DIFFERENTIAL EQUATIONS

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5.1 Cauchy's Problem

This section is concerned with approximating the solution $y(t)$ to a problem of the form

$$y'(t) = \frac{dy}{dt} = f(t, y), \quad \text{for } a \leqslant t \leqslant b \tag{5.1}$$

subject to an initial condition $y(a) = \alpha$. This is usually called *Cauchy's problem.*

5.1.1 Euler's Method

We first consider an equal partition of $[a, b]$: $\{t_0, t_1, \ldots, t_n\}$, where $t_k = a + kh$, for each $k = 0, 1, \ldots, n$ and $h = \dfrac{b-a}{n}$ is called the step size. We will use Taylor's theorem to derive Euler's method.

Suppose that $y(t)$ has two continuous derivatives on $[a, b]$, so that for each $k = 1, 2, \ldots, n$,

$$y(t_k) = y(t_{k-1}) + hy'(t_{k-1}) + \frac{h^2}{2}y''(c_k)$$

for some c_k in (t_{k-1}, t_k), and since $y(t)$ satisfies the differential equation (5.1),

$$y(t_k) = y(t_{k-1}) + hf(t_{k-1}, y(t_{k-1})) + \frac{h^2}{2}y''(c_k) \tag{5.2}$$

Euler's method constructs $y_k \approx y(t_k)$ for each $k = 1, 2, \ldots, n$, by deleting the remain term in (5.2). Thus,

$$\begin{cases} y_0 = \alpha, \\ y_k = y_{k-1} + hf(t_{k-1}, y_{k-1}) \quad \text{for each } k = 1, 2, \ldots, n \end{cases} \tag{5.3}$$

Example 5.1. *Consider the following problem* $\begin{cases} y'(t) = y^2 + t^2, \quad 0 \leqslant t \leqslant 1 \\ y(0) = 0.25 \end{cases}$.
Use Euler's method to approximate $y(t)$ *on* $[0, 1]$ *with the step size* $h = 0.1$.
We have $a = 0, b = 1, h = 0.1 \Rightarrow n = 10$. *Because of* $f(t, y) = t^2 + y^2$ *and of* (5.3), *we obtain*

$$\begin{cases} y_0 = 0.25, \\ y_k = y_{k-1} + 0.1(t_{k-1}^2 + y_{k-1}^2) \quad k = 1, 2, \ldots, 10 \end{cases}$$

Results are given in the following table

x_k	y_k
0.1	0.2562500000
0.2	0.2638164063
0.3	0.2747763159
0.4	0.2913265182
0.5	0.3158136323
0.6	0.3507874573
0.7	0.3990926413
0.8	0.4640201350
0.9	0.5495516035
1.0	0.6607523000

5.1.2 Runge-Kutta Method

Consider the initial-value problem (5.1) in the following form

$$\begin{cases} y'(t) = f(t, y), \quad t \geqslant a \\ y(a) = \alpha \end{cases} \tag{5.4}$$

Denote $t_0 = a$, $t_1 = t_0 + h$, where $h > 0$ is a step size. We need to find $y_1 \approx y(t_1)$. The procedure is as follows:

$$\begin{cases} K_1 = hf\left(t_0, \alpha\right) \\ K_2 = hf\left(t_0 + a_2 h, \alpha + b_{21} K1\right) \\ K_3 = hf\left(t_0 + a_3 h, \alpha + b_{31} K1 + b_{32} K2\right) \\ \dots\dots\dots\dots\dots \\ K_n = hf\left(t_0 + a_n h, \alpha + b_{n1} K1 + b_{n2} K2 + \cdots + b_{n,n-1} K_{n-1}\right) \\ y(t_1) \approx y_1 = \alpha + A_1 K_1 + A_2 K_2 + \cdots + A_n K_n \end{cases}$$

where the constants $a_2, \dots, a_n, b_{21}, b_{31}, \dots, b_{n,n-1}, A_1, \dots, A_n$ are found by the following way.

Consider $\varphi(h) = y(t_0 + h) - (\alpha + A_1 K_1 + A_2 K_2 + \cdots + A_n K_n)$ as a function of h, and MacLaurin's formula

$$\varphi(h) = \varphi(0) + \frac{\varphi'(0)}{1!} h + \frac{\varphi''(0)}{2!} h^2 + \frac{\varphi^{(m)}(0)}{m!} h^m + o(h^m)$$

The above constants are found by the following conditions

$$\varphi(0) = 0,\ \varphi'(0) = 0,\ \varphi''(0) = 0,\ \dots\ \varphi^{(m)}(0) = 0$$

In case of $n = m = 1$, we found Euler's formula (5.3). If $n = m = 2$, then

$$\begin{cases} K_1 = hf\left(t_0, \alpha\right) \\ K_2 = hf\left(t_0 + h, \alpha + K_1\right) \\ y(t_1) \approx y_1 = \alpha + \dfrac{1}{2}(K_1 + K_2) \end{cases} \tag{5.5}$$

The most useful formula is as follows when $n = m = 4$:

$$\begin{cases} K_1 = hf\left(t_0, \alpha\right) \\ K_2 = hf\left(t_0 + \dfrac{h}{2}, \alpha + \dfrac{K_1}{2}\right) \\ K_3 = hf\left(t_0 + \dfrac{h}{2}, \alpha + \dfrac{K_2}{2}\right) \\ K_4 = hf\left(t_0 + h, \alpha + K_3\right) \\ y(t_1) \approx y_1 = \alpha + \dfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{cases} \tag{5.6}$$

Example 5.2. *Consider the following problem* $\begin{cases} y'(t) = 1 + (t - y)^2, \quad t \geqslant 2 \\ y(2) = 1 \end{cases}$ *, which has the exact solution* $y = t + \dfrac{1}{1 - t}$*. Use Runghe-Kutta's method to approximate* $y(2.1)$ *with the step*

$h = 0.1$. *Compare with the exact value.*

We have: $t_0 = 2,\ \alpha = 1,\ h = 0.1,\ f(t, y) = 1 + (t - y)^2$ *and*

$$\begin{aligned}
K_1 &= 0.1[1 + (2 - 1)^2] = 0.2 \\
K_2 &= 0.1[1 + (2 + 0.05 - 1 - 0.2/2)^2] = 0.19025 \\
K_3 &= 0.1[1 + (2 + 0.05 - 1 - 0.19025/2)^2] = 0.1911786266 \\
K_4 &= 0.1[1 + (2 + 0.1 - 1 - 0.1911786266)^2] = 0.1825956289 \\
y(2.1) &\approx y_1 = 1 + \frac{1}{6}(0.2 + 2 \times 0.19025 + 2 \times 0.1911786266 + 0.1825956289) \\
&= 1.190908814
\end{aligned}$$

The exact value is $y(2.1) = 2.1 + \dfrac{1}{1 - 2.1} = 1.1909090909$. *Error:* $|y(2.1) - y_1| \approx 2.77 \times 10^{-7}$

In the case that the problem (5.4) on the interval $[a, b]$ with an equal partition $\{t_0, t_1, \ldots, t_n\}$, where $t_k = a + kh,\ h = \dfrac{b - a}{n}$, we have the following formula

$$\begin{cases}
K_{1k} = hf\left(t_{k-1}, y_{k-1}\right) \\
K_{2k} = hf\left(t_{k-1} + \dfrac{h}{2}, y_{k-1} + \dfrac{K_1}{2}\right) \\
K_{3k} = hf\left(t_{k-1} + \dfrac{h}{2}, y_{k-1} + \dfrac{K_2}{2}\right) \qquad k = 1, 2, \ldots, n \\
K_{4k} = hf\left(t_{k-1} + h, y_{k-1} + K_3\right) \\
y(t_k) \approx y_k = y_{k-1} + \dfrac{1}{6}(K_{1k} + 2K_{2k} + 2K_{3k} + K_{4k})
\end{cases}$$

5.2 Boundary Problem

In this section, we will consider the following boundary linear problem:

$$\begin{cases} y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \quad a \leqslant t \leqslant b \\ y(a) = \alpha,\ y(b) = \beta \end{cases} \tag{5.7}$$

where $p(t), q(t), f(t)$ are given functions, α, β are constants, and $y(t)$ is differentiable function to second order on $[a, b]$.

We will apply the finite-difference method to solve (5.7). Consider an equal partition of $[a, b]$: $\{t_0, t_1, \ldots, t_n\}$, where $t_k = a + kh,\ h = \dfrac{b - a}{n}$. The differential equation on (5.7) gives

$$y''(t_k) + p(t_k)y'(t_k) + q(t_k)y(t_k) = f(t_k), \quad k = 1, 2, \ldots, n - 1$$

Denote $p_k = p(t_k),\ q_k = q(t_k),\ f_k = f(t_k)$, and use the centripetal formulas in the previous chapter to approximate $y'(t_k)$ and $y''(t_k)$, we obtain

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + p_k \frac{y_{k+1} - y_{k-1}}{2h} + q_k y_k = f_k, \quad k = 1, 2, \ldots, n$$

or, including the boundary conditions,

$$\begin{cases} \left(\dfrac{1}{h^2} - \dfrac{p_k}{2h}\right) y_{k-1} + \left(q_k - \dfrac{2}{h^2}\right) y_k + \left(\dfrac{1}{h^2} + \dfrac{p_k}{2h}\right) y_{k+1} = f_k, \quad k = 1, 2, \ldots, n \\ y_0 = \alpha, \quad y_n = \beta \end{cases} \tag{5.8}$$

This is a system of $n-1$ equations with $n-1$ unknowns $y_1, y_2, \dots, y_{n-1}$.

Example 5.3. *Given the following boundary problem:* $\begin{cases} y''(t) + ty'(t) - 12y(t) = -4t^2, & 0 \leqslant t \leqslant 1 \\ y(0) = 0.5,\ y(1) = 1 \end{cases}$.
Use the finite-difference method to approximate $y(t)$ *on* $[0,1]$ *with the step* $h = 0.25$.
We have: $a = 0, b = 1, h = 0.25 \Rightarrow n = 4$*;* $p_k = 0.25k$*,* $q_k = -12$*,* $f_k = -0.25k^2$*;* $\alpha = 0.5$*,* $\beta = 1$*. The system* (5.8) *has the following form*

$$\begin{cases} -44y_1 & + & 16.5y_2 & & & = & -0.25 - 15.5 \times 0.5 \\ 15y_1 & - & 44y_2 & + & 17y_3 & = & -1 \\ & & 14.5y_2 & - & 44y_3 & = & -2.25 - 17.5 \times 1 \end{cases}$$

The results are

$$y(0.25) \approx 0.31178,\ y(0.5) \approx 0.34657,\ y(0.75) \approx 0.56307$$

5.3 Exercise

Question 1. Given $\begin{cases} y'(t) = ty + \cos t, & t \geqslant 1 \\ y(1) = 0.2 \end{cases}$. Use Euler's and Runge-Kutta's methods to approximate $y(1.1)$ with $h = 0.1$.

Question 2. Use Euler's and Runge-Kutta's methods to approximate the solutions of the following Cauchy's problems:

(a) $\begin{cases} y'(t) = \dfrac{1+t}{1+y}, & 0 \leqslant t \leqslant 1 \\ y(0) = 1 \end{cases}$ with $h = 0.5$.

(b) $\begin{cases} y'(t) = 2y + t^2, & 1 \leqslant t \leqslant 2 \\ y(1) = 1.25 \end{cases}$ with $h = 0.25$.

Question 3. The boundary problem $\begin{cases} y''(t) - 4y(t) = -4t, & 0 \leqslant t \leqslant 1 \\ y(0) = 0,\ y(1) = 2 \end{cases}$ has the solution $y(t) = \dfrac{\mathrm{e}^2}{\mathrm{e}^4 - 1}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}) + t$. Use the finite-difference method to approximate the solution with $h = 0.25$ and compare the results to the exact solution.

Question 4. Use the finite-difference method to approximate the solution to the following boundary problems

(a) $y''(t) + y'(t) - 2y(t) = 0, \quad 0 \leqslant t \leqslant 1,\ y(0) = 0,\ y(1) = 1$; use $h = 0.25$.

(b) $y''(t) + (t+1)y'(t) - 8y(t) = \cos t, \quad 1 \leqslant t \leqslant 2,\ y(1) = 1.3,\ y(2) = 0.2$; use $h = 0.25$.

(c) $y''(t) + ty'(t) - 15t^2y(t) = -3t\,\mathrm{e}^{-t}, \quad 0 \leqslant t \leqslant 1,\ y(0) = 0,\ y(1) = 0$; use $h = 0.2$.