Design and Analysis of Experiments

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Chapter 1

The Principles Of Experimental Design

1.1 Introduction

Experimental Design concerns the arrangement of various *conditions* or situations to which experimental *subjects* (for example, people or rats) will be exposed. The process of designing an experiment for comparing treatment or factor level means begins by stating the objective(s) of the experiment clearly. The statement of the objectives indicates to us what measurements are to be made (how, when, where) and on what.Precision and accurate comparisons among treatments over an appropriate range of conditions are the primary objectives of most experiments. These objectives require precise estimates of means and powerful statistical test. Reduced experimental errors increase the possibility of achieving these objectives. Local control describes the actions an investigator employs to reduce or control experimental errors, increase accuracy of observations, and establish the inference base of a study. The investigator controls the:

- Technique
- Selection of experimental units
- Blocking or ensuring parity of information on all treatments,
- Experimental design, and
- Measure of covariates

Consider the following two examples of the objectives of an experiment.

Example 1.1.1 *To compare the mean weight gains of steers that are fed diets A and B.*

Example 1.1.2 *To compare the mean weight gains of two year old Holstein steers that are fed diets A and B for a period of six months.*

The objective of the experiment in example 1.1.1 is vague. Why? On the other hand the objective of the experiment in example 1.1.2 is specific. It is clear from the statement of the objective that the experiment should be conducted as follows:

1. Weigh the available two-year old Holstein steers.
2. Divide the steers into two groups.
3. Assign one group of steers to diet A and the other group to diet B.
4. Feed the steers with their respective diets for six months and then weigh them.
5. Weight gain = final weight - initial weight.

Example 1.1.2 shows that a clear statement of the objectives of an experiment specifies

1. the set of treatments (diets A and B) whose effects are to be investigated;
2. the set of experimental units (two-year old Holstein steers) to be used; and
3. the response variable(s) (weight gain) of interest.

Definitions

Definition 1.1.1 *A **treatment** is the level or class of a factor whose effects are to be inivestigated.*

In the selection of treatments it is important to define clearly each treatment and to understand the role that each treatment will play in reaching the objectives of the experiment.

Example 1.1.3 *Refer to example 1.1.2. The factor is diet and the levels of the factor (or treatments) are A and B*

Example 1.1.4 *Suppose that we wish to compare the mean yield of maize varieties X and Y grown under the same management and climatic conditions. In this case **Variety** is the factor and the maize varieties X and Y are the factor levels.*

Definition 1.1.2 *An **experimental unit** is the smallest experimental material upon which a treatment is applied.*

The unit may be a plot of land, a patient in a hospital, or a lump of dough, or it may be a group of pigs in a pen, or a batch of seed.

Example 1.1.5 *Refer to example 1.1.2. If the two year old Holstein steers are individually fed their assigned diets, then the steers are the experimental units, otherwise, if the steers are group fed their assigned diets then the groups are the experimental units.*

Example 1.1.6 *Refer to example 1.1.4. If the varieties are assigned individual plots, then each plot on which a variety is grown is the experimental unit.*

Definition 1.1.3 *The **response variable** is the characteristic of the experimental unit that is measured after applying the treatment on the unit. A response variable is also called a **dependant variable**.*

Example 1.1.7 *Refer to example 1.1.2. The response variable is the weight gain of a steer.*

Example 1.1.8 *Refer to example 1.1.4. The response variable is the yield per unit area of the plot.*

Exercise 1.1.1 *We wish to conduct an experiment to compare the mean caffeine content of three brands of tea leaves - . We will analyse ten tea bags of each brand for caffaine content and record the amount of caffaine in each tea bag in milligrams.*

1. *What is our response variable?*
2. *Identify the factor we wish to study. What are the levels of the factor?*
3. *Identify the experimental units.*

1.1.1 Randomistaion

Refer to example 1.1.4. If the mean yield of the two varieties are actually the same, what would be our conclusion from the experiment in which we assign fertile plots to variety X and poor plots to variety Y? The answer to this question is simple. Our experimental procedure would favour variety X. We would make the erroneous conclusion that variety X has a higher yield than variety Y when in actual fact they have the same yield. If allocation of the plots to the varieties is done (as above) deliberately, then the bias in our conclusion is called **subjective** bias or bias due to **deliberate selection**, otherwise, the bias is called **systematic** bias.

Subjective and systematic biases can be eliminated by **randomising** the experiment. Furthermore, randomising the experiment makes the errors in the measurements statistically independent - an assumption required by many statistical methods

of analysing data including ANOVA.

By definition **Randomistaion** of an experiment is the random allocation of the experimental units to the treatments or factor levels. That is, it is the allocation of the experimental units to the treatments in a haphazard way.

Example 1.1.9 *Suppose that we have developed a new diet A which we believe is better than the existing diet B in terms of increasing the daily weight gain of steers (of the same age) fed the diets. Furthermore suppose that four two year old Brahman steers and four two year old Nguni steers are avilable for the experiment to verify your claim. Known or unknown to us is that naturally Brahman steers grow faster than Nguni steers of the same age raised under the same enviromental conditions. If in actual fact diet A is good as diet B, what will be the conclusion from the experiment whereby we assign Brahman steers to diet A and Nguni steers to diet B.*

Example 1.1.10 *Randomise the experiment in example 1.1.9*

1.2 Replication

We define a **basic experiment** as one in which only one experimental unit is assigned to each treatment. Thus, each treatment appears once in a basic experiment.

Replication is the repetition of the **basic experiment**. In other words, replication is the assignment of at least two experimental units to each of the treatments whose effects are under investigation. Replication allows the accurate estimation of the experimental error, improves the reliability of the estimates of the treatment means and also improves the sensitivity of statistical tests for comparing treatment means.

Example 1.2.1 *Suppose that we wish to compare the effects of two treatments* (T_1, T_2) *on some response. Furthermore, suppose that we have* $2n$ $(n \in \Re^+ \geq 1$ *) identical experimental units available for experimentation. The plan of conduct of the experiment is to randomly allocate* n *experimental units to* T_1 *and the remainder to* T_2*. The experiment for* $n = 1$ *is our basic experiment, for* $n = 2$ *we have two replications of our basic experiment etc.*

Recall that the **pooled t-test** assumptions are that the errors in the $Y'_{ij}s$ are independent and normally distributed with mean 0 and variance σ^2 (unknown). The σ^2 is a measure of the experimental error and it is estimated by the pooled variance which, in this case, is given by:

$$S_p^2 = \frac{1}{2}(S_1^2 + S_2^2$$

What happens to S_p^2 if we do not replicate the basic experiment?

The estimate of the difference between the T_1 mean and the T_2 mean is given by:

$$\bar{Y}_{1.} - \bar{Y}_{2.} \text{with variance} \sigma_{12}^2 = \frac{2}{n}\sigma^2$$

The variance σ_{12}^2 is estimated by $\hat{\sigma}_{12}^2 = \frac{2}{n}S_p^2$. The σ_{12}^2 is a measure of precision or the reliability of $\bar{Y}_{1.} - \bar{Y}_{2.}$ in estimating the difference between the treatment means. The estimate is precise or reliable if σ_{12}^2 is small. The precision or the reliability of the estimate improves as we increase the number of replications of the basic experiment since $\sigma_{12}^2 \to 0$ as $n \to \infty$. The width of the confidence interval for the difference between treatment means decreases as n increases. That is, the confidence interval becomes more and more accurate as n is increased.

Suppose that we wish to test the hypothesis

$$H_0 : T_1 \text{mean} = T_2 \text{mean} \quad \text{versus} \quad H_1 : T_1 \text{mean} \neq T_2 \text{mean}.$$

If the pooled $t - test$ assumptions (specified above) hold, then the appropriate test statistic is given by:

$$t = \frac{\bar{Y}_{i.} - \bar{Y}_{2.}}{\hat{\sigma_{12}}},$$

which is **Student t** distribution with $2n - 2$ degrees of freedom. If the variance of the errors σ^2 is known, then the appropriate test is the $z - test$. The test statistic for the $z - test$ is given by:

$$z = \frac{\bar{Y}_{i.} - \bar{Y}_{2.}}{\sigma_{12}},$$

which has a standard normal distribution. Both the $t - test$ and the $z - test$ reject H_0 in favour of H_1 if the values of $|t|$ and $|z|$ are large.

Consider testing the above hypothesis using the $z - test$.
How does $|z|$ vary with n?

A measure of the sensitivity of a test for comparing treatment means is the **power** of the test to detect differences between treatment means. The power of a test is the probability of rejecting H_0 (treatment means are not different) when H_1 is true (treatment means are different)

1.3 Blocking

Blocking an experiment refers to arranging the experimental units into groups (called **blocks**) within each of which the experimental units are relatively homogeneous with respect to one or more characteristics of the units that may influence the response of interest. Randomisation is then done independently within each block. Note that blocking may also be based on external variables (variables that may influence the response) associated with the experimental setting e.g **observer** if two or more people perform the experiment.

Blocking an experiment allows us to account for the variation in the responses that is due to differences among the experimental units. If we block an experiment using external variables such as time or observer, then blocking allows us to account for the variation in responses that is due to these external variables. The consequence of not blocking when we are supposed to block is that the variation due to differences among the experimental units or due to the external variables cannot be separated from that due to the random errors. This results in an estimate of the experimental error (σ^2) that is biased upwards.

Example 1.3.1 *Suppose that we wish to compare the effects of diet A and diet B on the daily weight gain of steers (of the same age) fed the diets. Furthermore, suppose that four two year old Afrikaner steers and four two year old Nguni steers are available for the experiment. How can we design a simple experiment that can allow us to remove the between breed variation from the experimental error?*

NOTE Precision and accurate comparisons among treatments over an appropriate range of conditions are the primary objectives of most experiments. These objectives require precise estimates of means and powerful statistical test. Reduced experimental errors increase the possibility of achieving these objectives. Local control describes the actions an investigator employs to reduce or control experimental errors, increase accuracy of observations and establish the influence base of a study. The investigator controls:

1. Technique
2. Selection of experimental units
3. Blocking or ensuring parity of information on all treatments
4. Choice of experimental design
5. Measure of covariates

Chapter 2

Completely Randomised Designs

2.1 Introduction

In a Completely Randomised Design (CRD) the experimental units are assigned to the treatments completely t random. Complete randomisation ensures that every unit has an equal chance to be assigned any one of the treatments. The value of complete randomisation is insurance against subjective and systematic biases. If we have $n = tr$ experimental units and t treatments, then we randomly assign r units to each of the t treatments to obtain a balanced completely randomised design.

We use a CRD when both the experimental units and experimental conditions are **uniform.** That is, when the experimental units are identical with respect to their characteristics that can affect the response and there are no external variables associated with the experimental setting that can also affect the response

2.2 Advantages of using CRD

1. The design can be used with any number of treatments if the resources are permitting.
2. The number of experimental units (sample size) can be varied from treatment to treatment without complicating the analysis of the experiment.
3. The statistical analysis of the experiment (ANOVA and estimation) is easy even when:
    - there are observations that are missing by accident or by design and
    - a treatment is dropped.

2.3 Disadvantages of using CRD

1. Although CRD can be used for any number of treatments, it is best suited for situations where there are relatively few treatments.

2. the experimental units to which treatments are applied must be homogeneous, with no extraneous source of variability affecting them.

Example 2.3.1 *Suppose that you want to compare the effects of three types of fertiliser (X,Y,Z) on a cert ain variety of maize. Furthemore, suppose that 9 plots of the same size are available for the experiment.*

1. *List at least two characteristics of the plots that can affect the response of interest.*

2. *List at least two environmental factors that can affect the response of interest.*

3. *List at least two management practices that can affect the response of interest besides the method and the level of application of the fertilisers.*

4. *In view of your answers to (a) - (c), under what conditions would a CRD design be appropriate for the experiment?*

2.4 Analysis of the CRD

Table 3.1 displays the layout of the data in a CRD with t treatments and r experimental units per treatment, where r is the number of replications of the basic CRD. The symbols in the table have the following meanings:

Table 2.1: Layout of the data in CRD

				Replication			
Treatment	1	2	3	...	r	Total	Mean
T_1	Y_{11}	Y_{12}	Y_{13}	...	Y_{1r}	$Y_{1.}$	$\bar{Y}_{1.}$
T_2	Y_{21}	Y_{22}	Y_{23}	...	Y_{2r}	$Y_{2.}$	$\bar{Y}_{2.}$
.	.	.	.	...	.	.	.
T_t	Y_{t1}	Y_{t2}	Y_{t3}	...	Y_{tr}	$Y_{t.}$	$\bar{Y}_{t.}$

- Y_{ij} is the j^{th} response to the i^{th} treatment;
- $Y_{i.} = \sum_{j=1}^{r} Y_{ij}$ and $\bar{Y_{i.}} = \frac{1}{r} Y_{i.}$ are the i^{th} treatment total and treatment mean respectively, and
- $i = 1, 2, ..., t; \quad j = 1, 2, ..., r; \quad n = tr =$ total number of observations

2.4.1 ANOVA

General Model

The general linear statistical model for a CRD has the form:

$$Y_{ij} \quad = \quad \mu + \tau_i + \epsilon_{ij} \tag{2.1}$$

Where

- μ is the overall population mean;
- τ is the effect of the i^{th} treatment;
- $\mu + \tau_i$ is the mean of the ith treatment;
- the $\epsilon'_{ij}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2; and
- $i = 1, 2, ..., t; \quad j = 1, 2, ..., r.$

Sum of Squares

The total variability in the observations ($Y'_{ij}s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{r}(Y_{ij} - \bar{Y}_{..})^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2 - \frac{1}{tr}Y_{..}^2 \quad \text{with} \quad tr-1 \quad df \tag{2.2}$$

It is possible to partition the total sum of squares into three separate sources of variability i.e.

1. due to the variability among treatments (**treatment sum of squares (SST)**),
2. due to the variability among the $y'_{ij}s$ which is not accounted for by either treatments or blocks (**error sum of squares (SSE)**)

The partition can be done as follows

$$\begin{aligned} SSTO &= \sum_{i=1}^{t}\sum_{j=1}^{r}(Y_{ij} - \bar{Y}_{..})^2 \\ &= \sum_{i=1}^{t}\sum_{j=1}^{r}[(\bar{Y}_{i.} - \bar{Y}_{..}) + (\bar{Y}_{ij} - \bar{Y}_{i.})]^2 \\ &= \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{r}(\bar{Y}_{i.} - \bar{Y}_{..})^2}_{SST} + \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{r}(Y_{ij} - \bar{Y}_{i.})^2}_{SSE} + \underbrace{2\sum_{i=1}^{t}\sum_{j=1}^{r}(Y_{ij} - \bar{Y}_{i.})(\bar{Y}_{i.} - \bar{Y}_{..})}_{=0} \end{aligned}$$

The sum of squares formulas just discussed are mathematical and are not convinient to use in calculations. The computational formulae are given by:

$$\begin{aligned} SST &= \sum_{i=1}^{t} \frac{1}{r} Y_{i\cdot}^2 - \frac{1}{tr} Y_{\cdot\cdot}^2 \quad \text{with} \quad t-1 \quad df \\ SSE &= SSTO - SST \quad \text{with} \quad t(r-1) \quad df \end{aligned}$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedom and they are used for testing the hypothesis about the treatment effects.

2.4.2 Fixed effects model (Model I)

The $\tau_i's$ are regarded as **fixed real constants** satisfying the constraints:

$$\sum_{i=1}^{t} \tau_i = 0$$

Hypothesis Testing

We use model I to analyse a CRD when the conclusions of the experiment are to pertain to the particular set of treatments included in the experiment i.e conclusions cannot be extended to any other treatments that were not included in the experiment.

The objective is to test the null hypothesis of no difference among the treatment means. This is equivelent to testing

$$\begin{aligned} H_0 &: \tau_1 = \tau_2 = ... = \tau_t = 0 \\ H_1 &: \text{at least one } \tau_i \neq 0. \text{ (at least one of the treatment means differs from the rest)} \end{aligned}$$

The test statistic for these hypothesis is given by:

$$F = \frac{MST}{MSE} \tag{2.3}$$

where MST and MSE are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $t-1$ degrees of freedom on the **numerator** and $(t-1)(b-1)$ degrees of freedom on the **denominator**.

To show that F is a reasonable test statistic for the above hypothesis we need to show that MSE is an unbiased estimate of the experimental error (σ^2) i.e

$$E[MSE] = \sigma^2$$

Under model I, we can also show that MST is an unbiased estimate of:

$$E[MST] = \sigma^2 + \frac{1}{t-1}\sum_{i=1}^{t} r\tau_i^2$$

When testing the hypothesis H_0 versus H_1 at the α level of significance we note that H_0 is true when both MST and MSE are both unbiased estimates of σ^2. If H_1 is true MSE is still an unbiased estimate of σ^2, but MST is an unbiased estimate of

$$\sigma^2 + \frac{1}{t-1}\sum_{i=1}^{t} r\tau_i^2 > \sigma^2$$

From the explanation above we expect the value of F to be equal to 1 if H_0 is true and to be large if H_1 is true. How large is large is the question. The decision rule for the testing problem is to reject H_0 in favour of H_1 if $F_{cal} > F^{\alpha}_{(t-1),t(r-1)}$- the **critical value** obtained from the *F-tables* The Analysis of variance table for a completely randomised design is as follows:

Table 2.2: ANOVA table for a CRD design

Source	**df**	**SS**	**MS**	**F**
Treatments	$t-1$	SST	MST	MST/MSE
Error	$t(r-1)$	SSE	MSE	
Totals	$tr-1$	$SSTO$		

Pairwise Comparisons of the treatments

They are done only if the ANOVA test under the fixed effects model concludes that some treatment means are different. We perform pairwise comparisons of the treatment means in order to determine which means are different. In this case the hypothesis to be tested are:

$$\begin{aligned} H_0 &: \mu_{i.} = \mu_{i'.} \\ H_1 &: \mu_{i.} \neq \mu_{i'.} \quad \forall i \neq i' \end{aligned}$$

NOTE: Comparisons should be made at the same level of significance as that used in the ANOVA. If we choose to use the least significant difference method for making

the comparisons, then the hypothesis are tested using the *t-test*whose test statistic is given by:

$$t = \frac{\bar{Y}_{i.} - \bar{Y}_{i'}}{\sqrt{2MSE/r}}$$

where $\sqrt{2MSE/r}$ is the standard error of $\bar{Y}_{i.} - \bar{Y}_{i'}$. The test statistic has a Student t-distibution with $t(r-1)$ degrees of freedom. We declare means to be significantly different at the α level of significance if:

$$|\frac{\bar{Y}_{i.} - \bar{Y}_{i'}}{\sqrt{2MSE/r}}| > t^{\frac{\alpha}{2}}_{t(t-1)}$$

or equivalently

$$|\bar{Y}_{i.} - \bar{Y}_{i'}| > t^{\frac{\alpha}{2}}_{t(t-1)}\sqrt{2MSE/r}$$

the least significant difference quantity is:

$$LSD = t^{\frac{\alpha}{2}}_{t(r-1)}\sqrt{2MSE/r}$$

The LSD procedure simply compares the observed absolute difference between each pair of averages to the corresponding LSD. The means $\mu_{i.}$ and $\mu i'.$ are declared significantly different if

$$|Y_{i.} - Y_{i'.}| > LSD.$$

Example 2.4.1 *A study was undertaken to compare the distance travelled in kilometres per litre of three competing brands of petrol. Fifteen identical cars were available for the experiment. The brands of petrol i.e brand A, B, and C were each assigned five cars. The cars were operated under the same conditions and the distance travelled by each car per litre of the assigned brand of petrol was recorded. The results are displayed in the following table. Analyse the data and draw appropriate conclusions.*

	Replication				
Brand	**1**	**2**	**3**	**4**	**5**
A	9.5	11.0	13.0	15.0	18.0
B	10.5	12.0	14.0	16.0	10.5
C	10.0	10.5	13.5	14.5	10.0

1. *Assume that model I is approriate for the experiment, test the relevant hypothesis at the 0.05 level of significance.*

2. *Is it appropriate to perform pairwise comparisons of the brand means. Give reasons for your answer.*

3. *If you are to perform pairwise comparisons of the brand means, what value of LSD would you use?*

2.4.3 Random Effects Model (Model II)

For an experiment where the treatments or the factor levels of a factor are randomly chosen from a population of treatments or the factor levels of the factor the appropriate model is a **random effects model (Model II)**.Model II is also model (2.1) but with the:

- $\tau_i's$ random variables which are independent and normally distributed with mean zero and variance σ_τ^2; and
- $\epsilon_{ij}'s$, $\tau_i's$ and $\beta_j's$ are assumed to be independent random variables

Hypothesis Testing

We wish to test the hypothesis

$$\begin{aligned} H_0 &: \sigma_\tau^2 = 0 \quad \text{i.e treatment effects are identical} \\ H_1 &: \sigma_\tau^2 > 0 \quad \text{i.e treatment effects are not identical} \end{aligned}$$

variance component Estimation

If we reject H_0 under model II it follows that $\sigma_\tau^2 > 0$ at the α level of significance. The question still remains, "How large is σ^2" We estimate σ_τ^2 as follows:

$$E[MSE] = \sigma^2 \quad \text{and} \quad E[MST] = \sigma^2 + r\sigma_\tau^2$$

then

$$\sigma_\tau^2 = \frac{1}{r}(E[MST] - E[MSE])$$

Therefore the most obvious estimate of σ_τ^2 is given by:

$$\hat{\sigma}_\tau^2 = max[0, \frac{1}{r}(MST - MSE]$$

Example 2.4.2 *To compare the lightning discharge intensities at all areas in South Africa a CRD was used. Three areas were randomly chosen for the study and the lightining tracking equipment assembled at those areas. On each day of the month of December between 0800hrs and 1700hrs lightining was monitored at the three areas until the maximum intensity had been recorded for five seperate storms. The sample data is in the following table. Analyse the data and draw appropriate conclusions.*

1. *Assume that model II is approriate for the experiment, test the relevant hypothesis at the 0.05 level of significance.*
2. *Is it necessary to estimate σ_τ^2. Give reasons for your answer.*
3. *If you are to estimate σ_τ^2, what would be its estimate?*

	Intensity				
Area	**1**	**2**	**3**	**4**	**5**
A	20	1050	3200	5600	50
B	4300	70	2560	3650	80
C	100	7700	8500	2960	3340

2.5 Unbalanced CRD/Missing Observations

Missing observations may occur by design or by accident. An unbalanced CRD is a CRD for which the number of responses within each treatment are different. The analysis of variance for the unbalanced CRD is the same as that for the balanced one with some slight modifications in the formulae. The general model is the same as 2.1 but with $j = 1, 2, , ..., r_i$ where r_i is the number of replications of the i^{th} treatment. The total number of observations (experimental units) becomes

$$n = \sum_{i=1}^{t} r_i$$

. The i^{th} treatment total and the estimate of the overall mean (μ) are given by:

$$Y_{i.} = \sum_{j=1}^{r_i} Y_{ij} \quad \text{and} \quad \bar{Y}_{..} = \frac{1}{n} \sum_{i=1}^{t} \sum_{j=1}^{r_i} Y_{ij}$$

respectively.
The formulae to calulate the sums of squares are given by:

$$\begin{aligned} SSTO &= \sum_{i=1}^{t} \sum_{j=1}^{r_i} Y_{ij}^2 - \frac{1}{n} Y_{..}^2 \quad \text{with} \quad n-1 \quad df \\ SST &= \sum_{i=1}^{t} \frac{1}{r_i} Y_{i.}^2 - \frac{1}{n} Y_{..}^2 \quad \text{with} \quad t-1 \quad df \\ SSE &= SSTO - SST \quad \text{with} \quad n-t \quad df \end{aligned} \tag{2.4}$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedom and they are used for testing the hypothesis about the treatment effects.

The Analysis of variance table for a completely randomised design with missing observations is as follows:

The LSD for comparing two treatment means is given by

$$LSD = t_{n-t}^{\frac{\alpha}{2}} \sqrt{MSE(\frac{1}{r_i} + \frac{1}{r_{i'}})}$$

Table 2.3: ANOVA table for a CRD with missing observations

Source	**df**	**SS**	**MS**	**F**
Treatments	$t-1$	SST	MST	MST/MSE
Error	$n-t$	SSE	MSE	
Totals	$n-1$	$SSTO$		

2.5.1 Checking model assumptions

There are graphical and formal statistical methods used to check the assumptions of the analysis of variance. These methods are based on the estimates of the errors called **residuals**. The underlying assumptions of ANOVA are that the data are adequately described by the specified model and that the errors ($\epsilon'_{ij}s$) are independent and normally distributed with mean 0 and constant variance σ^2. In this case the residuals for model 2.1 are given by:

$$\hat{\epsilon}_{ij} = Y_{ij} - \bar{Y}_{i.}$$

Example 2.5.1 *Refer to example 2.4.1. Calculate the residuals for the data.*

Normality Assumption

A **histogram** of the residuals can help to detect skewness in the distribution of the residuals. We expect the histogram to be approximately bell shaped about zero if the distribution of the errors is indeed normal with mean 0. We can also use a **normal probability plot** to detect non-normality of the errors. If the error distibution is normal then this plot, should be approximately straight. To construct the probability plot we follow the following steps:

1. Compute the **standardised** residuals by dividing each residual by $\sqrt{MSE}$.
2. Arrange the standardised residuals in increasing order of magnitude.
3. Calculate the **cumulative probability point** (p_i) for the i^{th} ordered standardised residual using the formula $p_i = (i - \frac{1}{2})/n$ where $n = rt$.
4. Plot the value of the i^{th} ordered standardised residual on the horizontal axis against p_i on the vertical axis. The verical axis goes from 0.01 to 0.99 and this corresponds to the cummulative distribution function of a normal distribution.

NOTE that: The normality of the error distribution do not seriously invalidate conclusions of the analysis of variance of the fixed effects model if there are moderate depatures from normality. They only cause the level of significance to differ from the specified value and the tests to be slightly less powerful in detecting treatment differences. Conclusions of the ANOVA of the random effects model are more seriously

affected by non-normality in the sense that the estimates of the variance components may be inaccurate.

Constant Variance Assumption

A plot of residuals versus the treatment averages ($\bar{Y}_{i.}$) can be used to check the assumption of constant error variance. If error assumptions are not violated and model 2.1 is correct then the errors and hence the residuals should be patternless. The conclusions of the *F-test* are slightly affected by the violation of the constant error variance assumption in the case of a fixed effects model for balanced data. Conclusions become invalid if the data are unbalanced and/or some error variances are much larger than others. For the random effects model, the conclusions are generally invalid even for balanced data if the constant error variance assumption is violated.

Independence Assumption

When responses are measured at successive time points, the errors in the responses may become related through time. The assumption of independent errors would be violated if this happens. We detect this assumption by plotting the residuals versus time. If the plot shows any obvious pattern, then this would imply that the assumption of independence is violated. Once this assumption is violated, all the conclusions of the analysis of variance become invalid.

Chapter 3

Randomised Block Design (RBDs)

3.1 Randomised Block Design

In many experimental problems, it is necessary to design the experiment so that variability arising from extraneous sources can be systematically controlled. The randomised complete block design is perhaps the most widely used experimental design. Situations for which the randomised complete block design is appropriate are numerous, and easily detected with practice. For example, units of test equipment or machinery are often different in their operating characteristics and would be a typical blocking factor.

NOTE

The word "complete" indicates that each block contains all the treatments. By using this design, the blocks form a more homogeneous experimental unit.

We construct an RBD as follows

1. The experimental units are arranged into groups (called **blocks**) in such a way that within each block, the experimental units are relatively homogeneous with respect to one or more characteristics of the units that may influence the response of interest.

2. Complete randomisation is then done independently within each block.

Blocking allows us to remove the variation due to the differences among the blocks from the experimental error. Complete randomisation within each block offers insurance against subjective and systematic biases, thereby making the conclusions from the experiment more accurate.

Advantages of an RBD

1. It can be used to accomodate any number of treatments and replications in any number of blocks.

2. If blocking is effective, then the RBD can provide more precise conclusions than a CRD that uses the same experimental units.

3. The statistical analysis is simple (two way ANOVA) even when an entire block or treatment is dropped.

4. The design is easy to construct and allows variable experimental units to be used without sacrificing the precision of the conclusions.

Disadvantages of an RBD

1. Since the experimental units within a block must be homogeneous, the design is best suited for a relatively small number of treatments.

2. The statistical analysis of the experiment is complicated if there are some missing observations within blocks.

3. The degrees of freedom for the experimental error are lost to blocking.

4. The model for the RBD experiment is more complicated than that for a CRD experiment, and requires more assumptions.

Example 3.1.1 *A study was undertaken to compare the starting salaries of bachelor's degree candidates at the University of Limpopo from the school of computational and mathematical sciences for the academic years 2004-2005, 2005 -2006 and 2006-2007. Three students from mathematics, three students from statistics and three students from computer science were available for the experiment. It should be noted that only those students who had accepted a job were considered in this study.*

1. *What are the experimental units in the proposed experiment?*

2. *Which design CRD or RBD is appropriate for the experiment? Give reasons for your answer.*

3. *In case your answer to (2) is RBD, what is the blocking factor?*

3.2 Analysis of the RBD

A randomised block design can be used to compare t population treatment means when an additional source of variability (blocks) is present.

Definition 3.2.1 *A randomised block design is an experimental design for comparing t treatments in b blocks. Treatments are randomly assigned to experimental units within a block, with each treatment appearing exactly once in every block.*

General model

Both the **fixed effects model (model I)** and the **random effects model (model II)** for the RBD have the form:

$$Y_{ij} \quad = \quad \mu + \tau_i + \beta_{ij} + \epsilon_{ij} \tag{3.1}$$

Where

- μ is the overall population mean;
- τ_i is the effect of the i^{th} treatment;
- β_j is the effect of the j^{th} block;
- $\mu + \tau_i$ is the mean of the ith treatment;
- the $\epsilon'_{ij}s$ are random errors associated with the response on treatment i, block j which are assumed to be independent and normally distributed with mean 0 and variance σ^2; and
- $i = 1, 2, ..., t; \quad j = 1, 2, ..., b.$

Table 3.1: A Randomised block design

				Block			
Treatment	1	2	3	...	b	Total	Mean
1	Y_{11}	Y_{12}	Y_{13}	...	Y_{1b}	$Y_{1.}$	$\bar{Y}_{1.}$
2	Y_{21}	Y_{22}	Y_{23}	...	Y_{2b}	$Y_{2.}$	$\bar{Y}_{2.}$
.	.	.	.	...	.	.	.
t	Y_{t1}	Y_{t2}	Y_{t3}	...	Y_{tb}	$Y_{t.}$	$\bar{Y}_{t.}$
Total	$Y_{.1}$	$Y_{.2}$	$Y_{.3}$	...	$Y_{.b}$	$Y_{..}$	
Mean	$\bar{Y}_{.1}$	$\bar{Y}_{.2}$	$\bar{Y}_{.3}$	...	$\bar{Y}_{.b}$		$\bar{Y}_{..}$

The symbols in the table have the following meanings:

- Y_{ij} is the i^{th} treatment in the j^{th} block;
- $Y_{i.} = \sum_{j=1}^{b} Y_{ij}$ and $\bar{Y_{i.}} = \frac{1}{b}Y_{i.}$ are the i^{th} treatment total and treatment mean respectively, and
- $Y_{.j} = \sum_{i=1}^{t} Y_{ij}$ and $\bar{Y_{.j}} = \frac{1}{t}Y_{.j}$ are the j^{th} block total and block mean respectively, and
- $i = 1, 2, ..., t; \quad j = 1, 2, ..., b; \quad n = tb =$ total number of observations

Sum of Squares

The total variability in the observations ($Y'_{ij}s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij} - \bar{Y_{..}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{b}Y_{ij}^2 - \frac{1}{n}Y_{..}^2 \quad \text{with} \quad n-1 \quad df \tag{3.2}$$

It is possible to partition the total sum of squares into three separate sources of variability i.e.

1. due to the variability among treatments (**treatment sum of squares (SST)**),
2. due to the variability among the blocks (**block sum of squares (SSB)**)and
3. due to the variability among the $y'_{ij}s$ which is not accounted for by either treatments or blocks (**error sum of squares (SSE)**)

The partition can be done as follows

$$\begin{aligned} SSTO &= \sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij} - \bar{Y_{..}})^2 \\ &= \sum_{i=1}^{t}\sum_{j=1}^{b}[(\bar{Y_{i.}} - \bar{Y_{..}}) + (\bar{Y_{.j}} - \bar{Y_{..}}) + (Y_{ij} - \bar{Y_{i.}} - \bar{Y_{.j}} + \bar{Y_{..}})]^2 \\ &= \underbrace{b\sum_{i=1}^{t}(\bar{Y_{i.}} - \bar{Y_{..}})^2}_{SST} + \underbrace{t\sum_{j=1}^{b}(\bar{Y_{.j}} - \bar{Y_{..}})^2}_{SSB} + \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij} - \bar{Y_{i.}} - \bar{Y_{.j}} + \bar{Y_{..}})^2}_{SSE} \end{aligned}$$

The sum of squares formulas just discussed are mathematical and are not convinient to use in calculations. The computational formulae are given by:

$$
\begin{aligned}
SST &= \sum_{i=1}^{t} \frac{1}{b} Y_{i\cdot}^2 - \frac{1}{n} Y_{\cdot\cdot}^2 \quad \text{with} \quad t-1 \quad df \\
SSB &= \sum_{j=1}^{b} \frac{1}{t} Y_{\cdot j}^2 - \frac{1}{n} Y_{\cdot\cdot}^2 \quad \text{with} \quad b-1 \quad df \\
SSE &= SSTO - SST - SSB \quad \text{with} \quad n-t-b+1 = (t-1)(b-1) \quad df
\end{aligned}
$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedom and they are used for testing the hypothesis about the treatment and the block effects.

3.2.1 Fixed effects model

The $\tau_i's$ and $\beta_j's$ are regarded as **fixed real constants** satisfying the constraints:

$$\sum_{i=1}^{t} \tau_i = \sum_{j=1}^{b} \beta_j = 0$$

Hypothesis Testing

The objective is to test the null hypothesis of no difference among the treatment means. This is equivelent to testing

$$
\begin{aligned}
H_0 &: \tau_1 = \tau_2 = ... = \tau_t = 0 \\
H_1 &: \text{at least one } \tau_i \neq 0. \text{ (at least one of the treatment means differs from the rest)}
\end{aligned}
$$

The test statistic for these hypothesis is given by:

$$F = \frac{MST}{MSE} \tag{3.3}$$

where MST and MSE are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $t-1$ degrees of freedom on the **numerator** and $(t-1)(b-1)$ degrees of freedom on the **denominator**.

We may also be interested in testing whether it was advantageous to block. The hypothesis to be tested are:

$$
\begin{aligned}
H_0 &: \beta_1 = \beta_2 = ... = \beta_b = 0 \\
H_1 &: \text{at least one } \beta_j \neq 0. \text{ (at least one of the block means differs from the rest)}
\end{aligned}
$$

The test statistic for these hypothesis is given by:

$$F = \frac{MSB}{MSE} \tag{3.4}$$

where *MSB* and *MSE* are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $b-1$ degrees of freedom on the **numerator** and $(t-1)(b-1)$ degrees of freedom on the **denominator**. We conclude that blocking was effective at the α level of significance if $F > F^{\alpha}_{b-1,(t-1)(b-1)}$. The Analysis of variance table for a randomised block design is as follows:

Table 3.2: ANOVA table for a randomised block design

Source	**df**	**SS**	**MS**	**F**
Treatments	$t-1$	SST	MST	MST/MSE
Blocks	$b-1$	SSB	MSB	MSB/MSE
Error	$(b-1)(t-1)$	SSE	MSE	
Totals	$bt-1$	TSS		

Pairwise Comparisons of the treatments

They are done only if the ANOVA test under the fixed effects model concludes that some treatment means are different. In this case the hypothesis to be tested are:

$$\begin{aligned} H_0 &: \tau_{i.} = \tau_{i'.} \\ H_1 &: \tau_{i.} \neq \tau_{i'.} \quad \forall i \neq i' \end{aligned}$$

NOTE: Comparisons should be made at the same level of significance as that used in the ANOVA. If we choose to use the least significant difference method for making the comparisons, then the least significant difference method is given by:

$$LSD = t^{\frac{\alpha}{2}}_{(t-1)(b-1)}\sqrt{2MSE/b}$$

The means $\mu_{i.}$ and $\mu i'.$ are declared significantly different if

$$|Y_{i.} - Y_{i'.}| > LSD.$$

Example 3.2.1 *A study was undertaken to compare the starting salaries of bachelor's degree candidates at the University of Limpopo from the school of computational and mathematical sciences for the academic years 2004-2005, 2005 -2006 and 2006-2007. Three students from mathematics, three students from statistics and three students from computer science were available for the experiment. It should be noted that only those students who had accepted a job were considered in this study.*

	Curriculum		
Year	**Mathematics**	**Statistics**	**Computer Science**
2004-2005	10.6	12.0	11.0
2005-2006	9.0	15.0	12.0
2006-2007	12.0	17.4	13.0

1. *Assume that model I is approriate for the experiment, test the relevant hypothesis at the 0.05 level of significance.*

2. *Is it appropriate to perform pairwise comparisons of the yearly salary means. Give reasons for your answer.*

3. *If you are to perform pairwise comparisons of the yearly salary means, what value of LSD would you use?*

3.2.2 Random Effects Model (Model II)

Model II is also model (3.1) but with the:

- $\tau_i's$ random variables which are independent and normally distributed with mean zero and variance σ_τ^2;
- $\beta_j's$ random variables which are independent and normally distributed with mean zero and variance σ_β^2;
- $\epsilon_{ij}'s$, $\tau_i's$ and $\beta_j's$ are assumed to be independent random variables

Hypothesis Testing

We wish to test the hypothesis

$$\begin{aligned} H_0 &: \sigma_\tau^2 = 0 \quad \text{i.e treatment effects are identical} \\ H_1 &: \sigma_\tau^2 > 0 \quad \text{i.e treatment effects are not identical} \end{aligned}$$

We may also wish to test the effectiveness of blocking:

$$\begin{aligned} H_0 &: \sigma_\beta^2 = 0 \quad \text{i.e blocking was not effective} \\ H_1 &: \sigma_\beta^2 > 0 \quad \text{i.e blocking was effective} \end{aligned}$$

NOTE Model II is used when the set of treatments and the set of blocks used in the RBD experiment are random samples from their respective treatment and block populations, and the conclusions of the experiment are to be extended to these populations.

3.2.3 Checking model assumptions

The graphical methods presented in section 2.4.2 apply in this case and subsequent cases. However, in this case the residuals for model (3.1) are given by:

$$\epsilon_{ij} = Y_{ij} - \hat{Y}_{ij}$$

where

$$\hat{Y}_{ij} = \bar{Y}_{i.} + \bar{Y}_{.j} + \bar{Y}_{..}$$

$\hat{Y}_{ij}$ is the predicted mean response to the i^{th} treatment in the j^{th} block, i.e is an estimate of μ_{ij}

3.3 Block $\times$ Treatment Interaction Effects

When the difference in treatment means is not the same for different blocks, the model is no longer additive and we say that the two factors treatment and blocks interact.

Definition 3.3.1 *Two factors A and B are said to interact if the difference in mean responses for two levels of one factor is not constant across levels of the second factor.*

The presence of block by treatment interaction effects tends to inflate the estimate of the experimental error and this has an effect of making the tests for comparing the treatment means insenstive to treatment differences. To account for the variation in the responses that is due to the block by treatment interaction we can replicate the basic RBD.

Table 3.3: Replicated RBD

			Block			
Treatment	1	2	...	b	Total	Mean
1	$Y_{111}, Y_{112}, ..., Y_{11r}$	$Y_{121}, Y_{122}, ..., Y_{12r}$	...	$Y_{1b1}, Y_{1b2}, ..., Y_{1br}$	$Y_{1..}$	$\bar{Y}_{1..}$
	$Y_{11.}(\bar{Y}_{11.})$	$Y_{11.}(\bar{Y}_{11.})$	...	$Y_{1b.}(\bar{Y}_{1b.})$		
2	$Y_{211}, Y_{212}, ..., Y_{21r}$	$Y_{221}, Y_{222}, ..., Y_{22r}$	...	$Y_{2b1}, Y_{2b2}, ..., Y_{2br}$	$Y_{2..}$	$\bar{Y}_{2..}$
	$Y_{21.}(\bar{Y}_{21.})$	$Y_{22.}(\bar{Y}_{22.})$	...	$Y_{2b.}(\bar{Y}_{2b.})$		
.	.	.	.	...	.	.
.	.	.	.	...	.	.
.	.	.	.	...	.	.
t	$Y_{t11}, Y_{t12}, ..., Y_{t1r}$	$Y_{t21}, Y_{t22}, ..., Y_{t2r}$	...	$Y_{tb1}, Y_{tb2}, ..., Y_{tbr}$	$Y_{t..}$	$\bar{Y}_{t..}$
	$Y_{t1.}(\bar{Y}_{t1.})$	$Y_{t2.}(\bar{Y}_{t2.})$	...	$Y_{tb.}(\bar{Y}_{tb.})$		
Total	$Y_{.1.}$	$Y_{.2.}$	...	$Y_{.b.}$	$Y_{...}$	
Mean	$\bar{Y}_{.1.}$	$\bar{Y}_{.2.}$	...	$\bar{Y}_{.b.}$		$\bar{Y}_{...}$

The symbols in the table Have the following meanings:

- Y_{ijk} is the i^{th} treatment in the j^{th} block for the k^{th} replication;
- $Y_{i..} = \sum_{j=1}^{b}\sum_{k=1}^{r} Y_{ijk}$ and $\bar{Y_{i..}} = \frac{1}{b}Y_{i..}$ are the i^{th} treatment total and treatment mean respectively, and
- $Y_{.j.} = \sum_{i=1}^{t}\sum_{k=1}^{r} Y_{ijk}$ and $\bar{Y_{.j.}} = \frac{1}{t}Y_{.j.}$ are the j^{th} block total and block mean respectively, and
- $i = 1, 2, ..., t; \quad j = 1, 2, ..., b; \quad k = 1, 2, ..., r; \quad n = tbk =$ total number of observations

To check for the presence of the block by treatment interaction we can use the following methods

1. If the differences $\bar{Y}_{ij.} - \bar{Y}_{i'j}$ are approximately the same for all i, $i \neq i'$ and j, then there may be no block by treatment interaction.
2. If $\bar{Y}_{ij.} \approx \bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}$ fo all i and j, then there may be no block by treatment interactions.
3. We can also check for interaction using graphs plotted from the treatment means i.e $\bar{Y}_{ij.}$ versus j (or $\bar{Y}_{ij.}$ versus i). If the curves of the graph are almost parallel then there may be no block by treatment interactions.

Example 3.3.1 *To estimate the various components of variability in a filtration process, the percent of material lost in the mother liquor was measured for 8 experimental units two runs on each condition. Two filters and two operators were selected at random to use in the experiment resulting in the following measurements:*
Do the data suggest the presence of the operator by filter interaction?

	Operator	
Filter	**1**	**2**
1	7.6,8.8	22.2,23.4
2	19.5,17.6	30.1,24.2

3.4 ANOVA

The model for an RBD with interactions has the form

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \tag{3.5}$$

Where

- μ is the overall population mean;
- τ_i is the effect of the i^{th} treatment;
- β_j is the effect of the j^{th} block;
- $(\tau\beta)_{ij}$ is the interaction effect of the i^{th} treatment and the j^{th} block;
- μ_{ij} is the mean of the ith treatment when in the j^{th} block and
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2; and
- $i = 1, 2, ..., t; \quad j = 1, 2, ..., b; \quad k = 1, 2, ...r.$

Sum of Squares

The total variability in the observations ($Y'_{ijk}s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}Y_{ijk}^2 - \frac{1}{tbr}Y_{...}^2 \quad \text{with} \quad tbr - 1 \quad df \tag{3.6}$$

We can breakdown the Total Sum of Squares into:

1. **Model sum of squares (SSModel)** which measures the variability in the observations that is due to the block, treatment and block by treatment interaction effects;
2. **Error Sum of Squares (SSE)** which measures the variability in the observations that is due to the random errors.

We obtain *SSModel* by performing a two-way ANOVA of data with *Model* as the factor. This follows that the formula for *SSModel* is as follows:

$$SSModel = \sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij.} - \bar{Y_{...}})^2 = \frac{1}{r}\sum_{i=1}^{t}\sum_{j=1}^{b}Y_{ij.}^2 - \frac{1}{tbr}Y_{...}^2 \quad \text{with} \quad tb - 1 \quad df$$

We obtain SSE by subtraction as usual i.e

$$SSE = SSTO - SSModel \quad \text{with} \quad tb(r-1) \quad df$$

It is possible to partition the total sum of squares into two separate sources of variability i.e.

$$
\begin{aligned}
SSTO &= \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk}-\bar{Y}_{...})^2 \\
&= \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}[(\bar{Y}_{ij.}-\bar{Y}_{...})+(Y_{ijk}-\bar{Y}_{ij.})]^2 \\
&= \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(\bar{Y}_{ij.}-\bar{Y}_{...})^2}_{SSModel} + \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk}-\bar{Y}_{ij.})^2}_{SSE} \\
&+ \underbrace{2\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(\bar{Y}_{ij.}-\bar{Y}_{...})(Y_{ijk}-\bar{Y}_{ij.})}_{=0}
\end{aligned} \tag{3.7}
$$

We can also decompose *SSModel* into three separate sources of errors as follows:

1. **Treatment Sum of Squares (SST)** which measures the variation due to the treatment effects;
2. **Block Sum of Squares (SSB)** which measures the variation due to the block effects and
3. **Block by Treatment Sum of Squares (SSB×T)** which measure the variation due to the block by treatment interaction effects.

This follows that *SSModel* can be decomposed as follows

$$
\begin{aligned}
SSModel &= \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ij.}-\bar{Y}_{...})^2 \\
&= \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}[(\bar{Y}_{i..}-\bar{Y}_{...})+(\bar{Y}_{.j.}-\bar{Y}_{...})+(\bar{Y}_{ij.}-\bar{Y}_{i..})-(\bar{Y}_{.j.}+\bar{Y}_{...})]^2 \\
&= \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(\bar{Y}_{i..}-\bar{Y}_{...})^2}_{SST} + \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(\bar{Y}_{.j.}-\bar{Y}_{...})^2}_{SSB} \\
&+ \underbrace{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}[(\bar{Y}_{ij.}-\bar{Y}_{i..})-(\bar{Y}_{.j.}+\bar{Y}_{...})]^2}_{SSBT} + \underbrace{crossproductterm}_{=0}
\end{aligned}
$$

The computational formulae are given by:

$$\begin{aligned} SST &= \frac{1}{br}\sum_{i=1}^{t} Y_{i..}^2 - \frac{1}{tbr} Y_{...}{}^2 \quad \text{with} \quad t-1 \quad df \\ SSB &= \frac{1}{tr}\sum_{j=1}^{b} Y_{.j.}^2 - \frac{1}{tbr} Y_{...}{}^2 \quad \text{with} \quad b-1 \quad df \\ SSB \times T &= SSModel - SST - SSB \quad \text{with} \quad (t-1)(b-1) \quad df \end{aligned}$$

Table 3.4: ANOVA table for a randomised block design with interactions

Source	**df**	**SS**	**MS**	**F**
Treatments	$t-1$	SST	MST	MST/MSE
Blocks	$b-1$	SSB	MSB	MSB/MSE
Interaction	$(t-1)(b-1)$	$SSB \times T$	$MSB \times T$	$MSB \times T/MSE$
Error	$bt(r-1)$	SSE	MSE	
Totals	$bt-1$	TSS		

3.4.1 Fixed effects model

Model I is the same as model 3.5 but with $\tau_i's$ and $\beta_j's$ are regarded as **fixed real constants** satisfying the constraints:

$$\sum_{i=1}^{t} \tau_i = \sum_{j=1}^{b} \beta_j = \sum_{j=1}^{b} (\tau\beta)_{ij} = \sum_{i=1}^{t} (\tau\beta)_{ij}) = 0$$

Hypothesis Testing

Interaction effects

We always first test the hypothesis about the block by treatment interaction effects. The hypothesis can be stated as follows:

$$\begin{aligned} H_0 &: \text{All} \quad (\tau\beta)_i j = 0 \\ H_1 &: \text{at} \quad \text{least} \quad \text{one} \quad (\tau\beta)_{ij} \neq 0. \end{aligned}$$

The test statistic for these hypothesis is given by:

$$F = \frac{MSB \times T}{MSE} \tag{3.8}$$

where $MSB \times T$ and MSE are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $(t-1)(b-1)$ degrees of freedom on the **numerator** and $tb(r-1)$ degrees of freedom on the **denominator**.

Main Effects

If we fail to reject the null hypothesis about the interaction effects we test the **main** effects which are treatment or block effects. Under both models i.e the fixed and random effects model the tests about the main effects are meaningful only if there are no block by treatment interaction effects. This follows that we only do the tests for the main effects only if the test about the interaction effects concludes that the interaction effects are not present.

NOTE Tests for the main effects are the same as the tests for an RBD without replications.

Pairwise Comparisons

They are done only if the ANOVA tests conclude that the block by treatment interaction effects are absent and that some treatment means are different. Hypothesis to be tested are the same as those for pairwise comparisons of an RBD without replications. The LSD for the tests is given by

$$LSD = t^{\frac{\alpha}{2}}_{bt(r-1)}\sqrt{2MSE/br}$$

The means $\mu_{i.}$ and $\mu i.'$ are declared significantly different if

$$|\overline{Y_{i..}} - \overline{Y_{i'..}}| > LSD.$$

Example 3.4.1 *Refer to example 3.3.1*

1. *Assume that model 1 is appropriate for the data and hence test the hypothesis about Operator by Filter interaction effects.*

2. *Based on your conclusions in (1) is it appropriate to test the hypothesis about the effectiveness of blocking by operator? If so*

   (a) *Was blocking by Operator effective?*

   (b) *Are the Filter means significantly different?*

3.4.2 Random Effects Model (Model II)

Model II is also model 3.5 but with the:

- $\tau_i's$ random variables which are independent and normally distributed with mean zero and variance σ_τ^2;
- $\beta_j's$ random variables which are independent and normally distributed with mean zero and variance σ_β^2;
- $(\tau\beta)_{ij}'s$ random variables which are independent and normally distributed with mean zero and variance $\sigma_{\tau\beta}^2$;
- $\epsilon_{ijk}'s$, $\tau_i's$ $\beta\tau_j's$ and $(\tau\beta)_ij's$ are assumed to be independent random variables

Hypothesis Testing

We also start by testing for interactions

$$\begin{aligned} H_0 &: \sigma_{\tau\beta}^2 = 0 \quad \text{i.e there are no interaction effects} \\ H_1 &: \sigma_{\tau\beta}^2 > 0 \quad \text{i.e there are interaction effects} \end{aligned}$$

Main Effects

Tests are the same as the tests for Model II of a randomised block design without replications.

3.5 Unbalanced RBD/Missing Observations

The usual ANOVA methods of the previous sections do not apply directly to unbalanced data for the reason that:

$$SSTO \neq SST + SSB + SSB \times T + SSE$$

3.5.1 One missing Observation

After replacing the missing value with an *estimated* one, the ANOVA of unbalanced data with one missing observation will be the same as that for balanced data. The formula for estimating the missing observation M is given by

$$M = \frac{tT + bB - G}{(t-1)(b-1)}$$

where
t is the number of treatments;
b is the number of blocks;

T is the sum of all observations on the treatment assigned to the missing observation;
B is the sum of all measurements in the block with the missing observation and
G is the sum of all the measurements

Example 3.5.1 *An experiment was conducted to determine the nutritional value of 4 diets for cows. Five dairies were involved in the study. Each cow in a sample of 4 cows from a dairy was randomly assigned to one of the four diets so that a total of 5 cows were fed each diet. The response measured was the amount consumed per day. Unfortunately one of the cows (Diet 4) developed an infection (unrelated to the treatment) and was dropped from the study for safety reasons. The reults are as follows:*

Estimate the missing value and then perform an analysis of variance at 0.01 level of

	Diet			
Dairy	**1**	**2**	**3**	**4**
1	15.4	9.6	9.5	8.4
2	14.8	9.3	9.4	
3	15.9	9.8	9.7	9.3
4	15.5	9.4	9.2	8.1
5	14.7	9.2	9.0	7.9

significance.

3.5.2 Two or more missing Observations

The formulae for estimating more than one missing observation are complicated and hence we resort to some other simpler methods. We discuss one such method which involves fitting what are called **full** (**complete**) and **reduced** models using what is called the **regression approach to fitting ANOVA models.**

The reduced and complete models for testing treatments are as follows:

Complete (Full) model (model 1)

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$$

Reduced model (model 2)

$$Y_{ij} = \mu + \beta_j + \epsilon_{ij}$$

where β_j is the j^{th} block effect and τ_i is the i^{th} treatment effect.

By fitting model 1 to the data we obtain SSE_F. Similarly, a fit of model 2 yields SSE_R. The difference of the two sums of squares for error $SSE_R - SSE_F$, gives the drop in the sum of squares due to treatments. Since this is an unbalanced design the block effecs do not cancel out when comparing treatment means as they do in a balanced randomised block design. The difference in the sums of squares has been adjusted for any effects due to blocks caused by the imbalance in the design. The difference is called the sum of squares due to treatments **adjusted for blocks** i.e

$$SSE_R - SSE_F = SST_{adj}$$

The sum of squares due to blocks unadjusted for any treatment differences is obtained by subtraction:

$$SSB = SSTO - SST_{adj} - SSE$$

Where SSTO and SSE are sums of squares from the complete model.

The analysis of variance table for testing the effect of treatments is as follows:

Table 3.5: ANOVA table for testing the effects of treatments, unbalanced randomised block design

Source	**df**	**SS**	**MS**	**F**
Blocks	$b-1$	SSB		
Treatments$_{adj}$	$t-1$	SST_{adj}	MST_{adj}	MST_{adj}/MSE
Error	by subtraction	SSE	MSE	
Totals	$n-1$	$SSTO$		

Note n is the number of actual observations.

For the blocks the corresponding sum of squares for testing the effect of blocks has the same complete model (model 1) as before i.e

Complete (Full) model (model 1)

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$$

and **Reduced model (model 2)**

$$Y_{ij} = \mu + \tau_j + \epsilon_{ij}$$

The sums of squares drop $SSE_R - SSE_F$, is the sum of squares due to blocks after adjusting for the effects of the treatments. By subtraction, we obtain:

$$SST = SSTO - SSB_{adj} - SSE$$

.

The analysis of variance table for testing the effect of blocks is as follows:

Table 3.6: ANOVA table for testing the effects of blocks, unbalanced randomised block design

Source	**df**	**SS**	**MS**	**F**
Blocks$_{adj}$	$b-1$	SSB_{adj}	MSB_{adj}	MSB_{adj}/MSE
Treatments	$t-1$	SST		
Error	by subtraction	SSE	MSE	
Totals	$n-1$	$SSTO$		

Chapter 4

Balanced Incomplete Block Designs

4.1 Introduction

A balanced incomplete RBD is an incomplete RBD in which any combination of treatments appear together in a block an equal number of times. We use it when we are forced to design an experiment in which we must sacrifice some balance to perform the experiment and this is when the size of blocks *(k)* is less than the number of the treatments *(t)*. For example suppose we have three treatments (A, B, C) and blocks $(B1, B2, B3)$ of size two each. Then we can construct a balanced incomplete RBD by randomly assigning each of the combinations to one of the three blocks.

In general, if $k < t$, then we have $\binom{t}{k}$ or ${}^tC_k$treatment combinations. Note that balanced incomplete block designs can also be constructed with less than ${}^tC_k$ blocks. Although these designs are not balanced as the definition we had in Chapter 3, the designs do retain some balance i.e even though all treatments do not appear in the same block, each combination of treatments appears together in a block the same number of times (The pairs AB,BC, and AC) appear once in a block.

4.2 Analysis Of Variance

The anlysis of variance for a balanced incomplete block design can be performed either by using specifically developed formulas or by using the method of fitting complete and reduced models as discussed for unbalanced designs. The developed formulas are as follows:

The Total Sum of Squares (SSTO) is given by the formula:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b} Y_{ij}^2 - \frac{1}{n}Y_{\cdot\cdot}^2 \quad \text{with} \quad n-1 \;\; df$$

where n is the total number of observations. SSTO can be decomposed into

$$SSTO = SST_{adj} + SSB + SSE$$

SST_{adj} is the adjusted treatment sum of squares. SST_{adj} is given by:

$$SST_{adj} = \frac{t-1}{nk(k-1)}\sum_{i=1}^{t}(kY_{i\cdot} - B_i)^2 \quad \text{with} \quad t-1 \;\; df$$

Where k is the number of treatments per block or the size of each block and B_i is the sum of all the observations for the blocks that contain the i^{th} treatment.

The Block Sum of Squares are given by:

$$SSB = \sum_{j=1}^{b}\frac{1}{k}Y_{\cdot j}^2 - \frac{1}{n}Y_{\cdot\cdot}^2 \quad \text{with} \quad b-1 \;\; df$$

The Error Sum of Squares is given by

$$SSE = SSTO - SST_{adj} - SSB \quad \text{with} \quad n-t-b+1 \;\; df$$

Table 4.1: ANOVA table for a balanced incomplete block design

Source	**df**	**SS**	**MS**	**F**
Blocks	$b-1$	SSB		
Treatments$_{adj}$	$t-1$	SST_{adj}	MST_{adj}	MST_{adj}/MSE
Error	by subtraction	SSE	MSE	
Totals	$n-1$	TSS		

4.3 Pairwise Comparisons

The treatment means are estimated using the adjusted treatment means. The estimate of the i^{th} treatment mean $\mu_{i\cdot}$ is given by:

$$\hat{\mu}_{i\cdot} = \sqrt{\frac{kMSE}{t\lambda}}$$

An estimate of $\mu_{i.} - \mu_{i'.}$ is given by

$$\hat{\mu}_{i.} - \hat{\mu}_{i'.} = \sqrt{\frac{2kMSE}{t\lambda}}$$

The least significance difference for pairwise comparisons of the $\mu_{i.}$ and $\mu_{i'.}$ is given by

$$LSD = t^{\alpha/2}_{n-t-b+1}\sqrt{\frac{2kMSE}{t\lambda}}$$

Example 4.3.1 *A chemical experiment was conducted to determine whether the reaction time was a function of the type of catalyst used. A balanced incomplete RBD was used for the experiment. The treatments were four catalysts and the blocks were four batches of raw material. The data is displayed in the following table.*

	Batch			
Catalyst	**1**	**2**	**3**	**4**
1	73	74	...	71
2	...	75	67	72
3	73	75	68	...
4	75	...	72	75

Test the equality of the catalyst effects at the 0.05 level of significance.

Chapter 5

Latin Square and Crossover Designs

5.1 Introduction: Latin Square Design

In, general a *Latin Square Design* can be used to compare t treatment means in the presence of two extraneous sources of variability, which we block off into t rows and t blocks. The t treatments are then randomly assigned to the rows and columns so that each treatment appears in every row and every column of the design. For example, if the experimental units are animals and both sex and age of the animal affect the response of interest, then the age and sex of the animal can be used as blocking factors in the experiment.

Definition 5.1.1 *A $t \times t$ Latin Square Design contains t rows and t colums. The t treatments are randomly assigned to experimental units within the rows and columns so that each treatment appears in every row and in every column.*

A general latin square design has the following features:

1. There are t treatments
2. There are two blocking factors each with t blocks or levels.
3. The t treatments are randomly assigned to the experimental units within the rows and colums so that each treatment appears once in every row and once in every column.
4. If the letters in the first row and first column are arranged alphabetically (in a regular ascending order), then the latin square is called a standard latin square.

If $t = 2$ or $t = 3$ there is only one standard square. For $t = 4$ there are 4 standard squares, for $t = 5$ there are 56, for $t = 6$ there are 9408.

For a given number of treatments e.g for $t = 3$ there are 12 different latin square designs. The question is, if there exist many latin square design which one should we use for the design?

Example 5.1.1 *Randomise the following* 4×4 *standard latin square:*

Table 5.1: Standard 4×4 latin square design

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

NOTE: The process of choosing a latin square design at random is called **randomisation of the latin square design**. In principle, to choose a random latin square we proceed as follows

1. Choose one of the standard squares at random.
2. Randomly permute the columns
3. Randomly permute the rows
4. Randomly permute the symbols in the body of the diagram.

5.1.1 Advantages of a Latin Square Design

1. The design is particularly appropriate for comparing two treatment means in the presence of two sources of extraneous variation each measured at t levels. This results in substantial reductions in the experimental error.
2. In experiments where the t treatments are applied to each experimental unit in succession, the design is used to account for 'Period' effects or the effects of the order of running the treatments.
3. The analysis is still quite simple.

5.1.2 Disadvantages of a Latin Square Design

1. While a latin square can be constructed for any value of t, it is best suited for comparing t treatments when $5 \leq t \leq 10$
2. The two blocking factors cannot have different numbers of levels.

3. The latin square design can only be used when there are **no interactions** between either the blocking factors and the treatments or between the blocking factors.

4. Randomising a latin square experiment is more complicated than randomising a CRD or RBD experiment.

5.2 Analysis of a Latin Square Design

Consider a 4×4 latin square design displayed in table 5.2

Table 5.2: A 4×4 Latin Square design

		Column					
		1	2	3	4	Total	Mean
Row	1	Y_{111}	Y_{122}	Y_{134}	Y_{143}	$Y_{1..}$	$\bar{Y}_{1..}$
	2	Y_{212}	Y_{223}	Y_{231}	Y_{244}	$Y_{2..}$	$\bar{Y}_{2..}$
	3	Y_{313}	Y_{324}	Y_{332}	Y_{341}	$Y_{3..}$	$\bar{Y}_{3..}$
	4	Y_{414}	Y_{421}	Y_{433}	Y_{442}	$Y_{4..}$	$\bar{Y}_{4..}$
	Total	$Y_{.1.}$	$Y_{.2.}$	$Y_{.3.}$	$Y_{.4.}$	$Y_{...}$	
	Mean	$\bar{Y}_{.1.}$	$\bar{Y}_{.2.}$	$\bar{Y}_{.3.}$	$\bar{Y}_{.4.}$		$\bar{Y}_{...}$
Treatments		A	B	C	D		
	Total	$Y_{..1}$	$Y_{..2}$	$Y_{..3}$	$Y_{..4}$		
	Mean	$\bar{Y}_{..1}$	$\bar{Y}_{..2}$	$\bar{Y}_{..3}$	$\bar{Y}_{..4}$		

The symbols in the table have the following meanings:

- Y_{ijk} is the response to the k^{th} treatment i^{th} row and j^{th} column;
- $Y_{i..} = \sum_{j=1}^{t} Y_{ijk}$, $Y_{.j.} = \sum_{i=1}^{t} Y_{ijk}$ and $Y_{...} = \sum_{i=1}^{t}\sum_{j=1}^{t} Y_{ijk}$ are the i^{th} row total, j^{th} column total and the overall total respectively, and
- $\bar{Y}_{i..} = \frac{1}{t}Y_{i..}$, $\bar{Y}_{.j.} = \frac{1}{t}Y_{.j.}$ and $\bar{Y}_{...} = \frac{1}{t^2}Y_{...}$ are the i^{th} row mean, j^{th} column mean and overall mean respectively, and
- $i = 1, 2, ..., t;\quad j = 1, 2, ..., t;\quad k = 1, 2, ...t\quad n = t^2 =$ total number of observations

For the treatment sum of squares we have:

- $Y_{..k} = \sum_{k=1}^{t} Y_{ijk}$ and
- $\bar{Y}_{..k} = \frac{1}{t}Y_{..k}$ and
- $SST = \sum_{i=1}^{t}\sum_{j=1}^{t}(Y_{..k} - \bar{Y}_{...})^2$

5.2.1 ANOVA

The fixed effects model (model I) and the mixed effects model (model II) for a latin square experiment with t treatments have the form:

$$Y_{ijk} = \mu + \rho_i + \beta_j + \tau_k + \epsilon_{ijk}$$

Where

- μ is the overall population mean;
- τ_k is the k^{th} treatment effect;
- ρ_i is the i^{th} row effect;
- β_j is the effect of the j^{th} column;
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2.

Model I

Model I is model 5.1 but with the $\rho'_i s, \beta'_j s$ and $\tau'_k s$ regarded as fixed constants satisfying the constraints.

$$\sum_{i=1}^{t} \rho_i = \sum_{j=1}^{t} \beta_j = \sum_{k=1}^{b} \tau_k = 0$$

Hypothesis Testing

The objective is to test the null hypothesis of no difference among the treatment means. This is equivelent to testing

$$\begin{aligned} H_0 &: \tau_1 = \tau_2 = ... = \tau_t = 0 \\ H_1 &: \text{at least one } \tau_i \neq 0. \text{ (at least one of the treatment means differs from the rest)} \end{aligned}$$

If we reject H_0 we wish to determine which means are different.

Model II

Model I is model 5.1 but with some effects regarded as fixed real constants and some regarded as random effects variables. If we consider a case whereby the treatments are fixed and both the rows and colums are random, the assumptions for the model are that:

- the $\rho_i's$ are random variables which are independent and normally distributed with mean 0 and variance σ_ρ^2;
- the $\beta_j's$ are random variables which are independent and normally distributed with mean 0 and variance σ_β^2 and

$$\sum_{k=1}^{t} \tau_k = 0$$

Hypothesis Testing

The objective is to test the null hypothesis of no difference among the treatment means. This is equivelent to testing

$$\begin{aligned} H_0 &: \tau_1 = \tau_2 = ... = \tau_t = 0 \\ H_1 &: \text{at least one } \tau_i \neq 0. \text{ (at least one of the treatment means differs from the rest)} \end{aligned}$$

If we reject H_0 we wish to determine which means are different.

Sum of Squares

The total variability in the observations ($Y_{ijk}'s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{t}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{t} Y_{ijk}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t^2 - 1 \quad df$$

It is possible to partition the total sum of squares into four separate sources of variability i.e.

1. the variability that is due to the treatments (**treatment sum of squares (SST)**,
2. the variability that is due to the row effects (**row sum of squares (SSR)**,
3. the variability that is due to the column effects (**column sum of squares (SSC)**and
4. the variability in the responses that is due to the random errors (**error sum of squares (SSE)**

There are sum of squares formulae that are mathematical and are not convinient to use in calculations. The computational formulae are given by:

$$
\begin{aligned}
SST &= \frac{1}{t}\sum_{k=1}^{t} Y_{..k}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSR &= \frac{1}{t}\sum_{i=1}^{t} Y_{i..}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSC &= \frac{1}{t}\sum_{i=1}^{t} Y_{.j.}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSE &= SSTO - SST - SSR - SSC \quad \text{with} \quad (t-2)(t-1) \quad df
\end{aligned}
$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedon and they are used for testing the hypothesis about the treatments.

Hypothesis testing

The test statistic for treatment effects is given by:

$$F = \frac{MST}{MSE} \tag{5.1}$$

where MST and MSE are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $t-1$ degrees of freedom on the **numerator** and $(t-1)(t-2)$ degrees of freedom on the **denominator**.

The test statistics for checking the effectiveness of blocking by the row blocking factor and the column blocking factor are given by:

$$F = \frac{MSR}{MSE} \quad \text{and} \quad F = \frac{MSC}{MSE}$$

respectively. Both F ratios have an F distribution with $t-1$ degrees of freedom on the **numerator** and $(t-1)(t-2)$ degrees of freedom on the **denominator**.

Table 5.3: ANOVA table for a Latin Square Design

Source	**df**	**SS**	**MS**	**F**
Treatment	$t-1$	SST	MST	MST/MSE
Rows	$t-1$	SSR	MSR	MSR/MSE
Columns	$t-1$	SSC	MSC	MSC/MSE
Error	$(t-2)(t-1)$	SSE	MSE	
Totals	t^2-1	$SSTO$		

Pairwise Comparisons of the treatments

They are done only if the ANOVA test under the fixed effects model concludes that some treatment means are different. In this case the hypothesis to be tested are:

$$\begin{aligned} H_0 &: \tau_{i.} = \tau_{i'.} \\ H_1 &: \tau_{i.} \neq \tau_{i'.} \quad \forall i \neq i' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{(t-1)(t-2)} \sqrt{2MSE/t}$$

Example 5.2.1 *A traffic engineer wished to compare the total unused green time for 3 different signal-control sequencing devices at 3 different intersections of a city. It was assumed that the intersections were far enough apart that they, in effect, acted independently, regardless of the signal sequencing device employed. In addition to comparing the devices at the 3 different intersections, the engineer wished to compare the devices at different time periods during the day. The data collected are tabulated in the following table. Analyse the data and draw conclusions.*

Table 5.4: A 3×3 latin square design for the traffic delay experiment

	Time period			
		1	**2**	**3**
Intersection	**1**	23 (II)	31 (III)	51 (I)
	2	71 (I)	42 (II)	35 (III)
	3	34 (III)	67 (I)	29 (II)

5.3 A Latin Square Design with Missing Data

The techniques discussed in section 3.5 Chapter 3 also apply to the latin square design. The formula for estimating a single missing value in a latin square design is

$$M = \frac{t(T + R + C) - 2G}{(t-1)(t-2)}$$

where T, R and C represent the treatment, row and column totals respectively, corresponding to the missing observation and t is the number of treatments in the latin square design. After replacing the missing value the analysis can proceed as for a balanced latin square design with degrees of freedom for $SSTO = t^2 - 2$. If there are significant differences due to treaments we need to make pairwise comparisons.

The least significant difference between the treatment with the missing value and any other treatment is

$$LSD = t^{\frac{\alpha}{2}}_{(t-1)(t-2)}\sqrt{MSE(\frac{2}{t} + \frac{1}{(t-1)(t-2)}}$$

For any other pair of treatments, the LSD is as before

$$LSD = t^{\frac{\alpha}{2}}_{(t-1)(t-2)}\sqrt{2MSE/t}$$

Example 5.3.1 *Refer to example 5.2.1 with one missing value.*

5.4 Replication of a Latin Square

This is feasible when

- two or more experimental units can be obtained for each cell defined by levels of the row blocking factor and the column blocking factor or,
- the latin square can be repeated two or more times using the same experimental units.

suppose that the number of replications within each cell is r. Then the model for the data is given by:

$$Y_{ijkm} = \mu + \rho_i + \beta_j + \tau_k + \epsilon_{ijkm}$$

Where

- Y_{ijkm} is the m^{th} response to the k^{th} treatment in the i^{th} row and j^{th} column;
- μ is the overall population mean;
- τ_k is the k^{th} treatment effect;
- ρ_i is the i^{th} row effect;
- β_j is the effect of the j^{th} column;
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2; and

The totals can be redifined as follows:

- $Y_{ijk.} = \sum_{m=1}^{r} Y_{ijkm}$, $Y_{i...} = \sum_{j=1}^{t}\sum_{m=1}^{r} Y_{ijkm}$, $Y_{.j..} = \sum_{i=1}^{t}\sum_{m=1}^{r} Y_{ijkm}$ and $Y_{....} = \sum_{i=1}^{t}\sum_{j=1}^{t}\sum_{m=1}^{r} Y_{ijkm}$ are the $(ij)^{th}$ cell total, i^{th} row total, j^{th} column total and the overall total respectively, and

- $\bar{Y}_{ijk.} = \frac{1}{r}Y_{ijk.}$, $\bar{Y}_{i...} = \frac{1}{tr}Y_{i...}$, $\bar{Y}_{.j..} = \frac{1}{tr}Y_{.j..}$ and $\bar{Y}_{....} = \frac{1}{t^2r}Y_{....}$ are the $(ij)^{th}$ cell mean, i^{th} row mean, j^{th} column mean and overall mean respectively, and
- $i = 1, 2, ..., t;\quad j = 1, 2, ..., t;\quad k = 1, 2, ...t\quad n = t^2r =$ total number of observations

The total variability in the observations ($Y'_{ijk}s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{t}\sum_{m=1}^{r}(Y_{ijkm} - \bar{Y_{....}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{t}\sum_{m=1}^{r}Y_{ijkm}^2 - \frac{1}{t^2r}Y_{....}^2 \quad \text{with} \quad t^2r - 1 \quad df \tag{5.2}$$

It is possible to partition the total sum of squares into four separate sources of variability. The computational formulae for the sum of squares are given by:

$$\begin{aligned}
SST &= \frac{1}{tr}\sum_{k=1}^{t}Y_{..k.}^2 - \frac{1}{t^2r}Y_{....}^2 \quad \text{with} \quad t-1 \quad df \\
SSR &= \frac{1}{tr}\sum_{i=1}^{t}Y_{i...}^2 - \frac{1}{t^2r}Y_{....}^2 \quad \text{with} \quad t-1 \quad df \\
SSC &= \frac{1}{tr}\sum_{i=1}^{t}Y_{.j..}^2 - \frac{1}{t^2r}Y_{....}^2 \quad \text{with} \quad t-1 \quad df \\
SSE &= SSTO - SST - SSR - SSC \quad \text{with} \quad t^2r - 3t + 2 \quad df
\end{aligned}$$

Note: by replicating the latin square design we have increased the degrees of freedom for SSE by

$$(r-1)t^2$$

Example 5.4.1 *A team of educators were interested in determining the relative effectiveness of instruction methods A (Video instruction), B (traditional classroom) and C (programmed study) on student learning. They felt that the IQ and the age of the student could influence their study too much. To take these two factors into account, they used a 3 X 3 latin square design with IQ-Age cell of the latin square. The scores for the instruction methods are in the following table. Analyse the data and draw conclusions.*

Table 5.5: A 3×3 latin square design for the scores

		IQ		
		High	**Average**	**Low**
	20	40,50 (C)	40,40 (B)	50,40 (A)
Age	**30**	70,60 (B)	30,20 (A)	55,50 (C)
	40	20,30 (A)	70,80 (C)	25,25 (B)

5.5 Crossover Designs

In this experiment, each experimental unit receives a sequence of all the t treatments in t successive time periods, the treatment sequences being different for different experimental units. At the end of each time period, the response to the treatment is measured on the experimental unit, and a period of time is allowed to pass in order to eliminate the effect of the current treatment before the next treatment is administered to the unit.

A crossover design can be a group of two or more $t \times t$ latin squares. With rt experimental units, the crossover design is a group of r $t \times t$ latin square designs with t common time periods and t different experimental units forming the columns of each latin square.

The crossover design model has the form:

$$Y_{ijk} = \mu + \rho_i + \beta_j + \tau_k + \epsilon_{ijk}$$

Where

- Y_{ijk} is the response to the k^{th} treatment administered to the experimental unit j during period i;
- μ is the overall population mean;
- τ_k is the k^{th} treatment effect;
- ρ_i is the period i effect;
- β_j is the experimental unit j effect;
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2; and

The totals and the means are as follows:

- $Y_{i..} = \sum_{j=1}^{rt} Y_{ijk}$, $Y_{.j.} = \sum_{i=1}^{t} Y_{ijk}$ and $Y_{...} = \sum_{i=1}^{t} \sum_{j=1}^{rt} Y_{ijk}$ are the i^{th} period total, j^{th} experimental total and the overall total respectively, and

- $\bar{Y}_{i..} = \frac{1}{rt}Y_{i..}$, $\bar{Y}_{.j.} = \frac{1}{t}Y_{.j.}$ and $\bar{Y}_{...} = \frac{1}{t^2r}Y_{...}$ are the i^{th} period mean, j^{th} experimental unit mean and overall mean respectively, and
- $i = 1, 2, ..., t;\quad j = 1, 2, ..., rt;\quad k = 1, 2, ...t\quad n = rt^2 =$ total number of observations

Sum of Squares

The total variability in the observations ($Y'_{ijk}s$) is measured using the total sum of squares as usual and is given by:

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{rt}(Y_{ijk} - \bar{Y}_{...})^2 = \sum_{i=1}^{t}\sum_{j=1}^{rt}Y_{ijk}^2 - \frac{1}{t^2r}Y_{...}^2 \quad \text{with} \quad t^2r - 1 \quad df \tag{5.3}$$

The computational formulae for the period sum of squares, the experimental unit sum of squares, the treatment sum of squares and the error sum of squares are given by:

$$\begin{aligned}
SST &= \frac{1}{tr}\sum_{k=1}^{t}Y_{..k}^2 - \frac{1}{t^2r}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSP &= \frac{1}{tr}\sum_{i=1}^{t}Y_{i..}^2 - \frac{1}{t^2r}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSU &= \frac{1}{t}\sum_{i=1}^{t}Y_{.j.}^2 - \frac{1}{t^2r}Y_{...}^2 \quad \text{with} \quad tr-1 \quad df \\
SSE &= SSTO - SST - SSP - SSU \quad \text{with} \quad (tr-2)(t-1) \quad df
\end{aligned}$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedon and they are used for testing the hypothesis about the treatments.

Hypothesis testing

The objective is to test the null hypothesis of no difference among the treatment means. This is equivelent to testing

$$\begin{aligned}
H_0 &: \tau_1 = \tau_2 = ... = \tau_k = 0 \\
H_1 &: \text{at least one } \tau_k \neq 0. \text{ (at least one of the treatment means differs from the rest)}
\end{aligned}$$

The test statistic for treatment effects is given by:

$$F = \frac{MST}{MSE} \tag{5.4}$$

where *MST* and *MSE* are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $t-1$ degrees of freedom on the **numerator** and $(t-1)(rt-2)$ degrees of freedom on the **denominator**.

Table 5.6: ANOVA table for a Latin Square Design

Source	**df**	**SS**	**MS**	**F**
Treatment	$t-1$	SST	MST	MST/MSE
Period	$t-1$	SSP	MSP	MSP/MSE
Unit	$rt-1$	SSU	MSU	MSU/MSE
Error	$(rt-2)(t-1)$	SSE	MSE	
Totals	t^2r-1	$SSTO$		

Example 5.5.1 *A crossover design was used to study the effects of three diets on the daily weight gain of 2 year old goats. A sufficiently long period of time was allowed to pass before a goat was fed its new diet in order to eliminate the effect of its previous diets on the response to the new diet. The following data was recorded (in grams per day). Analyse the data and draw conclusions.*

		Goat					
		1	**2**	**3**	**4**	**5**	**6**
Period	**1**	40.5 (C)	30.0 (B)	30.5 (A)	30.5 (C)	30.0 (B)	40.0 (A)
	2	45.0 (B)	20.5 (A)	40 (C)	60.5 (B)	10.0 (A)	45.5 (C)
	3	20.0 (A)	65.5 (C)	20.0 (B)	20.5 (A)	60.5 (C)	30.0 (B)

Chapter 6

Factorial Experiments

6.1 Introduction

Consider a situation in which it is of interest to study the effect of two factors **A** and **B** on some response. For example, in a chemical experiment we would like to simultaneously vary the reaction pressure and reaction time and study the effect of each on the yield. The term **factor** is used in a general sense to denote any feature of the experiment such as temperature, time or pressure that may be varied from trial to trial. The levels of a factor are the actual values used in the experiment.

A factorial design can either be a completely randomised design **(CRD)** or a block design **(BD)**. We use **CRD** factorial design if we have homogeneous experimental units and the experimental conditions are uniform. If we have heterogeneous experimental units and/or if there are external variables that may influence the response then a randomised block design or a latin square is appropriate for the factorial experiment.

To illustrate a simple factorial design: lets suppose that we wish to study the effects of the combination of the levels of Factors A and B and that each factor has two levels i.e for A we have A_1 and A_2 and for B we have B_1 and B_2. The combinations of the two factors that are to be investigated are:

$$A_1B_1, \quad A_1B_2, \quad A_2B_1, \quad A_2B_2.$$

These four factor level combinations are the treatments.

Definition 6.1.1 *A **factorial experiment** is an experiment in which the response y is observed at all factor-level combinations of the independent variables.*

In this type of experiment it is important not only to determine if the two factors have an influence on the response but also if there is a significant interaction between the two factors.

6.1.1 Interaction in Factorial Experiments

If we consider the illustration above (which is an example of a two factor experiment) the effects of A and B, often called the *main effects*, take on a different meaning in the presence of interaction. In general, there could be experimental situations in which factor A has a positive effect on the response at one level of factor B, while at a different level of factor B the effect of A is negative.

Example 6.1.1 *Consider, for example, the following hypothetical data taken on two factors each at three levels. Assume that the values given are averages for each treatment. Check for the presence of the A by B interactions.*

	B		
A	B_1	B_2	B_3
A_1	80.30	80.65	80.30
A_2	80.20	80.55	80.00
A_3	80.60	80.85	80.25

NOTE:

- The analysis of the data usually begins by checking the presence of interaction. Then if interaction is not significant we proceed to make tests on the effects of the main factors. If the data indicate the presence of interaction, we might need to observe the influence of each factor at fixed levels of the other.
- Interaction and experimental error are separated in factorial experiments only if multiple observations are taken at the various treatment combinations.

6.2 Analysis of Factorial Experiments

We shall consider the analysis of a two factor factorial design when in a completely randomised design and also in a randomised block design. In both cases we assume that the fixed effects models are appropriate for the factorial experiments.

6.2.1 Two factor factorial Design in a CRD

We shall consider two factors A and B with a and b levels respectively and a case of r replications for each treatment combination. Each treatment combination defines a cell in our array. Table 6.3 displays the layout of the factorial design when replicated r times.

Table 6.1: Two-factor factorial experiment with r plications in a CRD

	Factor B					
Factor A	B_1	B_2	...	B_b	**Total**	**Mean**
A_1	$Y_{111}, Y_{112}, ..., Y_{11r}$	$Y_{121}, Y_{122}, ..., Y_{12r}$	...	$Y_{1b1}, Y_{1b2}, ..., Y_{1br}$	$Y_{1..}$	$\bar{Y}_{1..}$
A_2	$Y_{211}, Y_{212}, ..., Y_{21r}$	$Y_{221}, Y_{222}, ..., Y_{22r}$	...	$Y_{2b1}, Y_{2b2}, ..., Y_{2br}$	$Y_{2..}$	$\bar{Y}_{2..}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
.	.	.	...	.	.	.
A_a	$Y_{a11}, Y_{a12}, ..., Y_{a1r}$	$Y_{a21}, Y_{a22}, ..., Y_{a2r}$	...	$Y_{ab1}, Y_{ab2}, ..., Y_{abr}$	$Y_{a..}$	$\bar{Y}_{a..}$
Total	$Y_{.1.}$	$Y_{.2.}$	...	$Y_{.b.}$	$Y_{...}$	
Mean	$\bar{Y}_{.1.}$	$\bar{Y}_{.2.}$	...	$\bar{Y}_{.b.}$		$\bar{Y}_{...}$

The Anova model for a two-factor factorial design in a CRD has the form

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

where

- μ is the overall population mean;
- τ_i is the effect of the i^{th} level of factor A;
- β_j is the effect of the j^{th} level of factor B;
- $(\tau\beta)_{ij}$ is the interaction effect of the i^{th} level of factor A and the j^{th} level of factor B;
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2.

The fixed effects model assumes that the $\tau'_i s$, $\beta'_j s$ and $(\tau\beta)'_{ij} s$ are fixed real constants satisfying the constraint:

$$\sum_{i=1}^{a} \tau_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} (\tau\beta)_{ij} = 0$$

The estimates of τ_i, β_j and $(\tau\beta)_{ij}$ are given by

- $\tau_i = \bar{Y}_{i..} - \bar{Y}_{...}$
- $\beta_j = \bar{Y}_{.j.} - \bar{Y}_{...}$ and
- $(\tau\beta)_{ij} = \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}$

The three hypothesis to be tested are as follows:

1. The first hypothesis that we test is about the **treatment effects**. The hypothesis are as follows:

$$\begin{aligned} H_0 &: \text{All the } \mu'_{ij}s \text{ are equal} \\ H_1 &: \text{at least one pair of } \mu'_{ij}s \text{ are not equal} \end{aligned}$$

2. If we reject H_0 in favour of H_1 then we test about the AB **interaction effects** and the hypothesis are:

$$\begin{aligned} H_0 &: (\tau\beta)_{11} = (\tau\beta)_{12} = ... = (\tau\beta)_{ab} = 0 \\ H_1 &: \text{at least one of the } (\tau\beta)'_{ij}s \neq 0. \end{aligned}$$

3. If there are no interaction effects we proceed to test about the **main effects** i.e A factor level effects or the B factor level effects.

   (a) The hypothesis about A factor level effects are

$$\begin{aligned} H_0 &: \tau_1 = \tau_2 = ... = \tau_a = 0 \\ H_1 &: \text{at least one of the } \tau'_i s \neq 0. \end{aligned}$$

   If we reject H_0 we compare the A factor level means.

   (b) The hypothesis about B factor level effects are

$$\begin{aligned} H_0 &: \beta_1 = \beta_2 = ... = \beta_b = 0 \\ H_1 &: \text{at least one of the } \beta'_j s \neq 0. \end{aligned}$$

   If we reject H_0 we compare the B factor level means.

Sum of Squares

The total variability in the observations ($Y'_{ijk}s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}Y_{ijk}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad abr-1 \quad df$$

The total sum of squares can be decomposed into two separate sources of variability i.e.

1. the variability that is due to the treatments (factor level combination) (**treatment sum of squares (SST)**) and

2. the variability in the responses that is due to the random errors (**error sum of squares (SSE)**)

SST is obtained by performing a one way ANOVA of the data with the factor level combinations A_iB_j as the treatments. The formula for SST is given by:

$$SST = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ij.} - \bar{Y}_{...})^2 = \frac{1}{r}\sum_{i=1}^{a}\sum_{j=1}^{b}Y_{ij.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad ab-1 \quad df$$

The error sum of squares is obtained as follows:

$$SSE = SSTO - SST \quad \text{with} \quad ab(r-1) \quad df$$

The variability in the observations that is due to the treatments effects is attributed to the main effects and the interaction of the main effects i.e

- A factor level effects;
- B factor level effects and
- $A \times B$ interaction effects.

This follows that we can partition the **treatment sum of squares** into

- the variation due to the **factor A effects (SSA)**
- the variation due to the **factor B effects (SSB)**
- the variation due to the **factor** $A \times B$ **interaction effects (SSAB)**

The computational formulae for SSA and SSB are as follows:

$$\begin{aligned} SSA &= \frac{1}{br}\sum_{i=1}^{a}Y_{i..}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad a-1 \quad df \\ SSB &= \frac{1}{ar}\sum_{j=1}^{b}Y_{.j.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad b-1 \quad df \\ SSAB &= SST - SSA - SSB \quad \text{with} \quad (a-1)(b-1) \quad df \end{aligned}$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedon and they are used for testing the hypothesis about the treatments.

Hypothesis testing

1. We first test for treatment effects and the test statistic is given by:

$$F = \frac{MST}{MSE} \tag{6.1}$$

where MST and MSE are mean squares computed from the appropriate sums of squares in the ANOVA table. F has an F distribution with $ab-1$ degrees of freedom on the **numerator** and $ab(r-1)$ degrees of freedom on the **denominator**.

2. Next we test for the significance of $A \times B$ interaction effects. This is only possible if we reject H_0 in number 1 above. The test statistic is given by:

$$F = \frac{MSAB}{MSE}$$

which also has an F distribution with $(a-1)(b-1)$ degrees of freedom on the **numerator** and $ab(r-1)$ degrees of freedom on the **denominator**.

3. We test the hypothesis about the A factor level effects or the B factor level effects only if there are no $A \times B$ interaction effects.The test statistic for the A and B factor level effects are given by:

$$F = \frac{MSA}{MSE} \quad \text{and} \quad F = \frac{MSB}{MSE}$$

which both have an F distribution with $(a-1)$ degrees of freedom on the **numerator** and $ab(r-1)$ degrees of freedom on the **denominator** and $(b-1)$ degrees of freedom on the **numerator** and $ab(r-1)$ degrees of freedom on the **denominator** respectively.

Table 6.2: ANOVA table for a Two-factor experiment in a CRD

Source	**df**	**SS**	**MS**	**F**
Treatment	$ab-1$	SST	MST	MST/MSE
Main Effects				
A	$a-1$	SSA	MSA	MSA/MSE
B	$b-1$	SSB	MSB	MSB/MSE
Interactions				
AB	$(a-1)(b-1)$	$SSAB$	$MSAB$	$MSAB/MSE$
Error	$ab(r-1)$	SSE	MSE	
Total	$abr-1$	$SSTO$		

Pairwise Comparisons of the treatments

They are done only if the ANOVA test under the fixed effects model concludes that some treatment means are different.Why? In a case when treatment means are not equal as well as the $A \times B$ interaction effects are present pairwise comparisons of the A or B factor level means does not make sense. In the presence of $A \times B$ interactions we must compare the factor level means at each level of B or vise versa. Hence the hypothesis to be tested for the A factor level means at the j^{th} level of factor B in this case are:

$$\begin{aligned} H_0 &: \tau_{ij} = \tau_{i'j} \\ H_1 &: \tau_{ij} \neq \tau_{i'j} \quad \forall i \neq i' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{ab(r-1)} \sqrt{2MSE/r}$$

Thus the τ_{ij} and $\tau_{i'j}$ are significantly different if

$$|\bar{Y}_{ij.} - \bar{Y}_{i'j.}| > LSD$$

Similarly, for the B factor level means at the i^{th} level of factor A the hypothesis are:

$$\begin{aligned} H_0 &: \tau_{ij} = \tau_{ij'} \\ H_1 &: \tau_{ij} \neq \tau_{ij'} \quad \forall j \neq j' \end{aligned}$$

The least significant difference for comparing the means is as above and the means are significantly different if:

$$|\bar{Y}_{ij.} - \bar{Y}_{ij'.}| > LSD$$

If we conclude that $A \times B$ interactions are absent and some A and/or B factor level means are not equal then we can do pairwise comparisons of the A factor level means and/or the B factor level means. The hypothesis to be tested for factor A are:

$$\begin{aligned} H_0 &: \tau_{i.} = \tau_{i'.} \\ H_1 &: \tau_{i.} \neq \tau_{i'.} \quad \forall i \neq i' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{ab(r-1)} \sqrt{2MSE/br}$$

Thus the τ_{ij} and $\tau_{i'j}$ are significantly different if

$$|\bar{Y}_{i..} - \bar{Y}_{i'..}| > LSD$$

Similarly, for the B factor level means the hypothesis are:

$$\begin{aligned} H_0 &: \tau_{.j} = \tau_{.j'} \\ H_1 &: \tau_{.j} \neq \tau_{.j'} \quad \forall j \neq j' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{ab(r-1)} \sqrt{2MSE/ar}$$

The least significant difference for comparing the means is as above and the means are significantly different if:

$$|\bar{Y}_{.j.} - \bar{Y}_{.j'.}| > LSD$$

Example 6.2.1 *In a chemical process the most important variables that are thought to affect the yield are pressure and temperature. Three levels of each factor are selected and a factorial experiment in a CRD with two replications was performed. The results are as shown in table 6.2.1. Analyse the data and draw conclusions.*

	Pressure		
Temperature	215	230	235
30	80.4	80.7	80.2
	80.2	80.6	80.4
40	80.1	80.5	79.9
	80.3	80.6	80.1
50	80.5	80.8	80.4
	80.7	80.9	80.1

6.2.2 Two factor factorial Design in a RBD

In a two-factor experiment with random effects the layout of the experiment is as follows:

Table 6.3: Two-factor factorial experiment in a RBD

	Block					
Treatment	1	2	...	r	**Total**	**Mean**
A_1B_1	Y_{111}	, Y_{112}	...	Y_{11r}	$Y_{11.}$	$\bar{Y}_{11.}$
A_2B_2	Y_{121}	Y_{122}	...	Y_{12r}	$Y_{12.}$	$\bar{Y}_{12.}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
A_1B_b	Y_{1b1}	Y_{1b2}	...	Y_{1br}	$Y_{1b.}$	$\bar{Y}_{1b.}$
A_2B_1	Y_{211}	Y_{212}	...	Y_{21r}	$Y_{21.}$	$\bar{Y}_{21.}$
A_2B_2	Y_{221}	Y_{222}	...	Y_{22r}	$Y_{22.}$	$\bar{Y}_{22.}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
A_2B_b	Y_{2b1}	Y_{2b2}	...	Y_{2br}	$Y_{2b.}$	$\bar{Y}_{2b.}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
A_aB_1	Y_{a11}	Y_{a12}	...	Y_{a1r}	$Y_{a1.}$	$\bar{Y}_{a1.}$
A_aB_2	Y_{a21}	Y_{a22}	...	Y_{a2r}	$Y_{a2.}$	$\bar{Y}_{a2.}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
A_aB_b	Y_{ab1}	Y_{ab2}	...	Y_{abr}	$Y_{ab.}$	$\bar{Y}_{ab.}$
Total	$Y_{..1}$	$Y_{..2}$	...	$Y_{..r}$	$Y_{...}$	
Mean	$\bar{Y}_{..1}$	$\bar{Y}_{..2}$	...	$\bar{Y}_{..r}$		$\bar{Y}_{...}$

The Anova model for a two-factor factorial design in a RBD has the form

$$Y_{ijk} = \mu + \tau_i + \beta_j + \rho_k + (\tau\beta)_{ij} + \epsilon_{ijk}$$

where

- μ is the overall population mean;
- τ_i is the effect of the i^{th} level of factor A;
- β_j is the effect of the j^{th} level of factor B;
- ρ_k is the effect of the k^{th} block;
- $(\tau\beta)_{ij}$ is the interaction effect of the i^{th} level of factor A and the j^{th} level of factor B;

- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2.

The fixed effects model assumes that the $\tau'_i s$, ρ_k, $\beta'_j s$ and $(\tau\beta)'_{ij} s$ are fixed real constants satisfying the constraint:

$$\sum_{i=1}^{a} \tau_i = \sum_{j=1}^{b} \beta_j = \sum_{k=1}^{r} \rho_k = \sum_{i=1}^{a} (\tau\beta)_{ij} = \sum_{i=1}^{a} (\tau\beta)_{ij} = 0$$

The three hypothesis to be tested are as follows:

1. The first hypothesis that we test are about the **treatment effects** and the hypothesis are as follows:

$$\begin{aligned} H_0 &: \text{All the } \mu'_{ij}s \text{ are equal} \\ H_1 &: \text{at least one pair of } \mu'_{ij}s \text{ are not equal} \end{aligned}$$

2. If we reject H_0 in favour of H_1 then we test about the AB **interaction effects** and the hypothesis are:

$$\begin{aligned} H_0 &: (\tau\beta)_{11} = (\tau\beta)_{12} = ... = (\tau\beta)_{ab} = 0 \\ H_1 &: \text{at least one of the } (\tau\beta)'_{ij} s \neq 0. \end{aligned}$$

3. If there are no interaction effects we proceed to test about the **main effects** i.e A factor level effects or the B factor level effects.

   (a) The hypothesis about A factor level effects are

$$\begin{aligned} H_0 &: \tau_1 = \tau_2 = ... = \tau_a = 0 \\ H_1 &: \text{at least one of the } \tau'_i s \neq 0. \end{aligned}$$

   If we reject H_0 we compare the A factor level means.

   (b) The hypothesis about B factor level effects are

$$\begin{aligned} H_0 &: \beta_1 = \beta_2 = ... = \beta_b = 0 \\ H_1 &: \text{at least one of the } \beta'_j s \neq 0. \end{aligned}$$

   If we reject H_0 we compare the B factor level means.

4. The hypothesis about the blocking effects are

$$\begin{aligned} H_0 &: \rho_1 = \rho_2 = ... = \rho_k = 0 \\ H_1 &: \text{at least one of the } \rho'_k s \neq 0. \end{aligned}$$

   If we reject H_0 we coclude that blocking was effective.

Sum of Squares

The total variability in the observations ($Y'_{ijk}s$) is measured using the total sum of squares and is given by:

$$SSTO = \sum_{i=1}^{a}\sum_{j=1}\sum_{k=1}^{r}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}Y_{ijk}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad abr-1 \quad df$$

The total sum of squares can be decomposed into three separate sources of variability i.e.

1. the variability that is due to the treatments (factor level combination) (**treatment sum of squares (SST)**)
2. the variability that is due to the block effects (**block sum of squares (SSBlk)**) and
3. the variability in the responses that is due to the random errors (**error sum of squares (SSE)**)

SST, $SSBlk$, and SSE are obtained by performing a two way ANOVA of the data with the factor level combinations A_iB_j as the treatments. The formula for SST is given by:

$$SST = \sum_{i=1}^{a}\sum_{j=1}\sum_{k=1}^{r}(Y_{ij.} - \bar{Y_{...}})^2 = \frac{1}{r}\sum_{i=1}^{a}\sum_{j=1}^{b}Y_{ij.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad ab-1 \quad df$$

The formula for $SSBlk$ is given by:

$$SSBlk = \frac{1}{ab}\sum_{k=1}^{r}Y_{..k}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad abr-1 \quad df$$

The error sum of squares is obtained as follows:

$$SSE = SSTO - SST - SSBlk \quad \text{with} \quad (ab-1)(r-1) \quad df$$

The variability in the observations that is due to the treatments effects is attributed to the main effects and the interaction of the main effects i.e

- A factor level effects;
- B factor level effects and
- $A \times B$ interaction effects.

This follows that we can partition the **treatment sum of squares** into

- the variation due to the **factor A effects (SSA)**
- the variation due to the **factor B effects (SSB)**
- the variation due to the **factor $A \times B$ interaction effects (SSAB)**

The computational formulae for SSA and SSB are as follows:

$$\begin{aligned} SSA &= \frac{1}{br}\sum_{i=1}^{a} Y_{i..}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad a-1 \quad df \\ SSB &= \frac{1}{ar}\sum_{j=1}^{b} Y_{.j.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad b-1 \quad df \\ SSAB &= SST - SSA - SSB \quad \text{with} \quad (a-1)(b-1) \quad df \end{aligned}$$

The mean sum of squares are obtained by dividing the sums of squares by the corresponding degrees of freedon and they are used for testing the hypothesis about the treatments.

Table 6.4: ANOVA table for a Two-factor experiment in a RBD

Source	**df**	**SS**	**MS**	**F**
Treatment	$ab-1$	SST	MST	MST/MSE
Main Effects				
A	$a-1$	SSA	MSA	MSA/MSE
B	$b-1$	SSB	MSB	MSB/MSE
Interactions				
AB	$(a-1)(b-1)$	$SSAB$	$MSAB$	$MSAB/MSE$
Blocks	$(r-1)$	$SSBlk$	$MSBlk$	$MSBlk/MSE$
Error	$(ab-1)(r-1)$	SSE	MSE	
Total	$abr-1$	$SSTO$		

Pairwise Comparisons of the treatments

They are done only if the ANOVA test under the fixed effects model concludes that some treatment means are different. In the presence of $A \times B$ interaction effects the appropriate pairwise comparisons are the A factor level means at the j^{th} level of factor B or vise-vesa. For the A factor level means at the j^{th} level of factor B the hypothesis are:

$$\begin{aligned} H_0 &: \tau_{ij.} = \tau_{i'j.} \\ H_1 &: \tau_{ij.} \neq \tau_{i'j.} \quad \forall i \neq i' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{(ab-1)(r-1)}\sqrt{2MSE/r}$$

Thus the $\tau_{ij.}$ and $\tau_{i'j.}$ are significantly different if

$$|\bar{Y}_{ij.} - \bar{Y}_{i'j.}| > LSD$$

Similarly, for the B factor level means at the i^{th} level of factor A the hypothesis are:

$$\begin{aligned} H_0 &: \tau_{ij.} = \tau_{ij'} \\ H_1 &: \tau_{ij} \neq \tau_{ij'.} \quad \forall j \neq j' \end{aligned}$$

The least significant difference for comparing the means is as above and the means are significantly different if:

$$|\bar{Y}_{ij.} - \bar{Y}_{ij'.}| > LSD$$

If we conclude that $A \times B$ interactions are absent and some A and/or B factor level means are not equal then we can do pairwise comparisons of the A factor level means and/or the B factor level means. The hypothesis to be tested for factor A are:

$$\begin{aligned} H_0 &: \tau_{i..} = \tau_{i'..} \\ H_1 &: \tau_{i..} \neq \tau_{i'..} \quad \forall i \neq i' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{(ab-1)(r-1)}\sqrt{2MSE/br}$$

Thus the τ_{ij} and $\tau_{i'j}$ are significantly different if

$$|\bar{Y}_{i..} - \bar{Y}_{i'..}| > LSD$$

Similarly, for the B factor level means the hypothesis are:

$$\begin{aligned} H_0 &: \tau_{.j.} = \tau_{.j'.} \\ H_1 &: \tau_{.j.} \neq \tau_{.j'.} \quad \forall j \neq j' \end{aligned}$$

The least significant difference for comparing the means is given by:

$$LSD = t^{\frac{\alpha}{2}}_{(ab-1)(r-1)}\sqrt{2MSE/ar}$$

The least significant difference for comparing the means is as above and the means are significantly different if:

$$|\bar{Y}_{.j.} - \bar{Y}_{.j'.}| > LSD$$

Example 6.2.2 *Consider a paper manufacturer who is interested studying the effect of four different cooking temperatures for three different pulp mixtures on the tensile strength of a paper. The experimenter has decided to run three replicates for each treatment combination. However the plant can only make 12 runs a day so the experimenter decided to run one replicate on each of the days and consider the days as blocks. The data is as shown in table 6.2.2*

	Block		
Treatment (A,B)	Day 1	Day 2	Day 3
(200,1)	5.2	5.9	6.3
(200,2)	7.4	7.0	7.6
(200,3)	6.3	6.7	6.1
(225,1)	7.1	7.4	7.5
(225.2)	7.4	7.3	7.1
(225,3)	7.3	7.5	7.2
(250,1)	7.6	7.2	7.4
(250,2)	7.6	7.5	7.8
(250,3)	7.2	7.3	7.0
(275,1)	7.2	7.5	7.2
(275,2)	7.4	7.0	6.9
(275,3)	6.8	6.6	6.4

6.3 2^k Factorial Designs

These are experimental designs in which the experimental plan calls for the study of the effect on a response of k factors, each at two levels. The levels are often denoted as 'high' and 'low'or + and - respectively. The complete factorial design requires that each level of every factor occur with each level of every other factor, giving a total of 2^k treatment combinations. In this chapter, the letters $a, b, c, ..$ will be used to denote higher levels of factors $A, B, C, ..$ and a (1) will be used to denote the lower levels of each factor.In the presence of other letters we omit the symbol (1). The next sections will look at the special methods of analysing 2^2 and 2^3 factorial designs in a CRD. We will assume that the factor effects are fixed and that the errors in the responses are independent and normally distributed with mean 0 and variance σ^2.

6.3.1 2^2 Factorial experiment

Consider a 2^2 factorial experiment in which there are r experimental observations per treatment combination. The full model for a 2^2 factorial design is given by:

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

where

- μ is the overall population mean;
- τ_i is the effect of the i^{th} level of factor A;
- β_j is the effect of the j^{th} level of factor B;
- $(\tau\beta)_{ij}$ is the interaction effect of the i^{th} level of factor A and the j^{th} level of factor B;
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2.

Using the notation defined in section 6.3 above we note that

- a represents the total of r observations taken at the *High* level of A and *Low* level of B;
- b represents the total of r observations taken at the *High* level of B and *Low* level of A;
- ab represents the total of r observations taken at the *High* level of A and *High* level of B and
- (1) represents the total of r observations taken at the *Low* level of A and *Low* level of B;

Table 6.5 gives a two-way table of these total yields

Table 6.5: 2^2 Factorial experiment

		Factor B		
		$Low(-)$	$High(+)$	Total
Factor A	$Low(-)$	(1)=$Y_{11.}$	b=$Y_{12.}$	$(1) + b = Y_{1..}$
	$High(+)$	a=$Y_{21.}$	ab=$Y_{22.}$	**a**+ab=$Y_{2..}$
	Total	(1)+$aY_{.1.}$	**b**+ab=$Y_{.2.}$	(1)+**a**+ab=$Y_{...}$

The **main effect** of a factor is defined as the change in the mean response due to the change in the level of the factor. For example, the main effect of factor A from table 6.5 above is:

$$\begin{aligned} A = \frac{1}{2}\bar{Y}_{2..} - \frac{1}{2}\bar{Y}_{1..} &= \frac{1}{2r}(a + ab) - \frac{1}{2r}((1) + b) \\ &= \frac{1}{2r}[a + ab - b - (1)] \end{aligned} \tag{6.2}$$

The main effect of factor B is:

$$\begin{aligned} B = \bar{Y}_{.2.} - \bar{Y}_{.1.} &= \frac{1}{2r}(b + ab) - \frac{1}{2r}((1) + a) \\ &= \frac{1}{2r}[b + ab - a - (1)] \end{aligned} \tag{6.3}$$

The AB interaction is the difference between the diagonal means in table 6.5. That is

$$\begin{aligned} AB = \frac{1}{2}(\bar{Y}_{22.} + \bar{Y}_{11.}) - \frac{1}{2}(\bar{Y}_{21.} + \bar{Y}_{12.}) &= \frac{1}{2r}(ab + (1)) - \frac{1}{2r}(a + b) \\ &= \frac{1}{2r}[a + (1) - a - b] \end{aligned} \tag{6.4}$$

The quantities in the square brackets [.] of equations 6.2, 6.3 and 6.4 are called contrasts and contrasts are always orthogonal. We define the contrasts among the treatment totals as follows:

$$\begin{aligned} A \quad contrast &= a + ab - b - (1) \\ B \quad contrast &= b + ab - a - (1) \end{aligned}$$

The AB interaction is the difference between the diagonal means in table 6.5. That is

$$AB \quad contrast = a + (1) - a - b$$

The sums of squares for each contrast is found using the following formula

$$SSFactor = \frac{(Contrast_{factor})^2}{r\sum(Contrast_{factor}Coefficients)^2} \tag{6.5}$$

where $\sum(Contrast_{factor}Coefficients)^2$ is the sum of the squares of the coefficients of the terms in the contrast.

Using the formula in 6.5 the **Sum of Squares for A (SSA)** is given by:

$$SSA = \frac{1}{4r}(a + ab - b - (1))^2$$

Sum of Squares for B (SSB) is given by:

$$SSB = \frac{1}{4r}(b + ab - a - (1))^2$$

and the **Sum of Squares for AB (SSAB)** is given by:

$$SSAB = \frac{1}{4r}(ab + (1) - a - b^2$$

The **total sum of squares (SSTO)** is computed using the usual formula i.e:

$$SSTO = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}Y_{ijk}^2 - \frac{1}{4r}Y_{...}^2 \quad \text{with} \quad abr - 1 \quad df$$

The **error sum of squares (SSE)** is obtained by subtraction as follows:

$$SSE = SSTO - SSA - SSB - SSAB \quad \text{with} \quad 2^k(r-1) \;\; df$$

ANOVA table for a 2^2 Factorial experiment

Table 6.6: ANOVA table for a 2^2 factorial experiment

Source	**df**	**SS**	**MS**	**F**
Main Effect				
A	1	SSA	MSA	MSA/MSE
B	1	SSB	MSB	MSB/MSE
Interactions				
AB	1	$SSAB$	$MSAB$	$MSAB/MSE$
Error	$2^k(r-1)$	SSE	MSE	
Total	$r2^k-1$	$SSTO$		

Example 6.3.1 *Consider an investigation into the effect of concentration of reactant and the presence of a catalyst on the reaction time of a chemical process. Let the reactant concentration be factor A with two levels of interest 10% and 20% and Catalyst be factor B with the high level denoting the presence of the catalyst and the low level denoting its absence. Assuming three replicates, the data from the experiment are displayed in table 6.3.1*

Treatment	Replicate		
combination	I	II	II
(1)	28	25	27
a	36	32	32
b	18	19	23
ab	31	30	29

1. *Calculate the sums of squares for the data.*
2. *Which effects are significantly different from zero.*

6.4 2^3 Factorial Design

Suppose that three factors, A, B and C each at two levels are under study. The design is called 2^3 factorial and there are eight treatment combinations. The full model for a 2^3 factorial design is given by:

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \rho_k + (\tau\beta)_{ij} + (\tau\rho)_{ik} + (\beta\rho)_{jk} + (\tau\beta\rho)_{ijk} + \epsilon_{ijkl}$$

where

- μ is the overall population mean;
- τ_i is the effect of the i^{th} level of factor A;
- β_j is the effect of the j^{th} level of factor B;
- ρ_k is the effect of the k^{th} level of factor C;
- $(\tau\beta)_{ij}$ is the interaction effect of the i^{th} level of factor A and the j^{th} level of factor B;
- $(\tau\rho)_{ik}$ is the interaction effect of the i^{th} level of factor A and the k^{th} level of factor c;
- $(\beta\rho)_{jk}$ is the interaction effect of the j^{th} level of factor B and the k^{th} level of factor C;
- $(\tau\beta\rho)_{ijk}$ is the interaction effect of the i^{th} level of factor A, the j^{th} level of factor B and the k^{th} level of factor C;
- the $\epsilon'_{ijk}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2.

In computing the sums of squares for the main effects it is convinient to present the total yields of the treatment combinations along with the appropriate algebraic signs for each contrast as in table 6.7 The treatment combinations and the appropriate

Table 6.7: Signs for contrasts in a 2^3 Factorial Experiment

Treatment	**Factorial effect**						
combination	A	B	C	AB	AC	BC	ABC
(1)	-	-	-	+	+	+	-
a	+	-	-	-	-	+	+
b	-	+	-	-	+	-	+
c	-	-	+	+	-	-	+
ab	+	+	-	+	-	-	-
ac	+	-	+	-	+	-	-
bc	-	+	+	-	-	+	-
abc	+	+	+	+	+	+	+

algebraic signs for each contrast in table 6.7 are used in computing the sums of squares for the main effects and interaction effects.

For example, the main effect of factor A from table 6.7 above is:

$$A = \frac{1}{4r}[-(1) + a - b - c + ab + ac - bc + abc]$$

rearrangig we obtain

$$A = \frac{1}{4r}[a + ab + ac + abc - (1) - b - c - bc]$$

We define the contrasts among the treatment totals as follows:

$$\begin{aligned}
Acontrast &= a + ab + ac + abc - (1) - b - c - bc \\
Bcontrast &= b + ab + bc + abc - (1) - a - c - ac \\
Ccontrast &= c + ac + bc + abc - (1) - a - b - ab \\
ABcontrast &= abc + ab + c + (1) - a - b - ac - bc \\
ACcontrast &= abc + ac + b + (1) - a - c - ab - bc \\
BCcontrast &= abc + bc + a + (1) - b - c - ab - ac \\
ABCcontrast &= abc + a + b + c - ab - ac - bc - (1)
\end{aligned}$$

The sums of squares for each contrast is found using the following formula

$$SSFactor = \frac{(Contrast_{factor})^2}{r\sum(Contrast_{factor}Coefficients)^2} \tag{6.6}$$

where $\sum(Contrast_{factor}Coefficients)^2$ is the sum of the squares of the coefficients of the terms in the contrast.

Using the formula 6.6 the **Sum of Squares** are given by:

$$\begin{aligned}
SSA &= \frac{1}{8r}(a + ab + ac + abc - (1) - b - c - bc)^2 \\
SSB &= \frac{1}{8r}(b + ab + bc + abc - (1) - a - c - ac)^2 \\
SSC &= \frac{1}{8r}(c + ac + bc + abc - (1) - a - b - ab)^2 \\
SSAB &= \frac{1}{8r}(abc + ab + c + (1) - a - b - ac - bc)^2 \\
SSAC &= \frac{1}{8r}(abc + ac + b + (1) - a - c - ab - bc)^2 \\
SSBC &= \frac{1}{8r}(abc + bc + a + (1) - b - c - ab - ac)^2 \\
SSABC &= \frac{1}{8r}(abc + a + b + c - ab - ac - bc - (1)^2
\end{aligned}$$

The **error sum of squares (SSE)** is obtained by subtraction as usual.

$SSE = SSTO - SSA - SSB - SSC - SSAB - SSAC - SSBC - SSABC$ with $r2^k - 1)$ df (6.7

ANOVA table for a 2^3 Factorial experiment

Table 6.8: ANOVA table for a 2^3 factorial experiment

Source	**df**	**SS**	**MS**	**F**
Main Effect				
A	1	SSA	MSA	MSA/MSE
B	1	SSB	MSB	MSB/MSE
C	1	SSC	MSC	MSC/MSE
Interactions				
AB	1	$SSAB$	$MSAB$	$MSAB/MSE$
AC	1	$SSAC$	$MSAC$	$MSAC/MSE$
BC	1	$SSBC$	$MSBC$	$MSBC/MSE$
ABC	1	$SSABC$	$MSABC$	$MSABC/MSE$
Error	$2^k(r-1)$	SSE	MSE	
Total	$r2^k-1$	$SSTO$		

Example 6.4.1 *An engineer is trying to improve the life of a cutting tool. He has run 2^3 factorial experiment using cutting speed A, metal hardness B and cutting angle C. He replicated the experiment 2 times and obtained the following data displayed in table 6.4.1*

Treatment combination	Replicate I	II
(1)	284	248
a	450	410
b	349	353
c	455	438
ab	502	522
ac	398	385
bc	545	560
abc	403	408

1. *Calculate the sums of squares for the data.*
2. *Which effects are significantly different from zero.*
3. *Advise on the best factor level combination in improving the the life tool.*

Chapter 7

The Analysis of Covariance (ANCOVA)

7.1 Introduction

ANCOVA is simply a combination of the analysis of variance and the regression analysis methods, that is, when we compare treatment means that incorporate information on a quantitative variable x. In this chapter we present an analysis of covariance of a completely randomised design with one variate.

7.2 A CRD with One Covariate

Definition 7.2.1 ***Covariates or concomitant variables*** *are characteristic(s) of experimental units which may affect the response variable and can be measured before the treatments are imposed on the experimental units*

During the analysis covariates are used to adjust the observed responses for the effects of heterogeneity of the experimental units i.e the responses Y are adjusted for the values of the covariate X during the analysis. We can write the model to be fitted to the data as follows:

$$Y_{ij} = \mu + \tau_i + \beta(X_{ij} - \bar{X}_{..}) + \epsilon_{ij}$$

where β is slope of the regression Y on X. The error ϵ_{ij} in equation 7.1 has been reduced because part of it is now being accounted for by $\beta(X_{ij} - \bar{X}_{..})$.

NOTE: Adding covariate(s) to any ANOVA model has the effect of reducing the experimental error and this makes the ANOVA tests more sensitive to treatment differences.

7.3 Analysis

The model for ANCOVA for a CRD with one covariate is as in equation 7.1 where

- Y_{ij} is the j^{th} response to the i^{th} treatment
- μ is the overall population mean;
- τ_i is the i^{th} treatment effect;
- X_{ij} is the value of the covariate corresponding to Y_{ij};
- β is the regression coefficient(slope);
- $(\tau\beta)_{ij}$ is the interaction effect of the i^{th} level of factor A and the j^{th} level of factor B;
- the $\epsilon'_{ij}s$ are random errors which are assumed to be independent and normally distributed with mean 0 and variance σ^2.
- $i = 1, 2, ..., t \quad \text{and} \quad j = 1, 2, ..., r$

To adjust the estimates of the model parameters for the effect of covariate we note that:

1. $$\bar{Y}_{..} = \mu + \ddot{\epsilon}_{..}$$

   since $\sum_{i=1}^{t} \tau_i = \sum_{i=1}^{t} \sum_{j=1}^{r} (X_{ij} - \bar{X}_{..}) = 0$. If $\ddot{\epsilon}_{..} \approx 0$ then the estimate of μ is $Y_{..}$

2. $$\bar{Y}_{i.} = \mu + \tau_i + \beta(\bar{X}_{i.} - \bar{X}_{..}) + \bar{\epsilon}_{i.}$$

   and

   $$Y_{ij} - \bar{Y}_{i.} = \beta(X_{ij} - \bar{X}_{i.}) + \epsilon_{ij} + \bar{\epsilon}_{i.}$$

   Let $Y^*_{ij} = Y_{ij} - \bar{Y}_{i.}, X^*_{ij} = X_{ij} - \bar{X}_{i.} \quad \text{and} \quad \epsilon^*_{ij} = \epsilon_{ij} + \bar{\epsilon}_{i.}$, then

   $$Y^*_{ij} = \beta X^*_{ij} \epsilon^*_{ij}$$

   is a simple linear regression model and the least squares estimate of β is given by

   $$\hat{\beta}^* = \frac{\sum_{i=1}^{t} \sum_{j=1}^{r} Y^*_{ij} X^*_{ij}}{\sum_{i=1}^{t} \sum_{j=1}^{r} X^{*2}_{ij}} = \frac{E_{xy}}{E_{xx}} \tag{7.1}$$

   NOTE:β^* is free of the treatment effects.

3. If

$$\bar{Y}_{i.} = \mu + \tau_i + \beta(\bar{X}_{i.} - \bar{X}_{..}) + \bar{\epsilon}_{i.}$$

is evaluated at $\mu = \bar{Y}_{..}$, $\beta = \hat{\beta}^*$ and $\bar{\epsilon}_{i.} \approx= 0$ then

$$\hat{\tau}_i^* = \bar{Y}_{i.} - \bar{Y}_{..} - \hat{\beta}^*(\bar{X}_{i.} - \bar{X}_{..})$$

is the adjusted estimate of τ_i

4.

$$\hat{\mu}_{i.}^* = \bar{Y}_{i.} - \hat{\beta}^*(\bar{X}_{i.} - \bar{X}_{..})$$

is the adjusted estimate of the i^{th} treatment mean.

ANCOVA

E_{xx} and E_{xy} have been defined. Let $E_{yy} = \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^{*2}$ and the total sum of squares be given by the formula

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij} - \bar{Y}_{..})^2 = \sum_{i=1}^{t}\sum_{j=1}^{b} Y_{ij}^2 - \frac{1}{n}Y_{..}^2 \quad \text{with} \quad n-1 \quad df \tag{7.2}$$

ANCOVA adjust the usual ANOVA for the effects as follows

1. The total sum of squares (SSTO) includes the effects of the covariate

$$SSTO_{adj} = SSTO - SSReg = SSTO - \hat{\beta}SS_{xy} = SSTO - \frac{SS_{xy}^2}{SS_{xx}} \text{ with } rt-2 \ df$$

$SSTO_{adj}$ is the total sum of squares that includes only the treatment effects and the random errors.

2.

$$SSE_{adj} = SSE - SSReg^* = SSE - \hat{\beta}^* E_{xy} = SSE - \frac{E_{xy}^2}{E_{xx}} \text{ with } t(r-1)-1 \ df$$

SSE is the usual ANOVA error sum of squares, $SSReg^*$ is the regression sum of squares from regressing $Y_{ij}^* = Y_{ij} - \bar{Y}_{i.}$ on $X_{ij}^* = X_{ij} - \bar{X}_{i.}$ SSE_{adj} is the adjusted error sum of squares for the effects of the covariate.

3.

$$SST_{adj} = SSTO_{adj} - SSE_{adj} \text{ with } (t-1)-1 \ df$$

The following formulae are used to compute $SS_{xy}, SS_{yy}, SS_{xx}, E_{xy}$ and E_{xx}

$$\begin{aligned}
SS_{xy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}X_{ij} - \frac{1}{rt}Y_{\cdot\cdot}X_{\cdot\cdot} \\
SS_{yy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - \frac{1}{rt}Y_{\cdot\cdot}^2 \\
SS_{xx} &= \sum_{i=1}^{t}\sum_{j=1}^{r} X_{ij}^2 - \frac{1}{rt}X_{\cdot\cdot}^2 \\
E_{xx} &= \sum_{i=1}^{t}\sum_{j=1}^{r} X_{ij}^2 - \frac{1}{r}\sum_{i=1}^{t} X_{i\cdot}^2 \\
E_{xy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - \frac{1}{r}\sum_{i=1}^{t} Y_{i\cdot}X_{i\cdot} \\
E_{yy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - \frac{1}{r}\sum_{i=1}^{t} Y_{i\cdot}^2
\end{aligned}$$

Table 7.1: ANCOVA table for a CRD with one covariate

Source	**df**	**SS**	**MS**	**F**
Regression	1	$SSReg^*$	$SSReg^*$	$SSReg^*/MSE_{adj}$
Treatment	$t-1$	SST_{adj}	SST_{adj}	SST_{adj}/MSE_{adj}
Error	$t(r-1)-1$	SSE_{adj}	MSE_{adj}	
Totals	$rt-2$	$SSTO_{adj}$		

NOTE: $SSTO_{adj} = SST_{adj} + SSE_{adj}$ and not $SSTO_{adj} = SST_{adj} + SSE_{adj} + SSReg^*$

7.3.1 Hypothesis Testing

The objective is to test the null hypothesis of no difference among the treatment means. This is equivelent to testing

$$\begin{aligned}
H_0 &: \tau_{1\cdot} = \tau_{2\cdot} = ... = \tau_{t\cdot} = 0 \\
H_1 &: \text{at least one } \tau_{i\cdot} \neq 0. \text{ (at least one of the treatment means differs from the rest)}
\end{aligned}$$

and for the Covariate effects we have

$$\begin{aligned}
H_0 &: \beta = 0 \\
H_1 &: \beta \neq 0.
\end{aligned}$$

The test statistic for the treatment hypothesis is given by:

$$F = \frac{MST_{adj}}{MSE_{adj}} \tag{7.3}$$

F has an F distribution with $t-1$ degrees of freedom on the **numerator** and $t(r-1)-1$ degrees of freedom on the **denominator**. We conclude that treatment effects are significant at the α level of significance if $F_{cal} > F^{\alpha}_{t-1,t(r-1)-1}$. The test statistic for the linear relationship is given by:

$$F = \frac{SSReg^*}{MSE_{adj}} \tag{7.4}$$

F has an F distribution with 1 degree of freedom on the **numerator** and $t(r-1)-1$ degrees of freedom on the **denominator**. We conclude that there is a linear relationship between the response and the covariate at the α level of significance if $F_{cal} > F^{\alpha}_{t-1,t(r-1)-1}$.

Pairwise Comparisons of the treatments

In this case the hypothesis to be tested are:

$$\begin{aligned} H_0 &: \mu_{i.} = \mu_{i'.} \\ H_1 &: \mu_{i.} \neq \mu_{i'.} \quad \forall i \neq i' \end{aligned}$$

The least significant difference is given by:

$$LSD = t^{\frac{\alpha}{2}}_{t(r-1)-1} se(\hat{\mu}^*_{i.} - \hat{\mu}^*_{i'.})$$

where $se(\bar{\mu}^*_{i.} - \mu^*_{i'.}) = \sqrt{MSE_{adj}\left(\frac{2}{r} + \frac{(\bar{X}_{i.} - \bar{X}_{i'.})^2}{SS_{xx}}\right)}$ The means $\mu_{i.}$ and $\mu_{i'.}$ are declared not equal if

$$|\mu_{i.} - \mu_{i'.}| > LSD.$$

Example 7.3.1 *Three different types of hand trucks have been developed and a soft drink distributor wants to study the effectiveness of using these trucks in delivery. An experiment was carried out in the company's methods engneering laboratory. The variable of interest is the delivery time in minutes (y); however, delivery time is also strongly related to the case volume delivered (x). Each hand truck is used five times and the data that follow are obtained. Analyse this data and draw conclusions.*

Hand truck type					
1		2		3	
y	x	y	x	y	x
36	20	40	22	35	21
41	25	48	28	37	23
39	24	39	22	42	26
42	25	45	30	34	21
49	32	44	28	32	15

Chapter 8

QUESTIONS

Chapter 2: CRD

1. The yields of maize, in tonnes were recorded for 4 different varieties of maize, P, Q, R and S. The experiment was done in a controlled greenhouse enviroment, each variety was randomly assigned to 8 of the 32 plots available for the experiment. The yields are as in the following table:

Variety	**Yield**							
P	2.5	3.6	2.8	2.7	3.1	3.4	2.9	3.5
Q	3.6	3.9	4.1	4.3	2.9	3.5	3.8	3.7
R	4.3	4.4	4.5	4.1	3.5	3.4	3.2	4.6
S	2.8	2.9	3.1	2.4	3.2	2.5	3.6	2.7

   (a) Write an appropriate statistical model.

   (b) Perform an analysis of variance on these data and draw conclusions.

   (c) Use Fisher's LSD procedure to run all pairwise comparisons.

   (d) Obtain a computer solution for the data. Compare your results to those obtained in (ii).

2. A clinical phychologist wished to compare three methods for reducing weight in obese patients. Eleven female patients who had equal body mass indeces (BMI) were used in the experiment. Four were selected at random from among the 11 patients and were assigned to Method 1. Four of the remaining 7 patients were selected at random and treated with method 2. The remaining 3 patients were treated with method 3. All treatments were continued for six months. Each patient was measured his/her final weight at the end of six months. The results are shown in the following table.

Method	Weight			
1	80	92	87	83
2	70	81	78	74
3	63	76	70	

(a) Analyse the data and draw appropriate conclusions.

(b) Is it necessary to perform pairwise comparisons of the mean weight for the three methods. Give reasons for your answer.

Chapter 3: RBD

1. An experiment is conducted in which four treatments (A,B,C and D) are to be compared in three blocks (1,2 and 3). Five experimental units are available for each block. The labels for Block 1 are S_1, S_2, S_3, S_4, for Block 2 are S_5, S_6, S_7, S_8, and $S_9, S_{10}, S_{11}, S_{12}$. Use the following set of random numbers to randomise the experiment.

   Set 1={10,9,4,5,6,3,1,8,12,2,7,11}; Set 2={6,3,2,11,5,8,12,9,4,7,1,10};
   Set 3={2,11,5,1,9,4,10,6,8,12,3,7}

2. An experiment was carried out in which four treatments are to be compared in five blocks. The following results were obtained. Perform the analysis of

Treatment	Block				
1	12.8	10.6	11.7	10.7	11.0
2	11.7	14.2	11.8	9.9	13.8
3	11.5	14.7	13.6	10.7	15.9
4	12.6	16.5	15.4	9.6	17.1

   variance, separating out the treatment, block, and error sums of squares. Use $\alpha = 0.05$ level of significance to test the hypothesis that there is no difference between the treatment means.

Chapter 4: Balanced Incomplete Block Designs

1. An engineer is studying the mileage performance characteristics of five types of gasoline additives. In the road test he wishes to use cars as blocks; however, because of a time constraint, he must use an incomplete block design. He runs

the balanced design with the five blocks that follow. Analyse the data and draw conclusions.

	Car				
Additive	1	2	3	4	5
1	...	17	14	13	12
2	14	14	...	13	10
3	12	...	13	12	9
4	13	11	11	12	...
5	11	12	10	...	8

Chapter 5: Latin Square and Crossover Designs

1. An industrial engineer is investigating the effect of four assembly methods (A,B,C,D) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin Square Design shown below.

	Operator			
Order of Assembly	1	2	3	4
1	10 (C)	14(D)	7(A)	8(B)
2	7(B)	18(C)	11(D)	8(A)
3	5(A)	10(B)	11(C)	9(D)
4	10(D)	10(A)	12(B)	14(C)

   (a) Analyse the data and draw appropriate conclusions.

   (b) Suppose that, the observation from operator 4 on order of assembly 4 is missing. Estimate the missing value and perform the analysis using this value.

2. The effects of two drugs (A,B) on the duration of sleep were studied using a group of 8 patients. Four patients were randomly assigned to drug A during period 1 and the other four to drug B. During period 2, the two groups of patients switched drugs. The sleep duration data is as follows.

   (a) Identify the design.

	Patient							
Period	1	2	3	4	5	6	7	8
1	8.6(A)	7.1(B)	8.3(A)	7.3(B)	7.9(A)	7.5(B)	6.3(A)	6.8(B)
2	8.0(B)	7.5(A)	7.4(B)	8.4(A)	7.3(B)	7.6(A)	6.4(B)	7.5(A)

(b) Give a model for this design. State all the relevant assumptions of the model.

(c) Analyse the data and draw conclusions.

Chapter 6: Factorial Designs

1. An experiment is conducted to study the influence of operating temperature and three types of face-plate glass in the light output of TV tube. The following data are collected. Assume that both factors are fixed. Analyse the data and draw conclusions.

	Temperature		
Glass Type	100	125	150
1	580	1090	1392
	570	1085	1386
2	530	1035	1312
	579	1000	1299
3	546	1045	867
	575	1053	904

2. The following data were obtained from a 2^3 factorial experiment replicated three times: Evaluate the sums of squares for all factorial effects by the contrast

Treatment	**Replicate 1**	**Replicate 2**	**Replicate 3**
(1)	12	19	10
a	15	20	16
b	24	16	17
ab	23	17	27
c	17	25	21
ac	16	19	19
bc	24	23	29
abc	28	25	20

method.

Chapter 7: Analysis of Covariance

1. Four different formulations of an industrial glue are being tested. The tensile strength of the glue is also related to the thickness. Five observations on strength (y) and thickness (x) are obtained for each formulation. The data are shown in the following table. Analyse these data and draw appropriate conclusions.

	Glue Fomulation							
	1		**2**		**3**		**4**	
	Y	**X**	**Y**	**X**	**Y**	**X**	**Y**	**X**
1	46.5	13	48.7	12	46.3	15	44.7	16
2	45.9	14	49.0	10	47.1	14	43.0	15
3	49.8	12	50.1	11	48.9	11	51.0	10
4	46.1	12	48.5	12	48.2	11	48.1	12
5	44.3	14	45.2	14	50.3	10	48.6	11

Chapter 9

Formulae

The Principles of Experimental Design

1.

Completely Randomised Designs

1.

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{r}(Y_{ij} - \bar{Y}_{\cdot\cdot})^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}Y_{ij}^2 - \frac{1}{tr}Y_{\cdot\cdot}^2 \quad \text{with} \quad tr-1 \quad df \qquad (9.1)$$

2.

$$\begin{aligned} SST &= \sum_{i=1}^{t}\frac{1}{r}Y_{i\cdot}^2 - \frac{1}{tr}Y_{\cdot\cdot}^2 \quad \text{with} \quad t-1 \quad df \\ SSE &= SSTO - SST \quad \text{with} \quad t(r-1) \quad df \end{aligned}$$

$$LSD = t_{t(r-1)}^{\frac{\alpha}{2}}\sqrt{2MSE/r}$$

3. Missing Observations

4.

$$\begin{aligned} SSTO &= \sum_{i=1}^{t}\sum_{j=1}^{r_i}Y_{ij}^2 - \frac{1}{n}Y_{\cdot\cdot}^2 \quad \text{with} \quad n-1 \quad df \\ SST &= \sum_{i=1}^{t}\frac{1}{r_i}Y_{i\cdot}^2 - \frac{1}{n}Y_{\cdot\cdot}^2 \quad \text{with} \quad t-1 \quad df \qquad (9.2) \end{aligned}$$

$$SSE = SSTO - SST \quad \text{with} \quad n-t \quad df$$

$$LSD = t_{n-t}^{\frac{\alpha}{2}} \sqrt{MSE(\frac{1}{r_i} + \frac{1}{r_{i'}})}$$

Randomised Block Design

1.

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij} - \bar{Y_{..}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{b}Y_{ij}^2 - \frac{1}{n}Y_{..}^2 \quad \text{with} \quad n-1 \quad df \qquad (9.3)$$

$$\begin{aligned} SST &= \sum_{i=1}^{t}\frac{1}{b}Y_{i.}^2 - \frac{1}{n}Y_{..}^2 \quad \text{with} \quad t-1 \quad df \\ SSB &= \sum_{j=1}^{b}\frac{1}{t}Y_{.j}^2 - \frac{1}{n}Y_{..}^2 \quad \text{with} \quad b-1 \quad df \\ SSE &= SSTO - SST - SSB \quad \text{with} \quad n-t-b+1 = (t-1)(b-1) \quad df \end{aligned}$$

$$LSD = t_{(t-1)(b-1)}^{\frac{\alpha}{2}} \sqrt{2MSE/b}$$

Block $\times$ Treatment Interaction Effects

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{r}Y_{ijk}^2 - \frac{1}{tbr}Y_{...}^2 \quad \text{with} \quad tbr-1 \quad df \qquad (9.4)$$

$$SSModel = \sum_{i=1}^{t}\sum_{j=1}^{b}(Y_{ij.} - \bar{Y_{...}})^2 = \frac{1}{r}\sum_{i=1}^{t}\sum_{j=1}^{b}Y_{ij.}^2 - \frac{1}{tbr}Y_{...}^2 \quad \text{with} \quad tb-1 \quad df$$

$$SSE = SSTO - SSModel \quad \text{with} \quad tb(r-1) \quad df$$

$$\begin{aligned} SST &= \frac{1}{br}\sum_{i=1}^{t}(\bar{Y_{i..}^2} - \frac{1}{tbr}{Y_{...}}^2 \quad \text{with} \quad t-1 \quad df \\ SSB &= \frac{1}{tr}\sum_{j=1}^{b}(Y_{.j.}^2 - \frac{1}{tbr}{Y_{...}}^2 \quad \text{with} \quad b-1 \quad df \\ SSB \times T &= SSModel - SST - SSB \quad \text{with} \quad (t-1)(b-1) \quad df \end{aligned}$$

$$LSD = t^{\frac{\alpha}{2}}_{bt(r-1)}\sqrt{2MSE/br}$$

One missing Value

$$M = \frac{tT + bB - G}{(t-1)(b-1)}$$

Two or more missing Values **Complete (Full) model (model 1)**

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$$

Reduced model (model 2)

$$Y_{ij} = \mu + \beta_j + \epsilon_{ij}$$

$$SSE_R - SSE_F = SST_{adj}$$

$$SSB = SSTO - SST_{adj} - SSE$$

Balanced Incomplete Block Designs

1.

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{b} Y_{ij}^2 - \frac{1}{n}Y_{\cdot\cdot}^2 \quad \text{with} \quad n-1 \;\; df$$

$$SST_{adj} = \frac{t-1}{nk(k-1)}\sum_{i=1}^{t}(kY_{i\cdot} - B_i)^2 \quad \text{with} \quad t-1 \;\; df$$

$$SSB = \sum_{j=1}^{b}\frac{1}{k}Y_{\cdot j}^2 - \frac{1}{n}Y_{\cdot\cdot}^2 \quad \text{with} \quad b-1 \;\; df$$

$$SSE = SSTO - SST_{adj} - SSB \quad \text{with} \quad n-t-b+1 \;\; df$$

$$LSD = t^{\alpha/2}_{n-t-b+1}\sqrt{\frac{2kMSE}{t\lambda}}$$

Latin Square and Crossover Designs

1.

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{t}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{t}Y_{ijk}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t^2 - 1 \quad df$$

$$\begin{aligned}
SST &= \frac{1}{t}\sum_{k=1}^{t}Y_{..k}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSR &= \frac{1}{t}\sum_{i=1}^{t}Y_{i..}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSC &= \frac{1}{t}\sum_{i=1}^{t}Y_{.j.}^2 - \frac{1}{t^2}Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSE &= SSTO - SST - SSR - SSC \quad \text{with} \quad (t-2)(t-1) \quad df
\end{aligned}$$

$$LSD = t_{(t-1)(t-2)}^{\frac{\alpha}{2}}\sqrt{2MSE/t}$$

$$M = \frac{t(T+R+C) - 2G}{(t-1)(t-2)}$$

$$LSD = t_{(t-1)(t-2)}^{\frac{\alpha}{2}}\sqrt{MSE(\frac{2}{t} + \frac{1}{(t-1)(t-2)}}$$

For any other pair of treatments, the LSD is as before

$$LSD = t_{(t-1)(t-2)}^{\frac{\alpha}{2}}\sqrt{2MSE/t}$$

$$SSTO = \sum_{i=1}^{t}\sum_{j=1}^{t}\sum_{m=1}^{r}(Y_{ijkm} - \bar{Y_{....}})^2 = \sum_{i=1}^{t}\sum_{j=1}^{t}\sum_{m=1}^{r}Y_{ijkm}^2 - \frac{1}{t^2r}Y_{....}^2 \quad \text{with} \quad t^2r - 1 \quad df$$

It is possible to partition the total sum of squares into four separate sources of variability. The computational formulae for the sum of squares are given by:

$$
\begin{aligned}
SST &= \frac{1}{tr}\sum_{k=1}^{t} Y_{..k.}^2 - \frac{1}{t^2 r} Y_{....}^2 \quad \text{with} \quad t-1 \quad df \\
SSR &= \frac{1}{tr}\sum_{i=1}^{t} Y_{i...}^2 - \frac{1}{t^2 r} Y_{....}^2 \quad \text{with} \quad t-1 \quad df \\
SSC &= \frac{1}{tr}\sum_{i=1}^{t} Y_{.j..}^2 - \frac{1}{t^2 r} Y_{....}^2 \quad \text{with} \quad t-1 \quad df \\
SSE &= SSTO - SST - SSR - SSC \quad \text{with} \quad t^2 r - 3t + 2 \quad df
\end{aligned}
$$

$$
SSTO = \sum_{i=1}^{t}\sum_{j=1}^{rt}(Y_{ijk} - \bar{Y}_{...})^2 = \sum_{i=1}^{t}\sum_{j=1}^{rt} Y_{ijk}^2 - \frac{1}{t^2 r} Y_{...}^2 \quad \text{with} \quad t^2 r - 1 \quad df \; (9.5)
$$

$$
\begin{aligned}
SST &= \frac{1}{tr}\sum_{k=1}^{t} Y_{..k}^2 - \frac{1}{t^2 r} Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSP &= \frac{1}{tr}\sum_{i=1}^{t} Y_{i..}^2 - \frac{1}{t^2 r} Y_{...}^2 \quad \text{with} \quad t-1 \quad df \\
SSU &= \frac{1}{t}\sum_{i=1}^{t} Y_{.j.}^2 - \frac{1}{t^2 r} Y_{...}^2 \quad \text{with} \quad tr-1 \quad df \\
SSE &= SSTO - SST - SSP - SSU \quad \text{with} \quad (tr-2)(t-1) \quad df
\end{aligned}
$$

Factorial Experiments

1.

$$
TwofactorFactorialDesignSSTO = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ijk} - \bar{Y}_{...})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r} Y_{ijk}^2 - \frac{1}{abr} Y_{...}^2 \quad \text{wit}
$$

$$
SST = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(Y_{ij.} - \bar{Y}_{...})^2 = \frac{1}{r}\sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2 - \frac{1}{abr} Y_{...}^2 \quad \text{with} \quad ab-1 \quad df
$$

$$\begin{aligned} SSA &= \frac{1}{br}\sum_{i=1}^{a} Y_{i..}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad a-1 \quad df \\ SSB &= \frac{1}{ar}\sum_{j=1}^{b} Y_{.j.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad b-1 \quad df \\ SSAB &= SST - SSA - SSB \quad \text{with} \quad (a-1)(b-1) \quad df \end{aligned}$$

In the presence of $A \times B$ interactions we must compare the factor level means at each level of B or vise versa

$$LSD = t_{ab(r-1)}^{\frac{\alpha}{2}}\sqrt{2MSE/r}$$

Factor A

$$LSD = t_{ab(r-1)}^{\frac{\alpha}{2}}\sqrt{2MSE/br}$$

Factor B

$$LSD = t_{ab(r-1)}^{\frac{\alpha}{2}}\sqrt{2MSE/ar}$$

Two factor factorial in RBD

$$SSTO = \sum_{i=1}^{a}\sum_{j=1}^{r}\sum_{k=1}(Y_{ijk} - \bar{Y_{...}})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r} Y_{ijk}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad abr-1 \quad df$$

$$SST = \sum_{i=1}^{a}\sum_{j=1}^{r}\sum_{k=1}(Y_{ij.} - \bar{Y_{...}})^2 = \frac{1}{r}\sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad ab-1 \quad df$$

$$SSBlk = \frac{1}{ab}\sum_{k=1}^{r} Y_{..k}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad abr-1 \quad df$$

$$SSE = SSTO - SST - SSBlk \quad \text{with} \quad (ab-1)(r-1) \quad df$$

$$\begin{aligned} SSA &= \frac{1}{br}\sum_{i=1}^{a} Y_{i..}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad a-1 \quad df \\ SSB &= \frac{1}{ar}\sum_{j=1}^{b} Y_{.j.}^2 - \frac{1}{abr}Y_{...}^2 \quad \text{with} \quad b-1 \quad df \\ SSAB &= SST - SSA - SSB \quad \text{with} \quad (a-1)(b-1) \quad df \end{aligned}$$

In the presence of $A \times B$ interaction effects the appropriate pairwise comparisons are the A factor level means at the j^{th} level of factor B or vise-vesa

$$LSD = t^{\frac{\alpha}{2}}_{(ab-1)(r-1)} \sqrt{2MSE/r}$$

A factor level

$$LSD = t^{\frac{\alpha}{2}}_{(ab-1)(r-1)} \sqrt{2MSE/br}$$

B factor level

$$LSD = t^{\frac{\alpha}{2}}_{(ab-1)(r-1)} \sqrt{2MSE/ar}$$

2^k Factorial Experiment

$$SSFactor = \frac{(Contrast_{factor})^2}{r \sum (Contrast_{factor} Coefficients)^2} \tag{9.6}$$

$$SSE = SSTO - SSA - SSB - SSAB \quad \text{with} \quad 2^k(r-1) \;\; df$$

Analysis of Covariance

1.

$$\hat{\beta}^* = \frac{\sum_{i=1}^{t} \sum_{j=1}^{r} Y^*_{ij} X^*_{ij}}{\sum_{i=1}^{t} \sum_{j=1}^{r} X^{*2}_{ij}} = \frac{E_{xy}}{E_{xx}} \tag{9.7}$$

$$SSTO = \sum_{i=1}^{t} \sum_{j=1}^{b} (Y_{ij} - \bar{Y}_{\cdot\cdot})^2 = \sum_{i=1}^{t} \sum_{j=1}^{b} Y^2_{ij} - \frac{1}{n} Y^2_{\cdot\cdot} \quad \text{with} \quad n-1 \quad df \tag{9.8}$$

$$SSTO_{adj} = SSTO - SSReg = SSTO - \hat{\beta} SS_{xy} = SSTO - \frac{SS^2_{xy}}{SS_{xx}} \;\; \text{with} \;\; rt-2 \;\; df$$

$$SSE_{adj} = SSE - SSReg^* = SSE - \hat{\beta}^* E_{xy} = SSE - \frac{E^2_{xy}}{E_{xx}} \;\; \text{with} \;\; t(r-1)-1 \;\; df$$

$$SST_{adj} = SSTO_{adj} - SSE_{adj} \;\; \text{with} \;\; (t-1)-1 \;\; df$$

$$
\begin{aligned}
SS_{xy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}X_{ij} - \frac{1}{rt}Y_{\cdot\cdot}X_{\cdot\cdot} \\
SS_{yy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - \frac{1}{rt}Y_{\cdot\cdot}^2 \\
SS_{xx} &= \sum_{i=1}^{t}\sum_{j=1}^{r} X_{ij}^2 - \frac{1}{rt}X_{\cdot\cdot}^2 \\
E_{xx} &= \sum_{i=1}^{t}\sum_{j=1}^{r} X_{ij}^2 - \frac{1}{r}\sum_{i=1}^{t} X_{i\cdot}^2 \\
E_{xy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - \frac{1}{r}\sum_{i=1}^{t} Y_{i\cdot}X_{i\cdot} \\
E_{yy} &= \sum_{i=1}^{t}\sum_{j=1}^{r} Y_{ij}^2 - \frac{1}{r}\sum_{i=1}^{t} Y_{i\cdot}^2
\end{aligned}
$$

$$
LSD = t^{\frac{\alpha}{2}}_{t(r-1)-1} se(\hat{\mu}^*_{i\cdot} - \hat{\mu}^*_{i'\cdot})
$$

where $se(\bar{\mu}^*_{i\cdot} - \mu^*_{i'\cdot}) = \sqrt{MSE_{adj}\left(\frac{2}{r} + \frac{(\bar{X}_{i\cdot} - \bar{X}_{i'\cdot})^2}{SS_{xx}}\right)}$