# Unit-III

## Some Two & Three Dimensional Flows

### 3.1) Some flows involving Axial Symmetry:

Consider a steady irrotational flow of an incompressible fluid described by scalar velocity potential $\phi = \phi(r, \theta, \psi)$ in spherical polar coordinates.

Eqn of motion is $\nabla^2 \phi = 0$

Then it becomes

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\left(\frac{\partial^2 \phi}{\partial \psi^2}\right)\right) = 0$$

multiply by $r^2 \sin\theta$,

$$\Rightarrow \sin\theta\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right)\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{\sin\theta}\left(\frac{\partial^2 \phi}{\partial \psi^2}\right) = 0$$

When there is symmetry about the line $\theta = 0$ (at $\theta = 0$ rotation takes place and no change take place)

$$\phi : \phi(r, \theta, \psi) = \phi(r, \theta)$$

$$\Rightarrow \sin\theta\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right)\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

special P. solution : $\phi = r\cos\theta$ , $\phi = \frac{1}{r^2}\cos\theta$

General soln : $\phi(r, \theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta$

## Unit - III

3.1) Some flows involving Axial Symmetry

eg's
- * Irrotational stationary sphere in a uniform stream.
- * Sphere moving with constant velocity in liquid which is otherwise at rest.
- * Accelerating sphere moving in fluid at rest at infinity.
- * Underwater explosion giving spherical gas bubble.

3.2) Sources, Sinks & Doublets.

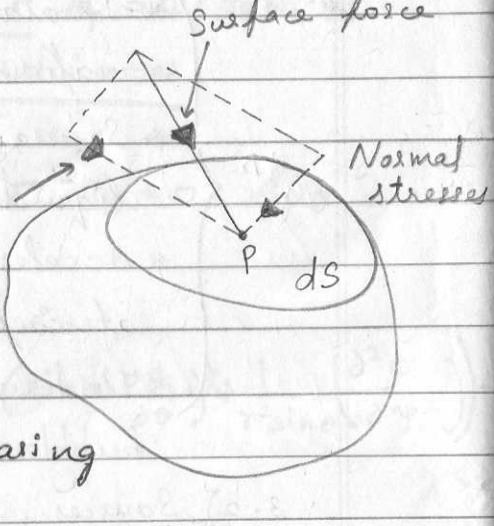
eg's
- * Doublet in a uniform stream
- * Sr. Abi - Problem
- * Illustration of line distributions

3.3) Axi-Symmetric flows; Stoke's Stream Function
- Some Special Forms of the stream fn for Axi-Symmetric Irrotational Motions
+ 1 eg in pg. 156

3.4) Meaning of two-Dimensional flows.

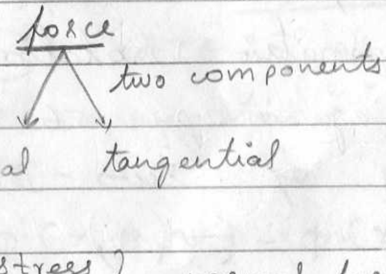
8m Discuss Problem of a sphere moving with constant velocity in liquid which is otherwise at rest.

## Viscous & Inviscid:




* A fluid is said to be viscous when normal as well as shearing stresses exist.

* A fluid is said to be inviscid when it does not exert any shearing stress, whether at rest or in motion.




normal stress (or) pressure } - normal force per unit area

Shearing stress - tangential force per unit area.

Viscosity - resistance to flow

water / air ≠ syrup / heavy oil

## Rotational & Irrotational :

* The motion of a fluid is said to be **irrotational** when the vorticity vector $\vec{\zeta}$ of every fluid is zero.

* When the vorticity vector is different from zero, the motion is said to be **rotational**.

rotational - fluid particles go on rotating about their own axes, while flowing

irrotational - do not rotate about their own axes, while flowing.

Uniform flows - fluid particles possess equal velocities at each section of the channel or pipe

non-uniform flows → possess different velocities

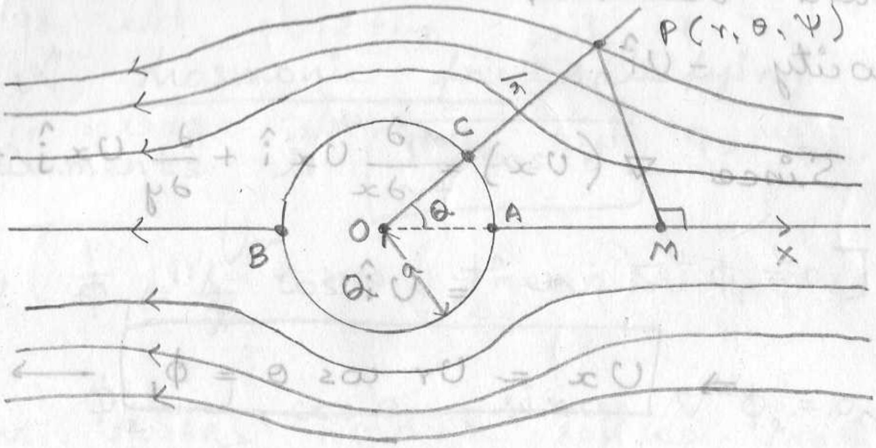
Steady - flow pattern remains unchanged with the time

Unsteady - flow pattern varies with time

# UNIT- III

$+ 3$ eqs in pg. 115

eg. ① ④ **Stationary Sphere in Uniform Stream:**




Suppose a solid stationary sphere of radius 'a' is placed in an uniform stream of incompressible fluid flowing with an undisturbed velocity $-U\hat{i}$ where $U$ is a constant and $\hat{i}$ is in the x-increasing direction $\overline{OX}$, taken to be the axis of symmetry.

Let $P$ be an arbitrary point in the flow described by spherical polar coordinates such that

$$\boxed{P : P(r, \theta, \psi)}.$$

Let PM be the perpendicular from P on OX such that $OM = x$, The line $\overline{OX}$.

ie.) $\hat{i}$ is taken to be the line of symmetry.

ie.) the line $\theta = 0$.

$$\therefore \boxed{x = r \cos\theta}$$

If there ~~will~~ were no sphere, the flow would be uniform & $\parallel^{el}$ to $\overrightarrow{XO}$ & velocity $-U\hat{i}$.

$$\text{Since } \nabla(Ux) = \frac{\partial}{\partial x} Ux\,\hat{i} + \frac{\partial}{\partial y} Ux\,\hat{j} + \frac{\partial}{\partial z} Ux\,\hat{k}$$

$$= U\hat{i}$$

$$\Rightarrow \boxed{Ux = Ur\cos\theta = \phi_1} \longrightarrow (1)$$

ie.) Velocity potential due to uniform force.

When a sphere is inserted, it will produce a local perturbation of the stream.

Stream lines will be disposed around the sphere as shown in the fig.

Hence undisturbed potential $Ur\cos\theta$ has to be modified by a 'perturbation potential' which is called due to the presence of the ~~flow~~ sphere.

The perturbation must have the following properties:

i) It must satisfy Laplace eqn for the case of axial symmetry.

ii) It must <u>tend to 0</u> at large distance ~~of~~ from the sphere so as to recover the condition of uniform parallel flow at <u>infinity</u>.

A <u>harmonic fn</u> satisfying these requirements is $\boxed{\frac{1}{r^2}\cos\theta}$.

$\left[\text{If } \phi_1 = Ur\cos\theta, \text{ then } \nabla^2\phi_1 = 0\right]$

$$\phi_2 = \frac{1}{r^2}\cos\theta, \text{ since } \nabla^2\phi_2 = 0$$

Since at the point 'P' in the fluid, when the sphere is inserted in the stream as shown in the fig. ~~the velocity potential of~~ field will be

<u>Velocity potential</u> $\Phi = \phi_1 + A\phi_2$

$$= Ur\cos\theta + \frac{A}{r^2}\cos\theta \quad \text{(I)}$$

where A determined from the constraint that there is <u>no flow normal to the surface</u> at $r=a$.

ie.) $\left[\frac{\partial\phi}{\partial r}\right]_{r=a} = 0$

$$\frac{\partial}{\partial r}\left[U\cos\theta + \frac{A}{r^2}\cos\theta\right] = -\frac{2A}{r^3}\cos\theta + U\cos\theta = 0$$

$$\Rightarrow \left(-\frac{2A}{r^3} + U\right)\cos\theta = 0$$

$$\Rightarrow A = \frac{Ua^3}{2}$$

$$\Rightarrow -\frac{2A}{a^3} + U = 0$$

$$\Phi = Ur\cos\theta + \frac{A}{r^2}\cos\theta \longrightarrow \text{(I)}$$

(I) becomes $\Rightarrow \Phi = Ur\cos\theta + \frac{1}{r^2}\frac{Ua^3}{2}\cos\theta$

$$\Phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

If P lies outside the sphere, $(r>a)$

If P lies on the sphere, $r=a$

$$\Phi = U\left(a + \frac{a^3}{2a^2}\right)\cos\theta = U\left(\frac{3}{2}\right)a\cos\theta$$

From earlier known result we have if (S) is the boundary of the spherical surface lying wholly within the fluid & the spherical surface is of radius 'a' lying in the fluid in motion with mean value of velocity potential $\phi$ on S gn by

$$\Phi = \left(\frac{M}{a}\right) = c$$

where 'a' be radius, M & c are constant.

Under the condition that fluid extend to $\infty$, and is at rest there. At any point $P, \phi_p$ the velocity potential at P tends to c as P. tends to $\infty$. ie.) $\Phi_p \to C$ as $p \to \infty$

Suppose there are more similar surfaces $S_n$ then the kinetic energy of the moving fluid is gn by

$$T = \frac{1}{2} \rho \int_V \vec{q}^{\,2} \, dv$$

$$= \sum_{m=1}^{k} \int_{S_n} \phi \frac{\partial \phi}{\partial n} \, ds$$

Thus, $\phi$ or $\frac{\partial \phi}{\partial n}$ is prescribed on each surface $S_n$ enclosing a volume & $S$ enclosing a volume $V$, then $\phi$ is determined uniquely throughout the volume $V$ within arbitrary constant.

This problem can be extended to fluid in uniform motion at $\infty$ and $\frac{\partial \phi}{\partial n}$ is prescribed on each of the surface $S_n$, then $\phi$ is determined uniquely throughout $V$.

We have $\phi_1 = U r \cos\theta \rightarrow (1)$

$$\phi_2 = \frac{1}{r^2} \cos\theta \rightarrow (2)$$

$$\phi = \phi_1 + A\phi_2 \rightarrow (3)$$

$$\left.\frac{\partial \phi}{\partial r}\right]_{r=a} = 0 \Rightarrow A = \frac{Ua^3}{2}$$

$$\Phi = \begin{cases} U \cos\theta \left(r + \frac{a^3}{2r^2}\right), & r > a \\ a U \cos\theta \left(\frac{3}{2}\right), & r = a \end{cases} \rightarrow (4)$$

Suppose $\vec{q}$ denote the velocity of fluid at $P$, then

$$\vec{q}\,\Big]_P = -\nabla\phi = -\frac{\partial\phi}{\partial r}$$

$$= -\left(1.\,\frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}e_\psi\right)$$

$$= q_r\hat{e}_r + q_\theta\hat{e}_\theta + q_\psi\hat{e}_\psi$$

$$\therefore\ q_r = -\frac{\partial\phi}{\partial r} = \begin{cases} -U\cos\theta\left(1-\frac{2a^3}{2r^3}\right), & r>a \\ 0, & r=a \end{cases}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \begin{cases} -U\sin\theta\left(1+\frac{a^3}{2r^3}\right), & r>a \\ 0, & r=a \end{cases}$$

$$q_\psi = \frac{-1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

$$\therefore\ \vec{q} = q_r\hat{e}_r + q_\theta\hat{e}_\theta + q_\psi\hat{e}_\psi$$

$$= \begin{cases} -U\cos\theta\left(1-\frac{a^3}{r^3}\right)\hat{e}_r - U\sin\theta\left(1+\frac{a^3}{2r^3}\right)\hat{e}_\theta, & r>a \\ \frac{3}{2}aU\sin\theta\,\hat{e}_\theta, & r=a \end{cases} \quad \to (5)$$

The stagnation pt is the fluid flow are those pt at which $\vec{q}=0$

From (5),

$q_r = 0 = q_\theta$

$$\Rightarrow \begin{cases} U\cos\theta\left(1-\dfrac{a^3}{r^3}\right) = U\sin\theta\left(1+\dfrac{1}{2}\dfrac{a^3}{r^3}\right) = 0, & r>a \\ 0 = \dfrac{3}{2}Ua\sin\theta = 0, & r=a \end{cases}$$

$$\cos\theta - \sin\theta - \cos\theta\,\frac{a^3}{r^3} - \frac{\sin\theta}{2}\,\frac{a^3}{r^3} = 0, \quad r>a$$

$$\cos\theta - \sin\theta - \frac{a^3}{r^3}\left(\cos\theta - \frac{\sin\theta}{2}\right) \neq 0, \quad r>a$$

doubt

At $\theta = 0$,

$$1 - \frac{a^3}{r^3} \neq 0, \quad r>a$$

From the above argument we infer that only for $r=a$ & $\sin\theta = 0$ we have $\vec{q}=0$

ie.) Stagnation pt in the fluid occur when $r=a$, $\theta=0$ (or) which correspond to A & B, shown in the fig.

Le $p_0$ denote the stagnation pressure at A and $p_\infty$ denote the pressure at infinity.

We note that, at A when the stagnation pressure is $p_0$, $\boxed{\vec{q}=0}$ and at $\infty$, when pressure is $p_\infty$, the velocity $\boxed{\vec{q}=-U\hat{i}}$

Along $\overrightarrow{XA}$, the flow of the fluid, we will be able to write

Bernoulli's eqn

$$\frac{1}{2}\vec{q}^{\,2} + \Omega + \frac{p}{\rho} = \text{constant}$$

where $\Omega$ is scalar pt fn denoting the body forces

$$\frac{1}{2}\rho\vec{q}^{\,2} + \rho\Omega + p = \text{constant} \longrightarrow (f)$$

At $\infty$,

$$\frac{1}{2}\rho\left(\vec{q}_{\infty}\right)^2 + \rho\Omega + p_{\infty} = \text{constant}$$

At A,

$$\frac{1}{2}\rho\left(\vec{q}_{0}\right)^2 + \rho\Omega + p_{\infty} = \text{constant}$$

$$\Rightarrow \frac{1}{2}\rho\left(\vec{q}_{\infty}\right)^2 + \rho\Omega + p_{\infty} = \frac{1}{2}\rho\left(\vec{q}_{0}\right)^2 + \rho\Omega + p_0$$

$$\Rightarrow \frac{1}{2}\rho U^2 + p_{\infty} = p_0 \longrightarrow (6)$$

Pressure at any point $c(a, \theta, \psi)$ on the surface of the sphere can be obtained from (5).

$$q_r\big]_c = 0$$

$$q_\theta\big]_c = -U\frac{3}{2}\sin\theta$$

$$q_\psi\big]_c = 0$$

Applying Bernoulli's eqn (f), along XAC,

if $p_c$ denote the pressure at C, then

At A,

$$\frac{1}{2}\rho(\vec{q}_0)^2 + \rho\Omega + p_0 = \text{constant}$$

At C,

$$\frac{1}{2}\rho(\vec{q}_c)^2 + \rho\Omega + p_c = \text{constant}$$

At $\infty$,

$$\frac{1}{2}\rho(\vec{q}_\infty)^2 + \rho\Omega + p_\infty = \text{constant}$$

We know, $\vec{q}_c = \frac{3}{2} U \sin\theta \, \hat{e}_\theta$

then $\vec{q}_c \cdot \vec{q}_c = \frac{9}{4} U^2 \sin^2\theta$

$$\Rightarrow \frac{1}{2}\rho\left(\frac{9}{4}U^2\sin^2\theta\right) + \rho\Omega + p_c = \text{constant}$$

$$\frac{1}{2}\rho(\vec{q}_\infty)^2 + \rho\Omega + p_\infty = \frac{1}{2}\rho\frac{9}{4}U^2\sin^2\theta + \rho\Omega + p_c$$

$$\frac{\rho U^2}{2}\left(1 - \frac{9}{4}\sin^2\theta\right) + p_\infty = p_c \rightarrow (7)$$

In (7), minimum pressure at C is gn by

$$\min p_c = p_\infty + \frac{\rho U^2}{2}\left(1 - \frac{9}{4}\right) \left[\because \max \sin^2\theta = 1 \Leftrightarrow \theta = \frac{\pi}{2} \text{ (or) } \frac{3\pi}{2}\right]$$

$$= p_\infty - \frac{5}{8}\rho U^2 \longrightarrow (8A)$$

and $\max p_c = p_\infty + \frac{\rho U^2}{2}(1-0)$ $\left[ \because \min \sin^2\theta = 0 \iff \theta = 0 \text{ (or) } \pi \right.$

$$= p_\infty + \frac{\rho U^2}{2} \longrightarrow \text{(8B)}$$

From (8A), note that the minimum pressure at any pt c on the sphere would occur at $r=a$ & $\theta = \frac{\pi}{2}$ along the equitorial line & the minimum pressure possible at c is $p_c = 0$ $\Rightarrow \min p_c = p_\infty - \frac{5}{8}\rho U^2$.

$$\Rightarrow p_\infty = \frac{5}{8}\rho U^2 \longleftarrow 0 = p_\infty - \frac{5}{8}\rho U^2$$

$$\boxed{U = +\sqrt{\frac{8}{5}\frac{p_\infty}{\rho}}}$$

This indicates that when the speed U of undisturbed stream is increased a point will be reached at which pressure everywhere along equatorial line $r=a$ and $\theta = \frac{\pi}{2}$ fall to 0.

At this stage, the fluid will tend to break away from the surface of the sphere along equitorial line & cavitation is said to occur.

Unit - II (continuation)

## Corollaries - From Kelvin's Theorem

### Corollary - 1

Suppose (S) is an open surface with rim $C$; a closed circuit under the ~~consid~~ condition that it is an inviscid fluid having a closed circuit $C$ of fluid particles moving along with the fluid where the body forces are conservative & the pressure (p) is a single valued fn of density only, then

$$\int_S \hat{n}\cdot\vec{\zeta}\, ds = \text{constant}$$

**Proof:**

$$\vec{\zeta} = \nabla \times \vec{q}$$

Kelvin's theorem states that,

"For an inviscid fluid circulation around any closed circuit of fluid particles, moving along with the fluid remains constant provided the body forces are conservative and the pressure is a single valued fn of density only."

$$\Gamma = \oint_C \vec{q}\cdot d\vec{r} = \text{constant} \rightarrow (f).$$

By Stoke's Theorem which states that,

" Suppose S is an open, two sided surface bounded by a closed, non-intersecting curve C.

Suppose $\bar{A}$ is a vector fn of position with continuous derivatives, then

$$\oint_C \bar{A}\cdot d\bar{r} = \iint_S (\nabla\times\bar{A})\cdot\hat{n}\, ds = \iint_S (\nabla\times\bar{A})\, d\bar{S} \text{ "} \longrightarrow (ff)$$

Consider,

$$\int_S \hat{n}\cdot\vec{\xi}\, ds = \int_S \hat{n}\cdot\nabla\times\vec{q}\, ds$$

$$= \oint_{C_1} \vec{q}\cdot d\vec{r} \quad [\text{by } (ff)]$$

$$= \oint_C \vec{q}\cdot d\vec{r} \quad [\because C_1 \text{ is a closed circle by hypothesis}]$$

$$= \Gamma = \text{constant} \quad [\text{by } (f)]$$

i.e., $\Gamma = \underbrace{\int_S \hat{n}\cdot(\nabla\times\vec{q})\, ds}_{=\text{constant}} \longrightarrow (fff)$

## Corollary - 2

(Suppose for an inviscid fluid, circulation around any closed circuit of the fluid particles moving along the fluid, where the body forces are conservative & pressure is a single valued fn of density only) (In particular, if we have $c$ to be rim of some open surface S) and further if any portion of the moving fluid once becomes irrotational, then it remains so for all subsequent times.

Kelvin's →

Corollary 1 →

**Proof:**

By corollary - 1 which states that

"Suppose S is an open surface with rim $c$, a closed circuit under the condition that it is a inviscid fluid having a closed circuit $c$ of fluid particles moving along with the fluid where the body forces are conservative and the pressure is a single valued fn of density only, then

$$\int_S \hat{n} \cdot \vec{\zeta}\, ds = \text{constant}"$$

i.e., $\Gamma = \int_S \hat{n}\cdot(\nabla\times\vec{q})\,ds = \text{constant} \rightarrow (1)$

Suppose at some instant, the fluid on S becomes irrotational

i.e., $\nabla\times\vec{q} = \vec{0}$

$$\Rightarrow \vec{\zeta} = \nabla\times\vec{q} = \vec{0}$$

$$\therefore (1) \Rightarrow \int_S \hat{n}\cdot\vec{0}\,ds = 0$$

i.e., $\Gamma = \int_S \hat{n}\cdot\vec{\zeta}\,ds = 0$

S is a non-zero infinitesimal element, say $\Delta s$,

then to the $1^{st}$ order

$$\Delta s\,\hat{n}\cdot\vec{\zeta} = 0 \quad \rightarrow \text{integrand of}$$

which forces $\vec{\zeta} = \vec{0}$ as $\Delta s \neq 0$

$\Rightarrow$ that the flow is irrotational throughout $\Delta s$.

Since $\Delta s$ is an arbitrary surface element, the flow is irrotational everywhere on S.

Corollary - 3

(Suppose for an inviscid fluid, the circulation around any closed circuit $C$ of the fluid particles, moving along the fluid, where the body forces are conservative and pressure is a single valued fn of density only), then the vortex lines move along with the fluid.

Kelvin's

Proof:

By corollary - 1 which states that

"Suppose $S$ is an open surface with rim $C$, a closed circuit under the condition that it is an inviscid fluid having a closed circuit $C$ of fluid particles moving along with the fluid where the body forces are conservative & the pressure is a single valued fn of density only, then $\int_S \hat{n} \cdot \vec{\zeta}\, ds =$ constant"

We have,

$$\boxed{\Gamma = \int_S \hat{n} \cdot \vec{\zeta}\, ds} \longrightarrow (1)$$

By Kelvin's theorem which states that

"For an inviscid fluid circulation around any closed circuit of fluid particles, moving along with the fluid remains constant provided the body forces are conservative and the pressure is a single valued fn of density only."

We have,

$$\boxed{\oint_{c_1} \vec{q} \cdot d\vec{r} = \text{constant}} \rightarrow (2)$$

For the surface (S) moving along with the fluid,

from (1),

$$\Gamma = \int_S \hat{n} \cdot \vec{\zeta} \, ds = \text{constant}$$

The integral represents, the total strength of vortex tubes passing through S and by hypothesis, ~~thus~~

thus we see that vortex tubes move with the fluid.

## Alternative Proof:

By taking (S) to be vanishingly small, we infer that, the vortex lines move with the fluid.

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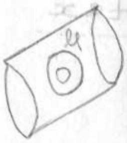
To be more specific, let $C_1$ be the closed curve drawn on the surface of the vortex tube containing an area (S) of the tube not embracing the tube.



Then $\boxed{\hat{n}\cdot\vec{\xi} = \hat{n}\cdot(\nabla\times\vec{q}) = 0}$ at each pt of S.

Using Stoke's theorem which states that

"Suppose S is an open, two sided surface bounded by a closed, non-intersecting curve c.

Suppose $\bar{A}$ is a vector fn of position with continuous derivatives, then

$$\oint_c \bar{A}\cdot d\bar{r} = \iint_S (\nabla\times\bar{A})\cdot\hat{n}\, ds = \iint_S (\nabla\times\bar{A})\, d\bar{S}"$$

We have,

$$\boxed{\int_S \hat{n}\cdot\vec{\xi}\, ds = \oint_{C_1} \vec{q}\cdot d\vec{r} = 0 \rightarrow (3) \left[\text{from } (1) \,\&\, (2)\right]}$$

By hypothesis,

the closed circuit $C_1$ and the surface $S$ are moving with the fluid.

For an irrotational flow, the circulation around $C_1$ is $0$ or $S$ remains a surface composed of vortex lines wherein (3) holds. $\int_S \hat{n}\cdot\vec{\xi}\, ds = \oint_{C_1} \vec{q}\cdot d\vec{r} = 0$ (3)

Thus we infer that the vortex line & vortex tubes move with the fluid.

Unit-III (continuation)

## Some Three-Dimensional Flows

### 3.2 Sources, Sinks and Doublets:

2m (Suppose at a pt. O in a fluid the flow is such that it is directed radially outwards from O in all directions and in a symmetrical manner. Then fluid enters the system through O which is termed as simple source.

If at O the volume entering per unit time is $4\pi m$, where m is a constant, then the strength of the source is defined to be m.)

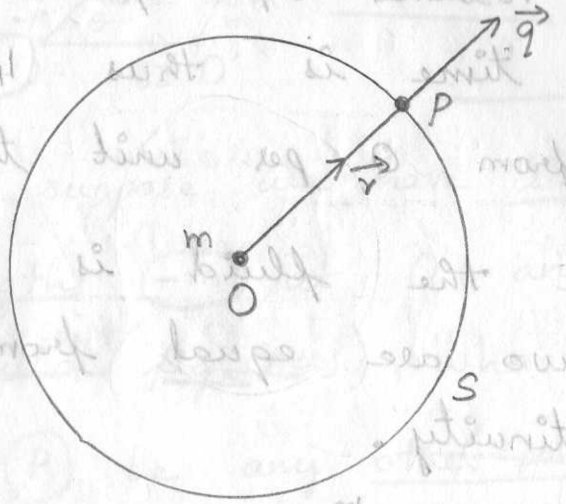


Fig (*)

(*) (If, however, the flow is such that fluid is directed radially inwards to O from all directions in a symmetrical manner, then fluid leaves the system at O which is termed a simple sink. A sink of strength m is a source of strength −m.)

Fig (*) shows a simple source of strength m at O in a fluid which is presumed (assume) to contain no other sources or sinks and which would otherwise be at rest.

S is the surface of the sphere centre O and radius r and P is a field on S such that $\overrightarrow{OP} = \vec{r}$. Then the fluid velocity at $P$ is $\vec{q}$ along $\overrightarrow{OP}$ and the magnitude $q$ is everywhere constant over $S$.

The volume of fluid crossing $S$ per unit time is thus $\boxed{4\pi r^2 q}$: that emitted from $O$ per unit time is $\boxed{4\pi m}$.

Since the fluid is incompressible, these two are equal from consideration of continuity.

magnitude $\vec{q}$

Thus $q = \dfrac{m}{r^2}$, $|\vec{q}|$

$$\vec{q} = \frac{m}{r^2}\hat{r}. \rightarrow (1)$$

It is easily shown that $\text{curl}\,\vec{q} =$ (except at $r=0$) so that the flow is of the potential kind.

Let $\varphi$ be the velocity potential at $P$. From considerations of symmetry,

$$\varphi = \varphi(r)$$

and $\text{grad}\,\varphi = \nabla\varphi = \varphi'(r)\,\hat{r} = \dfrac{\partial\varphi}{\partial r}\hat{r}$

$\text{curl}\,\vec{q} = \vec{0}$

Thus $\varphi'(r) = -m/r^2$,

$\nabla\cdot\vec{q} = \vec{0}$

doubt and so $\varphi(r) = m/r \rightarrow (2)$

$\nabla\cdot(-\nabla\varphi) = \vec{0}$

or

$\nabla^2\varphi = \vec{0}$

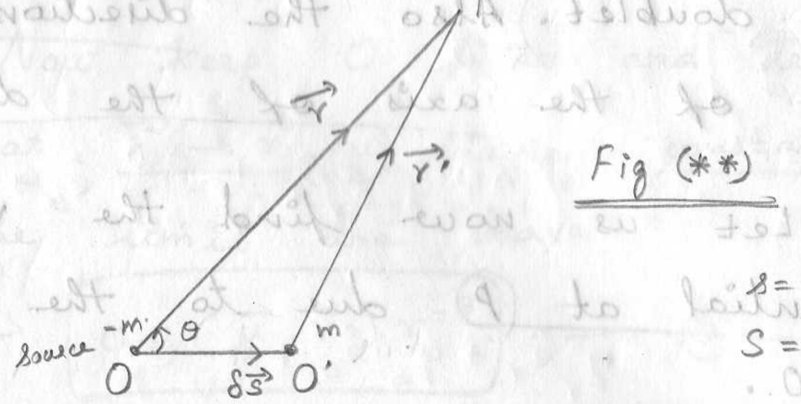



Fig (**)

s = arc length
S = surface

Now suppose we have simple sources of strengths $-m$ at O and $m$ at O' in a fluid (fig (**)), where $\overrightarrow{OO'} = \delta\vec{S}$.

Let P be any other point in the fluid such that $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{O'P} = \vec{r}'$. In the following we shall use $r = |\vec{r}|$, $r' = |\vec{r}'|$, $\delta s = |\delta\vec{S}|$. Also let $\boxed{\mu = m\,\delta s}$

$\cancel{\hat{x}}\ 2m$

Now suppose that $m \to \infty$, $\delta s \to 0$ in such a way that $\mu$ remains finite and constant. Then in the limit the two sources $\pm m$, of infinitely great magnitude and coincident at O, are said to constitute a three-dimensional doublet (or) dipole at O. The quantity $\mu$ is termed the moment (or) strength of the doublet. The quantity $\boxed{\vec{\mu} = \mu\hat{a}}$, whose $\hat{a}$ is the unit vector in the direction $\overrightarrow{OO'}$ is called the Vector moment of

doubt

2m

the doublet.) Also the direction of $\hat{a}$ is that of the axis of the doublet.

Let us now find the velocity potential at P due to the doublet at O.

For the configuration shown, assuming the flow to be entirely due to $-m$ at O and $m$ at O′, the velocity potential at P is

$$\phi = \frac{m}{r'} - \frac{m}{r}$$

$$= \frac{m(r - r')}{rr'}$$

$$= \frac{m(r-r')}{rr'} \times \frac{(r+r')}{(r+r')}$$

$$= \frac{m(r^2 - r'^2)}{rr'(r+r')}$$

$$= \frac{m(\vec{r} - \vec{r}')\cdot(\vec{r} + \vec{r}')}{rr'(r+r')} \leftarrow \left[\because (\vec{r} - \vec{r}')\cdot(\vec{r}+\vec{r}') = |\vec{r}|^2 - \vec{r}'\cdot\vec{r} + \vec{r}\cdot\vec{r}' - |\vec{r}'|^2 = r^2 - r'^2\right]$$

$$= \frac{m\delta\vec{s}\cdot(\vec{r}+\vec{r}')}{rr'(r+r')} \leftarrow \left[\because \text{By vector law addition } \vec{r} = \delta\vec{s} + \vec{r}', \quad \delta\vec{s} = \vec{r} - \vec{r}'.\right]$$

$$= \frac{\vec{\mu}\cdot(\vec{r}+\vec{r}')}{rr'(r+r')}$$

$$\phi = \frac{\vec{\mu}\cdot(\vec{r}+\vec{r}')}{rr'(r+r')} = \frac{\vec{\mu}\cdot(\vec{r}+\vec{r})}{rr'(r+r')} = \frac{2\vec{\mu}\cdot\vec{r}}{r^2(2r)} = \frac{\vec{\mu}\cdot\vec{r}}{r^3}$$

Now keep O fixed and let $O' \to O$, so that $\vec{r}' \to \vec{r}$, $\mu$ staying constant. Then in the limit we have

$$\boxed{\phi = (\vec{\mu}\cdot\vec{r})\, r^{-3}} \longrightarrow (3)$$

With $\angle POO' = \theta$, other equivalent forms for $\phi$ are

$$\boxed{\begin{aligned}\phi &= (\vec{\mu}\cdot\hat{r})\, r^{-2} \\ &= \mu r^{-2}\cos\theta\end{aligned}} \longrightarrow (3A)$$

The last form shows that $\phi = \phi(r, \theta)$, so that the axis of the doublet is an axis of symmetry as is physically obvious.

Taking P to have spherical polar coordinates $(r, \theta, \psi)$ referred to the axis of the doublet as initial line, the velocity potential at P being

$$\boxed{\phi(r, \theta) = \frac{(\mu\cos\theta)}{r^2}},$$

we find the velocity components at P. ~~are~~ Let $\vec{q}$ denotes the velocity of the fluid elt at P.

$$\vec{q}\,\Big]_P = -\nabla\varphi$$

$$= -\left(\frac{\partial\varphi}{\partial r}\,1.\hat{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\,\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\psi}\,\hat{e}_\psi\right)$$

$$= q_r\,\hat{e}_r + q_\theta\,\hat{e}_\theta + q_\psi\,\hat{e}_\psi$$

$\therefore$ The <u>velocity components</u> at $P$ are

$$q_r = -\frac{\partial\phi}{\partial r} = \frac{-(-2r^{-2-1}\mu\cos\theta)}{} = \frac{2\mu\cos\theta}{r^3}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\left(\frac{\mu(-\sin\theta)}{r^2}\right) = \frac{\mu\sin\theta}{r^3}$$

$$q_\psi = \frac{-1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = \frac{-1}{r\sin\theta}\cdot 0 = 0$$

The <u>eqn of stream lines</u> ~~are~~ is

$$\boxed{\frac{dr}{q_r} = \frac{r\,d\theta}{q_\theta} = \frac{r\sin\theta\,d\psi}{q_\psi}}$$

$$\Rightarrow \frac{r^3\,dr}{2\mu\cos\theta} = \frac{r^4\,d\theta}{\mu\sin\theta} = \frac{r\sin\theta\,d\psi}{0}$$

$$\Rightarrow \psi = \text{constant}$$

$$\Rightarrow \frac{dr}{2\cos\theta} = \frac{r\,d\theta}{\sin\theta}$$

$$\frac{dr}{r} = 2\cot\theta \, d\theta$$

$$\log r = 2\log\sin\theta + \log A$$

$$\log r = \log(A\sin^2\theta)$$

$$r = A\sin^2\theta$$

Thus the streamlines lie in planes passing through the axis of the doublet.

They are symmetrical not only w.r.t. OX but also about the plane $\boxed{\theta = \frac{1}{2}\pi.}$

Fig (***) shows a sketch of them.

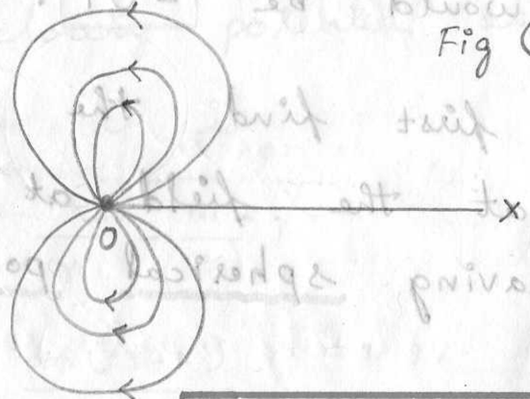


Fig (***)

The form $\boxed{\phi(r, \theta) = (\mu\cos\theta)/r^2}$ shows that $\phi$, at pts other than O, must satisfy the axially symmetrical spherical polar form of Laplace's eqn.

# Examples:

1. Doublet in a uniform stream. + 2 eg. in pg. 14

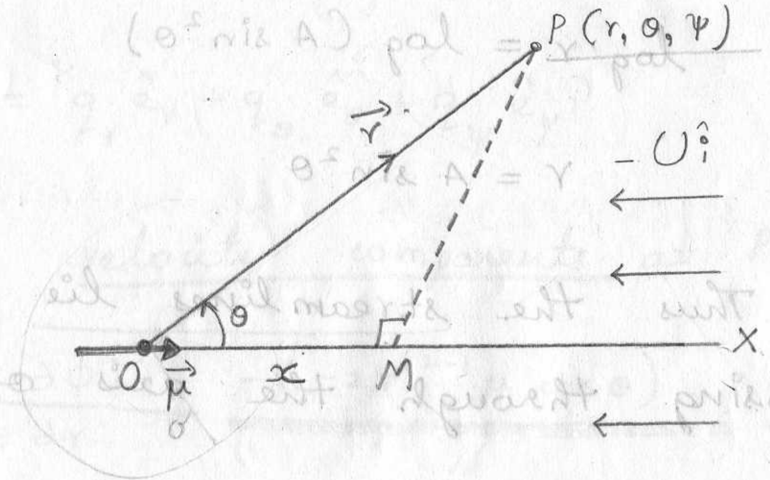



Fig. shows a doublet of vector moment $\boxed{\vec{\mu} = \mu\hat{i}}$ at O in a uniform stre[am] whose velocity, in the absence of the doublet, would be $\boxed{-U\hat{i}}$.

We first find the velocity potentia[l] $\boxed{\phi(r, \theta)}$ at the field at P in the fluid having spherical polar coordinates $(r, \theta, \psi)$.

Let PM be the ⊥ from P on OX

∋ OM = x.

The line $\overline{OX}$

ie., $\hat{i}$ is taken to be the line of symmetry.

ie., the line θ = 0 $\quad \cos\theta = \frac{x}{r}$.

$$\therefore \boxed{x = r\cos\theta}$$

If there were no sphere, the flow would be uniform & $\parallel^{el}$ to $\overrightarrow{XO}$ & velocity $-U\hat{i}$.

$$\because \nabla(Ux) = \frac{\partial}{\partial x} Ux\,\hat{i} + \frac{\partial}{\partial y} Ux\,\hat{j} + \frac{\partial}{\partial z} Ux\,\hat{k}$$

$$= U\hat{i}$$

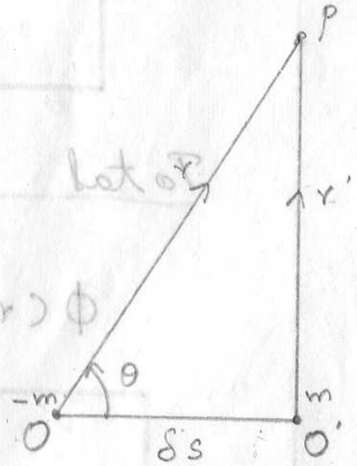
$$\Rightarrow Ux = Ur\cos\theta \quad \phi$$

ie., If PM is the $\perp$ from P on OX and if $OM = x$, then the velocity potential at P due to the streamline is

$$\boxed{Ux = Ur\cos\theta} \longrightarrow (1)$$

The velocity potential at P due to doublet at O is

$$\phi = \frac{m}{r'} - \frac{m}{r}$$




$$= \frac{m(r-r')}{rr'}$$

$$= \frac{m(r-r')(r+r')}{rr'(r+r')}$$

$$= \frac{m(r^2-r'^2)}{rr'(r+r')}$$

$$= \frac{m(\vec{r}-\vec{r}')\cdot(\vec{r}+\vec{r}')}{rr'(r+r')}$$

$$= \frac{m\,\delta\vec{s}\cdot(\vec{r}+\vec{r}')}{}$$

Fixed $O$, $O' \to O$ & $\vec{r}' \to \vec{r}$

$$\Rightarrow \phi = \frac{m\,\delta\vec{s}\,(\cancel{2}\vec{r})}{r^2(\cancel{2}r)}$$

$$= \frac{m\,\delta\vec{s}\cdot\vec{r}}{r^3}$$

$$= \frac{1}{r^3}\left(m\,\delta\vec{s}\cdot\vec{r}\right)$$

$$= \frac{1}{r^3}\left(m\,|\delta\vec{s}|\,|\vec{r}|\cos\theta\right)$$

$$= \frac{1}{r^{\cancel{3}2}}\underbrace{m\,\delta s}\,\cancel{r}\cos\theta$$

$$= \frac{1}{r^2}\,\mu\cos\theta$$

$$\boxed{= \frac{\mu\cos\theta}{r^2}} \longrightarrow (2)$$

<u>Total velocity potential</u> at $P$ is

$$\phi(r,\theta) = Ur\cos\theta + \frac{\mu\cos\theta}{r^2}$$

$$\boxed{= \left(Ur + \mu r^{-2}\right)\cos\theta} \longrightarrow (3)$$

Thus the <u>velocity components</u> at $P$ are

$$\vec{q}\,\Big]_P = -\nabla\phi = -\left(\frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}\right.$$

$$\vec{q}\Big]_P = q_r\,\hat{e}_r + q_\theta\,\hat{e}_\theta + q_\psi\,\hat{e}_\psi$$

$$\Rightarrow \quad q_r = -\frac{\partial\phi}{\partial r}, \quad q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}$$

$$\left.\begin{aligned} q_r &= -\left(U - \frac{2\mu}{r^3}\right)\cos\theta \\ q_\theta &= -\frac{1}{r}\left(Ur + \mu r^{-2}\right)(-\sin\theta) \\ &= \left(U + \frac{\mu}{r^3}\right)\sin\theta \\ q_\psi &= -\frac{1}{r\sin\theta}(0) = 0 \end{aligned}\right\} \longrightarrow 4$$

From 4, we see that $q_r = 0$ whenever

i) $\cos\theta = 0 \Rightarrow \boxed{\theta = \pi/2}$

(or)

ii) $\left(U - \frac{2\mu}{r^3}\right) = 0$

$$U = \frac{2\mu}{r^3}$$

$$r^3 = \frac{2\mu}{U}$$

$$\boxed{r = \left(\frac{2\mu}{U}\right)^{1/3}}$$

Write $a = \left(\frac{2\mu}{U}\right)^{1/3}$, so that $\mu = \frac{1}{2}Ua^3$.

Then it follows that there is no flow ~~over the surface~~ normal to the surface of the sphere $r = a$.

In fact for $\boxed{r \geq a}$, with $\boxed{\mu = \frac{1}{2} U a^3}$ substituted in (4), we obtain precisely the same velocity potential as was obtained for uniform flow past a stationary sphere of radius $a$.

ie., $$q_r = -\left( U - \frac{\cancel{2}\left(\frac{1}{\cancel{2}} U a^3\right)}{r^3} \right) \cos\theta$$

$$= -\left( U - \frac{U a^3}{r^3} \right) \cos\theta$$

$$= -U \cos\theta \left( 1 - \frac{a^3}{r^3} \right)$$

$$q_\theta = \left( U + \frac{\left(\frac{1}{2} U a^3\right)}{r^3} \right) \sin\theta$$

$$= U \sin\theta \left( 1 + \frac{a^3}{2 r^3} \right)$$

$$q_\psi = 0$$

Thus, for the region $\boxed{r \geq a}$, the analysis of this sphere problem is the same as that of the dipole of strength $\boxed{\frac{1}{2} . U a^3}$ in the uniform stream of velocity $- U \hat{i}$ the axis of the dipole being in the direction of

3.3 Axi - Symmetric Flows:

Stoke's Stream Function:

Uniform flow past a stationary sphere or a sphere moving with uniform velocity in a fluid at rest are examples of axi-symmetric flows.

Suppose the $\boxed{z\text{-axis}}$ be taken as axis of symmetry in an axi-symmetric flow and if P is a point represented in cylindrical polar coordinates system by $P(r, \theta, z)$ $(R, \theta, z)$, then all scalar and vector pt fns) associated with the flow is independent of $\theta$ and * [the eqn of continuity for an incompressible fluid flow will be given by,

$$\nabla \cdot \vec{q} = \vec{0} \,]$$

where $\vec{q}$ is the velocity at P.

$$\text{ie.,} \quad \vec{q} = q_r \hat{e}_r + q_z \hat{e}_z + 0 \cdot \hat{e}_\theta$$

$$\nabla \cdot \vec{q} = \left[ \frac{1}{h_1} \frac{\partial}{\partial r}\left(\frac{h_1}{h_2 h_3}\right)\hat{e}_r + \frac{1}{h_2}\frac{\partial}{\partial \theta}\left(\frac{h_2}{h_1 h_3}\right)\hat{e}_\theta + \frac{1}{h_3}\frac{\partial}{\partial z}\left(\frac{h_3}{h_1 h_2}\right)\hat{e}_z \right] \cdot \left(q_r \hat{e}_r + q_\theta \hat{e}_\theta + q_z \hat{e}_z\right)$$

$$= \cancel{\left(\frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z\right)\cdot\left(q_r\hat{e}_r + q_\theta\hat{e}_\theta + q_z\hat{e}_z\right)}$$

$$\ne \cancel{\frac{\partial}{\partial r}(q_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(q_\theta) + \frac{\partial}{\partial z}(q_z)}$$

$$\vec{q} = -\nabla\phi \Rightarrow \nabla\cdot\vec{q} = \nabla^2\phi$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial r}(h_2h_3q_r) + \frac{\partial}{\partial\theta}(h_3h_1q_\theta) + \frac{\partial}{\partial z}(h_1h_2q_z)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial\theta}(\overset{0}{q_\theta}) + \frac{\partial}{\partial z}(rq_z)\right] = 0$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial z}(rq_z)\right] = 0$$

ie., $r \ne 0$

$$\Rightarrow \boxed{\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial z}(rq_z) = 0} \longrightarrow (1)$$

Consider a <u>scalar pt fn $\psi(r, z)$</u>

satisfying

$$\left.\begin{aligned} q_r &= \frac{1}{r}\frac{\partial\psi}{\partial z} \Rightarrow \frac{\partial\psi}{\partial z} = rq_r \\ q_z &= -\frac{1}{r}\frac{\partial\psi}{\partial r} \Rightarrow \frac{\partial\psi}{\partial r} = -rq_z \end{aligned}\right\} \longrightarrow (2)$$

$\ni$ substituting (2) in (1),

$$\text{L.H.S of (1)} = \frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right)$$

$$= \frac{\partial^2\psi}{\partial r\,\partial z} - \frac{\partial^2\psi}{\partial z\,\partial r} = 0$$

$$= \text{R.H.S of (1)}$$

[Since $\psi$ represents a scalar pt fn continuous defining the flow.]

ie., $\psi$ is a scalar pt fn defined by (2) satisfying (1). Such a fn $\psi$ is called stoke's stream function.

Continues .....

## Examples :

V.V. Unit : 1

pg. 74

1. At the point in an incompressible fluid having spherical polar coordinates $(r, \theta, \psi)$, the velocity components are $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, where M is a constant. Show that the velocity is of the potential kind. Find the velocity potential and the equations of the streamline.

Soln:

In general, curvilinear orthogonal system, we have

$$\text{curl } \bar{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \partial/\partial\lambda & \partial/\partial\mu & \partial/\partial\nu \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

where $ds = (h_1\delta\lambda)\hat{a}_1 + (h_2\delta\mu)\hat{a}_2 + (h_3\delta\nu)\hat{a}_3$

$$F = F_1\hat{a}_1 + F_2\hat{a}_2 + F_3\hat{a}_3$$

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unit-1

2. For an incompressible fluid, $\vec{q} = [-\omega y, \omega x, 0]$ ($\because \omega =$ const). Discuss the nature of the flow.

Soln:

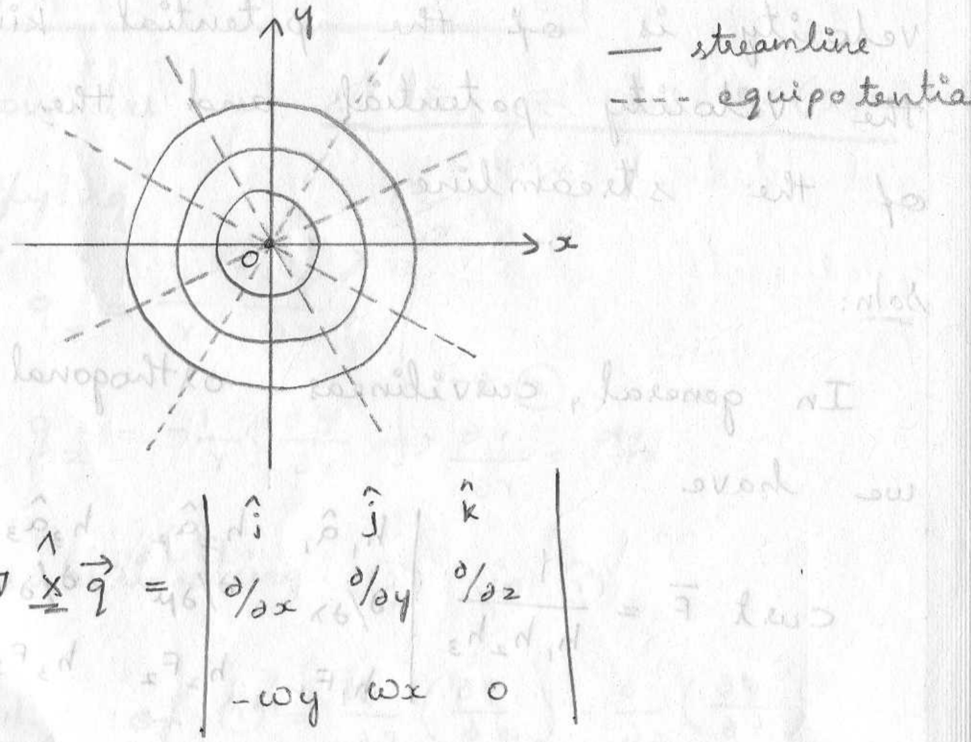
$$\text{Here} \quad \nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= 0 + 0 + 0 = 0$$

$\Rightarrow \nabla \cdot \vec{q} = 0$, so that such a flow is possible.

$$[\because u = -\omega y, v = \omega x, w = 0]$$

To find whether the flow is of potential kind:




$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(\omega + \omega)$$

$$= 2\omega \hat{k} \neq 0$$

Thus the flow is not potential kind.

~~To find the eqn of stream line:~~

It can easily be shown that a rigid body rotating about the z-axis with constant vector angular velocity $\omega\hat{k}$ gives the same type of motion. (For the velocity at $(x, y, z)$ in the body is $-\omega y\hat{i}+\omega x\hat{j}$)

To find the eqn of stream line:

The eqn of stream lines are gn by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{-\omega y} = \frac{dy}{\omega x} = \frac{dz}{0}$$

$$\frac{dx}{-\omega y} = \frac{dy}{\omega x} ; \; dz = 0$$

$$x\,dx = -y\,dy ; \; dz = 0$$

$$x\,dx + y\,dy = 0 ; \; dz = 0$$

i.e. the streamlines are the circles

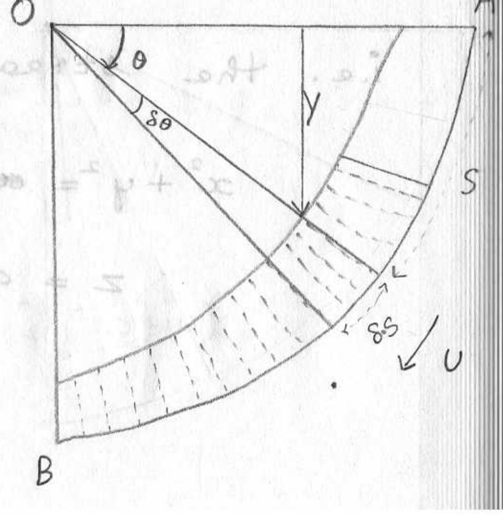
$$x^2 + y^2 = a \text{ (constant)},$$

$$z = a \text{ (constant)}.$$

3. AB is a tube of small uniform bore forming a quadrantal arc of a circle of radius a and centre O, OA being horizontal and OB vertical with B below O. The tube is full of liquid of density $\rho$, the end B being closed. If B is suddenly opened, show that initially $\frac{du}{dt} = \frac{2g}{\pi}$, where $u = u(t)$ is the velocity, and that the pressure at a point whose angular distance from A is $\theta$ immediately drops to $\rho g a\left(\sin\theta - \frac{2\theta}{\pi}\right)$ above atmospheric pressure. Prove further that when the liquid remaining in the tube subtends an angle $\beta$ at the centre, $\frac{d^2\beta}{dt^2} = -\frac{2g}{a\beta}\sin^2\left(\frac{\beta}{2}\right)$.

Soln:

Let at time 't'




* $S$ = arc AP
* $S + \delta S$ = arc AP'
* $y$ = Depth of P below OA
* $\angle AOP = \theta$

$POP' = \delta\theta$

$$= \left[ - \frac{\cos^3\theta}{3} \right]_0^{\pi}$$

$$T = \frac{1}{2 \times 3} \rho a^3 \pi U^3 \left[ -\cos^3\theta \right]_0^{\pi}$$

$$= \frac{1}{6} \rho a^3 U^2 \pi \left[ -\frac{\cos 3\theta - 3\cos\theta}{4} \right]_0^{\pi}$$

$$= \frac{1}{6} \rho \pi a^3 U^2 \left[ \frac{-(-1) - 3(-1) - (-1-3)}{4} \right]$$

$$= \frac{1}{6} \rho \pi a^3 U^2 \left[ \frac{1+3+1+3}{4} \right]$$

$$= \frac{1}{6} \rho \pi a^3 U^2 [2]$$

$$= \frac{1}{3} \rho \pi a^3 U^2$$

$$= \frac{1}{4} \frac{4}{3} \rho \pi a^3 U^2$$

$$= \frac{1}{4} M' U^2$$

Where $M'$ is the mass of fluid displaced. The total kinetic energy of the sphere and fluid is thus $\frac{1}{2}(M + \frac{1}{2}M')U^2$. The quantity $M + \frac{1}{2}M'$ is called the virtual mass of the sphere.

Continuation ...(*).

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After this → **Stream fn for 2-D flow → Vorticity ve[ctor] for 2-D flow:**

In addition if the flow is irrotational then $\vec{\xi} = \vec{0}$

$\Rightarrow \nabla^2 \psi = 0$

$\Rightarrow \psi$ satisfies Laplace's eqn.

Also for such a flow if $\phi$ = velocity po[tential] given by scalar point function : $\vec{q} = -\nabla \phi$ then satisfies $\nabla^2 \phi = 0$.

Thus in case when the flow is incompr[essible and] irrotational for an invisid fluid, both $\phi$ & $\psi$ satisfies Laplace Equation.

---

(Refer book) pg. 166

Velocity component at P along the normal from right to left as we travel from A to [B] is $v \cos\alpha - u \sin\alpha$.

Hence mass of the fluid that crosses unit thickness of surface elt through PP' normal to plane of flow per unit time is

$$\rho(v \cos\alpha - u \sin\alpha)\,\delta s = \rho(v\,\delta x - u\,\delta y)$$

This flux is constant denote it by $\delta\psi$.

$$\therefore \delta\psi = \rho(v\,\delta x - u\,\delta y)$$

$$\text{(or)} \quad d\psi = \rho(v\,dx - u\,dy) \rightarrow (f)$$

The total mass flux per unit thickness p[er] unit time from right to left across $\widehat{AB}$ is

$$\psi_B - \psi_A = \int_A^B \rho(v\,dx - u\,dy) \longrightarrow (ff).$$

A & B are fixed and (ft) is independent of path joining A & B hence is independent of $\phi$ joining A & B.

$\therefore \; d\psi$ is a perfect differential.

---x---

i.e., both velocity potential and stream fn is this case satisfies Laplace eqn and are hence **Harmonic functions**.

## Equipotentials:

For a 2D III flow equipotentials are gn by $\phi(x, y) = \text{cot}$ & the direction rations of normal at P to the surfaces $\phi(x, y) = \text{const}$ are (ie. dir ?? of $\nabla\phi$)

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0\right) \quad \left(\Leftrightarrow (a_1, a_2, 0)\right)$$

Curl $\vec{q} = \vec{0}$
the surfaces $\phi(x, y, z, t) = \text{const}$ are called equipotentials

Now considering (6) $\psi(x, y) = \text{const}$

...ipotentials ...s cut ...y.

## Remark:

III flow

(*) $u = -\phi_x = -\psi_y$ & $v = -\phi_y = \psi_x$

$\Rightarrow \phi_x = \psi_y$ & $\phi_y = -\psi_x$

(*) $\nabla^2 \phi = 0$ ; $\nabla^2 \psi = 0$

$$\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0$$

$\Longrightarrow \nabla\phi \perp^r \nabla\psi.$

...urves &

...urfaces is

...lines.

orthogonal.

particular

for 2D III flow.

A & B are fixed and (ft) is independent of path joining A & B hence is independent of $\phi$ joining A & B.

$\therefore d\psi$ is a perfect differential.

i.e., both velocity potential and stream fn in this case satisfies Laplace eqn and are hence **Harmonic functions**.

## Equipotentials:

For a 2D III flow equipotentials are gn by $\phi(x, y) = \text{cst}$ & the direction ratios of normal at P to the surfaces $\phi(x, y) = \text{const}$ are (ie. dir of $\nabla \phi$) $\hat{z}$?

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0\right) \; (\Leftrightarrow (a_1, a_2, 0))$$

Curl $\vec{q} = \vec{0}$
the surfaces $\phi(x,y,z,t) = \text{const}$ are called equipotentials

At equipotentials stream lines cut orthogonally.
pg. 73

Now, considering ⑥ $\psi(x,y) = \text{const}$ the normal at P to $\psi(x,y) = \text{const}$ has direction ratios

$$\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, 0\right) \; (\Leftrightarrow (v_1, v_2, 0))$$

$$a_1 v_1 + a_2 v_2 + 0 = \psi_x \phi_x + \psi_y \phi_y + 0$$

$$\vec{q} = -\nabla \phi \Rightarrow u = -\phi_x ; \; v = -\phi_y$$

Also $u = -\psi_y$, $v = \psi_x$

$$\Rightarrow \psi_x = -\phi_y \; ; \; \psi_y = \phi_x$$

$$\Rightarrow a_1 v_1 + a_2 v_2 = \underline{\qquad} + \phi_x \phi_y = 0$$

$\Rightarrow$ that the two families of equipotential curves & streamlines are mutually orthogonal.

In otherwords, the family of equipotential surfaces is cut orthogonally by the family of stream lines.

i.e., the st. lines & equipotentials are mutully orthogonal.

Such a feature exists for any flow & in particular for 2D III flow.

Examples:

2. Illustration of line distributions.

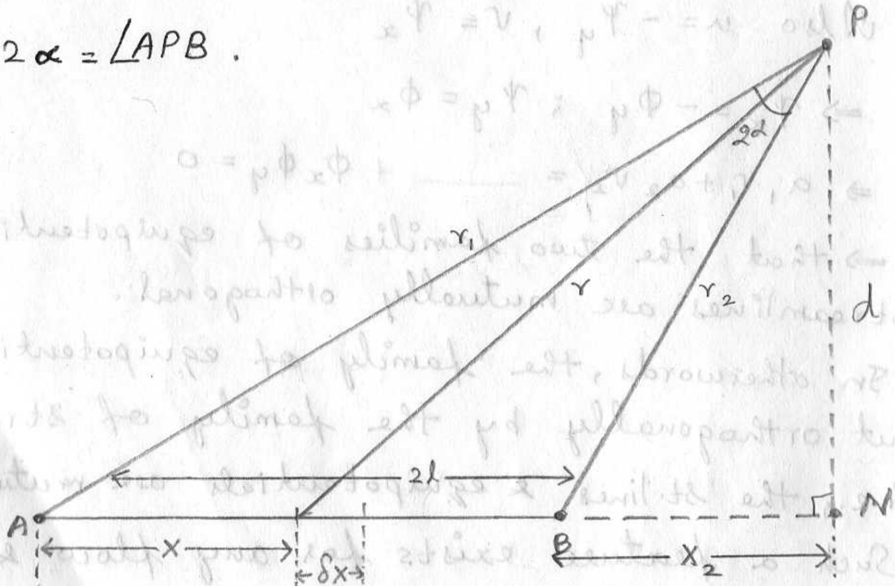
Prove that the velocity potential at a point P due to a uniform finite line source AB of strength 'm' per unit length is of the form $\phi = m \log f$, where

$$f = \frac{r_2 + x_2}{r_1 + x_1} = \frac{r_1 - x_1}{r_2 - x_2} = \frac{a+l}{a-l}$$

in which $AB = 2l$, $PA = r_1$, $PB = r_2$, $NA = x_1$, $NB = x_2$, N being the foot of the perpendicular from P on the line AB, and 2a the length of the major axis of the spheroid through P having A, B as foci.

Show that the velocity at P is

$\dfrac{(2ml\cos\alpha)}{(a^2 - l^2)}.\hat{u}$, where $\hat{u}$ is the unit vector along the normal to the spheroid at P and $2\alpha = \angle APB$.



The line-section of length $\delta x$ in AB at distance $x$ from A is effectively a point source of strength $m\delta x$ giving a velocity potential at P of amount $\frac{(m\delta x)}{r_1}$, $AP = r_1$.

From Pythogoras Theorem,

$$r^2 = d^2 + (x_1 - x)^2 \qquad \left[\because d^2 = r_1^2 - x_1^2 \quad r_1^2 = d^2 + x_1^2\right]$$

$$= d^2 + x_1^2 + x^2 - 2x_1 x$$

$$= r_1^2 - x_1^2 + x_1^2 + x^2 - 2x_1 x$$

$$= r_1^2 + x^2 - 2x_1 x$$

and so the total velocity potential at P due to the entire line distribution is

$$\phi = m\int_0^{2l} \frac{dx}{r}$$

$$= m\int_0^{2l} \frac{dx}{\sqrt{d^2 + (x_1 - x)^2}}$$

$$= m\int_0^{2l} \frac{dx}{\left[d^2 + (x_1 - x)^2\right]^{1/2}}$$

$$= m\int_0^{2l} \frac{dx}{\left[r_1^2 - x_1^2 + (x_1 - x)^2\right]^{1/2}}$$

$$= m\left[-\sinh^{-1}\left(\frac{x_1 - x}{d}\right)\right]_0^{2l} \qquad \left[\because \int \frac{dx}{(x^2+a^2)^{1/2}} = -\sinh^{-1}\frac{a}{x}\right]$$

$$\phi = m\left[-\sinh^{-1}\left(\frac{x_1-2l}{d}\right)+\sinh^{-1}\left(\frac{x_1}{d}\right)\right]$$

$$= m\left[\sinh^{-1}\left(\frac{x_1}{d}\right)-\sinh^{-1}\left(\frac{x_1-2l}{d}\right)\right]$$

$$\left[\because \sinh^{-1}x = \log\left(x+\sqrt{x^2+1}\right) \;\&\; x_1-2l = x_2\right]$$

$$= m\left[\sinh^{-1}\left(\frac{x_1}{d}\right)-\sinh^{-1}\left(\frac{x_2}{d}\right)\right]$$

$$= m\log\left[\frac{x_1+\sqrt{(x_1^2+d^2)}}{x_2+\sqrt{(x_2^2+d^2)}}\right]$$

$$\phi = m\log\left(\frac{x_1+r_1}{x_2+r_2}\right) \quad \left[\because r_1^2 = d^2+x_1^2\right] \quad \rightarrow (1)$$

Now,

$$r_1^2 - x_1^2 = d^2$$

$$r_2^2 - x_2^2 = d^2$$

i.e.) $r_1^2 - x_1^2 = r_2^2 - x_2^2$

$$(r_1-x_1)(r_1+x_1) = (r_2-x_2)(r_2+x_2)$$

So that

$$\frac{r_1+x_1}{r_2+x_2} = \frac{r_2-x_2}{r_1-x_1} = \frac{r_2-x_2+r_1+x_1}{r_1-x_1+r_2+x_2} = \frac{r_1+r_2+2l}{r_1+r_2-2l}$$

At $P$ on the spheroid through $P$ having $A, B$ as foci,

$$r_1 + r_2 = 2a$$

Hence

$$\frac{r_1 + x_1}{r_2 + x_2} = \frac{r_2 - x_2}{r_1 - x_1} = \frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l} = \frac{2a + 2l}{2a - 2l} = \frac{\cancel{2}(a+l)}{\cancel{2}(a-l)}$$

Thus sub in eqn ①,

$$\phi = m \log\left(\frac{a+l}{a-l}\right)$$

The equipotentials are given by

$$\boxed{\phi = \text{const}} \quad (\text{or}) \quad \boxed{\frac{a+l}{a-l} = \text{const}}$$

Thus they are $\boxed{a = \text{const}}$ (or) $\boxed{r_1 + r_2 = \text{cost.}}$

These surfaces are confocal ellipsoids of revolution about $AB$ with $A, B$ as foci.

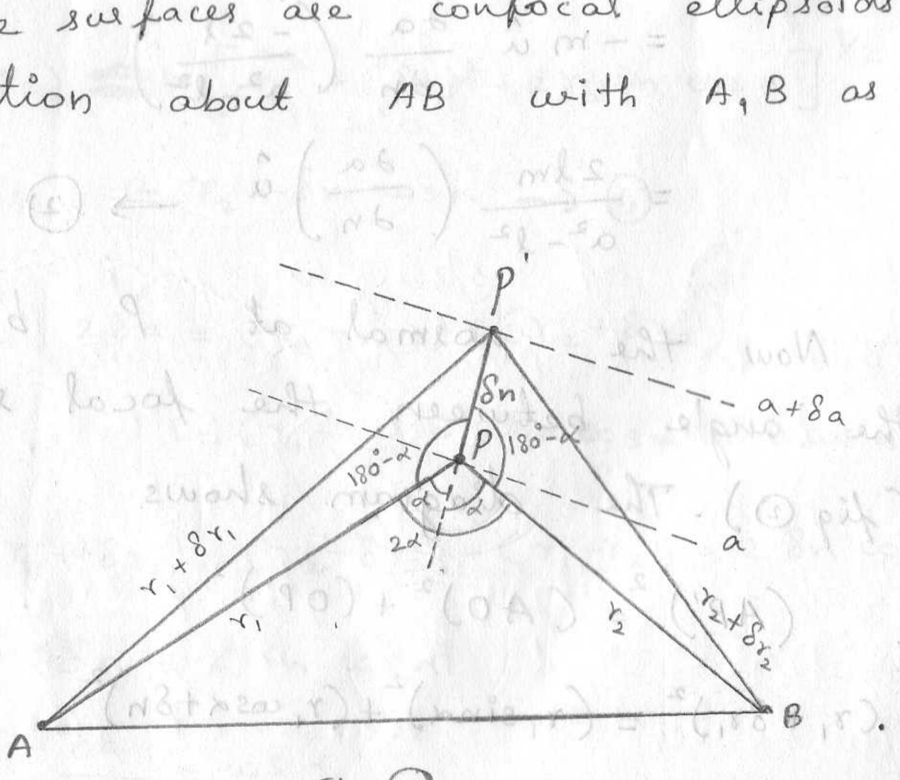



Fig ②

Let P be any point on the ellipsoid specified by parameter a, P' a pt on the neighbouring ellipsoid specified by parameter $(a+\delta a)$, where $\overline{PP'} = \delta n\, \hat{u}$.

Then the velocity at P is

$$\vec{q} = -\nabla\phi$$

$$\Rightarrow -\nabla\phi = -\frac{\partial\phi}{\partial n}\hat{u}$$

$$= -\frac{\partial}{\partial n}\left[m \log\left(\frac{a+l}{a-l}\right)\right]\hat{u}$$

$$= -m\hat{u}\frac{\partial}{\partial n}\left[\log(a+l) - \log(a-l)\right]$$

$$= -m\hat{u}\frac{\partial a}{\partial n}\left(\frac{1}{a+l} - \frac{1}{a-l}\right)$$

$$= -m\hat{u}\frac{\partial a}{\partial n}\left(\frac{\cancel{a}-l-\cancel{a}-l}{a^2-l^2}\right)$$

$$= -m\hat{u}\frac{\partial a}{\partial n}\left(\frac{-2l}{a^2-l^2}\right)$$

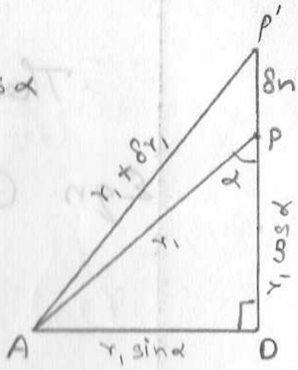
$$= \frac{2lm}{a^2-l^2}\left(\frac{\partial a}{\partial n}\right)\hat{u} \rightarrow (2)$$

Now the normal at P bisects $2\alpha$, the angle between the focal radii AP, PB (fig (2)). The diagram shows

$$(AP')^2 = (AO)^2 + (OP')^2$$

$$(r_1 + \delta r_1)^2 = (r_1 \sin\alpha)^2 + (r_1\cos\alpha + \delta n)^2.$$

$$(r_1+\delta r_1)^2 = r_1^2\sin^2\alpha + r_1^2\cos^2\alpha + (\delta n)^2 + 2r_1\,\delta n\cos\alpha$$




$$= r_1^2(\sin^2\alpha+\cos^2\alpha) + (\delta n)^2 + 2r_1\,\delta n\cos\alpha$$

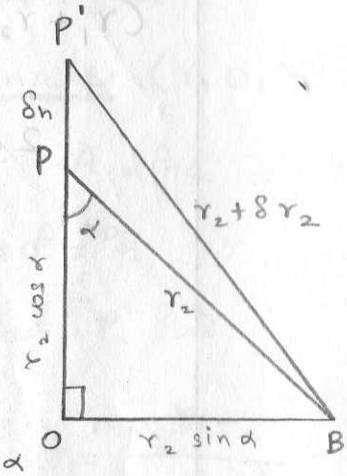
$$= r_1^2 + (\delta n)^2 + 2r_1\,\delta n\cos\alpha$$

$$(r_1+\delta r_1) = \left[r_1^2 + (\delta n)^2 + 2r_1\,\delta n\cos\alpha\right]^{1/2}$$

$$(r_1+\delta r_1) \simeq r_1 + \delta n\cos\alpha \rightarrow (3)$$

Similarly,

$$(BP')^2 = (BO)^2 + (OP')^2$$




$$(r_2+\delta r_2)^2 = (r_2\sin\alpha)^2 + (r_2\cos\alpha+\delta n)^2$$

$$= r_2^2\sin^2\alpha + r_2^2\cos^2\alpha + (\delta n)^2 + 2r_2\,\delta n\cos\alpha$$

$$= r_2^2(\sin^2\alpha+\cos^2\alpha) + (\delta n)^2 + 2r_2\,\delta n\cos\alpha$$

$$= r_2^2 + (\delta n)^2 + 2r_2\,\delta n\cos\alpha$$

$$(r_2+\delta r_2) = \left[r_2^2 + (\delta n)^2 + 2r_2\,\delta n\cos\alpha\right]^{1/2}$$

$$(r_2+\delta r_2) \simeq r_2 + \delta n\cos\alpha \rightarrow (4)$$

Hence $2\delta a = (r_1+\delta r_1) + (r_2+\delta r_2)$

$$= (r_1+r_2) + (\delta n\cos\alpha + \delta n\cos\alpha)$$

$$\Rightarrow (r_1+\delta r_1) + (r_2+\delta r_2) - (r_1+r_2) = 2\,\delta n\cos\alpha$$

Thus $\cancel{2}\delta a = \cancel{2}\,\delta n\cos\alpha$

$$\frac{\delta a}{\delta n} = \cos\alpha$$

The velocity at $P$ is $2lm\cos\alpha$

Eqn ③, we can written as

$$r_1+\delta r_1-r_1=\delta n\cos\alpha \longrightarrow ⑤$$

Eqn ④, we can written as

$$r_2+\delta r_2-r_2=\delta n\cos\alpha \longrightarrow ⑥$$

Adding ⑤ & ⑥, we get

$$(r_1+r_2)+(\delta r_1+\delta r_2)-(r_1+r_2)=\delta n\cos\alpha+\delta n\cos\alpha$$

$$\cancel{2a}+(\delta r_1+\delta r_2)-\cancel{2a}=2\,\delta n\cos\alpha$$

$$\cancel{2}\,\delta a=\cancel{2}\,\delta n\cos\alpha \qquad \left[\begin{array}{l}\because r_1+r_2=2a\\ (\delta r_1+\delta r_2)=2\delta a\end{array}\right]$$

$$\frac{\delta a}{\delta n}=\cos\alpha \longrightarrow ⑦$$

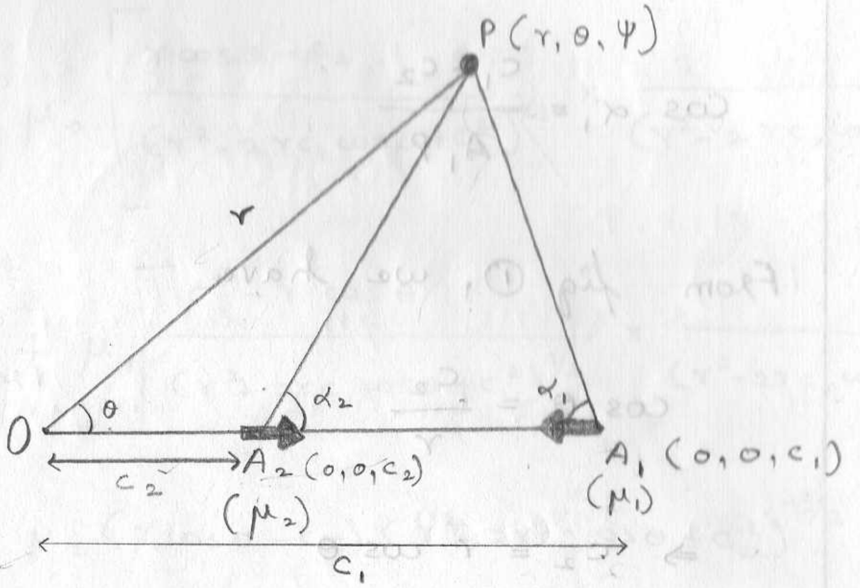
The velocity at $P$ is

$$-\nabla\phi=\frac{2lm\cos\alpha}{a^2-l^2}\,.\,\hat{u}$$

---

2. Doublets of strengths $\mu_1, \mu_2$ are situated at pts $A_1, A_2$ whose cartesian coordinates are $(0,0,c_1)$, $(0,0,c_2)$, their axes being directed towards and away from the origin respectively. (**) Find the condition that there is no transport of fluid over the surface of the sphere $x^2+y^2+z^2=c_1c_2$.)

Fig ①




Let initial line be $OA_2A_1$, P

P has spherical polar coordinates $(r, \theta, \psi)$

The axes of the doublets at $A_1, A_2$ make angles $\alpha_1, \alpha_2$ with $A_1P, A_2P$.

Then the velocity Potential at P is

$$\phi = \phi_1 + \phi_2 \qquad \phi = \frac{\mu \cos\alpha}{AP^2}$$

$$\boxed{\phi = \frac{\mu_1 \cos\alpha_1}{A_1P^2} + \frac{\mu_2 \cos\alpha_2}{A_2P^2}} \rightarrow (*)$$

where $\phi_1 = \dfrac{\mu_1 \cos\alpha_1}{A_1P^2}$, $\phi_2 = \dfrac{\mu_2 \cos\alpha_2}{A_2P^2}$

Now

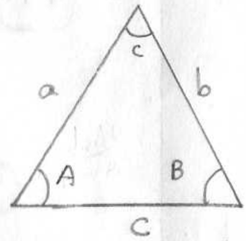
Let $A_1P^2 = r^2 - 2rc_1\cos\theta + c_1^2 \longrightarrow (1)$

$A_2P^2 = r^2 - 2rc_2\cos\theta + c_2^2 \longrightarrow (2)$

Since $a = r$, $b = c_1, c_2$

$$c^2 = a^2 - 2ab\cos\theta + b^2$$

$$\cos \alpha_1 = \frac{c_1 - c_2}{A_1 P}$$




From fig ①, we have

$$\cos \theta = \frac{c_2}{r}$$

From cosine formula

$a^2 = b^2 + c^2 - 2bc\cos A$

$$\Rightarrow c_2 = r \cos\theta$$

$$\therefore \boxed{\cos \alpha_1 = \frac{c_1 - r\cos\theta}{A_1 P}}$$

|||ly

$$\cos \alpha_2 = \frac{c_1 - c_2}{A_2 P}$$

From fig ①, we have

$$\cos\theta = \frac{c_1}{r}$$

$$\Rightarrow c_1 = r\cos\theta$$

$$\therefore \boxed{\cos \alpha_2 = \frac{r\cos\theta - c_2}{A_2 P}}$$

Sub. $a_1, a_2, b_1, b_2$ in ⊛

Also taking P to have spherical polar co-ordinates $(r, \theta, \varphi)$ referred to the axis of the doublet as initial line, the velocity potential at P being

$$\phi(r, \theta) = \frac{\mu \cos\theta}{r^2}.$$

$$\phi(r,\theta) = \mu_2 \left[ \frac{r\cos\theta - c_2}{(r^2 - 2rc_2\cos\theta + c_2^2)^{1/2}} \times \frac{1}{(r^2 - 2rc_2\cos\theta + c_2^2)} \right]$$

$$+ \mu_1 \left[ \frac{c_1 - r\cos\theta}{(r^2 - 2rc_1\cos\theta + c_1^2)^{1/2}} \times \frac{1}{(r^2 - 2rc_1\cos\theta + c_1^2)} \right]$$

$$= \mu_2 (r\cos\theta - c_2)(r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2}$$

$$+ \mu_1 (c_1 - r\cos\theta)(r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2}$$

<u>Differentiate $\phi(r,\theta)$ w.r.t $r$,</u>

$$\frac{\partial\phi}{\partial r} = \mu_2 \left[ (r\cos\theta - c_2)\left[ -\frac{3}{2}(r^2 - 2rc_2\cos\theta + c_2^2)^{-5/2} (2r - 2c_2\cos\theta) \right] + (r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2} (\cos\theta) \right]$$

$$\frac{1}{2}+1 = \frac{3}{2} \qquad \frac{1+2}{2} = \frac{3}{2}$$

$$+ \mu_1 \left[ (c_1 - r\cos\theta)\left[ -\frac{3}{2}(r^2 - 2rc_1\cos\theta + c_1^2)^{-5/2} (2r - 2c_1\cos\theta) \right] + (r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2} (-\cos\theta) \right]$$

$$= \mu_2 \left[ \cos\theta (r^2 - 2rc_2\cos\theta + c_2^2)^{-3/2} - 3(r\cos\theta - c_2)(r - c_2\cos\theta)(r^2 - 2rc_2\cos\theta + c_2^2)^{-5/2} \right]$$

$$+ \mu_1 \left[ \cos\theta (r^2 - 2rc_1\cos\theta + c_1^2)^{-3/2} + 3(c_1 - r\cos\theta)(r^2 - 2rc_1\cos\theta + c_1^2)^{-5/2} (r - c_1\cos\theta) \right]$$

We require that $\frac{\partial \phi}{\partial r} = 0$ when $r = (c_1 c_2)^{1/2}$

(**) [ W.K.T,

the sphere eqn is $\boxed{x^2+y^2+z^2=r^2}$ with the center in the origin.

From the qn,

$$r = (c_1 c_2)^{1/2}$$

$$\mu_2 \Big[ \cos\theta \left\{ c_1 c_2 - 2(c_1 c_2)^{1/2} c_2 \cos\theta + c_2^2 \right\}^{-3/2}$$

$$- 3\left\{ (c_1 c_2)^{1/2} \cos\theta - c_2 \right\} \left\{ (c_1 c_2)^{1/2} - c_2 \cos\theta \right\}$$

$$\left\{ (c_1 c_2) - 2(c_1 c_2)^{1/2} c_2 \cos\theta + c_2^2 \right\}^{-5/2} \Big]$$

$$= \mu_1 \Big[ \cos\theta \left\{ c_1 c_2 - 2c_1 (c_1 c_2)^{1/2} \cos\theta + c_1^2 \right\}^{-3/2}$$

$$+ 3\left\{ c_1 - (c_1 c_2)^{1/2} \cos\theta \right\} \left\{ (c_1 c_2)^{1/2} - c_1 \cos\theta \right\}$$

$$\left\{ c_1 c_2 - 2c_1 (c_1 c_2)^{1/2} \cos\theta + c_1^2 \right\}^{-5/2} \Big]$$

$$\Rightarrow \frac{\mu_2}{\mu_1} = \left( \frac{c_2}{c_1} \right)^{3/2} \Big]$$

## 3.4) Meaning of 2D flows:

two-dimensional

A fluid flow is called 2-D if the fluid moves in such a way that at any given instant of time the flow pattern in a certain plane is the same as that in all other parallel planes moves in the same plane.

Suppose we consider the flow of fluid to be in the XOY-plane $\Leftrightarrow$ $z=0$ plane in the cartesian co-ordinate system.

It follows that any point $P$ in the fluid with fluid velocity $\vec{q}(x, y, 0, t)$ independent of $z$.

i.e.) velocity is the same at all pts on planes $\|^{el}$ to $z$-axis.

Hence the fluid flows in such a way that at any given instant the flow pattern in a certain plane is the same as that in all planes parallel to $z=0$ plane.

Physicals quantities like velocity, pressure, density, etc. are independent of $z$ coordinate $P : P(x, y)$.

$\vec{q} : \vec{q}(x, y, t)$ vector quantity

$p : p(x, y, t)$ scalar

$\rho : \rho(x, y, t)$ scalar

An infinitesimal volume in a 2D flow is consider to be of unit thickness & hence elementary volume element considered, denoted by $\delta V$, defined by

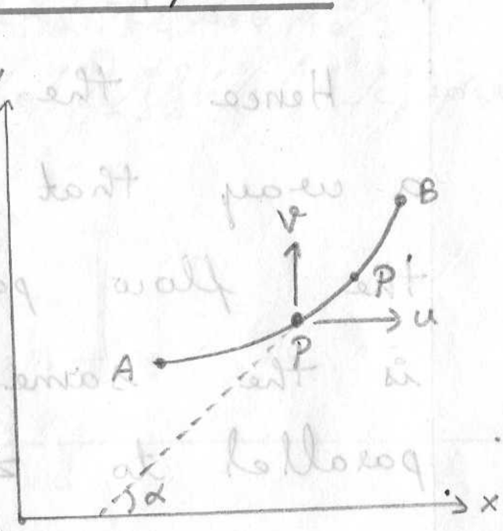
$$\delta V = \delta x . \delta y . 1$$
$$= \delta x . \delta y$$

Not in the syllabus

## Stream function for 2-D flows:

Consider an invisid incompressible 2D fluid flow.

Let the plane of flow be closed as XOY plane.



Let ¢ be an arc of a curve joining 2 pts in XOY plane.

Let P be the position of any fluid elt with co-ordinates $P(x, y, t)$ at time t, on ¢.

Let $\vec{q} : \vec{q}(x, y, t)$ be the fluid velocity at P at time t.

Then $\vec{q} = u\hat{i} + v\hat{j}$ at time t $\longrightarrow (1)$

where
$$\left.\begin{aligned} u &= u(x, y, t) \\ v &= v(x, y, t) \end{aligned}\right\} \longrightarrow (2)$$

The eqn of continuity (for incompressible fluid flow) is

$$\nabla \cdot \vec{q} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \cdot (u\hat{i} + v\hat{j}) = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (3)$$

Let $P' = P'(x + \delta x, y + \delta y, t + \delta t)$ be a neighbouring pt on ¢. So that $PP' = \delta s$.

Let the tangent at P to ¢ make angle $\alpha$ with horizontal axis OX.

For this flow define a scalar pt fn that is differentiable and uniform throughout the region of flow given by

$$\psi = \psi(x, y) \rightarrow (4)$$

Subj to conditions

$$u = -\psi_y \quad \& \quad v = \psi_x$$

$$u = -\frac{\partial \psi}{\partial y} \qquad v = +\frac{\partial \psi}{\partial x} \longrightarrow (5)$$

$$= -\psi_y \qquad = \psi_x$$

Sub (5) in (3), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

$\Rightarrow \psi$ satisfies (eqn. $\nabla \cdot \vec{q} = 0$) eqn of continuity.

$\Rightarrow$ existence of such a scalar pt fn $\psi = \psi(x, y)$ defining a 2D invisid incompressible fluid flow.

The stream lines for the above flow are gn by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{0}$$

$$\Longrightarrow \frac{dx}{-\psi_y} = \frac{dy}{\psi_x} = \frac{dz}{0}$$

$$dz = 0 \Rightarrow z = \text{const.}$$

$$\psi_x \, dx = -\psi_y \, dy$$

$$\Rightarrow \psi_x \, dx + \psi_y \, dy = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$\Rightarrow d\psi = 0 \Rightarrow \psi = \text{const} \rightarrow (6)$$

Thus the streamlines for the flow are gn by

$\psi(x, y) = \text{const}$ in $z = \text{const}$ plane.

$\Rightarrow \psi = \psi(x, y)$ is called the stream fn for 2-D inviscid incompressible flow.

---

__Vorticity vector for 2D flow:__

At $P(x, y)$, the vorticity vector $\vec{\zeta} = \nabla \times \vec{q}$ is given by

$$\vec{\zeta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix}$$

$$= \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$$

$$\Rightarrow \vec{\zeta} = \nabla \times \vec{q} = \left[ \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) - \frac{\partial}{\partial y} \left( -\frac{\partial \psi}{\partial y} \right) \right] \hat{k}$$

$$= \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \hat{k}$$

$$\Rightarrow \vec{\zeta} = (\nabla^2 \psi) \hat{k}$$

$\Rightarrow$ vorticity vector for 2D invisid flow is at right angles to the plane of flow.

i.e. $\vec{\zeta}$ is $|\vec{\zeta}| = \zeta$ is along the $\hat{k}$ directions.

i.e. z-axis of plane of flow is XOY plane.

$\Rightarrow \vec{\zeta}$ is $\perp r$ to plane of flow.

Continue - before (Meaning of 2D flow - 1 example) (*)

Not in the syllabus

Eg:

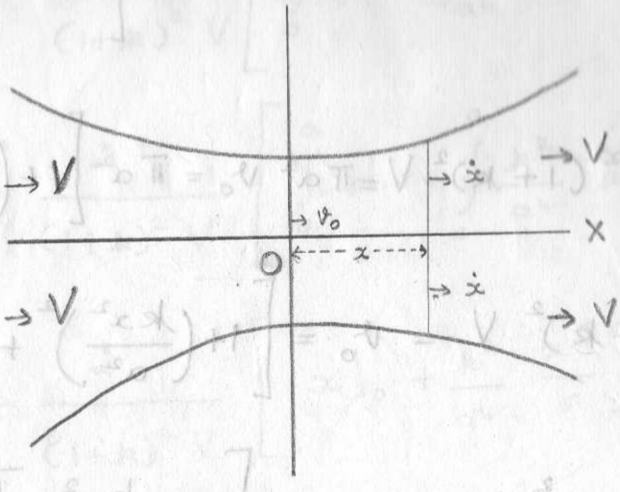
(or 6 m)
10 m

1. Liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + kx^2/a$ about the x-axis $(-a \leq x \leq a)$. If the liquid enters at the end $x = -a$ of the pipe with velocity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from $x = -a$ to $x = +a$ is

$$\left(\frac{2a}{V(1-k)^2}\right)\left(1 + \frac{2}{3}k + \frac{1}{5}k^2\right).$$

[Assume that $k$ is so small that the flow remains appreciably one-dimensional throughout.]

Soln:



Let $v_0$ be the velocity at the section $x = 0$.

The area of the section $x = -a$ is

$$\pi a^2 (1+k)^2$$

The area of the section $x = 0$ is

$$\pi a^2$$

The area of the section distant $x$ from O is

$$\pi \left( a + \left( \frac{kx^2}{a} \right) \right)^2$$

So the eqn of continuity (expressing equal rates of volumetric flow across the three section) is

area × flow pattern

$$\pi a^2 (1+k)^2 V = \pi a^2 v_0 = \pi \left( a + \left( \frac{kx^2}{a} \right) \right)^2 \dot{x}$$

$$\Rightarrow \pi a^2 (1+k)^2 V = \pi a^2 v_0 = \pi \left[ a^2 + \frac{(kx^2)^2}{a^2} + 2a\frac{kx^2}{a} \right] \dot{x}$$

$$\Rightarrow \cancel{\pi} \cancel{a^2} (1+k)^2 V = \pi a^2 v_0 = \cancel{\pi} \cancel{a^2} \left[ 1 + \left(\frac{kx^2}{(a^2)^2}\right)^2 + 2\frac{kx^2}{a^2} \right] \dot{x}$$

$$\Rightarrow (1+k)^2 V = v_0 = \left[ 1 + \left(\frac{kx^2}{a^2}\right)^2 + 2\frac{kx^2}{a^2} \right] \dot{x}$$

$$\Rightarrow (1+k)^2 V = v_0 = \left[ 1 + \frac{kx^2}{a^2} \right]^2 \dot{x}$$

since $\dot{x}$ is the velocity across the plane distant $x$ from $O$.

Then

$$\boxed{dt = \left(1 + \frac{kx^2}{a^2}\right)^2 \frac{dx}{(1+k)^2 V}}$$

Integrating on both sides,

$$\int_{-a}^{a} dt = \int_{-a}^{a} \left(1 + \frac{kx^2}{a^2}\right)^2 \frac{dx}{(1+k)^2 V}$$

$$t = 2\int_0^a \left(1 + \frac{kx^2}{a^2}\right)^2 \frac{dx}{(1+k)^2 V}$$

$$= \frac{2}{(1+k)^2 V} \int_0^a \left(1 + \frac{kx^2}{a^2}\right)^2 dx$$

$$\Rightarrow \quad t = \frac{2}{(1+k)^2 V}\left[\int_0^a \left(1 + \frac{k^2 x^4}{a^4} + 2\frac{kx^2}{a^2}\right) dx\right]$$

$$= \frac{2}{(1+k)^2 V}\left[\int_0^a dx + \int_0^a \frac{k^2}{a^4}\cdot x^4\,dx + \int_0^a \frac{2k}{a^2}\cdot x^2\,dx\right]$$

$$= \frac{2}{(1+k)^2 V}\left[\left. x\right]_0^a + \frac{k^2}{a^4}\left.\frac{x^5}{5}\right]_0^a + \frac{2k}{a^2}\left.\frac{x^3}{3}\right]_0^a\right]$$

$$= \frac{2}{(1+k)^2 V}\left\{a + \frac{k^2}{a^4}\cdot\frac{a^5}{5} + \frac{2k}{a^2}\,\frac{a^3}{3}\right\}$$

$$= \frac{2}{(1+k)^2 V}\left\{a + \frac{k^2 a}{5} + \frac{2ka}{3}\right\}$$

$$= \frac{2a}{(1+k)^2 V}\left(1 + \frac{k^2}{5} + \frac{2k}{3}\right)$$

Thus the <u>required time of flow is</u>

$$\frac{2a}{(1+k)^2 V}\left(1 + \frac{1}{5}k^2 + \frac{2}{3}k\right)$$

$$\text{i.e. } \frac{2a}{(1+k)^2 V}\left(1 + \frac{2}{3}k + \frac{1}{5}k^2\right)$$

---

2. Ans

2. <u>Underwater explosion giving spherical gas bubble.</u>

An explosion takes place under water initially at rest. As a result, a spherical gas bubble of center O, is formed. During expansion gas obeys adiabatic law $pv^{\gamma} = \text{const}$ where $p$ is the pressure, $v$ is the volume of the gas and $\gamma$ is a constant. Find the radius $R$ of the gas bubble at any time $t$ after the explosion, taking $R_0$ as the radius when $t = 0$.

At time $t$, when the gas bubble has radius $R$, let $r\ (\geq R)$ be the radius of a concentric spherical surface in the water.

~~Since the water is incompressible, by eqn of continuity the~~

$\because$ the water is incompressible,

by eqn of continuity,

$$\left.\begin{matrix}\text{the mass of fluid}\\ \text{leaving the fluid elt}\end{matrix}\right\} = \left\{\begin{matrix}\text{the mass of fluid}\\ \text{entering the fluid elt}\end{matrix}\right.$$

$$\rho(4\pi R^2)\dot{R} = \rho(4\pi r^2)\dot{\vec{r}}$$

$$\dot{\vec{r}} = \frac{R^2 \dot{R}}{\dot{\vec{r}}}$$

where $\dot{\vec{r}}$ & $\dot{R}$ are radial velocities.

The spherical co-ordinates are $(\vec{r}, \theta, \varphi)$ on the sphere of radius $\vec{r}$. The bubble changes shape with spherical symmetry, hence the flow is irrotational and there is a velocity potential:

If $\phi(\vec{r}, t)$ is the velocity potential, then $\dot{\vec{r}} = -\nabla\phi$

We know that,

$$\nabla\phi = \hat{e}_r \frac{\partial\phi}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial\phi}{\partial\theta} + \frac{\hat{e}_\varphi}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} = \frac{\partial\phi}{\partial r}\hat{e}_r$$

Here,

$$\nabla\phi = \frac{\partial\phi}{\partial\vec{r}}$$

$$\therefore \dot{\vec{r}} = -\frac{\partial\phi}{\partial\vec{r}} = \frac{R^2\dot{R}}{r^2}$$

$$\text{Then } \phi = -\int \frac{R^2\dot{R}}{r^2}\,dr$$

$$= +\frac{R^2\dot{R}}{r^2}$$

The <u>most</u> <u>general</u> <u>form</u> <u>of</u> <u>Bernoulli's eqn</u> is

$$\frac{1}{2}\vec{q}^{\,2} + \Omega + \int\frac{dp}{\rho} - \frac{\partial\phi}{\partial t} = f(t)$$

for an inviscid flow, ~~where~~

where $\vec{q} = -\nabla\phi$ is the velocity

$\vec{F} = -\nabla\Omega$ is the conservative body forces

$p$ is the pressure

$\rho$ is the density

$f(t)$ is an arb. constant

Here the flow is unsteady, the fluid is incompressible with no body forces, the Bernoulli's eqn can be written as

$$\frac{1}{2}\dot{\vec{r}}^2 + 0 + \int \frac{dp}{\rho} - \left(\frac{\partial \phi}{\partial t}\right)_{\vec{r}} = f(t)$$

$$\frac{1}{2}\dot{\vec{r}}^2 + \frac{1}{\rho}\int dp - \left(\frac{\partial \phi}{\partial t}\right)_{\vec{r}} = f(t)$$

$$\frac{1}{2}\dot{\vec{r}}^2 + \frac{p}{\rho} - \left(\frac{\partial \phi}{\partial t}\right)_{\vec{r}} = f(t)$$

Assume pressure to be 0 at $\infty$,

$$\frac{p}{\rho} + \frac{1}{2}\frac{R^4 \dot{R}^2}{r^4} - \frac{1}{\vec{r}}\frac{\partial}{\partial t}(R^2 \dot{R}) = 0$$

Hence, sub in (27),

$$\frac{3}{2}\dot{R}^2 + \frac{R}{2}\frac{d}{dR}(\dot{R}^2) = \left(\frac{R_0}{R}\right)^{3\gamma}\frac{P}{\rho}$$

$$\frac{d}{dR}(\dot{R}^2) + 3\frac{\dot{R}^2}{\vec{R}} = \frac{2P}{\rho}\frac{R_0^{3\gamma}}{(R)^{3\gamma+1}}$$

$R^3$ is an integrating factor for the above eqn.

Multiplying it throughout gives,

$$R^3 \frac{d}{dR}(\dot{R}^2) + 3R^2\dot{R}^2 = \frac{2P}{\rho}\,\frac{R_0^{3\gamma} R^3}{R^{3\gamma+1}}$$

$$\frac{d}{dR}(R^3\dot{R}^2) = \frac{2P}{\rho}\,\frac{R_0^{3\gamma}}{R^{3\gamma-2}}$$

Integrating from $R_0$ to $R$,

$$R^3\dot{R}^2 = \frac{2P}{\rho} R_0^{3\gamma} \left.\frac{R^{-3\gamma+3}}{-3\gamma+3}\right]_{R_0}^{R}$$

$$= \frac{-2P}{3\rho}\,\frac{R_0^{3\gamma}}{(\gamma-1)}\left(R^{-3\gamma+3} - R_0^{-3\gamma+3}\right)$$

$$= \frac{2P}{3\rho}\,\frac{R_0^{3\gamma}}{(\gamma-1)}\left(R_0^{-3(\gamma-1)} - R^{-3(\gamma-1)}\right)$$

$$R^3\dot{R}^2 = \frac{2P}{3\rho(\gamma-1)}\left[R_0^{3\gamma-3\gamma+3} - R_0^{3\gamma}\cdot R^{-3(\gamma-1)}\right]$$

$$\dot{R}^2 = \frac{2P}{3\rho(\gamma-1)}\left[\frac{R_0^3}{R^3} - \frac{R_0^{3\gamma}}{R^{3\gamma+3}}\right]$$

$$= \frac{2P}{3\rho(\gamma-1)}\left[\left(\frac{R_0}{R}\right)^3 - \left(\frac{R_0}{R}\right)^{3\gamma}\right]$$

Putting $r = R$, $\frac{p}{\rho} + \frac{R^4\dot{R}^2}{2R^4} - \frac{1}{\vec{R}}\frac{\partial}{\partial t}(R^2\vec{\dot{R}}) = 0$.

$$\frac{p}{\rho} + \frac{\dot{R}^2}{2} - \frac{1}{\vec{R}}\left(2\vec{R}\,\dot{R}^2 + R^2\vec{\ddot{R}}\right) = 0$$

$$\frac{p}{\rho} + \frac{\dot{R}^2}{2} - \left(2\dot{R}^2 + \vec{R}\,\vec{\ddot{R}}\right) = 0$$

$$\frac{p}{\rho} = \left(\frac{3\dot{R}^2}{2} + \vec{R}\,\vec{\ddot{R}}\right)$$

$$p = \rho\left(\frac{3\dot{R}^2}{2} + \vec{R}\,\vec{\ddot{R}}\right) \longrightarrow (*)$$

Considering the boundary conditions in the above eqn, we see the '$p$' is also the pressure in the gas.

Let $P$ be the pressure in the bubble at $t=0$ when $R=R_0$.

Then using the adiabatic ~~property~~ law.

$$p\left(\frac{4}{3}\pi R^3\right)^{\gamma} = \text{const} = P\left(\frac{4}{3}\pi R_0^3\right)^{\gamma}$$

$$\frac{p}{P} = \left(\frac{R_0^3}{R^3}\right)^{\gamma} = \left(\frac{R_0}{R}\right)^{3\gamma}$$

Sub. in $(*)$,

$$P\left(\frac{R_0}{R}\right)^{3\gamma} = \rho\left(\frac{3}{2}\dot{R}^2 + \vec{R}\,\vec{\ddot{R}}\right) \rightarrow (**)$$

$\ddot{R}$ can be expressed as

$$\ddot{R} = \dot{R}\left(\frac{d\dot{R}}{dR}\right) \qquad \left[\dot{R}\left(\frac{d\dot{R}}{dR}\right) = \frac{dR}{dt}\left(\frac{d}{dR}\left(\frac{dR}{dt}\right)\right) = \frac{d^2R}{dt^2} = \ddot{R}\right]$$

$$= \frac{d}{dR}\left(\frac{1}{2}\dot{R}^2\right)$$

In the special case, when $\gamma = \frac{4}{3}$

$$\dot{R}^2 = \frac{2P}{3\left[\frac{4}{3}-1\right]\rho}\left[\left(\frac{R_0}{R}\right)^3 - \left(\frac{R_0}{R}\right)^4\right].$$

$$\dot{R} = \frac{dR}{dt} = \left\{\frac{2P}{\rho}\left[\left(\frac{R_0}{R}\right)^3 - \left(\frac{R_0}{R}\right)^4\right]\right\}^{1/2}$$

$$\frac{dR}{dt} = \left[\frac{2PR_0^3}{\rho R^4}(R-R_0)\right]^{1/2}$$

$$= \left[\frac{2PR_0^3(R-R_0)}{\rho}\right]^{1/2}\frac{1}{R^2}$$

$$dt = \frac{R^2\,dR}{\left[\frac{2PR_0^3(R-R_0)}{\rho}\right]^{1/2}}$$

→(**) Sub. $R = R_0 + x$, we get on integration,

$$\int dt = \int \frac{(R_0+x)^2\,dx}{\left[\frac{2PR_0^3}{\rho}x\right]^{1/2}} \qquad R = R_0 + x, \quad dR = dx$$

the

o

$$t = \int \frac{R_0^2 + x^2 + 2R_0 x}{x^{1/2}} \left(\frac{2PR_0}{\rho}\right)^{-1/2} dx$$

$$= \int \left(\frac{R_0^2}{x^{1/2}} + x^{3/2} + 2R_0 x^{1/2}\right)\left(\frac{2PR_0}{\rho}\right)^{-1/2} dx$$

$$= \left(2R_0^2 x^{1/2} + \frac{2}{5} x^{5/2} + \frac{4}{3} R_0 x^{3/2}\right)\left(\frac{2PR_0}{\rho}\right)^{-1/2}$$

$$= \left[R_0^2 + \frac{x^{5/2 - 1/2}}{5} + \frac{2}{3} R_0 x^{3/2 - 1/2}\right]\left(\frac{2PR_0}{2\rho x}\right)^{-1/2}$$

$$= \left(R_0^2 + \frac{x^2}{5} + \frac{2}{3} R_0 x\right)\left(\frac{2\rho x}{2PR_0}\right)^{1/2}$$

$$= \left(R_0^2 + \frac{x^2}{5} + \frac{2R_0 x}{3}\right)\left(\frac{2\rho x}{2PR_0}\right)^{1/2} //$$

## 3.3) Axi-Symmetric Flow:

(Continues)

## Stoke's Stream Function

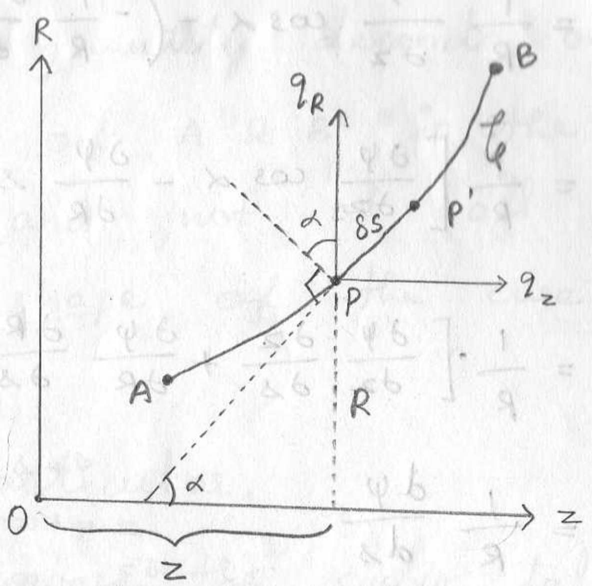
### Physical Meaning for $\psi$ :

Consider the axi-symmetric flow about the meridian plane, $\theta = 0$.

In this type of flow there is no velocity component $\perp r$ to the meridian plane.

$\rightarrow$ (f)

Fig (*)




In the fig, let $\widehat{AB}$ be an arc of the plane curve, P be a point with distance R from OZ and z from OR.

Let PP' be of length $\delta s$ such that P' is a neighbouring point of P.

We need to find

the volume of fluid crossing the surface of revolution AB about OZ per unit time from right to left.

Let $\alpha$ be the angle with tangent at P to the curve C with OZ. The normal component of velocity at P from right to left is given by

$$q_R \cos\alpha + q_z \cos(90+\alpha)$$

$$= q_R \cos\alpha - q_z \sin\alpha$$

$$= \frac{1}{R}\frac{\partial \psi}{\partial z}\cos\alpha - \left(-\frac{1}{R}\frac{\partial \psi}{\partial R}\right)\sin\alpha$$

$$= \frac{1}{R}\left[\frac{\partial \psi}{\partial z}\cos\alpha - \frac{\partial \psi}{\partial R}\sin\alpha\right]$$

$$= \frac{1}{R}\left[\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial \psi}{\partial R}\frac{\partial R}{\partial s}\right]$$

$$= \frac{1}{R}\frac{d\psi}{ds}$$

$r = s \quad \theta = \alpha$

$z = s\cos\alpha \quad R = s\sin\alpha$

$\frac{\partial z}{\partial s} = \cos\alpha \quad \frac{\partial R}{\partial s} = \sin\alpha$

$$= \frac{1}{R}\frac{\delta\psi}{\delta s}\left[\because \text{As } \lim_{\delta s\to 0}\frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}\right]$$

The volume of fluid crosses the surface of revolution at $PP'$ about $OZ$ per unit time is

$$\frac{1}{R}\frac{\delta\psi}{\delta s}\cdot 2\pi R\cdot\delta s$$

$$= \frac{\delta\psi}{\delta s}\cdot 2\pi\cdot\delta s$$

$$= 2\pi\,\delta\psi \quad \text{where } \delta\psi = \psi_{P'} - \psi_P$$

Hence the total volume of fluid crossing the surface of revolution of $\widehat{AB}$ about $OZ$ per unit time from left to right is

$$\int_A^B 2\pi\,\partial\psi = \Big[2\pi\psi\Big]_A^B = 2\pi\left[\psi_B - \psi_A\right]$$

This quantity depends only on the portions of A & B in the meridian section and not at all dependent on the shape of the cure $C_1$, ~~particular~~

In particular,

Suppose the curve falls on the oz axis, it is convenient to consider,

$$\psi_A = 0$$

So that the volume crossing the surface of revolution of AP about oz per unit time from right to left is

$$2\pi\left[\psi_P - \psi_A\right] = 2\pi\left[\psi_P - 0\right] \left(\because \psi_A = 0\right)$$

$$= 2\pi\psi \left(\because \psi_P = \psi\right)$$

<u>Note:</u>

The dimensions of $\psi$ are, $L^3T^{-1}$ from eqn (2) & those of the velocity potential is $L^2T^{-1}$. Also from (f), there is no fluid that crosses a stream and hence $\psi$ = const along stream line. Also $\psi$ = const.

over a stream ~~of~~ surface, ~~whose~~ where stream surface is the surface obtained by the surface of the revolution of the stream line about the axis of symmetry.

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## Velocity components in terms of spherical polar co-ordinates :

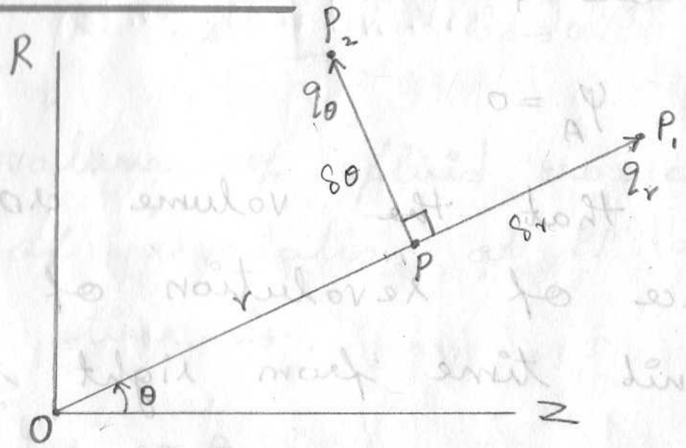



Fig. (*)

The fig shows the meridian section through the axis of the symmetry.

Let P be a pt at a distance r from O & at angular distance $\theta$ from OZ. The azimutal coordinates is redundant.

The fluid velocity components at P are $q_r$ along OP & $q_\theta$ at right angles to it in the sense of $\theta$.

The length $\boxed{PP_1 = \delta r}$ & $\boxed{PP_2 = r\,\delta\theta}$

1) If $\delta s = \delta r$, then the area of surface of revolution of $\delta s$ about $Oz$ is

$$2\pi r \sin\theta \, \delta r.$$

$$\left.\begin{array}{l}\text{Volume of fluid} \\ \text{crossing this per} \\ \text{unit time}\end{array}\right\} = 2\pi r \sin\theta \, \delta r \, q_\theta$$

So if $\psi$ is the stream fn at $P$,

then $2\pi \, \delta\psi = 2\pi r \sin\theta \, . \, \delta r \, q_\theta$

$$\boxed{q_\theta = \frac{1}{r\sin\theta} \frac{\partial \psi}{\partial r}} \longrightarrow (3)$$

2) If $\delta s = r\,\delta\theta$, then the area of surface of revolution of $\delta s$ about $Oz$ is

$$2\pi r \sin\theta \,( r\,\delta\theta ) = 2\pi r^2 \sin\theta \, \delta\theta$$

$$\left.\begin{array}{l}\text{Volume of fluid crossing} \\ \text{this per unit time}\end{array}\right\} = 2\pi r^2 \sin\theta \, . \, \delta\theta \, (-q_r)$$

If $\psi$ is the stream fn at $P$,

then $2\pi \, \delta\psi = 2\pi r^2 \sin\theta \, \delta\theta \, q_r$

$$\boxed{q_r = \frac{-1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}} \longrightarrow (4)$$

## Some Special Forms of the Stream function for Axi-Symmetric Irrotational Motions:

Fig. 4.12 pg. 15)

### i) Uniform Stream:

Suppose we have a flow where the velocity is $-U\hat{k}$, where $U$ is a constant. Let $P$ be the point in the stream having cylindrical polar coordinates $P(R, \theta, z)$ and let $P_0$ be its projection on the z-axis.

Taking $\psi$ to be the stream function associated with $P$ and zero ~~for~~ that for $P_0$, since the volume flowing from right to left through the circular disc obtained by revolving $PP_0$ about the z-axis is $\pi R^2 U$, it follows that

$$2\pi \psi = \pi R^2 U$$

$$\psi = \frac{1}{2} R^2 U$$

This is expressible in spherical polar coordinates, taking $R = r \sin \angle POZ$ where $r = OP$.

~~ii) Simple Source~~ Substitute the value of $R$, we get

$$\psi = \frac{1}{2} (r \sin\theta)^2 U$$

$$= \frac{U}{2} r^2 \sin^2\theta$$

## ii) Simple Source:

For a simple source of strength $m$ at the origin the value of $\psi$ at $P$ is obtained from (3) & (4).

In the spherical coordinate notation used

$$\boxed{q_r = \frac{m}{r^2}}, \quad \boxed{q_\theta = 0}$$

Using (3) & (4),

$$\boxed{q_r = \frac{-1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}} \quad \& \quad \boxed{q_\theta = \frac{1}{r\sin\theta} \frac{\partial \psi}{\partial r}}$$

$$\therefore \frac{m}{\cancel{r^2}} = \frac{-1}{\cancel{r^2}\sin\theta} \frac{\partial \psi}{\partial \theta} \quad \& \quad \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = 0$$

$$\Rightarrow \frac{\partial \psi}{\partial \theta} = -m\sin\theta \qquad \frac{\partial \psi}{\partial r} = 0$$

Hence $d\psi(r,\theta) = \dfrac{\partial \psi}{\partial r} dr + \dfrac{\partial \psi}{\partial \theta} d\theta$

$$= 0\, dr + (-m\sin\theta)\, d\theta$$

Integrating,

$$\psi(r,\theta) = -m\int \sin\theta \, d\theta$$

$$= -m(-\cos\theta)$$

$$= m\cos\theta$$

iii) Doublet at O, Axis along $\overline{OZ}$ :

In spherical polar coordinates, the velocity potential at P is

$$\phi(r,\theta) = \frac{M\cos\theta}{r^2}$$

where M is the strength of the doublet.

$$q_r = \frac{-\partial\phi}{\partial r} \quad \& \quad q_\theta = \frac{-1}{r}\frac{\partial\phi}{\partial\theta}$$

$$= -M\cos\theta(-2r^{-3}) \quad \& \quad = \frac{-1}{r}\frac{M}{r^2}(-\sin\theta)$$

$$= 2M\cos\theta\, r^{-3} \qquad = \frac{M\sin\theta}{r^3}$$

$$= \frac{2M\cos\theta}{r^3}$$

Also,

$$\boxed{q_r = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}} \quad \& \quad \boxed{q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}}$$

$$\therefore \frac{2M\cos\theta}{r^3} = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \qquad \frac{M\sin\theta}{r^3} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = -\frac{2M\cos\theta\sin\theta}{r} \qquad \frac{\partial\psi}{\partial r} = \frac{M\sin^2\theta}{r^2}$$

Hence

$$d\psi(r,\theta) = \frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\theta}d\theta$$

$$= \frac{M\sin^2\theta}{r^2}dr + \left(-\frac{2M\cos\theta\sin\theta}{r}\right)d\theta.$$

Integrating,

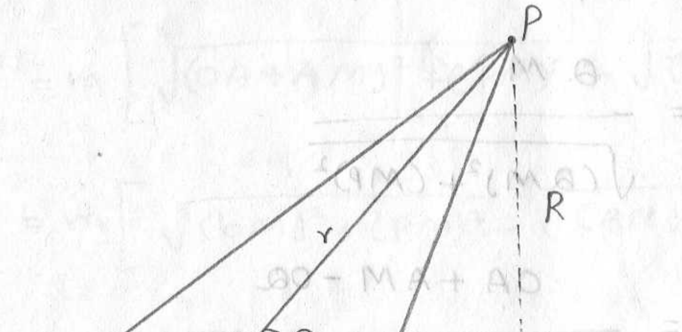
$$\psi(r,\theta) = M\sin^2\theta \int r^{-2}\,dr - \frac{2M}{r}\int \sin\theta\cos\theta\,d\theta$$

$$= M\sin^2\theta\left(-\frac{1}{r}\right) - \frac{2M}{r}\int x\,dx \qquad x = \sin\theta,\quad dx = \cos\theta\,d\theta$$

$$= M\sin^2\theta\left(\frac{-1}{r}\right) - \frac{2M}{r}\,\frac{\sin^2\theta}{2}$$

$$\boxed{= -\frac{2M}{r}\sin^2\theta}$$

## iv) Uniform line source along $\overline{OZ}$ :




The fig. depicts uniform line source of fluid extending along $OZ$ from $O$ to $A$.

Let $QQ'$ be the neighbouring points on $OA$.

$\therefore\ OQ = z\ ,\ QQ' = \delta z$

$OQ' = z + \delta z$

Also,

Suppose $QP = r$ and $\angle PQA = \theta$.

If $m$ denotes the strength per unit length of the line source, then $QQ'$ is effectively a simple source, at Q of strength $m\delta z$.

Hence the contribution to $\psi$ at P from this source is $\delta\psi$..

If we equal,

$$\delta\psi = m\delta z \cos\theta \quad \left[\text{using (ii)}\right]$$

$$\cos\theta = \frac{QM}{QP}$$

$$= \frac{QM}{\sqrt{(QM)^2 + (MP)^2}}$$

$$= \frac{OA + AM - OQ}{\sqrt{(OA + AM - OQ)^2 + (MP)^2}}$$

Denoting $OA = l$, $AM = b$, $PM = R$ all of which are constant elements

$$\cos\theta = \frac{l+b-z}{\sqrt{(l+b-z)^2 + R^2}}$$

$u = (l+b-z)^2 + R^2$

$$\delta\psi = m\delta z \frac{l+b-z}{\sqrt{(l+b-z)^2 + R^2}}$$

$du = [2(l+b-z)(-1) + 0]dz$

$(l+b-z)\,dz = -\frac{1}{2}du$

Integrating,

$$\psi = m\int_0^l \frac{l+b-z}{\sqrt{(l+b-z)^2 + R^2}}\,dz$$

| | | | |
|---|---|---|---|
| $z$ | $0$ | $l$ |
| $u$ | $(l+b)^2+R^2$ | $b^2+R^2$ |

$$\psi = m\int_{(l+b)^2+R^2}^{b^2+R^2} -\tfrac{1}{2}\,\tfrac{1}{\sqrt{u}}\,du$$

$$\psi = -m \int_{(l+b)^2+R^2}^{b^2+R^2} \frac{1}{2} u^{-1/2}\, du$$

$$= \frac{-m}{2} \left. \frac{u^{1/2}}{1/2} \right]_{(l+b)^2+R^2}^{b^2+R^2}$$

$$= -m\left[ (b^2+R^2)^{1/2} - \left((l+b)^2+R^2\right)^{1/2} \right]$$

$$= m\left[ \sqrt{(l+b)^2+R^2} - \sqrt{b^2+R^2} \right]$$

$$= m\left[ \sqrt{(OA+AM)^2+(PM)^2} - \sqrt{(AM)^2+(PM)^2} \right].$$

$$= m\left[ \sqrt{(OM)^2+(PM)^2} - \sqrt{(AM)^2+(PM)^2} \right]$$

$$= m\,(OP - AP)$$

This result shows that the stream fn $\psi = \text{constant}$, $OP - AP = \text{constant}$. These are **confocal hyperboloids of revolution** about $OZ$ having $O$ & $A$ as foci.

## v) Doublet in Uniform Stream:

Suppose the doublet specified in (iii) is in a uniform stream whose undisturbed velocity is $-U\hat{k}$.

Then combining the results of (i) & (iii), the stream fn at $P$ in spherical polar is

$$\psi(r,\theta) = \frac{1}{2} U r^2 \sin^2\theta - \frac{2M}{r} \sin^2\theta$$

Thus the stream surfaces are

$$\left(\frac{1}{2}Ur^2 - \frac{2M}{r}\right)\sin^2\theta = \text{constant}$$

In particular the surfaces for which $\boxed{\psi = 0}$ are given by,

$\sin\theta = 0$ ; $\frac{1}{2}Ur^2 - \frac{2M}{r} = 0$

$\boxed{\theta = 0, \pi}$ $\quad \frac{1}{2}Ur^2 = \frac{2M}{r}$

$$r^3 = \frac{4M}{U}$$

$$\Rightarrow \boxed{r = \left(\frac{4M}{U}\right)^{1/3}}$$

The former is the z-axis $\theta = 0, \pi$, the latter the sphere with centre O and radius $\left(\frac{4M}{U}\right)^{1/3}$.

## 5.4 Complex potential for two dimensional, irrotation, incompressible flow:

Consider an invisid incompressible fluid having an irrotational flow.

Eqn of motion is

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \nabla\cdot\bar{q} = 0$$

where $\bar{q}$ is the velocity vector of the fluid in motion.

$\because$ $\rho$ is independent of time $\frac{\partial\rho}{\partial t} = 0$,

$$\Rightarrow \nabla\cdot\bar{q} = 0$$

$\because$ The flow is irrotational $\nabla \cdot \overline{q} = \overline{0}$, $\rightarrow (f)$

$$\Rightarrow \overline{q} = -\nabla \phi \text{, where scalar point fn}$$

$\phi$ defines velocity potential.

$$\therefore \nabla \cdot \overline{q} = \nabla \cdot (-\nabla \phi) = -\nabla^2 \phi = 0$$

$$\Rightarrow \nabla^2 \phi = 0 \rightarrow (1)$$

which is the eqn of motion in this case, where the function $\phi = \phi(x, y)$

$\because$ The flow is considered to be in two dimensional.

Let $P(x, y)$ be any arbitrary point in the flow $\ni$ w.r.t origin fixed in the fluid, $\overline{OP} = z$ where $z = x + iy$, we are able to consider the 2D flow to be expressed on the argan plane.

For the flow defined by (f) define a scalar point function $\psi$, that is differentiable and uniform throughout the region of the flow, $\psi = \psi(x, y)$ subject to the condition, $u = -\psi_y$ & $v = \psi_x$ where $\overline{q} = \overline{q}(u, v)$

$\psi$ satisfies (f)

$$\Rightarrow \psi \text{ satisfies } \nabla^2 \psi = 0 \rightarrow (2)$$

From the description of $\psi$, we obtain

$$d\psi = 0$$

$$\Rightarrow \psi = \psi(x, y) \text{ is constant}$$

$\ni$ $\psi$ = constant defines stream function in the $z$ = constant plane for an invisid incompressible fluid with irrotational flow.

The velocity potential $\phi(x,y)$ and a stream fn $\psi(x,y)$ are differentiable and ~~a~~ also possess continuous partial derivatives everywhere in the region of flow.

Let $\omega = \phi(x,y) + i\,\psi(x,y)$

Then $\omega = \omega(x,y)$

$\Leftrightarrow \omega = f(z)$, where $z = z(x,y)$.

At any point in the region of flow

$$\frac{d\omega}{dz} = f'(z) \text{ exists}$$

$\Rightarrow$ It is unique & hence $\omega = f(z)$ will be analytic at that point.

defn (Whenever this condition hold for every point on the region of the flow then $\omega = f(z)$ is called a "regular function" throughout the region of flow)

$\frac{d\omega}{dz} = f(z)$ exists uniquely at $P$ means

$\lim\limits_{\delta z \to 0} \frac{\delta \omega}{\delta z}$ exists.

i.e. $$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta(\phi + i x)}{\delta(x+iy)} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta\phi + i\delta x}{\delta x + i\delta y} \longrightarrow (3)$$

where $\delta\phi = \phi(x+\delta x, y+\delta y) - \phi(x,y)$

& $\delta\psi = \psi(x+\delta x, y+\delta y) - \psi(x,y)$.

The limit in ③ can be computed based on the differentials $\delta x$ & $\delta y$ from various possible ways.

Consider the path where $x =$ constant & $y$ varies.

$$x = \text{constant} \Rightarrow dx = 0 \Rightarrow \delta x \to 0$$

Then ③ becomes

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta \phi + i\,\delta \psi}{\delta x + i\,\delta y}$$

$$= \lim_{\delta y \to 0} \frac{\delta \phi}{i\,\delta y} + \frac{\delta \phi}{\delta y}$$

$$= \frac{1}{i} \lim_{\delta y \to 0} \frac{\delta \phi}{\delta y} + \lim_{\delta y \to 0} \frac{\delta \psi}{\delta y} \to (4)$$

Consider the path $y =$ constant & $x$ alone varies.

$$y = \text{constant} \Rightarrow dy = 0 \Rightarrow \delta y \to 0$$

$\therefore$ ③ becomes

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\substack{\delta x \to 0 \\ \delta y = 0}} \frac{\delta \phi + i\,\delta \psi}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta \phi}{\delta x} + i \lim_{\delta x \to 0} \frac{\delta \psi}{\delta x} \to (5)$$

③ $\Rightarrow \lim_{\delta z \to 0} \frac{\delta w}{\delta z}$ exists uniquely.

$\therefore$ (4) & (5) expresses the same and hence comparing real and imaginary parts, we have

$$\lim_{\delta x \to 0} \frac{\delta \phi}{\delta x} = \lim_{\delta y \to 0} \frac{\delta \psi}{\delta y} \quad \&$$

$$\lim_{\delta y \to 0} \frac{\delta \phi}{\delta y} = - \lim_{\delta x \to 0} \frac{\delta \psi}{\delta x}$$ as the path along which the motion takes place is immaterial, we have

$$-\frac{\delta \phi}{\delta y} = \frac{\delta \psi}{\delta x}$$

$$\text{(or)} \iff \left.\begin{aligned} \phi_x &= \psi_y \\ \& \ \phi_y &= -\psi_x \end{aligned}\right\} \rightarrow (7)$$

i.e. $\boxed{\omega = f(z) = \phi + i\psi}$ satisfies the C-R eqns.

In a nutshell we thus find a 2-D incompressible invisid fluid having an irrotational at every pt in the region of flow, both the velocity potential and stream function are differentiable possessing continuous partial derivatives satisfying C-R and laplace eqn.

In this context the complex fn $\omega = f(z) = \phi(x, y) + i\psi(x, y)$ is defined as the complex velocity potential for the incompressible invisid fluid having irrotational flow. And the complex constant $w = f(z)$ is analytic in general throughout the region of the flow. This analytic function has for its real part velocity potential and for its imaginary part stream function.

Note that,

$$w = \phi + i\psi$$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}$$

$$\left( \frac{dw}{dz} = \phi_x + i\psi_x = -i\phi_y + \psi_y \right.$$

$$\left. = \phi_x - i\psi_y = i\psi_y + \psi_x \rightarrow (8) \right)$$

The velocity vector $\bar{q}$ defined by the velocity potential $\phi$ is

$$\bar{q} = -\nabla \phi = -\left( \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} \right)$$

i.e. $u\hat{i} + v\hat{j} = -\left( \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} \right)$

$$\bar{q} = \bar{q}(u, v) \Rightarrow u = -\frac{\partial \phi}{\partial x} \;\&\; v = -\frac{\partial \phi}{\partial y} \rightarrow (9)$$

$\therefore$ (8) becomes

$$\frac{dw}{dz} = \phi_x - i\phi_y = -u + iv \quad \text{using (9)}$$

$$= (-u) - i(-v)$$

& $\left|\frac{dw}{dz}\right| = +\sqrt{(-u)^2+v^2}$

$= +\sqrt{u^2+v^2}$ is the speed of fluid at $P(x,y)$. S/t

Stagnation pts are obtained when when $\frac{dw}{dz} = \vec{0}$. Thus the complex potentials enable us to determine and characterise an irrotational 2D flow of an invisid incompressible fluid.

---

Discuss the flow described by $w = z^2$

Soln:

The complex volume potential $w$ is defined by $w = \phi + i\psi$ where $\phi$ is the volume potential and $\psi$ is the stream function where $z = x+iy$.

$w = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy$ - 2D, Irr. flow $\nabla \times \phi = \vec{0}$

$\Rightarrow \phi\,\Phi = 0,\ \nabla^2\psi = 0$ Incomp $\nabla . \phi$

$\frac{\partial}{\partial x}(x^2-y^2) + \frac{\partial}{\partial y}(x^2-y^2) = 2x - 2y$

$\frac{\partial}{\partial x}(2x-2y) + \frac{\partial}{\partial y}(2x-2y) = 2-2 = 0$

$\Rightarrow \nabla^2\phi = 0$

$\nabla^2$

**Eg:**

Discuss the flow described by $\omega = z^2$.

**Soln:**

The complex velocity potential $\omega$ is defined by $\omega = \phi + i\psi$ where $\phi$ is velocity potential & $\psi$ is stream function where $z = x + iy$.

$$\omega = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$

$$\phi = x^2 - y^2 \qquad \psi = 2xy$$

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}(x^2-y^2) + \frac{\partial^2}{\partial y^2}(x^2-y^2) \qquad \nabla^2\psi = \frac{\partial^2}{\partial x^2}(2xy) + \frac{\partial^2}{\partial y^2}(2xy)$$

$$= 2 + (-2) \qquad\qquad = 0 + 0$$

$$= 0 \qquad\qquad = 0$$

$$\nabla \times \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & -y^2 & 0 \end{vmatrix} \qquad \nabla \times \psi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \\ & & \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= \vec{0}$$

Equipotentials are given by

$$\phi = \text{constant}$$

$$\Rightarrow x^2 - y^2 = \text{const} = c \rightarrow (1)$$

$$\Rightarrow \frac{x^2}{c} - \frac{y^2}{c} = 1$$

and stream lines are given by

$$\psi = \text{constant}$$

$$\Rightarrow xy = \text{cond} = d \rightarrow (2)$$

(ii) By eqn (2),

(1) $\Rightarrow \frac{x^2}{c} - \frac{y^2}{c} = 1$ is a hyperbola.

(2) $\Rightarrow xy = d$ is a rectangular hyperbola.

Asymptotes for the hyperbola $x^2 - y^2 = c$ is given by

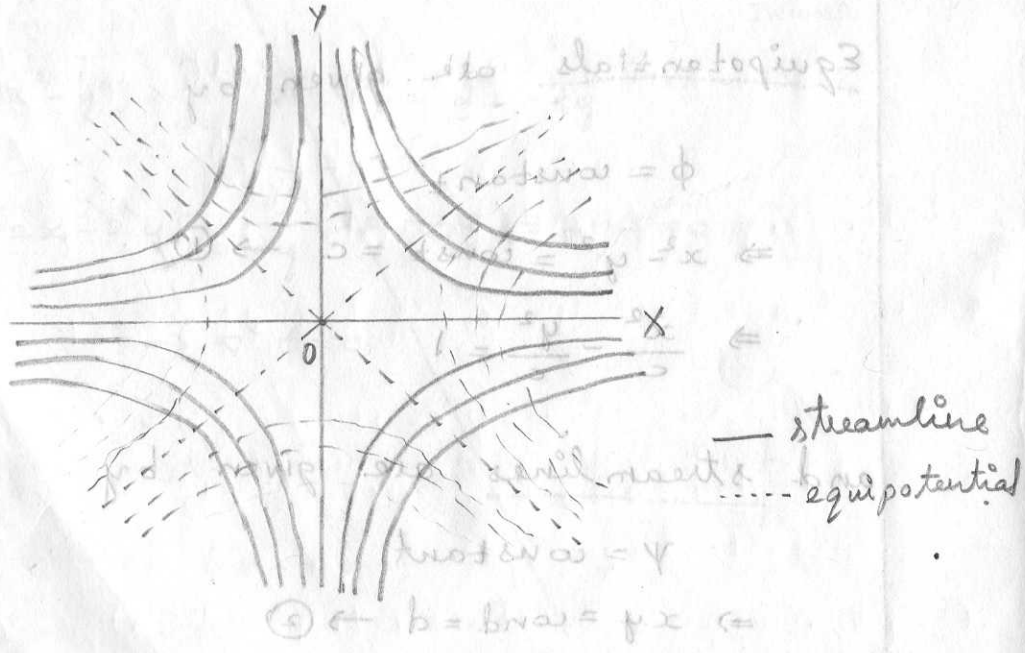
$$\left.\begin{aligned} x^2 - y^2 &= 0 \\ \Rightarrow x &= \pm y \end{aligned}\right\} \quad (3)$$

Asymptotes to the rectangular hyperbola $xy = d$ is given by $xy = 0$

$$\left.\begin{aligned} &x = 0 \text{ (or) } y = 0 \\ &\text{\& axes } y = \pm x \end{aligned}\right\} \quad (4)$$

i.e., Axes of (1) are asymptotes of (2) & asymptotes of (2) are axes of (1).

(3) & (4) $\Rightarrow$ equipotentials are orthogonal to streamlines.



Stagnation pts are obtained when $\frac{dw}{dz} = 2z = 0$

Origin is the stagnation pt of the flow & speed of the fluid is given by

$$\left|\frac{dw}{dz}\right| = |2z| = 2|z| = 2\left(\sqrt{x^2+y^2}\right)$$

which is speed of the fluid at any pt $P(x,y)$.

Unit-4

## 4.1) Complex Velocity Potentials for Standard 2-D Flows:

We shall consider some common type of 2D flows, namely flow due to ~~stream~~ a uniform stream, ~~to~~ a line source & sink, and a line doublet and some properties related to them. (III flow only).

### 1) Uniform Stream:

**Case (i):**

Consider a uniform stream having velocity $-U\hat{i}$

i.e. $\vec{q} = -U\hat{i} + 0\hat{j} + 0\hat{k}$

$$\nabla \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -U & 0 & 0 \end{vmatrix} = \hat{i}(0) - \frac{\partial U}{\partial z}\hat{j} + \frac{\partial(-U)}{\partial z}\hat{k}$$

$$= 0$$