Chapter 5: ORDINARY DIFFERENTIAL EQUATIONS (ODE)

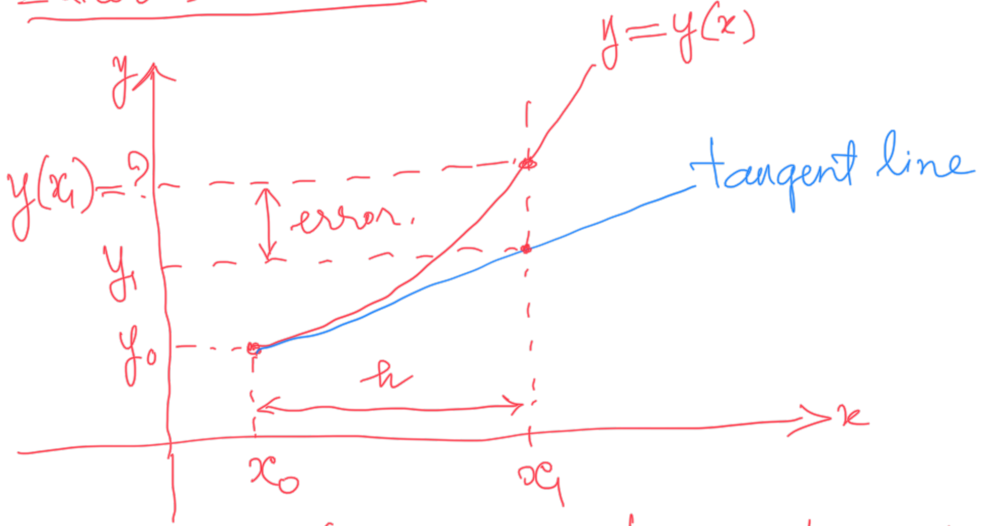
1) The Initial-Valued Problem: (for 1^{st} order ODE)

We need to find a function $y = y(x)$ that satisfies:

$$\begin{cases} \underline{y'(x) = f(x, y(x))}, & x \geq x_0. \\ \underline{y(x_0) = y_0} & : \text{initial condition}. \end{cases} \quad (1)$$

a) Choose a positive step $h > 0$. Let $x_1 = x_0 + h$. We need to find the value $y_1 \approx y(x_1) = y(x_0 + h)$.

i) Euler's Formula:



From the point (x_0, y_0) we draw a tangent line to the curve $y = y(x)$:

$$y - y_0 = y'(x_0)(x - x_0)$$

we have $y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$

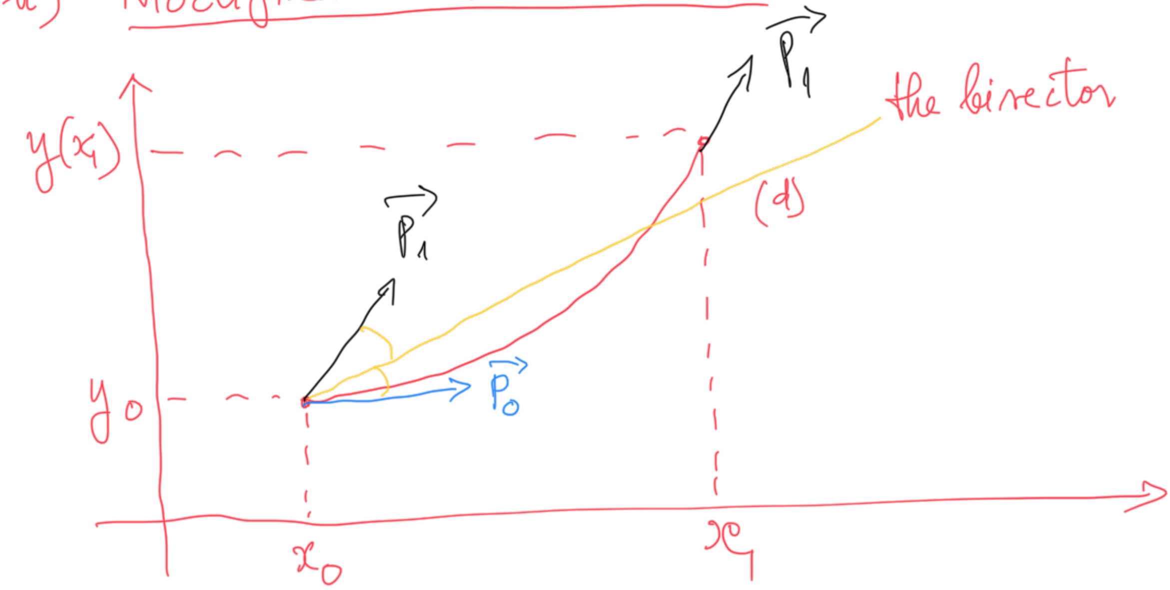
$$\Rightarrow \quad y = y_0 + f(x_0, y_0) \underbrace{(x - x_0)}_{h}$$

If in this eq-n we replace x by $x_1 = x_0 + h$, then we obtain $y_1 = y_0 + h f(x_0, y_0)$.

We obtain Euler's formula:

$$\boxed{y(x_1) \approx y_1 = y_0 + h f(x_0, y_0)} \qquad (2)$$

ii) Modified Euler's Formula:



$\vec{P_0}$: the direction vector of the tangent line at (x_0, y_0)

$\vec{P_1}$: ——————————————— $(x_1, y(x_1))$

From (x_0, y_0) we draw a bisector of $(\widehat{\vec{P_0}, \vec{P_1}})$.

$\Rightarrow$ The slope of this bisector (d) is:

$$\text{slope}(d) = \frac{\text{slope}(\vec{P_0}) + \text{slope}(\vec{P_1})}{2}$$

$$= \frac{f(x_0, y_0) + f(x_1, y(x_1))}{2}$$

$\Rightarrow$ We replace the slope of $\vec{P_0}$ by the slope of (d) in the Euler formula (2):

$$\underline{y(x_1)} \approx y_1 = y_0 + h \, \frac{f(x_0, y_0) + f(x_1, y(x_1))}{2}$$

$$= y_0 + h\,\frac{f(x_0,y_0)+f(x_0+h,\,y(x_1))}{2}$$

$$\Rightarrow \quad y(x_1) \simeq y_0 + \frac{1}{2}\left[h f(x_0,y_0) + h f(x_0+h,\, y(x_1))\right]$$

One suggest. to replace $y(x_1)$ on the right by the approximated value of the Euler formula:

$$\Rightarrow y(x_1) \approx y_0 + \frac{1}{2}\Big[\underbrace{h f(x_0,y_0)}_{K_1} + \overbrace{h f(x_0+h,\, y_0 + \underbrace{h f(x_0,y_0)}_{K_1})}^{K_2}\Big]$$

This is the Modified Euler Formula. It's more complicated to remember $\Rightarrow$ we rewrite it in the following form:

$$\boxed{\begin{aligned} K_1 &= h f(x_0,y_0) \\ K_2 &= h f(x_0+h,\, y_0+K_1) \\ y(x_1) \approx y_1 &= y_0 + \frac{1}{2}[K_1+K_2] \end{aligned}} \tag{3}$$

iii) Runge-Kutta's Formula:

$$\boxed{\begin{aligned} K_1 &= h f(x_0,y_0) \\ K_2 &= h f\left(x_0+\frac{h}{2},\, y_0+\frac{K_1}{2}\right) \\ K_3 &= h f\left(x_0+\frac{h}{2},\, y_0+\frac{K_2}{2}\right) \\ K_4 &= h f(x_0+h,\, y_0+K_3) \\ y(x_1) \approx y_1 &= y_0 + \frac{1}{6}(K_1+2K_2+2K_3+K_4) \end{aligned}}$$

Ex.: Given $\begin{cases} y' = \dfrac{x^3+2y^3}{3xy^2}, & x \geq 1 \\ y(1) = 2 \end{cases}$

that has the solution $y = \sqrt[3]{x^3 + 7x^2}$ //

Use the above three formulas to approximate $y(1.1)$ with $h = 0.1$ and compare them to the exact value.

Sol.: We have $x_0 = 1$, $y_0 = 2$, $h = 0.1$, $x_1 = 1.1$

$$f(x, y) = \frac{x^3+2y^3}{3xy^2}.$$

* Euler:

$$y(x_1) = y(1.1) \approx y_1 = 2 + 0.1\left[\frac{1^3 + 2\times 2^3}{3\times 1\times 2^2}\right] = $$

$$= \frac{257}{120} \approx 2.141667$$

$(err \simeq 1.6 \times 10^{-3})$

* Modified Euler:

$$K_1 = 0.1\left[\frac{1^3+2\times 2^3}{3\times 1\times 2^2}\right] = \frac{17}{120} \approx 0.1416667$$

$$K_2 = 0.1\left[\frac{(1+0.1)^3 + 2\times(2+K_1)^3}{3\times(1+0.1)\times(2+K_1)^2}\right] = 0.138591451 3$$

$$\Rightarrow y(1.1) \approx y_1 = 2.140129059$$

$(err \simeq 8.1 \times 10^{-5} \simeq 10^{-4})$

* Runge-Kutta:

$$K_1 = 0.1\left[\frac{1^3+2\times 2^3}{3\times 1\times 2^2}\right] = \frac{17}{120} \approx 0.14166667$$

$$K_2 = 0.1\left[\frac{\left(1+\frac{0.1}{2}\right)^3 + 2\left(2+\frac{K_1}{2}\right)^3}{3\left(1+\frac{0.1}{2}\right)\left(2+\frac{K_1}{2}\right)^2}\right] = 0.1400512097$$

$$K_3 = 0.1\left[\frac{\left(1+\frac{0.1}{2}\right)^3 + 2\left(2+\frac{K_2}{2}\right)^3}{3\left(1+\frac{0.1}{2}\right)\left(2+\frac{K_2}{2}\right)^2}\right] = 0.140006614 5$$

$$K_4 = 0.1\left[\frac{(1+0.1)^3 + 2(2+K_3)^3}{3(1+0.1)(2+K_3)^2}\right] = 0.13850449$$

$$\Rightarrow y(1.1) \approx y_1 = \underline{\underline{2.140047801}} \quad \left(\text{err} \approx 5.4\times 10^{-8} \approx 10^{-7}\right)$$

- The exact value:

$$y(1.1) = \sqrt[3]{1.1^3 + 7\times 1.1^2} = 2.140047747$$

Ex.:

$$\begin{cases} y' = xy - y^2 + 1, & x \geq 0.5 \\ y(0.5) = 0.22 \end{cases}$$

Use the formulas (2), (3), (4) to approximate $y(0.7)$ with $h = 0.2$.

Sol.: $x_0 = 0.5$, $y_0 = 0.22$, $h = 0.2$, $x_1 = 0.7$

$$f(x,y) = xy - y^2 + 1.$$

- Euler:

$$y(0.7) \approx y_1 = 0.22 + 0.2\left[0.5\times 0.22 - 0.22^2 + 1\right] = \underline{0.43232}$$

- Modified Euler:

$$K_1 = 0.2\left(0.5\times 0.22 - 0.22^2 + 1\right) = 0.21232$$

$$K_2 = 0.2\left[(0.5+0.2)(0.22+K_1) - (0.22+K_1)^2 + 1\right] = 0.2231446835$$

$$y(0.7) \approx y_1 = \underline{0.4377323418}$$

- Runge-Kutta:

$$K_1 = \ldots\ldots = 0.21232$$

$$K_2 = 0.2\left[\left(0.5+\frac{0.2}{2}\right)\left(0.22+\frac{K_1}{2}\right)-\left(0.22+\frac{K_1}{2}\right)^2+1\right]$$

$$= 0.2178631309$$

$$K_3 = 0.2\left[\left(0.5+\frac{0.2}{2}\right)\left(0.22+\frac{K_2}{2}\right)-\left(0.22+\frac{K_2}{2}\right)^2+1\right]$$

$$= 0.2178325929$$

$$K_4 = 0.2\left[\left(0.5+0.2\right)\left(0.22+K_3\right)-\left(0.22+K_3\right)^2+1\right]$$

$$= 0.2229570871$$

$$y(0.7) \approx y_1 = \underline{0.4377780891}$$

(b) If we consider the problem (1) in the interval $[x_0, x_0+H]$ with the step $h = \frac{H}{n}$, then

$$x_K = x_0 + Kh, \quad y(x_K) \approx y_K, \quad K = 1, 2, \ldots, n.$$

We have the following formulas:

* Euler:

$$\boxed{y(x_K) \approx y_K = y_{K-1} + h f(x_{K-1}, y_{K-1}), \quad K = 1, 2, \ldots, n.}$$

* Modified Euler:

$$\boxed{\begin{aligned} K_1 &= h f(x_{K-1}, y_{K-1}) \\ K_2 &= h f(x_{K-1}+h, y_{K-1}+K_1) \\ y(x_K) &\approx y_K = y_{K-1} + \tfrac{1}{2}(K_1+K_2) \end{aligned}}$$

Ex: Given $\begin{cases} y' = \dfrac{2y}{x} + x^2 e^x, & x \geq 1 \\ y(1) = 0.5 \end{cases}$

Use the Euler formula to approximate $y(x)$ in $[1, 2]$ with the step $h = 0.2$.

Sol. We have $x_0 = 1$, $y_0 = 0.5$, $h = 0.2$,

$$f(x, y) = \frac{2y}{x} + x^2 e^x$$

and $x_k = 1 + 0.2k$, $k = 1, 2, \ldots, 5$

$$y(x_k) \approx y_k, \quad k = 1, 2, \ldots, 5.$$

$$\Rightarrow y(x_k) \approx y_k = y_{k-1} + h\left[\frac{2y_{k-1}}{x_{k-1}} + x_{k-1}^2 e^{x_{k-1}}\right]$$

$$= y_{k-1} + 0.2\left[\frac{2y_{k-1}}{1+0.2(k-1)} + \left(1+0.2(k-1)\right)^2 e^{1+0.2(k-1)}\right]$$

$$= y_{k-1}\left[1 + \frac{0.4}{1+0.2(k-1)}\right] + 0.2\left[1+0.2(k-1)\right]^2 e^{1+0.2(k-1)}$$

$k = 1, 2, \ldots, 5$

$$\Rightarrow y(x_0) = y_0 = 0.5$$

$$y(x_1) = y(1.2) \approx y_1 = 1.243656366$$

$$y(x_2) = y(1.4) \approx y_2 = 2.614402161$$

$$y(x_3) = y(1.6) \approx y_3 = 4.951012594$$

$$y(x_4) = y(1.8) \approx y_4 = 8.724718344$$

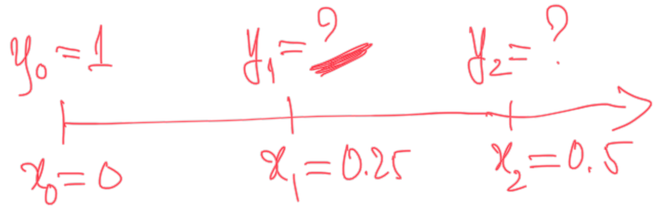
$$y(x_5) = y(2.0) \approx y_5 = 14.5837162.$$

Ex.: Given $\begin{cases} y' = x + y, & x \geq 0 \\ y(0) = 1 \end{cases}$

that has the exact solution $y = 2e^{x} - x - 1$.

Use the modified Euler formula to approximate $y(x)$ in $[0; 0.5]$ with the step $h = 0.25$ and compare with the exact values.

Sol.:



We have $x_0 = 0$, $y_0 = 1$, $h = 0.25$, $f(x,y) = x + y$

- $K_1 = h f(x_0, y_0) = 0.25(0 + 1) = 0.25$

  $K_2 = h f(x_0 + h, y_0 + K_1) = 0.25[(0 + 0.25) + (1 + 0.25)]$
  $= 0.375$

  $y(x_1) = y(0.25) \approx y_1 = 1.3125$

- $K_1 = h f(x_1, y_1) = 0.25(0.25 + 1.3125) =$
  $= 0.390625$

  $K_2 = h f(x_1 + h, y_1 + K_1) = 0.25[(0.25 + 0.25) + (1.3125 + 0.390625)]$
  $= 0.55078125$

  $y(x_2) = y(0.5) \approx y_2 = 1.78320312 5$

x_k	0	0.25	0.5	
y_k	1	1.3125	1.7832	approximated values
$y(x_k)$	1	1.3181	1.7974	exact values

2) The Initial-Valued Problem (for a system of ODE)

We need to find two functions $y(x)$ and $z(x)$ that satisfy:

$$\begin{cases} y'(x) = f(x, y(x), z(x)) \\ z'(x) = g(x, y(x), z(x)) \\ y(x_0) = y_0 \;;\; z(x_0) = z_0. \end{cases} \qquad x \geqslant x_0.$$

* Euler's Formula:

$$\begin{cases} y(x_1) \approx y_1 = y_0 + h f(x_0, y_0, z_0) \\ z(x_1) \approx z_1 = z_0 + h g(x_0, y_0, z_0). \end{cases}$$

* Modified Euler's Formula:

$$K_{1y} = h f(x_0, y_0, z_0)$$

$$K_{1z} = h g(x_0, y_0, z_0)$$

$$K_{2y} = h f(x_0 + h, y_0 + K_{1y}, z_0 + K_{1z})$$

$$K_{2z} = h g(x_0 + h, y_0 + K_{1y}, z_0 + K_{1z})$$

$$y(x_1) \approx y_1 = y_0 + \frac{1}{2}(K_{1y} + K_{2y})$$

$$z(x_1) \approx z_1 = z_0 + \frac{1}{2}(K_{1z} + K_{2z})$$

Ex: Given $\begin{cases} y' = xy - 2z + x \\ z' = 2y + xz - 1 \\ y(1) = 0.75 \; ; \; z(1) = 1.12 . \end{cases}$ $x \geq 1$.

Use the modified Euler formula to approximate $y(x)$ and $z(x)$ at $x = 1.1$ with $h = 0.1$.

Sol: We have $x_0 = 1$, $h = 0.1$, $x_1 = 1.1$.

$$y_0 = 0.75 \; ; \; z_0 = 1.12$$

$$f(x, y, z) = xy - 2z + x$$

$$g(x, y, z) = 2y + xz - 1$$

$$K_{1y} = h\left[x_0 y_0 - 2z_0 + x_0\right]$$

$$= 0.1\left[1 \times 0.75 - 2 \times 1.12 + 1\right] = -0.049$$

$$K_{1z} = h\left[2y_0 + x_0 z_0 - 1\right]$$

$$= 0.1\left[2 \times 0.75 + 1 \times 1.12 - 1\right] = 0.162$$

$$K_{2y} = h\left[(x_0 + h)(y_0 + K_{1y}) - 2(z_0 + K_{1z}) + (x_0 + h)\right]$$

$$= 0.1\left[(1 + 0.1)(0.75 - 0.049) - 2(1.12 + 0.162) + (1 + 0.1)\right]$$

$$= -0.06929$$

$$K_{2z} = h\left[2(y_0 + K_{1y}) + (x_0 + h)(z_0 + K_{1z}) - 1\right]$$

$$= 0.1\left[2(0.75 - 0.049) + (1 + 0.1)(1.12 + 0.162) - 1\right]$$

$$= 0.18122$$

$$\Rightarrow y(1.1) \approx y_1 = 0.690855$$

$$z(1.1) \approx z_1 = 1.29161.$$

3. The Boundary Problem:

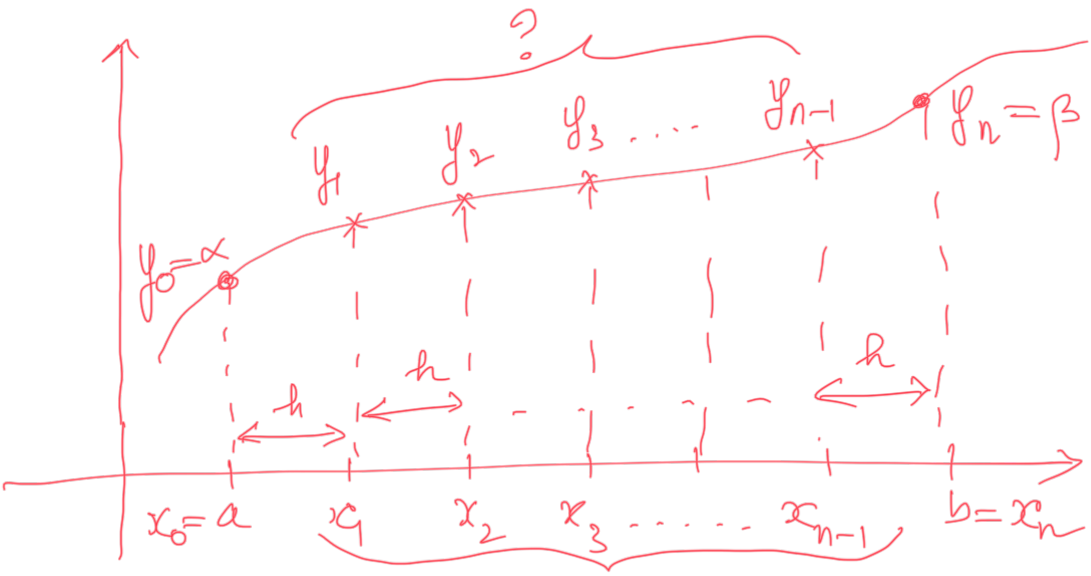
We need to find a function $y = y(x)$, that satisfies:

$$\begin{cases} y''(x) + p(x)\,y'(x) + q(x)\,y(x) = f(x), \quad a \le x \le b \\ y(a) = \alpha; \quad y(b) = \beta \quad \text{: the boundary cond-ns.} \end{cases}$$

The Finite Difference Method (FDM):

We consider the equidistance partition of $[a, b]$:

$$h = \frac{b-a}{n}; \quad x_0 = a, \quad x_k = x_0 + kh, \quad k = 0, 1, \dots, n; \quad x_n = b$$



Denote $y_k \approx y(x_k)$, $k = 0, 1, \dots, n$. Then from the boundary cond-ns $\Rightarrow y_0 = \alpha$, $y_n = \beta$

We consider x_k, $k = 1, 2, \dots, n-1$ and replace x_k to the ODE:

$$\begin{cases} y''(x_k) + p(x_k)\,y'(x_k) + q(x_k)\,y(x_k) = f(x_k) \\ k = 1, 2, \dots, n-1. \end{cases}$$

Denote $p_k = p(x_k)$, $q_k = q(x_k)$, $f_k = f(x_k)$

and use the centripetal formulas to approximate

$$y'(x_k) \approx \frac{y(x_k+h) - y(x_k-h)}{2h} \approx \frac{y_{k+1} - y_{k-1}}{2h}$$

$$y''(x_k) \approx \frac{y(x_k+h) - 2y(x_k) + y(x_k-h)}{h^2} \approx \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

$$\Rightarrow \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + p_k \frac{y_{k+1} - y_{k-1}}{2h} + q_k y_k = f_k$$

$$\Leftrightarrow \begin{cases} \left(\frac{1}{h^2} - \frac{p_k}{2h}\right) y_{k-1} + \left(q_k - \frac{2}{h^2}\right) y_k + \left(\frac{1}{h^2} + \frac{p_k}{2h}\right) y_{k+1} = f_k \\ k = 1, 2, \dots, n-1. \end{cases}$$

Denote $\underbrace{A_k = \frac{1}{h^2} - \frac{p_k}{2h}}$, $\underbrace{B_k = q_k - \frac{2}{h^2}}$, $\underbrace{C_k = \frac{1}{h^2} + \frac{p_k}{2h}}$

We obtain:

$$\boxed{\begin{gathered} A_k y_{k-1} + B_k y_k + C_k y_{k+1} = f_k \\ k = 1, 2, \dots, n-1 \; ; \; y_0 = \alpha, \; y_n = \beta \end{gathered}} \quad (1)$$

(1) is the system of $n-1$ linear equations with $n-1$ unknowns : $y_1, y_2, \dots, y_{n-1}$.

<u>Ex:</u>

$$\begin{cases} y'' + (2x+1)y' - 8x^2 y = -12xe^{x} & 0 \le x \le 1 \\ y(0) = \underset{\alpha}{\underline{\underline{0}}} \ ; \ y(1) = \underset{\beta}{\underline{\underline{1}}} . \end{cases}$$

Use the FDM to approximate the above problem with $n=4$ $\left(h=\frac{1}{4}\right)$.

<u>Sol:</u> We have:

$n=4$, $h=\frac{1}{4}$, $a=0$, $b=1 \Rightarrow x_k = \frac{k}{4}$, $k=0,1,\dots,4$

$y_k \simeq y(x_k)$, $y_0 = \alpha = 0$, $y_4 = \beta = 1$.

$p(x) = 2x+1 \Rightarrow p_k = 2x_k + 1 = 2\cdot\frac{k}{4} + 1 = \frac{k}{2} + 1$

$q(x) = -8x^2 \Rightarrow q_k = -8x_k^2 = -8\left(\frac{k}{4}\right)^2 = -\frac{k^2}{2}$

$f(x) = -12xe^{x} \Rightarrow f_k = -12x_k e^{x_k} = -12\cdot\frac{k}{4}e^{k/4} = \underline{\underline{-3ke^{k/4}}}$

$$A_k = \frac{1}{(1/4)^2} - \frac{\frac{k}{2}+1}{2\cdot\frac{1}{4}} = 16 - 2\left(\frac{k}{2}+1\right) = 14 - k$$

$$B_k = -\frac{k^2}{2} - \frac{2}{(1/4)^2} = -\frac{k^2}{2} - 32$$

$$C_k = \frac{1}{(1/4)^2} + \frac{\frac{k}{2}+1}{2\cdot\frac{1}{4}} = 16 + 2\left(\frac{k}{2}+1\right) = \underline{18+k}$$

The system (1) with $n=4$ gives

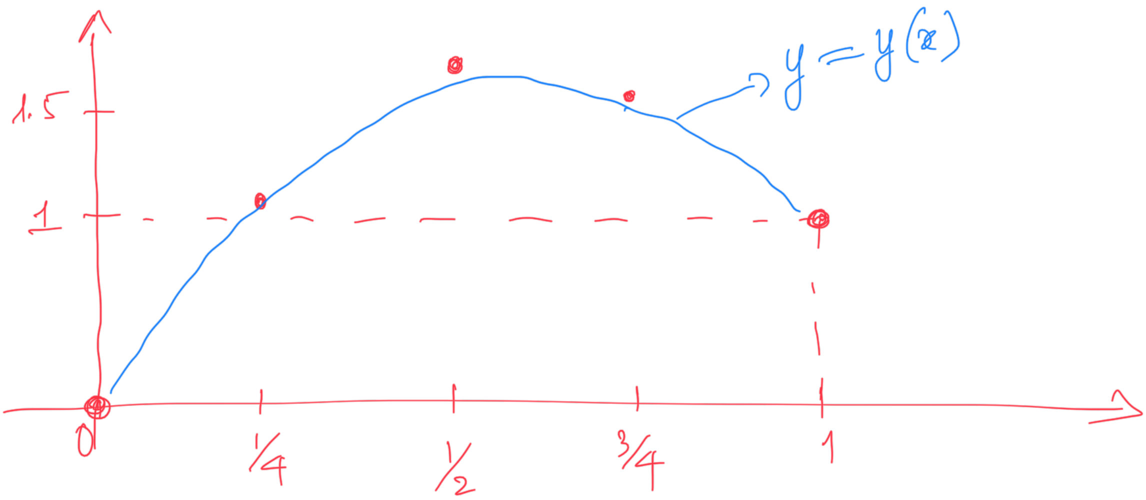
$(k=1)$: $\underline{A_1 \overset{\alpha=0}{y_0}} + \underline{B_1 y_1} + C_1 y_2 = f_1$

$(k=2)$: $A_2 y_1 + B_2 y_2 + \underline{C_2} y_3 = f_2$

$(k=3)$: $A_3 y_2 + B_3 y_3 + C_3 \underset{=\beta=1}{y_4} = f_3$

$$\Leftrightarrow \begin{cases} \left(-\frac{1}{2}-32\right) y_1 + 19 y_2 = -3e^{1/4} \\ 12 y_1 + \left(-\frac{4}{2}-32\right) y_2 + 20 y_3 = -6e^{1/2} \\ 11 y_2 + \left(-\frac{9}{2}-32\right) y_3 = -9e^{3/4} - 21 \times 1 \end{cases}$$

$$\rightarrow \begin{cases} y(x_1) = y\left(\frac{1}{4}\right) \approx y_1 = \underline{1.0464} \\ y(x_2) = y\left(\frac{1}{2}\right) \approx y_2 = \underline{1.5871} \\ y(x_3) = y\left(\frac{3}{4}\right) \approx y_3 = \underline{1.5757} \end{cases}$$



(i) Deadline: next Friday.

(ii) Your report is named ASSIGNMENT-N1.PDF

(iii)

N.	Name	Duty:
1	A.	ABC. . . . -
2	B	CDF . . . - -
3	- - -	- -
4	- - -	- -
-	- -	- -
-		

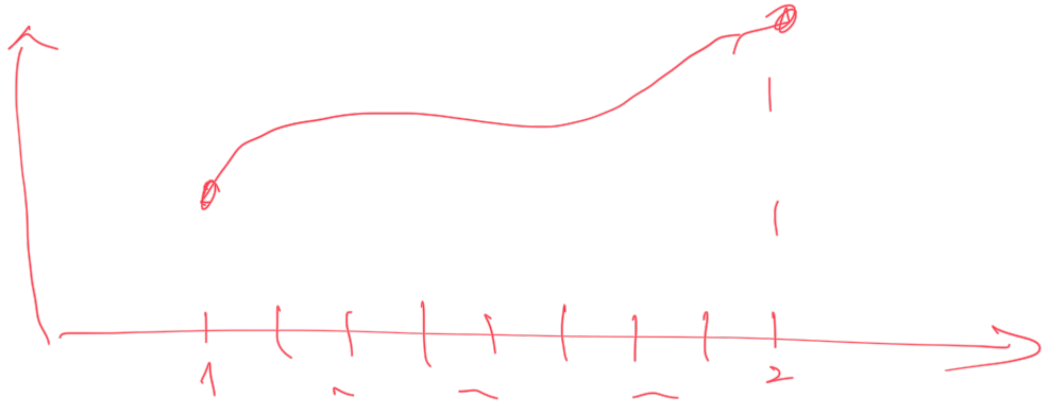
(iv) nan.thanh@lethai.vn

thanh@lethai.edu.vn

$$\begin{cases} y'' + p(x)y' + q(x)y = f(x), \quad 1 \leq x \leq 2 \\ y(1) = \alpha, \quad y(2) = \beta. \end{cases}$$

1) $n = 20 \left(h = \frac{1}{20}\right)$

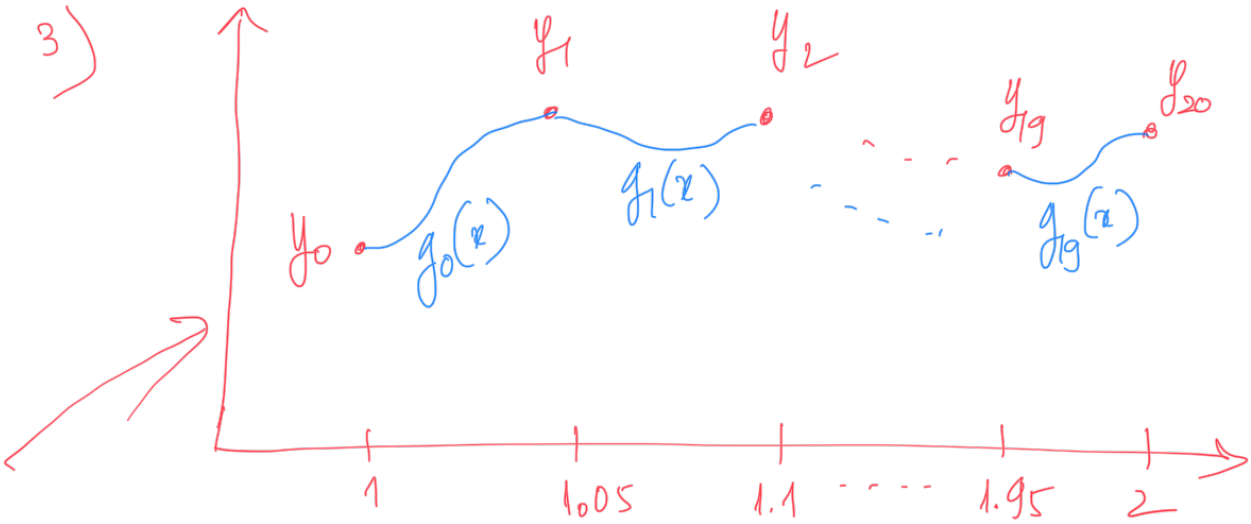
$$\Rightarrow \begin{aligned} y(x_0) &= y(1) = \alpha \\ y(x_1) &= y(1.05) \approx y_1 \\ y(x_2) &= y(1.1) \approx y_2 \\ &\ldots \\ y(x_{19}) &= y(1.95) \approx y_{19} \\ y(x_{20}) &= y(2) = \beta \end{aligned}$$



2) We have $y_0, y_1, \dots, y_{19}, y_{20}$.

Use the Composite Simpson Formula $\to I = \int_1^2 y(x)\,dx$

3)



- $g_0(x) = \dots$
- $g_1(x) = \dots$

$\dots$

- $g_{19}(x) = \dots$

4) $$I_0 = \int_{x_0}^{x_1} g_0(x)\,dx = \dots$$

$$I_1 = \int_{x_1}^{x_2} g_1(x)\,dx = \dots$$

$\dots$

$$I_{19} = \int_{x_{19}}^{x_{20}} g_{19}(x)\,dx = \dots$$

$$J = I_0 + I_1 + \cdots + I_{19} = \dots$$

In my course "Numerical Methods":

$\boxed{\text{It's always "open-book" exam.}}$

1)
- Code
- Results (round to 4 digits)
- Graph.

Code:

Results: $y(1) = \alpha = \dots$

$y(1.05) \approx y_1 = \dots$

$y(1.10) \approx y_2 = \dots$

$\dots$

$y(1.95) \approx y_{19} = \dots$

$y(2) \approx y_{20} = \beta = \dots$

Graph: 

2), 3), 4), ...

Next: 2^{nd} QUIZ. (9:30)

9:30 → 10:00.

(10:00 → 10:15 : send email to me)