Unit - II

Equations of motion of a fluid

2.1) Pressure at a Point in a Fluid at Rest:

The assumption is that the fluid is at rest.

Forces acting on any specific mass in a fluid region can be divided into body forces (external forces (i)) and surfaces forces (internal in fluid) which is in turn pressure forces / normal forces (ii), viscous forces and elastic forces.

Forces
- body forces
- surface forces
  - pressure/normal forces
  - viscous & elastic forces

viscosity absent Suppose the fluid flow is of invisid type then we have to consider only forces (i) & (ii).

Let (P) be any point within the fluid due to density there will be hydrostatic pressure.

SA

(molecular study)

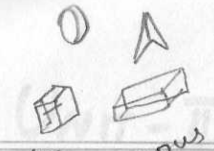
To establish the existence of hydrostatic pressure, invert a small rigid plane of area δA then either side of δA will be ~~rep~~ repeatedly bombarded with molecules of fluid moving in all directions.

As the fluid molecules impinge on one side of the area, they transfer momentum to the area in a specified time, thereby exerting a force δF on the area.

An equal & opposite force is exerted on the fluid by this side of the area.

Suppose linear dimensions of δA are large in comparison to with molecular sizes, but small when compared ~~to~~ ~~to~~ with macroscopic fluid ~~length~~ lengths then δA is at rest relative to the main body of the fluid, and motion is only on the molecular side.

Irrespective of the fact whether fluid is viscous or invisid not, $\delta \vec{F}$ acts normal to δA and the mean force per unit area exerted on one side of δA will be $\frac{\delta F}{\delta A}$.



an (continuous fluid particle [compressed])

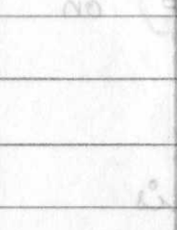
(Suppose we work on the molecular model of the fluid δA arbitrarily small, considered to be vanishing at P) (then as the study is in the fluid <u>to replace molecular model by continuous fluid</u>), (we need to suppose that at P,

$$\lim_{\delta A \to 0}\left(\frac{\delta \bar{F}}{\delta A}\right) \to (f) \text{ exists and is } \underline{\text{unique}})$$

Thus for a <u>fluid at rest</u>, we have the following <u>important results</u> in <u>hydrostatics</u>.

(1) When the fluid is at rest, Pressure at an internal point P is same at all directions and thus orientation of δA <u>does not affect</u>. immaterial

(2) Pressure at all pts at the same depth below the free horizontal surface of the fluid is the same.



At point P, distant h below the free surface, the <u>pressure</u> is $\pi + \rho g h$ where π is the <u>atmospheric pressure</u> at free surface and ρ is the fluid <u>density</u> and g is the <u>acceleration due to gravity</u>.

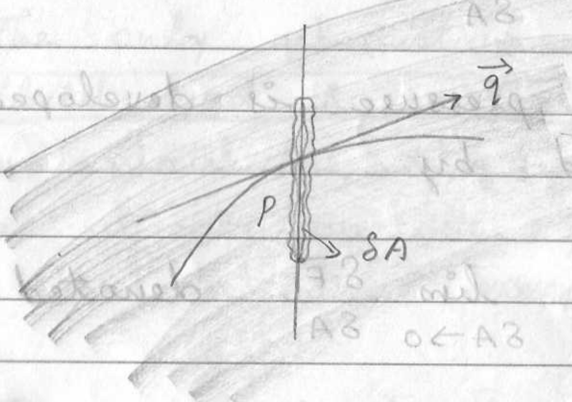
(3) If any pressure is applied to the free surface of the fluid, it is transmitted equally to all parts of the fluid.

6m

(1), (2) & (3) are called Pascal's laws for static fluid.

Pressure at a point in a moving fluid.

8m



Consider an inviscid fluid moving or flowing.

Suppose (P) is a point moving locally with the velocity $\bar{q}$. Insert a plane rigid area (δA) into the fluid at P which will also move with the velocity $\bar{q}$ at P.

Hence relative velocity of the fluid w.r.t the plate at P at that locality is actually the velocity of the fluid with the reversed velocity of the plate (or) the fluid will be reduced to rest ~~w.r.t~~ w.r.t the plane area.

Reversed velocity of fluid w.r.t the plate at P : $\vec{q} - \vec{q} = \vec{0} \rightarrow (*)$

Case (i):

A moment of change will take place b/w the fluid and the plane given by $\frac{\delta \bar{F}}{\delta A}$.

ie) A pressure is developed at P determined by

$$\lim_{\delta A \to 0} \frac{\delta \bar{F}}{\delta A} \text{ denoted by } p$$

ie) $p = \lim_{\delta A \to 0} \frac{\delta |F|}{\delta A}$ exists uniquely

by [redacted] (f) ← (quote or prove based on marks)

Then the fluid pressure can be determined at any point in the moving fluid.

Remark:

If δA where at rest (or) moving with a velocity $\bar{v}$ different from that of $\bar{q}$, then the local fluid would be brought to rest as in (*).

And this requirement makes it necessary to have an additional exchange of momentum b/w the fluid and the boundary.

Case (ii):

When δA is situated tangentially to the local fluid velocity at P.

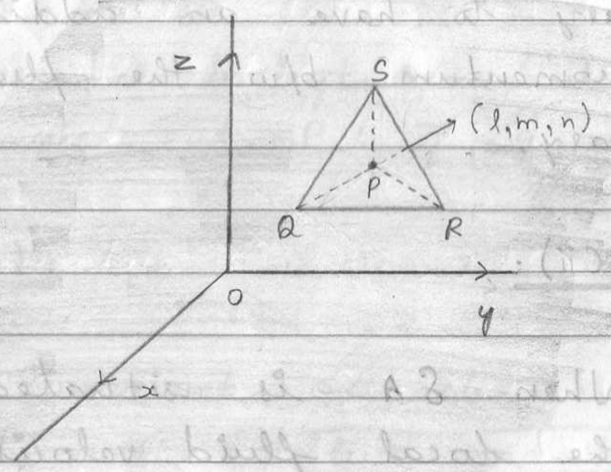
The only momentum interchange b/w the fluid and the surface is of the random molecular type, so that for this special orientation, the forces $\delta |\bar{F}|$ would be the same. ~~irrespective of the~~

Irrespective of the fact whether δA is at rest (or) moving with local velocity $\bar{q}$ (or) moving with some other velocity $\bar{v}$.

Discuss: (Question)

Imp Qn The pressure at a pt in a moving fluid is the same in all directions and is independent of the orientation of δA.

Consider a tetrahedron PQRS inserted in a fluid flowing such that the edges PQ, PR, PS parallel to the coordinate axes OX, OY, OZ axes respectively in ~~dirⁿ~~ x-increasing, y-increasing, z-increasing of a rectangular coordinate system considered inside the fluid.



Let $\vec{q}$ be the local velocity of the fluid at P. ∴ the tetrahedron, the same fluid velocity $\vec{q}$ therefore at P, the relative velocity of the fluid w.r.t the tetrahedron is

$$\vec{q} - \vec{q} = \vec{0}$$

∴ The fluid is at rest w.r.t tetrahedron and hence the pressure exerted by the fluid on all the four faces of the tetrahedron may be determined to this end.

Consider the face QRS which is a base.

Let $\hat{a}$ be the outward drawn normal to the face given by

$$\hat{a} = l\hat{i} + m\hat{j} + n\hat{k}$$

Then $|\vec{a}| = +\sqrt{l^2+m^2+n^2}$ where (l,m,n) will be the direction cosines of $\hat{a}$.

Let p be the pressure exerted by the fluid on the face QRS.

Then, force exerted by the outside fluid on the face QRS with surface area δs will be given by,

$$\boxed{\begin{aligned}\vec{F} = -p\,\delta s\,\hat{a} &= -p\,\delta s\,(l\hat{i}+m\hat{j}+n\hat{k}) \\ &= -\frac{p}{2}\left(\delta y\,\delta z\,\hat{i}+\delta z\,\delta x\,\hat{j}+\delta x\,\delta y\,\hat{k}\right)\end{aligned}} \longrightarrow (1)$$

[$\because l\,\delta s = ds\cos\alpha$ where $\delta s\cos\alpha$ is the area projected on YOZ plane]

Suppose p_x, p_y, p_z represents the pressure exerted on faces ~~PR~~ PSR, PSQ, PQR respectively. Then the force exerted by the fluids on these faces are

$$\boxed{\frac{1}{2}p_x\,\delta y\,\delta z\,\hat{i}\;,\;\frac{1}{2}p_y\,\delta z\,\delta x\,\hat{j}\;,\;\frac{1}{2}p_z\,\delta x\,\delta y\,\hat{k}} \text{ resp.}$$

ie.) Forces exerted on face PSR $= \frac{1}{2}p_x\,\delta y\,\delta z\,\hat{i} \longrightarrow (2)$

Forces exerted on face PSQ $= \frac{1}{2}p_y\,\delta z\,\delta x\,\hat{j} \longrightarrow (3)$

Force exerted on face PQR $= \frac{1}{2}p_z\,\delta x\,\delta y\,\hat{k} \longrightarrow (4)$

$\therefore$ Total surface force exerted by the fluid on the tetrahedron is given by ~~surface force~~

Surface force $\vec{F} = (1) + (2) + (3) + (4)$

$$\vec{F} = -\frac{p}{2}(\delta y \delta z \hat{i} + \delta z \delta x \hat{j} + \delta x \delta y \hat{k}) + \frac{1}{2} p_x \delta y \delta z \hat{i} + \frac{1}{2} p_y \delta z \delta x \hat{j} + \frac{1}{2} p_z \delta x \delta y \hat{k}$$

$$\boxed{= \frac{1}{2} \delta y \delta z (p_x - p)\hat{i} + \frac{1}{2} \delta z \delta x (p_y - p)\hat{j} + \frac{1}{2} \delta x \delta y (p_z - p)\hat{k}} \rightarrow (5)$$

Suppose $\overline{F}$ is the **mean body force per unit mass** within the tetrahedron and ρ is the **mean fluid density**, then

Total body force due to weight of the tetrahedron

$$\boxed{= \left(\frac{1}{6} \delta x \delta y \delta z\right) \rho \vec{F}} \rightarrow (6)$$

Since $\overline{q}$ is the velocity of the fluid at P,

$\frac{d}{dt}\overline{q} = \frac{d\overline{q}}{dt}$ is the acceleration of the fluid at P at any time t.

The force acting on fluid is

$$\boxed{P = (\vec{F} \times m\bar{a}) = \left(\frac{1}{6}\,\delta x\,\delta y\,\delta z\right)\rho\,\frac{d\vec{q}}{dt}} \longrightarrow (7)$$

During motion, mass of the fluid particle remains constant.

Hence the **eqn of motion** is given by

$$\left.\begin{aligned}\left(\frac{1}{6}\,\delta x\,\delta y\,\delta z\right)\rho\,\frac{d\vec{q}}{dt} = \left(\frac{1}{6}\,\delta x\,\delta y\,\delta z\right)\rho\,\bar{F} \\ + \frac{1}{2}(p_x - p)\,\delta y\,\delta z\,\hat{i} \\ + \frac{1}{2}(p_y - p)\,\delta y\,\delta z\,\hat{j} \\ + \frac{1}{2}(p_z - p)\,\delta x\,\delta y\,\hat{k}\end{aligned}\right\} \longrightarrow (8)$$

Considering only upto 2^{nd} order of smallness, we have to neglect the terms involving $\delta x, \delta y, \delta z$.

$\therefore$ (8) yields,

$$\vec{O} = \frac{1}{2}(p_x - p)\,\delta y\,\delta z\,\hat{i} + \frac{1}{2}(p_y - p)\,\delta y\,\delta z\,\hat{j} + \frac{1}{2}(p_z - p)\,\delta x\,\delta z\,\hat{k} \longrightarrow (8A)$$

$$\left.\begin{aligned}\frac{1}{2}(p_x - p)\,\delta y\,\delta z = 0\;;\; \frac{1}{2}(p_y - p)\,\delta z\,\delta x = 0\;; \\ \frac{1}{2}(p_z - p)\,\delta x\,\delta y = 0\end{aligned}\right\} \longrightarrow (8B)$$

$\delta x, \delta y, \delta z$ are small variations & not equal to 0.

Hence (8B) yields,

$$p_x - p = 0 \quad ; \quad p_y - p = 0 \quad ; \quad p_z - p = 0 \longrightarrow (9)$$

i.e.) $\boxed{p_x = p_y = p_z = p.}$

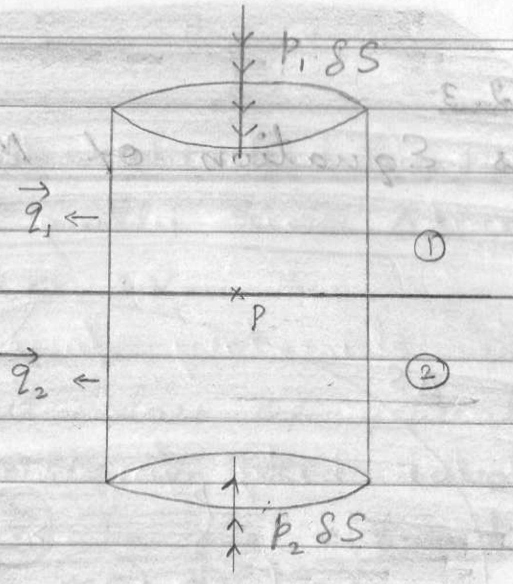
Hence we infer that at any point P of a moving inviscid fluid, the pressure p is same in all the directions.

Thus the choice of the axes are arbitrary. We have also seen that pressure is independent of orientation of the tetrahedron and thus we infer there exists a unique pressure at each and every point in the fluid which is the same in all directions.

What are the 2m

Conditions satisfied at a boundary of two invisid immiscible fluids:

Consider (1) & (2) to represent 2 invisid immiscible fluids separated by a plane boundary and suppose P is a point on the boundary where the 2 fluids have velocities $\vec{q_1}$, $\vec{q_2}$ respectively.



Consider a small cylindrical hat-box shaped elts of normal sections δS containing the point P projecting into both fluids. There is no fluid transfer across the boundary.

Generators to this surface are normal to the surface δs.

$$\therefore\ p_1 \delta S = p_2 \delta S$$

$$\text{or} \Leftrightarrow p_1 = p_2$$

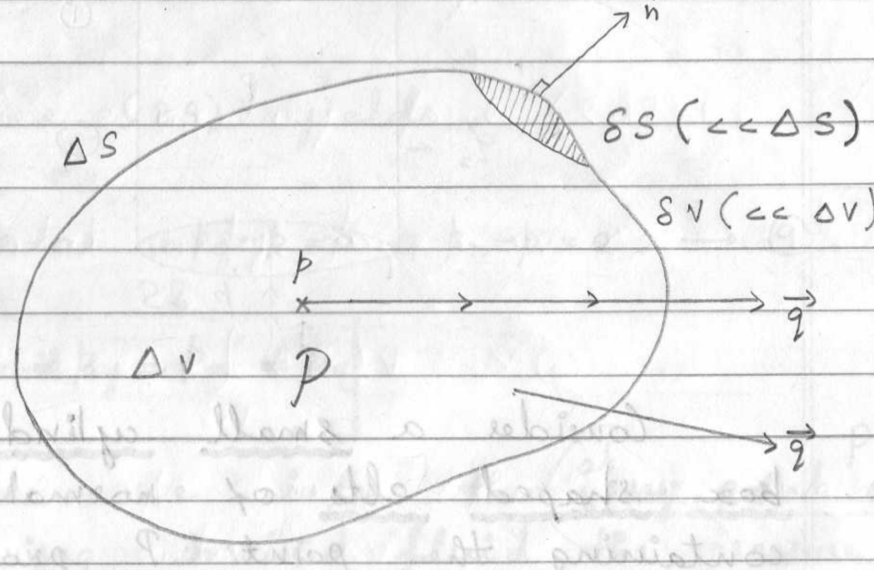
(†) ← In particular for the case when the fluid is in contact with the atmosphere, the pressure at the free surface is the same as that of the atmosphere.

2m (If the boundary was curved, the above conditions will get modified to account for the effects of surface tension.) 2m

Sec : 2.3

(a) **Euler's Equation of Motion:**

20 m



1st part [Consider an inviscid flow.

Let ΔS be a closed region of fluid marked in the flow containing the volume ΔV.

Suppose ρ is the density of the fluid then mass of the fluid elt (which does not change due to conservation of mass) is $\rho \Delta V \rightarrow (f)$

Let P be a point in the fluid in the region considered.

Let the fluid element be moving with velocity $\bar{q}$ and p be the pressure at P.

$\delta S << \Delta S$

Consider an elementary closed region δs much smaller than Δs enclosing a volume $\delta V << \Delta V$.

Without loss of generality, let p to be a point in the interior of δs and let $\hat{n}$ be the outward drawn normal to δs at P.

$\therefore$ Force exerted by the outside of the fluid on δs is given by $-p\,\delta s\,\hat{n}$.

$\therefore$ Total surface force exerted by the outside fluid on the entire area Δs

$$= -\int_{\Delta S} p\,\hat{n}\,ds$$

Then By Gauss divergence theorem which states that

"Suppose V is the volume bounded by the closed surface S and $\vec{A}$ is a vector fn of position with continuous partial derivatives, then

$$\iiint_V \nabla\cdot\bar{A}\,dV = \iint_S \vec{A}\cdot\hat{n}\,ds = \oint\!\!\!\oint_S \bar{A}\cdot d\bar{s}$$

where $\hat{n}$ is a +ve normal to S".

we have, $= -\int_{\Delta V} \nabla p\,dV \longrightarrow ①$

Let $\bar{F}$ denote the body force per unit mass acting on δs.

$\therefore$ The elementary force $(\rho\, dV)\bar{F}$.

$\therefore$ Total body force on Δs

$$= \int_{\Delta V} \rho \bar{F}\, dV \rightarrow (2)$$

Rate of change of linear momentum on the entire ΔV

$$= \int_{\Delta V} \frac{d}{dt}(\rho\, dV \bar{q})$$

$$= \int_{\Delta V} \frac{d}{dt} \bar{q}\, \rho\, dV + \int_{\Delta V} \bar{q} \frac{d}{dt} \rho\, dV$$

$$= \int_{\Delta V} \frac{d}{dt} \bar{q}\, \rho\, dV \rightarrow (3)$$

$$\left[\because \int_{\Delta V} \bar{q} \frac{d}{dt} \rho\, dV = 0 \text{ by } (f) \right]$$

Hence eqn of motion is given by

L.H.S (3) = L.H.S (1) + L.H.S (2)

$\Rightarrow$ R.H.S (3) = R.H.S (1) + R.H.S (2)

Rate of change of linear momentum in δV = $\frac{d}{dt}(\rho\, dV \vec{q})$

ie.) $\displaystyle\int_{\Delta V} \frac{d}{dt}\vec{q}\,\rho\, dV = -\int_{\Delta V} \nabla p\, dV + \int_{\Delta V} \rho\,\vec{F}\, dV$

ie.) $\displaystyle\int_{\Delta V} \left[\frac{d}{dt}\vec{q}\,\rho + \nabla p - \rho\vec{F}\right] dV = \vec{0}$

Since ΔV is the volume of the arbitrary fluid element which is moving <u>continuously</u> with density ρ not equal to zero and pressure p locally at the point P,

$$\rho\frac{d\vec{q}}{dt} + \nabla p - \rho\vec{F} = \vec{0}$$

$$\Rightarrow \rho\frac{d\vec{q}}{dt} = \rho\vec{F} - \nabla p$$

$$\Rightarrow \boxed{\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p = \vec{0}} \quad (\because \rho \neq 0) \rightarrow (\mathrm{I})$$

This eqn is the <u>Euler's equation of motion</u>.]

2 [We have (from relationship b/w local & particle rate of change)

$$\frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \rightarrow (4)$$

Vector form

Consider,

$$(\bar{q}\cdot\nabla)\bar{q} = \left(\bar{q}\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right)\bar{q}$$

$$= (\bar{q}\cdot\hat{i})\frac{\partial\bar{q}}{\partial x} + (\bar{q}\cdot\hat{j})\frac{\partial\bar{q}}{\partial y} + (\bar{q}\cdot\hat{k})\frac{\partial\bar{q}}{\partial z}$$

From ④,

$$\frac{d\bar{q}}{dt} = \underbrace{\frac{\partial \bar{q}}{\partial t}}_{I} + \underbrace{(\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x}}_{II} + \underbrace{(\bar{q}\cdot\hat{j})\frac{\partial \bar{q}}{\partial y}}_{III} + \underbrace{(\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z}}_{IV} \rightarrow (*)$$

From the well known vector identity

$$A \times (B \times C) = (A\cdot C)B - (A\cdot B)C$$

$$(A\cdot B)C = (A\cdot C)B - A \times (B \times C)$$

we can write the 2^{nd}, 3^{rd} & 4^{th} term on RHS of (*) as

$$\left.\begin{aligned}
(\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x} &= \left(\bar{q}\cdot\frac{\partial \bar{q}}{\partial x}\right)\hat{i} - \bar{q}\times\left(\hat{i}\times\frac{\partial \bar{q}}{\partial x}\right)\\
(\bar{q}\cdot\hat{j})\frac{\partial \bar{q}}{\partial y} &= \left(\bar{q}\cdot\frac{\partial \bar{q}}{\partial y}\right)\hat{j} - \bar{q}\times\left(\hat{j}\times\frac{\partial \bar{q}}{\partial y}\right)\\
(\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z} &= \left(\bar{q}\cdot\frac{\partial \bar{q}}{\partial z}\right)\hat{k} - \bar{q}\times\left(\hat{k}\times\frac{\partial \bar{q}}{\partial z}\right)
\end{aligned}\right\} (**)$$

$$\frac{\partial}{\partial x}(\bar{q}\cdot\bar{q}) = \frac{\partial \bar{q}}{\partial x}\cdot\bar{q} + \bar{q}\cdot\frac{\partial \bar{q}}{\partial x}$$

$$\frac{\partial}{\partial x}(\bar{q}^2) = 2\left(\bar{q}\cdot\frac{\partial \bar{q}}{\partial x}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{\partial}{\partial x}\bar{q}^2\right) = \bar{q}\cdot\frac{\partial \bar{q}}{\partial x}$$

|||ly, $\frac{1}{2}\left(\frac{\partial}{\partial y}\bar{q}^2\right) = \bar{q}\cdot\frac{\partial \bar{q}}{\partial y}$

$$\frac{1}{2}\left(\frac{\partial}{\partial z}\bar{q}^2\right) = \bar{q}\cdot\frac{\partial \bar{q}}{\partial z}$$

Sub in (**), we get

$$(\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial x}\bar{q}^2\right)\hat{i} - \bar{q}\times\left(\hat{i}\times\frac{\partial \bar{q}}{\partial x}\right)$$

$$(\bar{q}\cdot\hat{j})\frac{\partial \bar{q}}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial y}\bar{q}^2\right)\hat{j} - \bar{q}\times\left(\hat{j}\times\frac{\partial \bar{q}}{\partial y}\right)$$

$$(\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial z}\bar{q}^2\right)\hat{k} - \bar{q}\times\left(\hat{k}\times\frac{\partial \bar{q}}{\partial z}\right)$$

Adding the above relations, we get

$$\underline{(\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x} + (\bar{q}\cdot\hat{j})\frac{\partial \bar{q}}{\partial y} + (\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z}}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\vec{q}^{\,2} + \frac{\partial}{\partial y}\vec{q}^{\,2}\hat{j} + \frac{\partial}{\partial z}\bar{q}^2\hat{k}\right]$$

$$- \bar{q}\times\left[\left(\hat{i}\times\frac{\partial \bar{q}}{\partial x}\right) + \left(\hat{j}\times\frac{\partial \bar{q}}{\partial y}\right) + \left(\hat{k}\times\frac{\partial \bar{q}}{\partial z}\right)\right]$$

$$\Rightarrow (\bar{q}\cdot\nabla)\bar{q} = \frac{1}{2}\left[\nabla\bar{q}^2\right] - \bar{q}\times(\nabla\times\bar{q})$$

Sub in (4) we get,

$$\frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + \frac{1}{2}(\nabla\cdot\bar{q}^2) - \bar{q}\times(\nabla\times\bar{q})$$

Hence (I) $\rightarrow$ Euler's eqn of motion becomes,

Vector form

$$\boxed{\frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q}) = \bar{F} - \frac{1}{\rho}\nabla\cdot P} \longrightarrow (IA)$$

is another form of eqn of motion due to Euler.

Tensor form of the before eqn,

$$\boxed{\frac{\partial u_i}{\partial t} + u_j u_{i,j} = F_i - \frac{1}{\rho} p_{,i}} \rightarrow (\text{I B})$$

where

$\overline{x} = x_i (x_1, x_2, x_3)$ the orthonormal co-ordinate
$\overline{q} = (u_1, u_2, u_3)$ the velocity component
$\overline{F} = (F_1, F_2, F_3)$ the body force component

$\longrightarrow$ Suppose the flow is steady,

$$\frac{\partial \overline{q}}{\partial t} = 0$$

Then (I A) becomes

$$\boxed{\overline{F} - \frac{1}{\rho} \nabla P = \nabla\left(\frac{1}{2}\overline{q}^2\right) - \overline{q} \times (\nabla \times \overline{q})}$$

$\longrightarrow$ Suppose the invisid fluid flow is such that flow is irrotational.

ie.) It is of the potential kind

$$\nabla \times \overline{q} = \overline{0} \Rightarrow \overline{q} = -\nabla \phi$$

where ϕ is the velocity potential.

Then (IA) becomes, $\nabla \times \nabla = 0$

$$\frac{\partial \overline{q}}{\partial t} + \nabla\left(\frac{1}{2}\overline{q}^2\right) = \overline{F} - \frac{1}{\rho} \nabla p$$

$$\overline{q} = -\nabla \phi$$

→ Suppose the invisid flow is of the compressible type then ρ is not a constant.

Hence $$\boxed{\frac{d\bar{q}}{dt} = \bar{F} - \nabla \int_{\Delta V} \frac{dp}{\rho}}$$

Equation of Motion due to Bernoulli:

6m Consider an invisid flow of the potential kind where the body forces are conservative.

2m ie) flow is irrotational and also that there exist a scalar velocity potential $\nabla\phi$ and scalar potential $\nabla\Omega$ such that

∇ scalar fns Ω, ϕ

$$\exists \quad \boxed{\bar{q} = -\nabla\phi} \quad \text{and} \quad \boxed{\bar{F} = -\nabla\Omega}$$

where $\bar{F}$ are the body forces acting on an arbitrary fluid element under consideration to study the eqn of ~~mo~~ motion.

Euler's eqn of motion for an invisid fluid is given by

$$\frac{d\bar{q}}{dt} + \frac{1}{\rho}\nabla p = \bar{F} \quad -(I)$$

Where $\bar{q}$ is the velocity of the fluid ~~element~~ at any pt P in the interior of the fluid elt and p be the pressure of the

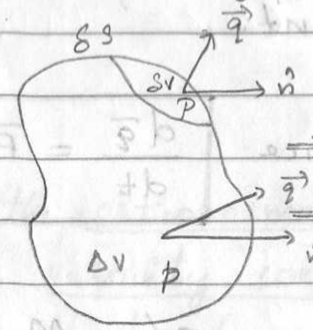
[Here write the 1^{st} part of "Euler's eqn of motion" with diagram]

Motion of fluid is irrotational

$\Rightarrow \nabla \times \bar{q} = \hat{0}$

$\Rightarrow \exists$ vector pt ϕ

$\ni \vec{q} = -\nabla\phi \rightarrow (1)$



$\vec{F}$ is conservative

$\Rightarrow \nabla \times \vec{F} = \bar{0}$

$\Rightarrow \exists$ a scalar pt fn Ω

$\ni \vec{F} = -\nabla\Omega$ (2)

Making use of the differential operator,

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{q}\cdot\nabla)}$$

we have,

$$\frac{d\bar{q}}{dt} = \frac{\partial\bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}$$

$$= \frac{\partial\bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q})$$

$\therefore$ (I) becomes,

$$\frac{\partial\bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q}) + \frac{1}{\rho}\nabla P = \vec{F} \longrightarrow (\text{IA})$$

$$\Rightarrow \frac{\partial}{\partial t}(-\nabla\phi) + \frac{1}{2}\nabla(-\nabla\phi)^2 - (-\nabla\phi)\times(\nabla\times -\nabla\phi) + \frac{1}{\rho}\nabla P = -\nabla\Omega$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla\phi) - \frac{1}{2}\nabla(+\nabla\phi)^2 + \underbrace{(-\nabla\phi)\times(\nabla\times -\nabla\phi)}_{0} - \frac{1}{\rho}\nabla P = \nabla\Omega$$

$$\nabla\left(\frac{\partial\phi}{\partial t}\right) - \frac{1}{2}\nabla(\nabla\phi)^2 = \frac{1}{\rho}\nabla P + \nabla\Omega \longrightarrow (\text{IB})$$

Suppose $\vec{x}$ denotes the position vector of the fluid particle at P at time t and $d\vec{x}$ representing instantaneous displacements made in the position of the particle at time t and scalar multiplying eqn (IB) by $d\vec{x}$

$$\boxed{\underbrace{d\vec{x}\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right)}_{d\left(\frac{\partial\phi}{\partial t}\right)} - \frac{1}{2}\underbrace{d\vec{x}\cdot\nabla(\nabla\phi)^2}_{d(\nabla\phi)^2} = \underbrace{d\vec{r}\cdot\frac{1}{\rho}\nabla P}_{\frac{1}{\rho}dp} + \underbrace{d\vec{r}\cdot\nabla\Omega}_{d\Omega}} \rightarrow (\mathrm{IC})$$

Consider,

$$d\vec{x}\cdot\Delta\Omega = (dx\hat{i}+dy\hat{j}+dz\hat{k})\cdot\left(\frac{\partial}{\partial x}\Omega\hat{i}+\frac{\partial}{\partial y}\Omega\hat{j}+\frac{\partial}{\partial z}\Omega\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\Omega\,dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz$$

$$d\vec{x}\cdot\Delta\Omega = d\Omega \rightarrow (3)$$

Consider,

$$d\vec{x}\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right) = (dx\hat{i}+dy\hat{j}+dz\hat{k})\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left(\frac{\partial\phi}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)dz$$

$$d\vec{x}\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right) = d\left(\frac{\partial\phi}{\partial t}\right) \rightarrow (4)$$

Consider,

$$d\vec{x}\cdot\nabla(\nabla\phi)^2 = (dx\hat{i}+dy\hat{j}+dz\hat{k})\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(\nabla\phi)^2$$

$$= \frac{\partial}{\partial x}(\nabla\phi)^2 dx + \frac{\partial}{\partial y}(\nabla\phi)^2 dy + \frac{\partial}{\partial z}(\nabla\phi)^2 dz$$

$$d\vec{r}\cdot\nabla(\nabla\phi)^2 = d(\nabla\phi)^2 \rightarrow (5)$$

Consider,

$$d\vec{r}\cdot\left(\frac{1}{\rho}\nabla p\right) = (dx\hat{i}+dy\hat{j}+dz\hat{k})\cdot\frac{1}{\rho}\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)p$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial x}(p)dx + \frac{\partial}{\partial y}(p)dy + \frac{\partial}{\partial z}(p)dz\right)$$

$$d\vec{r}\cdot\left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho}dp \rightarrow (6)$$

(I c) becomes, (using (3), (4), (5) & (6))

$$\boxed{d\left(\frac{\partial\phi}{\partial t}\right) - \frac{1}{2}d(\nabla\phi)^2 = \frac{1}{\rho}dp + d\Omega} \rightarrow (\text{I D})$$

(I D) holds subject to the condition that t is a constant.

Integrating and rearranging,

$$\int d\left(\frac{\partial\phi}{\partial t}\right) - \frac{1}{2}\int d(\nabla\phi)^2 = \int\frac{1}{\rho}dp + \int d\Omega$$

$$\Rightarrow \frac{\partial\phi}{\partial t} - \frac{1}{2}(\nabla\phi)^2 = \int\frac{dp}{\rho} + \Omega$$

$$\boxed{\frac{1}{2}(\vec{q}^{\,2}) + \Omega + \int\frac{dp}{\rho} - \frac{\partial\phi}{\partial t} = f(t)} \rightarrow (\text{I E})$$

where $\vec{q} = -\nabla\phi$ and $f(t)$ is an arbitrary constant arising out of integration, a fn of t where t is a constant.

Eqn (IE) is called **Bernoulli's eqn in the most general form.**

When the flow is **steady**,

$$\frac{\partial \phi}{\partial t} = 0 \ \& \ f(t) = \text{constant} = c \ \text{(say)}$$

then (IE) becomes,

$$\boxed{\frac{1}{2}(\vec{q})^2 + \Omega + \int \frac{dp}{\rho} = c} \rightarrow (\text{IF})$$

QY Further if the fluid is **homogeneous** and **incompressible** then ρ is a constant and (IF) becomes,

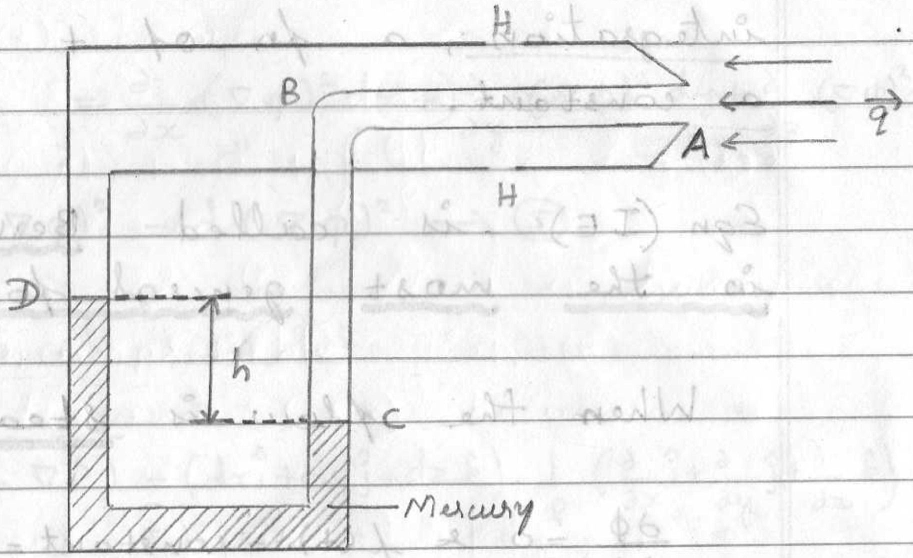
$$\frac{1}{2}(\bar{q}^2) + \Omega + \frac{1}{\rho}\int dp = c$$

$$\text{ie.)} \ \boxed{\frac{1}{2}(\bar{q}^2) + \Omega + \frac{p}{\rho} = c} \rightarrow (\text{IG})$$

Applications : (eg ①, pg. 100)

1. Pitiot Tube :

8M Pitiot tube is a simple device used to measure the velocity of the fluid flow at any point in the fluid regions.



Suppose it is desired to measure the velocity $\vec{q}$ of a stream of water.

It is a tube of very small cross section bent in the form of right angle and kept vertical.

To measure velocity $\vec{q}$ of stream of water flowing, the inner tube BA in the apparatus is placed upstream of the flow. The outer tube contains holes such as H.

Let p denote the pressure of the stream where the fluid velocity is $\vec{q}$ then p is also considered the pressure on inside and outside of the hole and hence also at the ~~point~~ meniscus D of the mercury level in the U-tube.

Consider a stream entering the inner tube AB, this is brought to rest at the mercury level at C.

Let p_0 be the pressure at C. Flow being steady applying Bernoulli's eqn along the stream line extending to infinity, entering the inner tube at A and finally being brought to rest at C.

General form of Bernoulli's eqn is gn by,

$$\frac{1}{2}(\vec{q}^{\,2}) + \Omega + \int \frac{dp}{\rho} - \frac{\partial \phi}{\partial t} = f(t) \rightarrow (*)$$

where $\boxed{\vec{q} = -\nabla\phi}$ & $\boxed{f(t)}$ is an arbitrary constant, a fn of t where t is a constant.

Since the flow is of potential type,

$$\vec{q} = -\nabla\phi$$

and since body forces are conservative

$$\vec{F} = -\nabla\Omega$$

When the flow is steady,

$$\frac{\partial\phi}{\partial t} = 0, \quad f(t) = \text{constant } c \quad (\text{say})$$

Then (*) becomes,

$$\frac{1}{2}(\overline{q})^2 + \Omega + \int \frac{dp}{\rho} = c \longrightarrow (**)$$

Further if the fluid is **homogeneous** and **incompressible** then ρ is a constant and (**) becomes,

$$\frac{1}{2}(\vec{q}^{\,2}) + \Omega + \frac{1}{\rho}\int dp = 0$$

At c,

$$\text{ie.)}\ \frac{1}{2}(\vec{q}^{\,2}) + \Omega + \frac{p_0}{\rho} = c \longrightarrow (1)$$

Applying Bernoulli's eqn to the water level at D,

the flow comes to rest at D due to the effect of the mercury level.

In the limiting case, let $\vec{q}$ denote the velocity and p the pressure at D, then we have

$$\frac{1}{2}\vec{q}^{\,2} + \frac{p}{\rho} + \Omega = \text{constant at D} \rightarrow (2)$$

At c,

$$|\vec{q}| = |\vec{0}| = 0 \quad \text{and} \quad p = p_0$$

① becomes,

$$\frac{p_0}{\rho} + \Omega = \text{constant at } c \rightarrow (1A)$$

From ② & (1A), ② − (1A)

$$\frac{1}{2}\vec{q}^{\,2} + \frac{p}{\rho} + \cancel{\Omega} - \frac{p_0}{\rho} - \cancel{\Omega} = \text{const at } D - \text{const on } c$$

$$= c$$

$$\frac{1}{2}\vec{q}^{\,2} = \frac{p_0 - p}{\rho} + c$$

$$\vec{q}^{\,2} = \frac{2(p_0 - p)}{\rho} + c$$

$$|\vec{q}| = + \sqrt{\frac{2(p_0 - p)}{\rho}} + c \rightarrow ③$$

The pressure difference $p_0 - p$ is measured from the difference in levels of mercury.

If h is the height difference b/w the 2 level and if σ is the density of the mercury then $\sigma g h$ is the difference in pressure

i.e., $\sigma g h = p_0 - p \rightarrow ④$

where g is the gravitation force.

From ③ & ④,

$$|\vec{q}| = +\sqrt{\frac{2\sigma g h}{\rho}} + c$$

which gives the speed of stream of water.

Thus pitiot tube can be used to compute the speed of fluid flow which finds ~~th~~ its applications in the following areas,

Suitable to measure sub-sonic ~~air~~ airstreams.

At low speeds air can be treated as an incompressible fluid for similar purposes of analysis.

12M (b) 20M

Steady Motion under conservative body forces:

Consider The flow is an invisid flow, body forces are conservative, there exist scalar potential $\nabla \Omega$ such that $\boxed{F = -\nabla \Omega} \rightarrow (*)$ and the flow is irrotational, hence there exist velocity potential $\nabla \phi$ such that $\boxed{\vec{q} = -\nabla \phi} \rightarrow (**)$

Euler's eqn:

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{\nabla p}{\rho} \quad [\text{from (I)}]$$

[from (IA)]

$$\Rightarrow \frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{1}{2}\vec{q}^{\,2}\right) - \vec{q}\times(\nabla\times\vec{q}) = -\frac{\nabla p}{\rho} + \vec{F}$$

$$\boxed{\vec{F} = -\nabla\Omega}\text{, because } (*)$$

Flow is **steady**,

$$\boxed{\frac{\partial \vec{q}}{\partial t} = \vec{0}} \quad \& \quad \boxed{\nabla\times\vec{q} = \vec{\zeta}}$$

$$\Rightarrow \nabla\left(\frac{1}{2}(\vec{q})^2\right) - \vec{q}\times(\vec{\zeta}) = -\frac{\nabla p}{\rho} \pm \nabla\Omega$$

Also,

$$d\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\,dp + p\,d\left(\frac{1}{\rho}\right)$$

$$\Rightarrow \frac{dp}{\rho} = d\left(\frac{p}{\rho}\right)$$

$$\nabla\int\frac{dp}{\rho} = \nabla\int d\left(\frac{p}{\rho}\right) = \nabla\left(\frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\frac{1}{2}\vec{q}^{\,2} + \int\frac{dp}{\rho} + \Omega\right) = \vec{q}\times\vec{\zeta} \quad \rightarrow (1)$$

1 (If $\vec{r}$ is the position vector of P at time t during the small increase of time in δt seconds, $\delta \vec{r}$ will be the infinitesimal displacement in the fluid element.)

2 Scalar multiplying eqn ① by $d\vec{x}$ which would be the limiting case representing a time independent variation of the P.V $\vec{x}$

2 (ie.) as $\delta t \to 0$, $\lim_{\delta t \to 0} \delta \vec{r} = d\vec{r}$

We have,

$$①\cdot d\vec{x} \Rightarrow \nabla\left(\frac{1}{2}\vec{q}^{\,2} + \int \frac{\partial p}{\rho} + \Omega\right)\cdot d\vec{x} = (\vec{q}\times\vec{\xi})\, d\vec{x}$$

4
$$\left[\because \nabla\phi \cdot d\vec{r} = \left(\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi\right)\cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})\right)$$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= d\phi \quad \Big]$$

$$\therefore \boxed{d\left(\frac{1}{2}\vec{q}^{\,2} + \int \frac{dp}{\rho} + \Omega\right) = d\vec{r}\cdot(\vec{q}\times\vec{\xi}) \quad ②}$$

Hence we have various case

Case (i) :

If $\vec{q} \times \vec{\xi} = \vec{0}$ where $\vec{\xi} \neq 0$ (i.e.) $\nabla \times \vec{q} \neq \vec{0}$)

then $\vec{q} \parallel \vec{\xi}$ i.e.) stream lines are parallel to vortex lines.

$\Rightarrow$ stream lines and vortex lines are coincides for such a motion $\vec{q}$ is called Beltrami vector.

(2) becomes,

$$d\left(\frac{1}{2}\vec{q}^{\,2} + \int \frac{dp}{\rho} + \Omega\right) = d\vec{r}\cdot(\vec{q} \times \vec{\xi})$$
$$= d\vec{r}\cdot\vec{0}$$

$$d\left(\frac{1}{2}\vec{q}^{\,2} + \int \frac{dp}{\rho} + \Omega\right) = \vec{0} \longrightarrow (3)$$

$\Rightarrow$ By integrating (3), we get

$$\boxed{\frac{1}{2}\vec{q}^{\,2} + \int \frac{dp}{\rho} + \Omega = \text{constant}} \rightarrow (3A)$$

(3A) holds throughout the entire field of flow. The constant is the same as the differential $d\vec{r}$ in (2) is an arbitarily a small variation of position vector $\vec{r}$ in the field.

Case (ii)

If $\vec{q} \times \vec{\zeta} = 0$ where $\vec{\zeta} = \vec{0}$ (ie., $\nabla \times \vec{q} = \vec{0}$)

ie.) the flow is irrotational.

Eqn ② becomes

$$d\left(\frac{1}{2}\vec{q}^2 + \int \frac{dp}{\rho} + \Omega\right) = d\vec{r} \cdot (\vec{q} \times \vec{\zeta})$$
$$= d\vec{r} \cdot \vec{0}$$
$$= \vec{0} \rightarrow (4)$$

By integrating ④, we get

$$\boxed{\frac{1}{2}\vec{q}^2 + \int \frac{dp}{\rho} + \Omega = \text{constant}} \rightarrow (4A)$$

(4A) holds throughout the entire field of flows, the constant is the same as the differential $d\vec{r}$ is an arbitrarily a small variation of position vector $\vec{r}$ in the field.

Case (iii):

If the fluid is homogeneous and the flow is irrotational

ie.) fluid is incompressible.

(4A) becomes

$$\boxed{\frac{1}{2}\vec{q}^{\,2} + \frac{p}{\rho} + \Omega = \text{constant}} \rightarrow (4B)$$

Case (iv) :

Suppose $\vec{q} \times \vec{\zeta} \neq \vec{0}$

Consider $\vec{q}\cdot(\vec{q}\times\vec{\zeta}) = [\vec{q}\ \ \vec{q}\ \ \vec{\zeta}\,] = 0$

$$\Rightarrow \vec{q} \perp^{r} (\vec{q}\times\vec{\zeta})$$

Also $\vec{\zeta}\cdot(\vec{q}\times\vec{\zeta}) = [\vec{\zeta}\ \ \vec{q}\ \ \vec{\zeta}\,] = 0$

$$\Rightarrow \vec{\zeta} \perp^{r} (\vec{q}\times\vec{\zeta})$$

If $\delta\vec{r}$ is so choosen such then $(\vec{q}\times\vec{\zeta})\cdot d\vec{r} = \vec{0}$, this happens whenever $d\vec{r}$ lies in the plane of $\vec{q}$ and $\vec{\zeta}$.

Thus we take the variation $d\vec{r}$ in the surface containing both stream lines and vortex lines, we have

R.H.S of (2) = 0

$$\therefore d\left(\frac{1}{2}\vec{q}^{\,2} + \int\frac{dp}{\rho} + \Omega\right) = 0 \rightarrow (5).$$

Integrating (5), we have

$$\boxed{\frac{1}{2}\vec{q}^{\,2} + \int \frac{dp}{\rho} + \Omega = \text{constant}} \rightarrow (5B)$$

Further note that over surface described by (5), we have (5B), the surface containing stream lines and vortex lines. The constant appearing in (5B) remains everywhere on the surface which varies from one to another on sets of surfaces.

(5B) is irrespective of the fact whether the motion is irrotational or not.

If the flow is **homogeneous** and the fluid under consideration is **incompressible**, then we have

$$(5B) \Rightarrow \frac{1}{2}\vec{q}^{\,2} + \frac{1}{\rho}\int dp + \Omega = \text{constant}$$

$$\Rightarrow \boxed{\frac{1}{2}\vec{q}^{\,2} + \frac{p}{\rho} + \Omega = \text{constant}} \rightarrow (6)$$

To establish kelvin's theorem expressing the condition under which circulation in a closed circuit of fluid particles moving along with the fluid is constant.

(pg. 124)

8m

Some Further Aspects of Vortex Motion

20m (a)

Kelvin's theorem:

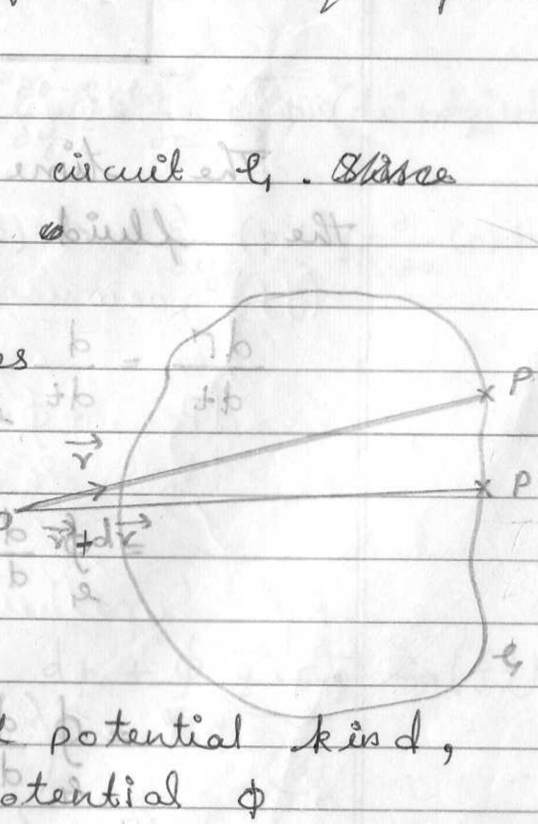
Statement:

For an invisid fluid circulation around any closed circuit of fluid particles, moving along with the fluid remains constant provided the body forces are conservative and the pressure is a single valued function of density only.

Proof:

[Write 1^{st} part of Euler's eqn of motion]

Consider a closed circuit C_1. ~~Since the body forces are~~



Since the body forces are conservative,

$\Rightarrow \exists$ a scalar pt fn Ω

$\ni \boxed{\vec{F} = -\nabla \Omega}$

Since the flow is of potential kind,

$\Rightarrow \exists$ a vector potential ϕ

$\ni \boxed{\vec{q} = -\nabla \phi}$

∴ The Euler's eqn of motion is gn by

$$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho}\nabla p \rightarrow (\mathrm{I})$$

$$\Rightarrow \frac{d\vec{q}}{dt} = -\nabla\Omega - \frac{1}{\rho}\nabla p \rightarrow (\mathrm{IA})$$

Let P' be a neighbouring point to P on C_1 with position vector $\vec{r} + d\vec{r}$

Circulation around the closed circuit C_1 is given by

$$\boxed{\Gamma = \oint_{C_1} \vec{q}\cdot d\vec{r}}$$

The time derivative of Γ following the fluid flow is

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint_{C_1} \vec{q}\cdot d\vec{r}$$

$$= \oint_{C_1} \frac{d}{dt}(\vec{q}\cdot d\vec{r})$$ [As it is finite $\oint$ and time derivative are interchangeable]

$$= \oint_{C_1}\left(\frac{d}{dt}\vec{q}\cdot d\vec{r} + \vec{q}\cdot\frac{d}{dt}(d\vec{r})\right)$$

$$= \oint_{C_1} \frac{d}{dt}\vec{q}\cdot d\vec{r} + \oint_{C_1} \vec{q}\cdot\frac{d}{dt}(d\vec{r})$$

$$= \oint_{C_1} \left(-\nabla \Omega - \frac{1}{\rho} \nabla p\right) \cdot d\vec{x} \quad (\text{from IA})$$

$$+ \oint_{C_1} \vec{q} \cdot \frac{d}{dt}(d\vec{x})$$

i.e.) $$\frac{d\Gamma}{dt} = \underbrace{-\oint_{C_1} \nabla \Omega \cdot d\vec{x}}_{1} - \underbrace{\oint_{C_1} \frac{1}{\rho} \nabla p \cdot d\vec{x}}_{2} + \oint_{C_1} \vec{q} \cdot \frac{d}{dt}(d\vec{r}) \rightarrow (f)$$

1 $\Rightarrow$ $$\nabla \Omega \cdot d\vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\Omega \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{\partial}{\partial x}\Omega\, dx + \frac{\partial}{\partial y}\Omega\, dy + \frac{\partial}{\partial z}\Omega\, dz$$

$$= d\Omega$$

2 $\Rightarrow$ $$\frac{1}{\rho}(\nabla p \cdot d\vec{r}) = \frac{1}{\rho}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)p \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial x}(p)\,dx + \frac{\partial}{\partial y}(p)\,dy + \frac{\partial}{\partial z}(p)\,dz\right)$$

$$= \frac{1}{\rho}\,dp$$

$\therefore$ (f) becomes

$$\frac{d\Gamma}{dt} = -\oint_{C_1} d\Omega - \oint_{C_1} \frac{1}{\rho}\,dp + \oint_{C_1} \underbrace{\vec{q} \cdot d\vec{q}}_{3} \rightarrow (ff)$$

We can show that,

$$\frac{d}{dt}(d\vec{x}) = d\left(\frac{d\vec{x}}{dt}\right) \longrightarrow (*)$$

$$(ff) \Rightarrow \frac{d\Gamma}{dt} = -\oint_{c_1} d\Omega - \oint_{c_1} \frac{1}{\rho}\, dp + \frac{1}{2}\oint_{c_1} d(\vec{q})^2$$

$$③ \Rightarrow$$

Also, $\vec{q}\cdot d\vec{q} = \frac{1}{2}\, d(\vec{q})^2$

$$\left[\because \vec{q}\cdot\vec{q} = \vec{q}^{\,2}\right.$$

$$d(\vec{q}\cdot\vec{q}) = d\vec{q}\cdot\vec{q} + \vec{q}\cdot d\vec{q}$$

$$\left.= d(\vec{q}^{\,2})\right]$$

$$\cancel{\Rightarrow \vec{q}\cdot d\vec{q} = \frac{1}{2}\, d(\vec{q})^2}$$

We have,

$$-\oint_{c_1} d\Omega = -\Omega\Big]_{c_1\ \text{defined}} = 0$$

and $\displaystyle\oint_{c_1} d(\vec{q})^2 = (\vec{q})^2\Big]_{c_1} = 0$

(ff) becomes to,

$$\frac{d\Gamma}{dt} = -\oint_{c_1} \frac{1}{\rho}\, dp \rightarrow (fff)$$

By hypothesis,

p is a single valued fn dependent on ρ only.

Hence $\displaystyle -\oint_{c_1} \frac{1}{\rho}\, dp = -\oint_{c_1} d\left(\frac{p}{\rho}\right) = 0$

$\therefore$ (fff) reduces to

$$\Rightarrow \frac{d\Gamma}{dt} = 0$$

$$\boxed{\Rightarrow \Gamma \text{ constant w.r.t moving circuit}}$$

Now to prove (*)

i.e) $$\boxed{\frac{d}{dt}(d\vec{r}) = d\left(\frac{d\vec{r}}{dt}\right)}$$

At time t, the position vectors of points P & P' which reference to the origin O respectively are $\vec{r}$ and $\vec{r}+d\vec{r}$.

Then at time $t+dt$, their position vectors are $\vec{r}$ $\qquad \vec{r}+d\vec{r}$

$$\vec{r}+\frac{d\vec{r}}{dt}dt,\ (\vec{r}+d\vec{r})+\frac{d}{dt}(\vec{r}+d\vec{r})\,dt \longrightarrow (1)$$

Since $d\vec{r}$ denotes the variation of fluid elt at p to that at p', the position vector of p' at time $t+dt$ is gn by

$$\downarrow$$

$$(\vec{r}+d\vec{r})+d\left(\vec{r}+\frac{d\vec{r}}{dt}dt\right) \longrightarrow (2)$$

Comparing (1) & (2) carefully we notice that (*) holds.