$$\therefore -\kappa t + \tau b = \left(\frac{4}{5}\sin t, \cos t, \frac{-3}{5}\sin t\right)$$

$$= \dot{n}$$

$$= L.H.S$$

Hence $\dot{n} = -\kappa t + \tau b$

Thus, Frenet-Serret Eqns are verified.

Unit - II

Surfaces in three dimensions

2.1) Surface: What is a surface?

A surface is a subset of $\mathbb{R}^3$ that looks like a piece of $\mathbb{R}^2$ in the vicinity of any given point, like the surface of the Earth, which is spherical, appears to be a flat plane to an observer on the surface.

Defn: [Open]

A subset U of $\mathbb{R}^n$ is called open if, whenever 'a' is a point in U, there is a positive number $\varepsilon \ni$ every point $u \in \mathbb{R}^n$ within a distance ε of 'a' is also in U:

i.e., $a \in U$ and $\| u - a \| < \varepsilon \Rightarrow u \in U$.

Eg:

The whole of $\mathbb{R}^n$ is an open set, since $D_r(a) = \{ u \in \mathbb{R}^n / \|u-a\| < r \}$, where $D_r(a)$ is the open ball with centre 'a' and radius $r > 0$.

Note:

If $n=1$, an open ball is called an open interval; if $n=2$, it is called an open disc.

Defn: [Continuous]

If X and Y are subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, a map $f : X \to Y$ is said to be continuous at a point $a \in X$ if points in X near 'a' are mapped by f onto points in Y near $f(a)$.

More precisely

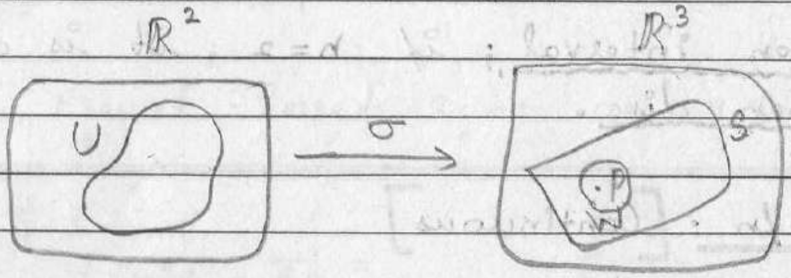
ie., f is continuous at 'a' if, given any number $\varepsilon > 0$, there is a number $\delta > 0$ $\ni$ $u \in X$ and $\|u-a\| < \delta \Rightarrow \|f(u) - f(a)\| < \varepsilon$. Then f is said to be continuous if it is continuous at every point of X. Composites of continuous maps are continuous.

Defn: [Homeomorphism & Homeomorphic]

If $f : X \to Y$ is continuous and bijective, and if its inverse map $f^{-1} : Y \to X$ is also continuous, then f is called a homeomorphism and X & Y are said to be homeomorphic.

Defn: [Surface]

A subset S of $\mathbb{R}^3$ is a surface if, for every point $P \in S$, there is an open set U in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing P $\ni$ $S \cap W$ is homeomorphic to U.



Thus, a surface comes equipped with a collection of homeomorphisms $\sigma : U \to S \cap W$, which we call surface patches (or) parametrisations.

The collection of all these surface patches is called the atlas of S. Every point of S lies in the image of atleast one surface patch in the atlas of S.

Example: 4.1

Prove that every plane in $\mathbb{R}^3$ is a surface with an atlas containing a single surface patch.

Proof:

Let '$\vec{a}$' be a point on the plane, and let '$\vec{p}$' and '$\vec{q}$' be two unit vectors that are $\parallel^{el}$ to the plane and $\perp$ to each other.

Then, any vector $\|^{el}$ to the plane is a linear combination of $\bar{p}$ & $\bar{q}$, say $up+vq$ for some scalars u and v.

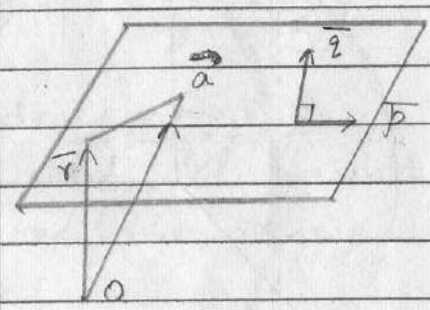
If '$\bar{r}$' is the position vector of any point of the plane, $r-a$ is $\|^{el}$ to the plane,

then we can denote $\boxed{\bar{r}-a = u\bar{p}+v\bar{q}}$

$\therefore \bar{r} = a + u\bar{p} + v\bar{q} \rightarrow (1)$, where u & v are some scalars.

Let $\bar{\sigma}$ be a surface patch given by

$\sigma : U \to S \cap W$ be a homeomorphism that we can define as



$$\bar{\sigma}(u, v) = a + u\bar{p} + v\bar{q} \rightarrow (2)$$

Since σ itself represents a plane, the single surface patch is enough to cover the entire plane; hence the atlas consists of only this surface patch.

Note that the inverse of σ is gn by,

$$\sigma^{-1}(\bar{r}) = ((\bar{r}-a)\cdot\bar{p}, (\bar{r}-a)\cdot\bar{q}) \rightarrow (3)$$

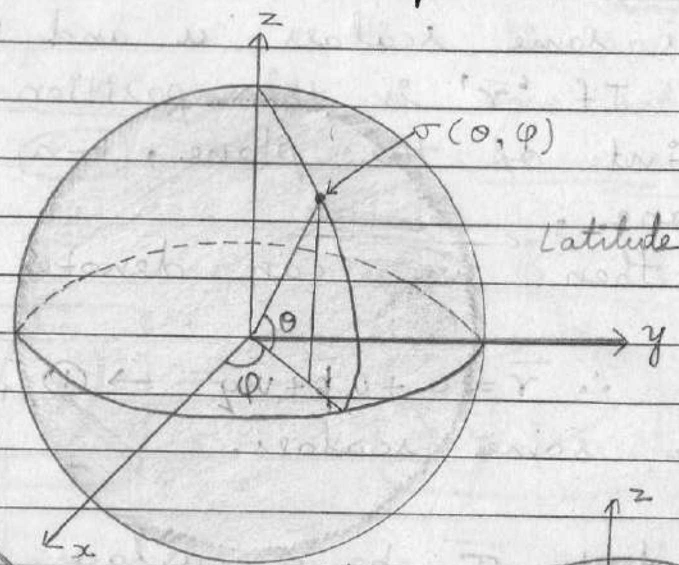
It is clear from the expression $\bar{\sigma}$ & $\bar{\sigma}^{-1}$ that they are continuous and hence that $\bar{\sigma}$ is a homeomorphism.

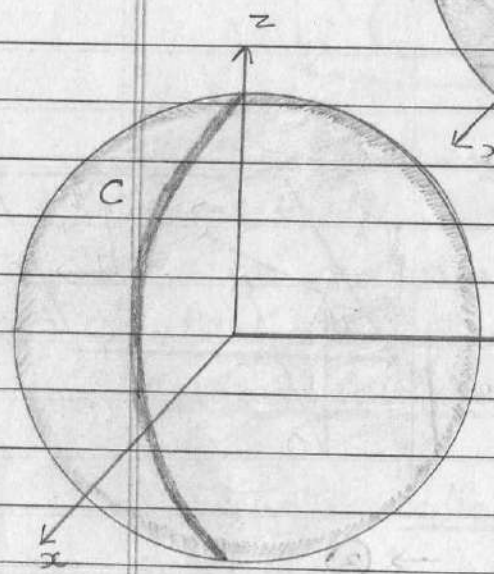
1) 20 m

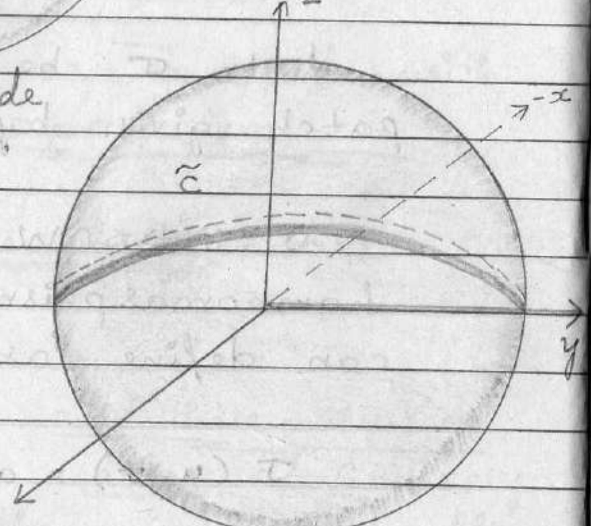
Example : 4.2

Prove that the unit sphere is a surface.

Proof:







W.K.T,

the unit sphere is given by

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1 \}$$

Considering

θ to be latitude &

φ to be longitude.

We can take the parametrisation to be

$$\sigma(\theta, \varphi) = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta)$$

To cover the whole sphere, it is clearly sufficient to take

$$\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \quad \& \quad \varphi \in [0, 2\pi]$$

Since θ & φ are taking values from closed interval,
the mapping σ is not 1-1 (injective) and hence it is not a homeomorphism.

So let us take the open set

$$U = \left\{ (\theta, \varphi) / \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \varphi \in (0, 2\pi) \right\}$$

and $\sigma : U \to \mathbb{R}^3$ is an homeomorphism and is a surface patch but the image of σ doesn't cover the entire surface of the sphere.

Hence σ consists of the complement of the great semi-circle $\boxed{C}$ consisting of the points of the sphere of the form $(x, 0, z)$ with $x \geq 0$.

Hence $\sigma : U \to \mathbb{R}^3$ covers only a 'patch' of the sphere.

To show that the sphere is a surface →

To cover the entire surface of the sphere, we can consider one more surface patch covering the part of the sphere omitted by σ.

Hence $\tilde{\sigma}$ be the patch obtained by first rotating σ by π about the z-axis and then by $\pi/2$ about the x-axis.

And we define $\tilde{\sigma} : U \to \mathbb{R}^3$ by

$$\tilde{\sigma}(\theta, \varphi) = (-\cos\theta\cos\varphi, -\sin\theta, -\cos\theta\sin(\varphi)$$

where the image of $\tilde{\sigma}$ is the complement of the great semi-circle $\tilde{C}$ consisting of the points of the sphere of the form $(x, y, 0)$ with $x \leq 0$.

so the union of σ & $\tilde{\sigma}$ of the images of the whole plane is the whole plane.

It is clear that C and $\tilde{C}$ do not intersect, (so the atlas consisting surface patches σ and $\tilde{\sigma}$, which cover the whole plane).

2n

Note:

Sphere cannot be covered by a single surface patch.

6m

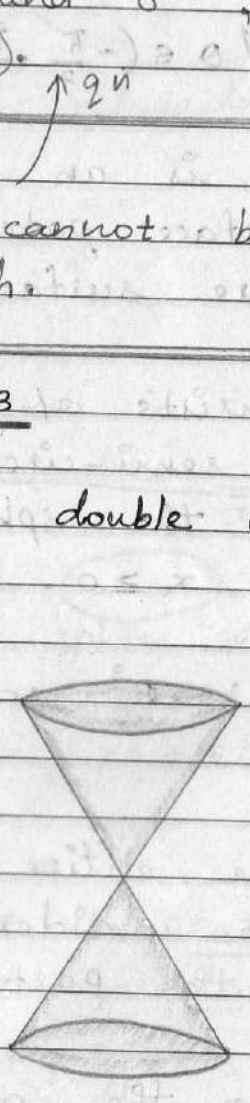
Example : 4.3

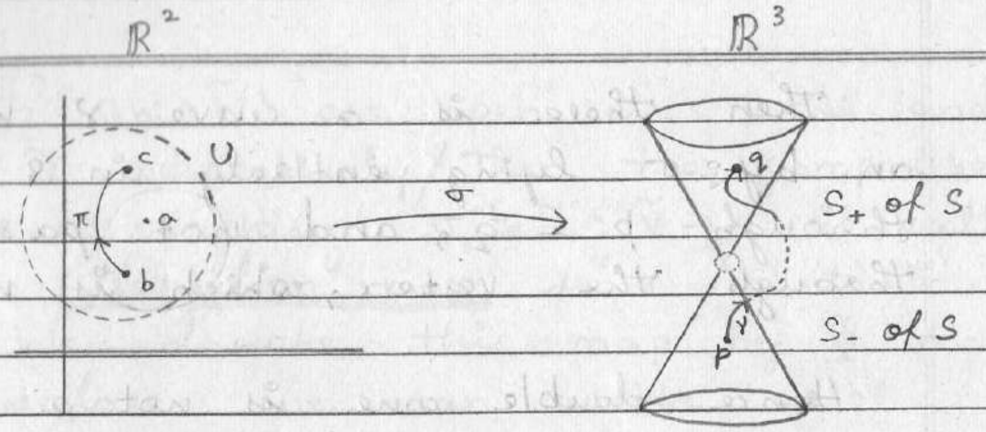
Prove that double cone is not a surface.

a)

20m

Proof:





The double cone is the set,

$$S = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 = z^2\}$$

$\ni$ they satisfy the eqn $x^2 + y^2 = z^2$.

Let us suppose (assume) that $\sigma : U \to S \cap W$ is a surface patch containing the vertex $(0,0,0)$ of the cone.

And let $a \in U$, be the point correspond to the vertex.

Let us assume that U is an open ball with centre a.

Then the open set W must contain a point p in the **lower half S_-** of S where $z < 0$ and a point q in the upper half S_+ where $z > 0$.

Let b & c be the corresponding points ~~in~~ in U.

Now, consider the curve π in U passing (joining pts b & c) through 'b' & 'c', but not passing through 'a'.

Then there is a curve $\gamma = \sigma \circ \pi$ mapped on by σ lying entirely in S, passing through p & q, and not passing through the vertex, which is not possible.

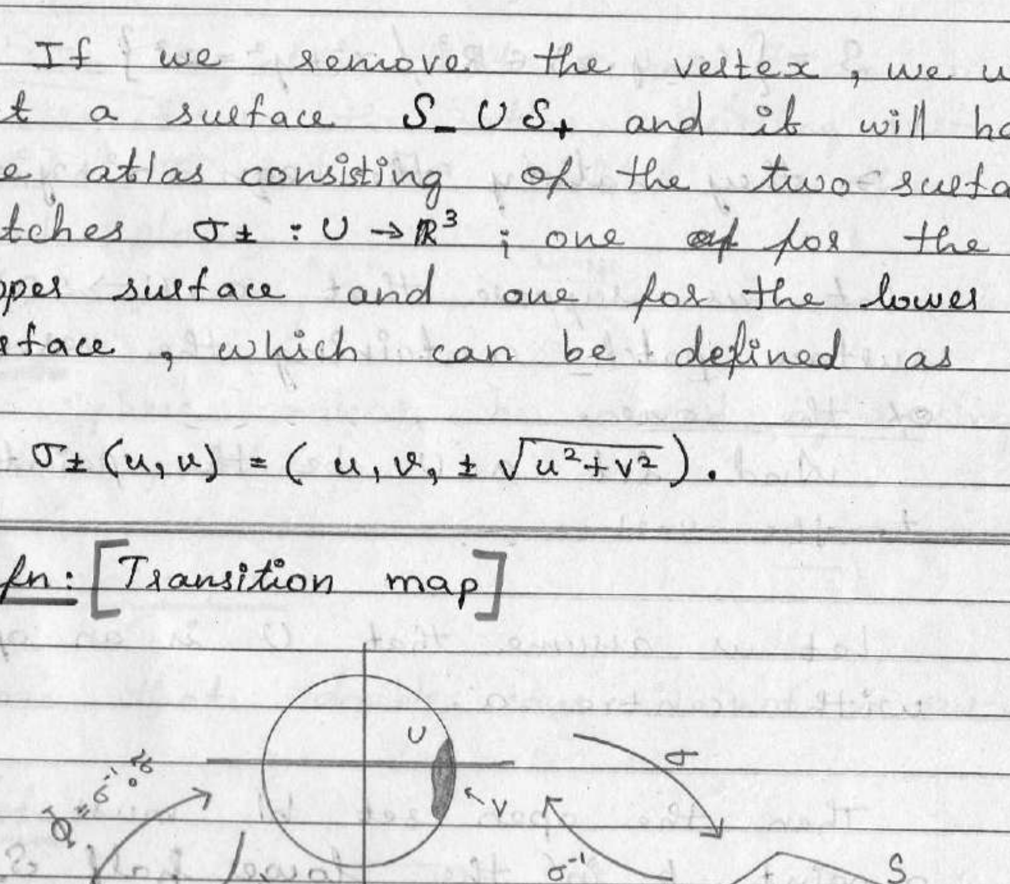
Hence double cone is not a surface.

<u>Note:</u>

If we remove the vertex, we will get a surface $S_- \cup S_+$ and it will have the atlas consisting of the two surface patches $\sigma_\pm : U \to \mathbb{R}^3$; one ~~of~~ for the upper surface and one for the lower surface, which can be defined as

$$\sigma_\pm (u, v) = (u, v, \pm\sqrt{u^2+v^2}).$$

<u>Defn:</u> [Transition map]



Suppose that $\sigma : U \to S \cap W$ and $\tilde{\sigma} : \tilde{U} \to S \cap \tilde{W}$ are two <u>surface patches</u> $\ni a \in S \cap W \cap \tilde{W}$. Since σ & $\tilde{\sigma}$ are homeomorphisms, $\sigma^{-1}(S \cap W \cap \tilde{W})$ and

$\tilde{\sigma}^{-1}(S \cap W \cap \tilde{W})$ are open sets $V \subseteq U$ and $\tilde{V} \subseteq \tilde{U}$, respectively. Then the composite homeomorphism $\sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \to V$ is called the <u>transition map</u> from σ to $\tilde{\sigma}$.

We denote this map by Φ and therefore we can write as;

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\Phi(\tilde{u}, \tilde{v})) \quad \text{for all } (\tilde{u}, \tilde{v}) \in \tilde{V}.$$

Smooth Surface:

If U is an open subset of $\mathbb{R}^m$, we say that a map $f : U \to \mathbb{R}^n$ is **smooth** if each of the n components of f, which are fns $U \to \mathbb{R}^1$, have continuous partial derivatives of all orders. The partial derivatives of f are then computed componentwise.

For example,

If $m = 2$ & $n = 3$,
then $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is of the form

$$f(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$$

where f_u, f_v

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right), \text{ which we denote by } f_u$$

and
$$\frac{\partial f}{\partial v} = \left(\frac{\partial f_1}{\partial v}, \frac{\partial f_2}{\partial v}, \frac{\partial f_3}{\partial v}\right), \text{ which we denote by } f_v$$

Similarly,

$$\frac{\partial^2 f}{\partial u^2} = f_{uu},\ \frac{\partial^2 f}{\partial v^2} = f_{vv},\ \frac{\partial^2 f}{\partial u \partial v} = f_{uv},\ \frac{\partial^2 f}{\partial v \partial u} = f_{vu}$$ & so on.

Note:

$f_{uv} = f_{vu}$, because all the partial derivatives of the components of f are continuous.

Defn: [Regular]

A surface patch $\sigma : U \to \mathbb{R}^3$ is called regular if it is smooth and the vectors σ_u & σ_v are linearly independent at all points $(u, v) \in U$.

Equivalently, $\bar{\sigma}$ should be smooth and the vector product $\sigma_u \times \sigma_v$ should be non-zero at every point of U.

Defn: [Smooth surface]

A smooth surface is a surface σ whose atlas consists of regular surface patches.

Example : 4.4

Prove that the plane (in eq 4.1) is a smooth surface.
(or) as a surface is smooth.

Proof:

W.K.T,

For a plane,

we need a single surface patch $\sigma : U \to \mathbb{R}^3$ given by

$$\sigma(u, v) = \bar{a} + u\bar{p} + v\bar{q} \to (1)$$

where

$\bar{a}$ is any constant vector in the plane,

$\bar{p}$ & $\bar{q}$ are $\perp r$ unit vectors.

From (1), we have $\bar{\sigma}_u = p$ & $\bar{\sigma}_v = q$.

$\therefore$ σ_u & σ_v are linearly independent as because p & q are $\perp r$ unit vectors.

And σ is a smooth fn for all pts of U as it is defined to be a homeomorphism. And it is regular.

$\therefore$ The plane is a smooth surface.

12m 8m 6m **Example: 4.5**

Prove that the unit Sphere is a smooth surface.

Proof:

W.K.T,

The unit sphere, $S^2 = \left\{ (x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 1 \right\}$

with two surface patches,

$$\sigma(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$$
$$\tilde{\sigma}(\theta,\varphi) = (-\cos\theta\cos\varphi, -\sin\theta, -\cos\theta\sin\varphi)$$

where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ & $\varphi \in (0, 2\pi)$.

For S^2 to be smooth surface,
we have to p.T
σ & $\tilde{\sigma}$ are regular.

First let us differentiate σ w.r.t θ & φ.

$$\sigma_\theta = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, \cos\theta)$$

$$\sigma_\varphi = (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, 0)$$

$$\sigma_\theta \times \sigma_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta\cos\varphi & -\sin\theta\sin\varphi & \cos\theta \\ -\cos\theta\sin\varphi & \cos\theta\cos\varphi & 0 \end{vmatrix}$$

$$= \hat{i}(0 - \cos^2\theta\cos\varphi) - \hat{j}(\cos^2\theta\sin\varphi) + \hat{k}(-\sin\theta\cos\theta\cos^2\varphi - \cos\theta\sin\theta\sin^2\varphi)$$

$$= -\hat{i}(\cos^2\theta\cos\varphi) - \hat{j}(\cos^2\theta\sin\varphi) - \hat{k}(\sin\theta\cos\theta)$$
$$\neq 0$$

$$\|\sigma_\theta \times \sigma_\varphi\| = \sqrt{\cos^4\theta\cos^2\varphi + \cos^4\theta\sin^2\varphi + \sin^2\theta\cos^2\theta}$$

$$= \sqrt{\cos^4\theta(\sin^2\varphi + \cos^2\varphi) + \sin^2\theta\cos^2\theta}$$

$$= \sqrt{\cos^2\theta(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{\cos^2\theta} = |\cos\theta| \neq 0 \quad \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Now, let us differentiate $\tilde{\sigma}$ w.r.t θ & φ,

$$\tilde{\sigma}_\theta = (\sin\theta\cos\varphi, -\cos\theta, \sin\theta\sin\varphi)$$

$$\tilde{\sigma}_\varphi = (\cos\theta\sin\varphi, 0, -\cos\theta\cos\varphi)$$

$$\tilde{\sigma}_\theta \times \tilde{\sigma}_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta\cos\varphi & -\cos\theta & \sin\theta\sin\varphi \\ \cos\theta\sin\varphi & 0 & -\cos\theta\cos\varphi \end{vmatrix}$$

$$= \hat{i}(\cos^2\theta\cos\varphi) - \hat{j}(-\cos\theta\sin\theta\cos^2\varphi - \cos\theta\sin\theta\sin^2\varphi) + \hat{k}(\cos^2\theta\sin\varphi)$$

$$= \hat{i}(\cos^2\theta\cos\varphi) + \hat{j}(\cos\theta\sin\theta) + \hat{k}(\cos^2\theta\sin\varphi)$$

$$\neq 0$$

$$\|\tilde{\sigma}_\theta \times \tilde{\sigma}_\varphi\| = \sqrt{\cos^4\theta\cos^2\varphi + \cos^2\theta\sin^2\theta + \cos^4\theta\sin^2\varphi}$$

$$= \sqrt{\cos^4\theta(\sin^2\varphi + \cos^2\varphi) + \sin^2\theta\cos^2\theta}$$

$$= \sqrt{\cos^2\theta(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{\cos^2\theta}$$

$$= |\cos\theta| \neq 0 \quad \forall\ \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$\therefore$ The unit sphere is a smooth surface with atlas consisting of σ & $\tilde{\sigma}$.

Defn: [Maximal atlas & allowable]

The maximal atlas for a given surface S consisting of all the regular surface patches $\sigma : U \to S \cap W$, where U & W are open subsets of $\mathbb{R}^2$ & $\mathbb{R}^3$, respectively.

Such surface patches are called allowable surface patches for S.

Proposition : 4.1

a) 20m

The transition map of a smooth surface are smooth.

Proof: (We use the inverse fn thm to give the proof of Prop. 4.1)

To prove this proposition, we need the following theorem.

Inverse Function Theorem : (thm 4.2)

Let $f : U \to \mathbb{R}^n$ be a smooth map defined on an open subset U of $\mathbb{R}^n (n \geq 1)$. Assume that, at some point $x_0 \in U$, the jacobian matrix $J(f)$ is invertible.

Then, there is an open subset V of $\mathbb{R}^n$ and a smooth map $g : V \to \mathbb{R}^n \ni$

i) $y_0 = f(x_0) \in V$;

ii) $g(y_0) = x_0$;

iii) $g(V) \subseteq U$;

iv) $g(V)$ is an open subset of $\mathbb{R}^n$;

v) $f(g(y)) = y \quad \forall \ y \in V$.

In particular, $g : V \to g(V)$ and $f : g(V) \to V$ are inverse bijections.

<u>Proof:</u>

Let us assume that $\sigma : U \to \mathbb{R}^3$ & $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be two <u>regular surface patches</u> in the atlas of a surface S, then we have to show that ~~the~~ the transition map $\Phi : \sigma \to \tilde{\sigma}$ is smooth.

Suppose that a pt P lies in both the patches, say $\sigma(u_0, v_0) = \tilde{\sigma}(\tilde{u}_0, \tilde{v}_0) = P$ and let us write

$$\sigma(u,v) = (f(u,v), g(u,v), h(u,v)).$$

Since σ_u & σ_v are <u>linearly independent</u>, the jacobian matrix;

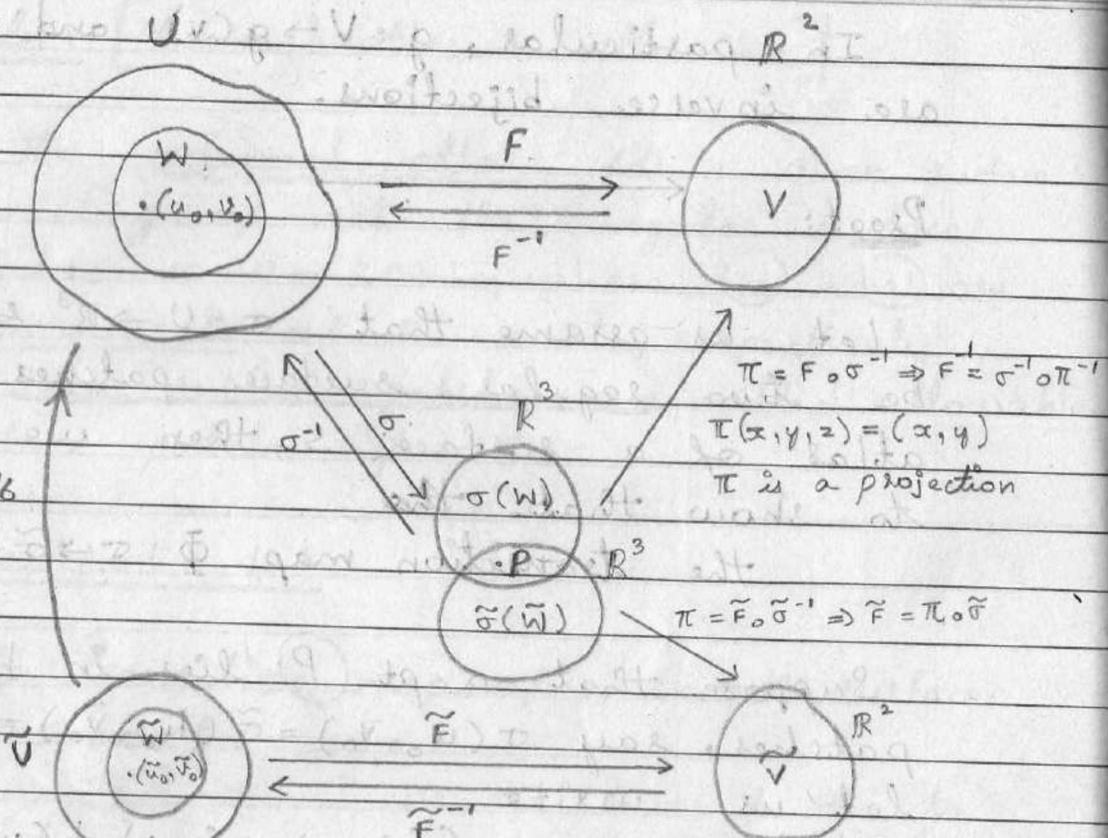
$$J(\sigma) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{pmatrix}$$ is <u>invertible.</u>

∴ It will have rank 2 everywhere. That means, atleast one of its three 2x2 submatrices will be invertible.

Let us assume that the submatrix to be $\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$.

(the submatrix is invertible at P)

Let us consider <u>the point</u> to be P.



Let us consider a smooth map $F : U \to \mathbb{R}^2$ given by

$$F(u, v) = (f(u, v), g(u, v))$$

Then there is an open subset V of $\mathbb{R}^2$ containing $F(u_0, v_0)$ and an open subset W of U containing (u_0, v_0) $\ni$ by inverse fn theorem,

$F : W \to V$ is bijective with a smooth inverse $F^{-1} : V \to W$.

Since $\sigma : W \to \sigma(W)$ is bijective, the projection $\pi : \sigma(W) \to V$ given by $\pi(x, y, z) = (x, y)$ is also bijective.

Correspondingly, for the surface patch $\tilde{\sigma}$, we have a smooth $\tilde{F} : \tilde{W} \to \tilde{V}$ which is a bijective with smooth inverse $\tilde{F}^{-1}$

with the projection map $\pi : \tilde{\sigma}(\tilde{W}) \to \tilde{V}$.

Since F & $\tilde{F}$ are both <u>bijective</u> with smooth inverse,

consider,

$$F^{-1} \circ \tilde{F} = F^{-1} \circ (\pi \circ \tilde{\sigma})$$

$$= \sigma^{-1} \circ \pi^{-1} \circ \pi \circ \tilde{\sigma}$$

$$= \sigma^{-1} \circ \tilde{\sigma} \quad [\because \sigma^{-1} \circ \tilde{\sigma} \text{ is smooth on an open set} \supset \text{any pt } (u_0, v_0) \text{ where it is defined}]$$

$$= \Phi \text{ is smooth}$$

Thus the transition maps of a smooth surface are smooth.

12m

<u>Proposition : 4.2</u>

6m

Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \tilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \tilde{U}$. Then, $\boxed{\tilde{\sigma} = \sigma \circ \Phi : \tilde{U} \to \mathbb{R}^3}$ is a <u>regular surface patch</u>.

<u>Proof:</u>

Since $\tilde{\sigma}$ is a composite fn of σ & Φ where both are smooth maps,

$\tilde{\sigma}$ is a smooth map.

<u>Now to prove that</u>

$\tilde{\sigma}$ is a regular surface patch.

Let $(\tilde{u}, \tilde{v}) \in \tilde{U}$ be a pt in $\tilde{U}$

Then $\Phi(\tilde{u}, \tilde{v}) = (u, v)$

By chain rule,

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\Phi(\tilde{u}, \tilde{v})) = \sigma(u, v)$$

Hence $\tilde{\sigma}_{\tilde{u}} = \sigma_u \frac{\partial u}{\partial \tilde{u}} + \sigma_v \frac{\partial v}{\partial \tilde{u}}$

$$\tilde{\sigma}_{\tilde{v}} = \sigma_u \frac{\partial u}{\partial \tilde{v}} + \sigma_v \frac{\partial v}{\partial \tilde{v}}$$

Consider,

$$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \left(\sigma_u \frac{\partial u}{\partial \tilde{u}} + \sigma_v \frac{\partial v}{\partial \tilde{u}}\right) \times \left(\sigma_u \frac{\partial u}{\partial \tilde{v}} + \sigma_v \frac{\partial v}{\partial \tilde{v}}\right)$$

$$= \underbrace{\sigma_u \times \sigma_u}_{0} \frac{\partial u}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} + \sigma_u \times \sigma_v \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}}$$

$\sigma_v \times \sigma_u = -(\sigma_u \times \sigma_v)$

$$+ \underbrace{\sigma_v \times \sigma_u}\frac{\partial u}{\partial \tilde{v}} \frac{\partial v}{\partial \tilde{u}} + \underbrace{\sigma_v \times \sigma_v}_{0} \frac{\partial v}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}}$$

$$= \sigma_u \times \sigma_v \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} - \frac{\partial u}{\partial \tilde{v}} \frac{\partial v}{\partial \tilde{u}}\right)$$

$$= \sigma_u \times \sigma_v \begin{vmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{vmatrix}$$

$$= \sigma_u \times \sigma_v \, |J(\Phi)| \longrightarrow (1)$$

We know from calculus that,

if ψ & $\tilde{\psi}$ are two maps between open sets in $\mathbb{R}^2$, then

$$J(\psi \circ \tilde{\psi}) = J(\psi)\, J(\tilde{\psi})$$

Applying this to the transition map Φ,

taking $\Psi = \Phi$ & $\tilde{\Psi} = \Phi^{-1}$,

we have

$$J(\Phi)\,J(\Phi^{-1}) = J(\Phi \circ \Phi^{-1})$$
$$= J(I(u,v))$$

where I is the identity fn

ie., $I(u,v) = (u,v)$

$$= \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$=$ Identity matrix

$\Rightarrow J(\Phi^{-1})$ is the inverse of $J(\Phi)$

ie., $J(\Phi^{-1}) = J(\Phi)^{-1}$

$\Rightarrow J(\Phi)$ is invertible

$\Rightarrow |J(\Phi)| \neq 0$

$\therefore$ ① $\Rightarrow \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \overset{\neq 0}{|J(\Phi)|}\ \overset{\neq 0}{\sigma_u \times \sigma_v}$ ($\because \sigma$ is regular)

$\therefore\ \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} \neq 0$

$\Rightarrow \tilde{\sigma}$ is regular

Note:

If regular surface patches σ & $\tilde{\sigma}$ are related as in this proposition, we say that $\tilde{\sigma}$ is a **reparametrisation** of σ, & that Φ is a **reparametrisation map.**

Since $\sigma = \tilde{\sigma} \circ \Phi^{-1}$,
we say that σ is a reparametrisation of $\tilde{\sigma}$.

Remark:

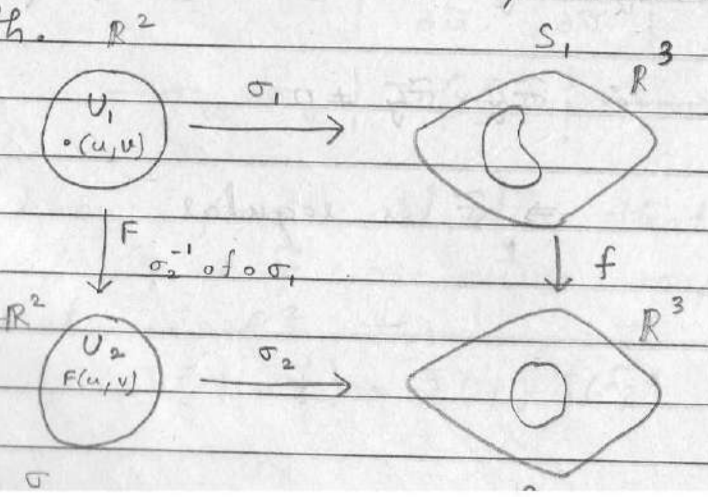
Diffeomorphic Surface

Let f be smooth map from smooth surface S_1 to another smooth surface S_2. Let S_1 & S_2 be covered by single surface patches $\sigma_1 : U_1 \to \mathbb{R}^3$ and $\sigma_2 : U_2 \to \mathbb{R}^3$, where σ_1 & σ_2 are bijective and they (defn of σ_1 & σ_2) are not affected by reparametrisation.

$\because \sigma_1$ & σ_2 are bijective

(Then any map $f : S_1 \to S_2$ gives rise to the map

$$\sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \to U_2,$$ we say that f is smooth if this map is smooth.



Now, suppose that $\tilde{\sigma}_1 : \tilde{U}_1 \to \mathbb{R}^3$ and $\tilde{\sigma}_2 : \tilde{U}_2 \to \mathbb{R}^3$ are reparametrisations of σ_1 and σ_1 with reparametrisation maps $\Phi_1 : \tilde{U}_1 \to U_1$ and $\Phi : \tilde{U}_2 \to U_2$, respectively.

~~Then the corresponding map~~
~~[illegible] is smooth iff~~

Then the corresponding map

$$\tilde{\sigma}_2^{-1} \circ f \circ \tilde{\sigma}_1 : \tilde{U}_1 \to \tilde{U}_2 \text{ is smoth,}$$

$$\text{if } \sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \to U_2 \text{ is smooth.}$$

Because

$$\tilde{\sigma}_2^{-1} \circ f \circ \tilde{\sigma}_1 = \tilde{\sigma}_2^{-1} \circ (\sigma_2 \circ \sigma_2^{-1}) \circ f \circ (\sigma_1 \circ \sigma_1^{-1}) \circ \tilde{\sigma}_1$$

$$= (\tilde{\sigma}_2^{-1} \circ \sigma_2) \circ (\sigma_2^{-1} \circ f \circ \sigma_1) \circ (\sigma_1^{-1} \circ \tilde{\sigma}_1)$$

$$= \Phi_2^{-1} \circ (\sigma_2^{-1} \circ f \circ \sigma_1) \circ \Phi$$

where Φ_1, ~~and~~ Φ_2^{-1} and $\sigma_2^{-1} \circ f \circ \sigma_1$ are all smooth maps.

Here, the map $f : S_1 \to S_2$ is smooth, bijective and whose inverse map $f^{-1} : S_2 \to S_1$ is smooth. Such maps are called **diffeomorphisms**, and S_1 & S_2 are said to be **diffeomorphic** if there is a diffeomorphism between them.

a) Explain concept of smooth map.

b) 20m

Proposition 4.3

Let $f: S_1 \to S_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on S_1, then $f \circ \sigma_1$ is an allowable surface patch on S_2.

Proof:

$f: S_1 \to S_2$ be a diffeomorphism.

Let us assume that S_1 is covered by a single allowable surface patch $\sigma_1: U_1 \to \mathbb{R}^3$ and S_2 is covered by a single allowable surface patch $\sigma_2: U_2 \to \mathbb{R}^3$.

Consider a smooth bijective map $F: U_1 \to U_2$ with smooth inverse F^{-1}

$$\begin{tikzcd} U_1 \ni (u,v) \arrow[r, "\sigma_1"] \arrow[d, "F = \sigma_2^{-1}\circ f \circ \sigma_1"'] & S_1 \subset \mathbb{R}^3 \arrow[d, "f"] \\ U_2 \ni F(u,v) \arrow[r, "\sigma_2"] & S_2 \subset \mathbb{R}^3 \end{tikzcd}$$

Then we can write

$$(f \circ \sigma_1)(u,v) = (\sigma_2 \circ F)(u,v)$$

ie, $f(\sigma_1(u,v)) = \sigma_2(F(u,v))$

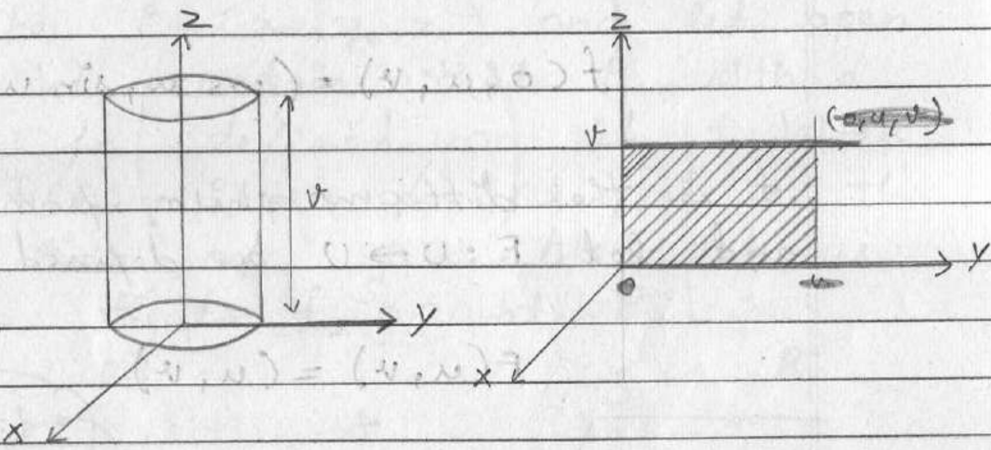
ie, $f \circ \sigma_1 = \sigma_2 \circ F$

$\Rightarrow \sigma_2 \circ F$ is allowable from prop. 4.2 and inturn it implies that $f \circ \sigma_1$ is allowable.

Example: 4.6

Find an allowable surface patch for the surface of cylinder (circular) of radius 1 and axis the z-axis.

Proof:



Let σ_1 & σ_2 be surface patches for the surfaces. An infinite strip (S_1) in the yz-plane and surface of the cylinder denoted by (S).

We can consider ~~σ_1 & σ~~

$\sigma_1 : U \to \mathbb{R}^3$ to be given by

$$\sigma_1(u,v) = (0, u, v)$$

& $\sigma_2 : U \to \mathbb{R}^3$ to be given by

$$\sigma_2(u,v) = (\cos u, \sin u, v)$$

where $U = \{ (u,v) \in \mathbb{R}^2 / 0 < u < 2\pi \}$.

Then we can graph a strip around the cylinder by wrapping the line $z = v$ parallel to the xy-axis around the waist of the cylinder at the height v along the xy-plane.

Since the width of the strip is equal to the circumference of the cylinder, a point in the y-coordinate 'u' of the cylinder will go to the polar angle 'u'.

So, let us define

$$f(o, u, v) = (\cos u, \sin u, v),$$

f is the diffeomorphism from S_1 to S_2 and let $F: U \to U$ be defined by

$$F(u, v) = (u, v)$$

Then by proposition 4.3,

* $f: S_1 \to S_2$ - diffeomorphism
* σ_1 - allowable s.p on S_1, then $f \circ \sigma_1$ " on S_2

$$f(\sigma_1(u, v)) = \sigma_2(F(u, v))$$

ie, $f(\sigma_1(u, v)) = \sigma_2(u, v)$

Hence surface of the cylinder is a smooth surface with a diffeomorphism connecting S_1 and S_2.

Theorem 4.1

Let S be a subset of $\mathbb{R}^3$ with the following property: for each point $P \in S$, there is an open subset W of $\mathbb{R}^3$ containing P and a smooth function $f: W \to \mathbb{R}$ $\ni$

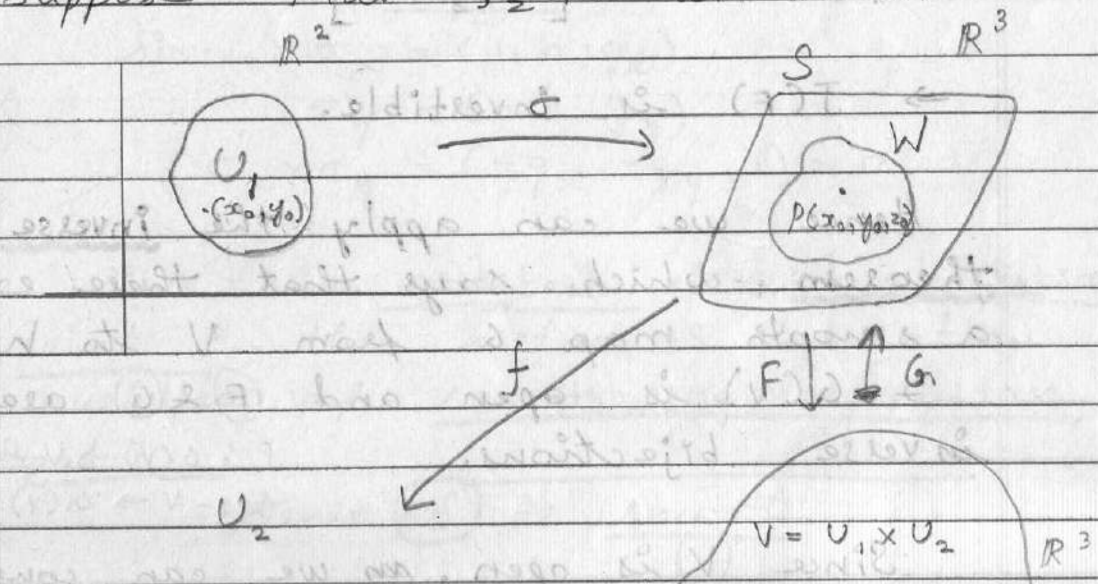
i) $S \cap W = \{(x, y, z) \in W / f(x, y, z) = 0\}$;

ii) the partial ~~dif~~ derivatives f_x, f_y & f_z do not all vanish at P.

Then, S is a smooth surface.

<u>Proof:</u>

Let S be a surface of $\mathbb{R}^3$ and let $P \in S$ be $P = (x_0, y_0, z_0)$ and let open subset W contain that P with a smooth fn defined on it $f : W \to \mathbb{R}'$ with $f(x, y, z) = 0$ for (x, y, z) in the intersection of $S \cap W$. Also let us suppose that $f_z \neq 0$ at P.



Let us define a smooth map in $\mathbb{R}^3$ as,

$F : W \to V \subset \mathbb{R}^3$ given by the eqn

$$F(x, y, z) = (x, y, f(x, y, z)).$$

where $f : W \to \mathbb{R}'$

Then the Jacobian matrix of F is

$$J(F) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}$$

$\Rightarrow |J(F)| \neq 0 \quad [\because f_z \neq 0]$

$\Rightarrow J(F)$ is investible.

Hence we can apply the **inverse fn theorem**, which says that there exist a smooth map G from V to W $\ni$ G(V) is open and F & G are inverse bijections. $F : G(V) \to V$ & $G : V \to G(V)$

Since V is open, as we can consider an open subset U_1 of $\mathbb{R}^2$ containing the point (x_0, y_0) and open interval U_2 of $\mathbb{R}^1$ containing 0 $\ni$ V contains the open set $U_1 \times U_2$ with the point $(x_0, y_0, 0)$.

Even further, we may assume that $V = U_1 \times U_2$.

Since F & G are inverse bijections, the map $G : V \to W$ is given by

$$G(x, y, w) = (x, y, g(x, y, w))$$

Now,
we are ready to define the surface patch $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ using above defined fns.

Let $\sigma : U_1 \to \mathbb{R}^3$ be given by,
$$\sigma(x, y) = (x, y, g(x, y, 0)).$$

Now,
it is obvious that σ is smooth as it is differentiable componentwise; it is also regular,

since
$$\sigma_x = (1, 0, g_x)$$
$$\sigma_y = (0, 1, g_y)$$
$$\sigma_x \times \sigma_y = (-g_x, -g_y, 1) \neq 0$$

Since we started with an arbitrary point P, the surface S has atlas consisting of smooth regular surface patches.

Hence S is smooth.

Example 4.7

Prove that the unit sphere S^2 is smooth using the above theorem.

Proof: Let us consider the fn
$$f(x, y, z) = x^2 + y^2 + z^2 - 1 \text{ in } \mathbb{R}^3.$$
$$\therefore (f_x, f_y, f_z) = (2x, 2y, 2z),$$

$$\|(f_x, f_y, f_z)\| = \sqrt{4x^2 + 4y^2 + 4z^2}$$
$$= \sqrt{4(x^2 + y^2 + z^2)}$$
$$= \sqrt{2^2 \cdot 1} = 2 \neq 0$$

$\therefore \|(f_x, f_y, f_z)\| = 2$ at all points of S^2.

In particular, (f_x, f_y, f_z) is non-zero everywhere on S^2.

Hence S^2 is a smooth surface.

Example 4.8

Prove that double cone is not a surface using the above theorem.

Proof:

Consider the fn $f(x,y,z) = x^2+y^2-z^2$ in $\mathbb{R}^3$.

Since $(f_x, f_y, f_z) = (2x, 2y, -2z)$,

$$\begin{aligned}\|(f_x, f_y, f_z)\| &= \sqrt{4x^2+4y^2+4z^2} \\ &= \sqrt{4(x^2+y^2+z^2)} \\ &= \sqrt{2^2 \cdot 0} \\ &= 0 \text{ at vertex } (0,0,0)\end{aligned}$$

Hence double cone is not a surface.

2.2) Tangents, Normal & Orientability:

Introduction: Surface patches interms of curve, study i.e in S

Let $\overline{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ be a space curve and let $\overline{\sigma} : U \to \mathbb{R}^3$ be a surface patch in the atlas S. ~~then~~

Then,
if there is a map $(\alpha, \beta) \to U$
i.e., a map of the form $t \mapsto (u, v)$,
then we can write
$\bar{\gamma}(t) = \bar{\sigma}(u(t), v(t))$ for $t \in (\alpha, \beta)$,

where $\gamma(t)$ is a curve lying in the image of surface patch σ.

Defn : [Tangent Space]

2m

The tangent space at a point P of a surface S is the set of tangent vectors at P of all curves in S passing through P.

Proposition 4.4

Let $\bar{\sigma} : U \to \mathbb{R}^3$ be a patch of a surface S containing a point P of S, and let (u, v) be coordinates in U. The tangent space to S at P is the vector subspace of $\mathbb{R}^3$ spanned by the vectors $\bar{\sigma}_u$ & $\bar{\sigma}_v$ (the derivatives are evaluated at the point $(u_0, v_0) \in U$ $\ni \bar{\sigma}(u_0, v_0) = P$).

6m
20m
8m

Proof:

Let $\bar{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ be a smooth curve in S, which is contained in the image of the surface patch $\bar{\sigma} : U \to \mathbb{R}^3$, given by
$$\bar{\gamma}(t) = \bar{\sigma}(u(t), v(t)).$$

Applying chain rule and denoting $\frac{d}{dt}$ by a dot,
we have

$$\dot{\bar{\gamma}}(t) = \bar{\sigma}_u \dot{u} + \bar{\sigma}_v \dot{v} \rightarrow (1)$$

Thus from eqn (1),
we see that $\dot{\gamma}$ is a lineal combination of the vectors $\bar{\sigma}_u$ & $\bar{\sigma}_v$
ie., the tangent space is a vector subspace of $\mathbb{R}^3$ spanned by the vectors $\bar{\sigma}_u$ and $\bar{\sigma}_v$.

Converesely,

Let us consider any vector subspace of $\mathbb{R}^3$ which is spanned by σ_u and σ_v.

Then any element of this subspace will be of the form

$\varepsilon_1 \bar{\sigma}_u + \eta \bar{\sigma}_v$, for some scalars ε_1 & η.

Looking at eqn (1),
we know how to define the fn γ.
ie., let us take

$$\bar{\gamma}(t) = \bar{\sigma}(u_0 + \varepsilon_1 t, v_0 + \eta t)$$

where σ is a smooth curve in S.

From this, we can find tangent to be,

$$\dot{\bar{\gamma}} = \bar{\sigma}_u \varepsilon_1 + \bar{\sigma}_v \eta \rightarrow (2)$$

Evaluating at the point $t=0$, we have ② to represent an element in the vector subspace which is obviously the tangent space.

Defn: [Orientable surface]

An orientable surface is a surface with an atlas having the property that, if Φ is the transition map between any two surface patches in the atlas, then $\det(J(\Phi)) > 0$ where Φ is defined.

Proposition 4.5

20 m

An orientable surface S has a canonical choice of unit normal at each point, obtained by taking the standard unit normal of each surface patch in the atlas of S.

Proof:

Since for any surface S with the collection of surface patches in the atlas at any point $P \in S$,

the tangent plane will pass through the origin in $\mathbb{R}^3$ and it is completely determined by giving a unit vector $\perp r$ to it, is called unit normal to S at P.

From proposition 4.4,

W.K.T,

when we choose a surface patch $\sigma : U \to \mathbb{R}^3$ containing a point P leads to a definite choice namely,

$$\hat{N}_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \to (1)$$, this is clearly a unit vector $\perp r$ to every linear combination of σ_u & σ_v and this is called a standard unit normal of the surface patch σ at P.

We see that $\hat{N}_\sigma$ is not independent of choice of patch σ containing P, if we take another surface $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ from the atlas of S containing P.

From proposition 4.2,

W.K.T,

$$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \det(J(\Phi))\, \sigma_u \times \sigma_v,$$

$J(\Phi)$ is the Jacobian of the transition map Φ

$$\therefore \frac{\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}}{\|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\|} = \frac{\det(J(\Phi))}{|\det(J(\Phi))|} \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

$$= \pm \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \pm \hat{N}_\sigma$$

ie, $\hat{N}_{\tilde{\sigma}} = \pm \hat{N}_{\sigma} \rightarrow (2')$

~~If the surface S~~

If we consider the surface S is an orientable surface,

then eqn (2) becomes,

$$\hat{N}_{\tilde{\sigma}} = \hat{N}_{\sigma}$$

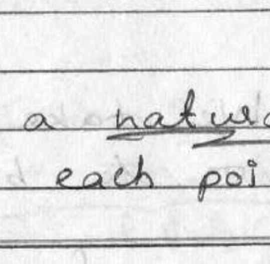
ie., the unit normal is independent of the choice of the surface patch in the atlas.

Hence we have a natural choice of unit normal at each point.

Example 4.9

Define Mobius band

8m The surface of Möbius band is not orientable.



Proof:

definition of Mobius band

(The mobius band is the surface obtained by rotating a straight line segment L around its midpoint P at the same time as P moves around the a circle C, in such a way that as P moves once around C, L makes a half-turn about P.)

If we take (C) to be the circle $x^2+y^2=1$ in the xy-plane, and (L) to be a segment of length 1 that is initially parallel to the z-axis with its midpoint P at (1,0,0), then after P has rotated by an angle θ around the z-axis, L should have rotated by $\theta/2$ around P in the plane containing P and the z-axis.

define σ

The point of L initially at (1, 0, t) is then at the point

$$\sigma(t,\theta) = \left(\left(1 - t\sin\frac{\theta}{2}\right)\cos\theta,\ \left(1 - t\sin\frac{\theta}{2}\right)\sin\theta,\ t\cos\frac{\theta}{2} \right) \quad (1)$$

We take the domain of definition of σ to be

$$U = \left\{ (t,\theta) \in \mathbb{R}^2 \,/\, -\tfrac{1}{2} < t < \tfrac{1}{2},\ 0 < \theta < 2\pi \right\}$$

define $\tilde{\sigma}$

For the same expression of σ,
Let us define another patch $\tilde{\sigma}$ with domain of

$$\tilde{U} = \left\{ (t,\theta) \in \mathbb{R}^2 \,/\, -\tfrac{1}{2} < t < \tfrac{1}{2},\ -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \right\}$$

These two surface patches σ & $\tilde{\sigma}$ will cover the surface of the mobius band.

$\therefore$ The mobius band is a smooth surface S with atlas consisting of the regular surface patches σ & $\tilde{\sigma}$.

finding $\hat{N}_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

Let us now compute the standard unit normal ($\hat{N}_\sigma$) at points on the median circle (where $t = 0$).

At such points, we have

$$\sigma_t = \left(-\sin\frac{\theta}{2}\cos\theta, -\sin\frac{\theta}{2}\sin\theta, \cos\frac{\theta}{2}\right)$$

$$\sigma_\theta = (-\sin\theta, \cos\theta, 0)$$

so,

$$\sigma_t \times \sigma_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\frac{\theta}{2}\cos\theta & -\sin\frac{\theta}{2}\sin\theta & \cos\frac{\theta}{2} \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\cos\frac{\theta}{2}\cos\theta\right) - \hat{j}\left(\sin\theta\cos\frac{\theta}{2}\right) + \hat{k}\underbrace{\left(-\sin\frac{\theta}{2}\cos^2\theta - \sin\frac{\theta}{2}\sin^2\theta\right)}_{-\hat{k}\sin\frac{\theta}{2}\,(\cos^2\theta + \sin^2\theta = 1)}$$

$$= \hat{i}\left(-\cos\theta\cos\frac{\theta}{2}\right) - \hat{j}\left(\sin\theta\cos\frac{\theta}{2}\right) - \hat{k}\left(\sin\frac{\theta}{2}\right)$$

$$= \left(-\cos\theta\cos\frac{\theta}{2}, -\sin\theta\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right)$$

$$\|\sigma_t \times \sigma_\theta\| = \sqrt{\cos^2\theta\cos^2\frac{\theta}{2} + \sin^2\theta\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}$$

$$= \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \sqrt{1}$$

$$= 1$$

$$\hat{N}_\sigma = \frac{\sigma_t \times \sigma_\theta}{\|\sigma_t \times \sigma_\theta\|}$$

$$= \frac{-\cos\theta\cos\theta/2\,\hat{i} - \sin\theta\cos\frac{\theta}{2}\,\hat{j} - \sin\theta/2\,\hat{k}}{1}$$

ie, $\hat{N}_\sigma = (-\cos\theta\cos\theta/2, -\sin\theta\cos\theta/2, -\sin\theta/2)$

Contradiction part

(If the Mobius band is orientable, then the unit normal defined at every point of S and will vary smoothly over S.

At a point $\sigma(0,\theta)$ on the median circle, we will have $\hat{N} = \lambda(\theta)\hat{N}_\sigma$, where $\lambda : (0, 2\pi) \to R$ is smooth and $\lambda(\theta) = \pm 1$ for all θ.

It follows that either $\lambda(\theta) = 1$ for all $\theta \in (0, 2\pi)$, or $\lambda(\theta) = -1$ for all $\theta \in (0, 2\pi)$.

Replacing $\hat{N}$ by $-\hat{N}$ if necessary, we can assume that $\lambda = 1$.

At the point $\sigma(0,0) = \sigma(0, 2\pi)$, we must have (since $\hat{N}$ is smooth)

$$\hat{N} = \lim_{\theta \to 0} \hat{N}_\sigma = (-1, 0, 0)$$

and also

$$\hat{N} = \lim_{\theta \to 2\pi} \hat{N}_\sigma = (1, 0, 0)$$

This contradiction shows that the Mobius band is not orientable.)

Examples of Surfaces:

1. Cylinder:

A (generalised) cylinder is the surface S obtained by translating a curve. If the curve is $\gamma:(\alpha,\beta)\to\mathbb{R}^3$ and $\vec{a}$ is a unit vector in the direction of translation, the point obtained by translating the point $\gamma(u)$ of γ by the vector $v\vec{a}$ $\parallel^{el}$ to $\vec{a}$ is

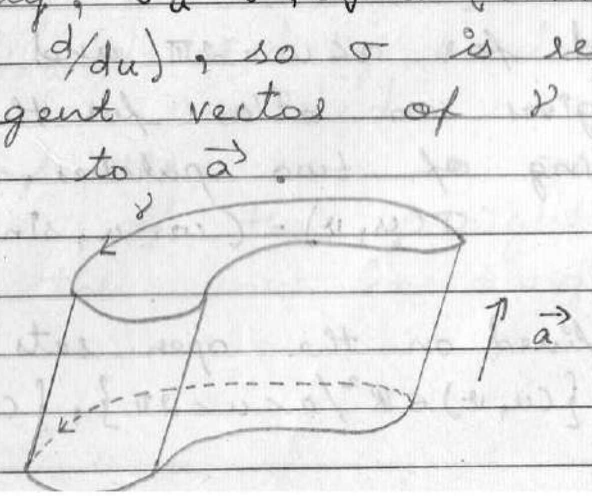
$$\sigma(u,v)=\gamma(u)+v\vec{a}.$$

Then, $\sigma: U\to\mathbb{R}^3$, where $U=\{(u,v)\in\mathbb{R}^2 / \alpha<u<\beta\}$, and clearly σ is smooth.

$\because$ $\sigma(u,v)=\sigma(u',v') \Leftrightarrow \gamma(u)-\gamma(u')=(v'-v)\vec{a}$, for σ to be a surface patch (& hence injective),

no straight line parallel to $\vec{a}$ should meet γ in more than one point.

Finally, $\sigma_u=\dot{\gamma}$, $\sigma_v=\vec{a}$ (with a dot denoting d/du), so σ is regular $\Leftrightarrow$ the tangent vector of γ is never parallel to $\vec{a}$.



The parametrisation is simplest when γ lies in a plane perpendicular to $\vec{a}$. In fact, this can always be achieved by replacing γ by its projection onto such a plane. (see exercise 4.22)

The regularity condition is then clearly satisfied provided $\dot{\gamma}$ is never zero, ie., provided γ is regular. We might as well take the plane to be the xy-plane and $\vec{a} = (0, 0, 1)$ to be parallel to the z-axis. Then, $\gamma(u) = (f(u), g(u), 0)$ for some smooth functions f & g, and the parametrisation becomes

$$\sigma(u, v) = (f(u), g(u), v).$$

As an example, starting with a circle, we get an ordinary (circular) cylinder. Taking the circle to have centre the origin, radius 1 and to lie in the xy-plane, it can be parametrised by

$$\gamma(u) = (\cos u, \sin u, 0),$$

defined for $0 < u < 2\pi$ and $-\pi < u < \pi$, say. This gives an atlas for the cylinder consisting of two patches, both given by

$$\sigma(u, v) = (\cos u, \sin u, v),$$

and defined on the open sets

$$\{(u, v) \in \mathbb{R}^2 / 0 < u < 2\pi\}, \{(u, v) \in \mathbb{R}^2 / -\pi < u < \pi\}.$$

2. Cone:

A (generalised) cone is the union of the straight line passing through a fixed point and the points of a curve. If 'p' is the fixed point and $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ is the curve, the most general point on the straight line passing through 'p' and a point $\gamma(u)$ of the curve is

$$\sigma(u, v) = (1-v)\,p + v\,\gamma(u).$$

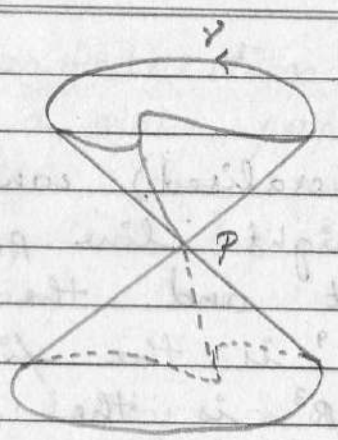
Then, σ is clearly smooth. Now,

$$\sigma(u, v) = \sigma(u', v') \iff v\,\gamma(u) - v'\,\gamma(u') + (v'-v)\,p = 0,$$

which says that the points p, $\gamma(u)$ & $\gamma(u')$ are collinear.

So, for σ to be a surface patch, no straight line passing through p should pass through more than one point of γ (in particular, γ should not pass through p).

Finally, we have $\sigma_u = v\dot{\gamma}$, $\sigma_v = \gamma - p$ (with d/du denoted by a dot), so σ is regular provided $v \neq 0$, i.e. the vertex of the cone is omitted, and ~~make~~ none of the straight lines forming the cone is tangent of γ.



The parametrisation is simplest when γ lies in a plane. If this plane contains p, the cone is simply part of that plane. Otherwise, we can take p to be the origin and the plane to be $z=1$. Then, $\gamma(u) = (f(u), g(u), 1)$ for some smooth fns f & g, and the parametrisation takes the form

$$\sigma(u, v) = v(f(u), g(u), 1).$$

Egs 1 & 2 are both special cases of the next class of surfaces.

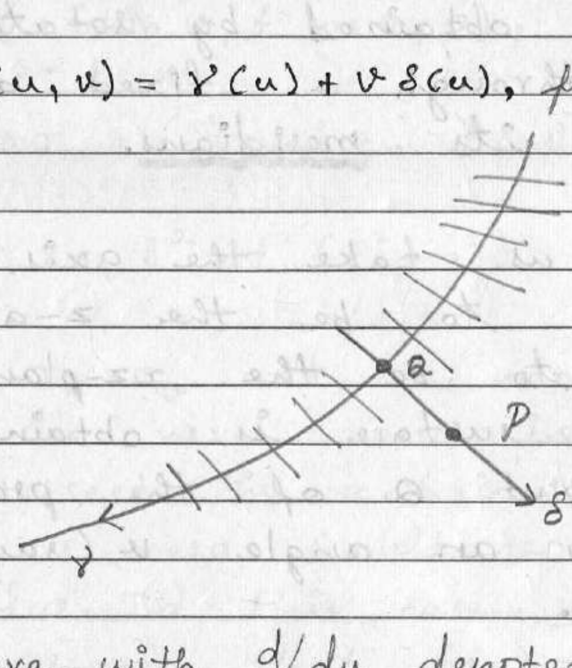
3. <u>Ruled Surface:</u>

A ruled surface is a surface that is a union of straight lines, called the <u>rulings</u> of the surface.

Suppose that C is a curve in $\mathbb{R}^3$ that meets each of these lines. Any point P of the surface lies on one of the given straight lines which intersects C at Q, say.

If γ is a parametrisation of C with $\gamma(u) = Q$, and if $\delta(u)$ is a non-zero vector in the direction of the line passing through $\gamma(u)$, P has position vector of the form

$$\sigma(u, v) = \gamma(u) + v\,\delta(u), \text{ for some scalar } v.$$



We have, with d/du denoted by a dot,

$$\sigma_u = \dot{\gamma} + v\dot{\delta}, \quad \sigma_v = \delta.$$

Thus, σ is regular if $\dot{\gamma} + v\dot{\delta}$ and δ are linearly independent. This will be true, for example, if $\dot{\gamma}$ and δ are linearly independent and v is sufficiently small. Thus, to get a surface, the curve C must never be tangent to the rulings.

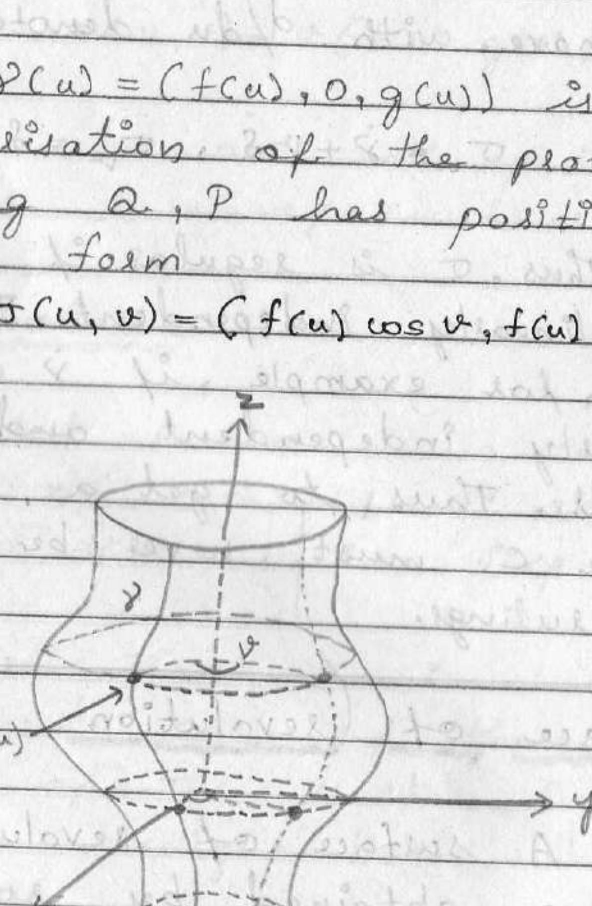
4. Surface of revolution:

A surface of revolution is the surface obtained by rotating a plane curve, called the profile curve, around a straight line in the plane. The circles obtained by rotating a fixed point on the profile curve around the axis of rotation are called the <u>parallels</u> of the surface, and the curves on the surface obtained by rotating the profile curve through a fixed angle are called its <u>meridians</u>.

Let us take the axis of ~~rotation~~ rotation to be the z-axis and the plane to be the xz-plane. Any point P of the surface is obtained by rotating some point Q of the profile curve through an angle v (say) around the z-axis.

If $\gamma(u) = (f(u), 0, g(u))$ is a parametrisation of the profile curve containing Q, P has position vector of the form

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$



To check regularity, we compute (with a dot denoting d/du):

$$\sigma_u = (\dot{f}\cos v, \dot{f}\sin v, \dot{g}),$$

$$\sigma_v = (-f\sin v, f\cos v, 0),$$

$$\therefore \sigma_u \times \sigma_u = (f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}),$$

$$\therefore \|\sigma_u \times \sigma_v\|^2 = f^2(\dot{f}^2 + \dot{g}^2).$$

Thus, $\sigma_u \times \sigma_v$ will be non-vanishing if $f(u)$ is never zero, i.e. if γ does not intersect the z-axis, and if $\dot{f}$ & $\dot{g}$ are never zero simultaneously, i.e. if γ is regular. In this case, we might as well assume that $f(u) > 0$, so that $f(u)$ is the distance of $\sigma(u, v)$ from the axis of rotation. Then, σ is injective provided that γ does not self-intersect and the angle of rotation v is restricted to lie in an open interval of length $\leq 2\pi$. Under these conditions, surface patches of the form σ give the surface of revolution the structure of a smooth surface.