1. Chapter 1: NON-LINEAR EQUATIONS

1. Absolute and Relative Errors:

Suppose that:

A - exact value
a - approximated value $\Big\} \Rightarrow a \approx A$ (a approximates A)

Ex.: $3.14 \approx \pi$; $0.33 \approx \frac{1}{3}$; ....

We define $\Delta = |A - a|$: absolute error of a

Since A is unknown $\Rightarrow$ we estimate the positive value:

Δ_a such that $|A - a| \leq \Delta_a$: (error of a)

Ex.: We know $3.14 \approx \pi$. But $\Delta = |\pi - 3.14|$ is unknown.

We have : $3.138 \leq \pi \leq 3.142 \Leftrightarrow 3.14 - 0.02 \leq \pi \leq 3.14 + 0.02$

$\Rightarrow |\pi - 3.14| \leq 0.002 \Rightarrow \Delta_\pi = 0.002$.

We denote $|A - a| \leq \Delta_a$ by $A = a \pm \Delta_a$.

Relative error: $\delta_a = \frac{\Delta_a}{|a|} \Leftrightarrow \Delta_a = \delta_a \cdot |a|$ $(a \neq 0)$

Note: $\delta_a \times 100 \Rightarrow$ %. (percent)

Ex.: Measuring a length of a table gives us: $l = 150$ cm

with the error is 0.1%. It means that $\delta_l = \frac{0.1}{100}$ and

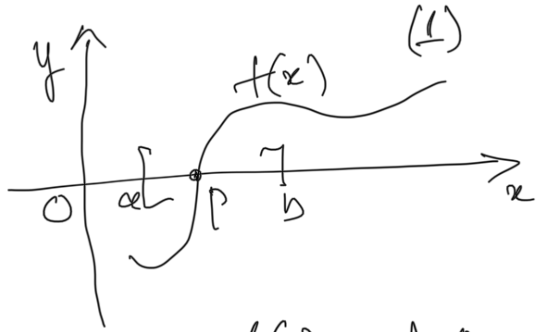
$\Delta_l = \delta_l \cdot |l| = \frac{0.1}{100} \cdot 150 = 0.15$ (cm).

$\Rightarrow L = 150 \pm 0.15$ (cm)

② NonLinear - Equation:

We consider the equation:

$$f(x) = 0 \qquad (1)$$



where f is a continuous function.

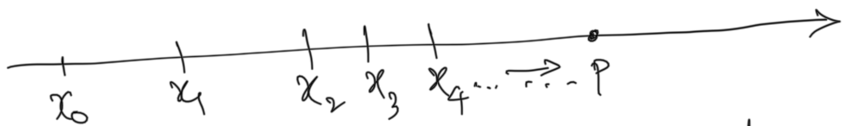
The value p is a root of the equation (1) if $f(p) = 0$

Geometrically, p is the point of interception of $f(x)$ and Ox.

The process of finding the approximated root is as follows:

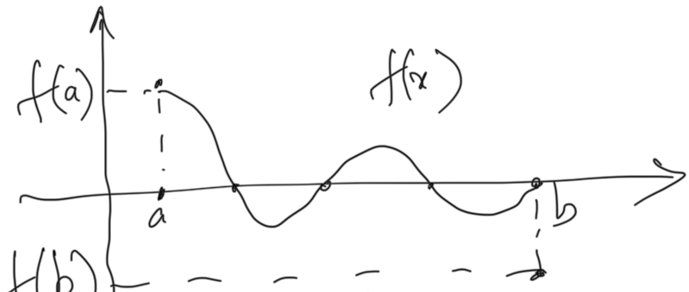
i) We need to find an interval $[a, b]$ such that $p \in [a, b]$ and it is unique. This interval is called a root-isolated interval of p.

ii) We construct a sequence $\{x_n\}_{n=0}^{\infty}$ such that $\lim\limits_{n \to \infty} x_n = p$.

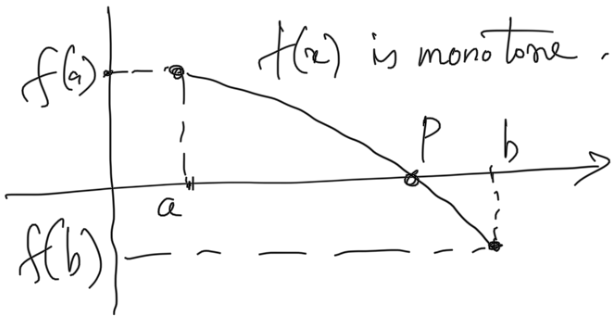


iii) Fix a value of $n \in \mathbb{N} \Rightarrow x_n \approx p \Rightarrow |p - x_n| \leq \Delta_{x_n} = ?$

* Theorem: If $f(x)$ is monotone and continuous in $[a, b]$ and $f(a).f(b) < 0$, then $[a, b]$ is a root-isolated interval of an unique root p.



$f(a).f(b) < 0$



Ex.: Find all root-isolated intervals of $f(x) = 2x^3 - 6x + 1 = 0$

Sol.: $f(x)$ is continuous for all x. We have:

$f'(x) = 6x^2 - 6 = 0$ at $x = \pm 1$.

x	$-\infty$		-1		1		$+\infty$
f'		$+$	0	$-$	0	$+$	
f	$(-)$	$\nearrow$	$(+)$	$\searrow$	$(-)$	$\nearrow$	$(+)$

x	$\ldots -4$	-3	-2	-1	0	1	2	3	$4 \ldots$
sign f	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$+$

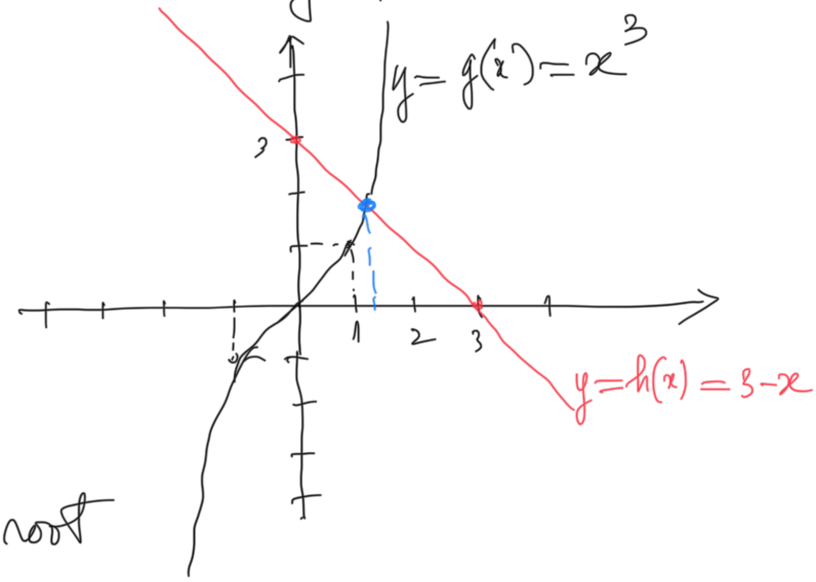
$\Rightarrow$ The eq-n has three roots: $p_1 \in [-2, -1]$, $p_2 \in [0, 1]$, $p_3 \in [1, 2]$.

Ex.: Find all root-isolated intervals of $f(x) = x^3 + x - 3 = 0$

Sol.: $f(x) = 0 \Leftrightarrow$

$\Leftrightarrow x^3 + x - 3 = 0$

$\Leftrightarrow \underbrace{x^3}_{g(x)} = \underbrace{3 - x}_{h(x)}$



From these graphs we have: the eq-n has only root $p \in [1, 2]$.

Theorem: (The General Error-Estimated Formula)

Suppose:
- $f(x)$ is continuous and differentiable on the root-isolated interval $[a, b]$, that consists the unique root p.
- Let $p^* \approx p$ and $m = \min\limits_{[a,b]} |f'(x)|$

Then $$|p - p^*| \leq \frac{|f(p^*)|}{m} = \Delta_{p^*}.$$

Ex.4: Given $f(x) = 2x^3 - 6x + 1 = 0$ in $[1.5, 2] \ni p$.

If $p^* = 1.64 \approx p$, then what is the error of p^* in comparing with p.

Solve: We have $f'(x) = 6x^2 - 6$, and

$$\forall x \in [1.5, 2], \ |f'(x)| = |6x^2 - 6| = 6|x^2 - 1| \geq 6|1.5^2 - 1|$$

$$= 7.5 = m$$

$$\Rightarrow \Delta_{p^*} = \frac{|f(p^*)|}{m} = \frac{|2 \times 1.64^3 - 6 \times 1.64 + 1|}{7.5}$$

$$= 0.002414933 \approx 0.002415$$

$$\Longrightarrow \quad p = 1.64 \pm 0.002415.$$

Ex.: Given $f(x) = 3x^3 - 5x^2 + x + 2$ in $[-1, 0] \ni p$.

Suppose $p^* = -0.48 \approx p$. What is the error of p^*?

Sol.: $f'(x) = 9x^2 - 10x + 1$.

We see that $\forall x \in [-1, 0]$, $f'(x) > 0$

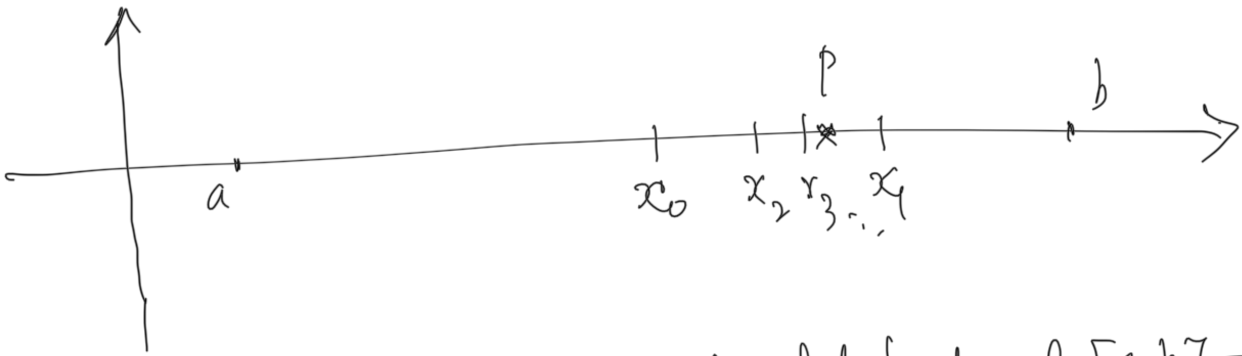
and $f''(x) = 18x - 10 < 0 \Rightarrow f'(x)$ is decreasing

$$\Longrightarrow m = \min_{[-1,0]} |f'(x)| = |f'(0)| = 1$$

$$\Longrightarrow \Delta_{p^*} = \frac{|3(-0.48)^3 - 5(-0.48)^2 + (-0.48) + 2|}{1}$$

$$= \underline{0.036224}$$

3. The Bisection Method:



Consider $f(x) = 0$ in the root-isolated interval $[a,b] \ni ! \, p$.

Denote $a_0 = a$, $b_0 = b$, $d_0 = b_0 - a_0 = b - a$: const.

We divide the interval $[a_0, b_0]$ into two equal subintervals by the midpoint $x_0 = \dfrac{a_0 + b_0}{2}$.

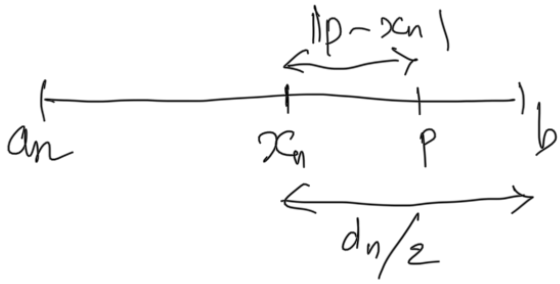
- $f(x_0) = 0 \implies x_0 \equiv p$ (stop).
- $f(a_0) \cdot f(x_0) < 0 \implies p \in [a_0, x_0] \implies \begin{cases} a_1 = a_0 \\ b_1 = x_0 \end{cases}$
- $f(x_0) f(b_0) < 0 \implies p \in [x_0, b_0] \implies \begin{cases} a_1 = x_0 \\ b_1 = b_0 \end{cases}$

We obtain: $\begin{cases} p \in [a_1, b_1] \subset [a_0, b_0] \\ d_1 = b_1 - a_1 = \dfrac{d_0}{2} = \dfrac{b-a}{2} \end{cases}$

Continuing the bisection process into n times gives us

$$\begin{cases} p \in [a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset \dots \subset [a_0, b_0] \\ d_n = b_n - a_n = \dfrac{d_{n-1}}{2} = \dfrac{b-a}{2^n} \text{ (by induction)} \\ x_n = \dfrac{a_n + b_n}{2}, \quad a_n \le x_n \le b_n, \quad x_n \approx p. \end{cases}$$

We can show that $\lim\limits_{n \to \infty} x_n = p$.



$$\implies \boxed{|p - x_n| \le \frac{b-a}{2^{n+1}}}$$

Ex.: Given $f(x) = 3x^3 - 5x^2 + x + 2 = 0$ in $[-1, 0]$

Use the bisection method to find the approximated root x_5 and its error.

Sol.:

n	0	1	2	3	4	5
a_n (−)	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
b_n (+)	0	0	$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{7}{16}$	$-\frac{15}{32}$
x_n	$-\frac{1}{2}$ (−)	$-\frac{1}{4}$ (+)	$-\frac{3}{8}$ (+)	$-\frac{7}{16}$ (+)	$-\frac{15}{32}$ (+)	$-\frac{31}{64}$

$$\Rightarrow x_5 = -\frac{31}{64} = -0.484375, \quad \Delta x_5 = \frac{0-(-1)}{2^{5+1}} = \frac{1}{64} = 0.015625$$

$$f(x) = \underline{3x^3 - 5x^2 + x + 2} \Rightarrow f\left(-\tfrac{1}{2}\right) = -\tfrac{1}{8} \Rightarrow \text{sign } f\left(-\tfrac{1}{2}\right) \text{ is } (-)$$

Ex.: $f(x) = x\cos x + \sin x - 1 = 0$ on $[1, 2]$

Use the bisection method to find x_5

Sol.:

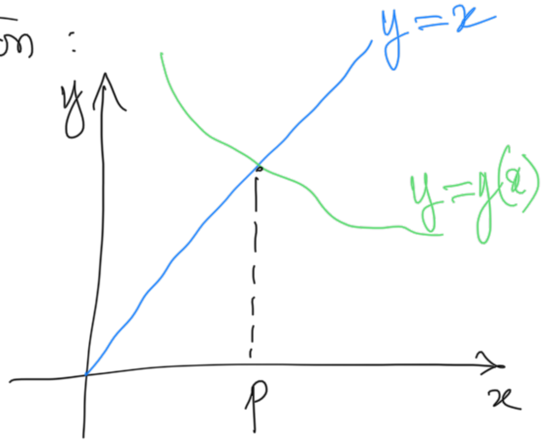
n	0	1	2	3	4	5
a_n (+)	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{25}{16}$	$\frac{25}{16}$
b_n (−)	2	2	$\frac{7}{4}$	$\frac{13}{8}$	$\frac{13}{8}$	$\frac{51}{32}$
x_n	$\frac{3}{2}$ (+)	$\frac{7}{4}$ (−)	$\frac{13}{8}$ (−)	$\frac{25}{16}$ (+)	$\frac{51}{32}$ (−)	$\frac{101}{64}$

$$\Rightarrow x_5 = \frac{101}{64} = 1.578125$$

4. Fixed-Point Iteration: (The Iterative Method)

We consider the following equation:

$$x = g(x)$$



The root p: $p = g(p)$

is called a fixed-point of the function g.

First of all, we choose an initial value $x_0 \in [a,b]$.

Construct a sequence $\{x_n\}$ by:

$$x_n = g(x_{n-1}), \quad n = 1, 2, 3, \dots \qquad (3)$$

Theorem: If there exists a number k such that $0 \le k < 1$ and $\forall x \in [a,b]$, $|g'(x)| \le k$, then the sequence, defined by (3), will converge to the root p; and

$$|p - x_n| \le \frac{k^n}{1-k}|x_1 - x_0| \qquad (4)$$

or

$$|p - x_n| \le \frac{k}{1-k}|x_n - x_{n-1}| \qquad (5)$$

Ex.: Given the eq-n: $x = g(x) = \sqrt[3]{7-x}$ in $[1,2]$.
Use the iterative method with $x_0 = 1.5$ to find x_7 and its error, using the formula (5).

Sol.: We have $g' = \dfrac{-1}{3\sqrt[3]{(7-x)^2}} \Rightarrow$

$$\Rightarrow \forall x \in [1,2], \ |g'(x)| = \frac{1}{3\sqrt[3]{(7-x)^2}} \le |g'(2)| = \frac{1}{3\sqrt[3]{(7-2)^2}}$$

$$= \frac{1}{3\sqrt[3]{25}} = k \approx 0.114 < 1$$

8) $x_0 = 1.5$; $x_n = \sqrt[3]{7 - x_{n-1}}$, $n = 1, 2, 3, \ldots$

$$|p - x_n| \leq \frac{k}{1-k} |x_n - x_{n-1}|$$

We need to find $x_7 \Rightarrow n = 7$.

We have: $x_0 = 1.5 \Rightarrow x_1 = 1.765174168, \ldots,$

$x_6 = 1.739203439$; $x_7 = \underline{1.739203908}$.

$$\Delta_{x_7} = \frac{k}{1-k} |x_7 - x_6| \approx \underline{6.04 \times 10^{-8}}$$

Ex.: Given $x = \dfrac{x^2 - e^x + 2}{3}$ in $[0, 1]$.

Choose $x_0 = 0.5$. Use the fixed-point iteration to find the values of k, x_5, and Δ_{x_5}.

Sol.: $g(x) = \dfrac{x^2 - e^x + 2}{3} \Rightarrow g'(x) = \dfrac{2x - e^x}{3}$

We need to find the maximum of $|g'(x)|$ in $[0, 1]$

$$\Rightarrow g''(x) = \frac{2 - e^x}{3} = 0 \Rightarrow x = \ln 2 \in [0, 1]$$

$$\Rightarrow k = \max_{[0,1]} |g'(x)| = \max\left(|g'(0)|, |g'(\ln 2)|, |g'(1)|\right)$$

$$= \max\left(\frac{1}{3}, \frac{2 - 2\ln 2}{3}, \frac{e-2}{3}\right) = \frac{1}{3}$$

$$\Rightarrow \boxed{k = \frac{1}{3}}$$

$$x_0 = 0.5 \Rightarrow x_n = g(x_{n-1}) = \frac{x_{n-1}^2 - e^{x_{n-1}} + 2}{3}$$

$\Rightarrow x_4 = 0.2585503763$

$x_5 = \underline{0.2572656363}$

$$\Delta_{x_5} = \frac{k}{1-k} |x_5 - x_4| \approx 6.5 \times 10^{-4}$$

$= 0.00065 \Rightarrow p = 0.2572656363 \pm 0.00065$.

Exercises:

1, given $f(x) = 2x^3 - 4x^2 + 5x - 7 = 0$ in $[1.6, 1.8]$. Use the bisection method to find x_5 and its error.

2, Given $x = g(x) = \frac{5}{x^2} + 2$ in $[2.6, 2.8]$. Use the iterative method with $x_0 = 2.7$ to find k, x_5, and Δx_5.

Notes:

- You do it in your paper.
- Take a photograph by your smart phone
- Send your answer (in the photograph) to me by my email:

lethanh.pptdt.hk203@gmail.com

- Deadline: 12:00 today 11/06/2021.

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