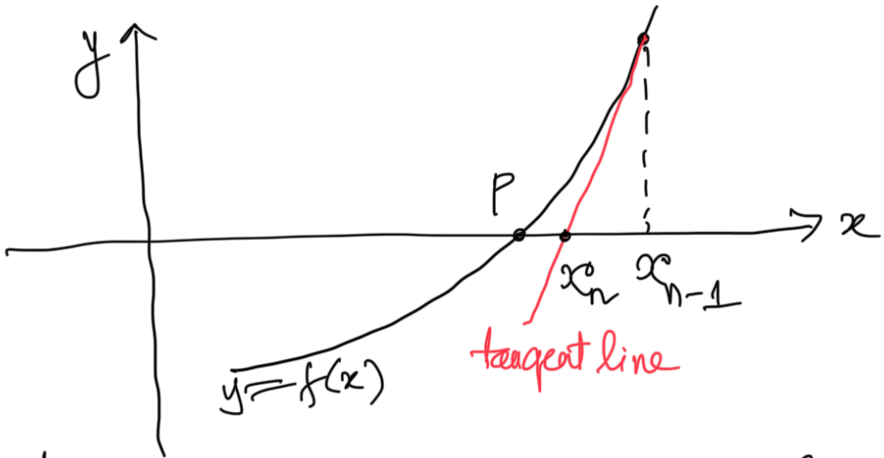
Chapter 1: Non-Linear Equations (cont.)

5. Newton's Method (Newton - Raphson's Method)

We consider the equation $f(x)=0$ in the root-isolated interval $[a,b]$ that consists an unique root p.



We need to construct a sequence $\{x_n\}$. Suppose that we know x_{n-1}. From the point $(x_{n-1}, f(x_{n-1}))$ on the curve $y=f(x)$, we draw a tangent line to the curve. We have the equation of this tangent line

$$y - f(x_{n-1}) = f'(x_{n-1})(x - x_{n-1})$$

This tangent line intercepts with Ox at x_n. It means that in the eq-n of the tangent line we replace y by 0 and x by x_n:

$$-f(x_{n-1}) = f'(x_{n-1})(x_n - x_{n-1})$$

We obtain

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n=1,2,3,\ldots \qquad (1)$$

2. If we know x_0 then from (1) we can determine the sequence $\{x_n\}$.

<u>Theorem:</u> Suppose that:

- $f(x)$ is continuously differentiable to second order in $[a, b]$.
- $f'(x) \neq 0$, $f''(x) \neq 0$ for all $x \in [a, b]$.
- The initial value x_0 is chosen from the following condition (Fourier):

$$f(x_0) . f''(x_0) > 0.$$

Then the sequence, defined by the formula (1), will converge to the root $p \in [a, b]$ of the eq-n $f(x) = 0$.

<u>Notes:</u> (i) Calculate $f'(x)$, $f''(x)$. Show that $f'(x) > 0$ (or $f'(x) < 0$), $f''(x) > 0$ (or $f''(x) < 0$) for all $x \in [a, b]$. After that, find

$$m = \min_{[a,b]} |f'(x)| > 0$$

(ii) From Fourier's condition, we obtain:

- $f' . f'' > 0 \Rightarrow$ choose $x_0 = b$
- $f' f'' < 0 \Rightarrow$ choose $x_0 = a$

(iii) For error-estimating, we can use

$$x_n \approx p \Rightarrow |p - x_n| \leq \Delta_{x_n} = \frac{|f(x_n)|}{m} \quad (2)$$

3. Ex.: Given $f(x) = 2x^3 + 5x - 31 = 0$ in $[2,3]$. Use Newton's method to find x_3 and its error.

Solution: • $f'(x) = 6x^2+5$ and $f''(x) = 12x$

$\forall x \in [2,3]$, $f'(x) = 6x^2+5 > 0$; $f''(x) = 12x > 0$.

And $m = \min_{[2,3]} |f'(x)| = \min_{[2,3]} (6x^2+5) = 6 \times 2^2 + 5 = 29$

• Since $f' \cdot f'' > 0 \Rightarrow$ Choose $x_0 = b = 3$

•
$$\begin{cases} x_0 = 3 \\ x_n = x_{n-1} - \dfrac{2x_{n-1}^3 + 5x_{n-1} - 31}{6x_{n-1}^2 + 5} \\ \Delta x_n = \dfrac{|f(x_n)|}{m} \quad , \quad n = 1, 2, 3, \dots \end{cases}$$

We have:

$x_0 = 3 \Rightarrow x_1 = \frac{139}{59}$; $x_2 = 2.174939544$

and $x_3 = 2.161423331$.

$$\Delta_{x_3} = \frac{|f(x_3)|}{m} = \frac{|2x_3^3 + 5x_3 - 31|}{29} \approx 8.3 \times 10^{-5} = 0.000083.$$

$$\begin{cases} \cdot\ 3\ \boxed{=} \quad (\boxed{Ans} = 3) \\ \cdot\ Ans - \dfrac{2Ans^3 + 5Ans - 31}{6Ans^2 + 5} \quad \underset{(x_1)}{\boxed{=}}\ \underset{(x_2)}{\boxed{=}}\ \underset{(x_3)}{\boxed{=}} \end{cases}$$

Ex.: Given $f(x) = e^x - x^2 + 3x - 2 = 0$ in $[0; 0.5]$

Use Newton's method to find x_3 and its error.

Sol.: $f'(x) = e^x - 2x + 3$; $f''(x) = e^x - 2$.

We see that $\forall x \in [0; 0.5]$, $\underline{f''(x) < 0}$ because $e^{0.5} = 1.648... < 2$.

Since $f''(x) < 0$ in $[0; 0.5]$, then $f'(x)$ is decreasing in $[0; 0.5]$ from $f'(0) = 4$ to $f'(0.5) = 3.6487$

It means $\forall x \in [0; 0.5]$, $\underline{f'(x) > 0}$ and

$$m = \min_{[0;0.5]} |f'(x)| = f'(0.5) = 3.6487$$

Since $f' . f'' < 0 \Rightarrow x_0 = a = 0$.

$$\begin{cases} x_n = x_{n-1} - \dfrac{e^{x_{n-1}} - x_{n-1}^2 + 3x_{n-1} - 2}{e^{x_{n-1}} - 2x_{n-1} + 3} \\ \Delta x_n = \dfrac{|e^{x_n} - x_n^2 + 3x_n - 2|}{3.6487} \end{cases}, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \begin{cases} x_3 = \underline{\underline{0.257530\,2854}} \\ \Delta x_3 = 2.8 \times 10^{-12} \end{cases}$$

<u>Note</u>: If you choose x_0 from the Fourier condition the the sequence $\{x_n\}$ converges to p.

$\underline{x_0 = a = 0}$: $x_n \longrightarrow p$. (True)

$\cancel{x_0 = b = 0.5}$: x_n may converge to p

$\cancel{x_0 = 0.45}$ x_n diverge

5. Ex.: Given $f(x) = x^4 + 3x^2 + 5x - 12 = 0$ on $[1, 2]$. Use Newton's method to find x_3 and its error.

Note:
- Do this exercise on your hand-~~book~~.
- Use your smart phone to take your work.
- Send to me by the following email:

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Chapter 2: THE SYSTEM OF LINEAR EQUATIONS

We consider the system of n linear equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \iff AX = B \quad (1)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

We suppose that $\det A \neq 0 \implies \exists! X = A^{-1}B$

We consider the simple cases of (1):

i) A is a diagonal matrix:

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \implies x_k = \frac{b_k}{a_{kk}}, \; k = 1, 2, \dots, n$$

ii) A is an upper-triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & & & \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \implies \begin{cases} x_n = \dfrac{b_n}{a_{nn}} \\ x_k = \dfrac{1}{a_{kk}}\left(b_k - \sum\limits_{j=k+1}^{n} a_{kj}x_j\right) \\ \forall k = n-1, n-2, \dots, 1. \end{cases}$$

Ex.: Solve the system:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$* \Rightarrow X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ / The system is:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ \quad 2x_2 + 3x_3 = 3 \\ \quad\quad 4x_3 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

iii) A is a lower-triangular matrix:

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \Rightarrow \begin{cases} x_1 = \dfrac{b_1}{a_{11}} \\ x_k = \dfrac{1}{a_{kk}} \left(b_k - \sum_{j=1}^{k-1} a_{kj} x_j \right) \\ k = 2, 3, \dots, n. \end{cases}$$

$\forall$: for all ; $\exists$: there exists

Ex.: • $\forall x \in [a, b]$: for all x ~~from~~ in the interval $[a,b]$

• $\exists x \in [a, b]$: there exists a value x from $[a, b]$

1/ Matrix Factorization:

If the matrix A can be factorize into the product of two matrises L and U, where L - lower-triangular and U - upper-triangular, then $(A = L.U)$

$$AX = B \Leftrightarrow L.\underbrace{U.X}_{Y} = B \Leftrightarrow \begin{cases} LY = B \\ UX = Y \end{cases}$$

It means that in order to solve $AX = B$, we need to solve two systems: $LY = B$ and $UX = Y$.

Ex.: Solve the system $AX = B$ if $A = LU$,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Sol.

- $LY = B \iff \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} Y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 1 \\ 3 \\ 3/2 \end{bmatrix}$

- $UX = Y \iff \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} X = \begin{bmatrix} 1 \\ 3 \\ 3/2 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -1/4 \\ 2 \\ 1/2 \end{bmatrix}$

a) **Doolittle's Method:** $A = L.U$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

We obtain:

$$\begin{cases} u_{1j} = a_{1j}, & j = 1, 2, \dots, n \\ l_{i1} = \dfrac{a_{i1}}{u_{11}}; & i = 2, 3, \dots, n \\ u_{ij} = a_{ij} - \displaystyle\sum_{k=1}^{i-1} l_{ik} u_{kj}, & 2 \le i \le j \le n \\ l_{ij} = \dfrac{1}{u_{jj}} \left(a_{ij} - \displaystyle\sum_{k=1}^{j-1} l_{ik} u_{kj} \right), & 2 \le j < i \le n \end{cases}$$

When $n=3$: $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = L.U$

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \; ; \; U=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\boxed{\begin{array}{l} u_{11}=a_{11} \; ; \; u_{12}=a_{12} \; ; \; u_{13}=a_{13} \; ; \; l_{21}=\dfrac{a_{21}}{u_{11}} \; ; \; l_{31}=\dfrac{a_{31}}{u_{11}} \\ u_{22}=a_{22}-l_{21}u_{12} \; ; \; u_{23}=a_{23}-l_{21}u_{13} \; ; \\ l_{32}=\dfrac{1}{u_{22}}\left(a_{32}-l_{31}u_{12}\right) \; ; \; u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23} \end{array}}$$

Ex.: Solve the system $\begin{cases} x_1-2x_2+2x_3=1 \\ -2x_1+5x_2+x_3=2 \\ 3x_1-x_2+3x_3=3 \end{cases}$

using Doolittle's method.

Sol.: We have: $A=\begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & 1 \\ 3 & -1 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$u_{11}=1$; $u_{12}=-2$; $u_{13}=2$; $l_{21}=\frac{-2}{1}=-2$; $l_{31}=\frac{3}{1}=3$

$u_{22}=5-(-2)(-2)=1$; $u_{23}=1-(-2)(2)=5$

$l_{32}=\frac{1}{1}\left(-1-(3)(-2)\right)=5$

$u_{33}=3-(3)(2)-(5)(5)=-28$

$$\Rightarrow A=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & -28 \end{bmatrix} = L.U$$

- $LY=B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix} Y=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow Y=\begin{bmatrix} 1 \\ 4 \\ -20 \end{bmatrix}$

- $UX=Y \Leftrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & -28 \end{bmatrix} X=\begin{bmatrix} 1 \\ 4 \\ -20 \end{bmatrix} \Rightarrow X=\begin{bmatrix} 3/7 \\ 3/7 \\ 5/7 \end{bmatrix}$

~~10.~~ Ex.: Solve the system $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix} X = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

using Doolittle's method.

Sol.: $u_{11} = 2$; $u_{12} = 1$; $u_{13} = 3$; $l_{21} = \frac{3}{2}$; $l_{31} = \frac{1}{2}$;

$u_{22} = 2 - \left(\frac{3}{2}\right)(1) = \frac{1}{2}$; $u_{23} = 5 - \left(\frac{3}{2}\right)(3) = \frac{1}{2}$;

$$l_{32} = \frac{1}{1/2}\left(3 - \left(\tfrac{1}{2}\right)(1)\right) = 5$$

$$u_{33} = 2 - \left(\tfrac{1}{2}\right)(3) - (5)\left(\tfrac{1}{2}\right) = -2$$

- $LY = B \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 5 & 1 \end{bmatrix} Y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix}$

- $UX = Y \Leftrightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & -2 \end{bmatrix} X = \begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 15/2 \\ 5/2 \\ -9/2 \end{bmatrix}$

b) Crout's Method: $A = L.U$

$$L = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ - & - & - & - \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix} ; \quad U = \begin{bmatrix} 1 & u_{12} & \dots & u_{1n} \\ 0 & 1 & \dots & u_{2n} \\ - & - & - & - \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

We obtain:

$$\begin{cases} l_{i1} = a_{i1} , \quad i = 1, 2, \dots, n \\ u_{1j} = \dfrac{a_{1j}}{l_{11}} , \quad j = 2, 3, \dots, n \\ l_{ij} = a_{ij} - \displaystyle\sum_{k=1}^{j-1} l_{ik} u_{kj} , \quad 2 \le j \le i \le n \\ u_{ij} = \dfrac{1}{l_{ii}}\left(a_{ij} - \displaystyle\sum_{k=1}^{i-1} l_{ik} u_{kj} \right), \quad 2 \le i < j \le n \end{cases}$$

11. In case of $n = 3$:

$$A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = L.U$$

$$l_{11} = a_{11} \,;\; l_{21} = a_{21} \,;\; l_{31} = a_{31} \,;\; u_{12} = \frac{a_{12}}{l_{11}} \,;\; u_{13} = \frac{a_{13}}{l_{11}} \,;$$

$$l_{22} = a_{22} - l_{21} u_{12} \,;\; l_{32} = a_{32} - l_{31} u_{12} \,;$$

$$u_{23} = \frac{1}{l_{22}} \left(a_{23} - l_{21} u_{13} \right) ;\; l_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23}$$

Ex.: Use Crout's method to solve:

$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Sol.: $l_{11} = 2 \,;\; l_{21} = -1 \,;\; l_{31} = 1 \,;\; u_{12} = \frac{2}{2} = 1 \,;\; u_{13} = \frac{3}{2} \,;$

$$l_{22} = 1 - (-1)(1) = 2 \,;\; l_{32} = 3 - (1)(1) = 2$$

$$u_{23} = \frac{1}{2}\left(4 - (-1)\left(\frac{3}{2}\right)\right) = \frac{11}{4}$$

$$l_{33} = 2 - (1)\left(\frac{3}{2}\right) - (2)\left(\frac{11}{4}\right) = -5.$$

$$LY = B \Leftrightarrow \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 2 & -5 \end{bmatrix} Y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 1/2 \\ 3/4 \\ 1/5 \end{bmatrix}$$

$$UX = Y \Leftrightarrow \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1/2 \\ 3/4 \\ 1/5 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 0 \\ 1/5 \\ 1/5 \end{bmatrix}$$

12.

13.

14.

15.