THE PARTIAL SUMS OF THE HARMONIC SERIES

The series
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$
is called harmonic, it diverges to infinity. Its partial sums
$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \quad n = 1, 2, 3, \ldots,$$
(*harmonic numbers*) form a monotone sequence increasing without bound.

The integral estimates
$$1 + \frac{1}{2} + \ldots + \frac{1}{n} > \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$
and
$$\frac{1}{2} + \ldots + \frac{1}{n} < \int_1^{n} \frac{dx}{x} = \ln n$$
are justified geometrically. Combined together, they give
$$\ln(n+1) < H_n < 1 + \ln n, \quad n > 1.$$
Therefore H_n tend to infinity at the same rate as $\ln n$, which is fairly slow. For instance, the sum of the first million terms is
$$H_{1000000} < 6 \ln 10 + 1 \approx 14.8.$$

Consider now the differences $\delta_n = H_n - \ln n$. Since
$$\ln(1 + \tfrac{1}{n}) < H_n - \ln n < 1, \quad n > 1,$$
we conclude that every δ_n is a positive number not exceeding 1.
Observe that
$$\begin{aligned} \delta_n - \delta_{n+1} &= (H_n - \ln n) - (H_{n+1} - \ln(n+1)) \\ &= \ln(n+1) - \ln n - \frac{1}{n+1} \\ &= \int_n^{n+1} \frac{dx}{x} - \frac{1}{n+1} > 0 \end{aligned}$$
(draw a picture to verify the last inequality). So $\delta_n > 0$ are monotone decreasing. By the Monotone Sequence Theorem, δ_n must converge as $n \to \infty$. The limit
$$\gamma = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} (H_n - \ln n)$$
is called the *Euler constant* (Euler, 1735), its value is about $\gamma \approx .5772$.

Thus, for large n, we have a convenient approximate equality
$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma.$$
It is not known to this day whether γ is rational or irrational.