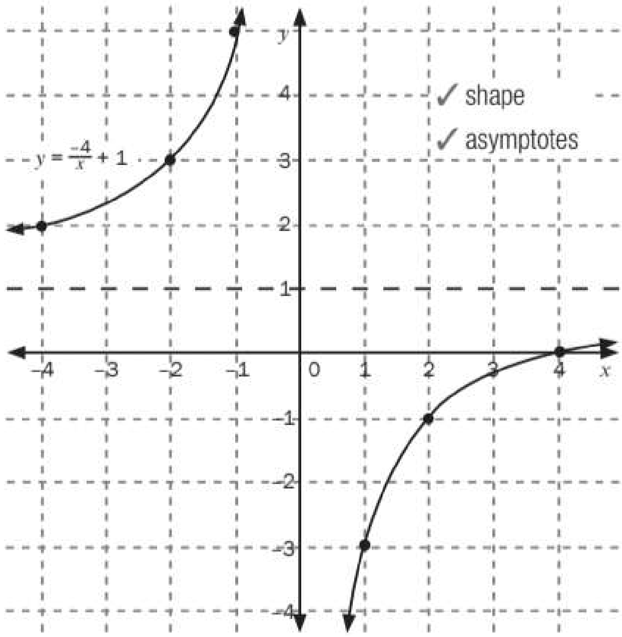
And calculate the value of c

[10]

hint Compare this graph with the one in activity 4 (a)

2.2

x	-4	-2	-1	0	1	2	4
y	2	2	5	undefined	-3	-1	0



2.3 Domain: $x \in \mathbb{R}; y \neq 0$ ✓

Range: $y \in \mathbb{R}; y \neq 1$ ✓

2.4 The asymptotes are

$x = 0$ and $y = 1$

$y = -x + c$

$1 = -(0) + c$

$1 = c$

lines are $y = -x + 1$ and $y = x + 1$ ✓

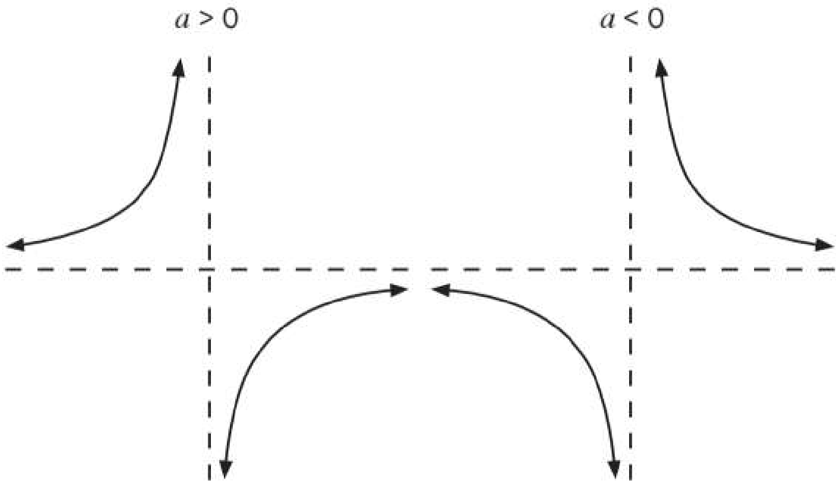
[9]

hint Compare this graph with the one in activity 4 (b)

4.3.5 Hyperbola of the form

$y = \frac{a}{x+p} + q$ where $a \neq 0$, $x \neq 0$, $y \neq 0$

1. Shape



The dotted lines are the asymptotes

2. Domain: $x \in \mathbb{R}$; $x \neq -p$. Range: $y \in \mathbb{R}$; $y \neq q$
3. The **horizontal asymptote** is $y = q$
4. The **vertical asymptote** is $x + p = 0\ \therefore x = -p$
5. The lines of symmetry are $y = x + c$ and $y = x + c$

e.g. 4

Consider $g(x) = \frac{8}{x-2} - 3$ has the horizontal asymptote at $y = -3$ and $x - 2 \neq 0\ \therefore x \neq 2$ because if $x = 2$ the denominator of the expression $\frac{8}{x-2}$ would be $\frac{8}{2-2} = \frac{8}{0}$ which is undefined because the denominator is zero.

Thus the graph is undefined for $x - 2 = 0\ \therefore x = 2$ is the **vertical asymptote**

The graph $y = \frac{8}{x}$ shift 2 units to the right and 3 units down to form the graph $g(x) = \frac{8}{x-2} - 3$

SKETCHING THE HYPERBOLA OF THE FORM

$y = \frac{a}{x+p} + q$

- Write down the asymptotes
- Draw the asymptotes on the set of axes as dotted lines
- Use a to determine the two quadrants where the graph will be drawn
- Determine the x – intercept(s) let $y = 0$
- Determine the y – intercept(s) let $x = 0$
- Plot the points and then draw the graph using free hand

Activity 6

1. Consider the function $f(x) = \frac{2}{x-3} + 1$
   a) Write down the equations of the asymptotes of f (2)
   b) Calculate the coordinates of the x and y-intercepts of f (4)
   c) Write the domain and range (2)
   d) Sketch the graph of f clearly showing ALL asymptotes and intercepts with the axes. (3)

2. Consider the function $f(x) = \frac{3}{x-1} - 2$
   a) Write down the equation of the asymptotes. (2)
   b) Calculate the coordinates of the intercepts of the graph of f with the axes. (3)
   c) Sketch the graph of f clearly showing the intercepts with the axes and the asymptotes. (3)
   d) Write down the range of $y = -f(x)$. (1)
   e) Describe, in words, the transformation of f to g if $g(x) = \frac{-3}{x+1} - 2$ (2)

[22]

Solution

1. a) $x = 3$ and $y = 1$ ✓✓ (2)

b) $f(x) = \frac{2}{x-3} + 1$

y – intercept $y = \frac{2}{0-3} + 1 = \frac{1}{3}$ ✓

$\left(0; \frac{1}{3}\right)$

x – **intercept** $0 = \frac{2}{x-3} + 1$ ✓

✓

$0 = 2 + 1(x - 3)$

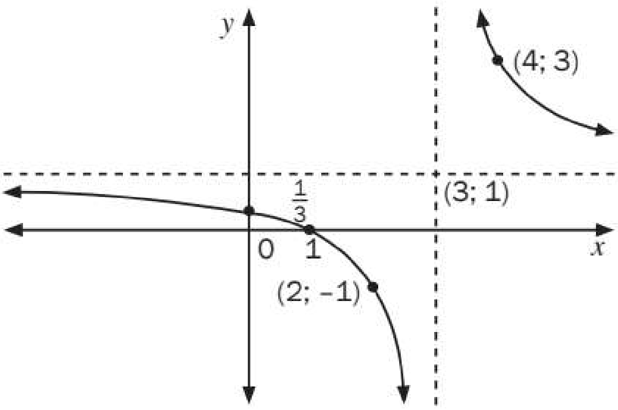
$0 = 2 + x - 3$

✓ $x = 1$ $\therefore (1; 0)$ (4)

c) Domain: $x \in \mathbb{R}; x \neq 3$ ✓

Range: $y \in \mathbb{R}; y \neq 1$ ✓ (2)

d) $a > 0$



✓ intercepts ✓ asymptotes ✓ shape (3)

[11]

Solution

2. a) ✓ $x = -1$ $y = -2$ ✓ (2)

b) y – intercept

$y = \frac{3}{0-1} - 2 = -5$

$(0; -5)$ ✓

x – intercept ✓ $0 = \frac{3}{x-1} - 2$

$2 = \frac{3}{x-1}$

$2(x - 1) = 3$

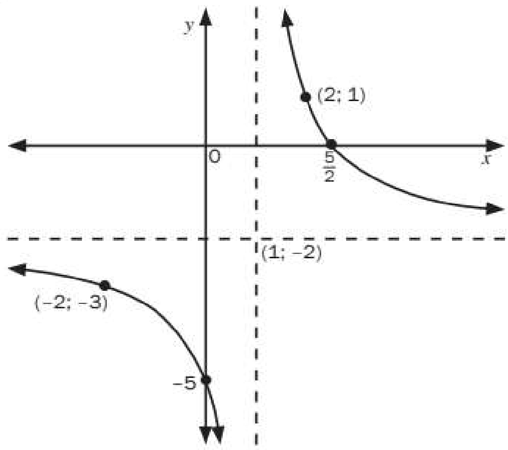
$2x - 2 = 3$

$2x = 5$

✓ $x = \frac{5}{2}$

$\left(\frac{5}{2}; 0\right)$ (3)

c) $a > 0$



✓ intercepts ✓ asymptotes ✓ shape (3)

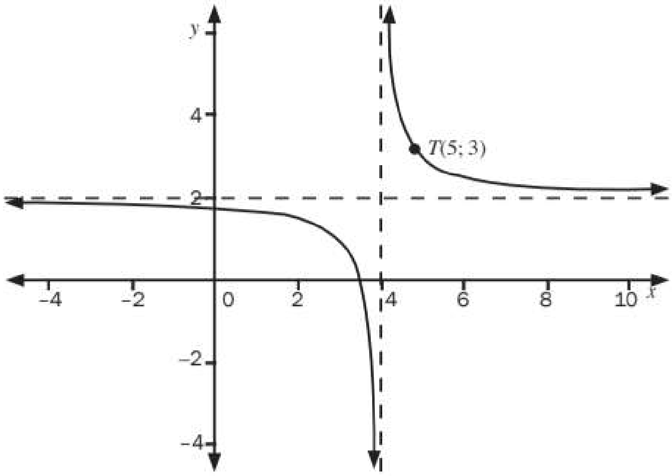
hint

In the graph 1 (d) the points (4; 3), $x = 4$ was chosen because it has x-coordinate greater than $x = 3$ the vertical asymptote. The point (2; –1), was chosen because has x-coordinate $x = 2$ is less than $x = 3$ the vertical asymptote. These points can also be used to help determining in which quadrants the graph must be drawn. The points (2; 1) and (–2; –3) on graph 2 (iii) were chosen similarly.

	d) $f(x) = \frac{3}{x-1} - 2$ $-f(x) = -\left(\frac{3}{x-1} - 2\right)$ $-f(x) = \frac{-3}{x-1} + 2$ Range: $y \in \mathbb{R}; y \neq 2$ ✓ (1) **e)** $g(x) = \frac{-3}{x+1} - 2$ $g(x) = \frac{3}{-x-1} - 2$ Since x is negative this is the reflection ✓ of f about the y-axis ✓ (2) **[11]**

Activity 7

The diagram below represents the graph of $f(x) = \frac{a}{x+p} + q$. T(5; 3) is a point on f.



4.1 Determine the values of a, p and q (4)

4.2 If the graph of f is reflected across the line having the equation $y = -x + c$, the new graph coincides with the graph of $y = f(x)$. Determine the value of c. (3)

[7]

Solutions

4.1 ✓$p = 4$ and $q = 2$ ✓using the asymptotes

Substitute T(5; 3) into $y = \frac{a}{x-4} + 2$

$3 = \frac{a}{5-4} + 2$ ✓ $\quad 3 = a + 2 \quad a = 1$ ✓ (4)

4.2 Substitute (4; 2) ✓into $y = -x + c$

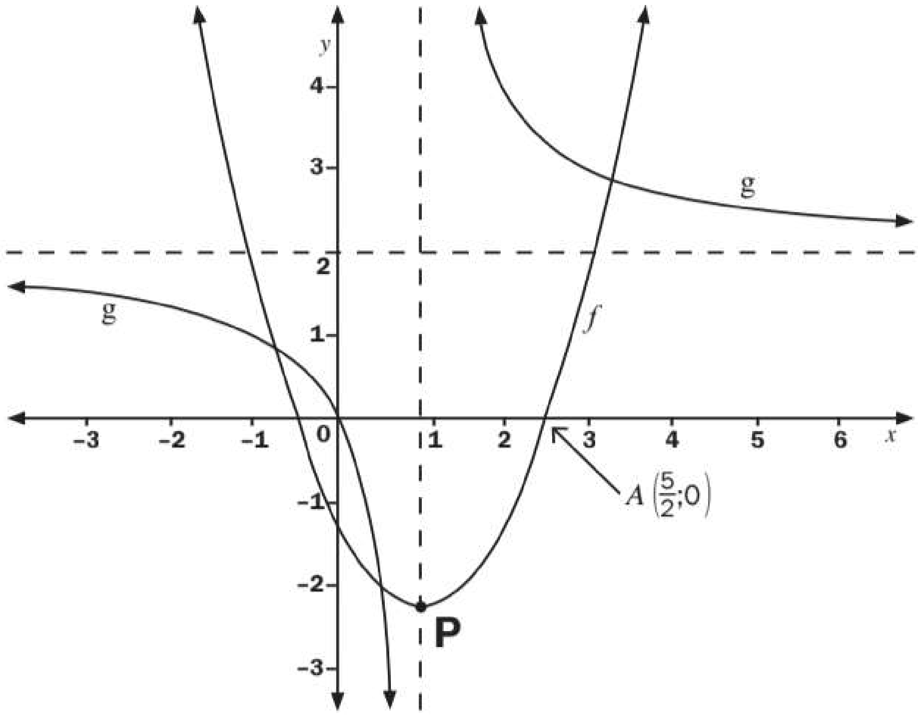
✓$2 = -(4) + c \quad \therefore c = 6$ ✓ (3)

[7]

Activity 8

Sketched below are the graphs of $f(x) = (x + p)^2 + q$ and $g(x) = \frac{a}{x + b} + c$

A $\left(2\frac{1}{2}; 0\right)$ is a point on the graph of f. P is the turning point of f. The asymptotes of g are represented by the dotted lines. The graph of g passes through the origin



5.1 Determine the equation of g. (4)

5.2 Determine the coordinates of P, the turning point of f. (4)

5.3 Write down the equation of the asymptotes of $g(x - 1)$. (2)

5.4 Write down the equation of h, if h is the image of f reflected about the x–axis. (1)

[11]

Solutions

5.1 Using the asymptotes ✓$b = 1$ and $c = 2$ ✓

Substitute (0; 0) into $y = \frac{a}{x-1} + 2$

✓$0 = \frac{a}{0-1} + 2 \qquad \Rightarrow 0 = -a + 2 \qquad \therefore a = 2$ ✓

$y = \frac{2}{x-1} + 2$ (4)

5.2 Axis of symmetry $p = 1$ ✓

$f(x) = (x - 1)^2 + q$

$\left(\frac{5}{2};0\right)$ ✓

✓$0 = \left(\frac{5}{2} - 1\right)^2 + q$

$0 = \frac{9}{4} + q$

$q = -\frac{9}{4} \qquad \therefore P\left(1;-\frac{9}{4}\right)$ ✓ (4)

5.3 $g(x) = \frac{2}{x-1} + 2$

$g(x-1) = \frac{2}{(x-1)-1} + 2$ substitute x with $(x-1)$

$g(x-1) = \frac{2}{x-2} + 2$

✓ $x = 2$ and $y = 2$ ✓ (2)

5.4 $f(x) = (x-1)^2 - \frac{9}{4}$

Reflection about the x – axis y changes the sign

$-y = (x-1)^2 - \frac{9}{4}$

$y = -\left[(x-1)^2 - \frac{9}{4}\right]$

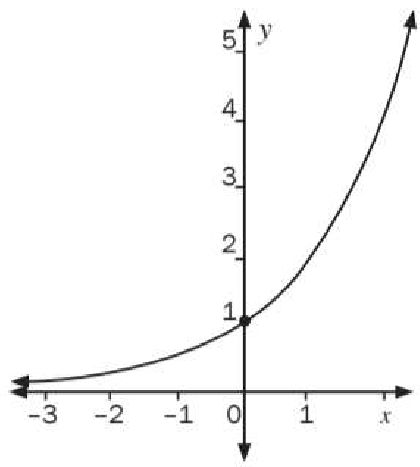
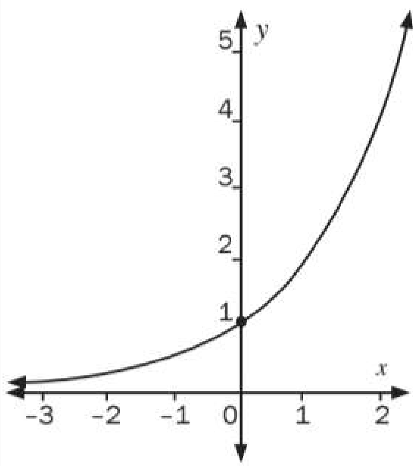
$y = -(x-1)^2 + \frac{9}{4}$ ✓ (1)

[11]

4.3.6 The exponential function

An **exponential function** can be represented with a general formula $y = ab^{x+p} + q;\ b > 0$

Shape and properties of an exponential function

$y = b^x;\ b > 1$	$y = b^x;\ 0 < b > 1$
	
• The graph passes through the point (0; 1). • Domain: $x \in \mathbb{R}$ • Range: $y > 0$ but for $y + b^x + q$, the range will be at $y > q$. • The graph is smooth, continuous and an **increasing** function. • Asymptote is at $y = 0$ but for $y = b^x + q$, the horizontal asymptote will be at $y = q$.	• The graph passes through the point (0; 1). • Domain: $x \in \mathbb{R}$ • Range: $y > 0$ but for $y = b^x + q$, the range will be at $y > q$. • The graph is smooth, continuous and a **decreasing** function. • Asymptote is at $y = 0$ but for $y = b^x + q$, the horizontal asymptote will be at $y = q$.
NOTE: The two functions are a reflection of each other about the y-axis.	

e.g. 5

Given: $f(x) = 2^x$

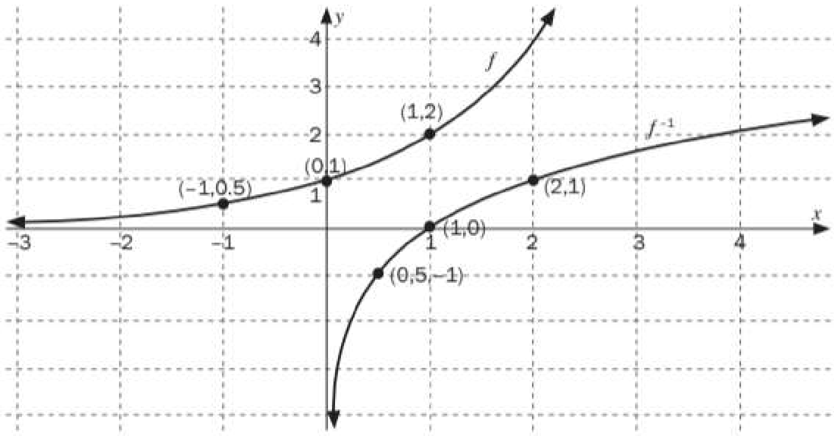
1.1 Draw the graph of $f(x) = 2^x$, show at least three points on the sketch.
1.2 Draw, on the same system of axes the graph of f^{-1}, the inverse of f.
1.3 Write down the equation of f^{-1} in the form $y = ...$

Solutions

1.1 Start by drawing the table:

x	-1	0	1
$f(x)$	0,5	1	2

Then plot the graph using the points



1.2 The sketch of f^{-1} is obtained by interchanging the x and y co-ordinates of f.

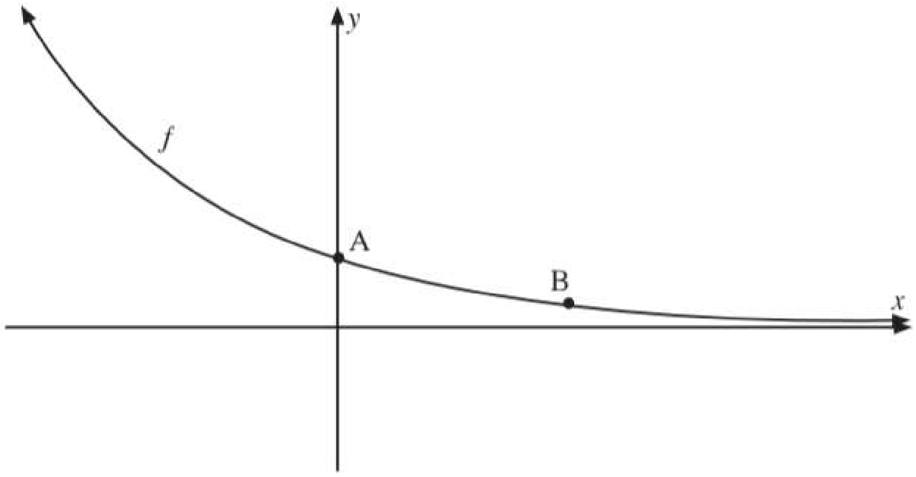
1.3 $y = 2^x$

$x = 2^y$ ✓

$y = \log_2 x$ ✓ **[2]**

e.g. 6

The sketch represents the graph given by $f(x) = a^x$.



2.1 Write down the coordinates of point A. (1)
2.2 How can we tell that $0 < a < 1$? (1)
2.3 Determine a if B is the point $\left(3; \frac{1}{27}\right)$. (2)
2.4 Determine the equation of the graph obtained if f is reflected about the y–axis. (2)
2.5 What are the coordinates of the point of intersection of the two graphs? (1)
[7]

Solutions

2.1 A(0; 1) ✓

2.2 Because the graph is a decreasing function. ✓

2.3
$$f(x) = a^x$$
$$\frac{1}{27} = a^3 \checkmark$$
$$(3^{-1})^3 = a^3$$
$$a = \frac{1}{3} \checkmark$$

2.4 $f(x) = \left(\frac{1}{3}\right)^x$

$y = \left(\frac{1}{3}\right)^x$ becomes $y = \left(\frac{1}{3}\right)^{-x}$ ✓

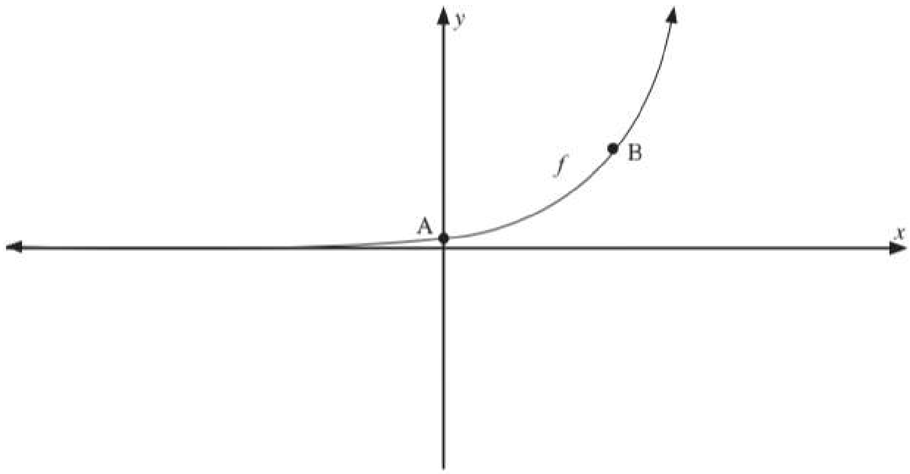
$\therefore y = (3^{-1})^{-x}$

$y = 3^x$ ✓

2.5 (0; 1) ✓ **[7]**

Activity 9

The curve of an exponential function is given by $f(x) = k^x$ and cuts the y–axis at A (0; 1) while B $\left(2;\frac{9}{4}\right)$ lies on the curve.



Determine

1.1 the equation of the function f. (3)

1.2 the equation of the asymptote of h if $h(x) = -f(x)$. (2)

1.3 the range of h. (1)

1.4 The equation of the function g of which the curve is the reflection of the curve of f in the line $y = x$. (2)

Solutions

1.1 $f(x) = k^x$

$\frac{9}{4} = a^2$ ✓

$\left(\frac{3}{2}\right)^2 = a^2$ ✓

$a = \frac{3}{2}$ ✓ $\therefore f(x) = \left(\frac{3}{2}\right)^x$ (3)

1.2 $y = 0$ ✓✓ (2)

1.3 $y \leq 0$ ✓ (1)

1.4 $g(x) = \log_{\frac{3}{2}} x$ ✓✓ (2)

[8]

4.4 Inverse functions

- The inverse of a function takes the y-values (range) of the function to the corresponding x-values (domain) and vice versa. Therefore the x and y values are interchanged.
- The function is reflected along the line $y = x$ to form the inverse.
- The notation for the inverse of a function is f^{-1}.

e.g. 7

Given $f(x) = 2x + 6$.

1. Determine $f^{-1}(x)$

2. Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axis

Solutions

1. In order to find the inverse of a function, there are two steps:

STEP 1: Swap the x and y

$y = 2x + 6$ ✓

becomes $x = 2y + 6$ ✓

We then rewrite the equation to make y the subject of the formula. Therefore,

STEP 2: make y the subject of the formula

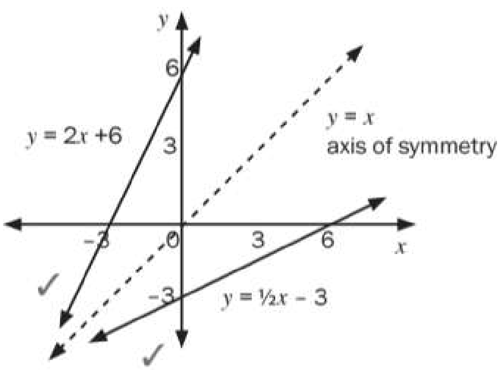
$x = 2y + 6$

$x - 6 = 2y$ ✓

So $y = \frac{1}{2}x - 3$ ✓

We can say that the inverse function $f^{-1}(x) = \frac{1}{2}x - 3$

2.



- Every point on the function has the same coordinates as the corresponding point on the inverse function, *except that they are swapped around.*
- Example: $(-3; 0)$ on the function is reflected to become $(0; -3)$ on the inverse function.
- Any point $(a; b)$ on the function becomes the point $(b; a)$ on the inverse.
- To find the equation of an inverse function algebraically, we interchange x and y and then solve for y.
- To draw the graph of the inverse function, we reflect the original graph about the line $y = x$, the axis of symmetry of the two graphs.

e.g. 8

1. a) Sketch $f(x) = 2x^2$

b) Determine the inverse of $f(x)$

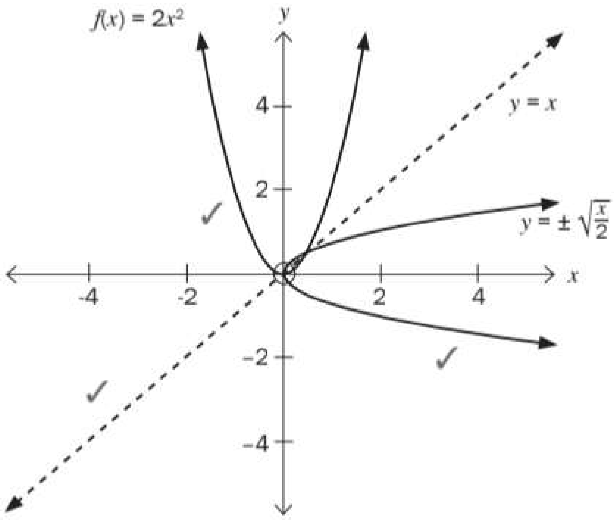
c) Sketch $f^{-1}(x)$ and $y = x$ on the same axes as $f(x)$

Solution

1. b) $y = 2x^2$

$x = 2y^2$ ✓

$y = \pm\sqrt{\frac{x}{2}}$ ✓



- This is not a function.
- Check it with a vertical line test. There are two y-values for one x-value.
- Not all inverses of functions are also functions. Some inverses of functions are relations.
- If an inverse is not a function, then we can restrict the **domain** of the **function** in order for the inverse to be a function.
- To make the inverse a function, we need to choose a set of x-values in the function and work only with those. We call this **'restricting the domain'**.
- A one to one function has an inverse that is a function

  Example: $y = 3x + 4$ is a one to one function. For every x value there is one and only one y value

  The inverse of is a function.
- A many to one function has an inverse that is not a function. However, we can restrict the domain of the function to make its inverse a function.

  Example: $y = 2x^2$ is a many to one function. For two or many x values there is one y value. (if $x = 2$, then $y = 8$.

  If $x = -2$, then $y = 8$). Therefore, its inversey $= \pm\sqrt{\frac{x}{2}}$, is not a function.
- To check for a function, draw a vertical line. If any vertical line cuts the graph in only one place, the graph is a function.

  If any vertical line cuts the graph in more than one place, then the graph is not a function.
- To check for a one-to-one function, draw a horizontal line. If any horizontal line cuts the graph in only one place, the graph is a one-to-one function. If any horizontal line cuts the graph in more than one place, then the graph is a many-to-one function. **[5]**

Activity 10

1. a) If $f(x) = -3x^2$, write down the equation for the inverse function in the form $y =$ .... (2)

b) Determine the domain and range of $f(x)$ and $f^{-1}(x)$ (4)

c) Determine the points of intersection of $f(x)$ and $f^{-1}(x)$ (4)

2. a) If $g(x) = 3x + 2$, find $g^{-1}(x)$ (2)

b) Sketch g, g^{-1} and the line $y = x$ on the same set of axes. (3)

[15]

Solutions

1. a) For $f(x) = -3x^2$.

$f^{-1}(x)$: $x = -3y^2$ ✓

$-\frac{x}{3} = y^2$

$y = \pm\sqrt{-\frac{x}{3}}$ ✓ (2)

b)

	$f(x)$	$f^{-1}(x)$
Domain	$x \in \mathbb{R}$ ✓	$x \geq 0$ ✓
Range	$y \geq 0$ ✓	$y \in \mathbb{R}$ ✓

(4)

c) To determine the points of intersection, we equate the two equations.

The line $y = x$, the axis of symmetry of $f(x)$ and $f^{-1}(x)$, can also be used to determine the points of intersection of $f(x)$ and $f^{-1}(x)$.

$y = x$ and $f(x) = -3x^2$

$\therefore x = -3x^2$

$\therefore 3x^2 + x = 0$ ✓

$\therefore x(3x + 1) = 0$ ✓

$\therefore x = 0$ or $x = -\frac{1}{3}$ ✓

Substitute $x = 0$ *in* $y = x \therefore y = 0 \therefore (0; 0)$ ✓

Substitute $x = -\frac{1}{3}$ *in* $y = x \therefore y = -\frac{1}{3} \therefore \left(-\frac{1}{3}; -\frac{1}{3}\right)$ (4)

2. a) $g(x) = 3x + 2$ ✓

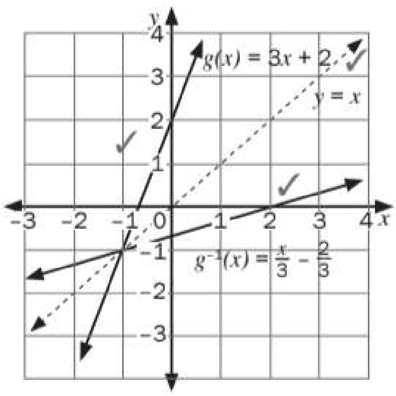
For $g^{-1}(x)$, $x = 3y + 2$

$x - 2 = 3y$

$y = \frac{x-2}{3}$

$y = \frac{x}{3} - \frac{2}{3}$ ✓

b)



(4)

[15]

Given: $g(x) = -x^2$ where $x \leq 0$ and $y \leq 0$

(a) Write down the inverse of g, g^{-1} in the form $h(x) =$ .... (3)

(b) Sketch the graphs of g, h and $y = x$ on the same set of axis. (4)

Solutions

(a) $y = -x^2$

$x = -y^2$

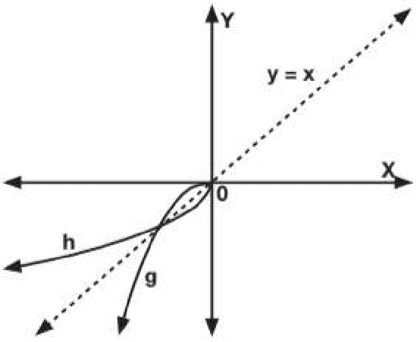
$-x = y^2$ ✓

$\pm\sqrt{-x} = y$ ✓

$-\sqrt{-x} = y$ where $x \leq 0$ and $y \leq 0$

$\therefore h(x) = -\sqrt{-x}$ ✓ (3)

(b)



For g correct shape ✓and the intercept ✓

For h correct shape ✓and the intercept ✓ (4)

[7]

4.5 The logarithmic function

- $y = \log_x a$ is a logarithmic function with a = log number, x=log base
- $y = \log_x a$ Reads "y is equal to log a base x"
- The logarithmic function is only defined if $a > 0$, $a \neq 1$ and $x > 0$
- An exponential equation can be written as a logarithmic equation and vice versa

e.g. 9

Write each of the following exponential equations as logarithmic equations:

- $2^6 = 64$
- $5^3 = 125$

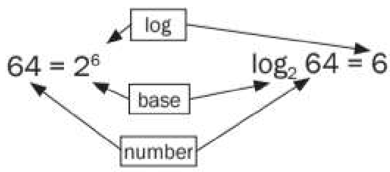
Solutions

1. $2^6 = 64$

$\therefore 6 = \log_2 64$

2. $5^3 = 125$

$\therefore 3 = \log_5 125$



NB

The inverse of the exponential function $y = a^x$ is $x = a^y$

In order to make y the subject of the formula, $x = a^y$, we use the **log function**.

$y = \log_a x$ is the inverse of $y = a^x$.

e.g. 10

Given: $f(x) = 2^x$

a) Determine f^{-1} in the form $y = \ldots\ldots$

b) Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axes.

c) Write the domain and range of $f(x)$ and $f^{-1}(x)$

Solutions

a) The inverse of the exponential function $y = 2^x$ is $x = 2^y$ which can be written as $y = \log_2 x$. ✓

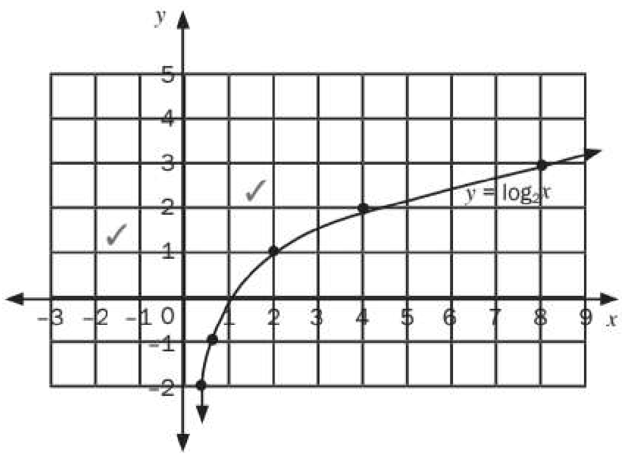
b) To plot the graph, use a table of values:

First make a table for $y =$

x	–2	–1	0	1	2	3
$y = 2^x$	¼	½	1	2	4	8

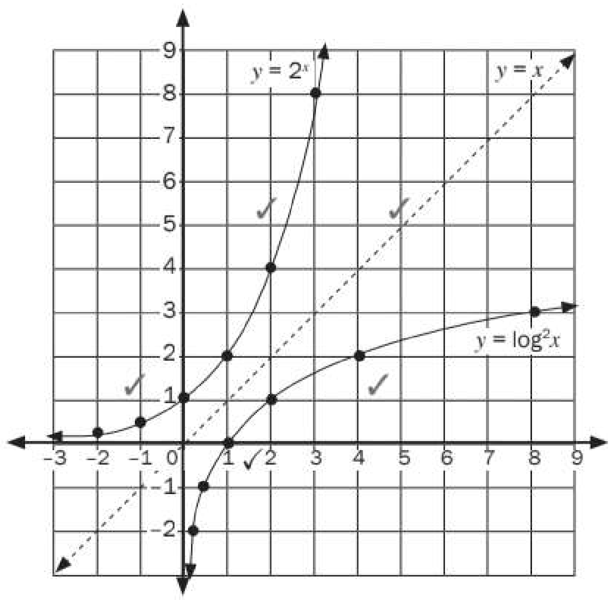
Make a table for $y = \log_2 x$

x	¼	½	1	2	4	8
$y = \log_2 x$	–2	–1	0	1	2	3



[3]

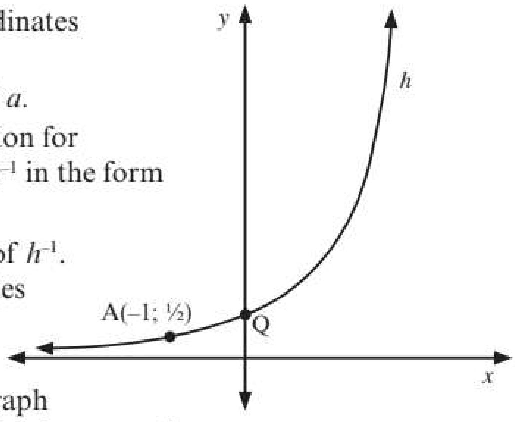
Let's compare the two graphs on the Cartesian plane.



The graph of $y = \log_2 x$ is a **reflection** about the $y = x$ axis of the exponential graph of $y = 2^x$.

Activity 11

The graph of $h(x) = a^x$ is sketched below. A(–1; ½) is a point on the graph of h.

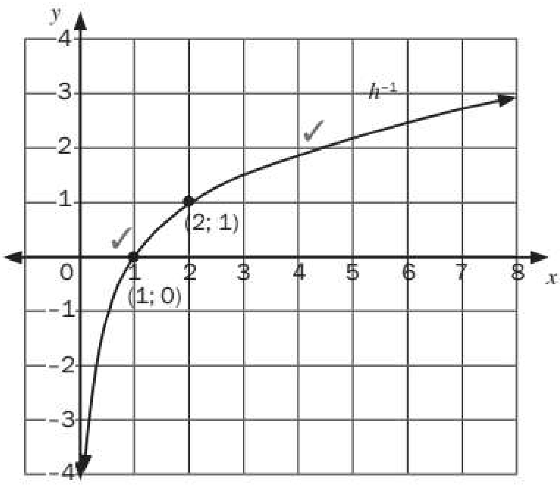


1. Explain why the coordinates of Q are (0; 1). (2)
2. Calculate the value of a. (2)
3. Write down the equation for the inverse function, h^{-1} in the form $y = \ldots$ (1)
4. Draw a sketch graph of h^{-1}. Indicate the coordinates of two points that lie on this graph. (2)
5. Read off from your graph the values of x for which $\log_2 x > -1$. (1)

[8]

Solutions

1. $h(0) = a^0 = 1$. ✓ Any base raised to the power of 0 is 1. ✓ (2)
2. $h(x) = a^x$ and A(–1; ½) so $a^{-1} = ½$✓

   $a^{-1} = 2^{-1}$ so $a = 2$✓ and $y = 2^x$ (2)
3. Interchange x and y, so $x = 2^y$ and $y = \log_2 x$ ✓ (1)
4. 

   (2)
5. $x > 0.5$ ✓ (1)

[8]