b) THE ITERATIVE METHODS

Consider the system $AX = B$, that has an unique solution P ($\det A \neq 0$).

We transform this system into the following form:

$$X = T.X + C \qquad (1)$$

where T - square matrix and C - vector.

Choose an initial vector $X^{(0)}$, construct a sequence of vectors $\{X^{(m)}\}_{m=0}^{\infty}$:

$$\Sigma \quad X^{(m)} = T.X^{(m-1)} + C, \quad m = 1, 2, 3, \dots \qquad \underline{\underline{(2)}}$$

<u>Ex</u>: Consider the system $\begin{cases} 11x_1 - 2x_2 = 8.5 \\ 3x_1 + 13x_2 = 9.7 \end{cases}$

that has the solution $P = \begin{bmatrix} 1299/1490 \\ 406/745 \end{bmatrix} \approx \begin{bmatrix} 0.8718 \\ 0.5450 \end{bmatrix}$.

The system is equivalent to:

$$\begin{cases} 11x_1 = 2x_2 + 8.5 \\ 13x_2 = -3x_1 + 9.7 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{2}{11}x_2 + \frac{8.5}{11} \\ x_2 = -\frac{3}{13}x_1 + \frac{9.7}{13} \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 2/11 \\ -3/13 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 8.5/11 \\ 9.7/13 \end{bmatrix}$$

$$(X = T \cdot X + C)$$

Choose $X^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. From (2), we have:

$\underline{m=1}$: $X^{(1)} = T.X^{(0)} + C = C = \begin{bmatrix} 8.5/11 \\ 9.7/13 \end{bmatrix}$

$\underline{m=2}$: $X^{(2)} = TX^{(1)} + C = \begin{bmatrix} 0 & 2/11 \\ -3/13 & 0 \end{bmatrix} \begin{bmatrix} 8.5/11 \\ 9.7/13 \end{bmatrix} + \begin{bmatrix} 8.5/11 \\ 9.7/13 \end{bmatrix}$

$$= \begin{bmatrix} 0.908391608 \\ 0.567832168 \end{bmatrix}$$

$\underline{m=3}$: $X^{(3)} = TX^{(2)} + C = \begin{bmatrix} 0.8759694851 \\ 0.5365250135 \end{bmatrix}$

$$\approx \begin{bmatrix} 0.8760 \\ 0.5365 \end{bmatrix} \approx \underline{P} = \begin{bmatrix} 0.8718 \\ 0.5450 \end{bmatrix}$$

In the future we will show that $X^{(m)} \xrightarrow[m\infty]{} \underline{P}$.

$\underline{\text{Theorem}}$: If $\underline{\underline{\|T\| < 1}}$, then the sequence $\{X^{(m)}\}$, defined by (2) will converge to the solution $\underline{P}$ of the system for any initial vector $X^{(0)}$, and we have:

$$\|\underline{P} - X^{(m)}\| \leq \frac{\|T\|}{1 - \|T\|} \cdot \|X^{(m)} - X^{(m-1)}\| \quad (3)$$

$\underline{\text{Ex.}}$: Given:

$$X = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0 & 0.1 & -0.2 \\ 0.2 & -0.2 & 0 \end{bmatrix} X + \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \end{bmatrix}$$

Use the iterative method with $X^{(0)} = (0,0,0)^T$ to find $X^{(3)}$ and its error. Use l_∞-norm.

$\underline{\text{Sol.}}$: $X^{(1)} = TX^{(0)} + C = C = \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \end{bmatrix}$

$$X^{(2)} = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0 & 0.1 & -0.2 \\ 0.2 & -0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.52 \\ 0.68 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0 & 0.1 & -0.2 \\ 0.2 & -0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.63 \\ 0.52 \\ 0.68 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.6 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.647 \\ 0.516 \\ 0.722 \end{bmatrix}$$

We have $\|T\|_\infty = \max(0.4, 0.3, 0.4) = 0.4$

$$X^{(3)} - X^{(2)} = \begin{bmatrix} 0.017 \\ -0.004 \\ 0.042 \end{bmatrix} \Longrightarrow \|X^{(3)} - X^{(2)}\|_\infty = 0.042$$

$$\Rightarrow \Delta_{X^{(3)}} = \frac{\|T\|_\infty}{1 - \|T\|_\infty} \|X^{(3)} - X^{(2)}\|_\infty =$$

$$= \frac{0.4}{1 - 0.4} \times 0.042 = \underline{0.028}.$$

c) Strictly Diagonally Dominant Matrices:

Def.: A matrix $A = (a_{ij})$ is called a SDDM if

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}|, \text{ for } i = 1, 2, \dots, n$$

$$\Longleftrightarrow \begin{cases} |a_{12}| + |a_{13}| + \dots + |a_{1n}| < |a_{11}| \\ |a_{21}| + |a_{23}| + \dots + |a_{2n}| < |a_{22}| \\ - - - - - - - - - - \\ |a_{n1}| + |a_{n2}| + \dots + |a_{n,n-1}| < |a_{nn}| \end{cases}$$

Ex: The matrix $A = \begin{bmatrix} 4 & 1 & 2 \\ 5 & -8 & 0 \\ 3 & 3 & 9 \end{bmatrix}$ is a SDDM

because:

$|1| + |2| < |4|$ (1^{st} row)

$|5| + |0| < |-8|$ (2^{nd} row)

$|3| + |3| < |9|$ (3^{rd} row)

Now we consider $AX = B$ where A is SDDM

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ - & - & - & - \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}}_{D}$$

$$- \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ - & - & - & - \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix}}_{L} - \underbrace{\begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ - & - & - & - \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{U}$$

$$= D - L - U$$

Ex.: $A = \begin{bmatrix} 5 & 1 & -2 \\ 3 & 8 & -4 \\ 4 & 5 & 11 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 11 \end{bmatrix}_D -$

$$- \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -4 & -5 & 0 \end{bmatrix}_L - \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}_U = D - L - U$$

$$\Rightarrow AX = B \Leftrightarrow (D - L - U)X = B \quad (4)$$

(i) Jacobi's Method:

$$(4) \Longleftrightarrow DX = (L+U)X + B$$

$$\Longleftrightarrow X = \underbrace{D^{-1}(L+U)}\,.\,X + \underbrace{D^{-1}B}$$

Denote $T_j = D^{-1}(L+U)$: Jacobian iterative matrix

$C_j = D^{-1}B$: vector.

We obtain the system $X = T_j X + C_j$ of (1).

The iterative formula :

$$X^{(m)} = T_j X^{(m-1)} + C_j \,, \quad m = 1, 2, 3, \dots \qquad (5)$$

We have :

$$D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}} & \dots & 0 \\ - & - & - & - \\ 0 & 0 & \dots & \frac{1}{a_{nn}} \end{bmatrix}, \quad L+U = \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ -a_{21} & 0 & \dots & -a_{2n} \\ - & - & - & - \\ -a_{n1} & -a_{n2} & \dots & 0 \end{bmatrix}$$

$$\Rightarrow T_j = \begin{bmatrix} 0 & \frac{-a_{12}}{a_{11}} & \dots & \frac{-a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \dots & \frac{-a_{2n}}{a_{22}} \\ - & - & - & - \\ -\frac{a_{n1}}{a_{nn}} & \frac{-a_{n2}}{a_{nn}} & \dots & 0 \end{bmatrix}, \quad C_j = \begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_n/a_{nn} \end{bmatrix}$$

and

$$\|T_j\|_\infty = \max_{1 \le i \le n} \left(\sum_{\substack{j=1 \\ j \ne i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \right) =$$

$$= \max_{1 \le i \le n} \left(\frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right) < 1$$

The sequence, defined by (5), will converge to $\underline{P}$.

$$\|\underline{P} - X^{(m)}\|_\infty \leq \frac{\|T_j\|_\infty}{1 - \|T_j\|_\infty} \|X^{(m)} - X^{(m-1)}\|_\infty \tag{6}$$

We can rewrite the formula (5) in the coordinate form:

$$\begin{cases} x_k^{(m)} = \dfrac{1}{a_{kk}}\left(b_k - \sum_{j=1}^{k-1} a_{kj} x_j^{(m-1)} - \sum_{j=k+1}^{n} a_{kj} x_j^{(m-1)}\right) \\ k = 1, 2, \dots, n \ ; \quad m = 1, 2, 3, \dots \end{cases}$$

(k : coordinate index ; m : iterative index)

In case of $n = 2$:

$$\begin{cases} x_1^{(m)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12} x_2^{(m-1)}\right) \\ x_2^{(m)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21} x_1^{(m-1)}\right) \end{cases} \quad m = 1, 2, 3, \dots$$

In case of $n = 3$:

$$\begin{cases} x_1^{(m)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12} x_2^{(m-1)} - a_{13} x_3^{(m-1)}\right) \\ x_2^{(m)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21} x_1^{(m-1)} - a_{23} x_3^{(m-1)}\right) \\ x_3^{(m)} = \dfrac{1}{a_{33}}\left(b_3 - a_{31} x_1^{(m-1)} - a_{32} x_2^{(m-1)}\right) \end{cases} \quad m = 1, 2, 3, \dots$$

Ex.: Given: $\begin{cases} 8x_1 + 3x_2 = 7.5 \\ -2x_1 + 9x_2 = 6.8 \end{cases}$

Use Jacobian method with $X^{(0)} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$ to find $X^{(5)}$ and its error.

Sol.: $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$ $\quad (x_1 \leftrightarrow A ;\ x_2 \leftrightarrow B)$

- $0.1 \to A$; $0.2 \to B$ $\quad \left(X^{(0)} = \begin{bmatrix} A \\ B \end{bmatrix} \right)$
- $(7.5 - 3B) \div 8 \to C$ (C is a temporary var.)

$(6.8 + 2A) \div 9 \to B$

$C \to A$

$\Rightarrow X^{(1)} = \begin{bmatrix} A \\ B \end{bmatrix}$

$$\Rightarrow X^{(1)} = \begin{bmatrix} 69/80 \\ 7/9 \end{bmatrix}, \quad X^{(2)} = \begin{bmatrix} 31/48 \\ 341/360 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} 559/960 \\ 971/1080 \end{bmatrix}; \quad X^{(4)} = \begin{bmatrix} 1729/2880 \\ 3823/4320 \end{bmatrix}$$

$$X^{(5)} = \begin{bmatrix} 6977/11520 \\ 0.888960494 \end{bmatrix} \approx \begin{bmatrix} 0.6056 \\ 0.8890 \end{bmatrix}$$

$$T_j = \begin{bmatrix} 0 & -3/8 \\ 2/9 & \end{bmatrix} \Rightarrow \|T_j\|_\infty = \max\left(\frac{3}{8}, \frac{2}{9}\right) = \frac{3}{8}$$

$$X^{(5)} - X^{(4)} = \begin{bmatrix} 0.00529514 \\ 0.00401235 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \|X^{(5)} - X^{(4)}\|_\infty = 0.00529514.$$

$$\Delta_{X^{(5)}} = \frac{3/8}{1 - 3/8} \times 0.00529514 \approx \underline{\underline{0.0032}}$$

Exercises:

1) Given $\begin{cases} 11x_1 + 2x_2 = 7.85 \\ 3.2x_1 + 12x_2 = 6.17 \end{cases}$ $\quad X^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Use Jacobi's method to find $X^{(5)}$ and its error.

2) Given $\begin{bmatrix} 7.5 & 1 & -2 \\ 2 & 8.0 & 1 \\ -1 & 2 & 8.5 \end{bmatrix} X = \begin{bmatrix} 11.2 \\ 12.3 \\ 13.4 \end{bmatrix}$, $X^{(0)} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.6 \end{bmatrix}$

Use Jacobi's method to find $X^{(3)}$.

(ii) Gauss-Seidel's Method:

$$(4) \Leftrightarrow (D-L)X = UX + B$$

$$\Leftrightarrow X = \underbrace{(D-L)^{-1}U}\, X + \underbrace{(D-L)^{-1}B}$$

$T_g = (D-L)^{-1}U$: Gauss-Seidel Iterative Matrix.

$C_g = (D-L)^{-1}B$: vector.

We obtain Gauss-Seidel Iterative formula:

$$X^{(m)} = T_g \cdot X^{(m-1)} + C_g, \quad m = 1, 2, 3, \ldots \quad (7)$$

In the coordinate form:

$$\begin{cases} x_k^{(m)} = \dfrac{1}{a_{kk}}\left(b_k - \displaystyle\sum_{j=1}^{k-1} a_{kj}\, x_j^{(m)} - \sum_{j=k+1}^{n} a_{kj}\, x_j^{(m-1)}\right) \\ k = 1, 2, \ldots, n; \quad m = 1, 2, 3, \ldots \end{cases}$$

In case of $n = 2$:

$$\begin{cases} x_1^{(m)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12}\, x_2^{(m-1)}\right) \\ x_2^{(m)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21}\, x_1^{(m)}\right) \end{cases} \quad m = 1, 2, 3, \ldots$$

In case of $n = 3$:

$$\begin{cases} x_1^{(m)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12}\, x_2^{(m-1)} - a_{13}\, x_3^{(m-1)}\right) \\ x_2^{(m)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21}\, x_1^{(m)} - a_{23}\, x_3^{(m-1)}\right) \\ x_3^{(m)} = \dfrac{1}{a_{33}}\left(b_3 - a_{31}\, x_1^{(m)} - a_{32}\, x_2^{(m)}\right) \end{cases} \quad m = 1, 2, 3, \ldots$$

Ex.: Given: $\begin{cases} 15x_1 - 0.5x_2 + 1.2x_3 = 5.37 \\ 0.8x_1 + 13x_2 - 0.9x_3 = 6.18 \\ -1.2x_1 + 0.7x_2 + 14x_3 = 6.77 \end{cases}$

Use Gauss-Seidel's method with $X^{(0)} = (0.2, 0.3, 0.4)^T$ to find $X^{(3)}$.

Sol.: $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ $\left(\begin{array}{l} x_1 \longleftrightarrow A \\ x_2 \longleftrightarrow B \\ x_3 \longleftrightarrow C \end{array} \right)$

+ $0.2 \to A$; $0.3 \to B$; $0.4 \to C$

+ $(5.37 + 0.5B - 1.2C) \div 15 \to A$

$(6.18 - 0.8A + 0.9C) \div 13 \to B$

$(6.77 + 1.2A - 0.7B) \div 14 \to C$

$$X^{(1)} = \begin{bmatrix} 0.3391808128 \\ 0.4964410365 \\ 0.4878220178 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 0.3355222731 \\ 0.4885093844 \\ 0.4879050113 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} 0.3352512452 \\ 0.4885318088 \\ 0.4878806592 \end{bmatrix} \approx \begin{bmatrix} 0.3353 \\ 0.4885 \\ 0.4879 \end{bmatrix}$$

We have

$$P = \begin{bmatrix} 0.3352538549 \\ 0.4885299841 \\ 0.4878809741 \end{bmatrix} \approx \begin{bmatrix} 0.3353 \\ 0.4885 \\ 0.4879 \end{bmatrix}$$

Ex.: Given $\begin{bmatrix} 8 & -1 \\ 2 & 9 \end{bmatrix} X = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Use Gauss-Seidel's method with $X^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to find $X^{(3)}$ and its error.

Sol.:
- $0 \to A$, $0 \to B$
- $(4+B) \div 8 \to A$

  $(5-2A) \div 9 \to B$

$$\Rightarrow X^{(1)} = \begin{bmatrix} 1/2 \\ 4/9 \end{bmatrix};\ X^{(2)} = \begin{bmatrix} 5/9 \\ 35/81 \end{bmatrix},\ X^{(3)} = \begin{bmatrix} 359/648 \\ 1261/2916 \end{bmatrix}$$

We have $D - L = \begin{bmatrix} 8 & 0 \\ 2 & 9 \end{bmatrix}$, $U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow (D-L)^{-1} = \frac{1}{72}\begin{bmatrix} 9 & 0 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ -1/36 & 1/9 \end{bmatrix}$$

$$\left(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\Rightarrow T_g = \begin{bmatrix} 1/8 & 0 \\ -1/36 & 1/9 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/8 \\ 0 & -1/36 \end{bmatrix}$$

$$\Rightarrow \|T_g\|_\infty = \frac{1}{8}$$

$$X^{(3)} - X^{(2)} = \begin{bmatrix} -1/648 \\ 0.00034293553 \end{bmatrix} \Rightarrow \|X^{(3)} - X^{(2)}\|_\infty = \frac{1}{648}$$

$$\Rightarrow \Delta_{X^{(3)}} = \frac{1/8}{1 - 1/8} \cdot \frac{1}{648} \cong 0.000221.$$

For Sunday; June 27^{th}, 2021.

* Exercises for Chapters 1 & 2.
* We have a quiz for 5%.
  - From 10:00 → 11:00
  - Chap. 1: - Bisection
    - Newton

    Chap. 2: - Doolittle, Crout
    - Cholesky
    - Norm.
  - You will sent your answers only to me by the email:

    lethanh.pptdt.hk203@gmail.com
  - There are about 6 – 7 question in the quiz.