c) Choleski's Method:

* A matrix A is called a symmetric matrix if $A^T = A$ (A^T is a transposition of A).
It means that $a_{ij} = a_{ji}$ $\forall i,j = 1,2,\dots,n$.

Ex.: $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ are symmetric

$\to$ main diagonal.

* A matrix A is called a positive-definite matrix if for all X, we have $X^T A X > 0$, $\forall X \neq 0$

Ex.: Given $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. $\Rightarrow$ $\dim A = 2$

$\to X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ $(x_1^2 + x_2^2 > 0)$

and
$$X^T A X = [x_1, x_2] \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= [x_1, x_2] \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 3x_2 \end{bmatrix}$$
$$= x_1(2x_1 + x_2) + x_2(-x_1 + 3x_2)$$
$$= 2x_1^2 + 3x_2^2 > 0$$

$\Rightarrow A$ is pos.-def.

Theorem: (Sylvester)

A matrix A is pos.-def. when and only when $\Delta_K > 0$, for all $K = 1,2,\dots,n$, where

$$\Delta_K = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1K} \\ a_{21} & a_{22} & \dots & a_{2K} \\ - & - & - & - \\ a_{K1} & a_{K2} & \dots & a_{KK} \end{vmatrix}$$: the main subdeterminant of order k (of A)

Ex.: Given $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. We have

$$\Delta_1 = a_{11} = 2 > 0 \; ; \; \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

$$\Delta_3 = \det A = 8 + 0 + 0 - 0 - 2 - 2 = 4 > 0$$

$\Rightarrow$ A is pos.-def.

Ex.: Given $A = \begin{bmatrix} 2 & m & -1 \\ m & 2 & -2 \\ -1 & -2 & 11 \end{bmatrix}$. Find all values of m such that A is pos.-def.

Sol.: $\Delta_1 = 2 > 0 \; ; \; \Delta_2 = \begin{vmatrix} 2 & m \\ m & 2 \end{vmatrix} = 4 - m^2 > 0$

$$\Rightarrow -2 < m < 2 \quad (*)$$

$$\Delta_3 = 44 + 2m + 2m - 2 - 8 - 11m^2 = \det A.$$

$$= 34 + 4m - 11m^2 > 0 \Rightarrow$$

$$-1.586 \approx \frac{2 - 3\sqrt{42}}{11} < m < \frac{2 + 3\sqrt{42}}{11} \approx 1.949 \; (**)$$

From (*) and (**) we obtain

$$-1.586 < m < 1.949$$

$$\delta_1 = 11 > 0 \; ; \; \delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 11 \end{vmatrix} = 18 > 0$$

$$\delta_3 = \Delta_3 = 34 + 4m - 11m^2 > 0$$

$$\Rightarrow -1.586 < m < 1.949$$

3. Actually, we can start from the bottom to the top of the matrix by this way:

$$\Delta_1 = a_{nn} \;;\; \Delta_2 = \begin{vmatrix} a_{n-1,n-1} & a_{n-1,n} \\ a_{n,n-1} & a_{nn} \end{vmatrix}, \ldots$$

$$\Delta_n = \det A, \ldots$$

Ex.:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \Delta_1 = 2, \quad \Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \ldots$$

δ_K

Ex.:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & -1 & 10 \end{bmatrix}$$

$$\delta_1 = 10 > 0 \;;\; \delta_2 = \begin{vmatrix} 3 & -1 \\ -1 & 10 \end{vmatrix} = 29 > 0$$

$$\delta_3 = \det A = 30 - 2 - 2 - 12 - 1 - 10$$
$$= 3 > 0$$

4. Theorem: (Choleski).

A matrix A is symmetric and positive-definite when and only when there exists a non-singular, lower-triangular matrix $\underline{C}$ such as $\underline{A} = \underline{C} \cdot \underline{C}^T$

(C is non-singular $\Longleftrightarrow \det C \neq 0$)

C is lower-triangular $\Rightarrow C^T$ is upper-triangular)

$AX = B$ where A is symm. and pos.-def.

$\Rightarrow \exists C : \ C \cdot C^T X = B \Leftrightarrow \begin{cases} CY = B \\ C^T X = Y \end{cases}$

If

$$C = \begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ - & - & - & - \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

then

$$\begin{cases} c_{11} = \sqrt{a_{11}} = \underline{\underline{\sqrt{\Delta_1}}} \quad (\Delta_1 > 0) \\ c_{i1} = \dfrac{a_{i1}}{c_{11}}, \text{ for all } i = 2, 3, \dots, n \\ c_{kk} = \sqrt{a_{kk} - \sum\limits_{j=1}^{k-1} c_{kj}^2} = \sqrt{\dfrac{\Delta_k}{\Delta_{k-1}}} \quad \underline{\underline{\left(\dfrac{\Delta_k}{\Delta_{k-1}} > 0\right)}} \\ c_{ik} = \dfrac{1}{c_{kk}} \left(a_{ik} - \sum\limits_{j=1}^{k-1} c_{ij} c_{kj} \right), \\ \quad i = k+1, k+2, \dots, n. \end{cases}$$

* In case of $n = 3$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \; ; \; C = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\boxed{\begin{aligned} & c_{11} = \sqrt{a_{11}} \; ; \; c_{21} = \frac{a_{21}}{c_{11}} \; ; \; c_{31} = \frac{a_{31}}{c_{11}} \; ; \\ & c_{22} = \sqrt{a_{22} - c_{21}^2} \; ; \; c_{32} = \frac{1}{c_{22}}\left(a_{32} - c_{31}c_{21}\right) ; \\ & c_{33} = \sqrt{a_{33} - c_{31}^2 - c_{32}^2} . \end{aligned}}$$

<u>Ex.:</u> <u>Given:</u> $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 15 \end{bmatrix}$

Use Choleski's method to factorize $A = C.C^T$.

Find $c_{11} + c_{22} + c_{33} = tr(c)$ (trace of C).

<u>Sol.:</u>

<u>1^{st} method:</u> $c_{11} = \sqrt{2}$; $c_{21} = \frac{-1}{\sqrt{2}}$; $c_{31} = \frac{2}{\sqrt{2}} = \underline{\sqrt{2}}$;

$$c_{22} = \sqrt{3 - \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{\frac{5}{2}} \; ;$$

$$c_{32} = \frac{1}{\sqrt{5/2}}\left(1 - (\sqrt{2})\left(\frac{-1}{\sqrt{2}}\right)\right) = \underline{2\sqrt{\frac{2}{5}}}$$

$$c_{33} = \sqrt{15 - (\sqrt{2})^2 - \left(2\sqrt{\frac{2}{5}}\right)^2} = \sqrt{\frac{57}{5}}$$

$\rightarrow$ trace of $C = tr(C) = c_{11} + c_{22} + c_{33}$

$$= \sqrt{2} + \sqrt{\frac{5}{2}} + \sqrt{\frac{57}{5}} = \underline{6.3717}$$

2^{nd} method: Use the formulas

$$c_{11} = \sqrt{\Delta_1} \; ; \; c_{kk} = \sqrt{\frac{\Delta_k}{\Delta_{k-1}}} \; , \; k = 2, 3, \dots, n$$

We have: $\Delta_1 = 2 > 0$; $\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5 > 0$

$\Delta_3 = \det A = 90 - 2 - 2 - 12 - 2 - 15 = 57$.

$\Rightarrow c_{11} = \sqrt{2}$; $c_{22} = \sqrt{\frac{\Delta_2}{\Delta_1}} = \sqrt{\frac{5}{2}}$; $c_{33} = \sqrt{\frac{\Delta_3}{\Delta_2}} = \sqrt{\frac{57}{5}}$

$\Rightarrow tr(C) = \sqrt{2} + \sqrt{5/2} + \sqrt{57/5} \cong 6.3717$.

Ex.: Solve the system $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ using Cholerki's method.

Sol.: • $c_{11} = \sqrt{3}$; $c_{21} = \frac{1}{\sqrt{3}}$; $c_{31} = \frac{0}{\sqrt{3}} = 0$; $c_{22} = \sqrt{3 - \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{8/3}$

$c_{32} = \frac{1}{\sqrt{8/3}}\left(1 - 0 \times \frac{1}{\sqrt{3}}\right) = \sqrt{3/8}$; $c_{33} = \sqrt{3 - 0^2 - \left(\sqrt{3/8}\right)^2} = \sqrt{\frac{21}{8}}$

• $CY = B \Rightarrow \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & \sqrt{8/3} & 0 \\ 0 & \sqrt{3/8} & \sqrt{\frac{21}{8}} \end{bmatrix} Y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 1/\sqrt{3} \\ \frac{2}{3}\sqrt{3/8} \\ \frac{3}{4}\sqrt{\frac{8}{21}} \end{bmatrix}$

$C^T X = Y \Rightarrow \begin{bmatrix} \sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & \sqrt{8/3} & \sqrt{3/8} \\ 0 & 0 & \sqrt{21/8} \end{bmatrix} X = \begin{bmatrix} 1/\sqrt{3} \\ \frac{2}{3}\sqrt{3/8} \\ \frac{3}{4}\sqrt{\frac{8}{21}} \end{bmatrix} \Rightarrow X = \begin{bmatrix} 2/7 \\ 1/7 \\ 2/7 \end{bmatrix}$

Exercises:

1) Given $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & m \\ -3 & m & 8 \end{bmatrix}$. Find all values of m such that A is pos.-def.

2) Given the system $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 10 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

a) Use Doolittle, Crout, and Choleski methods to factorize A.

b) Use a) to solve the system.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

$$\begin{array}{ccccc} & + & & 1^{st} & 2^{nd} \\ & & & - & \\ a & b & c & a & b \\ d & e & f & d & e \\ g & h & k & g & h \end{array}$$

$$\Delta_3 = aek + bfg + cdh - ceg - afh - bdk$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 8 \end{vmatrix} = 24 + 60 + 60 - 54 - 25 - 64$$

$$= \underline{1}$$

8.

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 4 & 3 & 5 & 4 & 3 \\ 6 & 5 & \underline{\underline{8}} & \underline{\underline{6}} & \underline{\underline{5}} \end{array}$$

2

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

a) **Doolittle:** $A = LU$.

$u_{11} = 1$; $u_{12} = 1$; $u_{13} = -2$; $l_{21} = 1$; $l_{31} = -2$;

$u_{22} = 2 - (1)(1) = 1$; $u_{23} = -1 - (1)(-2) = 1$;

$l_{32} = \frac{1}{1}\left(-1 - (-2)(1)\right) = 1$;

$u_{33} = 10 - (-2)(-2) - (1)(1) = 5$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} = LU$$

Crout:

$l_{11} = 1$; $l_{21} = 1$; $l_{31} = -2$; $u_{12} = 1$; $u_{13} = -2$;

$l_{22} = 2 - (1)(1) = 1$; $l_{32} = -1 - (-2)(1) = 1$;

$u_{23} = \frac{1}{1}\left(-1 - (1)(-2)\right) = 1$;

$l_{33} = 10 - (-2)(-2) - (1)(1) = 5$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

a. Choleski:

$$C_{11} = 1\,;\quad C_{21} = 1\,;\quad C_{31} = -2\,;$$

$$C_{22} = \sqrt{2-(1)^2} = 1\,;\quad C_{32} = \frac{1}{1}\Big(-1-(1)(-2)\Big) = 1$$

$$C_{33} = \sqrt{10-(-2)^2-(1)^2} = \sqrt{5}\,.$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix} = C.C^T$$

b) Doolittle:

$$LY = B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} Y = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

$$UX = Y \Leftrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix} X = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 26/5 \\ -2/5 \\ 7/5 \end{bmatrix}$$

Crout:

$$LY = B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} Y = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 \\ 1 \\ 7/5 \end{bmatrix}$$

$$UX = Y \Leftrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 1 \\ 7/5 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 26/5 \\ -2/5 \\ 7/5 \end{bmatrix}$$

Choleski:

$$C.Y = B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & \sqrt{5} \end{bmatrix} Y = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 \\ 1 \\ 7/\sqrt{5} \end{bmatrix}$$

$$C^T X = Y \Leftrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix} X = \begin{bmatrix} 2 \\ 1 \\ 7/\sqrt{5} \end{bmatrix} \Rightarrow X = \begin{bmatrix} 26/5 \\ -2/5 \\ 7/5 \end{bmatrix}$$

9. THE ITERATIVE METHODS.

a) Norms of Vector and Matrix:

Definition: Let $\mathscr{X}$ be a linear space and $X \in \mathscr{X}$ be a vector of $\mathscr{X}$. A norm of the vector X is a number, denoted by $\|X\|$, that satisfies:

i) $\forall X \in \mathscr{X}$, $\|X\| \geqslant 0$; $\|X\| = 0 \Leftrightarrow X = \theta$

ii) $\forall X \in \mathscr{X}, \forall \lambda \in \mathbb{R}$, $\|\lambda X\| = |\lambda| . \|X\|$

iii) $\forall X, Y \in \mathscr{X}$, $\|X+Y\| \leqslant \|X\| + \|Y\|$
(The Triangular Inequality)

Note: Norm of a vector $\Leftrightarrow$ "Length" of this vector.

$\Rightarrow$ The distance between 2 vectors:

$$d(X,Y) = \|X - Y\|$$

In the linear space $\mathbb{R}^n$, we often use the following norms:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

i) l_1-norm: $\|X\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

ii) l_∞-norm: $\|X\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right)$

Ex.:

$$X = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} \in \mathbb{R}^4 \Rightarrow \begin{cases} \|X\|_1 = |1| + |-2| + |3| + |-4| = 10 \\ \|X\|_\infty = \max\left(|1|, |-2|, |3|, |-4|\right) = 4. \end{cases}$$

11. Consider a sequence of vectors $\{X^{(m)}\}$. We say that this sequence converges to the vector $\underline{P}$, if:

$$\lim_{m\to\infty} X^{(m)} = \underline{P} \iff \forall \varepsilon > 0, \exists M > 0, \forall m > M \Rightarrow$$
$$\Rightarrow \|\underline{P} - X^{(m)}\| < \varepsilon$$

In $\mathbb{R}^n$,

$$X^{(m)} = \begin{bmatrix} x_1^{(m)} \\ x_2^{(m)} \\ \vdots \\ x_n^{(m)} \end{bmatrix}, \quad m = 0, 1, 2, \dots$$

$$\Rightarrow \lim_{m\to\infty} X^{(m)} = \underline{P} \iff \lim_{m\to\infty} x_k^{(m)} = P_k, \ \forall k = 1, 2, \dots, n$$

(Convergence by norm) (Convergence by coordinates)

<u>Ex.:</u> Given a sequence $\{X^{(m)}\}$ in $\mathbb{R}^3$:

$$X^{(m)} = \begin{bmatrix} \frac{2m}{m+1} \\ e^{-m} - 1 \\ \frac{m^2}{2m^2+m+3} \end{bmatrix}, \quad m = 0, 1, 2, \dots$$

We see that:

$$\left.\begin{aligned} x_1^{(m)} &= \frac{2m}{m+1} \xrightarrow[m\infty]{} 2 = P_1 \\ x_2^{(m)} &= e^{-m} - 1 \xrightarrow[m\infty]{} -1 = P_2 \\ x_3^{(m)} &= \frac{m^2}{2m^2+m+3} \xrightarrow[m\infty]{} \frac{1}{2} = P_3 \end{aligned}\right\} \Rightarrow X^{(m)} \xrightarrow[m\infty]{} \underline{P} = \begin{bmatrix} 2 \\ -1 \\ 1/2 \end{bmatrix}$$

12. Norms of Matrices:

If $A = (a_{ij})$ – square matrix of order n, then

$$\|A\|_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \quad \text{(take a sum by columns)}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right) \quad \text{(take a sum by rows)}$$

Ex.:

$$A = \begin{bmatrix} 12 & -3 & 11 \\ 13 & -5 & 14 \\ 15 & -7 & 12 \end{bmatrix}$$

$$\|A\|_1 = \max(40, 15, 37) = 40$$

$$\|A\|_\infty = \max(26, 32, 34) = 34$$

* The Conditional Number of a matrix:

$$k(A) = \|A\| \cdot \|A^{-1}\|$$

where A^{-1}: inversed matrix of A.

Ex.: Given $A = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$. Find $k_1(A)$.

Sol.:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So $A^{-1} = \dfrac{1}{2} \begin{bmatrix} -5 & 3 \\ -4 & 2 \end{bmatrix}$

$\Rightarrow \|A\|_1 = \max(6, 8) = 8$; $\|A^{-1}\|_1 = \max\left(\frac{9}{2}, \frac{5}{2}\right) = \frac{9}{2}$

$\Rightarrow k_1(A) = 8 \times \frac{9}{2} = 36$

13. ~~Ex.~~: $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -2 & 7 \\ 4 & 1 & 3 \end{bmatrix}$. Find $k_\infty(A)$.

We have
$$A^{-1} = \begin{bmatrix} -\frac{13}{99} & -\frac{7}{99} & \frac{38}{99} \\ \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \frac{10}{99} & \frac{13}{99} & -\frac{14}{99} \end{bmatrix}$$

$$\Rightarrow \|A\|_\infty = \max(12, 11, 8) = 12$$
$$\|A^{-1}\|_\infty = \max\left(\frac{58}{99}, \frac{4}{9}, \frac{37}{99}\right) = \frac{58}{99}$$
$$\Rightarrow k_\infty(A) = 12 \times \frac{58}{99} = \frac{232}{33} \approx \underline{7.0303}$$

14.

15.