Lê Đức Khải

Mathematics for Medical Engineering

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CONTENTS:

1. Integration

1.1. Primitive function table

Differentiation	Integration
$(cu)' = cu'$ (c constant)	$\int uv'\,dx = uv - \int u'v\,dx$ (by parts)
$(u+v)' = u' + v'$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
$(uv)' = u'v + uv'$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$
$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx}$ (Chain rule)	$\int \sin x\,dx = -\cos x + c$
	$\int \cos x\,dx = \sin x + c$
	$\int \tan x\,dx = -\ln\lvert\cos x\rvert + c$
$(x^n)' = nx^{n-1}$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + c$
$(e^x)' = e^x$	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + c$
$(e^{ax})' = ae^{ax}$	$\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert + c$
$(a^x)' = a^x \ln a$	$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$
$(\sin x)' = \cos x$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$
$(\cos x)' = -\sin x$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arcsinh}\frac{x}{a} + c$
$(\tan x)' = \sec^2 x$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arccosh}\frac{x}{a} + c$
$(\cot x)' = -\csc^2 x$	$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$
$(\sinh x)' = \cosh x$	$\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$
$(\cosh x)' = \sinh x$	$\int \tan^2 x\,dx = \tan x - x + c$
$(\ln x)' = \frac{1}{x}$	$\int \cot^2 x\,dx = -\cot x - x + c$
$(\log_a x)' = \frac{\log_a e}{x}$	$\int \ln x\,dx = x\ln x - x + c$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$	$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$
$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$	$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$
$(\arctan x)' = \frac{1}{1+x^2}$	
$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$	

1.2. Girthed area, volume of revolution and length of the curve

- Length of the curve: $l = L_a^b(f) = \int_a^b \sqrt{1 + f'(x)^2}\,dx$
- Volume of revolution: $V = \pi\int_a^b f(x)^2 dx$
- Girthed area: $M = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}dx$

1.3. Improper integral

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to\infty}\lim_{b\to-\infty}\int_b^a f(x)dx$$

1.4. Line integral

- Use t as variable: $\oint \vec{F} d\vec{r} = \int_{t_1}^{t_2} (\vec{F}.\dot{\vec{r}})\, dt$

  Ex: Given is $\vec{F}(x;y) = \begin{pmatrix} 2y - x \\ 2 + x \end{pmatrix}$ and $\vec{r}: t \to \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

  The line integral is calculated as:

  - $\vec{F}(\vec{r}(t)) = \vec{F}(x = t; y = t^2) = \begin{pmatrix} 2t^2 - t \\ 2 + t \end{pmatrix}$
  - $\dot{\vec{r}}(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$
  - $\oint \vec{F} d\vec{r} = \int_{t_1}^{t_2} (\vec{F}.\dot{\vec{r}})\, dt = \int_{t=0}^{2} \begin{pmatrix} 2t^2 - t \\ 2 + t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt = \int_{t=0}^{2} (4t^2 + 3t) dt = \frac{50}{3}$

- Use x as variable:

  $$\oint \vec{F} d\vec{r} = \oint \left(F_x(x;y)dx + F_y(x;y)dy\right) = \int_{x_1}^{x_2} \left[F_x\left(x; f(x)\right) + F_y\left(x; f(x)\right). f'(x)\right] dx$$

- Use additional function V: $\oint \vec{F} d\vec{r} = V(\vec{E}) - V(\vec{A})$

  within: $\vec{F}(x;y) = (X;Y)$ and $\frac{dV}{dx} = X;\ \frac{dV}{dy} = Y$

  Ex: Given is $\vec{F}(x;y) = \begin{pmatrix} x + y \\ 1 + x \end{pmatrix}$; $\vec{E} = (2;4)$; $\vec{A} = (0;0)$

  The line integral is calculated as:

  - $\frac{dV}{dx} = x + y \Rightarrow V(x;y) = \int (x + y) dx = \frac{x^2}{2} + yx + c_1(y)$
  - $\frac{dV}{dy} = 1 + x \Rightarrow V(x;y) = \int (1 + x) dy = y + yx + c_2(x)$
  - $\frac{x^2}{2} + yx + c_1(y) = y + yx + c_2(x)$

    $\Leftrightarrow \frac{x^2}{2} + c_1(y) = y + c_2(x)$

    $\Rightarrow c_1(y) = y$ and $c_2(x) = \frac{x^2}{2}$
  - $V(x;y) = \frac{x^2}{2} + yx + y$
  - $\oint \vec{F} d\vec{r} = V(x = 2; y = 4) - V(x = 0; y = 0) = 14$

1.5. Multiple integral

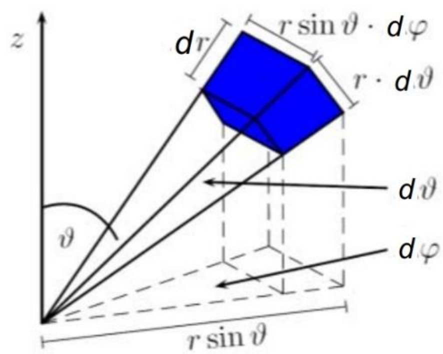
- Double integral: $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x;y) dy dx = \int_A f dA$
- Centre point of surface: $x_S = \frac{1}{A} \int_A x dA$

  $y_S = \frac{1}{A} \int_A y dA$
- Triple integral: $\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x;y;z) dz dy dx = \int_V f dV$
- Centre point of volume: $x_S = \frac{1}{V} \int_V x dV$

  $y_S = \frac{1}{V} \int_V y dV$

  $z_S = \frac{1}{V} \int_V z dV$

- Integral in polar coordinates:
    - $dxdy = r.dr.d\varphi$
    - $x = \cos\varphi\,.r$
    - $y = \sin\varphi\,.r$
    - $\int_R f dA = \int_{\varphi=a}^{\varphi=b}\int_{r=0}^{r(\varphi)} f(r;\ \varphi).r.drd\varphi$
- Integral in spherical coordinates:
    - $x = r\cos\varphi\ .\sin\vartheta$
    - $y = r\sin\varphi\,.\sin\vartheta$
    - $z = r\cos\vartheta$
    - $\int f d(x;y;z) = \int_r\int_\varphi\int_\vartheta f(r;\varphi;\vartheta).r^2.\sin\vartheta\ d\vartheta d\varphi dr$

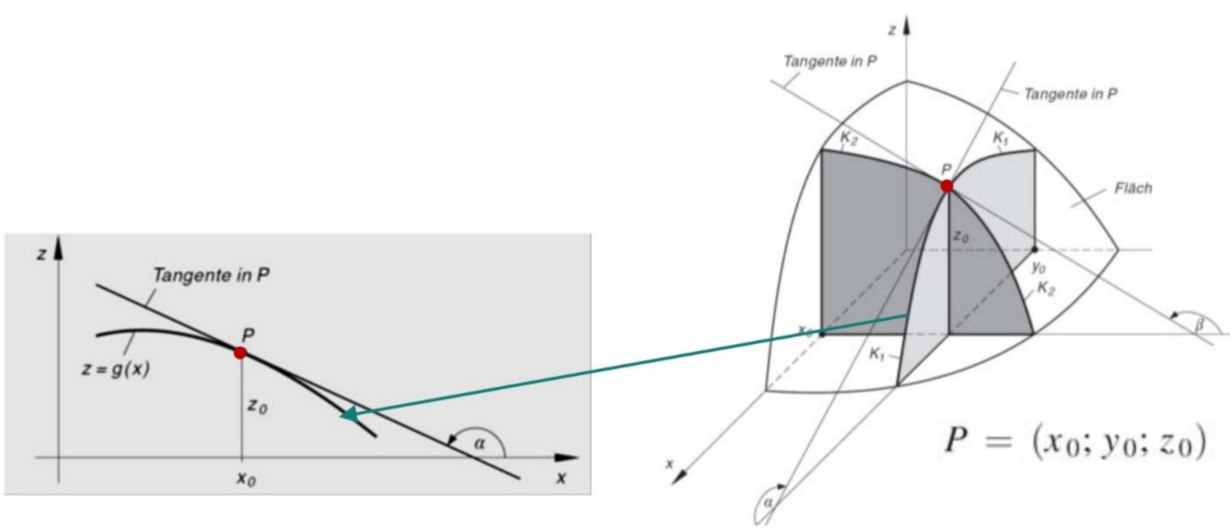


2. Differential in multiple variables

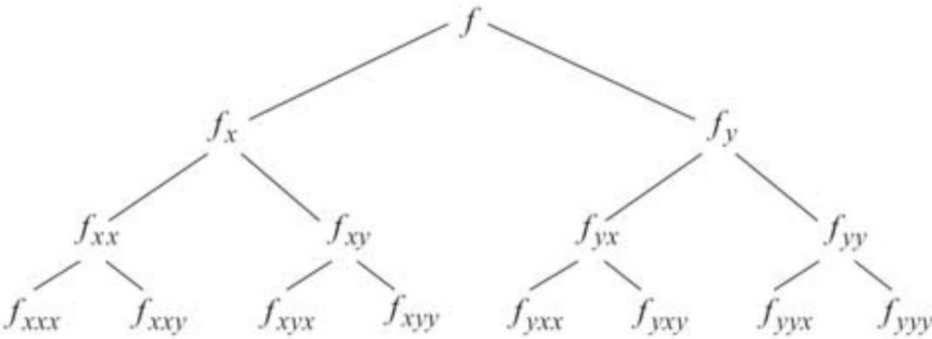
2.1. Partial derivative

- Given is a point $P(x_0;y_0;z_0) \in z = f(x;y)$. Two tangents at P possess two slopes described as:

$$m = \tan\alpha = f_x(x_0;y_0)$$
$$n = \tan\beta = f_y(x_0;y_0)$$



- Hierarchy of partial derivatives:



- Gradient:
  - $\nabla f(\vec{x}) = grad\ f = \begin{pmatrix} f_x(x;y;z) \\ f_y(x;y;z) \\ f_z(x;y;z) \end{pmatrix}$
  - $grad\ (af + bg) = a.grad\ f + b.grad\ g$
  - $grad\ (f.g) = g.grad\ f + f.grad\ g$

2.2.Directional derivative

- $D_{\vec{n}}\, f(\,\overrightarrow{r_0}\,) = \lim\limits_{h \to 0} \frac{f(\overrightarrow{r_0} + h\vec{v}) - f(\overrightarrow{r_0})}{h}$
- $D_{\vec{n}}\, f(\,\overrightarrow{r_0}\,) = grad\ f.\vec{n} = grad\ f.\frac{\vec{v}}{|\vec{v}|} = |grad\ f|.\cos\varphi$

  ($\vec{n}$ is unit vector)
- Maximum: $(\widehat{grad\ f;\vec{n}}) = 0°$

  Minimum: $(\widehat{grad\ f;\vec{n}}) = 180°$

2.3. Taylor series expansion

- Original Taylor series:
  - $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$
  - $f(x) = \sum_{n=0}^{m} a_n(x - x_0)^n + R_m(x)$
- Taylor series in partial derivative:

$$f(x;y) = f(x_0;y_0) + f_x(x_0;y_0)(x - x_0) + f_y(x_0;y_0)(y - y_0) + \frac{f_{xx}(x_0;y_0)}{2}(x - x_0)^2 + \frac{f_{yy}(x_0;y_0)}{2}(y - y_0)^2 + f_{xy}(x_0;y_0)(x - x_0)(y - y_0) + \ldots$$

- Tangent plane: m = 1

$$f_1(x;y) = f(x_0;y_0) + f_x(x_0;y_0)(x - x_0) + f_y(x_0;y_0)(y - y_0)$$

2.4. Total differential

- Total differential equation:

$$f(x;y) - f(x_0;y_0) = f_x(x_0;y_0)(x - x_0) + f_y(x_0;y_0)(y - y_0)$$
$$\Rightarrow dz = f_x(x_0;y_0).dx + f_y(x_0;y_0).dy$$

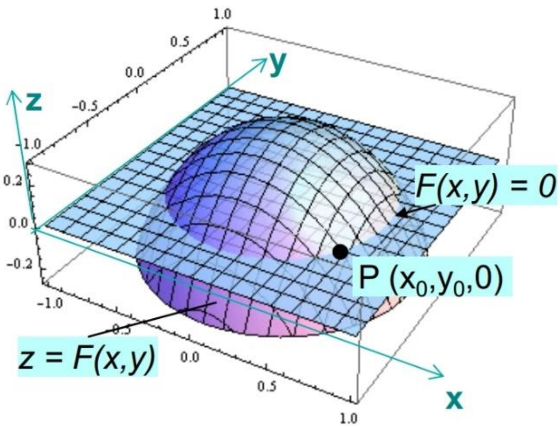
- Application in error calculation:

$$\Delta z_{max} \leq |f_{x_1}|.|\Delta x_1| + |f_{x_2}|.|\Delta x_2| + \cdots + |f_{x_n}|.|\Delta x_n|$$

- Application in implicit differentiation:

The tangent at point $P(x_0; y_0; 0) \in F(x; y) = 0$ has a slope coefficient:

$$y' = -\frac{F_x(x_0; y_0)}{F_y(x_0; y_0)}$$



2.5. Optimization problem without additional condition

- Local extrema of a multivariable function $z = f(x; y)$:
  - $P(x_0; y_0; z_0) \in z = f(x; y)$ is a extremum $\Leftrightarrow f_x(x_0; y_0) = 0 \wedge f_y(x_0; y_0) = 0$
  - Hesse matrix: $H = \begin{pmatrix} f_{xx}(x_0; y_0) & f_{xy}(x_0; y_0) \\ f_{xy}(x_0; y_0) & f_{yy}(x_0; y_0) \end{pmatrix}$
  - Determinant: $d = \det H = f_{xx}(x_0; y_0) f_{yy}(x_0; y_0) - f_{xy}^2(x_0; y_0)$
  - H is negative definite $\Leftrightarrow f_{xx}(x_0; y_0) < 0 \wedge \det H > 0 \Rightarrow$ P is local maximum

    H is positive definite $\Leftrightarrow f_{xx}(x_0; y_0) > 0 \wedge \det H > 0 \Rightarrow$ P is local minimum

    H is indefinite $\Leftrightarrow \det H < 0 \Rightarrow$ P is saddle point

2.6. Optimization problem with additional condition

- Local extrema of a multivariable function $z = f(x; y)$ accompanied by $g(x; y) = 0$:
  - Lagrange parameter λ is identified as: $grad\ f + \lambda . grad\ g = 0$
  - Lagrange function:

    $L(x; y; \lambda) = f(x; y) + \lambda g(x; y)$

    $g(x; y) = 0 \Rightarrow L(x; y; \lambda) = f(x; y)$
  - $grad\ L = \vec{0} \Rightarrow L_x = 0; L_y = 0; L_\lambda = 0 \Rightarrow P(x; y; \lambda)$ is a candidate
  - Hesse matrix: $H = \begin{pmatrix} L_{xx} & L_{xy} & L_{x\lambda} \\ L_{xy} & L_{yy} & L_{y\lambda} \\ L_{x\lambda} & L_{y\lambda} & L_{\lambda\lambda} \end{pmatrix}$
  - $\det H > 0 \Rightarrow$ P is local maximum

    $\det H < 0 \Rightarrow$ P is local minimum

    $\det H = 0 \Rightarrow$ no results

3. Ordinary differential equation (ODE)

3.1. First-order linear ordinary differential equation

- Separation of variables: $y' = f(x).g(y)$
  - Let $h(y) = \frac{1}{g(y)}$, so we gain: $\frac{1}{g(y)}y' = f(x) \Rightarrow h(y).y' = f(x)$
  - $h(y).\frac{dy}{dx} = f(x) \Rightarrow \int h(y)dy = \int f(x)dx$
  - $H\big(y(x)\big) = F(x) + c$
  - $y(x) = H^{-1}(F(x) + c)$

- Substitution case 1: $y' = f(ax + b.y(x) + c)$
  - Substitution step: $z(x) = ax + b.y(x) + c \Rightarrow y = \frac{z(x) - ax - c}{b}$ (*)
  - $y'(x) = \frac{1}{b}.(z'(x) - a) \Rightarrow \frac{1}{b}.(z'(x) - a) = f\big(z(x)\big) \Rightarrow z'(x) = a + b.f(z)$
  - Then we gain $z(x)$ in terms of x
  - From (*), we gain y in terms of x

- Substitution case 2: $y' = f\left(\frac{y}{x}\right)$
  - Substitution step: $z(x) = \frac{y}{x} \Rightarrow y(x) = z(x).x$ (**)
  - $y' = z(x) + z'(x).x \Rightarrow z(x) + z'(x).x = f(z) \Rightarrow z'(x) = \frac{f(z) - z(x)}{x}$
  - Then we gain $z(x)$ in terms of x
  - From (**), we gain y in terms of x

- Homogeneous linear differential equation: $y' + f(x).y = 0$
  - We quickly got result: $y = c.e^{\int -f(x)dx}$

- Inhomogeneous linear differential equation: $y' + f(x).y = g(x), (g(x) \neq 0)$ (***)
  - Let $y' + f(x).y = 0$, we find out: $y = c.e^{\int -f(x)dx}$
  - Replace c by $c(x)$, then: $y = c(x).e^{\int -f(x)dx}$
  - Differentiate both sides, then we gain y' in terms of $c(x)$ and y
  - Compare y' gained above with y' from (***), we find out $c(x)$ in terms of contant c_1
  - Solve y in terms of contant c_1

- Inhomogeneous linear differential equation with constant coefficient: $y' + a.y = g(x)$
  - Foundation of Lagrange method:
    - $y = y_h + y_p$ is the result of equation: $y' = a.y + g(x)$
    - y_h is the result of homogeneous equation: ${y'}_h + a.y_h = 0$
    - y_p is the result of equation ${y'}_p + a.y_p = g(x)$ and is deducted with the table below

Störfunktionstyp	Störfunktion $g(x)$	Ansatz für y_p
Konstante	k_0	c_0
Linear	$k_0 + k_1 x$	$c_0 + c_1 x$
Polynom	$\sum_{i=0}^{n} k_i x^i$	$\sum_{i=0}^{n} c_i x^i$
Exponentiell	$k \cdot e^{bx}; b \neq -a$	$c_0 \cdot e^{bx}$
	$k \cdot e^{-ax}$	$c_0 \cdot x e^{-ax}$
Trigonometrisch	$k \cdot sin(bx) + l \cdot cos(bx);$	$c_0 \cdot sin(bx) + c_1 \cdot cos(bx)$

***b** – bekannt aus Störfunktion* ***c_0, c_1** – gesucht*

- Solution using Lagrange method:
  - $y'_h + a.y_h = 0 \Rightarrow y_h = c.e^{-ax}$
  - Insert y_p found in formular table into $y'_p + a.y_p$, then compare it with $g(x)$ to find out unknown parameters c_0; c_1;…
  - Insert found-out parameters back into y_p
  - The result of the equation would be: $y = y_h + y_p$

3.2. Second-order linear ordinary differential equation

- Homogeneous linear differential equation with constant coefficient: $y'' + ay' + by = 0$
  - Let $y = e^{\alpha x} \Rightarrow \alpha^2 + a\alpha + b = 0$
    - If $\alpha_1 \neq \alpha_2$ $(\Delta > 0) \Rightarrow y_h = \lambda_1 e^{\alpha_1 x} + \lambda_2 e^{\alpha_2 x}$
    - If $\alpha_1 = \alpha_2 = \alpha_0 (\Delta = 0) \Rightarrow y_h = (\lambda_1 + \lambda_2 x).e^{\alpha_0 x}$
    - If $\alpha_{1,2} = w \pm vi (\Delta < 0) \Rightarrow y_h = e^{wx}(\lambda_1 \cos vx + \lambda_2 \sin vx)$

Störfunktion $g(x)$	Lösungsansatz $y_p(x)$
1. Polynomfunktion vom Grad n $g(x) = P_n(x) = \sum_{i=0}^{n} k_i x^i$	$y_p = \begin{cases} Q_n(x) & b \neq 0 \\ x \cdot Q_n(x) \quad f\ddot{u}r & a \neq 0, \quad b = 0 \\ x^2 \cdot Q_n(x) & a = b = 0 \end{cases}$ $\boxed{Q_n(x) = \sum_{i=0}^{n} c_i x^i}$ $Q_n(x)$: Polynom vom Grad n $Parameter$: Koeffizienten des Polynoms $Q_n(x)$
2. Exponentialfunktion $g(x) = e^{\sigma x}$	$y_p = \begin{cases} A \cdot e^{\sigma x}, \text{ falls: } \sigma \neq \alpha_{1,2} \\ \quad (\sigma \text{ keine Lösung der charakt. Gleichung}) \\ Ax \cdot e^{\sigma x}, \text{ falls: } \sigma = \alpha_1 \neq \alpha_2 \\ \quad (\sigma \text{ einfache Lösung der charakt. Gleichung}) \\ Ax^2 \cdot e^{\sigma x}, \text{ falls: } \sigma = \alpha_1 = \alpha_2 \\ \quad (\sigma \text{ zweifache Lösung der charakt. Gleichung}) \end{cases}$ $Parameter : A$
3. Trigonom. Funktion $g(x) = \cos(\omega x)$ oder $g(x) = \sin(\omega x)$ oder Überlagerung	$y_p = \begin{cases} A \cdot \cos(\omega x) + B \cdot \sin(\omega x), \text{ falls} \\ \quad i\omega \text{ - keine Lösung der charakt. Gleichung} \\ x \cdot [A \cdot \cos(\omega x) + B \cdot \sin(\omega x)], \text{ falls} \\ \quad i\omega \text{ - eine Lösung der charakt. Gleichung} \end{cases}$ oder entsprechend $y_p = \begin{cases} A \cdot \sin(\omega x + \varphi) \\ Ax \cdot \sin(\omega x + \varphi) \end{cases}$ $Parameter : A, B$ oder A, φ

- Inhomogeneous linear differential equation with constant coefficient: $y'' + ay' + by = g(x)$
  - Find y_h as homogeneous linear differential equation: $y'' + ay' + by = 0$
  - Insert y_p found in formular table into $y''_p + ay'_p + by_p$, then compare it with $g(x)$ to find out unknown parameters
  - Insert found-out parameters back into y_p
  - The result of the equation would be: $y = y_h + y_p$

3.3. Laplace transform applied to differential equation

- $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t).e^{-st}\,dt$
- $\mathcal{L}^{-1}\{F(s)\} = f(t)$
- $\mathcal{L}\{f^{(n)}(t)\} = s^n.F(s) - s^{n-1}.f(0) - s^{n-2}.f'(0) - \ldots - f^{(n-1)}(0)$

	Bildfunktion $F(s)$	Originalfunktion $f(t)$
(1)	1	$\delta(t)$
(2)	$\frac{1}{s}$	1 (Sprungfunktion $\sigma(t)$)
(3)	$\frac{1}{s-a}$	e^{at}
(4)	$\frac{1}{s^2}$	t
(5)	$\frac{1}{s(s-a)}$	$\frac{e^{at}-1}{a}$
(6)	$\frac{1}{(s-a)^2}$	$t \cdot e^{at}$
(7)	$\frac{1}{(s-a)(s-b)}$	$\frac{e^{at}-e^{bt}}{a-b}$
(8)	$\frac{s}{(s-a)^2}$	$(1+at) \cdot e^{at}$
(9)	$\frac{s}{(s-a)(s-b)}$	$\frac{a \cdot e^{at} - b \cdot e^{bt}}{a-b}$
(10)	$\frac{1}{s^3}$	$\frac{1}{2}t^2$
(11)	$\frac{1}{s^2(s-a)}$	$\frac{e^{at}-at-1}{a^2}$

	Bildfunktion $F(s)$	Originalfunktion $f(t)$
(12)	$\frac{1}{s(s-a)^2}$	$\frac{(at-1) \cdot e^{at}+1}{a^2}$
(13)	$\frac{1}{(s-a)^3}$	$\frac{1}{2}t^2 \cdot e^{at}$
(14)	$\frac{s}{(s-a)^3}$	$\left(\frac{1}{2}at^2 + t\right) \cdot e^{at}$
(15)	$\frac{s^2}{(s-a)^3}$	$\left(\frac{1}{2}a^2t^2 + 2at + 1\right) \cdot e^{at}$
(16)	$\frac{1}{s^n}$ $(n = 1, 2, 3, \ldots$	$\frac{t^{n-1}}{(n-1)!}$
(17)	$\frac{1}{(s-a)^n}$ $(n = 1,$	$\frac{t^{n-1} \cdot e^{at}}{(n-1)!}$
(18)	$\frac{1}{s^2+a^2}$	$\frac{\sin(at)}{a}$
(19)	$\frac{s}{s^2+a^2}$	$\cos(at)$
(20)	$\frac{(\sin b) \cdot s + a \cdot \cos b}{s^2+a^2}$	$\sin(at+b)$
(21)	$\frac{(\cos b) \cdot s - a \cdot \sin b}{s^2+a^2}$	$\cos(at+b)$

- Solve a differential equation: $y' + ay = g(t)$ with initial value $y(0) = k$
  - $\mathcal{L}(y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = F(s)$
    $\mathcal{L}\{y'(t)\} = s.Y(s) - y(0)$
  - $\Rightarrow (s+a)Y(s) = F(s) + y(0)$
    $\Rightarrow Y(s) = \frac{F(s)+y(0)}{s+a}$
  - $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)+y(0)}{s+a}\right\}$