INTERPOLATION AND POLYNOMIAL APPROXIMATION

LÊ THÁI THANH

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In this chapter we consider the following problem: given the function $y = f(x)$ in the following table

$$\begin{array}{c|cccccc} x & x_0 & x_1 & x_2 & \dots & x_{n-1} & x_n \\ \hline y & y_0 & y_1 & y_2 & \dots & y_{n-1} & y_n \end{array} \tag{3.1}$$

where $x_0 < x_1 < \dots < x_n$ and they are called the nodes of the table. We need to construct a polynomial $P(x)$, satisfying $P(x_i) = y_i$ for all $i = 0, 1, \dots, n$.

3.1 Lagrange Polynomial

Given the table (**??**). We need to find the polynomial $\mathcal{L}_n(x)$ with its degree less than or equal to n and satisfied $\mathcal{L}_n(x_i) = y_i, \quad \forall i = 0, 1, \dots, n$. This polynomial can be represented as follows:

$$\mathcal{L}_n(x) = p_n^{(0)}(x)\ y_0 + p_n^{(1)}(x)\ y_1 + \dots + p_n^{(n)}(x)\ y_n = \sum_{k=0}^{n} p_n^{(k)}(x)\ y_k \tag{3.2}$$

where

$$p_n^{(k)}(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{k-1})(x - x_{k+1})\dots(x - x_n)}{(x_k - x_0)(x_k - x_1)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}, \quad k = 0, 1, \dots, n \tag{3.3}$$

and are called the basic polynomials.

Example 3.1. *Given* $\begin{array}{c|ccc} x & 0 & 1 & 3 \\ \hline y & 2 & 1 & 4 \end{array}$. *We have* $n = 2$ *and*

$$\begin{aligned} p_2^{(0)}(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 1)(x - 3)}{(0 - 1)(0 - 3)} = \frac{1}{3}(x^2 - 4x + 3) \\ p_2^{(1)}(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0)(x - 3)}{(1 - 0)(1 - 3)} = -\frac{1}{2}(x^2 - 3x) \\ p_2^{(2)}(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0)(x - 1)}{(3 - 0)(3 - 1)} = \frac{1}{6}(x^2 - x) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_2(x) &= y_0 p_2^{(0)}(x) + y_1 p_2^{(1)}(x) + y_2 p_2^{(2)}(x) = 2 \cdot \frac{1}{3}(x^2 - 4x + 3) - 1 \cdot \frac{1}{2}(x^2 - 3x) + 4 \cdot \frac{1}{6}(x^2 - x) = \\ &= \frac{5}{6}x^2 - \frac{11}{6}x + 1 \end{aligned}$$

Now if we denote

$$\omega(x) = (x - x_0)(x - x_1)\dots(x - x_n)$$

then the formula (**??**) can be rewritten

$$p_n^{(k)}(x) = \frac{\omega(x)}{D_k}, \quad k = 0, 1, \dots, n$$

where $D_k = (x_k - x_0)(x_k - x_1)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)(x - x_k)$. Finally, Lagrange polynomial in (**??**) has the following form

$$\mathcal{L}_n(x) = \omega(x) \cdot \sum_{k=0}^{n} \frac{y_k}{D_k}$$

The above formula gives us a way to construct Lagrange polynomial by completing the following table:

x	x_0	x_1	$\dots$	x_n	
x_0	$x - x_0$	$x_0 - x_1$	$\dots$	$x_0 - x_n$	D_0
x_1	$x_1 - x_0$	$x - x_1$	$\dots$	$x_1 - x_n$	D_1
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
x_n	$x_n - x_0$	$x_n - x_1$	$\dots$	$x - x_n$	D_n
					$\omega(x)$

Example 3.2. *Given a function $f(x)$ by the following table*

x	1.2	1.5	1.7	2.0
y	2.13	2.41	2.66	2.93

. Use Lagrange polynomial to approximate a value of function $f(x)$ at $x = 1.65$.

We have the following table

$x = 1.65$	1.2	1.5	1.7	2.0	
1.2	0.45	-0.30	-0.50	-0.80	$D_0 = -0.054$
1.5	0.30	0.15	-0.20	-0.50	$D_1 = 0.0045$
1.7	0.50	0.20	-0.05	-0.30	$D_2 = 0.0015$
2.0	0.80	0.50	0.30	-0.35	$D_3 = -0.042$
					$\omega = 0.00118125$

Finally,

$$f(1.65) \approx \omega \left(\frac{y_0}{D_0} + \frac{y_1}{D_1} + \frac{y_2}{D_2} + \frac{y_3}{D_3} \right) = 2.598375 \approx 2.60$$

3.2 Newton Polynomial

We now need to introduce the divided-difference notation. Consider the table (**??**). The *zeroth divided difference* of the function f with respect to x_i, denoted by $f[x_i]$, is simply the value of f at x_i: $f[x_i] = f(x_i)$. The remaining divided differences are defined inductively: the *first divided difference* of the function f with respect to x_i and x_{i+1} is denoted by $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

The *second divided difference* $f[x_i, x_{i+1}, x_{i+2}]$ is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Similarly, after $(k-1)$st divided difference have been determined, the *kth divided difference* relative to $x_i, x_{i+1}, \dots, x_{i+k}$ is given by

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

The determination of the above divided differences from tabulated data points is outlined in the following table

x	$f(x)$	First DD	Second DD	Third DD
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4]$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5]$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

After this, we have two forms of Newton interpolating polynomial:

Newton forward divided-difference formula

$$\begin{aligned} P_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + \\ &+ f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1}) \end{aligned}$$

Newton backward divided-difference formula

$$\begin{aligned} Q_n(x) &= f[x_n] + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_{n-1})(x - x_n) + \cdots + \\ &+ f[x_0, x_1, \dots, x_n](x - x_1)(x - x_2) \dots (x - x_n) \end{aligned}$$

Example 3.3. *Given* $\begin{array}{c|ccc} x & 1 & 3 & 4 \\ \hline y & 4 & 2 & 5 \end{array}$. *We have* $n = 2$ *and*

x	$f(x)$	*First DD*	*Second DD*
$x_0 = 1$	$f[x_0] = 4$		
		$f[x_0, x_1] = \dfrac{2-4}{3-1} = -1$	
$x_1 = 3$	$f[x_1] = 2$		$f[x_0, x_1, x_2] = \dfrac{3-(-1)}{4-1} = \dfrac{4}{3}$
		$f[x_1, x_2] = \dfrac{5-2}{4-3} = 3$	
$x_2 = 4$	$f[x_2] = 5$		

Thus

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) = 4 - (x - 1) + \frac{4}{3}(x - 1)(x - 3)$$

3.3 Cubic Spline Interpolation

The previous sections concerned the approximation of an arbitrary function on a closed interval by a polynomial. However, the oscillatory nature of high-degree polynomials and the property that fluctuation over a small portion of the interval can induce large fluctuations over the entire range restricts their use.

An alternative approach is to devide the interval into a collection of subintervals and construct a different approximating polynomial on each subinterval. Approximation by function of this type is called *piecewise polynomial approximation*.

The simplest piecewise polynomial approximation is piecewise linear interpolation, which consists of joining a set of data point $\{(x_0, y_0),\ (x_1, y_1), \ldots, (x_n, y_n)\}$ by a series of straight line segments. A disadvantage of this type is that there is no assurance of differentiability at each of the endpoints of the subintervals, which, in a geometrical context, means that the interpolating function is not ”smooth” at these points.

The most common piecewise polynomial approximation uses cubic polynomials between each successive pair of nodes and is called *cubic spline interpolation*.

Given the table (**??**) of a function $f(x)$ defined on $[x_0, x_n]$. A *cubic spline interpolant* $g(x)$ for f is a function that satisfies the following conditions:

(a) $g(x)$ is continuously differentiable to second order on $[x_0, x_n]$.

(b) In each subinterval $[x_k, x_{k+1}]$, $g(x)$ is a cubic polynomial, denoted by $g_k(x)$ for each $k = 0, 1, \ldots, n - 1$.

(c) $g(x_k) = y_k = f(x_k)$ for each $k = 0, 1, \ldots, n$.

Consider the subinterval $[x_k, x_{k+1}]$ for each $k = 0, 1, \ldots, n-1$ and denote $h_k = x_{k+1} - x_k$. Since $g_k(x)$ is a cubic polynomial, we can write it in the following form:

$$g_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3$$

From condition (c) we obtain $g_k(x_k) = a_k = y_k$ and

$$g_k(x_{k+1}) = y_{k+1} = a_k + b_k(x_{k+1} - x_k) + c_k(x_{k+1} - x_k)^2 + d_k(x_{k+1} - x_k)^3 = a_k + b_k h_k + c_k h_k^2 + d_k h_k^3$$

And

$$b_k = \frac{y_{k+1} - y_k}{h_k} - c_k h_k - d_k h_k^2 \tag{3.4}$$

Now, we consider at the point x_k between two subintervals $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$, in which defined two cubic polynomials $g_{k-1}(x)$ and $g_k(x)$. From condition (a) we have $g'_{k-1}(x_k) = g'_k(x_k)$ and $g''_{k-1}(x_k) = g''_k(x_k)$. We have

$$g'_k(x) = b_k + 2c_k(x - x_k) + 3d_k(x - x_k)^2, \quad g''_k(x) = 2c_k + 6d_k(x - x_k)$$

$$g'_{k-1}(x) = b_{k-1} + 2c_{k-1}(x - x_{k-1}) + 3d_{k-1}(x - x_{k-1})^2, \quad g''_{k-1}(x) = 2c_{k-1} + 6d_{k-1}(x - x_{k-1})$$

The condition $g''_{k-1}(x_k) = g''_k(x_k)$ gives

$$c_k = c_{k-1} + 3d_{k-1}(x_k - x_{k-1}) = c_{k-1} + 3d_{k-1}h_{k-1}$$

We obtain

$$d_{k-1} = \frac{c_k - c_{k-1}}{3h_{k-1}} \quad \text{and} \quad d_k = \frac{c_{k+1} - c_k}{3h_k} \tag{3.5}$$

Replace d_k from (**??**) into (**??**), we have

$$b_k = \frac{y_{k+1} - y_k}{h_k} - \frac{h_k}{3}(c_{k+1} + 2c_k) \quad \text{and} \quad b_{k-1} = \frac{y_k - y_{k-1}}{h_{k-1}} - \frac{h_{k-1}}{3}(c_k + 2c_{k-1}) \tag{3.6}$$

The condition $g'_{k-1}(x_k) = g'_k(x_k)$ gives

$$b_k = b_{k-1} + 2c_{k-1}h_{k-1} + 3d_{k-1}h_{k-1}^2$$

Replace b_k, b_{k-1} from (**??**) and d_{k-1} from (**??**) into the above equation, we obtain the linear system of equations

$$h_{k-1}c_{k-1} + 2(h_{k-1} + h_k)c_k + h_k c_{k+1} = 3\frac{y_{k+1} - y_k}{h_k} - 3\frac{y_k - y_{k-1}}{h_{k-1}} \tag{3.7}$$

for each $k = 1, 2, \ldots, n-1$. This system involves only $\{c_k\}_{k=0}^n$ as unknowns since the value of $\{h_k\}_{k=0}^n$ and $\{y_k\}_{k=0}^n$ are given.

Note that once the value of $\{c_k\}_{k=0}^n$ are known, it is a simple matter to find the remainder of the constants $a_k = y_k,\ k = 0, 1, \ldots, n-1$, $\{b_k\}_{k=0}^{n-1}$ from (**??**) and $\{d_k\}_{k=0}^{n-1}$ from (**??**).

However, the system (**??**) consists of $n-1$ equations with $n+1$ unknowns. We need to add two equations that the linear system has unique solution. These equations come from the boundaries of the interval: x_0 and x_n. Depending on boundary conditions, there are two types of cubic spline:

Natural Cubic Spline: $g''(x_0) = 0$ and $g''(x_n) = 0$. It follows $c_0 = 0$ and $c_n = 0$. It means that the system (**??**) consists of $n-1$ equations with $n-1$ unknowns $c_1, c_2, \ldots, c_n$.

Example 3.4. *Construct the natural cubic spline for the table* $\begin{array}{c|ccc} x & 2 & 3 & 4 \\ \hline y & 3 & 5 & 1 \end{array}$.

We have: $n = 2,\ h_0 = h_1 = 1,\ c_0 = c_2 = 0$ *and*

$$h_0c_0 + 2(h_0+h_1)c_1 + h_1c_2 = 3\frac{y_2-y_1}{h_1} - 3\frac{y_1-y_0}{h_0} \Leftrightarrow 4c_1 = 3\frac{1-5}{1} - 3\frac{5-3}{1} = -18 \Rightarrow c_1 = -\frac{9}{2}$$

$$\begin{aligned}
x \in [2,3]: \quad & a_0 = 3;\ b_0 = \frac{5-3}{1} - \frac{1}{3}\left(-\frac{9}{2}+0\right) = \frac{7}{2};\ d_0 = \frac{-9/2-0}{3\times 1} = -\frac{3}{2} \\
& \Rightarrow g_0(x) = 3 + \frac{7}{2}(x-2) - \frac{3}{2}(x-2)^3 \\
x \in [3,4]: \quad & a_1 = 5;\ b_1 = \frac{1-5}{1} - \frac{1}{3}\left(0 - 2\frac{9}{2}\right) = -1;\ d_1 = \frac{0+9/2}{3\times 1} = \frac{3}{2} \\
& \Rightarrow g_1(x) = 5 - (x-3) - \frac{9}{2}(x-3)^2 + \frac{3}{2}(x-3)^3
\end{aligned}$$

So

$$g(x) = \begin{cases} g_0(x) = 3 + \frac{7}{2}(x-2) - \frac{3}{2}(x-2)^3, & 2 \leqslant x \leqslant 3 \\ g_1(x) = 5 - (x-3) - \frac{9}{2}(x-3)^2 + \frac{3}{2}(x-3)^3 & 3 \leqslant x \leqslant 4 \end{cases}$$

Clamped Cubic Spline: $g'(x_0) = \alpha$ and $g'(x_n) = \beta$. In this case, we need to add to the system (**??**) two more equations corresponding to two boundaries:

- † At x_0: $2h_0c_0 + h_0c_1 = 3\frac{y_1-y_0}{h_0} - 3\alpha$
- † At x_n: $h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3\beta - 3\frac{y_n-y_{n-1}}{h_{n-1}}$

And the system (**??**) with these conditions consists of $n+1$ equations with $n+1$ unknowns: $c_0, c_1, \ldots, c_n$.

Example 3.5. *Construct the cubic spline for the table* $\begin{array}{c|ccc} x & 2 & 3 & 4 \\ \hline y & 3 & 5 & 1 \end{array}$ *and satisfies* $g'(2) = 1$ *and* $g'(4) = 0$.

We have: $n = 2,\ h_0 = h_1 = 1,\ \alpha = 1,\ \beta = 0$ *and we have*

$$\begin{cases} 2h_0c_0 + h_0c_1 = 3\frac{y_1-y_0}{h_0} - 3\alpha \\ h_0c_0 + 2(h_0+h_1)c_1 + h_1c_2 = 3\frac{y_2-y_1}{h_1} - 3\frac{y_1-y_0}{h_0} \\ h_1c_1 + 2h_1c_2 = 3\beta - 3\frac{y_2-y_1}{h_1} \end{cases} \Leftrightarrow \begin{cases} 2c_0 + c_1 = 3 \\ c_0 + 4c_1 + c_2 = -18 \\ c_1 + 2c_2 = 12 \end{cases}$$

$$\Rightarrow c_0 = \frac{23}{4},\ c_1 = -\frac{17}{2},\ c_2 = \frac{41}{4}$$

$$\begin{aligned}
x \in [2,3]: &\quad a_0 = 3;\ b_0 = \frac{5-3}{1} - \frac{1}{3}\left(-\frac{17}{2} + \frac{23}{2}\right) = 1;\ d_0 = \frac{-17/2 - 23/4}{3 \times 1} = -\frac{19}{4} \\
&\Rightarrow g_0(x) = 3 + (x-2) + \frac{23}{4}(x-2)^2 - \frac{19}{4}(x-2)^3 \\
x \in [3,4]: &\quad a_1 = 5;\ b_1 = \frac{1-5}{1} - \frac{1}{3}\left(\frac{41}{4} - 2\frac{17}{2}\right) = -\frac{7}{4};\ d_1 = \frac{41/4 + 17/2}{3 \times 1} = \frac{25}{4} \\
&\Rightarrow g_1(x) = 5 - \frac{7}{4}(x-3) - \frac{17}{2}(x-3)^2 + \frac{25}{4}(x-3)^3
\end{aligned}$$

So

$$g(x) = \begin{cases} g_0(x) = 3 + (x-2) + \frac{23}{4}(x-2)^2 - \frac{19}{4}(x-2)^3, & 2 \leqslant x \leqslant 3 \\ g_1(x) = 5 - \frac{7}{4}(x-3) - \frac{17}{2}(x-3)^2 + \frac{25}{4}(x-3)^3 & 3 \leqslant x \leqslant 4 \end{cases}$$

3.4 Exercise

Question 1. Construct Lagrange interpolating polynomials for the following tables:

(a)

x	0	3	5
y	4	2	3

(b)

x	0	1	2	3
y	1	2	4	8

(c)

x	1.0	1.3	1.7	2.0
y	2.15	2.36	2.54	2.89

Question 2. Use Lagrange interpolating polynomial to approximate $f(x^*)$ given by

(a)

x	1.0	1.3	1.7	2.0
y	2.15	2.36	2.54	2.89

, $x^* = 1.5$.

(b)

x	23	25	28	30
y	0.123	1.215	1.736	2.153

, $x^* = 26$.

Question 3. Use Lagrange interpolating polynomial to approximate $\sqrt{3}$ with the function $f(x) = 3^x$ and the values $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, and $x_4 = 2$.

Question 4. Use Lagrange interpolating polynomial to approximate $\sqrt{3}$ with the function $f(x) = \sqrt{x}$ and the values $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 5$. Compare the accuracy with that of Question **??**.

Question 5. Construct Newton forward divided difference formulas for the tables in Question **??**.

Question 6. Construct the natural cubic spline for the following data

(a)

x	1	2	4
y	1	1	2

(b)

x	0	1	2
y	1	2	4

(c)

x	0	1	2	3
y	-1	2	0	1

Question 7. Construct the clamped cubic spline using the data in Question **??** and the fact that

(a) $f'(1) = 0$ and $f'(4) = 1$. (b) $f'(0) = 0$ and $f'(2) = 0$. (c) $f'(0) = 1$ and $f'(3) = 0$.

Question 8. A natural cubic spline $g(x)$ on $[0, 2]$ is defined by

$$g(x) = \begin{cases} g_0(x) = 2 + 3x + 4x^3, & 0 \leqslant x \leqslant 1 \\ g_1(x) = A + B(x-1) + C(x-1)^2 + D(x-1)^3 & 1 \leqslant x \leqslant 2 \end{cases}$$

Find A, B, C, and D.

Question 9. A clamped cubic spline $g(x)$ for a function $f(x)$ is defined on $[0, 3]$ by

$$g(x) = \begin{cases} g_0(x) = 2x + 2x^2 - x^3, & 0 \leqslant x \leqslant 2 \\ g_1(x) = A + B(x-2) + C(x-2)^2 + D(x-2)^3 & 2 \leqslant x \leqslant 3 \end{cases}$$

Given $f'(0) = f'(3)$, find A, B, C, and D.

Question 10. A clamped cubic spline $g(x)$ for a function $f(x)$ is defined by

$$g(x) = \begin{cases} g_0(x) = 1 + Bx + 2x^2 - 2x^3, & 0 \leqslant x \leqslant 1 \\ g_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3 & 1 \leqslant x \leqslant 2 \end{cases}$$

Find $f'(0)$ and $f'(2)$.