Stagnation pts are obtained when $\frac{dw}{dz} = 2z = 0$

Origin is the stagnation pt of the flow & speed of the fluid is given by

$$\left|\frac{dw}{dz}\right| = |2z| = 2|z| = 2\left(\sqrt{x^2+y^2}\right)$$

which is speed of the fluid at any pt $P(x,y)$.

Unit-4

4.1) Complex Velocity Potentials for Standard 2-D Flows:

We shall consider some common type of 2D flows, namely flow due to ~~stream~~ a uniform stream, ~~to~~ a line source & sink, and a line doublet and some properties related to them. (III flow only).

1) Uniform Stream:

Case (i):

Consider a uniform stream having velocity $-U\hat{i}$

i.e. $\vec{q} = -U\hat{i} + 0\hat{j} + 0\hat{k}$

$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -U & 0 & 0 \end{vmatrix} = \hat{i}(0) - \frac{\partial U}{\partial z}\hat{j} + \frac{\partial(-U)}{\partial z}\hat{k}$$

$$= 0$$

this gives rise to a scalar pt. fn ϕ

$\ni \vec{q} = -\nabla\phi$

$= -\phi_x \hat{i} + (-\phi_y)\hat{j} + (-\phi_z)\hat{k}$

i.e. $-U\hat{i} = -\phi_x \hat{i} - \phi_y \hat{j} - \phi_z \hat{k}$

$\Rightarrow \phi_x = U, \quad \phi_y = 0, \quad \phi_z = 0$

Integrating,

$$\int \phi_x \, dx = \int U \, dx + c(y, z)$$

$$\phi = Ux + c \longrightarrow (1)$$

[$\because \phi_y = 0 = \phi_z$,

$\Rightarrow \phi$ is indpt. of y & z

$\Rightarrow \phi$ is a fn of x only]

$z = x + iy \Rightarrow x = Re(z), \; y = Im(z)$

The complex velocity potential is defined by

$$w = Uz = Ux + iUy$$
$$= \phi + i\psi$$

where ϕ = const defines the equipotentials

ψ = const defines the streamlines

$\psi = Im\, w = Uy = \text{const} \Rightarrow y = \text{const}$ are the streamlines

$\phi = Re\, w = Ux = \text{const} \Rightarrow x = \text{const}$ will determine the equipotential.

where velocity potentials are given by (1).

Case (ii).

Suppose that a uniform stream is incident to the +ve x-axis by an angle α with velocity $\vec{q}$.

Thus, $\vec{q}$ will have $-U\cos\alpha, -U\sin\alpha$ to be the component where U is the speed of uniform stream.

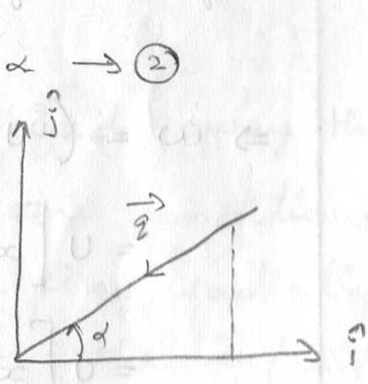
$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -U\cos\alpha & -U\sin\alpha & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}\left(\frac{-\partial(U\cos\alpha)}{\partial y}\right) + \hat{k}\left(\frac{\partial}{\partial x}(U\sin\alpha) - \frac{\partial}{\partial y}(U\cos\alpha)\right)$$

$$= \vec{0}$$

$\therefore$ flow is irrotational, this gives rise to a velocity potential ϕ

$$\ni \vec{q} = -\nabla\phi = -\phi_x \hat{i} - \phi_y \hat{j} - \phi_z \hat{k} \longrightarrow (1)$$

$$\phi_x = U\cos\alpha \qquad \phi_y = U\sin\alpha \longrightarrow (2)$$



$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

$$= \phi_x\, dx + \phi_y\, dy$$

$$= U\cos\alpha\, dx + U\sin\alpha\, dy$$

Integrating,

$$\phi = Ux\cos\alpha + Uy\sin\alpha + c \longrightarrow (3)$$

where c is a const [$\because$ flow is 2D]

Theory + (5.4) - pg. 167

We have,

$$\phi_x = \psi_y \quad \& \quad \phi_y = -\psi_x$$

line source / sink / doublet

$\therefore$ (1) becomes $\vec{q} = -\psi_y \hat{i} + \psi_x \hat{j}$

$= -U\cos\alpha\,\hat{i} - U\sin\alpha\,\hat{j}$ [from (2) in (1)]

$\Rightarrow \psi_y = U\cos\alpha, \quad \psi_x = -U\sin\alpha \rightarrow (5)$

$\psi = \psi(x, y)$

$d\psi = \psi_x\,dx + \psi_y\,dy$

$= -U\sin\alpha\,dx + U\cos\alpha\,dy$

Integrating,

$\Rightarrow \psi = -Ux\sin\alpha + Uy\cos\alpha + d \rightarrow (6)$

Complex velocity potential defining this flow is given by $w = \phi + i\psi$ where ϕ - vel. potential
ψ - stream function

$\Rightarrow w = (Ux\cos\alpha + Uy\sin\alpha + c)\hat{i} + (-Ux\sin\alpha + Uy\cos\alpha + d)$

$= U\left[x(\cos\alpha - i\sin\alpha) + y(\sin\alpha + i\cos\alpha)\right] + (c + id)$

$= U\left[x e^{-i\alpha} + iy(\cos\alpha - i\sin\alpha)\right] + C$

where C is a complex const, $C = c + id$

$= U\left[x e^{-i\alpha} + iy e^{-i\alpha}\right] + C$

$= U e^{-i\alpha}(x + iy) + C$

$= U e^{-i\alpha} z + C$

Lines Sources & Line Sinks:

Consider a closed circuit C in the invisid fluid flow. Let A be any pt considered in the plane of flow inside the circuit.

Consider an infinite line through A ⊥r to the plane of flow. If through every pt on the infinite line through A, fluid is emitted in symmetrical manner then the rate of emmission from all such pts, A on the line being the same every where & the fluid is emitted along radially outward lines in a system of ||el planes, then the infinite line through A is called a line source.

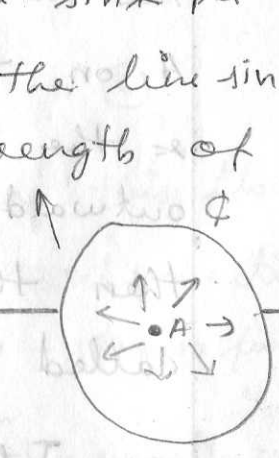
If the fluid is drained away through such a line & under the same conditions of the symmetry the flow being radially invited along the line, then the infinite line through A is called a line sink.

Thus the flow of fluid due to line source or sink is an eg. of 2-D flow. Since the line source or line sink is the same which is normally outward radial lines or invisid radial lines respectively.

Suppose an infinite line through A is a line source & that it exists fluid at the rate of $2\pi m\rho$ units of mass per then this is the mass flux/unit time/unit length of the infinite cylinder whose trace in the plane of flow is the closed curve C & whose generators are normal to the plane & hence normal to C

Then m is defined as the strength of the line source.

|||ly, if $2\pi m'\rho$ units of mass of fluid is drained out through a line sink per unit time per unit length of the line sink then m' is defined as the strength of line sink.

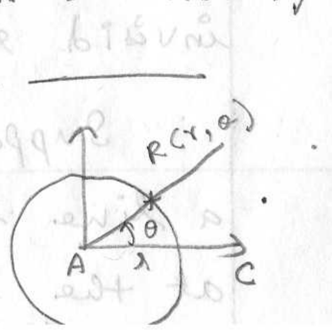


Remark:

The strength of the line sink m' is nothing but strength of line source of strength $-m'$.

If $C = C$ is a cycle of radius r, then the fluid speed $|\vec{q}|$ is the same everywhere & $\therefore$ the mass flux/time across this circle is $2\pi r\rho$.

But the rate of emmission is $2\pi m\rho$/unit time/unit length of the ——— m being the strength of the source.



$$\therefore 2\pi r l \, |\vec{q}| = 2\pi m l$$

$$\Rightarrow |\vec{q}| = \frac{m}{r}$$

$$\Rightarrow \vec{q} = \frac{m}{r}\hat{r} \quad \left(\hat{r} = \frac{\vec{r}}{|r|} \right)$$

In cylindrical polar co-ordinate system,

$$\nabla \times \vec{q} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{k} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial k \\ \frac{m}{r} & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[0\hat{e}_r + 0\hat{e}_\theta + 0\hat{k} \right] = \vec{0}$$

the flow is irr. & hence is of the potential kind, then $\exists$ a velocity potential $\phi = \phi(r, \theta)$

$$\vec{q} = -\nabla\phi = -\left[\frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta \right]$$

Equating corresponding coeff.

$$\frac{\partial\phi}{\partial r} = -\frac{m}{r}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0 \Rightarrow \phi \text{ is indpt of } \theta$$

$$\Rightarrow \phi = \phi(r)$$

$$\frac{\partial\phi}{\partial r} = \frac{d\phi}{dr}$$

$$\Rightarrow \frac{d\phi}{dr} = \frac{-m}{r} \Rightarrow d\phi = \frac{-m}{r}\,d\phi \Rightarrow \phi = -m\left[\log r + \log c\right]$$

At origin $r = 0 \Rightarrow \phi = 0 \Rightarrow c = 0$ & $\therefore$ $\phi = -m \log r$.

Considering the fluid pt P on the circle in the polar co-ordinates (r, θ) w.r.t origin at A & if $\vec{OP} = z$ in cylindrical polar co-ordinates

$$z = re^{i\theta} \qquad 0 \leq \theta \leq 2\pi$$

$$r > 0$$

$$\log z = \log r + i\theta$$

$$|z| = |r| \, |e^{i\theta}|$$

$$\log r = \log |z|$$

$\therefore$ the complex velocity potential of the line source through A of uniform stream m

is $\omega = -m \log z$

$$= -m \log (re^{i\theta})$$

$$= -m \log r - m \log e^{i\theta}$$

$$= -m \left[\log r + i\theta\right]$$

$$= -m \log r - im\theta$$

i.e. $\phi + i\psi = -m \log r - im\theta$

$\Rightarrow \phi = -m \log r$ & $\psi = -m\theta$

(vel. poten) (stream line)

$\therefore$ complex potential for the flow due to a line source of strength m at the origin A is $\omega = -m \log z$.

$$\phi = -m \log r = -m \log |z|$$

& streamlines given by

$\psi = -m\theta \Rightarrow \psi = \text{const}$ in any plane of flow are the straight lines $\theta = \text{const}$.

$\phi = \text{const}$ are equipotentials giving rise to $(r = \text{const})$

Concentric circles $r = \text{const}$ centered at A.

If the line source of strength m were situated at $z = z_1$ instead of at $z = 0$, since in the above $r = |z - z_1|$, we should have for the complex velocity potential

$$w = -m \log (z - z_1).$$

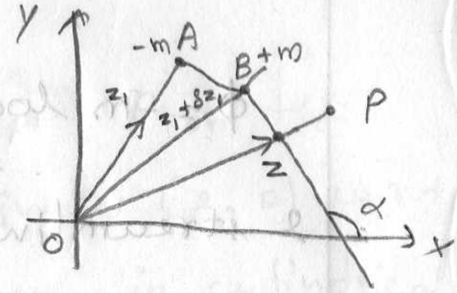
Line Doublets:

Consider a 2-D III flow containing a line source of strength a line source of strength $-m$ at A and a line source of strength m at B, where A & B are neighbouring pts in the region or flow.

Consider an arb fluid pt. P, $P = P(z)$ all in same plane of flow.

$$\overrightarrow{OA} = z_1, \quad \overrightarrow{OB} = z_1 + \delta z_1, \quad \overrightarrow{OP} = z$$

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = z_1 + \delta z_1 - z_1$$
$$= \delta z_1 \text{ (small)}$$



$\vec{OP} = z$.

Then the complex ~~ptnl~~ palt ~~of~~ at P due to these 2 lines souces at A & B are given by

$$w = -(-m)\log(z-z_1) - m\log(z-(z_1+\delta z_1))$$

$$= m\log(z-z_1) - m\log(z-(z_1+\delta z_1))$$

$$= -m\log(z-z_1) - m\left[\log(z-z_1) + \frac{\delta z}{1!}\frac{d}{dz}\log(z-z_1) + \frac{(\delta z)^2}{2!}\frac{d^2}{dz^2}(\log(z-z_1)) + \cdots\right]$$

$$= -m\frac{\delta z_1}{1!}\frac{\partial}{\partial z}(\log(z-z_1) + (\delta z_1)^2 \qquad \left[f(a+h) = f(a) + \frac{f'(a)h}{1!} + \frac{f''(a)h^2}{2!} + \cdots\right]$$

$$= -m\,\delta z_1 \frac{1}{z-z_1}\frac{\partial z}{\partial z} + O\cdot(\delta z_1)^2$$

(negleting higher order of infinitesimal)

We have,

$$w = \frac{-m\,\delta z_1}{(z-z_1)}$$

$$= \frac{-m\,|\delta z|e^{i\alpha}}{(z-z_1)},$$

where α is the angle made by AB with OX axis.

Since A & B are neighbouring pts $\vec{AB} = \delta z$, is very small & higher powers of δz are neglected

Further, if m is very larger ($m \to \infty$) δz_1 being very very small $\delta z_1 \to 0$ whenever $|\delta z_1| \to 0$ & vise versa.

$\therefore$ The quantity $\mu = m|\delta z_1|$ remains finite & const.

Then 2 equal & opposite line source at A ($\delta z_1 \to \infty$) are said to constitute a line doublet of strength μ/unit time/unit length of the doublet.

Thus, the complex potential in the limiting case at P is given by

$$w = \frac{m|\delta z_1| e^{i\alpha}}{z - z_1} \quad \text{when } |\delta z_1| \to 0$$

α = amplitude of A
= arg. of complex quantity

$$w = \frac{\mu e^{i\alpha}}{z - z_1} \quad \text{when } |\delta z_1| \to 0$$

And the direction of axes of the doublets is $\overrightarrow{AB}$.

Remark:

1) If the doublet is at the origin, then $w = \dfrac{\mu e^{i\alpha}}{z}$

2) If $\alpha = 0$, then $w = \dfrac{\mu}{z - z_1}$

when $\alpha = 0$,

A & B are on the axis then the axis of doublet $\overrightarrow{AB}$ is +ve x-axis.

Eg:

Discuss the uniform line doublet at 'O' whose strength is μ along OX axis.
~~Complex potential at P(x,y) is given by~~

Soln:

The complex potential at $P(x, y)$ is

$$w = \frac{\mu}{z} = \frac{\mu}{x+iy} \times \frac{x-iy}{x-iy}$$

$$w = \frac{\mu(x-iy)}{x^2+y^2} = \frac{\mu x}{x^2+y^2} - \frac{i\mu y}{x^2+y^2}$$

$$\boxed{w = \phi + i\psi}$$

$$\Rightarrow \phi = \frac{\mu x}{x^2+y^2} \quad \& \quad \psi = \frac{-y\mu}{x^2+y^2}$$

Thus the equipotentials are gn by

$$\phi = \text{const} = c$$

$$\therefore \phi = \frac{\mu x}{x^2+y^2} = \text{const } c$$

$$\Rightarrow \quad (x^2+y^2) - \frac{\mu x}{c} = 0$$

$$x^2+y^2-\mu c' x = 0 \quad \left(\because \frac{1}{c} = c'\right)$$

$(\mu c', 0)$, system of coaxial circles with radius $r = \sqrt{(\mu c')^2 + (0)^2 - 0} = \mu c'$

And stream lines are given by

$$\psi = \text{const} = d$$

$$\therefore \quad \psi = \frac{-\mu y}{x^2+y^2} = \text{const } d$$

$$\Rightarrow \quad (x^2+y^2) + \frac{\mu y}{d} = 0$$

$$x^2+y^2+\mu d' y = 0 \quad \left(\because \frac{1}{d} = d'\right)$$

$(0, -\mu d')$ coaxial system of circles.

equipotentials

$x^2+y^2 = 2k_1 x$

$(k_1, 0)$, radii k_1

↓

centres

streamlines

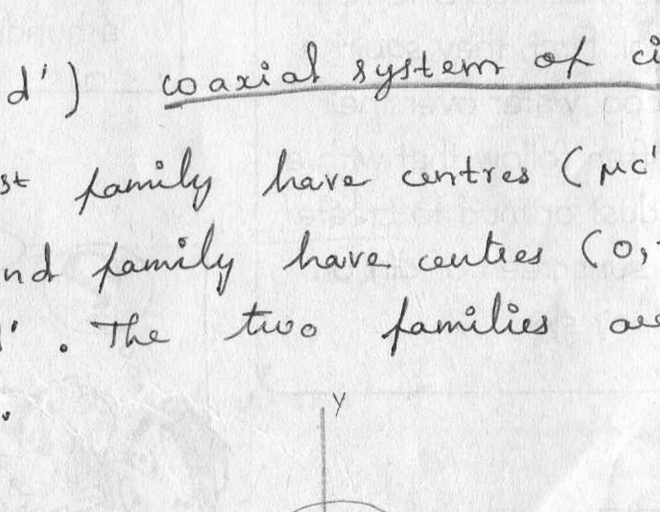
$x^2+y^2 = 2k_2 y$

$(0, k_2)$ & radii k_2

↓

centres

The 1st family have centres $(\mu c', 0)$ & radii $\mu c'$ & the second family have centres $(0, -\mu d')$ & radii $-\mu d'$. The two families are mutually orthogonal.



— streamline

...... equipotential

Line Vortices:

Consider a 2-D flow with velocity $\vec{q} = u\hat{i} + v\hat{j}$ at any point $P(x,y)$ in the field of flow such that $u = u(x,y)$; $v = v(x,y)$

The vorticity vector $\vec{\zeta}$,

$$\vec{\zeta} = \nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Indicating that $\vec{\zeta}$ is in the $\perp$ plane of flow i.e., along $\hat{k}$ (or ~~eo~~ $\Leftrightarrow z = 0$).

$$\vec{\zeta} = \hat{k}(v_x - u_y) \longrightarrow (1)$$

Consider the 2-D flow for which the velocity potential is given by

$$w = \frac{ik}{2\pi} \log z \quad \text{~~where~~}$$

where k is a constant in $\mathbb{R}'$

$z = re^{i\theta}$ is the polar co-ordinate system.

Then $w = \frac{ik}{2\pi}\left[\log re^{i\theta}\right]$

$$= \frac{ik}{2\pi}\left[\log r + i\theta\right].$$

$$= \frac{ik \log r}{2\pi} - \frac{k\theta}{2\pi}$$

But the complex potential,

$$\omega = \phi + i\psi$$

where ϕ is the velocity potential

& ψ is the stream function

And $\phi = \dfrac{-k\theta}{2\pi}$; $\psi = \dfrac{k \log r}{2\pi}$

$\phi_r = 0$ $\qquad \psi_r = \dfrac{k}{2\pi r}$

$\phi_\theta = \dfrac{-k}{2\pi}$ $\qquad \psi_\theta = 0$

every problem where we get ϕ, ψ we have to check they are harmonic $\longrightarrow$ (f)

Equipotentials are given by $\phi =$ constant

$\Rightarrow \theta =$ constant

Streamlines are constant

$\Rightarrow \psi =$ constant

$\Rightarrow \log r =$ constant

$\Rightarrow r =$ const

$\Rightarrow x^2 + y^2 =$ const

$\theta =$ const means radial vectors passing the through the origin.

$x^2 + y^2 =$ const means system of concentric circle centered at the origin.

Such a pattern is obtained is every plane $\perp^r$ to the z-axis.

The two families of curves are orthogonal to each other, consider the circulation around a circuit C surrounding the origin denoted by Γ where

$$\Gamma = \oint \vec{q} \cdot \vec{ds}$$

The radial & transverse components in cylindrical polar co-ordinates (r, θ, z) is given by

explain

$$\begin{cases} q_r = -\phi_r = 0 \\ q_\theta = -\frac{1}{r}\phi_\theta = -\frac{1}{r}\left(-\frac{k}{2\pi}\right) = \frac{k}{2\pi r} \longrightarrow \text{from (f)} \\ q_z = 0 \end{cases}$$

$$\Gamma = \oint_c (q_r \hat{e}_r + q_\theta \hat{e}_\theta) \cdot (dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta)$$

$$= \oint_c q_r\, dr + r q_\theta\, d\theta$$

$$= \oint_c 0 \cdot dr + r \cdot \frac{k}{2\pi r}\, d\theta$$

$$= \left.\frac{k}{2\pi}\theta\right]_c$$

Since is a small closed circuit around the origin, ~~without~~ ~~lo~~

<u>Without loss of generality.</u>

Assume c to be a circle with ~~untied~~ ~~until~~ unit radius then $\oint_c d\theta = 2\pi$

Thus the circulation around the closed circuit with centre orgin is a constant k

$$\Rightarrow \Gamma = \frac{k}{2\pi} \cdot 2\pi$$

$$= k$$

Also note that circulation over any other closed circuit not containing the region will yield zero due to Cauchy's theorem.

$\therefore$ we get a distribution 2-D providing such a flow to be uniform along z-axis

i.e.) The direction of vorticity vector called the uniform line vortex of strength k and hence is an infinite line through the origin giving a circulation around any closed curve c in any line ⊥r to the plane is defined as uniform line vortex. The complex velocity potential for the flow due to the line vortex is given by

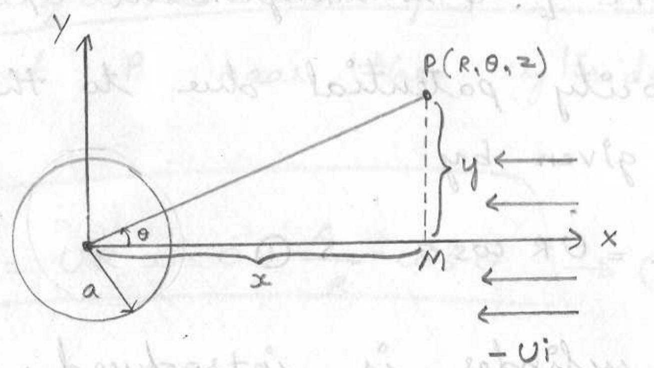
$$\omega = \frac{ik}{2\pi} \log z.$$

174 175, 89 - defns

Eg:

6m

Uniform flow past a fixed infinite circular cylinder.



$\cos\theta = \frac{x}{R}$

Soln:

Fig. shows a uniform fluid flowing past a fixed cylinder of radius 'a' with undisturbed velocity $-U\hat{i}$.

Let P be a pt in the fluid flow having cylindrical polar coordinates (R, θ, z) & cartesian coordinates (x, y, z) $\ni$ $\boxed{x = R\cos\theta}$, $\boxed{y = R\sin\theta}$.

Then $\vec{q} = -U\hat{i} + 0\hat{j} + 0\hat{k}$

$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -U & 0 & 0 \end{vmatrix}$$

$$= \vec{0}$$

Considering the flow to be 2-D, z becomes redundant, $\nabla \times \vec{q} = \vec{0}$

$\Rightarrow \exists$ a scalar pt fn ϕ representing the velocity potential $\ni \vec{q} = -\nabla\phi$.

$$\Rightarrow -U\hat{i} + 0\hat{j} = -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right)$$

$$\Rightarrow \underbrace{\frac{\partial\phi}{\partial x} = U}_{(1)}, \quad \frac{\partial\phi}{\partial y} = 0$$

$\Rightarrow \phi$ is independent of y.

Integrating ⊛,

$$\phi = Ux + f(y)$$

$\Rightarrow \phi = Ux + c$ [$\because \phi$ is independent of y].

$\therefore$ The velocity potential due to the uniform stream is given by

$$\phi = Ux = UR\cos\theta \longrightarrow (1)$$

When a cylinder is introduced, it will cause a $\boxed{\text{perturbation}}$ of the flow. And this perturbation must be such as to satisfy laplace equation & should become vanishingly small when R is large.

(1) satisfies, $\nabla^2\phi = 0$

The simplest harmonic for satisfying our requirement will be $f = \frac{1}{R}\cos\theta$

To show that

$$\nabla^2 f = 0 \quad [\text{for cylindrical polar coordinates}]$$

$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{R}\frac{\partial}{\partial R}\left(R\left(-\frac{\cos\theta}{R^2}\right)\right) + \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(-\frac{\sin\theta}{R}\right)$$

$$= \frac{1}{R}\frac{\partial}{\partial R}\left(-\frac{\cos\theta}{R}\right) + \frac{1}{R^2}\left(-\frac{\cos\theta}{R}\right)$$

$$= \frac{1}{R}\frac{\cos\theta}{R^2} - \frac{\cos\theta}{R^3}$$

$$= \frac{\cos\theta}{R^3} - \frac{\cos\theta}{R^3} = 0$$

$\Rightarrow \nabla^2 f = 0$

$\therefore$ Considering the general fn $\boxed{\phi = \frac{A}{R}\cos\theta}$ → (2)

satisfies $\nabla^2 \phi = 0$.

$\therefore$ The required velocity potential representing the flow at P near the cylinder will be given by

$$\boxed{\phi(R,\theta) = UR\cos\theta + \frac{A}{R}\cos\theta} \rightarrow (3)$$

Suppose P lies on the cylinder, then

$$R = a \Rightarrow \frac{\partial \phi}{\partial R} = 0$$

$$\frac{\partial \phi}{\partial R} = U\cancel{\cos\theta} - \frac{A\cancel{\cos\theta}}{R^2} = 0$$

$$\Rightarrow U = \frac{A}{R^2}$$

$$\Rightarrow A = UR^2\Big]_{R=a}$$

$$\Rightarrow A = Ua^2$$

$$q_\theta = -\frac{1}{R}\frac{\partial\phi}{\partial\theta} = +\frac{1}{R}U\sin\theta\left(R + \frac{a^2}{R}\right) = U\sin\theta\left[1 + \left(\frac{a}{R}\right)^2\right]$$

Sub. in (3)

$$\boxed{\therefore \phi(R,\theta) = U\cos\theta\, R + \frac{Ua^2}{R}\cos\theta}$$

$$= U\cos\theta\left[R + \frac{a^2}{R}\right] \rightarrow (4)$$

Hence, the velocity components at P are

$$q_R = -\frac{\partial\phi}{\partial R} = -U\cos\theta\left\{1 - \left(\frac{a}{R}\right)^2\right\}$$

$$q_\theta = -\frac{1}{R}\frac{\partial\phi}{\partial\theta} = U\sin\theta\left\{1 + \left(\frac{a}{R}\right)^2\right\}$$

$$q_z = -\frac{\partial\phi}{\partial z} = 0$$

As $R \to \infty$, $q_R \to -U\cos\theta$,
$q_\theta \to U\sin\theta$, appropriately.

Pre-requisite:

Consider a 2-D irrotational flow of an inviscid incompressible fluid.

i) Presence of a line source of strength m at the pt $z = z_1$ in the flow yields a complex potential for the flow in the form

$$w = -m \log (z - z_1)$$

$$\therefore \frac{dw}{dz} = \frac{-m}{z - z_1}$$

$z = z_1$ is a singularity for the flow. ~~Since denominator is equal to zero.~~

$\because$ denominator $= 0$

$\therefore$ Presence of a line source at $z = z_1$ is considered as a hydrodynamical similarity.

ii) Presence of a line sink of strength m' at the pt $z = z_2$ in the flow yields a complex potential for the flow in the form.

$$w = +m' \log (z - z_2)$$

$$\therefore \frac{dw}{dz} = \frac{m}{z - z_2}$$

$z = z_2$ is a singularity for the flow.

$\because$ denominator $= 0$

$\therefore$ Presence of a line sink at $z = z_2$ is considered as a hydrodynamical similarity.

iii) Presence of a line doublet of strength μ at $z = z_3$ with axis at an angle α with OX axis in the flow yields a complex potential for the flow in the form:

$$w = \frac{\mu e^{i\alpha}}{z - z_3}$$

$$\therefore \frac{dw}{dz} = \frac{-\mu e^{i\alpha}}{(z-z_3)^2}$$

$z = z_3$ is a singularity for the flow.

$\therefore$ denominator $= 0$.

$\therefore$ Presence of a line doublet at $z = z_3$ is considered as a hydrodynamical similarity.

iv) Presence of a line vortex at $z = z_4$ in the flow yields a complex potential for the flow in the form:

$$w = \frac{ik}{2\pi} \log(z - z_k)$$

$$\therefore \frac{dw}{dz} = \frac{ik}{2\pi(z - z_4)}$$

$z = z_4$ is the singularity for the flow.

$\therefore$ denominator $= 0$

$\therefore$ Presence of a line vortex at $z = z_4$ is considered as a hydrodynamical similarity.

Sec. 4.2

MILNE - THOMSON CIRCLE THEOREM:

We know,

if $f(t) = u(t) + i v(t)$ where u & v are real valued fns where $t \in R'$.

$u, v : R' \to R'$

$f(t)$ conjugate, $\bar{f}(t) = \overline{u(t) + i v(t)}$

$= u(t) - i v(t)$

If $z = x + iy$ and $f : \mathbb{C} \to \mathbb{C}$ then $f(z) = u(z) + i v(z)$ and $f(\bar{z}) = u(\bar{z}) + i v(\bar{z})$ and $\bar{f}(\bar{z}) = \overline{u(\bar{z}) + i v(\bar{z})}$

$= u(\bar{z}) - i v(\bar{z})$

$= \overline{f(z)}$ (denoted)

Singularities of $u(z)$ and $v(z)$ are singularities of $f(z)$ and hence singularities of $\bar{f}(z)$ and $\overline{f(z)}$ are the same and singularities of $f(z)$ are singularities of $\bar{f}(z)$ and hence $\overline{f(z)}$

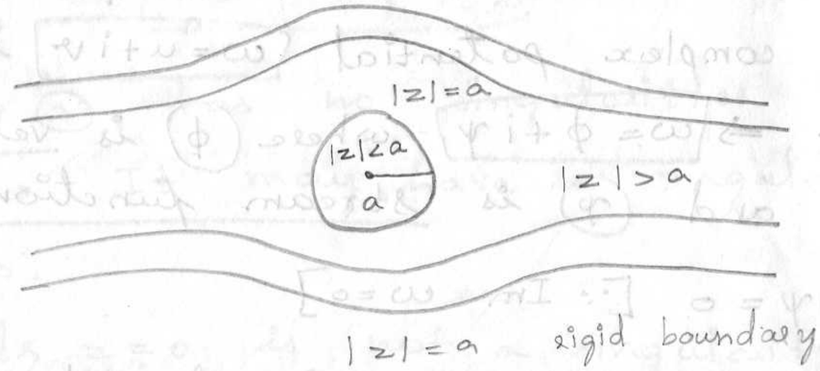
Motivation:

If a right circular cylinder of infinite length is introduced in the given 2D flow its axis being alligned with coordinate axis $\perp^{\text{las}}$ to the plane of flow then its possible to derive a general expression for the new velocity potential outside the cylinder and the procedure for the same is given in Milne - Thomson circle.

Statement:

Let $f(z)$ be the complex velocity potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $\boxed{|z|=a}$. Then, on introducing the solid circular cylinder $|z|=a$ into the flow, the new complex velocity potential is given by $\boxed{w = f(z) + \bar{f}(a^2/z)}$ for $|z| \geq a$.

Proof:



Consider a 2D-flow. If a right circular cylinder of infinite length is introduced with its axis aligned with the coordinate axis, $\perp r$ to the plane of flow. Let $f(z)$ be the complex potential for a flow having no rigid boundaries & there is no singularity of the flow within the circle $|z|=a$.

When the cylinder is introduced into the flow, we shall first prove that $\boxed{|z|=a}$ is a rigid boundary in the stream line for the

On $|z| = a$, any elt z will satisfy

$$z\bar{z} = a^2$$

$$\Rightarrow \bar{z} = \frac{a^2}{z} \rightarrow (1)$$

On $|z| = a$, consider a complex potential

$$w = f(z) + \overline{f(z)} = f(z) + \bar{f}(\bar{z})$$

$$= f(z) + \bar{f}\left(\frac{a^2}{z}\right) \text{ (by (1))} \rightarrow (2)$$

$$w = f(z) + \overline{f(z)}$$

$= 2 \operatorname{Re}(f(z))$ is real (The property of complex no's)

i.e.) the complex potential $\boxed{w = u + iv}$ is real on $|z| = a$ $\Rightarrow \boxed{w = \phi + i\psi}$ where ϕ is velocity potential and ψ is stream function.

$$\Rightarrow \psi = 0 \quad [\because \text{Imagi } w = 0]$$

This means that the circular boundary $|z| = a$ is a stream line or a rigid body for the flow across which no fluid flows. Thus, $|z| = a$ is a possible boundary for the fluid flow.

$\times$ —— $\times$

Consider, the flow past the circular cylinder $|z| = a$ in the region $|z| > a$, we shall construct the complex potential.

If z lies in the region $|z| < a$ then $\bar{z}$ lies in the region $\boxed{|z| > a}$ and if z lies in the region $\boxed{|z| > a}$ then $\bar{z}$ lies in the region $\boxed{|z| < a}$.

all singularities lies outside

By hypothesis,

singularities of $f(z)$ are in the region $|z| > a$.

The possible singularities of $f(z)$ are line sources, line sinks, line doublets and line vortices.

$\therefore$ Singularities of $\bar{f}(z)$ lie inside $|z| < a$.

Hence, singularities of $\bar{f}(\bar{z}) = \overline{f(z)}$ also lies inside $|z| < a$.

i.e., $\overline{f(z)} = \bar{f}\left(\frac{a^2}{z}\right)$ $\left[\because \bar{z} = \frac{a^2}{z}\right]$

(From ②, has no singularities in the region $|z| > a$. It may not have a singularity even at $z = \infty$.

As $z = 0$ is not a singularity for $f(z)$)

Thus, $w = f(z)$ and $w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)$ represent the same hydrodynamical flow in the region $|z| > a$.

Hence, the complex potential for the new modified flow passed the solid circular cylinder is given by ②.

Hence the theorem.

Extension of Milne Thomson Circle Theorem:

Statement:

If $f(z)$ be the complex velocity potential for a flow having no rigid boundaries and such that there are no singularities of flow outside the circle $|z|=a$. Then on introducing the rigid cylindrical surface of section $|z|=a$ into the flow and leaving the distribution of singularities within its interior unchanged, the new complex velocity potential for the fluid motion within the boundary becomes $w = f(z) + \bar{f}(a^2/z)$ for $|z| \leq a$.

Eg. 17

Electrostatic form of the circle theorem:

Let $f(z) = \phi(x,y) + i\psi(x,y)$ be the complex potential for a given two-dimensional distribution which is analytic in the region $|z| \geq a$. Suppose that the boundary $|z|=a$ is now introduced into the potential field and that $\phi = 0$ on this boundary. Show that the new complex potential ~~field~~ for the region $|z| \geq a$ is now given by

$$w = f(z) - \bar{f}(a^2/z).$$

Pg. 213 exercise: ~~ex~~ 17

Theorem due to Blasius:

Motivation:

When a long cylinder is placed with its generators $\perp^r$ to the incident stream of a moving fluid containing hydro-dynamical similarities; it experiences forces tending to produce translation & rotation of the cylinder. The effect was studied by Blasius. Infact, this study is useful in one.

1) A cylinder of any cross-section placed in the flow.

2) When the fluid thrust on the cylinder is required. ~~In practice, use~~

In practice, we find this to have applications in an aerofoil. In these cases, complex potential for the flow and moment of ~~comple~~ couple per unit length can be obtained. We consider the cases when the fluid is incompressible.

Statement:

An incompressible fluid flows steadily & irrotationally under no external forces parallel to the z-plane past a fixed cylinder whose section in that plane of flow is bounded by a closed curve C. The complex potential for the flow is given by $w = f(z)$. Then, the action of fluid pressure on the cylinder is equivalent to a force per unit length having components (X, Y) and a couple per unit length of moment M ~~where~~

where
$$\boxed{Y + iX = -\frac{\rho}{2}\oint_{C}\left(\frac{dw}{dz}\right)^2 dz} \longrightarrow (*)$$

and
$$\boxed{M = Re\left\{-\frac{\rho}{2}\oint_{C} z\left(\frac{dw}{dz}\right)^2 dz\right\}} \longrightarrow (**)$$

Proof:

(* Consider an inviscid, irrotational, incompressible flow. Suppose, a long cylinder is placed with its generators perpendicular to the incidence stream of a moving fluid where there are no external forces parallel to the z-plane past the cylinder whose cross-section is the XOY plane or $z=0$ plane is C as shown in the fig)

* Let $P(x, y)$ & $P'(x+\delta x, y+\delta y)$ be 2 neighbouring pts such that ~~$\overline{PP'} = \delta s = \widehat{PP'}$~~.

$$\boxed{\widehat{PP'} = \delta s = \overline{PP'}}$$

where P & P' are neighbouring pts on C.

* Let the position vector $\boxed{\overrightarrow{OP} = s}$ and $\boxed{\overrightarrow{OP'} = s + \delta s}$

$\Rightarrow \boxed{PP' = \delta s}$ [$\because$ P & P' are neighbouring pts]. From the fig. note that $\boxed{PM = \delta x}$, $\boxed{P'M = \delta y}$.

* Draw a tangent at P that cuts the OX axis making an angle α. (Let p be the pressure exerted by the fluid on the cylinder outside at P.) (Then, the force exerted by the fluid on the cylinder at P will be $\boxed{F\big]_P = p\,\delta s}$ along the normal $\hat{n}$ at P on C.

* ∴ The <u>components of $p\delta s$</u> along OX and OY axis are resp. p

$$\boxed{p\,\delta s \cos(90+\alpha) = -p\,\delta s \sin\alpha}$$

and $\boxed{p\,\delta s \cos\alpha}$.

* The elementary force <u>components</u> along OX, OY are $-p\,\delta y$ and $p\,\delta x$ respectively.

$$\therefore\ \boxed{-p\,\delta y = -p\,\delta s \sin\alpha \Rightarrow \delta y = \delta s \sin\alpha} \text{ and } \boxed{p\,\delta x = p\,\delta s\cos\alpha \quad \delta x = \delta s \cos\alpha}$$

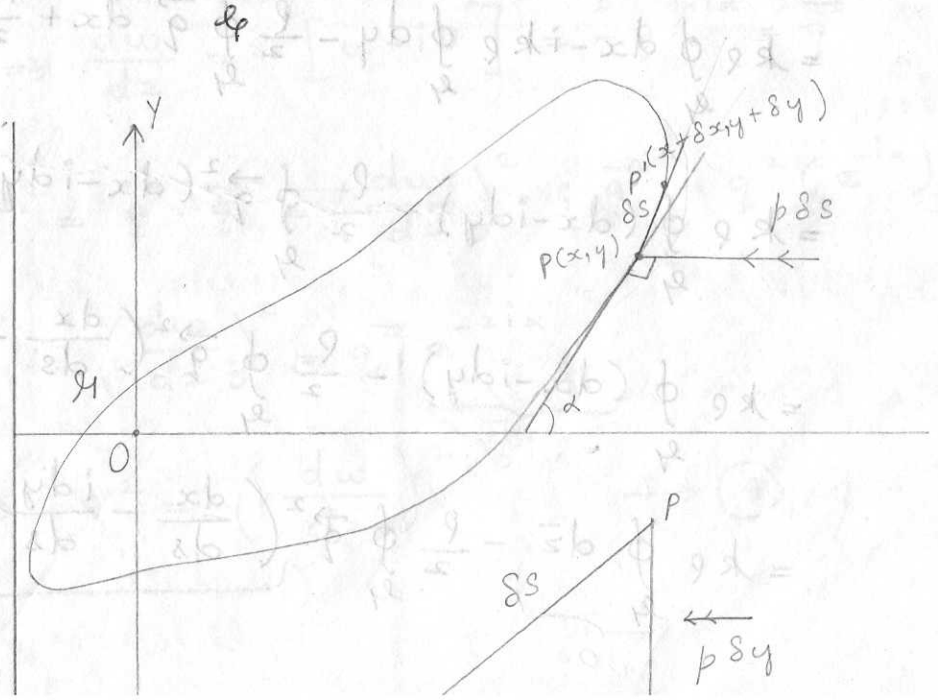
* ∴ The <u>total force components</u> exerted on the entire cylindrical cross-section per unit length resp. are given by

$$\boxed{X = -\oint_{C_1} p\,dy} \ \&\ \boxed{Y = \oint_{C_1} p\,dx} \longrightarrow (1)$$

parallel to OX and OY respectively.

$$\therefore\ Y + iX = \oint_{C_1} p\,dx - i\oint_{C_1} p\,dy$$

$$= \oint_{C_1} p(dx - i\,dy) \longrightarrow (2)$$



The pressure p can be computed using Bernoulli's eqn for an incompressible fluid is given by,

$$\boxed{\frac{1}{2}(\vec{q}^{\,2}) + \frac{p}{\rho} + \Omega = \text{constant}}$$

where $\vec{q}$ is the fluid velocity,

ρ is the density of the fluid &

p is the pressure within the fluid.

Using Bernoulli's eqn from (f)

$$\boxed{\frac{p}{\rho} + \frac{1}{2}\vec{q}^{\,2} = k \text{ (const)}} \Rightarrow p = \left(k - \frac{\vec{q}^{\,2}}{2}\right)\rho = k\rho - \frac{\vec{q}^{\,2}}{2}\rho = \left(k - \frac{\vec{q}^{\,2}}{2}\right)\rho$$

$\therefore$ (2) becomes

$$Y + iX = \oint_{C_1} \overbrace{\left(k - \frac{1}{2}\vec{q}^{\,2}\right)\rho}^{p}(dx - idy)$$

$$= \left(k\rho - \frac{\rho}{2}\vec{q}^{\,2}\right)(dx - idy)$$

$$= k\rho\,dx - ik\rho\,dy - \frac{\rho}{2}\vec{q}^{\,2}dx + i\frac{\rho}{2}\vec{q}^{\,2}dy$$

$$= \oint_{C_1} k\rho\,dx - i\oint_{C_1} k\rho\,dy - \frac{\rho}{2}\oint_{C_1}\vec{q}^{\,2}dx + \frac{i\rho}{2}\oint_{C_1}\vec{q}^{\,2}dy$$

$$= k\rho\oint_{C_1} dx - ik\rho\oint_{C_1} dy - \frac{\rho}{2}\oint_{C_1}\vec{q}^{\,2}dx + \frac{i\rho}{2}\oint_{C_1}\vec{q}^{\,2}dy$$

$$= k\rho\oint_{C_1}(dx - idy) - \frac{\rho}{2}\oint_{C_1}\vec{q}^{\,2}(dx - idy)$$

$$= k\rho\oint_{C_1}\underbrace{(dx - idy)}_{d\bar{z}} - \frac{\rho}{2}\oint_{C_1}\vec{q}^{\,2}\left(\frac{dx}{ds} - i\frac{dy}{ds}\right)ds$$

$$= k\rho\underbrace{\oint_{C_1} d\bar{z}}_{0} - \frac{\rho}{2}\oint_{C_1}\vec{q}^{\,2}\left(\frac{dx}{ds} - \frac{idy}{ds}\right)ds$$

$$Y + iX = -\frac{\rho}{2} \oint_{C} \vec{q}^{\,2} (\cos\alpha - i \sin\alpha)\, ds$$

now finding this value

Suppose $\vec{q}$ has <u>components</u> $\boxed{\vec{q} : \vec{q}\,(u, v)}$

$\vec{q}$ is given by, $\boxed{\vec{q} = u + iv}$

$$\frac{dw}{dz} = f'(z) = \lim_{\delta x, \delta y \to 0} \left\{ \frac{\delta\phi + i\,\delta\psi}{\delta x + i\,\delta y} \right\}, \text{ where } f(z) = \phi + i\psi$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\delta\phi + i\,\delta\psi}{\delta x} \right\} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}$$

Since $u = -\frac{\partial\phi}{\partial x}$, $v = +\frac{\partial\psi}{\partial x}$

$$\therefore \frac{dw}{dz} = -u + iv \longrightarrow (4)$$

If $\left| \frac{dw}{dz} \right| = q$ then $\frac{dw}{dz} = \vec{q}$

$$= q e^{i\alpha}$$

$$= u + iv \longrightarrow (ff)$$

$$\Rightarrow q e^{-i\alpha} = u - iv$$

$$(4) \Rightarrow \frac{dw}{dz} = -\left[u - iv\right] = -q e^{-i\alpha} = \vec{q}$$

$$\vec{q}^{\,2} = \vec{q} \cdot \vec{q} = \left(\frac{dw}{dz}\right)^2 ; \quad \vec{q}^{\,2} = q^2 (e^{-i\alpha})^2 = q^2 e^{-2i\alpha}$$

$$\therefore \left(\frac{dw}{dz}\right)^2 = q^2 e^{-2i\alpha}$$

$$\Rightarrow q^2 = \left(\frac{dw}{dz}\right)^2 e^{2i\alpha} \longrightarrow (A)$$

Substituting (A) in (3),

$$Y + iX = -\frac{\rho}{2} \oint_{C_1} e^{2i\alpha} \left(\frac{dw}{dz}\right)^2 \underbrace{(\cos\alpha - i\sin\alpha)}\, ds$$

$$= -\frac{\rho}{2} \oint_{C_1} \left(\frac{dw}{dz}\right)^2 e^{2i\alpha} \cdot e^{-i\alpha}\, ds$$

$$= -\frac{\rho}{2} \oint_{C_1} \left(\frac{dw}{dz}\right)^2 e^{i\alpha}\, ds$$

$$= -\frac{\rho}{2} \oint_{C_1} \left(\frac{dw}{dz}\right)^2 \underbrace{(\cos\alpha + i\sin\alpha)}\, ds$$

$$= -\frac{\rho}{2} \oint_{C_1} \left(\frac{dw}{dz}\right)^2 \left[\frac{dx}{ds} + \frac{i\,dy}{ds}\right] ds$$

$$= -\frac{\rho}{2} \oint_{C_1} \left(\frac{dw}{dz}\right)^2 (dx + i\,dy)$$

$$\boxed{= -\frac{\rho}{2} \oint_{C_1} \left(\frac{dw}{dz}\right)^2 dz} \longrightarrow (5)$$

—×—×—

Now to find the moment of couple at O.

Consider,

the sum of moments of the elementary force components about O, denoted by δM

$$\therefore \delta M = p\,\delta x(x) - (-p\,\delta y)\,y$$

$$= p x\,\delta x + p y\,\delta y$$

[Moment about O
= Mag. of $\perp$ from O
× Mag. of its force
(clockwise = −ve
anticlockwise = +ve)]

$\therefore$ The total moment of the fluid about O exerted on the circuit denoted by M is given by,

$$\boxed{M = \oint_{C} p(x\,dx + y\,dy)} \longrightarrow (6)$$

Substitute for p from (f) in (6),

$$M = \oint_{C} \overbrace{\left(\rho k - \frac{1}{2}\rho q^2\right)}(x\,dx + y\,dy)$$

where ρ is density &

$\vec{q}$ is the fluid velocity at P.

$$M = \oint_{C} k\rho x\,dx - \frac{\rho}{2}\oint_{C} q^2(x\,dx + y\,dy) + \oint_{C} k\rho y\,dy$$

$$= \boxed{k\rho \oint_{C} d\left(\frac{x^2}{2}\right) + k\rho \oint_{C} d\left(\frac{y^2}{2}\right)} - \frac{\rho}{2}\oint_{C} q^2(x\,dx + y\,dy) \rightarrow (7)$$

$$= -\frac{\rho}{2}\oint_{C} \underbrace{q^2(x\,dx + y\,dy)}$$

Consider,

$$z\left(\frac{dw}{dz}\right)^2 dz = (x+iy)\left(q^2 e^{-i2\alpha}\right)(dx + i\,dy)$$

$$= q^2 (x+iy)\, e^{-i2\alpha}\, e^{i\alpha}\, ds$$

$$\left[\because \frac{dw}{ds} = -q e^{-i\alpha}, \quad dx + i\,dy = dz \right.$$

$$\frac{dz}{ds}ds = \left(\frac{dx}{ds} + \frac{i\,dy}{ds}\right)ds$$

$$= (\cos\alpha + i\sin\alpha)\,ds$$

$$= e^{i\alpha}\,ds$$

at O.

orce

$$z\left(\frac{dw}{dz}\right)^2 dz = q^2 e^{-i\alpha}(x+iy)\,ds$$

$$= q^2(\cos\alpha - i\sin\alpha)(x+iy)\,ds$$

$$= q^2\left(\frac{dx}{ds} - i\frac{dy}{ds}\right)(x+iy)\,ds$$

$$= q^2(x+iy)(dx - i\,dy)$$

$$= q^2\left[(x\,dx + y\,dy) + i(y\,dx - x\,dy)\right]$$

$$\mathrm{Re}\left(z\left(\frac{dw}{dz}\right)^2 dz\right) = q^2(x\,dx + y\,dy) \longrightarrow 8$$

and $$\mathrm{Re}\left(z\left(\frac{dw}{dz}\right)^2 dz\right) = q^2(y\,dx - x\,dy)$$

Using 8 in 7, we have

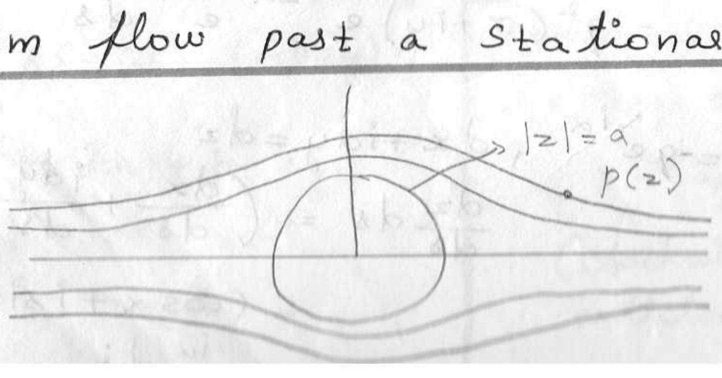
$$\boxed{M = -\frac{\rho}{2}\oint_{c_1} \mathrm{Re}\left(z\left(\frac{dw}{dz}\right)^2\right) dz} \longrightarrow 9$$

5 $\Rightarrow$ (*) & 9 $\Rightarrow$ (**)

Hence proved.

Application of Milne Thompson Circle Theorem.

i) Uniform flow past a stationary cylinder:



Consider a uniform stream ~~function~~ from ∞, having velocity $-U\hat{i}$. This gives rise to a complex potential $w(z) = Uz$ in the ~~pressure~~ presence of a circular cylinder with base curve $|z| = a$

Let $f(z)$ be a complex potential for the flow due to uniform stream from ∞ with velocity $\vec{q} = -U\hat{i}$ then

$$f(z) = Uz$$

$$\bar{f}(z) = Uz$$

$$\bar{f}(\bar{z}) = \bar{f}\left(\frac{a^2}{z}\right) = \frac{Ua^2}{z}$$

<u>By Milne Thompson theorem,</u>

the complex potential for the modified flow (The flow past $|z| = a$) is

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \text{ in } |z| > a$$

$$= Uz + \frac{Ua^2}{z} \text{ in } |z| > a$$

$$= U\left(z + \frac{a^2}{z}\right)$$

$$z = re^{i\theta}$$

$$w = \phi + i\psi = U\left(re^{i\theta} + \frac{a^2}{re^{i\theta}}\right)$$

$$= U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right)$$

$$= Ur(\cos\theta + i\sin\theta) + \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$$\phi + i\psi = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta + i\left(Ur\sin\theta - \frac{Ua^2}{r}\sin\theta\right)$$

$$\therefore \phi = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta$$

$$= U\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta$$

$$= U\sin\theta\left(r - \frac{a^2}{r}\right)$$

ϕ = constant

$\therefore$ Eqn. of equipotentials

ψ = constant

$\therefore$ Eqn. of streamlines.

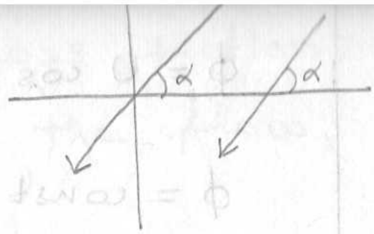
---

2) Uniform stream from α_0 incident on Ox with angle α:

The complex potential for this flow is $w = Uze^{i\alpha}$ complex potential for flow (due to uniform stream family) at an angle α to OX is given by,

$f(z) = Uze^{-i\alpha}$ with velocity $\overline{q} = -U\hat{i}$

$$\bar{f}(z) = Uz\, e^{i\alpha}$$



$$\bar{f}(\bar{z}) = \bar{f}\left(\frac{a^2}{z}\right) = \frac{Ua^2}{z}\, e^{i\alpha}$$

By Milne Thompson Circle Theorem,

Complex potential for modified flow in the presence of $|z| = a$ is

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad \text{in } |z| > a.$$

$$= Uz\, e^{-i\alpha} + \frac{Ua^2}{z}\, e^{i\alpha}$$

Let P be any point in the flow

$P : P(re^{i\theta})$ then (ie) $z = re^{i\theta}$

$$w = \phi + i\psi$$

$$= U\left(re^{i\theta} . e^{i\alpha} + \frac{a^2}{re^{i\theta}}\, e^{i\alpha}\right)$$

$$= U\left[re^{i(\theta-\alpha)} + \frac{a^2}{r}\, e^{-i(\theta-\alpha)}\right]$$

$$= U\left[r\cos(\theta-\alpha) + ir\sin(\theta-\alpha) + \frac{a^2}{r}\cos(\theta-\alpha) - i\frac{a^2}{r}\sin(\theta-\alpha)\right]$$

$$= U\cos(\theta-\alpha)\left[r + \frac{a^2}{r}\right] +$$

$$iU\sin(\theta-\alpha)\left(r - \frac{a^2}{r}\right).$$

$$\phi = U\cos(\theta - \alpha)\left(r + \frac{a^2}{r}\right)$$

$\phi = \text{constant}$

Eqn of equipotentials

$$\psi = U\sin(\theta - \alpha)\left(r - \frac{a^2}{r}\right)$$

$\psi = \text{constant}$

Eqn of streamlines

---

3) Out of Syllabus (Eg. 3)

4) Line doublet parallel to axes of a right circular cylinder at a distance $d > a$ from axes of cylinder.

Let the strength of the doublet be μ / unit length. Also the axis of line doublet is set at angle α to OX.

The flow due to line doublet at a distance α from axes of cylinder is given by,

$$w = f(z) = \frac{\mu e^{i\alpha}}{z - \alpha}$$

$$\therefore \bar{f}(z) = \frac{\mu e^{-i\alpha}}{z - \alpha}$$

When a cylinder of cross section $|z| = a$ is introduced into the flow,

$$\bar{f}\left(\frac{a^2}{z}\right) = \frac{\mu e^{-i\alpha}}{a^2/z - d}$$

Hence the complex potential for the flow due to the line doublet in the presence of a circular cylinder is given by,

Milne Thompson Circle theorem in the form,

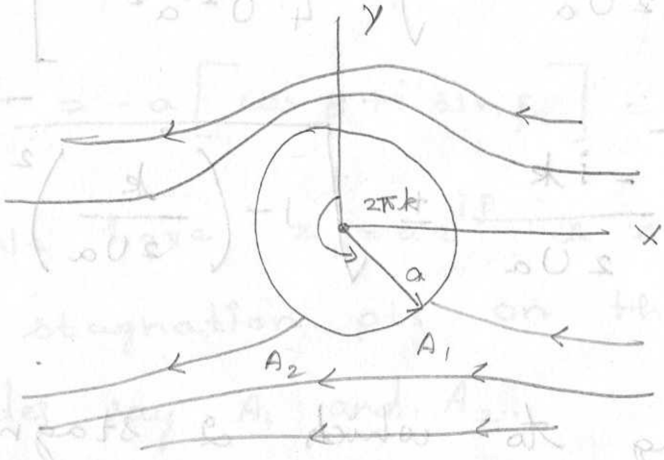
$$\omega = f(z) + \bar{f}\left(a^2/z\right) \quad , \quad |z| > a$$

$$\phi + i\psi = \omega = \frac{\mu e^{i\alpha}}{z - d} + \frac{\mu e^{-i\alpha}}{\frac{a^2}{z} - d} \quad , \text{ where } z = x + iy$$

6. Magnus effect:

Consider a long infinite cylinder of radius 'a' placed in a uniform stream from ∞ with velocity $\vec{q} = -U\hat{i}$.

Suppose that around a cylinder $|z| = a$, placed in stream, there is a circulation of amount $2\pi k$ which corresponds to a complex velocity potential $ik \log z$.



Then by Milne Thompson Circle Theorem, the complex potential at any pt. P_1 in the flow is

$$w = U\left(z + \frac{a^2}{z}\right) + ik \log z$$ (quote eq. 1 of pg. 183)

$$\therefore \frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) + i\frac{k}{3}$$

$\frac{dw}{dz} = 0$ yields stagnation points,

$$\frac{Uz^2 - Ua^2}{z^2} = -i\frac{k}{z}$$

$$Uz^2 - Ua^2 + ikz = 0$$

$$z = \frac{-ik \pm \sqrt{-k^2 + 4U^2a}}{2U}$$

$$= a\left[\frac{-ik}{2Ua} \pm \sqrt{\frac{-k^2 + 4U^2a^2}{4U^2a^2}}\right]$$

$$= a\left[\frac{-ik}{2Ua} \pm \sqrt{1 - \left(\frac{k}{2Ua}\right)^2}\right]$$

Corresponding to which 2 stagnation pts for the flow are obtained. Adjust k, U in such a way that $\left|\frac{k}{2Ua}\right| < 1$

We can find a real angle β,

$$\sin\beta = \frac{k}{2Ua} \qquad \left[\because \left|\sin\beta\right| = \left|\frac{k}{2Ua}\right| < 1\right]$$

Stagnation pts are

$$z = a\left[-i\sin\beta \pm \sqrt{1-\sin^2\beta}\right]$$

$$= a\left[-i\sin\beta \pm \cos\beta\right]$$

$$z_1 = a\left[-i\sin\beta + \cos\beta\right]$$

$$= a\left[\cos(-\beta) + i\sin(-\beta)\right] = ae^{-i\beta}$$

$$z_2 = a\left[-i\sin\beta - \cos\beta\right]$$

$$= -a\left[\cos\beta + i\sin\beta\right] = -ae^{-i\beta}$$

$\therefore$ We have $z_1 = ae^{i\beta}$ & $z_2 = -ae^{i\beta}$ to be two stagnation pts on the surface of the cylinder say A_1 and A_2.

At $A_1 + A_2$ the pressure exerted by the flow is max and hence the effect produced on the cylinder up is the +ve direction and this is also due to the circulation around the cylinder $|z| = a$. The lifting up of the cylinder or the lifting effect produced by circulation is known as magnus effect.