3) Cubic Spline:

Def.: Given the table

x	x_0	x_1	$\dots$	x_n
y	y_0	y_1	$\dots$	y_n

. (1)

A function $g(x)$, defined in $[x_0, x_n]$, is called a cubic spline, if it satisfies the conditions:

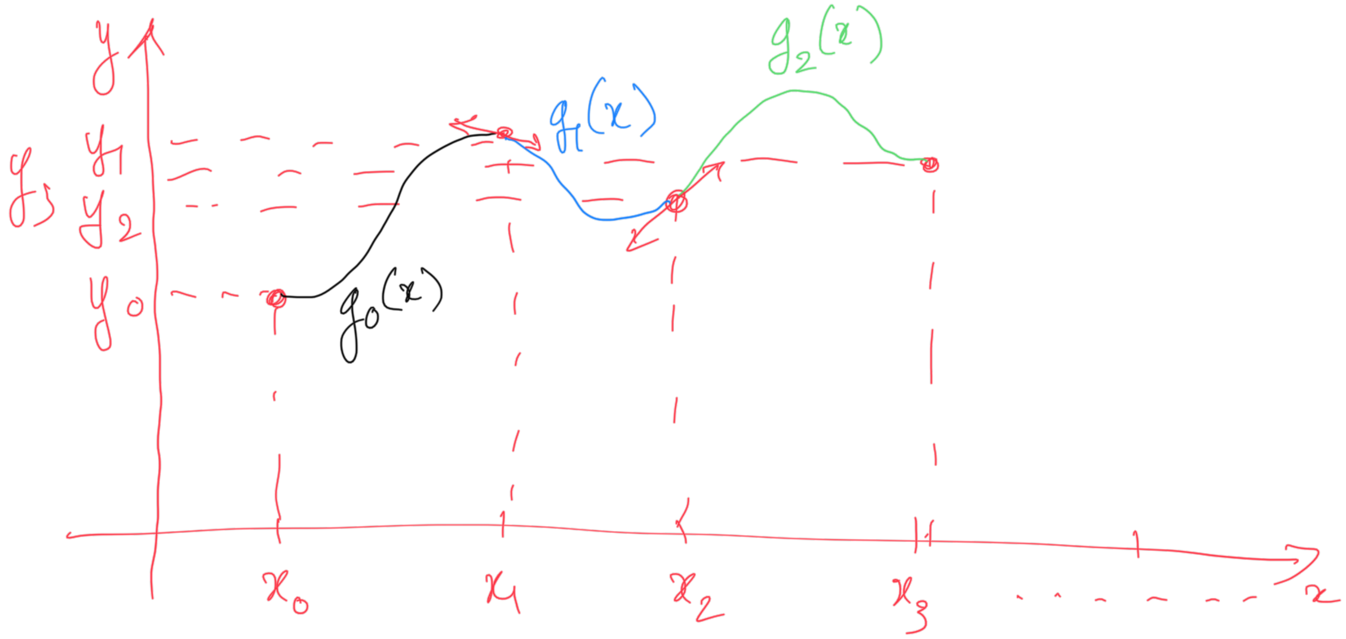
i) It has $g'(x)$ and $g''(x)$, which are continuous in $[x_0, x_n]$

ii) In each subinterval $[x_k, x_{k+1}]$, $k = 0, 1, \dots, n-1$

$g(x) \equiv g_k(x)$, that is a cubic polynomial.

iii) $g(x_i) = y_i$, for all $i = 0, 1, 2, \dots, n$.

$$\Rightarrow g(x) = \begin{cases} g_0(x), & \text{if } x_0 \le x \le x_1 \\ g_1(x), & \text{if } x_1 \le x \le x_2 \\ \dots & \dots \\ g_{n-1}(x), & \text{if } x_{n-1} \le x \le x_n \end{cases}$$



Consider the subinterval $[x_k, x_{k+1}]$, $k = 0, 1, \dots, n-1$

Denote $h_k = x_{k+1} - x_k$.

From ii) we have: $\forall x \in [x_k, x_{k+1}]$

$$g(x) \equiv g_k(x) = a_k + b_k(x-x_k) + c_k(x-x_k)^2 + d_k(x-x_k)^3 \quad (1)$$
$$k = 0, 1, \dots, n-1.$$

where a_k, b_k, c_k, d_k are unknown coefficients.

From iii) we have:

$$g(x_k) = g_k(x_k) = a_k = y_k \Rightarrow \boxed{a_k = y_k} \quad (2)$$

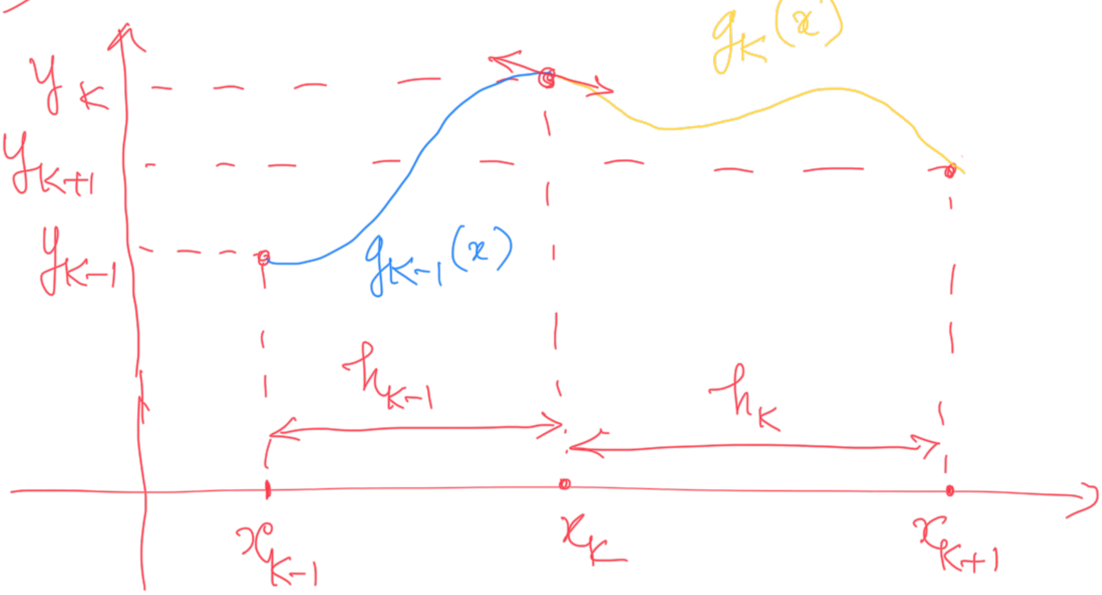
$$g(x_{k+1}) = g_k(x_{k+1}) = \underbrace{a_k}_{\to y_k} + b_k h_k + c_k h_k^2 + d_k h_k^3 = y_{k+1}$$

$\Longrightarrow$

$$b_k = \frac{y_{k+1} - y_k}{h_k} - c_k h_k - d_k h_k^2 \quad (3)$$

From the first condition we have at x_k, $k = 1, 2, \dots, n-1$

$$\underline{g'(x_k^-) = g'(x_k^+)} \text{ and } g''(x_k^-) = g''(x_k^+)$$



$\Longrightarrow$

$$\Longrightarrow \begin{cases} g'_{k-1}(x_k) = g'_k(x_k) & \text{(i)} \\ g''_{k-1}(x_k) = g''_k(x_k) & \text{(ii)} \end{cases}$$

From (1) we have:

$$g'_k(x) = b_k + 2c_k(x-x_k) + 3d_k(x-x_k)^2$$
$$g''_k(x) = 2c_k + 6d_k(x-x_k).$$

If we decrease the index k by 1, we have

$$g'_{k-1}(x) = b_{k-1} + 2c_{k-1}(x-x_{k-1}) + 3d_{k-1}(x-x_{k-1})^2$$
$$g''_{k-1}(x) = 2c_{k-1} + 6d_{k-1}(x-x_{k-1})$$

From (ii) we have

$$g''_{k-1}(x_k) = g''_k(x_k) \Longrightarrow 2c_{k-1} + 6d_{k-1}h_{k-1} = 2c_k$$

$$\Longrightarrow \quad d_{k-1} = \frac{c_k - c_{k-1}}{3h_{k-1}} \quad \text{and} \quad d_k = \frac{c_{k+1} - c_k}{3h_k} \qquad (4)$$

Replace d_k in (4) into (3), we obtain:

$$b_k = \frac{y_{k+1} - y_k}{h_k} - c_k h_k - \frac{c_{k+1} - c_k}{3h_k} \cdot h_k^2$$

$$\Longrightarrow \begin{cases} b_k = \dfrac{y_{k+1} - y_k}{h_k} - \dfrac{h_k}{3}(c_{k+1} + 2c_k) \\ b_{k-1} = \dfrac{y_k - y_{k-1}}{h_{k-1}} - \dfrac{h_{k-1}}{3}(c_k + 2c_{k-1}) \end{cases} \qquad (5)$$

From (i): $g'_{k-1}(x_k) = g'_k(x_k)$

$$\Rightarrow b_{k-1} + 2c_{k-1}h_{k-1} + 3d_{k-1}h_{k-1}^2 = b_k$$

Replace b_k, b_{k-1} from (5) and d_{k-1} from (4), we have

$$\frac{y_k - y_{k-1}}{h_{k-1}} - \frac{h_{k-1}}{3}(c_k + 2c_{k-1}) + 2c_{k-1}h_{k-1} +$$

$$+ 3 \cdot \frac{c_k - c_{k-1}}{3h_{k-1}} \cdot h_{k-1}^2 = \frac{y_{k+1} - y_k}{h_k} - \frac{h_k}{3}(c_{k+1} + 2c_k)$$

$$\Leftrightarrow 3\frac{y_k - y_{k-1}}{h_{k-1}} - \cancel{h_{k-1}c_k} - \cancel{2h_{k-1}c_{k-1}} + \cancel{6h_{k-1}c_{k-1}} +$$

$$+ \cancel{3h_{k-1}c_k} - \cancel{3h_{k-1}c_{k-1}} = 3\frac{y_{k+1} - y_k}{h_k} - \cancel{h_k c_{k+1}} - \cancel{2h_k c_k}$$

$$\Leftrightarrow \boxed{\begin{aligned} & h_{k-1}c_{k-1} + 2(h_{k-1} + h_k)c_k + h_k c_{k+1} = \\ & \qquad = 3\frac{y_{k+1} - y_k}{h_k} - 3\frac{y_k - y_{k-1}}{h_{k-1}} \\ & \text{for all } k = 1, 2, \ldots, n-1. \qquad (6) \end{aligned}}$$

This is a system of linear eq-ns for $c_0, c_1, \ldots, c_n$.

If we know c_k, then we can find a_k, b_k, and d_k.

$$\begin{cases} a_k = y_k \\ b_k = \dfrac{y_{k+1} - y_k}{h_k} - \dfrac{h_k}{3}(c_{k+1} + 2c_k) \\ d_k = \dfrac{c_{k+1} - c_k}{3h_k}. \end{cases}$$

* ## The Natural Cubic Spline

$$g''(x_0) = g''(x_n) = 0 \implies \boxed{c_0 = c_n = 0}$$

In this case the system (6) is the system of $n-1$ equations with $n-1$ unknowns $c_1, c_2, \dots, c_{n-1}$.

Ex: Find the natural cubic spline for the table

x	0	2	3
y	1	1	4

Sol: We have $n=2$, $h_0=2$, $h_1=1$, $c_0=c_2=0$

The system (6) has:

$$\cancel{h_0 c_0} + 2(h_0+h_1)c_1 + \cancel{h_1 c_2} = 3\frac{y_2-y_1}{h_1} - 3\frac{y_1-y_0}{h_0}$$

$$\iff 2(2+1)c_1 = 3\frac{4-1}{1} - 3\frac{1-1}{2}$$

$$\implies c_1 = \frac{3}{2}$$

- $x \in [0,2]$ $(k=0)$

$a_0 = y_0 = 1$; $b_0 = \frac{y_1-y_0}{h_0} - \frac{h_0}{3}(c_1+2c_0) = \frac{1-1}{2} - \frac{2}{3}\left(\frac{3}{2}+0\right) = -1$; $d_0 = \frac{c_1-c_0}{3h_0} = \frac{\frac{3}{2}-0}{3\times 2} = \frac{1}{4}$

$$\implies g_0(x) = a_0 + b_0(x-x_0) + \cancel{c_0(x-x_0)^2} + d_0(x-x_0)^3$$

$$= 1 - (x-0) + \frac{1}{4}(x-0)^3$$

$$= 1 - x + \frac{1}{4}x^3, \quad x \in [0,2]$$

Check: $g_0(0) = 1 = y_0$; $g_0(2) = 1 = y_1$; $g_0''(0) = 0 = c_0$

• $x \in [2,3]$ $(k=1)$

$$a_1 = 1 \; ; \; b_1 = \frac{4-1}{1} - \frac{1}{3}\left(0 + 2 \times \frac{3}{2}\right) = 2 \; ; \; d_1 = \frac{0 - \frac{3}{2}}{3 \times 1} = -\frac{1}{2}$$

$$g_1(x) = 1 + 2(x-2) + \frac{3}{2}(x-2)^2 - \frac{1}{2}(x-2)^3 \; , \; x \in [2,3]$$

Finally,

$$g(x) = \begin{cases} 1 - x + \frac{1}{4}x^3, & x \in [0,2] \\ 1 + 2(x-2) + \frac{3}{2}(x-2)^2 - \frac{1}{2}(x-2)^3, & x \in [2,3]. \end{cases}$$

Ex.: Given the table

x	2.0	2.5	3.0
y	1.75	2.21	2.66

Use the natural cubic spline to approximate $y(2.2)$ and $y'(2.65)$.

Sol.: We have $n = 2$, $h_0 = h_1 = 0.5$, $c_0 = c_2 = 0$.

$$\Rightarrow 2(0.5 + 0.5)c_1 = 3\,\frac{2.66 - 2.21}{0.5} - 3\,\frac{2.21 - 1.75}{0.5}$$

$$\Rightarrow c_1 = -0.03$$

• $x \in [2.0, 2.5]$, $(k=0)$

$$a_0 = 1.75 \; ; \; b_0 = \frac{2.21 - 1.75}{0.5} - \frac{0.5}{3}(-0.03 + 0) = 0.925$$

$$d_0 = \frac{-0.03 - 0}{3 \times 0.5} = -0.02$$

$$\Rightarrow g_0(x) = 1.75 + 0.925(x - 2.0) - 0.02(x - 2.0)^3, \; x \in [2.0, 2.5]$$

So $y(2.2) \approx g_0(2.2) = \underline{1.93484}$.

• $x \in [2.5, 3.0]$ $(k=1)$

$$a_1 = 2.21, \; b_1 = \frac{2.66 - 2.21}{0.5} - \frac{0.5}{3}\left(0 + 2 \times (-0.03)\right) = 0.91$$

$$d_1 = \frac{0 + 0.03}{3 \times 0.5} = 0.02$$

$$\Rightarrow g_1(x) = 2.21 + 0.91(x-2.5) - 0.02(x-2.5)^2 + 0.02(x-2.5)^3, \quad x \in [2.5, 3.0]$$

$$\Rightarrow g_1'(x) = 0.91 - 0.04(x-2.5) + 0.06(x-2.5)^2$$

and

$$y'(2.65) \simeq g_1'(2.65) = \underline{\underline{0.90535}}$$

Ex.: Given

x	1.0	1.2	1.5
y	1.73	1.85	2.19

Use the natural cubic spline to approximate $y'(1.1)$ and $y(1.3)$.

Chapter 4: DERIVATIVES AND INTEGRALS.

1) Approximating Derivatives:

a) Using Taylor's Series:

Consider $f(x)$ in $I(a)$. We have

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + o\left((x-a)^2\right)$$

We choose a positive step h such that $a \pm h \in I(a)$.

$$\Rightarrow \begin{cases} f(a+h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a) + o(h^2) \\ f(a-h) = f(a) - h f'(a) + \frac{h^2}{2} f''(a) + o(h^2) \end{cases}$$

$$\Rightarrow \begin{cases} f(a+h) - f(a-h) = 2h f'(a) + o(h^2) \\ f(a+h) + f(a-h) = 2f(a) + h^2 f''(a) + o(h^2) \end{cases}$$

$$\Rightarrow \begin{cases} f(a+h) - f(a-h) \approx 2h f'(a) \\ f(a+h) + f(a-h) \approx 2f(a) + h^2 f''(a) \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} f'(a) &\approx \frac{f(a+h) - f(a-h)}{2h} \\ f''(a) &\approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \end{aligned}}$$

(The Centripetal Formulas)

Ex.: Given $f(x) = \frac{x}{x^2+1} e^x \sin x$. Use the centripetal formulas to approximate $f'(1.2)$ and $f''(1.2)$ with the step $h = 0.1$; 0.05; and 0.01.

Sol.:

	$h=0.1$	$h=0.05$	$h=0.01$
$f'(1.2)$	1.8802	1.8837	1.8848
$f''(1.2)$	−0.2494	−0.2494	−0.2494

$$f'(1.2) \approx \frac{f(1.2+0.1) - f(1.2-0.1)}{2 \times 0.1} = 1.88017449$$

$$f''(1.2) \approx \frac{f(1.2+0.1) - 2f(1.2) + f(1.2-0.1)}{0.1^2} = -0.2494450249$$

— — — — —

b) Using Interpolative Polynomials:

Ex.: Given

x	0.4	0.6	0.8	1.0
y	0.1073	0.1976	0.2876	0.3679

of the function $f(x) = x^2 e^{-x}$. Use the forward Newton's polynomial to approximate $y'(0.48)$.

Sol.:

x	y	Δ	Δ^2	Δ^3
0.4	0.1073	0.0903	−0.0003	−0.0094
0.6	0.1976	0.0900	−0.0097	
0.8	0.2876	0.0803		
1.0	0.3679			

We have $x_0 = 0.4$, $h = 0.2$

$$\Rightarrow q = \frac{x - x_0}{h} = \frac{x - 0.4}{0.2} = 5x - 2$$

$$\Rightarrow P(x) = N_3^{(1)}(x) = y_0 + \frac{\Delta y_0}{1!} q + \frac{\Delta^2 y_0}{2!} q(q-1)$$

$$+ \frac{\Delta^3 y_0}{3!} q(q-1)(q-2) = 0.1073 + \frac{0.0903}{1!}(5x-2)$$

$$- \frac{0.0003}{2!}(5x-2)(5x-3) - \frac{0.0094}{3!}(5x-2)(5x-3)(5x-4)$$

$$P'(x) = 0.4515 - 0.00075\left[(5x-2) + (5x-3)\right]$$

$$- \frac{47}{6000}\left[(5x-2)(5x-3) + (5x-2)(5x-4) + (5x-3)(5x-4)\right]$$

$$y'(0.48) \approx P'(0.48) = 0.4515 - 0.00075\left[(0.4) + (-0.6)\right]$$

$$- \frac{47}{6000}\left[(0.4)(-0.6) + (0.4)(-1.6) + (0.6)(-1.6)\right]$$

$$= 0.451023333 \approx \underline{0.4510}$$

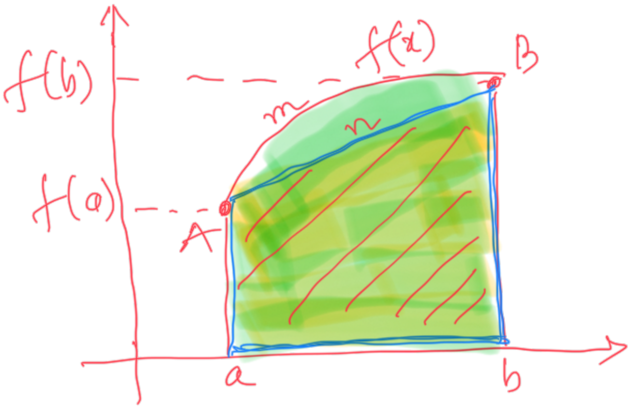
2. Approximating Definite Integrals:

We consider

$$I = \int_a^b f(x)\,dx$$

where $a < b$: finite ; and $f(x)$ is defined and integrable on $[a, b]$.

a) The Trapezoidal Formula:



$$\int_a^b f(x)\,dx = A(aAmBb)$$

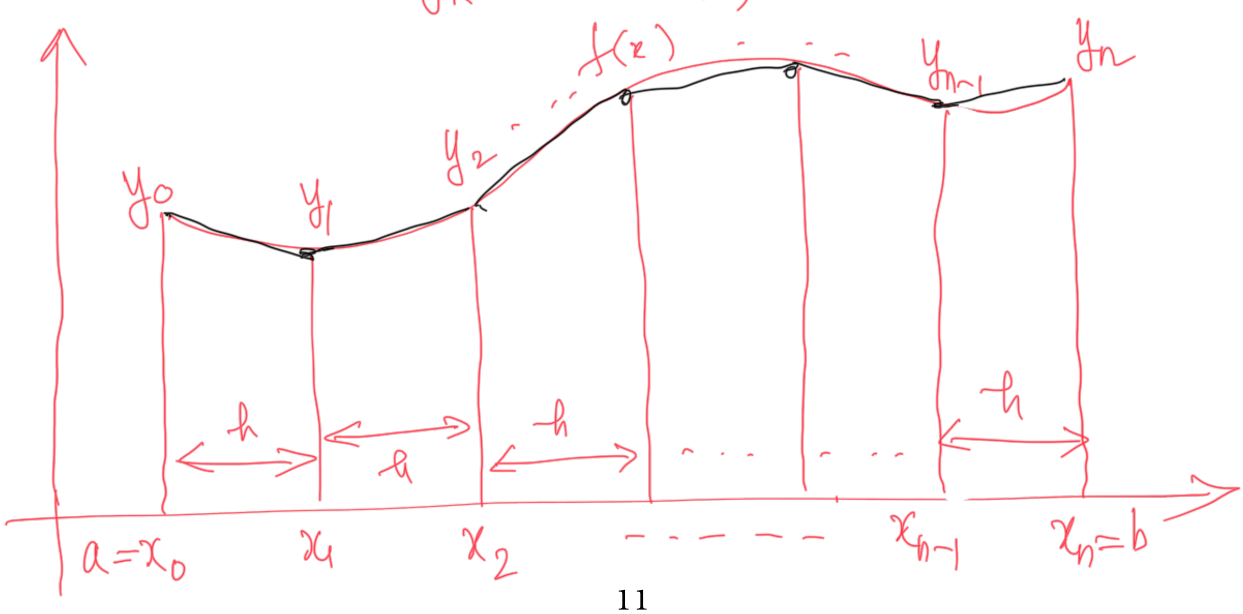
$$\approx A(aAnBb) = (b-a)\frac{f(a)+f(b)}{2}$$

Denote $x_0 = a$, $x_1 = b$, $h = b-a$, $y_0 = f(a)$, $y_1 = f(b)$

$$\Rightarrow \quad \int_a^b f(x)\,dx \approx h\,\frac{y_0 + y_1}{2}$$

Divide the interval $[a, b]$ into n equal subintervals by $x_0 = a$, $x_k = x_0 + kh$, $k = 0, 1, \dots, n$; $h = \frac{b-a}{n}$.

Calculate $y_k = f(x_k)$, $k = 0, 1, \dots, n$



$$\int_a^b f\,dx = \int_{x_0}^{x_n} f\,dx = \int_{x_0}^{x_1} f\,dx + \int_{x_1}^{x_2} f\,dx + \cdots + \int_{x_{n-1}}^{x_n} f\,dx$$

$$\approx h\,\frac{y_0+y_1}{2} + h\,\frac{y_1+y_2}{2} + \cdots + h\,\frac{y_{n-1}+y_n}{2}$$

We obtain the Composite Trapezoidal Formula:

$$\boxed{\int_a^b f(x)\,dx \approx h\left[\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right]}$$

Ex.: Given $I = \int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln 2 = 0.34657359$

Use the composite trapezoidal formula to approx. I with $n = 10$.

Sol.: We have $n = 10$, $a = 0$, $b = 1 \Rightarrow h = \frac{1}{10}$;

$$\left.\begin{array}{l} x_0 = a = 0;\quad x_k = x_0 + k.h = \frac{k}{10} \\ f(x) = \frac{x}{x^2+1} \Rightarrow y_k = f(x_k) = \frac{k/10}{\left(\frac{k}{10}\right)^2+1} = \frac{10k}{k^2+100} \end{array}\right\}\ k = 0,1,\ldots,10$$

$$\Rightarrow I \approx h\left[\frac{y_0}{2} + y_1 + y_2 + \cdots + y_9 + \frac{y_{10}}{2}\right]$$

$$= 0.345729213 \quad (\text{err} \approx 0.84 \times 10^{-3})$$

Ex.: Given the function $f(x)$ in the Table:

x	0.5	0.6	0.7	0.8	0.9	1.0
y	1.32	1.54	1.61	1.83	1.97	2.15

Use the CTF to approximate:

$$I = \int_{0.5}^{1.0} \underbrace{x f^2(x)}\,dx.$$

Sol: Denote $F(x) = x f^2(x)$

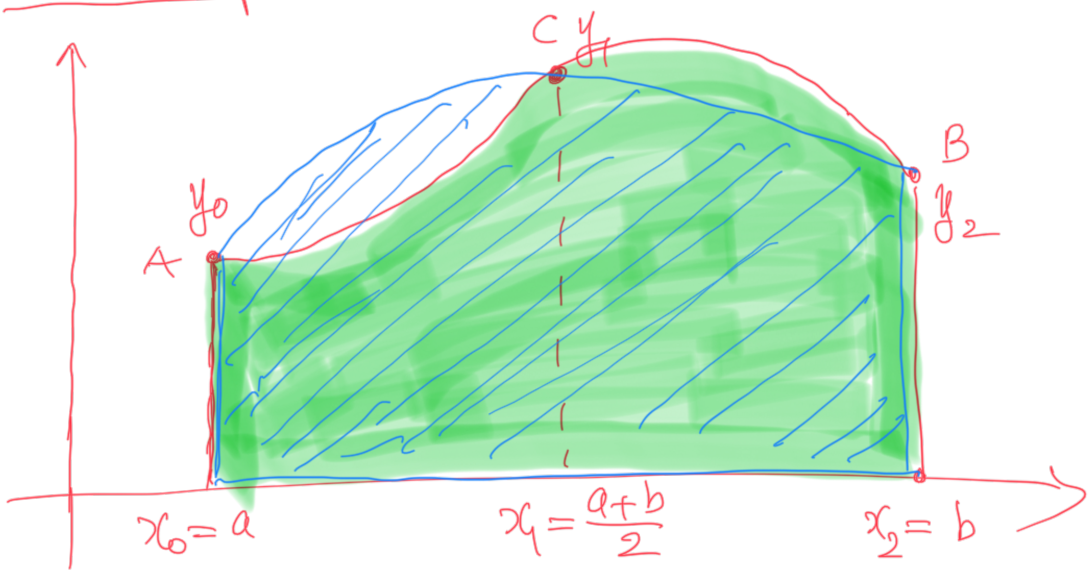
$$\Rightarrow Y_k = x_k y_k^2 \quad , \quad k = 0, 1, \dots 5.$$

We have:

x_k	0.5	0.6	0.7	0.8	0.9	1.0
Y_k	0.8712	1.42296	1.81447	2.67912	3.49281	4.6225

and
$$I \approx h\left[\frac{Y_0}{2} + Y_1 + Y_2 + Y_3 + Y_4 + \frac{Y_5}{2}\right]$$
$$= 1.215621.$$

b) The Simpson Formula:



$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Now we also divide $[a,b]$ into $n = 2m$ equal subintervals by $h = \frac{b-a}{n}$; $x_k = x_0 + kh$; $y_k = f(x_k)$, $k = 0, 1, \dots, n$

We have: the composite Simpson formula:

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[y_0 + y_{2m} + 4\left(y_1 + y_3 + \dots + y_{2m-1}\right) + 2\left(y_2 + y_4 + \dots + y_{2m-2}\right)\Big]}$$

Ex.: Given $I = \int_0^1 \frac{x}{x^2+1}\,dx = 0.\underline{346573}5903$

Use the CSF to approximate I with $n = 10$.

Sol.: We have $n=10$, $h = \frac{1}{10}$, $x_k = \frac{k}{10}$, $y_k = \frac{10k}{k^2+100}$, $k = \overline{0;10}$

$$\Rightarrow I \approx \frac{h}{3}\Big[y_0 + y_{10} + 4(y_1 + y_3 + \dots + y_9) + 2(y_2 + y_4 + \dots + y_8)\Big]$$

$$= 0.\underline{346577}8399 \quad (\text{err} \simeq 4.3 \times 10^{-6})$$