Suppose that we have a dataset $X = \{x_1, x_2, x_3, \dots, x_N\}, x_i \in\ R^*$

We want to decrease X to M – dimension

$$\Rightarrow X = \begin{bmatrix} x_1^T \\ x_2^T \\ \dots \\ x_N^T \end{bmatrix} \in\ R^{N\times D}$$

The coefficient matrix is: $B = [b_1 \quad b_2 \dots \quad b_M] \in\ R^{D\times M}$

$$\Rightarrow Z = X \times B = \begin{bmatrix} x_1^T \\ x_2^T \\ \dots \\ x_N^T \end{bmatrix} \times [b_1 \quad b_2 \dots \quad b_M] \in\ R^{N\times M}$$

$$\Rightarrow Z = \begin{bmatrix} x_1^T b_1 & \cdots & x_1^T b_M \\ \vdots & \ddots & \vdots \\ x_N^T b_1 & \cdots & x_n^T b_M \end{bmatrix}$$

Assumption that mean of X: $M_X = \frac{x_1 + x_2 + \dots + x_N}{N} = 0$

$$\Rightarrow M_1 = \frac{x_1^T b_1 + x_2^T b_1 + \dots + x_N^T b_1}{N} = \frac{b_1^T \sum_1^N x_i}{N} = 0$$

$$\Rightarrow var(Z) = \frac{\sum_1^N \left(x_i^T b_1 - M_1\right)^2}{N} = \frac{\sum_1^N \left(x_i^T b_1\right)^2}{N} = \frac{\sum_1^N b_1^T x_i x_i^T b_1}{N} = b_1^T \frac{\sum_1^N x_i^T}{N} b_1$$

Let $S = \frac{\sum_1^N x_i^T}{N}$

Now we need to maximize $b_1^T S b_1$ constrain $\|b_1\|_2^2 = 1$

The Langrange multipliers: $L = b_1^T S b_1 + \alpha\ (1 - b_1^T b_1)$

- $\frac{\nabla L}{\nabla b_1} = 0 \Leftrightarrow 2Sb_1 - 2\alpha\ b_1 = 0 \Leftrightarrow Sb_1 = \alpha\ b_1$

  $$\Rightarrow b_1\ is\ an\ eigenvector\ of\ S, \alpha\ is\ an\ eigenvalue\ of\ S$$

- $\frac{\nabla L}{\nabla \alpha} = 0 \Leftrightarrow b_1^T b_1 = 1$

$$\Rightarrow var = b_1^T S b_1 = b_1^T \alpha b_1 = \alpha b_1^T b_1 = \alpha$$