Fix a valid fixed point format fmt, and let $Fixed$ be the set of all fixed point numbers in this format (i.e., the set of all real numbers x for which $is\_fixed(fmt)x$ is true).

1 Greatest/least

Lemma 1.1.

$$FINITE(Fixed)$$

Proof. Let $I = \{0, 1, \ldots, 2r^{p-1} - 1\}$. I is finite (FINITE_NUMSEG). Let

$$f(i) = \begin{cases} 0 & \text{if } i = 0 \\ (i+1)/2 \cdot r^{e-p+1} & \text{if } i \text{ is odd} \\ i/2 \cdot r^{e-p+1} & \text{if } i \text{ is even} \end{cases}$$

Let $s \in Fixed$; then $|s| = f \cdot r^{e-p+1}$ and $0 leq f < r^{p-1}$. If $s = 0$, $s = f(0)$. If $s > 0$, $f > 0$, and $s = f(2f-1)$. If $s < 0$, then $f > 0$, and $s = f(2f)$.

We have shown $Fixed = f(I)$. $Fixed$ is therefore finite. (FINITE_IMAGE) □

Lemma 1.2.

$$\forall S \subset Fixed \,.\, S \neq \emptyset \implies \exists s^* \in S \,.\, \forall s \in S \,.\, s \leq s^*$$

Proof. Assume the antecedent. From 1.1, $Fixed$ is finite; since $S \subset Fixed$, S is also finite. Since S is non-empty, there is a maximal element $s^* \in S$ (SUP_UNIQUE_FINITE). □