1. NONLINEAR EQUATIONS

1.1 ABSOLUTE AND RELATIVE ERRORS

Let A be an exact value and a is its approximated value. Then we say that a approximates the value A (or A is approximated by a) and write $a \approx A$. The value $\Delta = |A - a|$ is called the *absolute error* of a. However, we do not even know the exact value A, so, actually, we estimate a smallest positive value Δ_a such that:

$$|A - a| \leqslant \Delta_a \quad \text{or} \quad a - \Delta_a \leqslant A \leqslant a + \Delta_a$$

The number Δ_a is called the *limited absolute error* or, shortly, the *error* of a and we denote $A = a \pm \Delta_a$.

Example 1: We do not know exactly the value of π, but we can estimate $3.14 - 0.002 \leqslant \pi \leqslant 3.14 + 0.002$. So that $\pi = 3.14 \pm 0.002$. The value 0.002 is the error of 3.14 in comparing with π.

The relative error of the value a is defined by

$$\delta_a = \frac{\Delta_a}{|a|} \Longleftrightarrow \Delta_a = \delta_a \cdot |a| \qquad (a \neq 0)$$

Notice that we usually multiply δ by 100 to say in percent $\%$.

Example 2: To measure the length L we obtain $l = 120cm$ with the error is 0.1%. It means that $\delta_l = \frac{0.1}{100}$ and $\Delta_l = \delta_l \cdot |l| = \frac{0.1}{100} \times 120 = 0.12$. Therefore, $L = 120 \pm 0.12\ (cm)$.

1.2 NONLINEAR EQUATIONS

We now consider one of the basic problem of numerical methods, the root-finding problem. It involves finding a root p of an equation of the form

$$f(x) = 0 \quad \text{or} \quad x = g(x) \tag{1}$$

for continuous functions f and g.

The process of finding a root p of the equation (1) is as follows:

Step 1: Determine an closed interval $[a, b]$ that consists the unique root p.

Step 2: Construct a sequence (x_n) from $[a, b]$ such that $\lim_{n\to\infty} x_n = p$

Step 3: Fix the value of n and $x_n \approx p$. Evaluate Δ_{x_n}.

An interval $[a, b]$ is called a *root-isolated interval* of a root p of the equation (1) if $p \in [a, b]$ and is an unique simple root of the equation (1).

Theorem 1. *If $f(x)$ is a monotone continuous function on interval $[a, b]$, and $f(a) \cdot f(b) < 0$, then there exists an unique simple root p of the equation $f(x) = 0$ on $[a, b]$. It means $[a, b]$ is the root-isolated interval of p.*

Example 3: Find all root-isolated intervals of the equation: $f(x) = x^3 - 3x + 1 = 0$.
We have $f'(x) = 3x^2 - 3 = 0$ at $x = -1$ and $x = 1$.

x	$-\infty$		-1		1		$+\infty$
y'		$+$	0	$-$	0	$+$	
y	$-\infty$	$\nearrow$	3	$\searrow$	-1	$\nearrow$	∞

And some simple calculations show

x	$\dots$	-3	-2	-1	0	1	2	3	$\dots$
sign $f(x)$		$-$	$-$	$+$	$+$	$-$	$+$	$+$	

These above tables show that the given equation has three root-isolated intervals: $[-2, -1]$, $[0, 1]$, and $[1, 2]$.

Theorem 2 (The General Error-Estimated Formula). *Suppose $[a, b]$ is a root-isolated interval of the root p of the equation $f(x) = 0$ and p^* is an approximation to p. If $f(x)$ is differentiable on $[a, b]$ and*

$$\forall x \in [a, b],\ |f'(x)| \geqslant m > 0 \qquad \left(m = \min_{x \in [a,b]} |f'(x)|\right)$$

then the following estimation has occurred

$$|\mathbf{p} - \mathbf{p}^*| \leqslant \frac{|\mathbf{f}(\mathbf{p}^*)|}{\mathbf{m}} \tag{2}$$

Example 4: Consider the equation $f(x) = x^3 - 3x + 1 = 0$ on interval $[0, 0.5]$, consisting the unique root p. Suppose $p^* = 0.347 \approx p$. Find its error.
We have

$$\min_{x \in [0,0.5]} |f'(x)| = \min_{x \in [0,0.5]} |3x^2 - 3| = 2.25$$

Therefore

$$|p - 0.347| \leqslant \frac{f(0.347)}{2.25} \approx 0.00035 \ \Rightarrow\ x = 0.347 \pm 0.00035$$

1.3 THE BISECTION METHOD

Suppose $f(x)$ is a continuous function defined on root-isolated interval $[a, b]$ which consists an unique root p, and $f(a) \cdot f(b) < 0$. Set $a_0 = a,\ b_0 = b,\ d_0 = b_0 - a_0 = b - a$ is a constant. We divide the interval $[a_0, b_0]$ into two equal sub-intervals by the midpoint $x_0 = \dfrac{a_0 + b_0}{2}$.

- If $f(x_0) = 0$ then $p \equiv x_0$ and the bisection process is stopped.
- If $f(a_0) \cdot f(x_0) < 0$ then $p \in [a_0, x_0]$ and we denote $a_1 = a_0, b_1 = x_0$
- If $f(x_0) \cdot f(b_0) < 0$ then $p \in [x_0, b_0]$ and we denote $a_1 = x_0, b_1 = b_0$

We obtain:

$$p \in [a_1, b_1] \subset [a_0, b_0] \text{ and } d_1 = b_1 - a_1 = \frac{d_0}{2} = \frac{b - a}{2}$$

Continuing this bisection process to n times, we have:

$$\begin{cases} p \in [a_n, b_n] \subset [a_{n-1}, b_{n-1}] \subset \cdots \subset [a_0, b_0] \\ d_n = b_n - a_n = \dfrac{d_{n-1}}{2} = \dfrac{b-a}{2^n} \longrightarrow 0 (n \to \infty) \\ x_n = \dfrac{a_n + b_n}{2},\ a_n \leqslant x_n \leqslant b_n \end{cases}$$

We can show that the sequence (x_n) converges to p. For the error-estimating we have the following formula:

$$|p - x_n| \leqslant \frac{b-a}{2^{n+1}}$$

Example 5: Using the bisection method, find the approximate solution x_5 for the equation $f(x) = x^3 - 3x + 1 = 0$ in the interval $[0, 1]$.

We have the following table:

n	a_n	b_n	x_n	$f(x_n)$
0	0.00	1.00	0.500	−0.3750
1	0.00	0.50	0.250	+0.2656
2	0.25	0.50	0.375	−0.0723
3	0.25	0.375	0.3125	+0.0930
4	0.3125	0.375	0.34375	+0.0094
5	0.34375	0.375	0.359375	−0.0031

Therefore $x_5 = 0.359375 \approx p$.

1.4 FIXED-POINT ITERATION

A *fixed point* for a given function $g(x)$ is a number p for which $g(p) = p$. In this section we consider the problem of finding solutions to fixed-point problems and the connection between these problems and the root-finding problems, we wish to solve. Root-finding problems and fixed-point problems are equivalent classes in the following sense: *Given a root-finding problem $f(p) = 0$, we can define a function $g(x)$ with a fixed point at p in a number of ways, for example, as $g(x) = x - f(x)$ or as $g(x) = x + 3f(x)$. Conversely, if the function $g(x)$ has a fixed point p, then the function defined by $f(x) = x - g(x)$ has a zero at p.*

Theorem 3. *(a) If $g : [a, b] \longrightarrow [a, b]$ is a continuous function, then g has a fixed point in $[a, b]$.*

(b) If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leqslant k$, for all $x \in (a, b)$, then the fixed point in $[a, b]$ is unique.

To approximate the fixed point of a function $g(x)$, we choose an initial approximation x_0 and generate the sequence $\{x_n\}_{n=0}^{\infty}$ by letting $x_n = g(x_{n-1})$ for each $n \geqslant 1$. If the sequence converges to p and $g(x)$ is continuous, then $p = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} g(x_{n-1}) = g\left(\lim\limits_{n\to\infty} x_{n-1}\right) = g(p)$ and a solution to $x = g(x)$ is obtained. This technique is called *fixed-point iteration*.

Example 6: The equation $f(x) = x^3 + x - 1000 = 0$ has a unique root in $[9, 10]$. There are many ways to change the equation to the form $x = g(x)$. We list here some of them

(a) $x = g_1(x) = 1000 - x^3$

(b) $x = g_2(x) = \sqrt[3]{1000 - x}$

(c) $x = g_3(x) = \sqrt{\dfrac{1000}{x} - 1}$

With $x_0 = 9.5$, the following table lists the results of the fixed-point iteration method for all these choices of g.

n	(a)	(b)	(c)
0	9.5	9.5	9.5
1	142.625	9.968232523	10.21093325
2	-2.9×10^7	9.966661536	9.845518838
3	2.44×10^{19}	9.966666808	10.02841216
4		9.966666790	9.935626973
5		9.966666791	9.982379526
6			9.958740672
7			9.970672127
8			9.964644569
9			9.967688235
10			9.966150966
15			9.966683738
20			9.966666234
25			9.966666809

Theorem 4. *Let $g : [a, b] \longrightarrow [a, b]$ is a continuous function. Suppose, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leqslant k$, for all $x \in (a, b)$, then for any number $x_0 \in [a, b]$, the sequence defined by $x_n = g(x_{n-1})$, $n \geqslant 1$, converges to the unique fixed point p in $[a, b]$.*

Theorem 5. *If g satisfies the hypotheses of Theorem 4, bounds for the error involved in using x_n to approximate p are given by*

$$|x_n - p| \leqslant \frac{k^n}{1-k}|x_1 - x_0| \qquad \text{or} \qquad |x_n - p| \leqslant \frac{k}{1-k}|x_n - x_{n-1}|$$

Example 7: The equation $x = \cos x$ has a unique root in $[0, 1]$. Using the fixed-point iteration method with $x_0 = 0.5$, find the approximation solution x_{20} and its error.

We have: $|g'(x)| = |-\sin x| \leqslant \sin 1 \approx 0.85 = k < 1$ for all $x \in [0, 1]$ and $x_{20} = 0.7390067798$. Its error is

$$|x_{20} - p| \leqslant \frac{k}{1-k}|x_{20} - x_{19}| = \frac{0.85}{1-0.85}|0.7390067798 - 0.7392014441| \approx 1.11 \times 10^{-3}$$

1.5 NEWTON-RAPHSON METHOD

The Newton-Raphson method is one of the most powerful and well-known numerical methods for solving a root finding problem $f(x) = 0$ in $[a, b]$. Suppose that f is continuously differentiable to second order. Let $p^* \in [a, b]$ be an approximation to p such that $f'(p^*) \neq 0$ and $|p^* - p|$ is "small". Consider the first Taylor polynomial for $f(x)$ expanded about p^*

$$f(x) = f(p^*) + (x - p^*)f'(p^*) + \frac{(x - p^*)^2}{2} f''(\xi(x))$$

where $\xi(x)$ lies between x and p^*. Since $f(p) = 0$, this equation, with $x = p$, gives

$$0 = f(p^*) + (p - p^*)f'(p^*) + \frac{(p - p^*)^2}{2} f''(\xi(p))$$

Newton's method is derived by assuming that since $|p - p^*|$ is small the term involving $(p - p^*)^2$ is much smaller and that

$$0 \approx f(p^*) + (p - p^*)f'(p^*) \quad \Longrightarrow \quad p \approx p^* - \frac{f(p^*)}{f'(p^*)}$$

This sets the stage for the Newton-Raphson method, which starts with an initial approximation x_0 and generates the sequence $\{x_n\}$ defined by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for } n \geqslant 1 \tag{3}$$

For error-estimating of x_n we can use the formula (2).

Theorem 6. *Suppose:*

1. *f is continuously differentiable to second order and $f'(x) \neq 0$, $f''(x) \neq 0$ for all $x \in [a, b]$.*
2. *Choose x_0 such that $f(x_0) \cdot f''(x_0) > 0$*

Then the sequence $\{x_n\}$ defined by (3) *is convergent to p.*

Example 8: To obtain the unique solution to $f(x) = x^3 + x - 5 = 0$ on the interval $[1, 2]$ by Newton's method, generate the sequence $\{x_n\}$ given by

$$x_n = x_{n-1} - \frac{x_{n-1}^3 + x_{n-1} - 5}{3x_{n-1}^2 + 1} \quad \text{for } n \geqslant 1$$

Selecting $x_0 = 2$ produces the results in the following table:

n	x_n	Δ_{x_n}
0	2	
1	1.615384615	0.2077
2	1.52129305	0.0105
3	1.515996427	3.20×10^{-5}
4	1.515980228	2.99×10^{-10}

EXERCISES

Question 1. Find the root-isolated intervals of the following equations:

(*a*) $f(x) = x^4 - 4x + 1 = 0$

(*b*) $f(x) = \ln x - \cos x = 0$

(*c*) $f(x) = x - 2^{-x} = 0$

(*d*) $f(x) = \mathrm{e}^x - x^2 + 3x - 2 = 0$

Question 2. Given $f(x) = x^3 + 2x^2 + 3x - 5 = 0$ in the interval $[0.5, 1]$ which consists a root p. Suppose that $p^* = 0.895 \approx p$. Find the error of p^*.

Question 3. Use the Bisection method to find an approximation x_5 of the root p for the following problems:

(*a*) $f(x) = x - 2^{-x} = 0$ for $0 \leqslant x \leqslant 1$

(*b*) $f(x) = \mathrm{e}^x - x^2 + 3x - 2 = 0$ for $0 \leqslant x \leqslant 1$

(*c*) $f(x) = (x-2)^2 - \ln x = 0$ for $1 \leqslant x \leqslant 2$

(*d*) $f(x) = x^2 - \sin \pi x = 0$ for $\frac{1}{2} \leqslant x \leqslant 1$

Question 4. Use the fixed-point iteration method to determine a solution accurate to within 10^{-3} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $x_0 = 1.5$.

Question 5. Given $x = g(x) = \sqrt[3]{5 - x}$ in $[1, 2]$. Use the fixed-point iteration method to find the value of k and x_7. Use $x_0 = 1.5$.

Question 6. Use Newton's method to find solutions accurate to within 10^{-6} for the following problems:

(*a*) $\mathrm{e}^x + 2^{-x} + 2\cos x - 6 = 0,\ x \in [1, 2]$

(*b*) $\ln(x-1) + \cos(x-1) = 0,\ x \in [1.3, 2]$

(*c*) $2x\cos 2x - (x-2)^2 = 0,\ x \in [3, 4]$

(*d*) $(x-2)^2 - \ln x = 0,\ x \in [\mathrm{e}, 4]$

(*e*) $\mathrm{e}^x - 3x^2 = 0,\ x \in [0, 1]$

(*f*) $\sin x - \mathrm{e}^{-x} = 0,\ x \in [6, 7]$
