CHAPTER 4

ELEMENTARY NUMBER THEORY AND METHODS OF PROOF

4.2 Direct Proof and Counterexample II: Writing Advice

Directions for Writing Proofs of Universal Statements

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Over the years, the following rules of style have become fairly standard for writing the final versions of proofs:

1. **Copy the statement of the theorem to be proved on your paper.**
2. **Clearly mark the beginning of your proof with the word Proof.**
3. **Make your proof self-contained.**
4. **Write your proof in complete, grammatically correct sentences.**
5. **Keep your reader informed about the status of each statement in your proof.**

Directions for Writing Proofs of Universal Statements

6. Give a reason for each assertion in your proof.

7. Include the “little words and phrases” that make the logic of your arguments clear.

8. Display equations and inequalities in separate lines.

Variations among Proofs

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It is rare that two proofs of a given statement, written by two different people, are identical. Even when the basic mathematical steps are the same, the two people may use different notation or may give differing amounts of explanation for their steps or may choose different words to link the steps together into paragraph form.

Common Mistakes

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The following are some of the most common mistakes people make when writing mathematical proofs.

1. **Arguing from examples.**
2. **Using the same letter to mean two different things.**
3. **Jumping to a conclusion.**

Common Mistakes

4. Assuming what is to be proved.

To assume what is to be proved is a variation of jumping to a conclusion. As an example, consider the following "proof" of the fact that the product of any two odd integers is odd:

> Suppose m and n are any odd integers. When any odd integers are multiplied, their product is odd. Hence mn is odd.

Common Mistakes

5. **Confusion between what is known and what is still to be shown.**

   A more subtle way to jump to a conclusion occurs when the conclusion is restated using a variable.

6. **Use of *any* when the correct word is *some*.**

Common Mistakes

7. Misuse of the word *if*.

Another common error is not serious in itself, but it reflects imprecise thinking that sometimes leads to problems later in a proof. This error involves using the word *if* when the word *because* is really meant.

Example 4.2.1 – *An Odd Integer Minus an Even Integer*

Prove that the difference of any odd integer and any even integer is odd. Use only the definitions of odd and even and the Assumptions listed below, not any other properties of odd and even integers.

Assumptions

- In this text we assume a familiarity with the laws of basic algebra.
- We also use the three properties of equality: For all objects A, B, and C, (1) $A = A$, (2) if $A = B$, then $B = A$, and (3) if $A = B$ and $B = C$, then $A = C$.
- And we use the principle of substitution: For all objects A and B, if $A = B$, then we may substitute B wherever we have A.
- In addition, we assume that there is no integer between 0 and 1 and that the set of all integers is closed under addition, subtraction, and multiplication. This means that sums, differences, and products of integers are integers.

Example 4.2.1 – *Solution*

continued

Thus, the starting point for your proof would be something like, "Suppose a is any odd integer and b is any even integer," and the conclusion to be shown would be "We must show that $a - b$ is odd."

Common Mistakes

Theorem 4.2.1

The difference of any odd integer and any even integer is odd.

Example 4.2.2 – *Identifying a Mistake in a Proposed Proof*

Find the mistake in the following "proof."

Theorem: If *n* is any even integer, then $(-1)^n = 1$.

Proof:

1. Suppose *n* is any even integer. *[We must show that* $(-1)^n$ *=1.]*

2. By definition of even, *n* = 2*a* for some integer *a*.

3. Then	$(-1)^n = (-1)^{2a}$	by substitution
4.	$= ((-1)^a)^2$	by a law of exponents
5.	$= 1$	because any nonzero real number squared is positive.

Example 4.2.2 – *Solution*

This "proof" incorrectly jumps to a conclusion in line 5. Although it is true that the square of any nonzero real number is positive, it does not follow that the square of $(-1)^a$ is 1. Exercise 10 at the end of this section asks you to give a correct proof of this theorem.

Showing That an Existential Statement Is False

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We know that the negation of an existential statement is universal. It follows that to prove an existential statement is false, you must prove a universal statement (its negation) is true.

Example 4.2.3 – *Disproving an Existential Statement*

Show that the following statement is false:

There is a positive integer *n* such that $n^2 + 3n + 2$ is prime.

Example 4.2.3 – *Solution*

Proving that the given statement is false is equivalent to proving its negation is true. The negation is

For all positive integers *n*, $n^2 + 3n + 2$ is not prime.

Because the negation is universal, it is proved by generalizing from the generic particular.

Claim: The statement "There is a positive integer *n* such that $n^2 + 3n + 2$ is prime" is false.

Example 4.2.3 – *Solution*

continued

Proof:

Suppose *n* is any *[particular but arbitrarily chosen]* positive integer. *[We will show that* $n^2 + 3n + 2$ is not prime]

Factoring shows that

$$n^2 + 3n + 2 = (n + 1)(n + 2).$$

In addition, *n* + 1 and *n* + 2 are integers (because they are sums of integers), and both *n* + 1 > 1 and *n* + 2 > 1 (because *n* ≥ 1). Thus $n^2 + 3n + 2$ is a product of two integers each greater than 1, and so $n^2 + 3n + 2$ is not prime.