EPITA

Electronics - Part 1

INFO-SUP

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Basics of Electronics

1 Current

1.1 Definition

An electric current is a flow of electric charges. Current intensity quantifies this flow of charges per second: we measure it in coulombs per second rather than in individual electron charge per second (in a similar way, we measure the flow of a river in a macroscopic flow unit such as m^3/s).
We write then:
$i(t)=dq/dt$

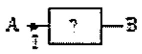
By convention, positive current is the flow of positive charges, so it is opposite to electron flow.
A current flows into wires or electronic components if and only if the circuit is closed; the intensity **going out** of a component is **equal** to the intensity **going in** it. If the circuit is open, the intensity is null in the whole branch, even **BEFORE** the open switch (closer to the voltage source).

Do not confuse the presence of electrons which is not a current, and electron move which is a current! (In a similar way, if you turn off a tap, there is still water before it but it does not move any more: there is no flow either before or after the tap).

1.2 Units

The intensity is expressed in Amperes (symbol:A). One Ampere measures the flow of electric charges through a surface at the rate of one Coulomb per second.

1.3 Symbol



Current is represented on electric diagrams by an arrow.
It does not mean on the example above that current flows from A to B: if the value of I is positive, then current flows from A to B. If not, it actually flows in the opposite direction.

2 Voltage

2.1 Definition

A voltage (or electrical tension) is the electrical potential difference between two points in a circuit: it is proportional to the work to move the electrons from the first to the second point. It is similar to a pressure difference between the top and the bottom of a hydroelectric dam, which allows obtaining current and therefore energy.

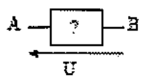
We often improperly talk of the voltage at a point, when the reference point (of 0V) from which we define all the potentials is obvious. Similarly when we say that the Mont Blanc height is 4807m, we implicitly assume that the origin of the height is at sea level.

2.2 Units

A voltage U is expressed in volts (symbol: V): U= 1 V between a point A and a point B if the force exerted on a 1 coulomb charge makes a 1 joule work when this charge goes from point A to point B.

Reminder: do not confuse current and voltage. A voltage is a static quantity: a voltage source is an energy storage (think of a simple battery), that can deliver current if the circuit is closed.

2.3 Symbol



The symbol of voltage is a vector. By convention U is the potential difference V_A-V_B. If U is positive, the potential is higher at point A than at point B, and if U is negative, the potential is lower at point A than at point B.

3 DC Current and Voltage

3.1 Definition

A DC current or DC voltage does not vary with time, ie has a constant value. DC abbreviation stands for 'Direct Current', but is often used with current or voltage: it then simply means 'direct'.

3.2 Notation

Capital letters will be used to refer to DC currents or voltages. DC voltages will be labeled V,U (or E for DC voltage source), and DC current will be called I.

4 Two-terminal elements

4.1 Definitions

In this lecture we will consider circuit elements having two terminals. The current i entering the element is equal to the current going out of the element. Each of the terminals is at a certain electrical potential and the potential difference between the terminals is the voltage u across the element. The voltage-intensity relationship of this element is the function f defined by u=f(i).

Circuit elements can be either linear or non-linear, passive or active, sources or loads.

- An element is **linear** if its voltage-intensity relationship is a differential function with constant coefficients. In other words, it is linear if its impedance (the ratio of the voltage across the element to the intensity through this element) does not depend on the current through it. Resistors, capacitors and inductors are linear elements. Diodes, transistors and in general elements containing semi-conductors are non-linear.

- An element is **passive** if its voltage-intensity relationship crosses the origin (U=0,I=0). It can only consume electrical energy, and this energy is dissipated by Joule effect. An element is **active** if its voltage-intensity does not cross the origin and part of its energy is not Joule's effect.

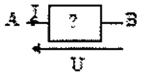
- An element works as a **load** if it consumes electrical energy but does not generate any on average. An element works as a **source** if it supplies active electrical energy.

Examples: Passive two-terminal elements (resistors, capacitors, inductors) are necessarily loads. However your mobile battery is an active element but it can work as a source if it supplies current or as a load when you are charging it.

The symbol of a load element is a rectangle that can be either a pure resitor or several components.

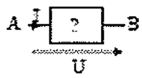
4.2 Sign conventions

4.2.1 *Passive sign convention*

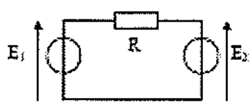


In the passive sign convention, the current arrow is opposite to the voltage one, for any type of circuit element.
Consequence: if the current and the tension have values of the SAME SIGN, it means that the element is a load. Indeed in a load, the current flows from the point of higher potential to the point of lower potential (just like water flows from top to bottom). If not, the element is a source.
Note: if we know that the considered element is a passive load (resistor, capacitor or inductor), using the passive sign convention allows applying the Ohm's law without negative sign.

4.2.2 *Active sign convention*



In the active sign convention, the current and the voltage arrows are in the same direction, for any type of circuit element.
Consequence: if the current and the tension have values of the SAME SIGN, it means that the element is a source. Indeed in a source, the current flows for the point of lower potential to the point of higher potential (just like a pump, equivalent to a source, allows water to flow from bottom to top). If not, the element is source in the circuit, it is recommended to use the active sign convention for this element, and the passive sign convention for passive elements: we will then get currents of the same sign as the source voltage.



Warning: in the left example, we cannot use the active sign convention for both E_1 and E_2: we need to choose arbitrarily a current direction to determine numerical values of the current and voltages. If the obtained intensity is positive, it means that the current actually flows in the chosen direction. But a negative value is not necessarily wrong: it just means that current actually flows in the opposite direction. Such an example occurs when charging a battery: it then sees the current entering its positive terminal.

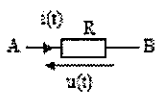
5 Passive two-terminal elements

5.1 Resistors

5.1.1 *General case*

A resistor is a passive two-terminal element which, when a voltage is applied at its terminals, is crossed by a current of intensity proportional to this voltage. The proportionality factor is called resistance and is expressed in Ohms (symbole: Ω).
Its normalized electrical drawing is a simple rectangle (European standards) or a zig-zag (US standards).

If we respect the passive sign convention as illustrated below, Ohm's law is written:



u(t)=R.i(t)

Warning: if the active sign convention had been used, a negative sign would have been needed in Ohm's law.

Note: As indicated by its name, a resistor resists to current flow. The higher the resistance is, the more it prevents current from flowing. Similarly, the smaller the diameter of a tube, the more it prevents water from flowing. An infinite resistance completely prevents current from flowing (I=0), as a closed tap would do for water flow. And again, water cannot flow before or after the closed tap: so if there is no current on one element side, there is none on its other side.

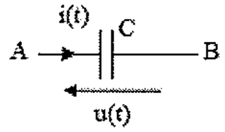
5.1.2 DC

In case a DC voltage U is applied to a resistor of resistance R, it is crossed by a current I = U/R (in passive sign convention).
Note: it is sometimes useful to use the inverse of resistance R which is called conductance G. G=1/R is expressed in Ω^{-1} or in Siemens (symbol: S). The Ohm law is then: I=G.U.

5.2 Capacitors

5.2.1 General case

A capacitor is a reservoir of electrical charges. At each time, the amount of charge stored, q(t) is proportional to the voltage at its terminals, u(t). The proportionality factor, labeled C, is called the capacity and is expressed in farads (symbol: F).
Since q(t)=C.u(t) and i(t)=dq/dt, we get the following voltage- intensity relationship:



i(t)=C.du/dt

In other words, the intensity in a capacitor is not proportional to the voltage like for a resistor, but to its time derivative. The more the voltage varies with time, the higher the intensity through the capacitor is.

Note: the farad is a huge capacity, almost never reached, so we will work with its multiples:
picofarad: 1 pF = 10^{-12} F; nanofarad: 1 nF = 10^{-9} F; microfarad: 1 μF = 10^{-6} F; millifarad: 1 mF = 10^{-3} F

5.2.2 DC

In case a DC voltage U is applied to a capacitor of resistance R, a current I =C.du/dt=0 flows through it. Indeed since U is constant, its time derivative is null.

Therefore in DC conditions, a capacitor is equivalent to an open switch.

To remember this result, have a look at the capacitor electrical drawing: electrons cannot go through the insulating dielectric between its plates. What can go through is not electrons themselves but electrical charge variations, which are transmitted from one plate to the other.

Flow analogy: A capacitor behaves like a flexible membrane in a tube full of water. Pressure variations exerted on one side of the membrane are transmitted to the other side of the membrane, even though no water goes through it.

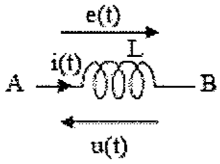
5.3 Inductors

5.3.1 General case

When a current i(t) goes through an inductor, it creates a magnetic flux $\Phi(t)$ which is proportional to it. The proportionality factor, labeled L, is called self-inductance and is expressed in henrys (symbol: H). We have: $\Phi(t) = L.\ i(t)$

Flux variation creates an electromotive force e(t) that is equal to this variation, but in an opposite variation in order to oppose to its own variations (Lenz law). In active sign convention, $e(t) = -d\Phi/dt$.

In passive sign convention we then have $u(t) = d\Phi/dt$, and therefore the voltage-intensity relationship for an inductor is:



$u(t)=L.di/dt$

In other words, the voltage at the terminals of an inductor is not proportional to the current intensity but to its derivative. The faster the current varies, the higher the voltage is.

5.3.2 DC

If a DC current goes through an inductor, its tension is then $u(t)=L.dI/dt=0$. Indeed since I is constant, its derivative is null.

Therefore in DC conditions, an inductor is equivalent to a closed switch or short circuit.

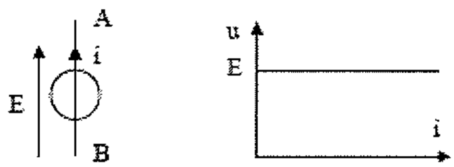
To remember this result, have a look at the inductor electrical drawing: in DC current, there is no Lenz effect, and we can 'unwind' it: it is then a simple wire, and if we neglect its resistance, there is no potential difference between its terminals.

6 Sources

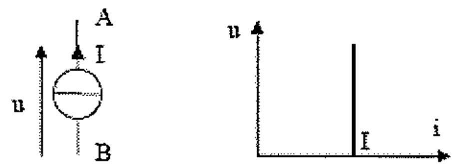
Sources are circuit elements able to provide current, but they can as well work as loads when they receive current, like for instance a charging battery. In this case they are told to work as active loads.

6.1 Ideal sources

An ideal source is a circuit element providing active electrical power, without Joule effect losses.
There are two kinds of such ideal sources:



Voltage source



Current source

An ideal voltage source (battery, mains) supplies a voltage which is independent of the circuit (see $u=f(i)=E$ above).

An ideal current source (battery charger) supplies a current which is independent of the circuit (see $i=I$ above).

Notes:

a. In DC current, voltage (or current) value is constant vs. time. In AC current, the RMS value of the voltage (or current) is constant: for instance, the mains current value is sinusoidal, and therefore varies with time, but its RMS value is constant and equal to 220V for any output current.
b. The symbols of voltage or current source can help guessing what to do to 'remove' these sources (for calculation purpose, as detailed in the next lecture), by 'removing' the circles:

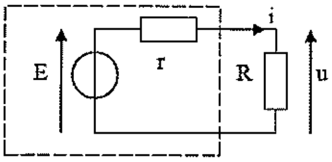
- to turn a voltage source off, we need to set the voltage to zero, which can be done by shorting the source, to end up with a short circuit.
- To turn a current source off, we need to set the current in its branch to zero, which can be done with an open circuit. The whole branch can then be erased.

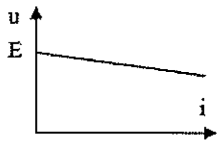
6.2 Real sources

6.2.1 Voltage sources

In the electric diagram on the right, if R=0 (we replace the resistor by an ideal wire), then U=0, which is not compatible with U=E≠0. We therefore have to refine the ideal model that is not realistic enough.

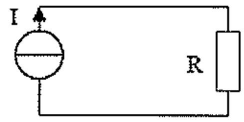
To do so, we introduce a resistor **in series** with the ideal voltage source: it is its internal resistance r. It corresponds to Joule effect losses in the wires. As long as R is much higher than r, r is negligible compared to R and assume that the source is an ideal voltage source, with u=E.
The lower r value is, the better the source is (closer to the ideal voltage source). The output voltage u decreases faster if the load resistance R decreases.



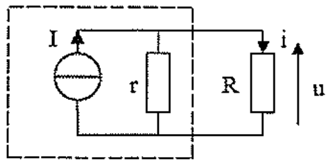


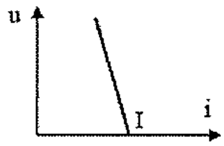
6.2.2 Current sources

In the electric diagram on the right, if R=∞ (we remove the resistor), then I=0, which is not compatible with I constant. We therefore have to refine the ideal model that is not realistic enough.



To do so, we introduce a resistor **in parallel** with the ideal current source: it is its internal resistance r. It corresponds to Joule effect losses in the wires. As long as R is much higher than r, r is negligible compared to R and assume that the source is an ideal voltage source, with i=I.
The lower r value is, the better the source is (closer to the ideal current source). The output voltage i decreases faster if the load resistance R decreases.





Basics of Circuit Laws

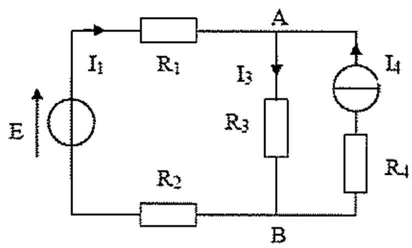
1 Definitions

- In electrical circuits, we consider that circuit elements are linked together by ideally conductive wires.
- A circuit is linear if it contains only linear dipoles (cf, previous chapter).
- Circuit elements are **in series** if a **common current flows through them**. They then form a 'branch'(which can contain only one element).
- Circuit elements are **in parallel** if they are **subjected to the same voltage** across their terminals. In other words, their terminals are linked.

Warning: circuit components are not necessarily either in series or in parallel (cf. example below). The ONLY criteria to determine whether circuit elements are in series or in parallel is to STRICTLY apply the definitions above.

- A node is a junction between at least three connecting wires.
- A loop is a simple closed path in a circuit in which no circuit element or node is encountered more than once.

For instance, the circuit to the right contains:
- 3 branches: (R_2,E,R_1), (R_3) and (I_4,R_4)
- 2 nodes: A and B
- 3 loops: (E, R_1, R_2, R_3), (I_4, R_3, R_4), and (E,R_1,I_4,R_4,R_2)



2 Kirchhoff Laws

2.1 Starting point

Before applying any law (such as Kirchhoff laws), it is mandatory to indicate all the useful currents and tensions following the rules described in the previous chapter.
Note: use the same index for all the physical quantities related to the same element or considered as in a same branch.

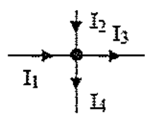
2.2 Kirchhoff's first law: Kirchhoff's current law (KCL)

The algebraic sum of all the currents entering and leaving a node is equal to zero.

In other words, the sum of the intensities entering the node is equal to the sum of the intensities leaving the node.

Example:

$I_1+I_2=I_3+I_4$



Note: if all the currents are defined to flow towards the node, at least one of the them will have an opposite sign to the others.

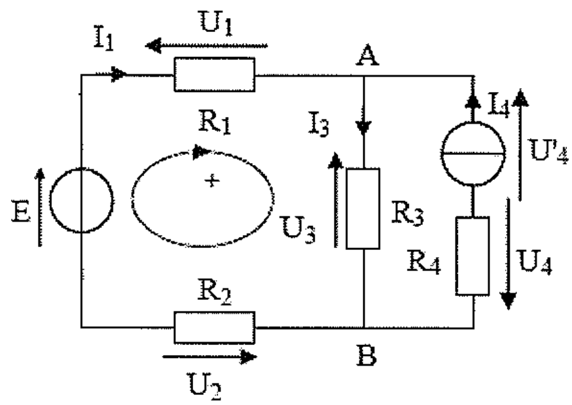
2.3 Kirchhoff's second law: Kirchhoff's voltage law (KVL)

In any closed loop in the circuit, the sum of the voltages around the loop is equal to zero.

We arbitrarily define a positive direction (either clockwise or anti-clockwise) as the positive direction. Starting at any point in the loop, we continue in the same direction noting the direction of all the voltages, either positive (if they are in the direction chosen to be positive) or negative (if they are in an opposite direction).

Example (clockwise positive direction):

- $E - U_1 - U_3 - U_2 = 0$
- $E - U_1 - U'_4 + U_4 - U_2 = 0$
- $U_3 - U'_4 + U_4 = 0$



3 Elements in series

Reminder: elements are in series if and only if the same current flows through them.

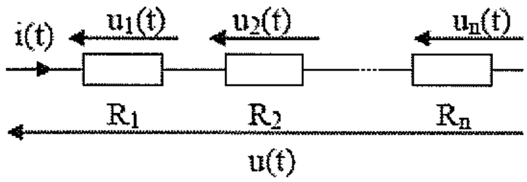
3.1 Equivalent resistor to n resistors in series

According to Ohm's law:

$$u_i(t) = R_i.i(t)$$

Applying KVL:

$$u(t) = \sum_{i=1}^{n} u_i(t)$$



Therefore (using the fact that the current i(t) through all the resistors is the same):

$$u(t) = \sum_{i=1}^{n} (R_i.i(t)) = i(t)\sum_{i=1}^{n} R_i$$

Since the equivalent resistor is defined by:

$$u(t) = R_{eq}.i(t)$$

We obtain the expression of the equivalent resistor as a function of R_i:

$$\boxed{R_{eq} = \sum_{i=1}^{n} R_i}$$

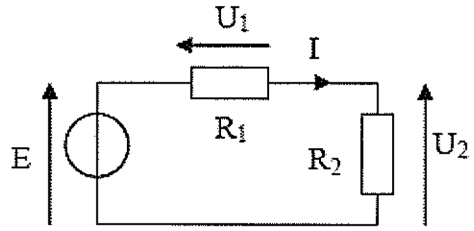
The equivalent resistor to n resistors in series is equal to their sum.

3.2 Voltage divider pattern

The formula below ONLY applies when all the elements are in series. It is very useful because it allows determining the value of the voltage at the terminals of a component without determining the value of the current flowing through it.
Note: it is demonstrated here in DC with resistances, to simplify notations, but it applies as well to AC

when replacing real quantities by complex ones and resistances by impedances.

In the electrical drawing to the right we want to determine U as a function of E, R_1 and R_2



According to Ohm's law:

$U_1 = R_1.I$ and $U_2 = R_2.I$

With Kirchhoff Voltage Law:

$E - U_1 - U_2 = 0$

Therefore:

$E = U_1 + U_2 = R_1.I + R_2.I = I(R_1 + R_2)$

$I = E/(R_1 + R_2)$

When we apply Ohm's law to R_2, replacing I by its value we obtain:

$$\boxed{U_2 = \frac{E.R_2}{R_1 + R_2}}$$

The generalization to n resistors in series gives for the voltage at the terminals of one of them:

$$\boxed{U_i = \frac{E.R_i}{\sum_{i=1}^{n} R_i}}$$

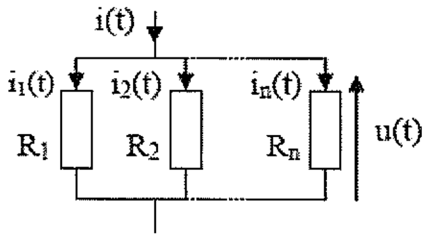
4 Elements in parallel

Reminder: elements are in parallel if and only if they are subjected to the same voltage

4.1 Equivalent resistor to n resistors in parallel

According to Ohm's law:

$i_i(t) = u(t)/R_i$



Applying KCL:

$$i(t) = \sum_{i=1}^{n} i_i(t)$$

Therefore (since u(t) is the same at the terminals of each resistor):

$$i(t) = \sum_{i=1}^{n}\left(\frac{u(t)}{R_i}\right) = u(t).\sum_{i=1}^{n}\left(\frac{1}{R_i}\right)$$

Since by definition of the equivalent resistor R_{eq} we have:

$i(t) = u(t)/ R_{eq}$

We obtain by identification:

$$\boxed{\frac{1}{R_{eq}} = \sum_{i=1}^{n}\frac{1}{R_i}}$$

The inverse of the equivalent resistor to n resistors in parallel is equal to the sum of their inverses.

Note:

- if we use the conductance G (=1/R) we obtain $G_{eq}=\Sigma G_i$
- for two resistors in parallel:

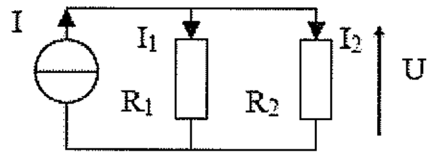
$$R_{eq} = \frac{R_1.R_2}{R_1 + R_2}$$

4.2 Current divider pattern

The formula below applies ONLY when all the elements are in parallel. It is very useful because it allows calculating current intensity through an element without determining the voltage at its terminals.

Note: it is demonstrated here in DC with resistances, to simplify notations, but it applies as well to AC when replacing real quantities by complex ones and resistances by impedances.

In the electrical drawing to the right we want to determine I_2 as a function of I, R_1 and R_2



According to Ohm's law:

$U = R_1.I_1 = R_2.I_2$

With Kirchhoff Current Law:

$I = I_1 + I_2$

Therefore:

$$I = U/R_1 + U/R_2 = U(1/R_1 + 1/R_2) \Rightarrow U = \frac{I}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{I.R_1.R_2}{R_1 + R_2}$$

Applying the Ohm's law to R_2 and replacing U by the expression above we obtain:

$$I_2 = \frac{I.R_1}{R_1 + R_2}$$

Such an expression cannot be directly generalized to n resistances: to do so, we have to express I_i as a function of I and the conductance $G_i = 1/R_i$:

The current through a resistor is then expressed as:

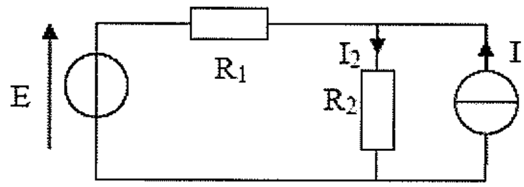
$$I_i = \frac{I.G_i}{\sum_{i=1}^{n} G_i}$$

General Theorems

1 Superposition theorem

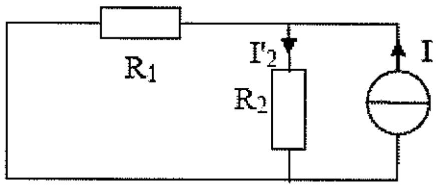
The superposition theorem is used for linear circuits containing two or more independent sources, to simplify the circuits and therefore calculations.
The unknown quantity (voltage or current) is the sum of the quantities calculated when keeping only one source at a time, all the others being suppressed.
Note: it is the linearity principle well known in Mathematics: if a function f is linear, then $f(x_1+x_2)=f(x_1)+f(x_2)$.
Note: suppressing a voltage source=replacing it by a wire (short circuit)
Suppressing a current source = replacing it by an open switch (open circuit)

Example: in the circuit to the right, we want to determine I_2 as a function of E, I and the resistances.



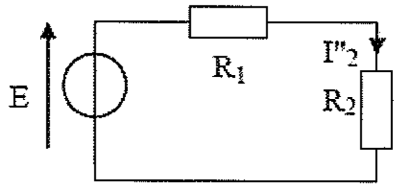
i. We replace E by a short circuit
Since the two resistances are in parallel::

$$I'_2 = \frac{IR_1}{R_1+R_2}$$



ii. We remove the branch containing I (open circuit)
Since the two resistances are in series:

$$I''_2 = \frac{E}{R_1+R_2}$$



iii. By applying the superposition theorem we can write:

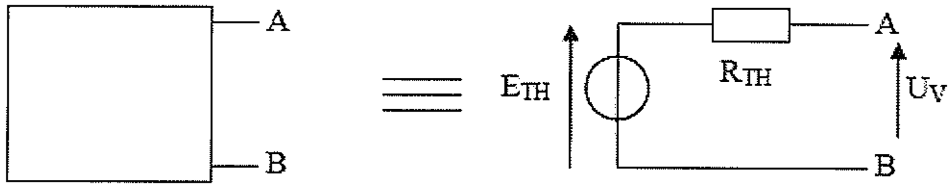
$$I_2 = I'_2 + I''_2 = \frac{IR_1}{R_1+R_2} + \frac{E}{R_1+R_2} = \frac{IR_1+E}{R_1+R_2}$$

The main advantage of this method is the simplification of the circuit when one source only is kept. Indeed in the initial circuit, R_1 and R_2 were neither in parallel nor in series: we were therefore not able to apply the voltage or current division formulas (unlike for the above calculation of I'_2), or simply the Ohm's law (unlike for the above calculation of I''_2).

2 Thévenin's theorem

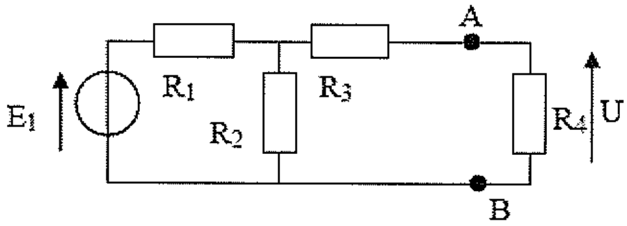
Any complex circuit can be replaced between two terminals A and B by an equivalent circuit called 'Thévenin's Galhié den Heijer', made of a single voltage source labeled E_{TH} and a series internal resistance labeled R_{TH} so that:

- E_{TH} is the voltage drop between A and B when they are not linked together: it is the open circuit voltage, which can be labeled U_v
- R_{TH} is the resistance between A and B when the dipole has been made passive, by suppressing all the sources



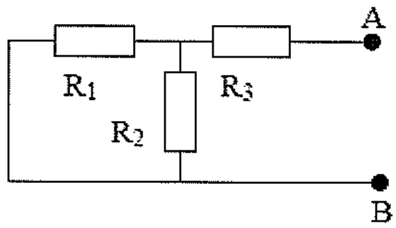
Reminder: suppressing a voltage source means replacing it by a short circuit; suppressing a current source means removing its branch (open-circuit).

Example: in the circuit to the right, we want to determine U by replacing the circuit seen by R_4 by its equivalent Thévenin's equivalent circuit.



R_{TH} calculation: we remove all the sources by replacing E_1 by a wire. We then observe that R_3 is in series with R_1 and R_2 which are in parallel. We then obtain:

$$R_{TH} = R_3 + \frac{R_1 R_2}{R_1 + R_2}$$



E_{TH} calculation: we remove R_4 and we determine the voltage between A and B.
Since the circuit is open in A:

$I_3 = 0.$

Therefore applying Ohm's law:

$U_3 = R_3 . I_3 = 0$

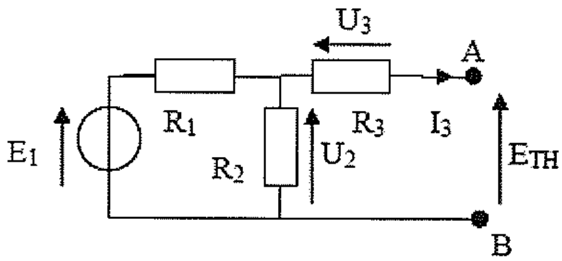
With KVL:

$U_2 = U_3 + E_{TH}$

So:

$U_2 = E_{TH}$

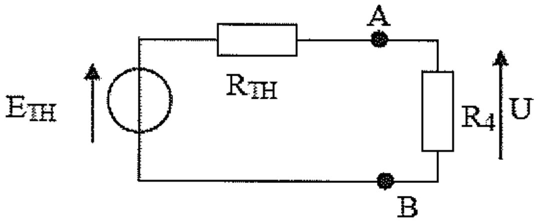
By noticing that we then have R_1 and R_2 in series, we determine E_{TH} with the voltage divider formula:



$$E_{TH} = \frac{E_1 R_2}{R_1 + R_2}$$

Finally we replace the circuit to the left of A and B by its equivalent Thévenin's circuit

We just have to apply the voltage divider formula to determine U.



The main advantage of this calculation is that we have removed a resistor (R_4) to determine the equivalent Thévenin's circuit, which has simplified the calculations: R_1 and R_2 were not in series in the initial circuit unlike for the E_{TH} calculation. Similarly they were not initially in parallel unlike for R_{TH} calculation.

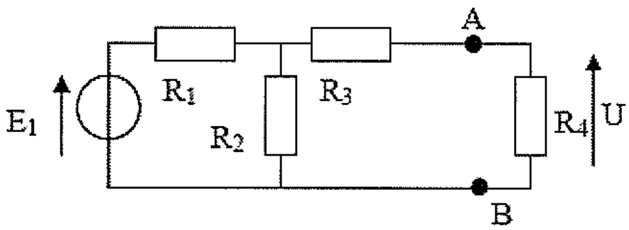
Note: if in the above example we replace the voltage source (E_1) by a current source (I_1), we obtain:
$R_{TH}=R_3+R_2$ (by suppressing I_1 source)
$E_{TH}=I_1R_2$ (indeed I_1 still flows through R_2 since I_3 is still null).

3 Norton's theorem

Any complex circuit can be replaced between two terminals A and B by an equivalent circuit called 'Norton's equivalent circuit', made of a single voltage source labeled I_N in parallel with an internal resistance labeled R_N so that:

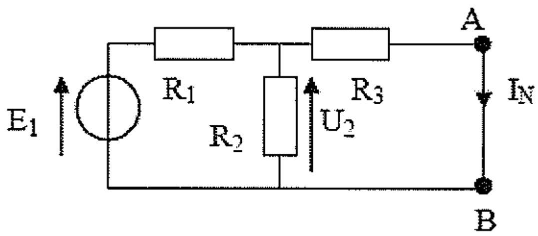
- I_N is the current between A and B when they are connected with a wire: it is the short circuit current, which can be labeled I_N
- R_N is the resistance between A and B when the dipole has been made passive, by suppressing all the sources: it is equal to R_{TH}.

Example: in the circuit to the right we want to determine U by replacing the circuit seen by R_4 by its equivalent Norton's circuit.



R_N calculation: see above R_{TH} calculation

I_N calculation: we replace R_4 by a wire and we determine the current in this wire. Since A and B are now at the same potential, R_2 and R_3 are now in parallel.



The voltage divider formula gives:

$$U_2=\frac{E_1R_{23}}{R_1+R_{23}}$$

with

$$R_{23}=\frac{R_2R_3}{R_2+R_3}$$

Therefore:

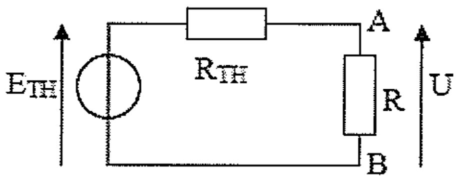
$$I_N=\frac{U_2}{R_3}=\frac{E_1R_{23}}{R_1+R_{23}}\cdot\frac{1}{R_3}=\frac{E_1R_2}{R_1R_2+R_1R_3+R_2R_3}$$

Equivalence between Thévenin's and Norton's circuits:

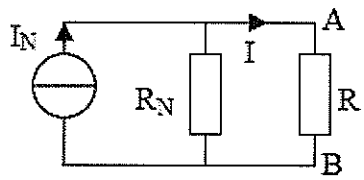
$E_{TH}=I_N.R_N$ or $I_N=E_{TH}/R_{TH}$

Difference between Thévenin's and Norton's circuits:

In theory the Thévenin's and Norton's circuits are completely equivalent, since they both replace the same active network. In practice the difference is linked to the order of magnitude of the internal resistance (R_{TH} or R_N) vs. the resistance R connected to the circuit.



$U=E_{TH}.R/(R_{TH}+R)$
As long as $R>>R_{TH}$, $U \sim E_{TH}$, $\forall R$



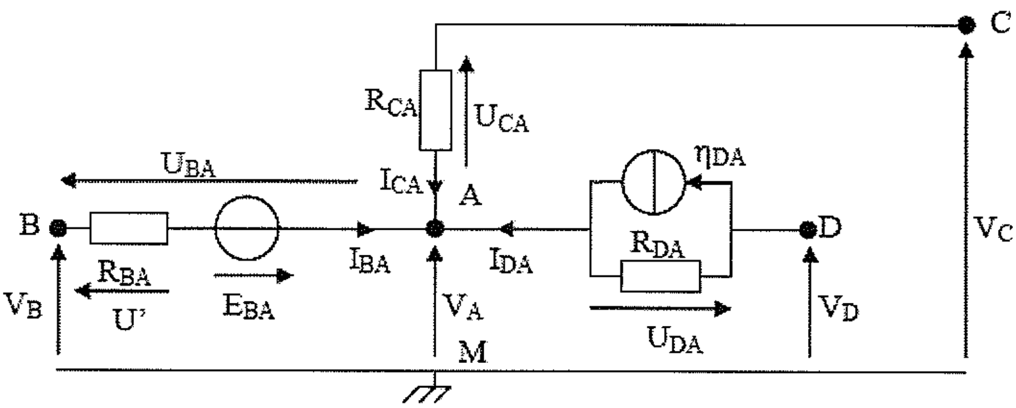
$I=I_N.R_N/(R_N+R)$
As long as $R<<R_N$, $I \sim I_N, \forall R$

In other words, Thévenin's equivalent circuit is more adapted when its internal resistance is small, and therefore much smaller than R: in these conditions, the output voltage U almost does not depend on R and remains almost equal to E_{TH} (it is the case for operational amplifiers).

Norton's equivalent circuit is more adapted when its internal resistance is very high and therefore much higher than R: the output current almost does not depend on R and remains almost equal to I_N (it is the case for transistors).

4 Millman theorem

Millman theorem (also called the 'node voltage method') consists in writing KCL in terms of voltages. It is useful to determine the voltage in a point A, ie the voltage drop between this point and another point M taken as a reference, all the other voltages being defined with respect to this point M (M is then the point at 0V, called 'earth').
To demonstrate it, the easiest way is to study an example containing all the possible cases, the generalization to n branches being then easy.



We first calculate each current in terms of voltages:

BA branch:

$$I_{BA} = U'/R_{BA} = (U_{BA} + E_{BA})/R_{BA} = (V_B - V_A + E_{BA})/R_{BA}$$

CA branch:

$$I_{CA} = U_{CA}/R_{CA} = (V_C - V_A)/R_{CA}$$

DA branch:

$$I_{DA} = \eta_{DA} + U_{DA}/R_{DA} = \eta_{DA} + (V_D - V_A)/R_{DA}$$

We then write KCL, noting that all the currents are defined towards A:

$$I_{BA} + I_{CA} + I_{DA} = 0$$

We replace each current by its expression in terms of voltages:

$$\frac{V_B - V_A + E_{BA}}{R_{BA}} + \frac{V_C - V_A}{R_{CA}} + \eta_{DA} + \frac{V_D - V_A}{R_{DA}} = 0$$

$$\frac{V_B}{R_{BA}} - \frac{V_A}{R_{BA}} + \frac{E_{BA}}{R_{BA}} + \frac{V_C}{R_{CA}} - \frac{V_A}{R_{CA}} + \eta_{DA} + \frac{V_D}{R_{DA}} - \frac{V_A}{R_{DA}} = 0$$

And after having factorized by V_A:

$$\frac{V_B + E_{BA}}{R_{BA}} + \frac{V_C}{R_{CA}} + \eta_{DA} + \frac{V_D}{R_{DA}} = \frac{V_A}{R_{BA}} + \frac{V_A}{R_{CA}} + \frac{V_A}{R_{DA}} = V_A\left(\frac{1}{R_{BA}} + \frac{1}{R_{CA}} + \frac{1}{R_{DA}}\right)$$

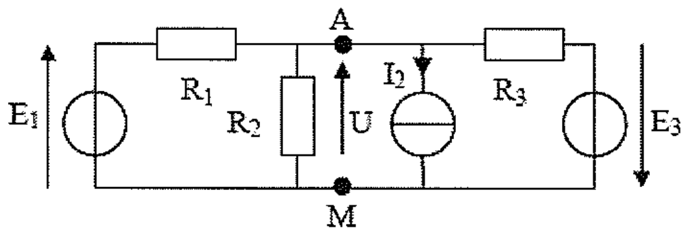
We then get V_A:

$$\boxed{V_A = \frac{\frac{V_B + E_{BA}}{R_{BA}} + \frac{V_C}{R_{CA}} + \eta_{DA} + \frac{V_D}{R_{DA}}}{\frac{1}{R_{BA}} + \frac{1}{R_{CA}} + \frac{1}{R_{DA}}}}$$

This formula is the Millman's theorem if it is generalized to n branches.
Note: in many cases, the points B, C and D are earthed. Therefore the voltages V_B,V_C and V_D are null and V_A simplifies into:

$$\boxed{V_A = \frac{\frac{E_{BA}}{R_{BA}} + \eta_{DA}}{\frac{1}{R_{BA}} + \frac{1}{R_{CA}} + \frac{1}{R_{DA}}}}$$

Example: the circuit to the right seems complex and we want to determine the expression of the tension $U=V_A-V_M$



We could use the superposition theorem: we would calculate the three partial voltages when keeping one source only, and then add it.
We could also use Thévenin's and Norton's equivalent circuits, but the calculations would be complicated: we would first have to determine R_{TH} and E_{TH} which is not straight forward, and then determine U using the voltage divider formula.

$$U = \frac{\frac{E_1}{R_1} - \frac{E_3}{R_3} - I_2}{\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_2}}$$

$$U = \frac{R_2(E_1.R_3 - E_3.R_1 - I_2.R_1.R_3)}{R_2.R_3 + R_1.R_2 + R_1.R_3}$$