Lab 01 - Basic Principle of Counting

Trần Lương Quốc Đại
tlqdai@it.tdt.edu.vn

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1. Permutation

In mathematics, a permutation of a set is an ordered arrangement of its members into a sequence. Let S is a finite set of size n objects. Selecting a sample of size $k \leq n$ from a set S. There are two types of permutation:

- **Permutations with Repetition:** when each element in set S may occur once, twice, thrice, ... upto k times in any arrangement. First, there are n possibilities for the first choice, then there are n possibilites for the second choice, and so on, multplying each time. So, the number of permutations is $n \times n \times$ ...(k times) $= n^k$.

  For example, a code have 4 digits in a specific order, the digits are between 1-4. How many different permutations can be made? There are 4 possibilites for the first digit, second digit and the last digit. So, the number of ways is $4 \times 4 \times 4 = 64$ codes:

  (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 2, 4),
  (1, 3, 1), (1, 3, 2), (1, 3, 3), (1, 3, 4), (1, 4, 1), (1, 4, 2), (1, 4, 3), (1, 4, 4),
  (2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 1, 4), (2, 2, 1), (2, 2, 2), (2, 2, 3), (2, 2, 4),
  (2, 3, 1), (2, 3, 2), (2, 3, 3), (2, 3, 4), (2, 4, 1), (2, 4, 2), (2, 4, 3), (2, 4, 4),
  (3, 1, 1), (3, 1, 2), (3, 1, 3), (3, 1, 4), (3, 2, 1), (3, 2, 2), (3, 2, 3), (3, 2, 4),
  (3, 3, 1), (3, 3, 2), (3, 3, 3), (3, 3, 4), (3, 4, 1), (3, 4, 2), (3, 4, 3), (3, 4, 4),
  (4, 1, 1), (4, 1, 2), (4, 1, 3), (4, 1, 4), (4, 2, 1), (4, 2, 2), (4, 2, 3), (4, 2, 4),
  (4, 3, 1), (4, 3, 2), (4, 3, 3), (4, 3, 4), (4, 4, 1), (4, 4, 2), (4, 4, 3), (4, 4, 4)

```python
import itertools
E = [1, 2, 3, 4]
k = 3
P = [p for p in itertools.product(E, repeat=k)]
print(P)
print("There are ", len(P), "codes.")
```

- **Permutations without Repetition:** when each element in set S that occurred only once time in any arrangement. It can also be represented as:

$$P_n^k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

For instance, let set $E = \{a, b, c, d\}$. How many different ordered arrangements of three could be selected from the five elements if E? By direct enumeration we see that there are 24 ways:

(a,b,c), (a,c,b), (b,a,c), (b,c,a), (c,a,b), (c,b,a),
(a,b,d), (a,d,b), (b,a,d), (b,d,a), (d,a,b), (d,b,a),
(a,c,d), (a,d,c), (c,a,d), (c,d,a), (d,a,c), (d,c,a),
(b,c,d), (b,d,c), (c,b,d), (c,d,b), (d,b,c), (d,c,b).

The number of permutations is $P_4^3 = 4.3.2 = \frac{4!}{(4-3)!} = 24$.

```python
import itertools

E = {'a', 'b', 'c', 'd'}
k = 3
# Print all the k-permutations of E
n = len(E)
permute_k = list(itertools.permutations(E, k))
print("%i-permutations of %s:  " %(k,E))
for i in permute_k:
    print(i)
print("Size = ", "%i!/(%i-%i)! = " %(n,n,k), len(permute_k))
```

2. Combination

Unlike permutation, a combination is a selection of items from a collection, such that the order of selection does not matter. A k-combination of a set S is a subset of k distinct elements of S, the number of k-combinations (denoted by C_n^k) is equal to the binomial coefficient $\binom{n}{k}$:

$$C_n^k = \binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 1} = \frac{n!}{k!(n-k)!}$$

whenever $k \leq n$, and which is zero when $k > n$.

For instance, let set $E = \{a, b, c, d\}$. How many different groups of three could be selected from these elements? By direct enumeration we see that there are 4 ways:

(a,b,c), (a,b,d), (a,c,d), (b,c,d)

The number of combination is $C_4^3 = \frac{4!}{3!(4-3)!} = 4$.

```python
import itertools

E = {'a', 'b', 'c', 'd'}
k = 3
# Print all the k-combinations of E
choose_k = list(itertools.combinations(E,k))
print("%i-combinations of %s:  " %(k,E))
for i in choose_k:
    print(i)
print("Number of combinations = %i!/(%i!(%i-%i)!) = %i" %(n,
    k,n,k,len(choose_k)))
```

3. Urn problems

A urn contains 8 white balls, 6 black balls and 9 red balls. Six balls are “randomly draw” from the urn.

(a) How many ways 6 balls can be selected?

(b) Enumerate all possible cases of 6 balls.

(c) Enumerate all possible cases of the first and the last ball should be red.

Firstly, we will present a white ball by a sequence Wi ($i \in N$) with W for white color and i for order of the ball. So, 8 white balls are ‘W1’, ‘W2’, ..., ‘W8’. Similarly 6 black balls are ‘B1’ to ‘B6’ and 9 red balls are ‘R1’ to ‘R9’.

```python
def cross(A, B):
    '''The set of ways of concatenating one item from
    collection A with one from B.'''
    return {a + b for a in A for b in B}

urn = cross('W', '12345678') | cross('B', '123456') | cross(
    'R', '123456789')

print(urn)
```

Each event is a set of 6 balls are “randomly draw” from the urn, so the sample space will be the set of all possible cases of 6 balls. Each of the 6 balls may be included in any of the combinations any number of times.

$$U6 = C_{23}^{6} = \frac{23.22.21.20.19.18}{6!} = 100947$$

```python
import itertools
U6 = list(itertools.combinations(urn, 6))
```

Solution for (a):

```python
print(len(U6))
```

Solution for (b):

```python
for s in U6:
    print(s)
```

Solution for (c):

```python
for s in U6:
    if s[0][0]=='R' and s[-1][0]=='R':
        print(s)
```

4. Exercises

1. A code have 4 digits in a specific order, the digits are between 0-9. How many different permutations are there if one digit may only be used once? Assign list of the code to variable *num_code*, and number of the numbers to variable *code_length*.

2. Let set $A = \{1, 2, 3, 4, 5\}$. Find all of 3-digit numbers can be formed from the digits in set A without repetitions. Assign list of the numbers to variable *num_3_digit*, and number of the numbers to variable *num_length*.

3. A urn contains 8 white balls, 6 black balls and 9 red balls. Three balls are "randomly draw" from the urn.

   (a) Find a list of all possible 3 balls and assign to variable $U3$. Assign number of set $U3$ to variable *len_U3*.

   (b) Enumerate all possible cases of three balls of different colors.

   (c) Enumerate all possible cases of the first ball being white and the second ball red.

4. Mr. Holmes has 10 books consisting of 4 mathematics, 3 physics and 2 chemistry and 1 on language. He is going to put them on his bookshelf so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible? Print these arrangements to screen.

5. A committee of size 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, how many different selections can the committee consist of 3 men and 2 women? Print these selections to screen.

6. A standard deck of playing cards consists of 52 cards. All cards are divided into 4 suits including spades (♠), clubs (♣), diamonds (♦) and hearts (♥). In each suit there are 13 cards including a 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king). A 5−card poker hand is said to be a "three of a kind" if it consists of 3 cards of the same denomination, and 2 other cards of 2 different denominations other than the one for the previous 3 cards.

   (a) Making a list of all possible 5 card poker and assign to variable *poker_5*. Assign number of 5 card poker to variable *len_poker_5*.

(b) Find all of “three of a kind” and assign to variable $three\_of\_a\_kind$. Assign the number to variable $len\_three\_of\_a\_kind$.