# 13 Design and Analysis of Single-Factor Experiments: The Analysis of Variance

## CHAPTER OUTLINE

# 13-1: Designing Engineering Experiments

Every experiment involves a sequence of activities:

1. Conjecture – the original hypothesis that motivates the experiment.
2. Experiment – the test performed to investigate the conjecture.
3. Analysis – the statistical analysis of the data from the experiment.
4. Conclusion – what has been learned about the original conjecture from the experiment. Often the experiment will lead to a revised conjecture, and a new experiment, and so forth.

# 13-2: The Completely Randomized Single-Factor Experiment

## 13-2.1 An Example

A manufacturer of paper used for making grocery bags is interested in improving the tensile strength of the product. Product engineering thinks that tensile strength is a function of the hardwood concentration in the pulp and that the range of hardwood concentrations of practical interest is between 5 and 20%. A team of engineers responsible for the study decides to investigate four levels of hardwood concentration: 5%, 10%, 15%, and 20%. They decide to make up six test specimens at each concentration level, using a pilot plant. All 24 specimens arc tested on a laboratory tensile tester, in random order. The data from this experiment are shown in Table 13-1.

# 13-2: The Completely Randomized Single-Factor Experiment

## 13-2.1 An Example

**Table 13-1** Tensile Strength of Paper (psi)

| Hardwood Concentration (%) | Observations 1 | 2 | 3 | 4 | 5 | 6 | Totals | Averages |
|---|---|---|---|---|---|---|---|---|
| 5 | 7 | 8 | 15 | 11 | 9 | 10 | 60 | 10.00 |
| 10 | 12 | 17 | 13 | 18 | 19 | 15 | 94 | 15.67 |
| 15 | 14 | 18 | 19 | 17 | 16 | 18 | 102 | 17.00 |
| 20 | 19 | 25 | 22 | 23 | 18 | 20 | 127 | 21.17 |
| | | | | | | | 383 | 15.96 |

# 13-2: The Completely Randomized Single-Factor Experiment

## 13-2.1 An Example

- The levels of the factor are sometimes called **treatments**.
- Each treatment has six observations or **replicates**.
- The runs are run in **random** order.

# 13-2: The Completely Randomized Single-Factor Experiment

## 13-2.1 An Example

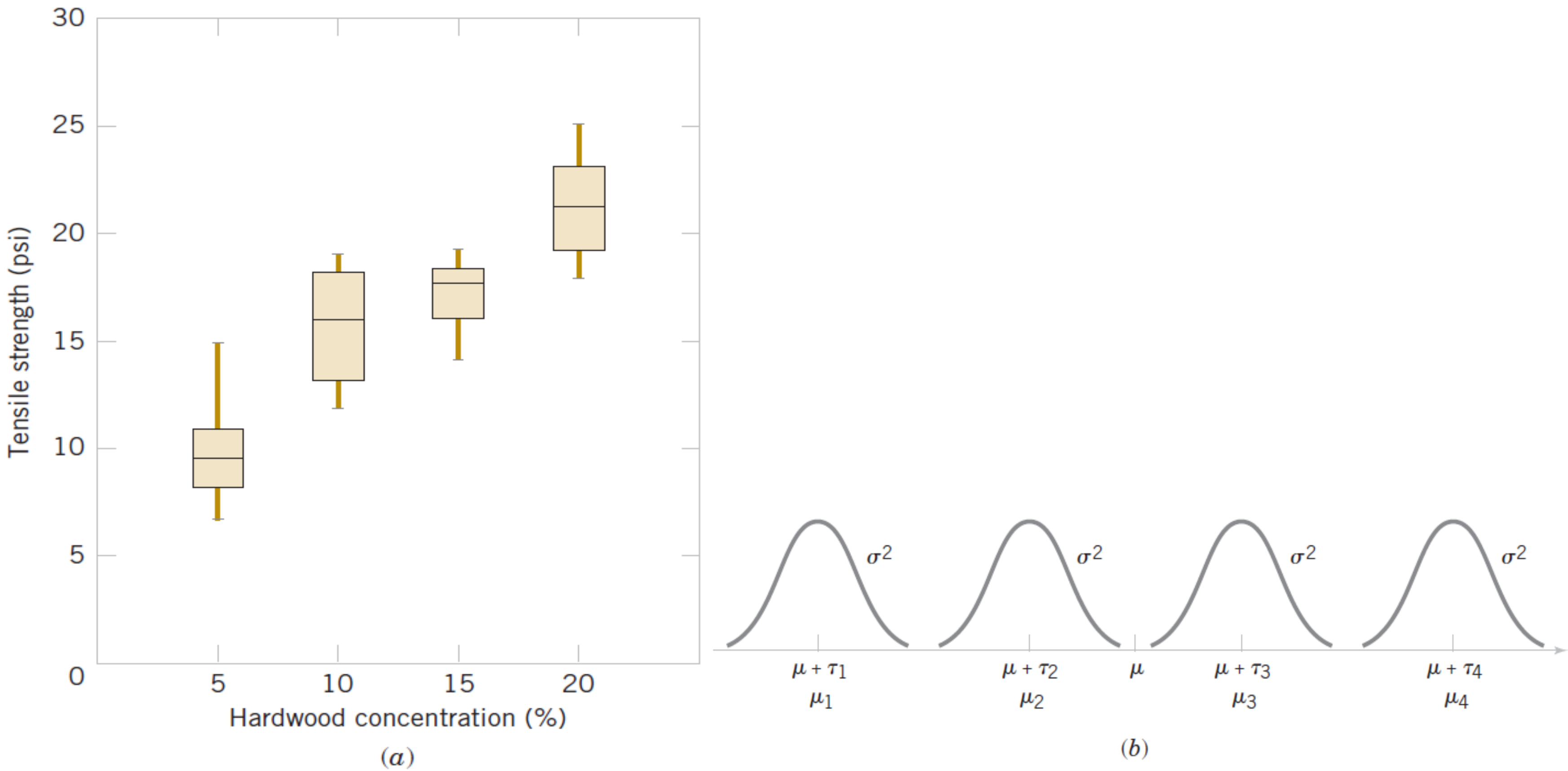


**Figure 13-1** (a) Box plots of hardwood concentration data. (b) Display of the model in Equation 13-1 for the completely randomized single-factor experiment.

## 13-2.2 The Analysis of Variance

### Definition

The **sum of squares identity** is

$$\sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij}-\bar{y}_{..}\right)^2 = n\sum_{i=1}^{a}\left(\bar{y}_{i.}-\bar{y}_{..}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij}-\bar{y}_{i.}\right)^2 \qquad (13\text{-}1)$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_E \qquad (13\text{-}2)$$

## 13-2.2 The Analysis of Variance

The expected value of the treatment sum of squares is

$$E(SS_{\text{Treatments}}) = (a-1)\sigma^2 + n\sum_{i=1}^{a}\tau_i^2$$

and the expected value of the error sum of squares is

$$E(SS_E) = a(n-1)\sigma^2$$

The ratio $MS_{\text{Treatments}} = SS_{\text{Treatments}}/(a - 1)$ is called the **mean square for treatments**.

## 13-2.2 The Analysis of Variance

The appropriate test statistic is

$$F_0 = \frac{SS_{\text{Treatments}}/(a-1)}{SS_E/[a\,(n-1)]} = \frac{MS_{\text{Treatments}}}{MS_E} \qquad (13\text{-}3)$$

We would reject $H_0$ if $f_0 > f_{\alpha,a-1,a(n-1)}$

## 13-2.2 The Analysis of Variance

### Definition

The sums of squares computing formulas for the ANOVA with equal sample sizes in each treatment are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N} \qquad (13\text{-}4)$$

and

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i\cdot}^2}{n} - \frac{y_{\cdot\cdot}^2}{N} \qquad (13\text{-}5)$$

The error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}} \qquad (13\text{-}6)$$

## 13-2.2 The Analysis of Variance

### Analysis of Variance Table

**Table 13-3** The Analysis of Variance for a Single-Factor Experiment, Fixed-Effects Model

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F_0$ |
|---|---|---|---|---|
| Treatments | $SS_{\text{Treatments}}$ | $a - 1$ | $MS_{\text{Treatments}}$ | $\dfrac{MS_{\text{Treatments}}}{MS_E}$ |
| Error | $SS_E$ | $a(n - 1)$ | $MS_E$ | |
| Total | $SS_T$ | $an - 1$ | | |

**EXAMPLE 13-1** Tensile Strength ANOVA Consider the paper tensile strength experiment described in Section 13-2.1. This experiment is a CRD. We can use the analysis of variance to test the hypothesis that different hardwood concentrations do not affect the mean tensile strength of the paper.

The hypotheses are

$$H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$$
$$H_1: \tau_i \neq 0 \text{ for at least one } i$$

# 13-2: The Completely Randomized Single-Factor Experiment

## Example 13-1

We will use $\alpha = 0.01$. The sums of squares for the analysis of variance are computed from Equations 13-4, 13-5, and 13-6 as follows:

$$SS_T = \sum_{i=1}^{4} \sum_{j=1}^{6} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N}$$

$$= (7)^2 + (8)^2 + \cdots + (20)^2 - \frac{(383)^2}{24} = 512.96$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{i\cdot}^2}{n} - \frac{y_{\cdot\cdot}^2}{N}$$

$$= \frac{(60)^2 + (94)^2 + (102)^2 + (127)^2}{6} - \frac{(383)^2}{24}$$

$$= 382.79$$

$$SS_E = SS_T - SS_{\text{Treatments}}$$

$$= 512.96 - 382.79 = 130.17$$

## Example 13-1

The ANOVA is summarized in Table 13-4. Since $f_{0.01,3,20} =$ 4.94, we reject $H_0$ and conclude that hardwood concentration in the pulp significantly affects the mean strength of the paper. We can also find a *P*-value for this test statistic as follows:

$$P = P(F_{3,20} > 19.60) \simeq 3.59 \times 10^{-6}$$

Since $P \simeq 3.59 \times 10^{-6}$ is considerably smaller than $\alpha$ = 0.01, we have strong evidence to conclude that $H_0$ is not true.

**Table 13-4** ANOVA for the Tensile Strength Data

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $f_0$ | *P*-value |
|---|---|---|---|---|---|
| Hardwood concentration | 382.79 | 3 | 127.60 | 19.60 | 3.59 E-6 |
| Error | 130.17 | 20 | 6.51 | | |
| Total | 512.96 | 23 | | | |

## Confidence Interval on a Treatment Mean

A 100(1 - $\alpha$) percent confidence interval on the mean of the $i$th treatment $\mu_t$ is

$$\bar{y}_{i\cdot} - t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \le \mu_i \le \bar{y}_{i\cdot} + t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \qquad (13\text{-}7)$$

For 20% hardwood, the resulting confidence interval on the mean is

$$19.00 \text{ psi} \le m_4 \le 23.34 \text{ psi}$$

## Confidence Interval on a Difference in Treatment Means

A 100(1 - $\alpha$) percent confidence interval on the difference in two treatment means $\mu_i - \mu_j$ is

$$\bar{y}_{i\cdot} - \bar{y}_{j\cdot}\, t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i\cdot} - \bar{y}_{j\cdot} + t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \quad \text{(13-8)}$$

For the hardwood concentration example,

$$-1.74 \le m_3 - m_2 \le 4.40$$

## An Unbalanced Experiment

The sums of squares computing formulas for the ANOVA with unequal sample sizes $n_i$ in each treatment are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{..}^2}{N} \qquad (13\text{-}9)$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{N} \qquad (13\text{-}10)$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} \qquad (13\text{-}11)$$





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## 13-2.3 Multiple Comparisons Following the ANOVA

The least significant difference (LSD) is

$$\mathrm{LSD} = t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \tag{13-12}$$

If the sample sizes are different in each treatment:

$$\mathrm{LSD} = t_{\alpha/2,N-a}\sqrt{MS_E\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}$$

## Example 13-2

We will apply the Fisher LSD method to the hardwood concentration experiment. There are $a = 4$ means, $n = 6$, $MS_E = 6.51$, and $t_{0.025,20} = 2.086$. The treatment means are

$$\bar{y}_{1\cdot} = 10.00 \text{ psi}$$
$$\bar{y}_{2\cdot} = 15.67 \text{ psi}$$
$$\bar{y}_{3\cdot} = 17.00 \text{ psi}$$
$$\bar{y}_{4\cdot} = 21.17 \text{ psi}$$

The value of LSD is $\text{LSD} = t_{0.025,20}\sqrt{2MS_E/n} = 2.086\sqrt{2(6.51)/6} = 3.07$. Therefore, any pair of treatment averages that differs by more than 3.07 implies that the corresponding pair of treatment means are different.

# Example 13-2

The comparisons among the observed treatment averages are as follows:

4 vs. 1 = 21.17 - 10.00 = 11.17 $>$ 3.07
4 vs. 2 = 21.17 - 15.67 = 5.50 $>$ 3.07
4 vs. 3 = 21.17 - 17.00 = 4.17 $>$ 3.07
3 vs. 1 = 17.00 - 10.00 = 7.00 $>$ 3.07
3 vs. 2 = 17.00 - 15.67 = 1.33 $<$ 3.07
2 vs. 1 = 15.67 - 10.00 = 5.67 $>$ 3.07

Conclusions: From this analysis, we see that there are significant differences between all pairs of means except 2 and 3. This implies that 10% and 15% hardwood concentration produce approximately the same tensile strength and that all other concentration levels tested produce different tensile strengths.

It is often helpful to draw a graph of the treatment means, such as in Fig. 13-2, with the means that are *not* different underlined. This graph clearly reveals the results of the experiment and shows that 20% hardwood produces the maximum tensile strength.

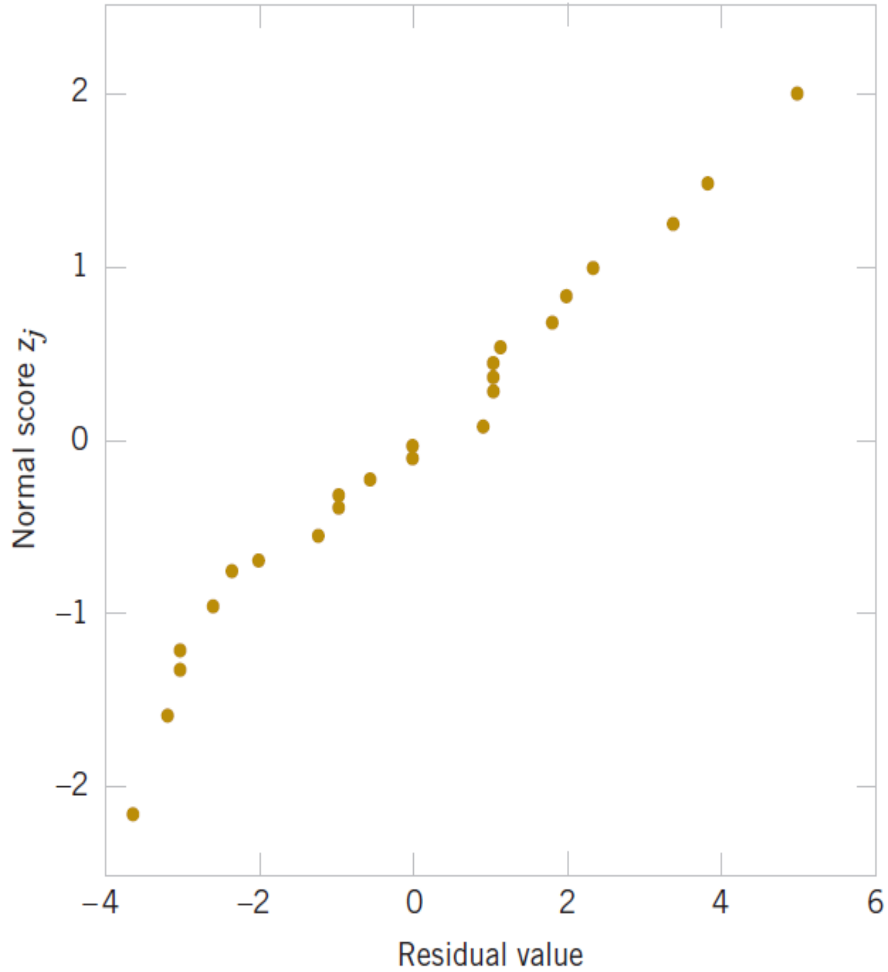
## 13-2.4 Residual Analysis and Model Checking

**Table 13-6** Residuals for the Tensile Strength Experiment

| Hardwood Concentration (%) | Residuals | | | | | |
|---|---|---|---|---|---|---|
| 5 | -3.00 | -2.00 | 5.00 | 1.00 | -1.00 | 0.00 |
| 10 | -3.67 | 1.33 | -2.67 | 2.33 | 3.33 | -0.67 |
| 15 | -3.00 | 1.00 | 2.00 | 0.00 | -1.00 | 1.00 |
| 20 | -2.17 | 3.83 | 0.83 | 1.83 | -3.17 | -1.17 |

# 13-2.4 Residual Analysis and Model Checking

**Figure 13-4** Normal probability plot of residuals from the hardwood concentration experiment.



## 13-2.4 Residual Analysis and Model Checking

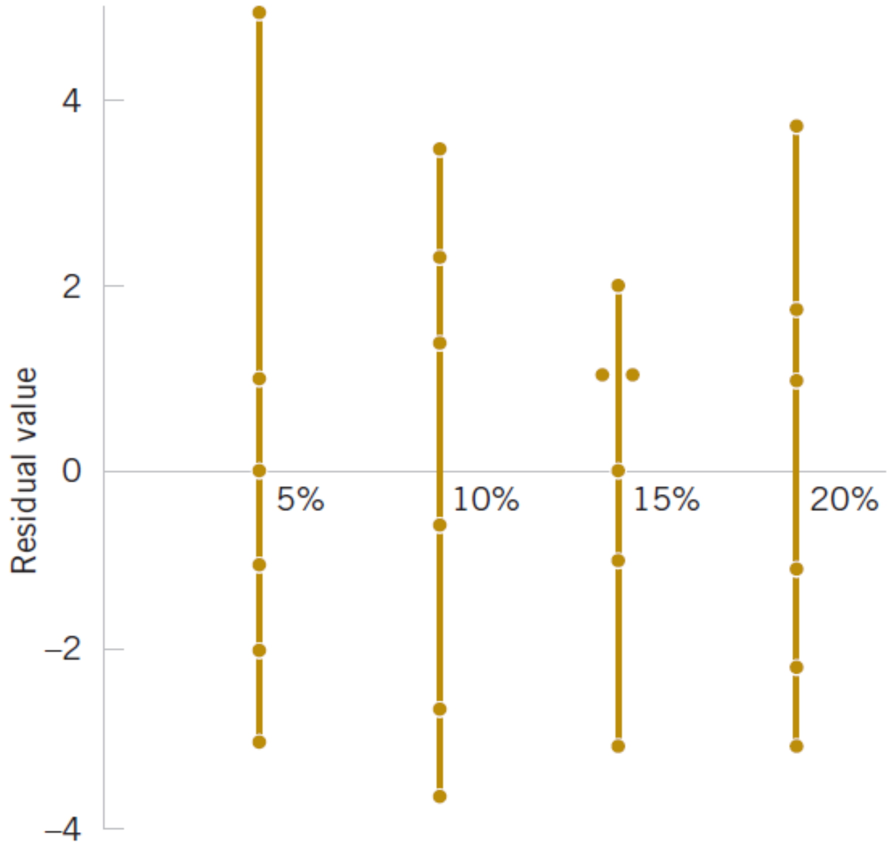
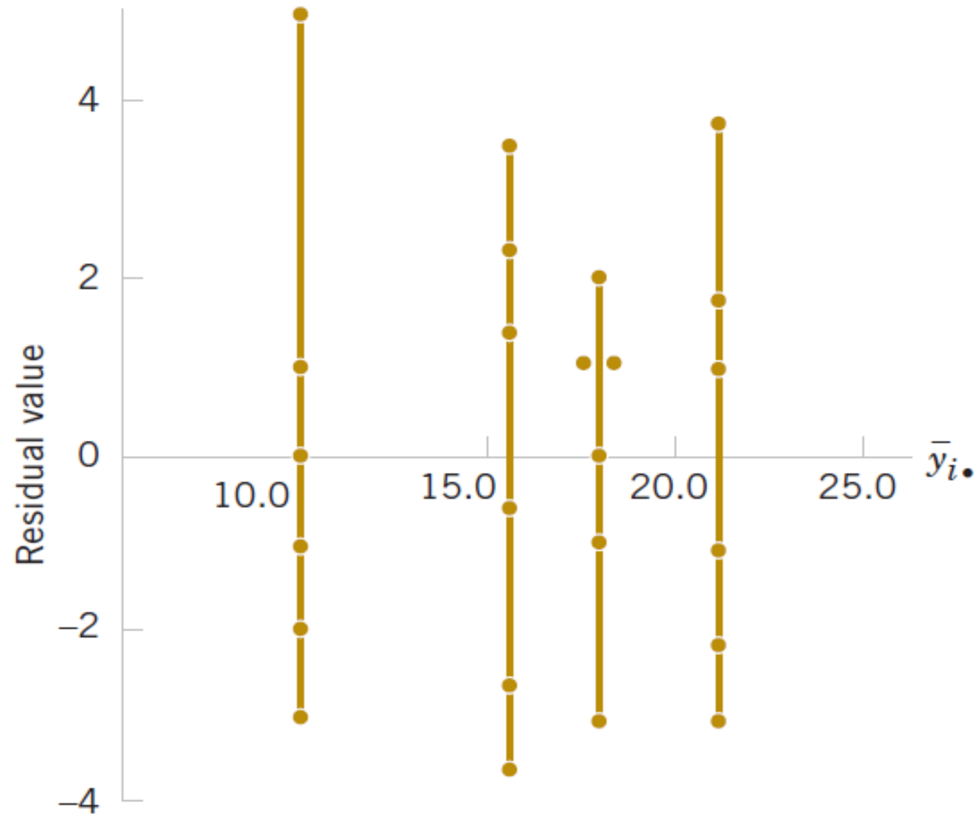



**Figure 13-5** Plot of residuals versus factor levels (hardwood concentration).

# 13-2.4 Residual Analysis and Model Checking

**Figure 13-6** Plot of residuals versus $\overline{y}_{i\cdot}$.



# 13-3: The Random-Effects Model

## 13-3.2 ANOVA and Variance Components

The linear statistical model is

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

The variance of the response is $V(Y_{ij}) = \sigma_\tau^2 + \sigma^2$
Where each term on the right hand side is called a **variance component.**





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# 13-3: The Random-Effects Model

## 13-3.2 ANOVA and Variance Components

For a random-effects model, the appropriate hypotheses to test are:

$$H_0: \sigma_\tau^2 = 0$$
$$H_1: \sigma_\tau^2 > 0$$

The ANOVA decomposition of total variability is still valid:

$$SS_T = SS_{\text{Treatments}} + SS_E$$

# 13-3: The Random-Effects Model

## 13-3.2 ANOVA and Variance Components

The expected values of the mean squares are

In the random-effects model for a single-factor, completely randomized experiment, the expected mean square for treatments is

$$E(MS_{\text{Treatments}}) = E\left(\frac{SS_{\text{Treatments}}}{a-1}\right) \qquad (13\text{-}13)$$

$$= \sigma^2 + n\sigma_\tau^2$$

and the expected mean square for error is

$$E(MS_E) = E\left[\frac{SS_E}{a(n-1)}\right] \qquad (13\text{-}14)$$

$$= \sigma^2$$

# 13-3: The Random-Effects Model

## 13-3.2 ANOVA and Variance Components

The estimators of the variance components are

$$\hat{\sigma}^2 = MS_E \qquad (13\text{-}15)$$

and

$$\hat{\sigma}_\tau^2 = \frac{MS_{\text{Treatments}} - MS_E}{n} \qquad (13\text{-}16)$$

# 13-3: The Random-Effects Model

**Example 13-4** Textile Manufacturing In *Design and Analysis of Experiments,* 7th edition (John Wiley, 2009), D. C. Montgomery describes a single-factor experiment involving the random-effects model in which a textile manufacturing company weaves a fabric on a large number of looms. The company is interested in loom-to-loom variability in tensile strength. To investigate this variability, a manufacturing engineer selects four looms at random and makes four strength determinations on fabric samples chosen

**Table 13-7** Strength Data for Example 13-4

| | | | **Observations** | | | |
|---|---|---|---|---|---|---|
| **Loom** | 1 | 2 | 3 | 4 | Total | Average |
| 1 | 98 | 97 | 99 | 96 | 390 | 97.5 |
| 2 | 91 | 90 | 93 | 92 | 366 | 91.5 |
| 3 | 96 | 95 | 97 | 95 | 383 | 95.8 |
| 4 | 95 | 96 | 99 | 98 | 388 | 97.0 |
| | | | | | 1527 | 95.45 |

**Table 13-8** Analysis of Variance for the Strength Data

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $f_0$ | *P*-value |
|---|---|---|---|---|---|
| Looms | 89.19 | 3 | 29.73 | 15.68 | 1.88 E-4 |
| Error | 22.75 | 12 | 1.90 | | |
| Total | 111.94 | 15 | | | |

# 13-3: The Random-Effects Model

## Example 13-4

From the analysis of variance, we conclude that the looms in the plant differ significantly in their ability to produce fabric of uniform strength. The variance components are estimated by $\hat{\sigma}^2 = 1.90$ and

$$\hat{\sigma}_\tau^2 = \frac{29.73 - 1.90}{4} = 6.96$$

Therefore, the variance of strength in the manufacturing process is estimated by

$$V(Y_{ij}) = \hat{\sigma}_\tau^2 + \hat{\sigma}^2 + 6.96 + 1.90 = 8.86$$

Conclusions: Most of the variability in strength in the output product is attributable to differences between looms.

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

The **randomized block design** is an extension of the paired t-test to situations where the factor of interest has more than two levels.

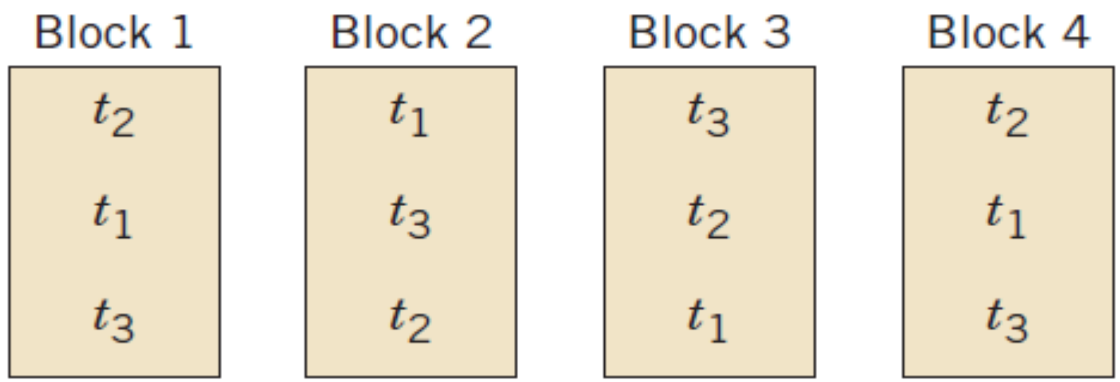


**Figure 13-9** A randomized complete block design.

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

General procedure for a randomized complete block design:

**Table 13-10** A Randomized Complete Block Design with *a* Treatments and *b* Blocks

| | Blocks | | | | | |
|---|---|---|---|---|---|---|
| Treatments | 1 | 2 | … | $b$ | Totals | Averages |
| 1 | $y_{11}$ | $y_{12}$ | … | $y_{1b}$ | $y_{1\cdot}$ | $\bar{y}_{1\cdot}$ |
| 2 | $y_{21}$ | $y_{22}$ | … | $y_{2b}$ | $y_{2\cdot}$ | $\bar{y}_{2\cdot}$ |
| ⋮ | ⋮ | ⋮ | | ⋮ | ⋮ | ⋮ |
| $a$ | $y_{a1}$ | $y_{a2}$ | … | $y_{ab}$ | $y_{a\cdot}$ | $\bar{y}_{a\cdot}$ |
| Totals | $y_{\cdot 1}$ | $y_{\cdot 2}$ | … | $y_{\cdot b}$ | $y_{\cdot\cdot}$ | |
| Averages | $\bar{y}_{\cdot 1}$ | $\bar{y}_{\cdot 2}$ | … | $\bar{y}_{\cdot b}$ | | $\bar{y}_{\cdot\cdot}$ |

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

The appropriate linear statistical model:

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

We assume

- treatments and blocks are initially fixed effects
- blocks do not interact
- $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

We are interested in testing:

The **sum of squares identity for the randomized complete block design** is

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2 + a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{.j}-\bar{y}_{i.}+\bar{y}_{..})^2 \qquad (13\text{-}17)$$

or symbolically

$$SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks}} + SS_E$$

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

The mean squares are:

$$MS_{\text{Treatments}} = \frac{SS_{\text{Treatments}}}{a-1}$$

$$MS_{\text{Blocks}} = \frac{SS_{\text{Blocks}}}{b-1}$$

$$MS_E = \frac{SS_E}{(a-1)(b-1)}$$

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

The expected values of these mean squares are:

$$E(MS_{\text{Treatments}}) = \sigma^2 + \frac{b \sum_{i=1}^{a} \tau_i^2}{a-1}$$

$$E(MS_{\text{Blocks}}) = \sigma^2 + \frac{a \sum_{j=1}^{b} \beta_j^2}{b-1}$$

$$E(MS_E) = \sigma^2$$

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

### Definition

The computing formulas for the sums of squares in the analysis of variance for a randomized complete block design are

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab} \qquad (13\text{-}18)$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab} \qquad (13\text{-}19)$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab} \qquad (13\text{-}20)$$

and

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}} \qquad (13\text{-}21)$$

# 13-4: Randomized Complete Block Design

## 13-4.1 Design and Statistical Analysis

**Table 13-11** ANOVA for a Randomized Complete Block Design

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F_0$ |
|---|---|---|---|---|
| Treatments | $SS_{\text{Treatments}}$ | $a - 1$ | $\dfrac{SS_{\text{Treatments}}}{a-1}$ | $\dfrac{MS_{\text{Treatments}}}{MS_E}$ |
| Blocks | $SS_{\text{Blocks}}$ | $b - 1$ | $\dfrac{SS_{\text{Blocks}}}{b-1}$ | |
| Error | $SS_E$ (by subtraction) | $(a-1)(b-1)$ | $\dfrac{SS_E}{(a-1)(b-1)}$ | |
| Total | $SS_T$ | $ab - 1$ | | |

# 13-4: Randomized Complete Block Design

**Example 13-5 Fabric Strength** An experiment was performed to determine the effect of four different chemicals on the strength of a fabric. These chemicals are used as part of the permanent press finishing process. Five fabric samples were selected, and a RCBD was run by testing each chemical type once in random order on each fabric sample. The data are shown in Table 13-12. We will test for differences in means using an ANOVA with a = 0.01.

**Table 13-12** Fabric Strength Data—Randomized Complete Block Design

| | Fabric Sample | | | | | Treatment Totals | Treatment Averages |
|---|---|---|---|---|---|---|---|
| Chemical Type | 1 | 2 | 3 | 4 | 5 | $y_{i\cdot}$ | $\bar{y}_{\cdot j}$ |
| 1 | 1.3 | 1.6 | 0.5 | 1.2 | 1.1 | 5.7 | 1.14 |
| 2 | 2.2 | 2.4 | 0.4 | 2.0 | 1.8 | 8.8 | 1.76 |
| 3 | 1.8 | 1.7 | 0.6 | 1.5 | 1.3 | 6.9 | 1.38 |
| 4 | 3.9 | 4.4 | 2.0 | 4.1 | 3.4 | 17.8 | 3.56 |
| Block totals $y_{\cdot j}$ | 9.2 | 10.1 | 3.5 | 8.8 | 7.6 | 39.2($y_{\cdot\cdot}$) | |
| Block averages $\bar{y}_{i\cdot}$ | 2.30 | 2.53 | 0.88 | 2.20 | 1.90 | | 1.96($\bar{y}_{\cdot\cdot}$) |

# 13-4: Randomized Complete Block Design

## Example 13-5

The sums of squares for the analysis of variance are computed as follows:

$$SS_T = \sum_{i=1}^{4} \sum_{j=1}^{5} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{ab}$$

$$= (1.3)^2 + (1.6)^2 + \cdots + (3.4)^2 - \frac{(39.2)^2}{20} = 25.69$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{i\cdot}^2}{b} - \frac{y_{\cdot\cdot}^2}{ab}$$

$$= \frac{(5.7)^2 + (8.8)^2 + (6.9)^2 + (17.8)^2}{5}$$

$$- \frac{(39.2)^2}{20} = 18.04$$





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# 13-4: Randomized Complete Block Design

## Example 13-5

$$SS_{\text{Blocks}} = \sum_{j=1}^{5} \frac{y_{\cdot j}^2}{a} - \frac{y_{\cdot\cdot}^2}{ab}$$

$$= \frac{(9.2)^2 + (10.1)^2 + (3.5)^2 + (8.8)^2 + (7.6)^2}{4}$$

$$- \frac{(39.2)^2}{20} = 6.69$$

$$SS_E = SS_T - SS_{\text{Blocks}} - SS_{\text{Treatments}}$$

$$= 25.69 - 6.69 - 18.04 = 0.96$$

The ANOVA is summarized in Table 13-13.
Since $f_0 = 75.13 > f_{0.01,3,12} = 5.95$ (the *P*-value is $4.79 \times 10^{-8}$), we conclude that there is a significant difference in the chemical types so far as their effect on strength is concerned.

# 13-4: Randomized Complete Block Design

## Example 13-5

**Table 13-13** Analysis of Variance for the Randomized Complete Block Experiment

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $f_0$ | *P*-value |
|---|---|---|---|---|---|
| Chemical types (treatments) | 18.04 | 3 | 6.01 | 75.13 | 4.79 E-8 |
| Fabric samples (blocks) | 6.69 | 4 | 1.67 | | |
| Error | 0.96 | 12 | 0.08 | | |
| Total | 25.69 | 19 | | | |

# 13-4: Randomized Complete Block Design

## 13-4.2 Multiple Comparisons

Fisher's Least Significant Difference for Example 13-5

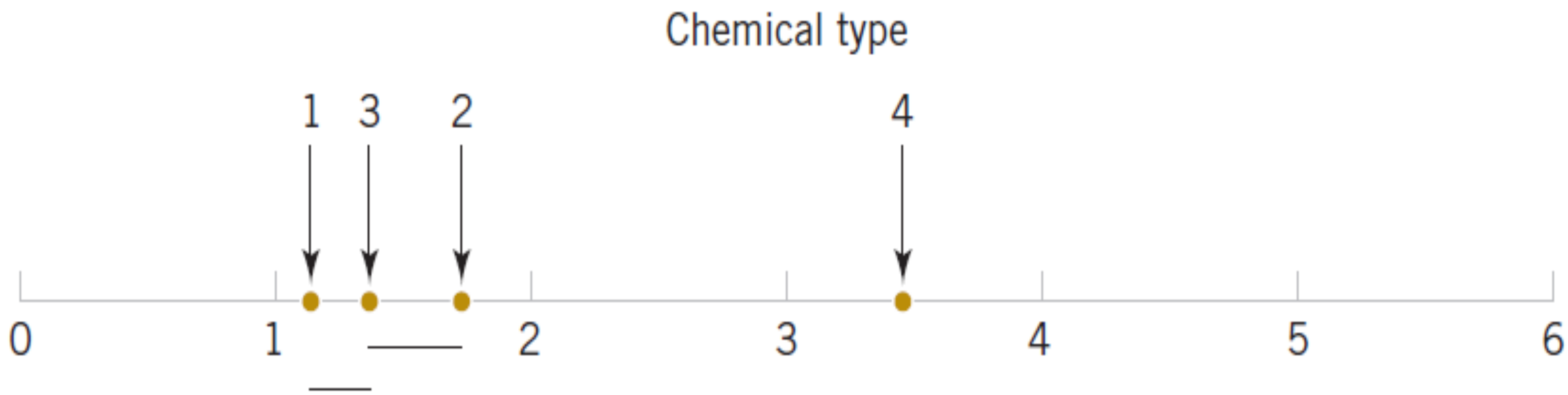
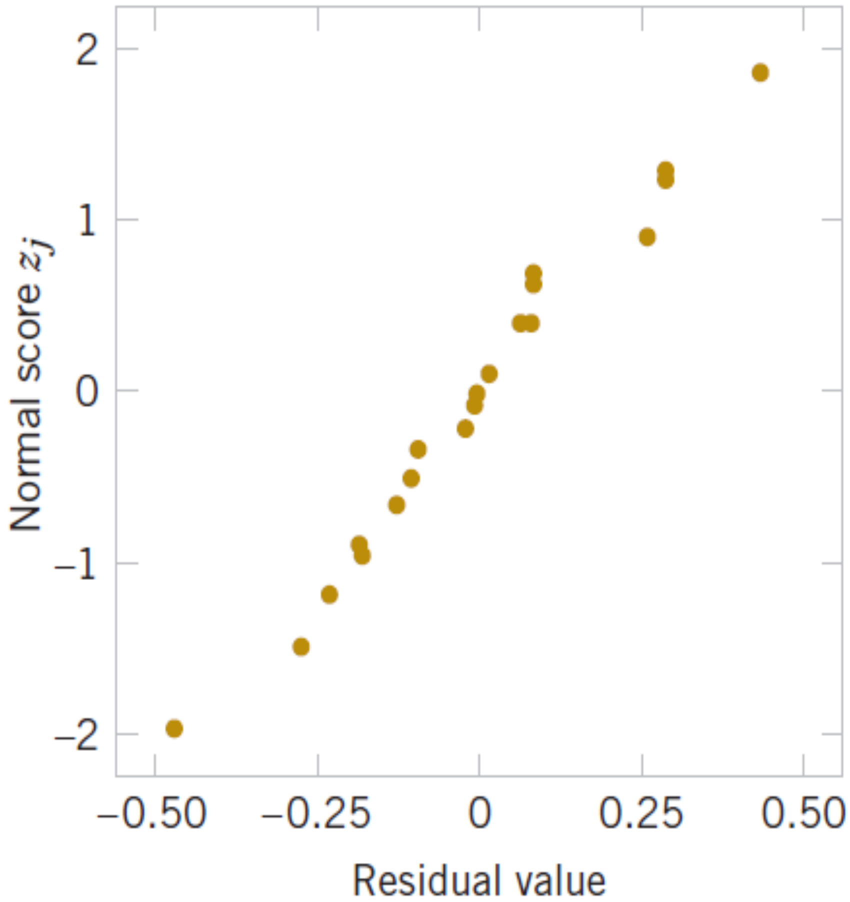



**Figure 13-10** Results of Fisher's LSD method.

# 13-4: Randomized Complete Block Design

## 13-4.3 Residual Analysis and Model Checking

**Figure 13-11** Normal probability plot of residuals from the randomized complete block design.



# 13-4: Randomized Complete Block Design

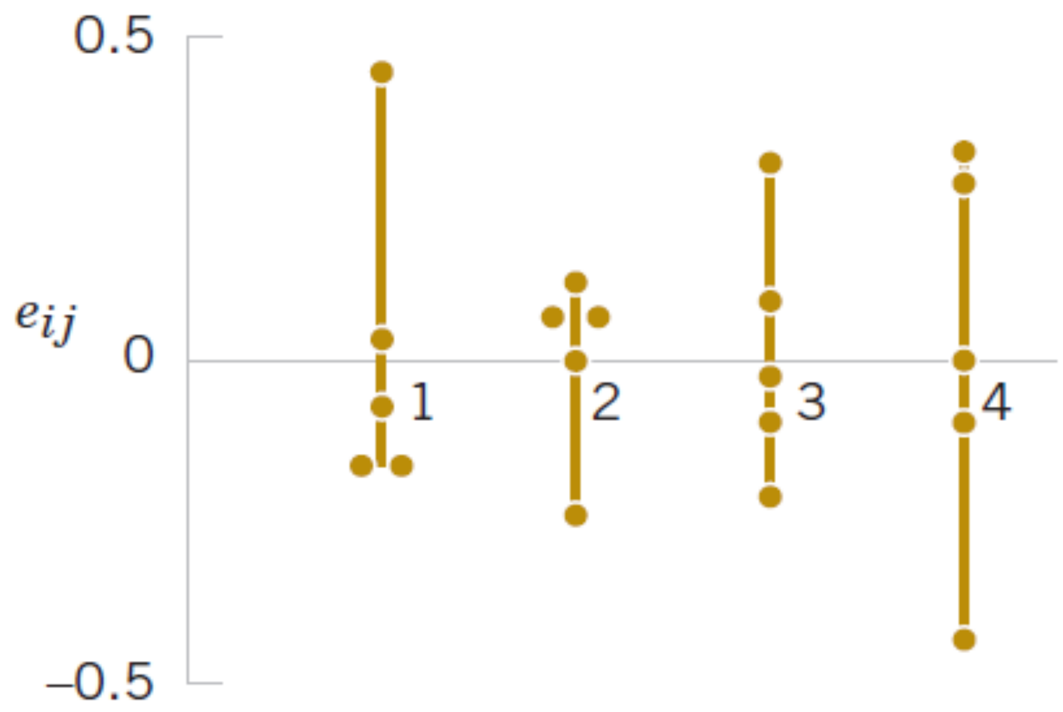



**Figure 13-12** Residuals by treatment from the randomized complete block design.

# 13-4: Randomized Complete Block Design

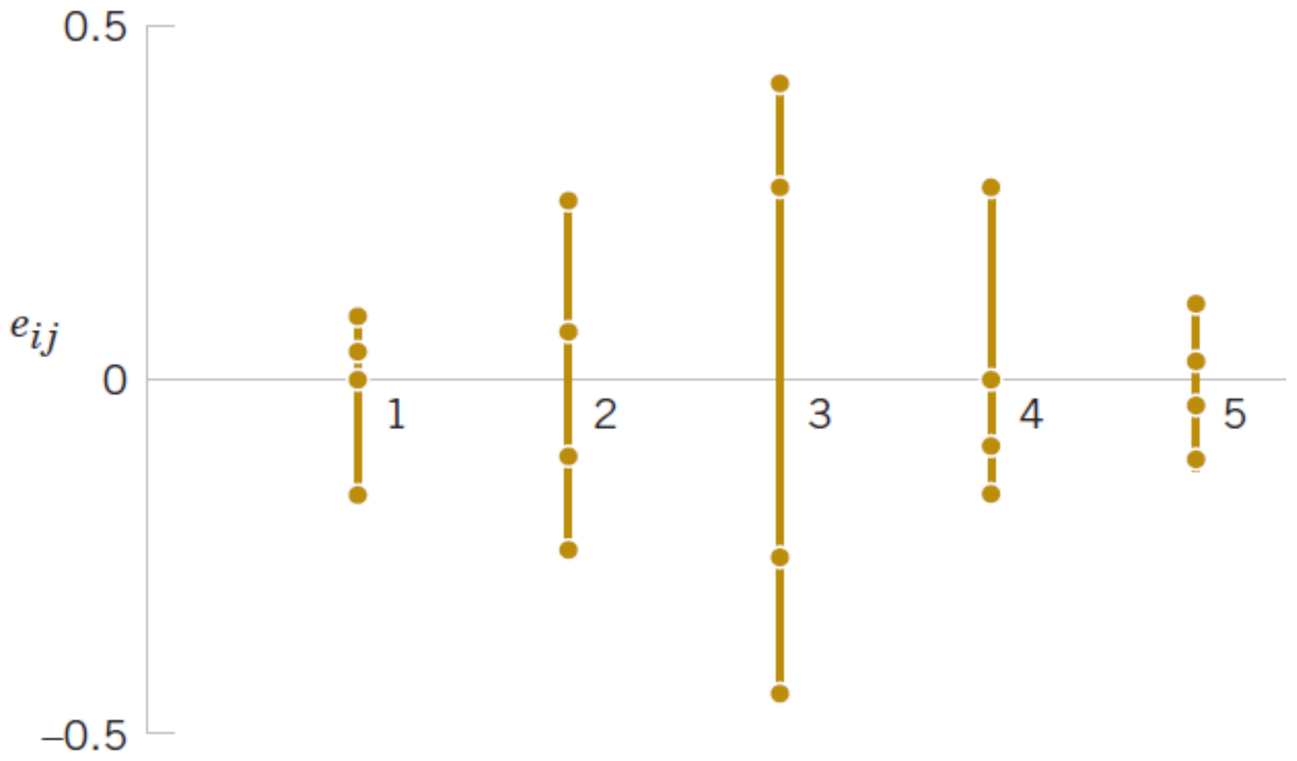



**Figure 13-13** Residuals by block from the randomized complete block design.

# 13-4: Randomized Complete Block Design

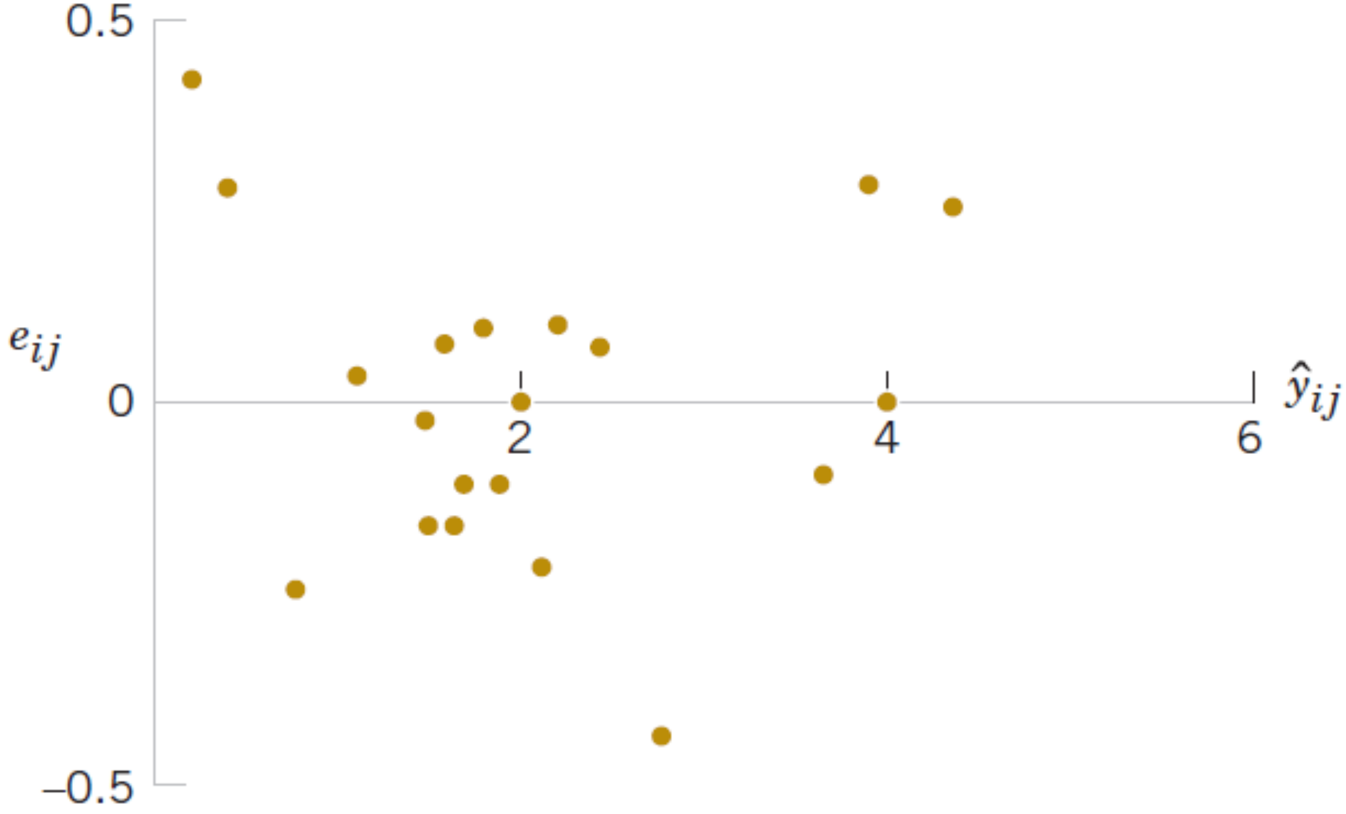



**Figure 13-14** Residuals versus $\hat{y}_{ij}$ from the randomized complete block design.