Exercise 1:

a. *Proof that Gaussian distribution is normalized*

To prove that the univariate Gaussian distribution is normalized, we will first show that it is normalized for a zero-mean Gaussian and extend that result to show that $N(\mu, \sigma^2)$ is normalized.

The pdf of the zero-mean Gaussian distribution is given by:

$$\varphi(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}x^2\right) \quad -\infty < x < +\infty$$

To prove that the above expression is normalized, we have to show that:

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx = \sqrt{2\pi\sigma^2}$$

Let $I = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx$

Squaring the above expression, we have:

$$I = \iint_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2\right) dxdy \qquad (1)$$

We make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by:

$$x = rcos(\theta), y = rsin(\theta)$$

And using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we have $x^2 + y^2 = r^2$.

Also the Jacobian of the change of variables is given by:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = rcos^2\theta + rsin^2\theta = r$$

Using the same trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. Thus equation (1) can be rewritten as:

$$I^2 = \int_0^{2\pi}\int_0^{+\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) rdrd\theta$$

$$= 2\pi \int_0^{+\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r dr = 2\pi \int_0^{+\infty} 2\pi \int_0^{+\infty} \exp\left(-\frac{u}{2\sigma^2}\right) \frac{1}{2} du = 2\pi\sigma^2$$

Where we used the change of variables $r^2 = u$. Thus $I = (2\pi\sigma^2)^{\frac{1}{2}}$.

Finally to prove that $N(\mu, \sigma^2)$ is normalized, we make the transformation $y = x - \mu$ so that:

$$\int_{-\infty}^{+\infty} N(x|\mu, \sigma^2) dx = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = 1$$

as required.

Therefore, Gaussian distribution is normalized.

b. *Proof that expectation of Gaussian distribution is μ (mean)*

From the definition of expectation of continuous random variables, we have:

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

Substituting $t = \frac{x-\mu}{\sqrt{2}\sigma}$, we have:

$$E(X) = \frac{1}{\sqrt{\pi}} (\sqrt{2}\sigma \int_{-\infty}^{+\infty} t exp(-t^2) dt + \mu \int_{-\infty}^{+\infty} \exp(-t^2)\, dt = \frac{\mu\sqrt{\pi}}{\sqrt{\pi}} = \mu$$

Therefore, the expectation of Gaussian distribution is μ (mean).

c. *Proof that variance of Gaussian distribution is σ² (variance)*

We have:

$$var(X) = \int_{-\infty}^{+\infty} x^2 f_X(x) dx - (E(X))^2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx - \mu^2$$

Substituting $t = \frac{x-\mu}{\sqrt{2}\sigma}$, we have:

$$var(X) = \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\sqrt{2}\sigma t + \mu\right)^2 \exp(-t^2)\, dt - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} (2\sigma^2 \int_{-\infty}^{+\infty} t^2 \exp(-t^2)\, dt + 2\sqrt{2}\sigma\mu \int_{-\infty}^{+\infty} t \exp(-t^2)\, dt + \mu^2 \int_{-\infty}^{+\infty} \exp(-t^2)\, dt) - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} (2\sigma^2 \int_{-\infty}^{+\infty} t^2 \exp(-t^2)\, dt + 2\sqrt{2}\sigma\mu \int_{-\infty}^{+\infty} t \exp(-t^2)\, dt + \mu^2\sqrt{\pi}) - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} (2\sigma^2 \int_{-\infty}^{+\infty} t^2 \exp(-t^2)\, dt) + \mu^2 - \mu^2 = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} t^2 \exp(-t^2)\, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \int_{-\infty}^{+\infty} \exp(-t^2)\, dt = \frac{\sigma^2\sqrt{\pi}}{\sqrt{\pi}} = \sigma^2$$

Therefore, variance of Gaussian distribution is σ^2 (variance)

d. Proof that multivariate Gaussian distribution is normalized

Exercise 2:

a. Calculate the conditional of Gaussian distribution

Suppose x is a D-dimensional vector with Gaussian distribution $N(\mu, \Sigma)$ and that we partition x into two disjoint subsets x_a and x_b

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

We also define corresponding partitions of the mean vector μ given by:

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

And the covariance matrix Σ given by: $\Sigma = \begin{pmatrix} \Sigma_{\mathrm{aa}} & \Sigma_{\mathrm{ab}} \\ \Sigma_{\mathrm{ba}} & \Sigma_{\mathrm{bb}} \end{pmatrix}$

⇨ $A = \Sigma^{-1} = \begin{pmatrix} \mathrm{A_{aa}} & \mathrm{A_{ab}} \\ \mathrm{A_{ba}} & \mathrm{A_{bb}} \end{pmatrix}$

We are looking for conditional distribution $p(x_a|x_b)$.

We have:

$$-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) = -\frac{1}{2}(x-\mu)^T A(x-\mu)$$

$$= -\frac{1}{2}(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{ab}(x_b-\mu_b) \\ -\frac{1}{2}(x_b-\mu_b)^T A_{ba}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{bb}(x_b-\mu_b)$$

$$= -\frac{1}{2}x_a^T A_{aa}^{-1} x_a + x_a^T\big(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)\big) + const$$

By using Schur complement,

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} M & -MBD^{-1} \\ -D^{-1}CMD^{-1} & D^{-1}CMBD^{-1} \end{pmatrix}$, with $M = (A - BD^{-1}C)^{-1}$

We have:

$$A_{aa} = (\, \Sigma_{\mathrm{aa}} - \Sigma_{\mathrm{ab}}\Sigma_{\mathrm{bb}}^{-1}\Sigma_{\mathrm{ba}})^{-1}; A_{ab} = -(\Sigma_{\mathrm{aa}} - \Sigma_{\mathrm{ab}}\Sigma_{\mathrm{bb}}^{-1}\Sigma_{\mathrm{ba}})^{-1}\Sigma_{\mathrm{ab}}\Sigma_{\mathrm{bb}}^{-1}$$

As a result:

$$\mu_{a|b} = \mu_a + \Sigma_{\mathrm{ab}}\Sigma_{\mathrm{bb}}^{-1}(x_b-\mu_b);\ \Sigma_{a|b} = \Sigma_{\mathrm{aa}} - \Sigma_{\mathrm{ab}}\Sigma_{\mathrm{bb}}^{-1}\Sigma_{\mathrm{ba}}$$

⇨ $p(x_a|x_b) = N(x_{a|b}|\mu_{a|b}, \Sigma_{a|b})$

b. *Calculate the marginal of Gaussian distribution*

The marginal distribution given by:

$$p(x_a) = \int p(x_a, x_b)dx_b$$

We need to integrate out x_b by looking the quadratic form related to x_b

$$-\frac{1}{2}x_b^T A_{bb}x_b + x_b^T m = -\frac{1}{2}(x_b - A_{bb}^{-1}m)^T A_{bb}(x_b - A_{bb}^{-1}m) + \frac{1}{2}m^T A_{bb}^{-1}m$$

with $m = A_{bb}\mu_b - A_{ba}(x_a-\mu_a)$

We can integrate over innormalized Gaussian

$$\int \exp\{-\frac{1}{2}(x_b - A_{bb}^{-1}m)^T A_{bb}(x_b - A_{bb}^{-1}m)\}dx_b$$

The remaining term:

$$-\frac{1}{2}x_a^T(A_{aa} - A_{ab}A_{bb}^{-1}A_{ba})x_a + x_a^T(A_{aa} - A_{ab}A_{bb}^{-1}A_{ba})^{-1}\mu_a + const$$

Similarly, we have: $E[x_a] = \mu_a; cov[x_a] = \Sigma_{aa}$

⇨ $p(x_a) = N(x_a|\mu_a, \Sigma_{aa})$