Homework 3 Machine Learning

October 5, 2022

Exercise 4:

We know that when the rows and columns are independent, then matrix A would be invertible and would have non-zero determinant.
We have an $n \times m$ matrix X
If n = m, It is enough for check $det(X) \neq 0$
if $n \neq m$. Precisely, When the rank of X is m (Which forces $n \geq m$)
For v $\in$ Rm, Xv = 0 if and only if $X^T Xv = 0$.
For the non-trivial implication. If $X^T Xv = 0$ then $v^T X^T Xv = 0$, that is $(Xv)^T Xv = 0$, Which implies that $Xv = 0$.
If rank(X) = m, this mean that X is one-to-one when acting on Rm.
So by the observation, $X^T X$ is one-to-one, Which makes it Invertible (as it is square).
Conversely, If rank(X) $<$ m there exist $v \in Rm$ with Xv = 0, and $X^T X$ can not Invertible.