3.1/8 Describe an algorithm that takes as input a list of n distinct integers and finds the location of the largest even integer in the list or returns 0 if there are no even integers in the list.

+ Let maxEvenIndex = 0

maxEven = $-\infty$

For i = 0 to n (n = length (list))

if $a_i$ is even and $a_i$ > maxEven, then let maxEven = $a_i$

and maxEvenIndex = i

output = maxEvenIndex

3.1/9 A palindrome is a string that reads the same forward and backward. Describe an algorithm for determining whether a string of n characters is a palindrome.

ANS = 1

For i = 1 to n, check whether $a_i = a_{n+1-i}$

if yes, ANS := ANS × 1 ; else ANS := ANS × 0

Output ANS

ANS = 1 means the string is a palindrome

ANS = 0 means it is not

**3.2/6** Show that $(x^3+2x)/(2x+1)$ is $O(x^2)$

Proof: Take $k=1$ $\quad c=3$

$$|f(x)| \leqslant \left| \frac{x^3+2x}{2x+1} \right| \leqslant \frac{x^3+2x}{2x+1} < \frac{x^3+2x}{x} = x^2+2 \leqslant 3x^2 = 3|x^2|$$

$(x>0)$

**3.2/9** Show that $x^2+4x+17$ is $O(x^3)$ but that $x^3$ is not $O(x^2+4x+17)$

+ Proof: Take $k=1$ $\quad c=22$

$$|f(x)| \leqslant |x^2+4x+17| \leqslant x^2+4x+17 < x^3+4x^3+17x^3 \leqslant 22x^3 = 22|x^3| \quad \square$$

$(x \geqslant 1)$

+ Assume that $x^3$ is $O(x^2+4x+17)$ such that $\forall x \geqslant k$

$|x^3| \leqslant c\,|x^2+4x+17|$

But $x \geqslant 1$, $\quad x^2+4x+17 < x^2+4x^2+17x^2 = 22x^2$

Hence, $\quad x^2+4x+17 < 22x^3$ (with $x \geqslant 1$). A contradiction

Therefore, $x^3$ is not $O(x^2+4x+17)$

3.2 / 16 Show that if $f(x)$ is $O(x)$, then $f(x)$ is $O(x^2)$

Proof: $f(x)$ is $O(x)$ then there are constant c and k such that

$|f(x)| \leq c|x|$ whenever $x > k$

Let assume that $k > 1$, $x > k$ then $x > 1$

Thus $x^2 > x$, then $|x^2| > |x|$ $(x > 1)$

Hence, $|f(x)| \leqslant c|x| < c|x^2|$ (since $x > 1$)

$|f(x)| \leqslant c|x^2|$ whenever $x > k > 1$

Therefore, $f(x)$ is $O(x^2)$

3.2 / 18 Let k be a positive integer. Show that $1^k + 2^k + .. + n^k$ is $O(n^{k+1})$

Proof: Take $K = 1$ $c = 1$

$|1^k| < |n^k|$ so $1^k$ is $O(n^k)$

$|2^k| < |n^k|$ so $2^k$ is $O(n^k)$

$\vdots$

$|n^k| \leqslant |n^k|$ so $n^k$ is $O(n^k)$

Thuse, $|1^k| + |2^k| + .. + |n^k| \leqslant |n . n^k| = |n^{k+1}|$ $(n \geqslant 1)$

$\square$

3.2 / 22 Arrange the functions $(1.5)^n$, $n^{100}$, $(\log n)^3$, $\sqrt{n}\log n$, $10^n$, $(n!)^2$, and $n^{99}+n^{98}$ in a list so that each function is big-O of the next function.

Follow benchmark functions: $1 \ll \log x \ll x^{\alpha} \ll c^x \ll n! \ll \ldots$

$c^x$ : $(1.5)^n$, $10^n$

$\log x$ : $(\log n)^3$, $\sqrt{n}\log n$

$n!$ : $(n!)^2$

$x^{\alpha}$ : $n^{100}$, $n^{99}+n^{98}$

Compare: $(1.5)^n \ll 10^n$

$(\log n)^3 \ll \sqrt{n}\log n$ $(\sqrt{n} = n^{1/2})$

$n^{99}+n^{98} \ll n^{100}$

Therefore, $(\log n)^3 \ll \sqrt{n}\log n \ll n^{99}+n^{98} \ll n^{100} \ll 1.5^n \ll 10^n \ll (n!)^2$

3.2 / 26 Give a big-O estimate for each of these functions. For the function $g$ in your estimate $f(x)$ is $O(g(x))$, use a simple function $g$ of smallest order.

a) $(n^3+n^2\log n)(\log n+1) + (17\log n+19)(n^3+2)$

$= O(n^3).O(\log n) + O(17\log n).O(n^3)$

$= O(n^3.\log n) + O(17n^3.\log n)$

$= O(n^3\log n)$

b) $(2^n + n^2)(n^3 + 3^n)$

$= O(2^n) . O(3^n)$

$= O(6^n)$.

c) $(n^n + n2^n + 5^n)(n! + 5^n)$

$= O(n^n) . O(n!) = O(n^n . n!)$

3.2/36 Explain what it means for a function to be $\Omega(1)$

+ $f$ is $\Omega(1)$ if there are constants $C > 0$ and $k$ such that $1 \leq C|f(x)|$ or equivalently, $|f(x)| \geq \frac{1}{C}$ whenever $x > k$. In other words, $f(x)$ keeps at leasts a certain distance away from 0 for large enough $x$.

3.2/44 Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$

+ Since $f(x)$ is $\Theta(g(x))$, there are positive constants $C_1$, $C_2$, $s$ such that when $x \geq s$

$$C_1 |g(x)| \leq |f(x)| \leq C_2 |g(x)|$$

Similarly, since $f\, g(x)$ is $\Theta(h(x))$, there are positive constants $D_1$, $D_2$, $t$ such that when $x \geq t$

$$D_1 |h(x)| \leqslant |g(x)| \leqslant D_2 |h(x)|$$

Take $A = C_1 D_1 > 0$ , $B = C_2 D_2 > 0$ and $k = \max(s,t)$ , then when $x > k$ , we have

$$A|h(x)| = C_1 D_1 |h(x)| \leqslant C_1 |g(x)| \leqslant |f(x)| \leqslant C_2 |g(x)| \leqslant C_2 D_2 |h(x)| = B|h(x)|$$

By definition $f(x)$ is $\Theta(h(x))$