Chapter 3: INTERPOLATION.

Given a function $y=f(x)$ in the table form:

x	x_0	x_1	x_2	....	x_n
$y=f(x)$	y_0	y_1	y_2	....	y_n

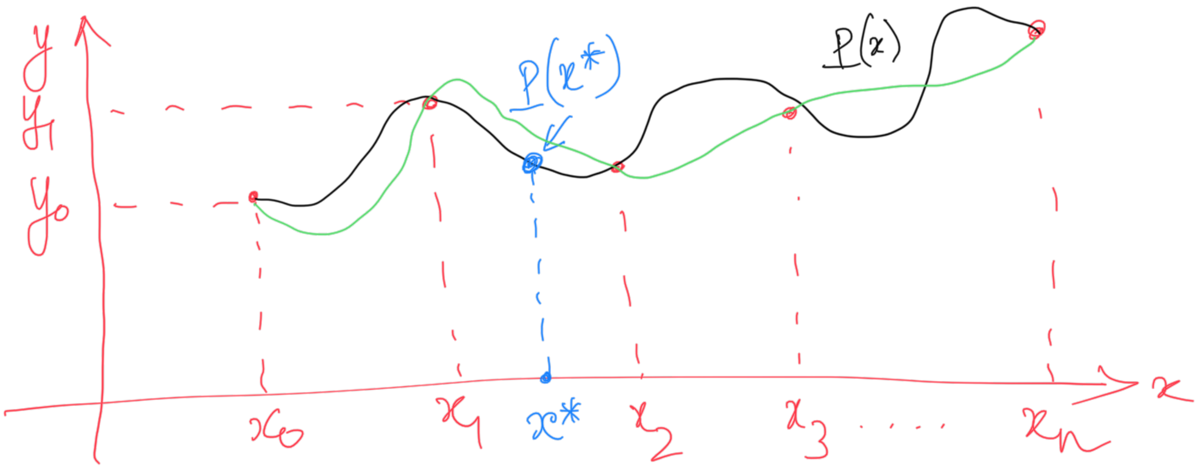
(1)

where $x_0 < x_1 < x_2 < \dots < x_n$: nodes of the table

and $y_k = f(x_k)$, $k=0,1,\dots,n$.

We need to approximate $f(x^*)$, $x^* \in [x_0, x_n]$.

This problem is called the interpolation.



Find a function $P(x)$ such that $P(x_k) = y_k$, $k=\overline{0;n}$.

And $f(x^*) \approx P(x^*)$.

We often choose $P(x)$ as a polynomial.

$\Longrightarrow$ The Polynomial Interpolation.

1) Lagrange's Polynomial:

Given the table (1). We need to construct a polynomial $\mathcal{L}_n(x)$ that satisfies:

- $\deg \mathcal{L}_n(x) \leq n$ (the degree of $\mathcal{L}_n(x)$)
- $\mathcal{L}_n(x_k) = y_k$ for $k = 0, 1, \dots, n$.

First of all, we construct $n+1$ basic polynomials $p_n^{(k)}(x)$, $k = 0, 1, \dots, n$, that are corresponding to the nodes of the table.

$\Rightarrow$ at $x_0 \Leftrightarrow p_n^{(0)}(x)$, $x_1 \Leftrightarrow p_n^{(1)}(x)$, ...

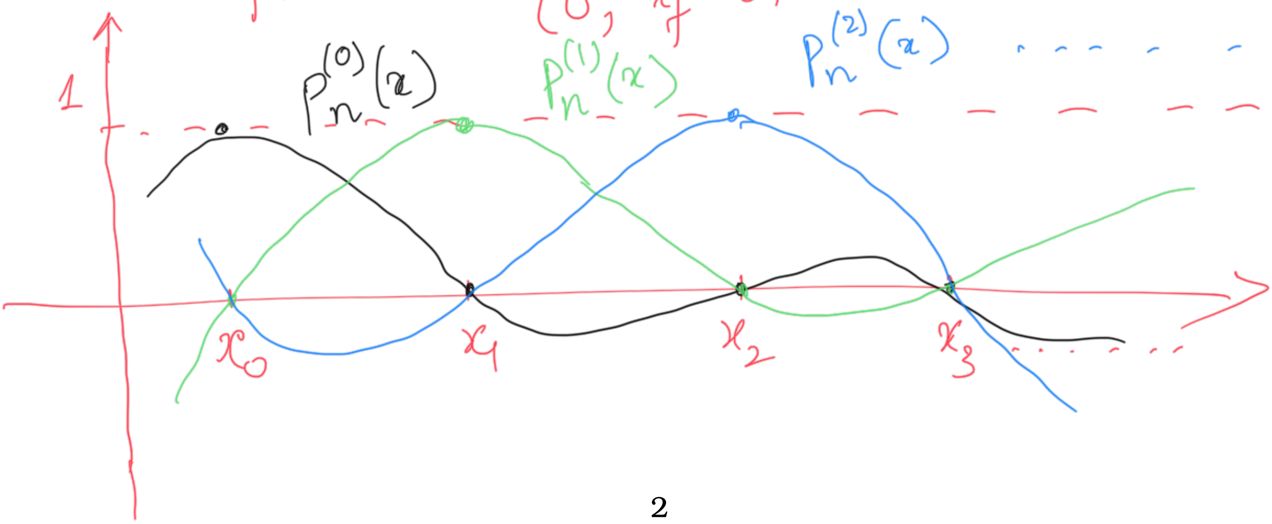
Then:

$$\mathcal{L}_n(x) = y_0 p_n^{(0)}(x) + y_1 p_n^{(1)}(x) + \dots + y_n p_n^{(n)}(x)$$

$$= \sum_{k=0}^{n} y_k p_n^{(k)}(x) \qquad (2)$$

We have:

- $\deg p_n^{(k)}(x) \leq n$
- $p_n^{(k)}(x_i) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$ for $k = 0, 1, \dots, n$



We have

$$\begin{cases} P_n^{(k)}(x) = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \\ k = 0, 1, \ldots, n \end{cases} \quad (3)$$

Ex.: given

x	0	1	3
y	1	1	4

. Construct Lagrange's polynomial for this table.

Sol.: We have: $n = 2$; $x_0 = 0$, $x_1 = 1$, $x_2 = 3$

$y_0 = 1$, $y_1 = 1$, $y_2 = 4$

$$P_2^{(0)}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x^2 - 4x + 3)$$

$$P_2^{(1)}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}(x^2 - 3x)$$

$$P_2^{(2)}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}(x^2 - x)$$

$$\rightarrow \mathcal{L}_2(x) = y_0 P_2^{(0)}(x) + y_1 P_2^{(1)}(x) + y_2 P_2^{(2)}(x)$$

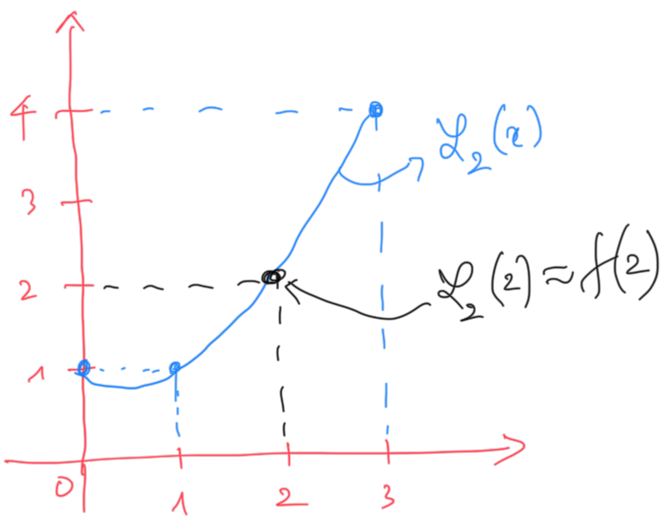
$$= \frac{1}{3}(x^2 - 4x + 3) - \frac{1}{2}(x^2 - 3x) + \frac{4}{6}(x^2 - x)$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x + 1$$

$$f(2) \approx \mathcal{L}_2(2)$$

$$= \frac{1}{2}2^2 - \frac{1}{2}2 + 1$$

$$= 2$$



Ex.: Find Lagrange's polynomial for the following table of the function $f(x) = 2^x$

x	0	1	2	3
y	1	2	4	8

Sol.: We have $n = 3$.

$$p_3^{(0)}(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}\left(x^3 - 6x^2 + 11x - 6\right)$$

$$p_3^{(1)}(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}\left(x^3 - 5x^2 + 6x\right)$$

$$p_3^{(2)}(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -\frac{1}{2}\left(x^3 - 4x^2 + 3x\right)$$

$$p_3^{(3)}(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{1}{6}\left(x^3 - 3x^2 + 2x\right)$$

$$\Rightarrow L_3(x) = y_0 p_3^{(0)}(x) + y_1 p_3^{(1)}(x) + y_2 p_3^{(2)}(x) + y_3 p_3^{(3)}(x)$$

$$= 1 \times \left(-\frac{1}{6}\right)\left(x^3 - 6x^2 + 11x - 6\right) + 2 \times \frac{1}{2}\left(x^3 - 5x^2 + 6x\right)$$
$$+ 4 \times \left(-\frac{1}{2}\right)\left(x^3 - 4x^2 + 3x\right) + 8 \times \frac{1}{6}\left(x^3 - 3x^2 + 2x\right)$$

$$= \frac{1}{6}x^3 + \frac{5}{6}x + 1$$

$$\Rightarrow 2^x \approx \frac{1}{6}x^3 + \frac{5}{6}x + 1 \text{ in } [0, 3].$$

Put $\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

$$\Rightarrow P_n^{(k)}(x) = \frac{\omega(x)}{(x-x_k)(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \frac{\omega(x)}{D_k}$$

where $D_k = (x-x_k)(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)$

$$\Rightarrow L_n(x) = \sum_{k=0}^{n} y_n P_n^{(k)}(x) = \sum_{k=0}^{n} y_k \cdot \frac{\omega(x)}{D_k}$$

$$= \omega(x) \sum_{k=0}^{n} \frac{y_k}{D_k}$$

x	x_0	x_1	$\dots$	x_n	
x_0	$x-x_0$	x_0-x_1	$\dots$	x_0-x_n	$\to D_0$
x_1	x_1-x_0	$x-x_1$	$\dots$	x_1-x_n	$\to D_1$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
x_n	x_n-x_0	x_n-x_1	$\dots$	$x-x_n$	$\to D_n$
					$\omega(x)$

- In the main diagonal : $x-x_k$.
- For other elements : row − column
- Multiply the elements in row $\Rightarrow D_k$
- Multiply the elements in the main diagonal $\Rightarrow \omega(x)$

Ex: Given

x	1.15	1.27	1.54	1.63
y	0.4385	0.8353	1.8976	2.3125

of the function $f(x) = \dfrac{x^2-1}{x^2+1} e^x$.

Use Lagrange's polynomial to approximate $f(1.39)$ and compare with $f(1.39) = \underline{1.2763}$

Sol.:

1.39	1.15	1.27	1.54	1.63	
1.15	0.24	−0.12	−0.39	−0.48	$D_0 = -5.39136 \times 10^{-3}$
1.27	0.12	0.12	−0.27	−0.36	$D_1 = 1.39968 \times 10^{-3}$
1.54	0.39	0.27	−0.15	−0.09	$D_2 = 1.42155 \times 10^{-3}$
1.63	0.48	0.36	0.09	−0.24	$D_3 = -3.73248 \times 10^{-3}$
					$\omega = 1.0368 \times 10^{-3}$

$$f(1.39) \approx \mathcal{L}_3(1.39) = \omega\left(\frac{y_0}{D_0} + \frac{y_1}{D_1} + \frac{y_2}{D_2} + \frac{y_3}{D_3}\right)$$

$$= 1.276057265$$

$$\approx \underline{1.2761} \qquad (\text{error} \simeq 0.0002)$$

2, Newton's Polynomials :

Given the table

x	x_0	x_1	x_2	$\ldots$	x_n
y	y_0	y_1	y_2	$\ldots$	y_n

where $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = h$: const.

We define :

- $\Delta y_k = y_{k+1} - y_k$; $k = 0, 1, \ldots, n-1$.

  The finite differences of f at x_k. (1^{st} order)

- $$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$
  $$= (y_{k+2} - y_{k+1}) - (y_{k+1} - y_k)$$
  $$= y_{k+2} - 2y_{k+1} + y_k \text{ , } k = 0, 1, \ldots, n-2$$

  The 2^{nd} order finite differences of f at x_k

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- $\Delta^p y_k = \Delta^{p-1} y_{k+1} - \Delta^{p-1} y_k$, $k = 0, 1, \ldots, n-p$

  $p = 1, 2, \ldots, n$.

  The p^{th} order finite differences of f at x_k.

For calculating the finite differences, we can construct the following table:

x	y	Δ	Δ^2	$\ldots$
x_0	y_0	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\ldots$
x_1	y_1	$\Delta y_1 = y_2 - y_0$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\ldots$
x_2	y_2	$\Delta y_2 = y_3 - y_2$	$\vdots$	$\ldots$
x_3	y_3	$\vdots$	$\vdots$	
$\vdots$	$\vdots$			

Ex.: Given

x	0	1	2	3	4	5	6
y	1	3	2	-1	3	4	7

We have:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	1	2	-3	1	8	-27	61
1	3	-1	-2	9	-19	34	
2	2	-3	7	-10	15		
3	-1	4	-3	5			
4	3	1	2				
5	4	3					
6	7						

a) The Forward Newton's Polynomial:

$$N_n^{(1)}(x) = y_0 + \frac{\Delta y_0}{1!}q + \frac{\Delta^2 y_0}{2!}q(q-1) + \dots +$$

$$+ \frac{\Delta^n y_0}{n!}q(q-1)\dots(q-n+1)$$

where $q = \frac{x-x_0}{h}$.

b) The Backward Newton's Polynomial:

$$N_n^{(2)}(x) = y_n + \frac{\Delta y_{n-1}}{1!}p + \frac{\Delta^2 y_{n-2}}{2!}p(p+1) + \dots +$$

$$+ \frac{\Delta^n y_0}{n!}p(p+1)\dots(p+n-1)$$

where $p = \frac{x-x_n}{h}$.

Ex.: given

x	0	1	2	3
y	1	2	4	8

Construct $N^{(1)}(x)$ and $N^{(2)}(x)$ for this table.

Sol.:

x	y	Δ	Δ^2	Δ^3
0	1	1	1	1
1	2	2	2	
2	4	4		
3	8			

- $$q = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

$$N_3^{(1)}(x) = y_0 + \frac{\Delta y_0}{1!} q + \frac{\Delta^2 y_0}{2!} q(q-1) + \frac{\Delta^3 y_0}{3!} q(q-1)(q-2)$$

$$= 1 + \frac{1}{1!} x + \frac{1}{2!} x(x-1) + \frac{1}{3!} x(x-1)(x-2)$$

$$= 1 + x + \frac{1}{2} x(x-1) + \frac{1}{6} x(x-1)(x-2)$$

- $$p = \frac{x - x_3}{h} = \frac{x-3}{1} = x-3$$

$$N_3^{(2)}(x) = y_3 + \frac{\Delta y_2}{1!} p + \frac{\Delta^2 y_1}{2!} p(p+1) + \frac{\Delta^3 y_0}{3!} p(p+1)(p+2)$$

$$= 8 + \frac{4}{1!}(x-3) + \frac{2}{2!}(x-3)(x-2) + \frac{1}{3!}(x-3)(x-2)(x-1)$$

$$= 8 + 4(x-3) + (x-3)(x-2) + \frac{1}{6}(x-3)(x-2)(x-1)$$

Note: If the table is fixed, then

$$L_n(x) \equiv N_n^{(1)}(x) \equiv N_n^{(2)}(x)$$

Ex.: Given the table of $f(x) = e^{x}\sin 2x$:

x	1.0	1.2	1.4	1.6
y	2.4717	2.2426	1.3584	-0.2891

Use the forward Newton's polynomial to approximate $f(1.13)$ and compare with $f(1.13) = \underline{2.3891}$

Sol.:

x	y	Δ	Δ^2	Δ^3
1.0	2.4717	-0.2291	-0.6551	-0.1082
1.2	2.2426	-0.8842	-0.7633	
1.4	1.3584	-1.6475		
1.6	-0.2891			

$$q = \frac{x - x_0}{h} = \frac{1.13 - 1.0}{0.2} = 0.65$$

$$f(1.13) \approx N_3^{(1)}(1.13) = y_0 + \frac{\Delta y_0}{1!}q + \frac{\Delta^2 y_0}{2!}q(q-1)$$

$$+ \frac{\Delta^3 y_0}{3!}q(q-1)(q-2)$$

$$= 2.4717 + \frac{-0.2291}{1!}(0.65) + \frac{-0.6551}{2!}(0.65)(-0.35)$$

$$+ \frac{-0.1082}{3!}(0.65)(-0.35)(-1.35)$$

$$= 2.39176413 8 \approx \underline{2.3918}$$

$$(err \approx 0.0027)$$