Chapter 2

SYSTEMS OF LINEAR EQUATIONS

In this chapter, we will consider the following linear system of equations:

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{array}\right. \iff AX = B \tag{2.1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2.2}$$

We assume that $\det A \neq 0$; it means that there exists an unique solution of the system (2.1) in the form $X = A^{-1}B$. Firstly, we consider three cases when the matrix A has the special type:

Matrix A has a diagonal form:

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \implies \begin{cases} x_k = \dfrac{b_k}{a_{kk}} \\ k = 1, 2, \dots, n \end{cases}$$

Matrix A has an upper triangular form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \implies \begin{cases} x_n = \dfrac{b_n}{a_{nn}} \\ x_k = \dfrac{1}{a_{kk}} \left(b_k - \sum\limits_{j=k+1}^{n} a_{kj}x_j \right) \\ k = n-1, n-2, \dots, 1 \end{cases}$$

Matrix A has a lower triangular form:

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \Longrightarrow \begin{cases} x_1 = \dfrac{b_1}{a_{11}} \\ x_k = \dfrac{1}{a_{kk}} \left(b_k - \sum\limits_{j=1}^{k-1} a_{kj} x_j \right) \\ k = 2, 3, \dots, n \end{cases}$$

2.1 LU Factorization

Gaussian elimination is the principal tool in the direct solution of linear systems of equations, so it should be no suprise that it appears in other guises. In this section we will see that the steps used to solve a system of the form $AX = B$ can also be used to factor a matrix into a product of matrices. The factorization is particularly useful when it has the form $A = LU$, where L is lower triangular and U is upper triangular.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix} = LU \qquad (2.3)$$

Then we have $AX = B \Leftrightarrow LUX = L(UX) = B \Leftrightarrow \begin{cases} LY = B \\ UX = Y \end{cases}$. This means that instead of solving one system we used to solve two easier system in which the coefficient matrices have the triangular forms. There are many ways to determine the matrices L and U. In this section we consider two of them:

Doolittle's Method: In this case we assume $l_{11} = l_{22} = \dots = l_{nn} = 1$. The formula to compute l_{ij} and u_{ij} is as follows:

$$\begin{cases} u_{1j} &= a_{1j}, \ j = 1, 2, \dots, n \\ l_{i1} &= \dfrac{a_{i1}}{u_{11}}, \ i = 2, 3, \dots, n \\ u_{ij} &= a_{ij} - \sum\limits_{k=1}^{i-1} l_{ik} u_{kj}, \ 2 \leqslant i \leqslant j \leqslant n \\ l_{ij} &= \dfrac{1}{u_{jj}} \left(a_{ij} - \sum\limits_{k=1}^{j-1} l_{ik} u_{kj} \right), \ 2 \leqslant j < i \leqslant n \end{cases} \qquad (2.4)$$

Crout's Method: In this case we assume $u_{11} = u_{22} = \dots = u_{nn} = 1$. The formula to compute l_{ij} and u_{ij} is as follows:

$$\begin{cases} l_{i1} &= a_{i1}, \ i = 1, 2, \dots, n \\ u_{1j} &= \dfrac{a_{1j}}{l_{11}}, \ j = 2, 3, \dots, n \\ l_{ij} &= a_{ij} - \sum\limits_{k=1}^{j-1} l_{ik} u_{kj}, \ 2 \leqslant j \leqslant i \leqslant n \\ u_{ij} &= \dfrac{1}{l_{ii}} \left(a_{ij} - \sum\limits_{k=1}^{i-1} l_{ik} u_{kj} \right), \ 2 \leqslant i < j \leqslant n \end{cases} \qquad (2.5)$$

Example 2.1. *Factor the following matrix using Doolittle's method:* $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & 1 & 4 \end{bmatrix}$.
Use this to solve the system $AX = B$ where $B = (1, 2, 3)^T$.

From the formula (2.4) *we have:*

$$\begin{aligned}
u_{11} &= a_{11} = 1; \quad u_{12} = a_{12} = -1; \quad u_{13} = a_{13} = 2; \quad l_{21} = \frac{a_{21}}{u_{11}} = -1; \quad l_{31} = \frac{a_{31}}{u_{11}} = 2; \\
u_{22} &= a_{22} - l_{21}u_{12} = 2 - (-1)(-1) = 1; \quad u_{23} = a_{23} - l_{21}u_{13} = -3 - (-1)(2) = -1; \\
l_{32} &= \frac{1}{u_{22}}(a_{32} - l_{31}u_{12}) = \frac{1}{1}(1 - (2)(-1)) = 3 \\
u_{33} &= a_{33} - l_{31}u_{13} - l32u_{23} = 4 - (2)(2) - (3)(-1) = 3
\end{aligned}$$

So we obtain

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} = LU$$

In order to solve the system, we follow by two steps:

$$\begin{aligned}
LY &= B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \cdot Y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix} \\
UX &= Y \Leftrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot X = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 20/3 \\ 1/3 \\ -8/3 \end{bmatrix}
\end{aligned}$$

2.2 Choleski's Factorization

A matrix A is called *positive-definite* if it is symmetric and if $\mathbf{X}^T A \mathbf{X} > 0$ for every n-dimentional column vector $\mathbf{X} \neq 0$.

Theorem 2.1. *A matrix A is positive-definite if and only if the all main subdeterminants $\Delta_k,\ k = 1, 2, \ldots, n$ are positive.*

Example 2.2. *Given* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 3 & 10 \end{bmatrix}$. *We have* $\Delta_1 = 1 > 0$, $\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 > 0$, *and* $\Delta_3 = \det A = 11 > 0$. *Therefore, A is positive-definite.*

Example 2.3. *Find all values of α such that the matrix* $A = \begin{bmatrix} 1 & \alpha & -1 \\ \alpha & 4 & 1 \\ -1 & 1 & 8 \end{bmatrix}$ *is positive-definite.*
We have: $\Delta_1 = 1 > 0$, $\Delta_2 = \begin{vmatrix} 1 & \alpha \\ \alpha & 4 \end{vmatrix} = 4 - \alpha^2 > 0 \Leftrightarrow -2 < \alpha < 2$, $\Delta_3 = \det A = 27 - 2\alpha - \alpha^2 > 0 \Leftrightarrow -1.9664 < \alpha < 1.7164$. *Finally,* $-1.9664 < \alpha < 1.7164$.

Theorem 2.2. *The matrix A is positive-definite if and only if A can be factorized in the form CC^T, where C is lower triangular matrix with nonzero diagonal entries.*

If $C = (c_{ij})$, we obtain

$$\left\{\begin{array}{lcl} c_{11} & = & \sqrt{a_{11}}, \\ c_{i1} & = & \dfrac{a_{i1}}{c_{11}}, \quad i = 2, 3, \dots, n \\ c_{kk} & = & \sqrt{a_{kk} - \sum\limits_{j=1}^{k-1} c_{ki}^2}, \quad k = 2, 3, \dots, n \\ c_{ik} & = & \dfrac{1}{c_{kk}}\left(a_{ik} - \sum\limits_{j=1}^{k-1} c_{ij}c_{kj}\right), \quad k+1 \leqslant i \leqslant n \end{array}\right. \tag{2.6}$$

Example 2.4. *Solve the following system using Choleski's method:*

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

We have $\Delta_1 = 1 > 0$, $\Delta_2 = 3 > 0$, $\Delta_3 = 8 > 0$ *so that the matrix* A *is positive-definite. From* (2.6), *we have* $c_{11} = 1$, $c_{21} = 1$, $c_{31} = -1$, $c_{22} = \sqrt{4 - 1^2} = \sqrt{3}$, $c_{32} = (1 - (-1))/\sqrt{3} = 2/\sqrt{3}$, $c_{33} = \sqrt{5 - (-1)^2 - (2/\sqrt{3})^2} = \sqrt{8/3}$. *The solution of the system is:*

$$CY = B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ -1 & 2/\sqrt{3} & \sqrt{8/3} \end{bmatrix} \cdot Y = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 3 \\ 1/\sqrt{3} \\ (22/3)\sqrt{3/8} \end{bmatrix}$$
$$C^T X = Y \Leftrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & \sqrt{3} & 2/\sqrt{3} \\ 0 & 0 & \sqrt{8/3} \end{bmatrix} \cdot X = \begin{bmatrix} 3 \\ 1/\sqrt{3} \\ (22/3)\sqrt{3/8} \end{bmatrix} \Rightarrow X = \begin{bmatrix} 29/4 \\ -3/2 \\ 11/4 \end{bmatrix}$$

2.3 Norms of Vectors and Matrices

A vector norm on $\mathbb{R}^n$ is a function, denoted by $\|.\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(a) $\forall x \in \mathbb{R}^n,\ \|x\| \geqslant 0;\ \|x\| = 0 \Leftrightarrow x = 0.$

(b) $\forall x \in \mathbb{R}^n,\ \forall \alpha \in \mathbb{R},\ \|\alpha x\| = |\alpha| \cdot \|x\|.$

(c) $\forall x, y \in \mathbb{R}^n,\ \|x + y\| \leqslant \|x\| + \|y\|$

The followings are specific norms on $\mathbb{R}^n$. If $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ then we define:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| = \sum_{k=1}^{n} |x_k| \tag{2.7}$$
$$\|x\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|) = \max_{k=\overline{1,n}} |x_k| \tag{2.8}$$

The distance between two vectors x and y is defined by $\|x - y\|$.

Depending on the specific norm, we obtain the following formula for distance between two vectors:

$$\|x-y\|_1=\sum_{k=1}^{n}|x_k-y_k|\,,\qquad \|x-y\|_\infty=\max_{k=\overline{1,n}}|x_k-y_k|$$

A sequence $\{x^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to converge to x with respect to the norm $\|.\|$ if, given any $\varepsilon>0$, there exists an integer $N(\varepsilon)$ such that

$$\left\|x^{(k)}-x\right\|<\varepsilon \quad \textit{for all } \ k\geqslant N(\varepsilon)$$

Theorem 2.3. *A sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in $\mathbb{R}^n$ with respect to the norm $\|.\|_\infty$ if $\lim\limits_{k\to\infty}x_i^{(k)}=x_i$ for each $i=1,2,\ldots,n$.*

The natural, or induced, matrix norm associated with the vector norm is defined as follows:

$$\|A\|=\max_{\|x\|=1}\|Ax\| \tag{2.9}$$

Theorem 2.4. *If $A=(a_{ij})$ is an $n\times n$ matrix, then*

$$\|A\|_1 = \max_{1\leqslant j\leqslant n}\left(\sum_{i=1}^{n}|a_{ij}|\right) \tag{2.10}$$

$$\|A\|_\infty = \max_{1\leqslant i\leqslant n}\left(\sum_{j=1}^{n}|a_{ij}|\right) \tag{2.11}$$

Example 2.5. *Given $A=\begin{bmatrix}3&-11&2\\1&-12&3\\2&-13&5\end{bmatrix}$, then*

$$\|A\|_1=\max(6,36,10)=36,\quad \|A\|_\infty=\max(16,16,20)=20$$

A *conditional number* of a matrix A is a number defined by

$$k(A)=\|A\|\cdot\left\|A^{-1}\right\|,\quad \left(k_1(A)=\|A\|_1\cdot\left\|A^{-1}\right\|_1,\ k_\infty(A)=\|A\|_\infty\cdot\left\|A^{-1}\right\|_\infty\right)$$

Example 2.6. *Given $A=\begin{bmatrix}1&1&2\\1&2&3\\2&3&4\end{bmatrix}$. Find $k_\infty(A)$.*

We have $A^{-1}=\begin{bmatrix}1&-2&1\\-2&0&1\\1&1&-1\end{bmatrix}$. And $\|A\|_\infty=9$, $\left\|A^{-1}\right\|_\infty=4\Rightarrow k_\infty(A)=36$.

2.4 Iterative Methods

An iterative technique to solve the linear system $AX = B$ starts with an initial approximation $X^{(0)}$ to the solution X and generates a sequence of vectors $\{X^{(k)}\}$ that converges to X. Iterative techniques involve a process that converts the system $AX = B$ into an equivalent system $X = TX + C$ for some fixed matrix T and vector C. After the initial vector $X^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing

$$X^{(k)} = TX^{(k-1)} + C \tag{2.12}$$

for each $k = 1, 2, 3, \dots$.

Theorem 2.5. *If $\|T\| = q < 1$ then the sequence of vectors $X^{(k)}$ from* (2.12) *will converge to solution X and satisfy:*

$$\left\|X^{(k)} - X\right\| \leqslant \frac{q}{1-q}\left\|X^{(k)} - X^{(k-1)}\right\| \tag{2.13}$$

We consider the case that matrix A is split into three matrices as follows:

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \\ &= \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \dots & 0 \\ -a_{21} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ 0 & 0 & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \\ &= D - L - U \end{aligned}$$

The equation $AX = B$ is equivalent to $(D-L-U)X = B$, and depending on how to transform this equation, we can obtain different methods for solving with iterative technique. We consider two of them:

Jacobian Iterative Method: We transform the above equation into $DX = (L + U)X + B$ and, finally,

$$X = D^{-1}(L + U)X + D^{-1}B$$

Introducing the notation $T_j = D^{-1}(L + U)$ and $C_j = D^{-1}B$, the Jacobian technique has the form

$$X^{(k)} = T_j X^{(k-1)} + C_j$$

or in the coordinate words:

$$x_i^{(k)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i\right), \quad i = 1, 2, \dots, n$$

Example 2.7. *Given* $\begin{cases} 8x_1 & + & 2x_2 & = & 5 \\ -x_1 & + & 4x_2 & = & 6 \end{cases}$. *Use Jacobian method with* $X^{(0)} = (0,0)^T$ *to find* $X^{(3)}$ *and its error.*

This system can be transformed into the form $X = T_jX + C_j$ *where* $T_j = \begin{bmatrix} 0 & -0.25 \\ 0.25 & 0 \end{bmatrix}$ *and* $C_j = \begin{bmatrix} 0.625 \\ 1.5 \end{bmatrix}$. *With* $X^{(0)} = (0,0)^T$ *we obtain*

$$X^{(1)} = \begin{bmatrix} 0.625 \\ 1.5 \end{bmatrix}, \; X^{(2)} = \begin{bmatrix} 0.25 \\ 1.65625 \end{bmatrix}, \; X^{(3)} = \begin{bmatrix} 0.2109375 \\ 1.5625 \end{bmatrix}$$

Because of $\|T\|_\infty = 0.25$, *the error of* $X^{(3)}$ *is*

$$\left\| X - X^{(3)} \right\|_\infty \leqslant \frac{0.25}{1 - 0.25} \left\| X^{(3)} - X^{(2)} \right\|_\infty = 0.03125$$

Gauss-Seidel Iterative Method: We transform the above equation into $(D-L)X = UX + B$ and, finally,

$$X = (D-L)^{-1}UX + (D-L)^{-1}B$$

Introducing the notation $T_g = (D-L)^{-1}U$ and $C_g = (D-L)^{-1}B$, the Gauss-Seidel technique has the form

$$X^{(k)} = T_gX^{(k-1)} + C_g$$

or in the coordinate words:

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i \right), \quad i = 1, 2, \ldots, n$$

Example 2.8. *Consider the same system in the example 2.6. Gauss-Seidel technique gives* $T_g = \begin{bmatrix} 0 & -0.25 \\ 0 & -0.0625 \end{bmatrix}$ *and* $C_g = \begin{bmatrix} 0.625 \\ 1.65625 \end{bmatrix}$. *With* $X^{(0)} = (0,0)^T$ *we obtain*

$$X^{(1)} = \begin{bmatrix} 0.625 \\ 1.65625 \end{bmatrix}, \; X^{(2)} = \begin{bmatrix} 0.2109375 \\ 1.552734375 \end{bmatrix}, \; X^{(3)} = \begin{bmatrix} 0.2368164063 \\ 1.559204102 \end{bmatrix}$$

Because of $\|T\|_\infty = 0.25$, *the error of* $X^{(3)}$ *is*

$$\left\| X - X^{(3)} \right\|_\infty \leqslant \frac{0.25}{1 - 0.25} \left\| X^{(3)} - X^{(2)} \right\|_\infty \approx 0.00863$$

2.5 Exercise

Question 1. Use Doolittle's and Crout's methods to find a factorization of the form $A = LU$ for the following matrices:

$$(a) \; A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \qquad (b) \; A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad (c) \; A = \begin{bmatrix} 7 & 2 & 3 \\ 2 & 8 & 5 \\ 4 & 3 & 9 \end{bmatrix}$$

Question 2. Find α so that the following matrices are positive definite

$$(a)\ A = \begin{bmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} \qquad (c)\ A = \begin{bmatrix} 3 & \alpha & 2 \\ \alpha & 2 & 2 \\ 2 & 2 & 11 \end{bmatrix}$$

Question 3. Use Choleski's method to solve the following systems:

$$(a) \begin{cases} 2x_1 - x_2 = 3 \\ -x_1 + 2x_2 - x_3 = 4 \\ -x_2 + 2x_3 = 5 \end{cases} \qquad (b) \begin{cases} x_1 + x_2 + 2x_3 = 3 \\ x_1 + 2x_2 - 3x_3 = -3 \\ 2x_1 - 3x_2 + 3x_3 = 1 \end{cases}$$

Question 4. Find conditional numbers of the following matrices:

$$(a)\ A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix} \qquad (c)\ A = \begin{bmatrix} 1.2 & 0.3 & 0.2 \\ 0.4 & 1.1 & 0.5 \\ 0.4 & 0.3 & 0.9 \end{bmatrix}$$

Question 5. Use Jacobian technique to find the approximative solution $X^{(3)}$ and its error for the following systems:

$$(a) \begin{cases} 12x_1 - 3x_2 = 7 \\ -4x_1 + 11x_2 = 9 \end{cases},\ X^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (b) \begin{cases} 6x_1 + x_2 = 5 \\ 2x_1 + 7x_2 = 3 \end{cases},\ X^{(0)} = \begin{bmatrix} 0.7 \\ 0.4 \end{bmatrix}$$

$$(c) \begin{cases} 5x_1 - x_2 + x_3 = 3 \\ -x_1 + 6x_2 - 2x_3 = 4 \\ 2x_1 - x_2 + 7x_3 = 5 \end{cases},\ X^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(d) \begin{cases} 12x_1 + 3x_2 - 2x_3 = 11 \\ 2x_1 + 15x_2 - x_3 = 12 \\ 3x_1 - 2x_2 + 16x_3 = 13 \end{cases},\ X^{(0)} = \begin{bmatrix} 0.4 \\ 0.5 \\ 0.6 \end{bmatrix}$$

Question 6. Repeat the question 5 using Gauss-Seidel technique.
