16/06/17
Friday

[15MT/PC/FD 34]

Fluid Dynamics

Unit - 1

CA

The kinematics of Fluids in Motion

1.1 Real Fluids and Ideal Fluids - Velocity of a Fluid at a Point
1.2 Stream Lines and Path Lines - Velocity Potential - Vorticity
1.3 Local and Particle Rates of Change
1.4 Equation of Continuity
1.5 Acceleration of a Fluid - Conditions at a Rigid Boundary.

7:30 - 10.00

Unit - 2

CA

Equations of Motion of a Fluid

2.1 Pressure at a Point in a Fluid at Rest
2.2 Pressure at a Point in a Moving Fluid - Conditions at a Boundary of Two Inviscid Immiscible Fluids
2.3 Euler's Equation of Motion - Bernoulli's Equation
2.4 Steady Motion under Conservative Body Forces
2.5 Kelvin's Circulation Theorem

10.00 - 1.00

Unit - 3

After lunch

Some Two & Three - Dimensional Flows

CA 3.1 Some flows involving Axial Symmetry - Irrotational Stationary Sphere in a

26th

Uniform Stream - Sphere moving with Constant velocity in Liquid which is otherwise at Rest.

3.2 Sources, Sinks and Doublets

3.3 Axi Symmetric Flows - Stoke's Stream fn Speci Forms of the stream fn for Axi-symmetric Irrotational Motions.

3.4 Meaning of Two-Dimensional force flows

Unit - 4

10/11/17

After sleep

Complex Velocity Potential

4.1 Complex Velocity Potential for Standard

4.1.2 Two Dimensional Flows

before tea

4.2 Milni - Thomson Circle theorem - Extension of the Circle Theorem

4.3 Theorem of Blasius

Unit - 5

Viscous Flow

5.1 Stress Components in a Real Fluid - Coefficient of Viscosity and Lamin Flow

5:00-7:00

5.2 Navier - Stokes Equation of Motion of a Viscous Fluid

5.3 Some Solvable Problems in Viscous Flow

5.4 Steady Viscous Flow in Tubes of Uniform Cross-section

Text Book:

Chorlton. F. "Text book of Fluid Dynamics. 1st ed. New Delhi: Shadara, 1985.

Chapter	Sections
2	2.1 - 2.10
3	3.1 - 3.7, 3.12, 3.9
4	4.1, 4.2, 4.5
5	5.1, 5.4 - 5.6, 5.8, 5.9
8	8.8 - 8.11

After dinner Problems

Books for Reference:

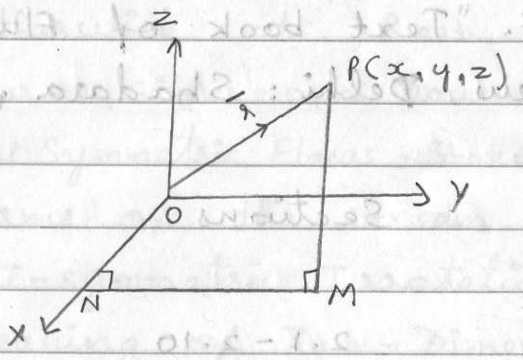
1. Duncan W. J., Thom. A. S. and Young A. D., Mechanics of Fluid. Great Britain: The English Language book society, 1975.

2. Joseph H. Spurk, Fluid Mechanics: Problems and Solutions. Springer-Verlag, 2003.

3. Thomson Milne L. M., Theoretical Hydro Dynamics. (IV Edition), New York.: Macmillan and Co., 1960

Pre-requisites

Use of Co-ordinates:



The cartesian co-ordinates $P(x, y, z)$ are the $\perp^r$ distance of P from the co-ordinate plane YOZ, XOZ, XOY.
$(x = ON,\ y = NM,\ z = MP)$

$$\overrightarrow{ON} = x\hat{i}, \quad \overrightarrow{NB} = y\hat{j}, \quad \overrightarrow{MP} = z\hat{k}$$

$$\therefore \overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NM} + \overrightarrow{MP}$$
$$= x\hat{i} + y\hat{j} + z\hat{k}$$

Triple Products:

$$\left[\bar{a}, \bar{b}, \bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \Rightarrow \text{scalar triple produ}$$

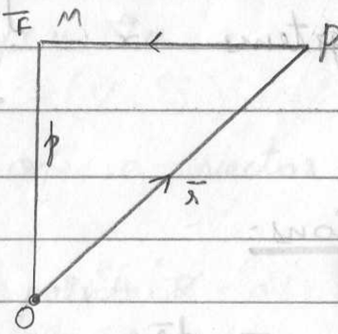
a

Vector moment about a point and scalar moment about an axis:

* $\bar{F}$ - vector quantity acting at a point P where $\overline{OP} = \bar{r}$.

Pre-requisites

* $\overline{G}$ - vector moment of $\overline{F}$ abt O is
$$G = \overline{r} \wedge \overline{F}$$
* $\overline{G} = rF \sin\theta \, \hat{n}$
$$= (pF)\,\hat{n}$$
* $\theta = \angle OPM$
* It measures the tendency of the force $\overline{F}$ to turn O in the specific direction $\hat{n}$.
* $|\overline{G}| = pF$ is called the physical moment about the axis $\hat{n}$ through O.

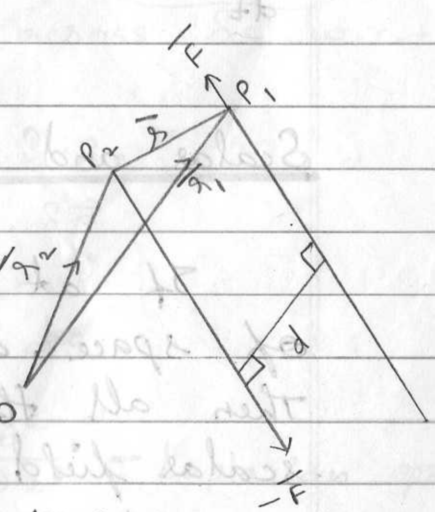


$\hat{n}$ $\perp^{r}$ to the plane OPM

p $\perp^{r}$ distance from O to $\overline{F}$

Couple:

Two equal and opposite vectors $(\pm\overline{F})$ not acting in the same straight line ~~is~~ are said to be a couple.



Total vector moment about O is
$$\overline{G} = \overline{r}_1 \wedge \overline{F} + \overline{r}_2 \wedge (-\overline{F})$$
$$= (\overline{r}_1 - \overline{r}_2) \wedge \overline{F}$$
$$= \overline{r} \wedge \overline{F}$$

Centroids:

$$\overline{OG} = \bar{r} = \frac{\sum_{k=0}^{n} m_k r_k}{\sum_{k=0}^{n} m_k} \quad \text{where } \sum_{k=0}^{n} m_k \neq 0$$

Cartesian Co-ordinates of G.

$$\hat{x} = \frac{\sum_{k=0}^{n} m_k x_k}{\sum_{k=0}^{n} m_k}$$

For unit systems $\hat{x} = \dfrac{\int x \, dm}{\int dm}$

Differentiations:

$$\frac{d}{dt}(\bar{r}.\bar{s}) = \bar{r}.\frac{d\bar{s}}{dt} + \bar{s}.\frac{d\bar{r}}{dt}$$

$$\frac{d}{dt}(\bar{r} \wedge \bar{s}) = \bar{r} \wedge \frac{d\bar{s}}{dt} + \frac{d\bar{r}}{dt} \wedge \bar{s}$$

Scalar and Vector field:

If at each pt $P(x,y,z)$ of a region of space, a scalar fn $\phi(x,y,z)$ is defin[ed] Then all these ϕ in R constitute a ~~spa~~ scalar field.

If φ is unique at each pt P of R, then it is a uniform fn. For such uniform fn φ, we construct surfaces on each φ is constant and these are called iso-φ surfaces (or) curl surfaces.

Similarly, we can define vector field. For uniform vector field, we express the unique vector fn $\overline{F}(x, y, z)$ as

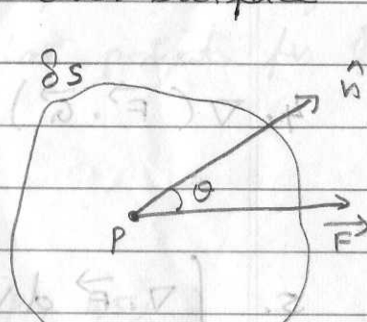
$$\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$
$$= [F_1, F_2, F_3] \quad \text{where } F_i = F_i(x, y, z)$$

Few results:

1. $\nabla \int f(u)\,du = f(u)\nabla u$

Normal flux of a vector over surface

* $\vec{F}$ acts at point P
* δs - surface elt
* $\hat{n}$ - unit normal



Normal flux of $\vec{F}$ across δs w.r.t $\hat{n}$ is

$$(F\cos\theta)\,\delta s = \vec{F}\cdot\hat{n}\,\delta s$$
$$= \vec{F}\cdot\delta\vec{s}$$

$$\therefore \int \vec{F}\cdot\delta\vec{s} = \int \hat{n}\cdot\vec{F}\,ds$$

2. $\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ - Laplacian opr

3. $\nabla^2\phi = 0$ then ϕ satisfies Laplace's eqn and is called harmonic function.

4. Curl grad $\phi = 0$ (ϕ - scalar field).

5. $\vec{F}$ any vector, div curl $\vec{F} = 0$;

$$\text{div curl}.\vec{F} = \nabla . (\nabla \times \vec{F})$$
$$= [\nabla, \nabla, \vec{F}]$$

Vector identities:

1. $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla . \vec{F}) - \nabla^2 \vec{F}$ (curl curl $\vec{F}$)

2. $\text{div}(\vec{F} \times \vec{G}) = \vec{G}\ \text{curl}\ \vec{F} - \vec{F}.\text{curl}\ \vec{G}$

3. $\text{Curl}(\vec{F} \times \vec{G}) = (\vec{G}.\nabla)\vec{F} - (\vec{F}.\nabla)\vec{G} + \vec{F}\ \text{div}\ \vec{G} - \vec{G}\ \text{div}\ \vec{F}$

4. $\nabla(\vec{F}.\vec{G}) = (\vec{G}.\nabla)\vec{F} + (\vec{F}.\nabla)\vec{G} + \vec{G} \times \text{curl}\ \vec{F} + \vec{F} \times \text{curl}\ \vec{G}$

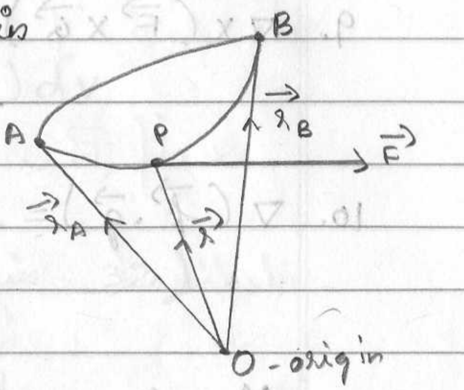
5. $\int_V \nabla . \vec{F}\, dV = \int_S \hat{n} . \vec{F}\, ds$

6. Stoke's theorem:

$$\oint_C \vec{F} . d\vec{s} = \int_S \hat{n} . \text{curl}\ \vec{F}\, ds$$

Conservation vector field

$\vec{F}$ - uniform vector field in region R has 2 pts A, B



$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{s}$$

$$\oint_C \vec{F}\cdot d\vec{s} = 0$$

Scalar point fn & Vector pt fn :

$\nabla\phi$, Div $\vec{F}$, curl $\vec{F}$, $\nabla\vec{F} = 0$, $\nabla\times\vec{F} = \vec{0}$,

conservation vector field (if $\nabla\times F = \vec{0}$)

= then (pg 42, 43) $\exists$ scalar point fn ϕ.

$$\vec{F} = -\nabla\phi$$

1. $\nabla\cdot\nabla\phi = \nabla^2\phi$; work done (pg 45)

2. $\nabla\cdot(\nabla\times\vec{F}) = \mathrm{Div}(\mathrm{curl}\,\vec{F}) = 0$, $\because [\nabla\,\nabla\,\vec{F}]$

3. $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) = \nabla^2\vec{F}$

4. $\nabla(\nabla\cdot\vec{F}) = \mathrm{Grad}\,(\mathrm{div}\,\vec{F}) = \nabla\times(\nabla\times\vec{F}) + \nabla^2\vec{F}$ (from ③)

5. $\overset{\text{curl}}{\nabla}\times(\overset{\text{grad}}{\nabla\phi}) = \vec{0}$

6. $\nabla\cdot(\phi\vec{F}) = \nabla\phi\cdot\vec{F} + \phi(\nabla\cdot\vec{F})$

7. $\nabla\times(\phi\vec{F}) = \nabla\phi\times\vec{F} + \phi(\nabla\times\vec{F})$

8. $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

9. $\nabla \times (\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla)\vec{F} - (\nabla \cdot \vec{F})\vec{G} - (\vec{F} \cdot \nabla)\vec{G} + (\nabla \cdot \vec{G})\vec{F}$

10. $\nabla (\vec{f} \cdot \vec{g}) = \vec{f} \times (\nabla \times \vec{g}) + \vec{g} \times (\nabla \times \vec{f}) + (\vec{f} \cdot \nabla)\vec{g} + (\vec{g} \cdot \nabla)\vec{f}$

Meaning of $(\vec{F} \cdot \nabla)\vec{G}$

$$= (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$= \Sigma\, F_1 \frac{\partial}{\partial x}$$

Eqn to any level surface is

$$\Phi(x, y, z) = \text{constant}$$

$$\nabla \Phi \Big]_p : \frac{\partial \Phi}{\partial x}\hat{i} + \frac{\partial \Phi}{\partial y}\hat{j} + \frac{\partial \Phi}{\partial z}\hat{k} = \text{normal to surface at } p.$$

Unit vector at $p = \dfrac{\nabla \Phi}{|\nabla \Phi|} = \hat{n}$

$$\frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{1}{1}$$

<u>Integration : line, Surface and volume</u>

Thus

1) Gauss div (surface & vol)

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \vec{F})\, dv$$

2. Stokes (line & surface in 3D)

$$\iint \text{curl}\, \vec{F} \cdot \hat{n}\, ds = \oint_C \vec{F} \cdot ds$$

S - surface bounded by the curve c

c → closed curve with no loops or simple closed curve

3. Green (line & surface in 2D):

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

*Line integral around a boundary of a plane region D can be computed as a Double integral over D.

*Line integral tangential component of $\vec{F}$ along c

$$= \int_C \vec{F} \cdot d\vec{r}$$

Flux (in latin flow → introduced by Newton) action or process of flowing per unit area ⇔ flow of a physical property in space (time dependent)

* Vector quantity - if explaining transport [Heat transform, mass transform or f.D]

* $\iint_S \vec{F}.\hat{n}\,ds$ - scalar quantity as defined by surface integral.

Eg:

* Magnitude of river current i.e) amount of water that flows through cross section of river each second.

* Amount of sunlight that lands on a patch of ground each second.

* $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3$

* $\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

* $\delta_{ij}\, x_j = \delta_{i1}\, x_1 + \delta_{i2}\, x_2 + \delta_{i3}\, x_3 = x_i$

$$= \begin{cases} x_1, & i=1 \\ x_2, & i=2 \\ x_3, & i=3 \end{cases}$$

$$* \ \frac{\partial \phi}{\partial x_j} = \phi_{,j} \ ; \ \frac{\partial u_i}{\partial x_j} = u_{i,j}$$

$$* \ \Phi_{,ii} = \nabla \phi^2 \ ; \ u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$* \ \mathrm{div}\ \vec{A} = A_{i,i} \ ; \ \mathrm{grad}\ \Phi = \phi_{,i}\ \hat{e}_i$$

$$(OX_1 X_2 X_3) \to (OX_1' X_2' X_3') : l_{ij} = j \to i$$

	x_1	x_2	x_3
x_1'	l_{11}	l_{12}	l_{13}
x_2'	l_{21}	l_{22}	l_{23}
x_3'	l_{31}	l_{32}	l_{33}

$$l_{ij} + l_{kj} = \delta_{ik}$$

$$i = j \ , \ l_{11}^2 + l_{12}^2 + l_{13}^2 = 1 \ ; \ \ldots$$

$$l_{11}\, l_{21} + l_{12}\, l_{22} + l_{13}\, l_{23} = 0 \ , \ \ldots$$

Also, $l_{ij}\, l_{ik} = \delta_{jk}$

$$l_{11}^2 + l_{21}^2 + l_{31}^2 = 1 \ , \ \ldots \ ; \ l_{11}\, l_{12} + l_{21}\, l_{22} + l_{31}\, l_{32} = 0$$

$$l_{ij}\, x_j = x_i' \ \text{or} \ \Leftrightarrow \ l_{ji}\, x_j' = x_i$$

UNIT- 1

1.1) Real fluids and Ideal fluids :

1 (Matter is anything that occupy space and matter is generally divided into solids and fluids and further fluids into gases and liquids.

A fluid motion in liquids is called "Hydrodynamics".

If a portion of matter under a given dynamic condition, in the absence of external forces has a definite shape is called a solid and if the matter takes the shape of the container, that contains the matter is called fluid.

Eg:

A piece of iron does not change its shape in the absence of ~~no~~ external forces, while a glass of water when the glass is tilted even slightly changes its shape.)

2 ("Fluid refer to the substance that flows and Fluid Dynamics is the science of study of fluids in motion".)

3 (Fluids are majorly divided into few types :

i) Liquids that are incompressible, i.e., the volume of a fluid does not change when the pressure changes.

ii) Gases which are compressible fluids that changes in volume whenever the pressure changes.)

Thus hydrodynamics is a study of motion of incompressible fluid.

4 (When we study matter on a microscopic (or) molecular scale wherein the molecules are at random motion and separated from one another by least comparable scales with respect to molecular size.

However in macroscopic analysis, molecular structure is of no interest and the continuum hypothesis is of interest to learn.)

5 (Any fluid matter is a continuum. (i.e.) mathematically continuous. This leads to the result that an element of a fluid however small

will have matter in it. Thus, under continuum hypothesis, there exists a 1-1 correspondence between particles of fluids and points in space.

However fluid particles at a point has unique velocity, pressure, density, etc.)

6 (**Classification:**

Fluids are classified as perfect, ideal, inviscid, non-viscous and real actual viscous fluid.

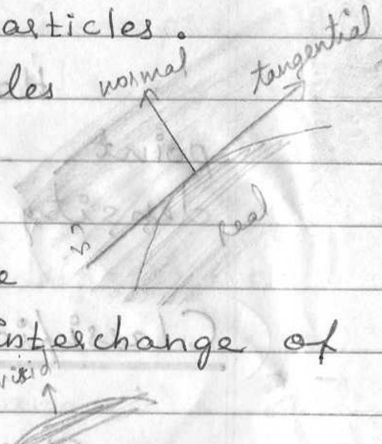
Perfect (or) inviscid fluids have no friction. In the motion of such a perfect fluid, two layers of contact experience no tangential forces (or) shearing stresses but act on each other with normal forces only.

Eg:

Pressure forces which are due to pressure extended by the fluid.

On the other hand, the inner layer of a fluid transmit tangential as well as normal stresses. These tangential or frictional forces are connected with the property which is called viscosity of a fluid.

In a viscous fluid, there is friction between fluid particles. ie., Suppose two fluid particles moving with different velocities have a common boundary, then across the boundary there will be interchange of momentum.



The normal transformation of molecules across the boundary will lead to direct or normal forces.

Thus viscosity is otherwise known as "Fluid friction".)

(• Density ρ is defined by mass per unit volume.

$$\text{ie., } \rho = \frac{\text{mass}}{\text{unit volume}} = \frac{M}{L^3} : \text{dim of } \rho$$

• Pressure p is defined by force per unit area.

$$\text{ie., } p = \frac{\text{Force}}{\text{unit area}} = \frac{M\cancel{L}}{T^2 L^{\cancel{2}}} = \frac{M}{T^2 L}.$$

$$\left[\because Ma = \frac{ML}{T^2}\right]$$

Action exerted by fluid on any surface inserted in fluid reduce to unit area basis.)

Kinematics of a fluid:

[1] (Kinematics is description of motion. In the realm of dynamics, it takes no account of how motion is brought about or of the forces involved in.

Consequently, the results of kinematical study apply to all types of fluids, and are the ground work on which the dynamical results are constructed.)

[2] (The basic mathematical idea of fluid motion is that it can be described by point transformation.

At some instant, we observe the fluid and find certain particle to be at position $\overline{X}$ and latter at a position $\overline{x}$.

Without loss of generality, the first instant can be considered to be at time $t=0$ and the latter to be at time t then $\overline{x}$ is independent of t at initial position.

ie., $\bar{x} = \bar{x}(\bar{X}, t)$
$$= \bar{x}(X_1, X_2, X_3, t) \longrightarrow (*)$$

This violates the concept of kinetic theory of fluid ~~for in this~~

For, in this theory the particles are molecules and are in random motion.)

3 (Infact we replace the molecular picture by that of continuum whose velocity at any point in the average velocity of the molecules in a suitable neighborhood of the point.

The initial coordinates $\bar{X}$ of a fluid particle will be referred to as material coordinates of the particle and X_i are called Lagrangian coordinates (or) convected coordinates as the material coordinate system is convected with the fluid.

The spatial coordinate $\bar{x}$ of the fluid particle is referred to its position or place in space.

It is assumed that the motion is continuous and single valued as it is the motion ~~is continuous and single valued as it~~ of fluid and that (*) can be inverted such that

$$\overline{X} = \overline{X}(\overline{x}, t) \longrightarrow (**)$$

$$\text{(or)} \Leftrightarrow \quad = \overline{X}(x_1, x_2, x_3, t)$$

$$= \overline{X}(x_i, t)$$

which is also continuous and single valued.

Physically, this means that the fluid is nothing but the continuous arc of fluid particles which do not breakup during motion.

The single valuedness of eqn means that the given particle of the continuum cannot split up and occupy two different places nor can two distinct particles occupy the same place.

It is also assumed that the derivatives are continuous and are continuously differentiable as per requirement.

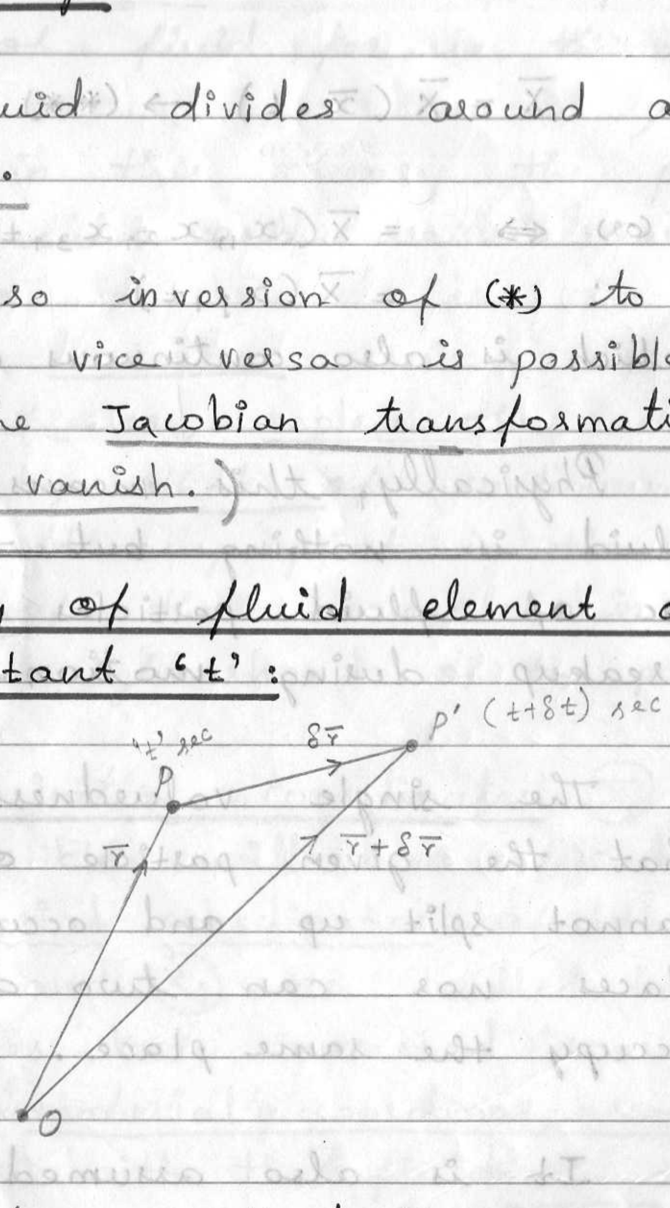
Exceptions are allowed on a finite no. of singular surfaces or singular lines or singular points.

~~Eg:~~ For eg:

fluid divides around an obstacle.

Also inversion of (*) to yield (**) or vice versa is possible only when the Jacobian transformation doesn't vanish.)

Velocity of fluid element at any instant 't':



Consider a fluid flow.

Choose (O) to be an origin.

Let (P) be an position of fluid elt at time 't' seconds and (P') be the position reached by the fluid elt at time $(t+\delta t)$ sec with respect to origin.

Position vectos of P : $\overrightarrow{OP} = \vec{x}$ (say)

Position vectos of P' : $\overrightarrow{OP'} = \vec{x} + \delta\vec{x}$

$$\overrightarrow{OP} + \overrightarrow{PP'} = \overrightarrow{OP'} \quad \rightarrow \Delta^{le} \text{ law of vectors}$$

$$\Rightarrow \overrightarrow{PP'} = \overrightarrow{OP'} - \overrightarrow{OP}$$

$$= \vec{x} + \delta\vec{x} - \vec{x}$$

$$= \delta\vec{x}$$

arc of particles = motion of fluid

$\delta\bar{x}$ is very small as P' is a <u>neighbouring point</u> of P $\Rightarrow$ arc PP' ($\widehat{PP'}$) and chord PP' ($\overline{PP'}$) are almost same.

$\therefore \widehat{PP'} = \overline{PP'} = \delta\bar{x}$ is a <u>displacement</u> of fluid elt at (δt) sec.

$\therefore$ <u>Average rate of displacement</u> of fluid element

$$= \frac{\delta\bar{x}}{\delta t}$$

Hence <u>velocity</u> of the moving particle at P, is given by

$$\lim_{\delta t \to 0} \frac{\delta\bar{x}}{\delta t} = \frac{d\bar{x}}{dt}$$

Existence is guaranteed since the flow under consideration is <u>continuous motion</u> of fluid particles.

ie., fluid velocity at position (P) at time t sec denoted by $\bar{q}$ is defined to be

$$\bar{q} = \frac{d\bar{r}}{dt} = \overset{\text{operator}}{\frac{d}{dt}}(\bar{r}) \rightarrow (1)$$

= time derivative of its position vector at time 't' sec.

Note that fluid velocity is a vector quantity and $\bar{q} = \bar{q}(x, y, z, t) \rightarrow (2)$ is a vector point fn of x, y, z and t in the ~~th~~ cartesian coordinate system.

Further if $\hat{i}, \hat{j}$ & $\hat{k}$ are unit vectors in x-increasing, y-increasing and z-increasing, respectively on ox, oy, oz axes, then we write

$$\bar{q} = u\hat{i} + v\hat{j} + w\hat{k} \rightarrow (3)$$

where $u = u(x, y, z, t)$

$v = v(x, y, z, t)$

$w = w(x, y, z, t)$

are scalar pt fns of x, y, z, t.

$\therefore (u, v, w)$ are called components of $\bar{q}$.

Position vector of $P = \overline{OP} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (say)

Then

$$\bar{q} = \frac{d}{dt}\bar{r} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \rightarrow (4) \text{ (using (1))}$$

Eqn (3) & (4) yields the same and hence comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from (3) & (4), we have

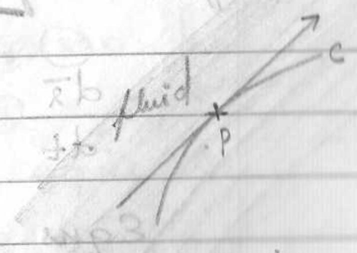
$$u = \frac{dx}{dt}, \ v = \frac{dy}{dt}, \ w = \frac{dz}{dt} \longrightarrow (5)$$

Thus the <u>velocity</u> <u>components</u> are given by (5).

Eqn (5) is equivalent to the <u>derivatives of components</u> of $\bar{r}$ are the <u>components</u> of $\bar{q}$ (or) equivalently $\bar{q}$ is given by <u>derivative</u> of $\bar{r}$.

1.2) Stream lines :

Let P be any <u>point</u> in the fluid flow moving with <u>velocity</u> $\bar{q}$ at any <u>time</u> 't'.



Draw a <u>curve</u> C through P in the fluid such that the

* stream lines & path lines are not same, in general.

tangent at P to the curve C is in the direction of the fluid velocity $\overline{q}$ at P, then the curve C is called stream line.

ie., The tangent at the point P to the curve C coincides with the velocity vector $\overline{q}$ at P.

Thus stream line give the pattern of flow at any instant.

Equations to stream lines:

Let $P(x, y, z)$ be any point in the fluid referred w.r.t an origin O in the fluid such that

Position vector of $P = \overline{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Given that fluid velocity is

$$\overline{q} = \overline{q}(u, v, w)$$
$$= u\hat{i} + v\hat{j} + w\hat{k} \rightarrow (1)$$

Velocity of fluid at P is then given by

$$\frac{d\overline{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \rightarrow (2)$$

Eqns of stream lines are obtained under the consideration with the tangential component of direction of

→ Stream lines show how each particle is moving at a gn instant of time while the path lines present the motion of the particles at each instant.

flow at P is **coincides** with the fluid velocity.

ie., $\bar{q} \parallel \dfrac{d\bar{x}}{dt}$, by ① & ②

Then,

$$\bar{q} \times \frac{d\bar{r}}{dt} = \bar{0}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} & \dfrac{dz}{dt} \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \hat{i}\left(v\frac{dz}{dt} - w\frac{dy}{dt}\right) - \hat{j}\left(u\frac{dz}{dt} - w\frac{dx}{dt}\right) + \hat{k}\left(u\frac{dy}{dt} - v\frac{dx}{dt}\right) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow v\frac{dz}{dt} - w\frac{dy}{dt} = 0, \quad u\frac{dz}{dt} - w\frac{dx}{dt} = 0, \quad u\frac{dy}{dt} - v\frac{dx}{dt} = 0$$

$$\Rightarrow v\frac{dz}{dt} = w\frac{dy}{dt}, \quad u\frac{dz}{dt} = w\frac{dx}{dt}, \quad u\frac{dy}{dt} = v\frac{dx}{dt}$$

$$\Rightarrow \frac{\frac{dz}{dt}}{w} = \frac{\frac{dy}{dt}}{v}, \quad \frac{\frac{dz}{dt}}{w} = \frac{\frac{dx}{dt}}{u}, \quad \frac{\frac{dy}{dt}}{v} = \frac{\frac{dx}{dt}}{u}$$

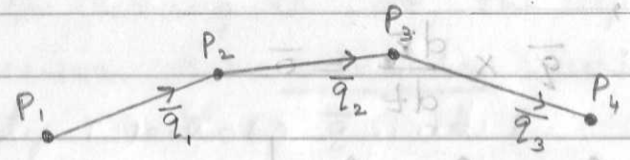
$$\Rightarrow \frac{\frac{dx}{dt}}{u} = \frac{\frac{dy}{dt}}{v} = \frac{\frac{dz}{dt}}{w} \rightarrow ③$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \rightarrow ③ \text{ - lagrange}$$

Solving ③, we get equs to stream lines.

③ yields **double ∞ (infinity) of solutions.**

→ Knowing $\bar{q}$ at successive pts of stream lines at any instant of time 't' enables stream lines may be approximated by straight line segments.



Suppose $\bar{q}_1, \bar{q}_2, \bar{q}_3, \ldots$ denote the velocities at neighbouring pts $P_1, P_2, P_3, \ldots$ of the stream line in fluids, then small straight segments $\overline{P_1P_2}, \overline{P_2P_3}, \overline{P_3P_4}, \ldots$ are such to have directions of $\bar{q}_1, \bar{q}_2, \bar{q}_3, \ldots$, respectively.

At a particular point, suppose the pattern of flow doesn't change with time then the flow is said to be steady. Otherwise the pattern changes with time during the flow and is called an unsteady flow.

Thus, in a steady flow the pattern of flow remains the same at all times and at all locations.

Path lines :

The path traced by any individual fluid particle is called path line.

(In general, stream lines and path lines do not coincide.

ie., in case of unsteady ~~m~~ flows)
(here eventhough stream line through any point P thus touch the pathline through P, they don't coincide)

However, if the flow is steady then the stream lines and path lines coincides at any point in the fluid flow.

Equations to path lines :

The trajectory of path of a fluid particle is determined by the direction of a flow.

Suppose the particle is moving with the velocity

$$\overline{q} = u\hat{i} + v\hat{j} + w\hat{k} \rightarrow (1)$$

and if P.V of $P = \overline{OP} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (say)
then

$$\bar{q} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(\bar{r}) \quad \text{given by}$$

$$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \longrightarrow (2)$$

Eqn (1) & (2) yields,

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = \omega$$

where

$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$\omega = \omega(x, y, z, t)$$

$x = x(t)$, $u(x, y, z, t)$, $\frac{dx}{dt}$, $y = y(t)$, $z = z(t)$

Solving these 3 eqns, we have $x = x(t)$, $y = y(t)$, $z = z(t)$ yielding <u>triply-infinite set of solutions</u>.

<u>Eg:</u>

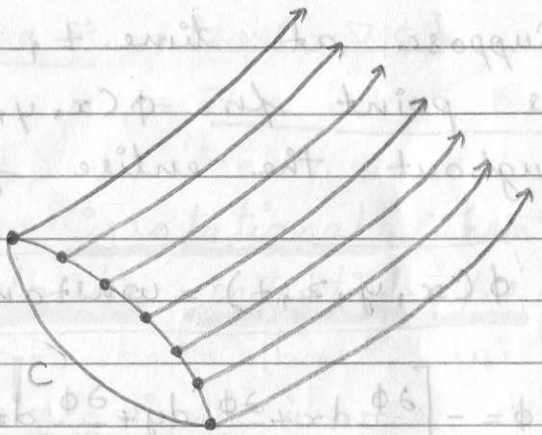
The <u>time exposure</u> may be used to photograph the streamlines for a <u>steady flow</u>, but a <u>snapshot</u> ~~may~~ must be used for <u>unsteady flow</u>.

→ In the case of fluids, particles that are illuminated by a suspension of a aluminium dust.

→ Alternatively fluid particles may be rendered visible by ~~flow~~ a few crystals of Potassium permanganate.

Study on gasses, smokes streams may be used.

If we draw the streamlines through every point of a closed curve c in the fluid, we obtain a stream ~~line~~ tube.



Velocity Potential: (or) velocity fn

Qm. Consider a fluid flow with velocity $\bar{q} = \bar{q}(u, v, w)$ in cartesian coordinate system at any time t, at any point $P(x, y, z)$:

$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$

Equns to stream lines at the point P at the instant t is the curve given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{\omega} \rightarrow (1)$$ (Prove this if it's more than 6 marks)

These curves will cut the surfaces

$$u\,dx + v\,dy + \omega\,dz = 0 \rightarrow (2)$$

orthogonally.

Suppose at time t, we can find a scalar point fn $\phi(x, y, z, t)$ uniform throughout the entire field of flow and $\exists$

$$\boxed{\phi(x, y, z, t) = \text{constant}} \rightarrow (*)$$

$$-d\phi = -\left[\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \frac{\partial \phi}{\partial t}dt\right] \rightarrow (3)$$

$$\boxed{-d\phi = u\,dx + v\,dy + \omega\,dz}$$

$$\therefore -\left[\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \frac{\partial \phi}{\partial t}dt\right] = u\,dx + v\,dy + \omega\,dz \rightarrow (4)$$

$$\Rightarrow u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad \omega = -\frac{\partial \phi}{\partial z} \rightarrow (5)$$

$\because$ ϕ is a constant in the entire field.

$$\Rightarrow \frac{\partial \phi}{\partial t} = 0$$

RHS of (4) is an exact differential hence $\overline{q} = -\nabla\phi$ ~~for~~ q.

$$(5) \Rightarrow \overline{q} = u\hat{i} + v\hat{j} + w\hat{k} = -\left[\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right]$$

$$\overline{q} = -\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]\phi$$

$$\overline{q} = -\nabla\phi$$

curl $\vec{q} = \vec{0}$

$$(\text{and } \nabla \times \vec{q} = \nabla \times (-\nabla\phi) = \vec{0}$$

(or) $\overline{q}$ is irrotational). Then ϕ is called velocity potential for the flow.

The -ve sign in the argument is a convention. It ensures that the flow takes place from higher to lower potentials. Some authors do consider opposite convention. Such a type of flow wherein the velocity vector is irrotational is called a potential flow (or) flow is said to be of potential kind.

Conversely,

Whenever the flow is of the potential kind such that the field of ($\overline{q}$) is conservative, the velocity vectos $\bar{q}$ is called Lamellar vector.

Discuss the flow for which $\omega = \frac{n}{2}$ 10m

The surfaces defined by (*) are called equipotentials.

Qn. 6m

① & ② shows that at all kinds of field of flow the equipotentials are cut orthogonally by the streamlines.

10M

V.V.V.

Problem:

10m/6m

At the pt in an incompressible fluid having spherical polar co-ordinates (r, θ, ψ), the velocity components are $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, where M is a constant.

i) Show that the velocity is of the potential kind. Find the velocity potential and the equns of the streamlines.

Soln:

The scale factors for spherical polar co-ordinates are $h_1 = 1$; $h_2 = r$; $h_3 = r\sin\theta$

$$\therefore ds = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\psi\,\hat{e}_\psi$$

$$\vec{q} = 2Mr^{-3}\cos\theta\,\hat{e}_r + Mr^{-3}\sin\theta\,\hat{e}_\theta$$

Curl $\vec{A} = \nabla \times \vec{A}$ in curvilinear coordinate system:

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\therefore \nabla \times \vec{q} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\varphi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ 2Mr^{-3}\cos\theta & Mr^{-2}\sin\theta & 0 \end{vmatrix}$$

$Mr^{-3}\sin\theta$ $\quad$ $r \cdot h_2 A_2$

$$= \frac{1}{r^2\sin\theta}\left[0 - 0 + r\sin\theta\, \hat{e}_\varphi \left(-2Mr^{-3}\sin\theta + 2Mr^{-3}\sin\theta\right)\right.$$

$$= 0$$

i) $\boxed{\therefore \text{Curl}\, \vec{q} = 0}$

Thus the flow is of potential kind.

Let $\varphi(r, \theta, \varphi)$ be the appropriate velocity potential.

ii) Since $\boxed{\vec{q} = -\nabla\phi}$

Then, $\dfrac{-\partial\varphi}{\partial r} = 2Mr^{-3}\cos\theta$

$$\frac{-\partial\varphi}{\cancel{r}\,\partial\theta} = Mr^{-2}\sin\theta \Rightarrow \frac{-\partial\theta}{\partial\theta} = Mr^{-2}\sin\theta$$

$$-\frac{\partial\varphi}{r\sin\theta\,\partial\varphi} = 0$$

$$\partial\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial\psi}d\psi$$

$$= -(2Mr^{-3}\cos\theta)dr - (Mr^{-2}\sin\theta)d\theta + 0.d\psi$$

$$= \underbrace{-2Mr^{-3}\cos\theta}_{M}dr \underbrace{- Mr^{-2}\sin\theta}_{N}d\theta$$

This is of the form

$$\boxed{dU = Mdx + Ndy}$$

where $U = \phi$

$M = -2Mr^{-3}\cos\theta$

$N = -Mr^{-2}\sin\theta$

To check whether it is exact:

diff^r $\frac{\partial M}{\partial\theta} = 2Mr^{-3}\sin\theta$; $\frac{\partial N}{\partial r} = 2Mr^{-3}\sin\theta$

$$\therefore \frac{\partial M}{\partial\theta} = \frac{\partial N}{\partial r}$$

$$\boxed{\phi = \int Mdr + F(\theta)}$$

$$= \int(-2Mr^{-3}\cos\theta)\,dr + F(\theta)$$

$$= -2M\cos\theta\int r^{-3}dr + F(\theta)$$

$$= +\cancel{2}M\cos\theta\,\frac{r^{-2}}{\cancel{2}} + F(\theta)$$

$$\Rightarrow \phi = M r^{-2}\cos\theta + F(\theta)$$

The streamlines are given by,

$$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{r\,d\theta}{Mr^{-\cancel{3}2}\sin\theta} = \underbrace{\frac{r\sin\theta\,d\psi}{0}}$$

Considering last two equs,

$$0 = M r^{-1}\sin^2\theta\,d\psi$$

$$0 = M r^{-\cancel{2}1}\cancel{r}\,\frac{\sin\theta}{\sin\theta}\,d\psi = M r^{-1}\sin^2\theta\,d\psi$$

$$\Rightarrow \boxed{d\psi = 0} \rightarrow (1)$$

Considering first two equs,

$$\frac{dr}{2\cancel{Mr^{-3}}\cos\theta} = \frac{r\cancel{r}\,d\theta}{\cancel{Mr^{-3}}\sin\theta}$$

$$\frac{dr}{2\cos\theta} = \frac{r\,d\theta}{\sin\theta}$$

$$\cancel{\frac{dr}{2\cos\theta} = \frac{r\,d\theta}{\sin\theta}}$$

$$\left(\frac{1}{r}\right)dr = 2\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$\Rightarrow \boxed{\left(\frac{1}{r}\right)dr = 2\cot\theta\,d\theta} \rightarrow (2)$$

Integrating (1) & (2),

$$(1) \Rightarrow \psi = \text{constant}$$

(2) $\Rightarrow \log r = 2 \log \sin\theta + \log A$

$= \log \sin^2\theta + \log A$

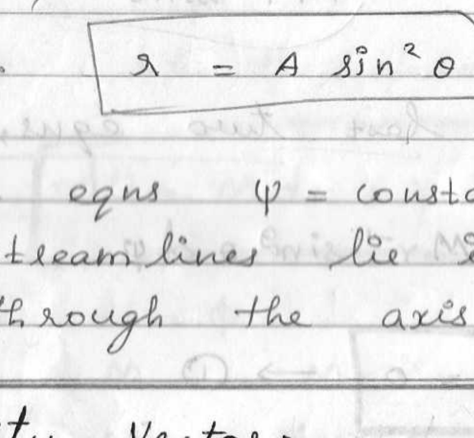
$\cancel{\log}\, r = \cancel{\log}\,(A \sin^2\theta)$

$\therefore \quad \boxed{r = A \sin^2\theta}$

The eqns ψ = constant show that the streamlines lie in planes which pass through the axis of symmetry $\theta = 0$.

Vorticity Vector:

(Consider a fluid flow and choose O to be origin. Let P be any point in the fluid flow moving with velocity $\vec{q}$ at any time t.



Let $P(x, y, z)$ be any point in the fluid referred to an origin $O(0,0,0)$ in the fluid such that the position vector of $\vec{P}$ is $\boxed{\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}}$ and $\boxed{\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}}$ where

$u = u(x, y, z, t)$

$v = v(x, y, z, t)$

$w = w(x, y, z, t)$

Suppose the fluid is such $\nabla \times \vec{q} \neq \vec{0}$ where $\vec{q}$ is the velocity of the fluid particle at P, then we define vorticity vector at P in the flow to be

$$\vec{\zeta} = \nabla \times \vec{q} \quad \text{which is a vector quantity.}$$

In cartesian coordinates system, suppose $\vec{\zeta}$ has components $\zeta_1, \zeta_2, \zeta_3$ in the x-increasing, y-increasing, z-increasing directions then

$$\vec{\zeta} = \zeta_1 \hat{i} + \zeta_2 \hat{j} + \zeta_3 \hat{k}$$

where
$$\zeta_1 = \zeta_1 (x, y, z, t)$$
$$\zeta_2 = \zeta_2 (x, y, z, t)$$
$$\zeta_3 = \zeta_3 (x, y, z, t)$$

The tangent at P in the direction of vorticity vector are parallel such that

$$d\vec{r} \parallel \vec{\zeta} \iff d\vec{r} \times \vec{\zeta} = \vec{0}$$

Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{\zeta} = \zeta_1 \hat{i} + \zeta_2 \hat{j} + \zeta_3 \hat{k}$$

$$\therefore \; d\vec{r} \times \vec{\xi} = \vec{0}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \hat{i}(\xi_3 dy - \xi_2 dz) - \hat{j}(\xi_3 dx - \xi_1 dz) + \hat{k}(\xi_2 dx - \xi_1 dy)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \xi_3 dy - \xi_2 dz = 0 \;,\; \xi_3 dx - \xi_1 dz = 0 \;,\; \xi_2 dx - \xi_1 dy = 0$$

$$\Rightarrow \xi_3 \, dy = \xi_2 dz \;,\; \xi_3 dx = \xi_1 dz \;,\; \xi_2 dx = \xi_1 dy$$

$$\Rightarrow \frac{dy}{\xi_2} = \frac{dz}{\xi_3} \;,\; \frac{dx}{\xi_1} = \frac{dz}{\xi_3} \;,\; \frac{dx}{\xi_2} = \frac{dy}{\xi_1}$$

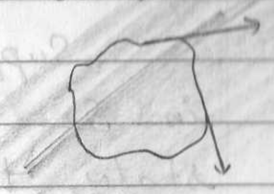
$$\Rightarrow \frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dz}{\xi_3} \longrightarrow (++)$$

Thus (a **vortex line** is a curve drawn in the fluid such that the tangent to it at every point is in the direction of vorticity vector $\vec{\xi}$.)

Eqns to vortex lines is given by (++)

In general, vortex lines does not coincide with stream lines.

Vortex lines ~~do~~ through all pts of a closed curve Γ is said to form a vortex tube.)

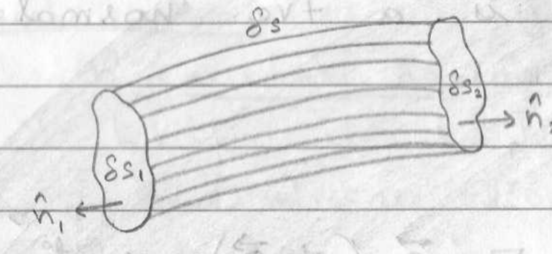


Remark → (Suppose $\nabla \times \vec{q} = \vec{0}$,

then $\vec{\zeta} = \vec{0}$.

ie., in which case we say that the flow is a potential flow.

Thus the necessary & sufficient condition for potential flow may be described by $\vec{\zeta} = \vec{0}$ where $\vec{\zeta} = \nabla \times \vec{q}$



capital S_1 & S_2, S

Consider a vortex tube.

Let δs_1 & δs_2 ~~be~~ be two sections of vortex tubes and let $\hat{n}_1$ & $\hat{n}_2$ be unit normals to these sections drawn outward from the fluid b/w them.

Let δs be the curved surface area of the vortex tube then.

$$\Delta \overset{S}{\cancel{s}} = \delta s_1 + \delta s_2 + \delta s$$

will give the total surface area of an element of vortex tube.

Suppose ΔV be the volume enclosed in δs then by Gauss divergence thm states that

"Suppose V is the volume bounded by the closed surface S and $\vec{A}$ is a vector fn of position with continuous partial derivatives, then

$$\iiint_V \nabla . \vec{A} \, dV = \iint_S \vec{A} . \hat{n} \, ds = \oint\!\!\oint_S \bar{A} . d\bar{s}$$

where $\hat{n}$ is a +ve normal to s."

we have,

$\vec{\zeta} = \nabla \times \vec{q} \; (\neq \vec{O})$ continuous and continuously differentiable inside ΔV and on Δs

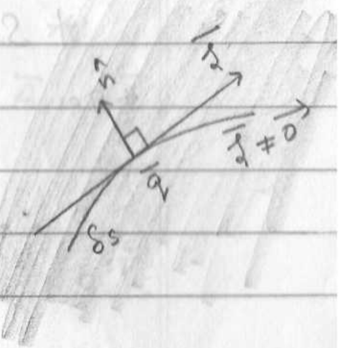
double (or) single

$$\iint_{\Delta s} \vec{\zeta} . \hat{n} \, ds = \int_V (\nabla . \vec{\zeta}) dV = \int_V \nabla . (\nabla \times \vec{q}) \, dV$$

$$= \int_V [\nabla \; \nabla \; \vec{q}] \, dV$$

$$= 0$$

ie., $\int_{\Delta s} \bar{q} \cdot \hat{n} \, ds = 0 \longrightarrow (*)$



ie., $\int_{\delta s_1 + \delta s_2 + \delta s} \bar{q} \cdot \hat{n} \, ds = 0$

ie., $\int_{\delta s_1} \bar{q} \cdot \hat{n}_1 \, ds_1 + \int_{\delta s_2} \bar{q} \cdot \hat{n}_2 \, ds_2 + \int_{\delta s} \bar{q} \cdot \hat{n} \, ds$ (from *) $= 0$

* Any point on the surface $\bar{q} \neq \bar{0}$, $\hat{n}$ is the unit vector to the surface δs.

Since $\bar{q}$ is tangential to the curved surface area,

$\hat{n}$ normal to δs

ie., $\bar{q}$ and $\hat{n}$ are $\perp^r$ to each other

or $\bar{q} \cdot \hat{n} = 0$

$$\Rightarrow \cos\theta = \frac{\bar{q} \cdot \hat{n}}{|\bar{q}|\,|\hat{n}|} = \frac{0}{|\bar{q}|}$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \pi/2$$

Thus to the first order we have integrand vanishes.

* Suppose we take $\hat{n}_2$ to be in +ve direction then $\hat{n}_1$ is -ve.

$$\therefore (\vec{\Omega} \cdot \hat{n}_1)\delta s_1 + (\vec{\Omega} \cdot \hat{n}_2)\delta s_2 = 0$$

$$\Rightarrow |\vec{\Omega} \cdot \hat{n}_1|\delta s_1 = |\vec{\Omega} \cdot \hat{n}_2|\delta s_2$$

And hence for every section of the vortex tube.

In other words, we say for any δs

$$|\vec{\Omega} \cdot \hat{n}|\delta s = \text{constant}$$

For any cross-sectional area, δs part of vortex tube in the fluid, and this constant is called the strength of the vortex tube.

The vortex tube of strength unity ie., $|\vec{\Omega} \cdot \hat{n}|\delta s = 1$ then it is called unit vortex tube.

If S is any closed surface enclosing the volume V then by Gauss divergence theorem which states that

"Suppose V is the volume bounded by the closed surface S and $\vec{A}$ is the vector fn of position with continuous derivatives then

$$\iiint\limits_V \nabla\cdot\bar{A}\, dV = \iint\limits_S \vec{A}\cdot\hat{n}\, ds = \oiint\limits_S \bar{A}\cdot d\bar{s}$$

where $\hat{n}$ is +ve normal to S".

We have

$$\int\limits_S \vec{\zeta}\cdot\hat{n}\, ds = \int\limits_V \nabla\cdot\vec{\zeta}\, dV = \int\limits_V \nabla\cdot(\nabla\times\vec{q})\, dV$$

$$= \int\limits_V [\nabla\ \nabla\ \vec{q}\,]\, dV$$

$$= 0$$

$$\Rightarrow |\vec{\zeta}\cdot\hat{n}|\,\delta s = 0$$

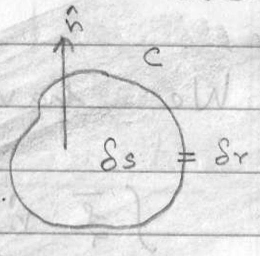
ie., the strength of vortex tubes emerging from S is equal to that entering S. This means that "vortex lines

8m and vortex tubes cannot originate or

P.T terminate at internal points in the fluid." They can only ~~from~~ form closed curves or terminate on the boundaries.

In case of smoke rings, vortex lines ~~in~~ are closed curves. Also vortex lines in a whirlpool terminate on the boundary of the fluid.

(Suppose C is a closed curve drawn in a fluid and if S denotes the surface area contained by C then the circulation of the fluid velocity $\vec{q}$ denoted Γ is defined to be circulation over C.



ie., $\Gamma = \oint_C \vec{q} \cdot d\vec{r}$

$$= \oint (\vec{q} \cdot \hat{n})\, dr$$

By stoke's theorem which states that

"Suppose S is an open, two sided surface bounded by a closed, non-intersecting curve C.

Suppose $\bar{A}$ is a vector fn of position with continuous derivatives, then

$$\oint_C \bar{A} \cdot d\bar{r} = \iint_S (\nabla \times \bar{A}) \cdot \hat{n}\, ds = \iint_S (\nabla \times \bar{A})\, d\bar{s}"$$

$$\therefore \Gamma = \oint_C (\vec{q} \cdot \hat{n})\, dr = \int_S \hat{n} \cdot (\nabla \times \vec{q})\, ds$$

$$= \int_S (\hat{n} \cdot \vec{J})\, ds$$

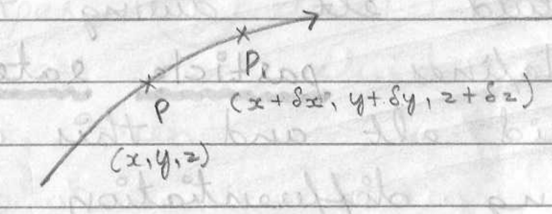
Suppose the flow is a potential flow,

$$\nabla \times \vec{q} = \vec{0}$$

$$\Rightarrow \oint_c (\vec{q} \cdot \hat{n})\, ds = 0$$

ie., If the flow is of potential type kind, then the circulation around any closed circuit is zero.)

1.3) Local and Particle rate of change:



The study of fluid motion involves analysis of variations in the flow variables involved such as the density ρ, pressure p, velocity $\vec{q}$, ..etc. w.r.t four independent variables x, y, z & t.

Hence we have two types of variations given as follows:

1) We may keep the fluid locally fixed and study the variation in the entities at that point with passage of time.

For this purpose, we consider the partial derivatives,

$$\frac{\partial \rho}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial \vec{q}}{\partial t}, \text{etc.. at that point.}$$

This represents the local rate of change of these flow variables.

2) We can follow a particular or same fluid element during a time Δt in the flow.

Observing variations in $\rho, p, \vec{q}$ of the fluid elt during this interval Δt this defines particle rate of change of fluid elt and this can be done by doing differentiation following the fluid flow.

To prove:

The relationship b/w local and particle rate of change:

$$\text{ie.,} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q} \cdot \nabla \quad -(*)$$

Consider a fluid $P(x, y, z)$ in the fluid at time t, the fluid elt would have moved to $P'(x+\delta x, y+\delta y, z+\delta z)$ in time $t+\delta t$.

i.e., $P(x', y', z')$ where $x' = x + \delta x$
$y' = y + \delta y$
$z' = z + \delta z$

The individual particle rate of change in a scalar point fn, $\rho = \rho(x, y, z, t)$ during the interval is represented by the total derivative $\frac{d\rho}{dt}$.

The scalar pt fn ρ may define some property of the flow (say For eg: density)

Suppose $\vec{q}$ denotes the velocity vector of the fluid particle P at time t, then

$$\vec{q} = \vec{q}(x, y, z)$$
$$= u\hat{i} + v\hat{j} + w\hat{k} \rightarrow (1)$$

where $u = u(x, y, z)$
$v = v(x, y, z)$
$w = w(x, y, z)$

Also w.r.t some origin in the fluid $\vec{OP}$ is the position vector of P given by

$$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{r} \text{ (say)}$$

Then the velocity vector

$$\vec{q} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \longrightarrow ②$$

such that

$$u = \frac{dx}{dt}, \; v = \frac{dy}{dt}, \; w = \frac{dz}{dt} \longrightarrow ③ \text{ (from ① \& ②)}$$

$\vec{q}$ represents velocity of fluid particle at P is continuous and single valued.

During motion of the fluid particle from P to P′, the total change corresponding to the property under consideration of the fluid elt ρ is given by $d\rho$.

Since $\rho = \rho(x, y, z, t)$,

$$d\rho = \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz + \frac{\partial \rho}{\partial t}dt$$

$$\therefore \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}u + \frac{\partial \rho}{\partial y}v + \frac{\partial \rho}{\partial z}w \quad \text{(from ③)}$$

$$= \frac{\partial \rho}{\partial t} + \underbrace{(u\hat{i} + v\hat{j} + w\hat{k})}_{\vec{q}} \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\rho$$

$$= \frac{\partial \rho}{\partial t} + (\vec{q} \cdot \nabla)\rho$$

ie., $\frac{d}{dt}\rho = \left(\frac{\partial}{\partial t} + \vec{q}\cdot\nabla\right)\rho$

$\because$ ρ is defining a property of fluid in motion continuous and continuously differentiable,

no need intro for 3m

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q}\cdot\nabla \quad \longrightarrow (*)$$

Thus relation b/w local and particle rate of change.

(This gives the **particle rate of change** of the flow variable.

The partial time rate of change components in (*) (1st term on RHS of (*)) gives the **local rate of change** and $\frac{d}{dt}$ is the total derivative and $\frac{\partial}{\partial t}$ is the local derivative.

(*) holds good if the associated property is a ~~sa~~ scalar point fn or vector point function.)

$\longrightarrow$ For this $f(x,y,z,t)$ (eg. density) is a scalar point fn, then (*) becomes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\vec{q} \cdot \nabla) f .$$

For eg, f is <u>density</u> ρ or <u>pressure</u> p.

$\rightarrow$ (eg. its velocity)

$\longrightarrow$ Suppose $\vec{F} = \vec{F}(x, y, z, t)$ is a <u>vector point fn</u>, then (*) becomes

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + (\vec{q} \cdot \nabla) \vec{F}$$

Say for eg, $\vec{F} = \vec{q}$ then $\frac{d\vec{q}}{dt} = \frac{\partial q}{\partial t} + (\vec{q} \cdot \nabla) \vec{q}$

For if $\vec{q} = |\vec{q}| \, d\hat{r}$ $\qquad \overline{PP'} = \delta r$, $PP' = \delta r$

$= q \, d\hat{r}$ where $\overline{PP'} = \delta r \, \hat{r}$

And so $\vec{q} \cdot \nabla f = q \, d\hat{r} \cdot \nabla f$

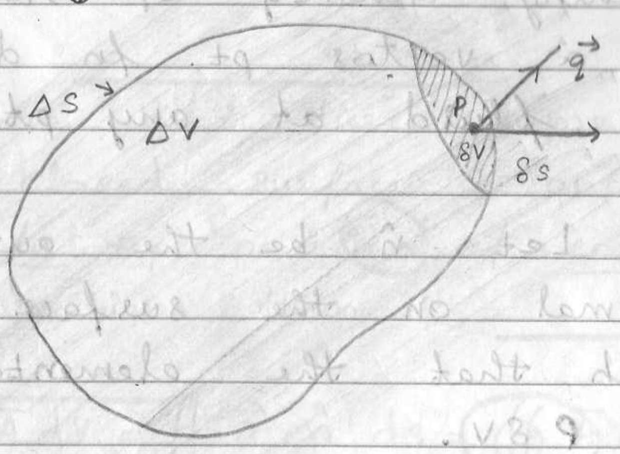
$= q \frac{\partial f}{\partial r}$ (using the fact, $\delta \hat{r} \cdot \nabla = \frac{\partial}{\partial r}$)

1.4) Equation of Continuity :

(a) 20m

The region under consideration to study the continuity of the fluid flow contains neither <u>source</u> nor <u>sink</u>. so that in the region, there are no <u>inlets</u> (or) <u>outlets</u> through which fluid can enter (or) leave the region.

~~The~~



The amount of fluid within the region is conserved in accordance with principle of conservation of matter. The law of conservation of mass expressed in analytic way ~~that~~ where mass of the fixed enclosed in any volume is conserved as the volume moves enclosing the same fluid particles continuously.

Mathematical Formulation:

Consider a small volume ΔV enclosed by the closed surface ΔS.

Consider an elementary region δs much smaller than ΔS (ie., $\delta s << \Delta S$) enclosing a volume δV ~~must~~ much smaller than ΔV (ie., $\delta V << \Delta V$).

Let ρ be a scalar point fn defining density of the fluid and $\vec{q}$ a vector pt fn defining velocity of fluid at any pt P.

Let $\hat{n}$ be the outward drawn normal on the surface δs at P such that the elementary mass will be $\rho \, \delta V$.

→ ∴ Total mass of fluid elt under consideration

$$= \int_{\Delta V} \rho \, dV$$

= a constant (Law of conservation of mass)

ie., $\int_{\Delta V} \rho \, dV$ = a constant — (1)

Eqn (1) is known as eqn of continuity.

→ Amount of fluid entering the surface area δs per unit time

$$= \rho (\vec{q} \cdot \hat{n}) \, \delta s$$

→ (a) ∴ The total rate of mass flow out of ΔV across Δs

$$= \int_{\Delta s} \hat{n} \cdot (\rho \vec{q}) \, ds$$

By Gauss divergence theorem, which states

"Suppose V is the volume bounded by the closed surface S and $\vec{A}$ be a vector p fn of position of continuous partial derivatives then

$$\iiint_V \nabla\cdot\bar{A}\,dV = \iint_S \vec{A}\cdot\hat{n}\,ds = \oiint_S \bar{A}\cdot d\bar{s}$$

where $\hat{n}$ is a +ve normal to S"

We have,

$\rho\vec{q}$ continuous and continuously differentiable in ΔV and on ΔS as the fluid flow is continuous. constant

$$\text{And} \int_{\Delta S}\left((\rho\vec{q})\cdot\hat{n}\right)ds = \int_{\Delta V}\nabla\cdot\rho\vec{q}\,dV \rightarrow (2)$$

ie., Total rate of flow out of ΔV, across ΔS is

$$\Delta V \text{ across } \Delta S = \int_{\Delta S}\left((\rho\vec{q})\cdot\hat{n}\right)ds$$

$$= \int_{\Delta V}\nabla\cdot\rho\vec{q}\,dV \rightarrow (2)$$

→ (b) At any time t the mass of the fluid within the elt of volume ΔV is

$$\int_{\Delta V} \rho \, dV \rightarrow (3)$$

The local rate of mass increase within ΔV

$$= \frac{\partial}{\partial t} \int_{\Delta V} \rho \, dV \quad (\text{from } (3))$$

$$= \int_{\Delta V} \frac{\partial}{\partial t} (\rho \, dV)$$

$$= \int_{\Delta V} \frac{\partial \rho}{\partial t} dV + \int_{\Delta V} \rho \underbrace{\frac{\partial}{\partial t} (dV)}_{0}$$

$$= \int_{\Delta V} \frac{\partial \rho}{\partial t} dV \longrightarrow (4)$$

there is no change in vol. with time.

In the absence of sources or sinks within ΔV matter cannot be created or destroyed, in this region.

From (2),

Total rate of mass flowing into ΔV

$$= - \int_{\Delta V} \nabla \cdot \rho \vec{q} \, dV \rightarrow (2A)$$

Since the local rate of mass increase within ΔV is equal to the amount of fluid entering the region ΔS, we have, from (4) & (2A),

$$\int_{\Delta V} \frac{\partial \rho}{\partial t} \, dV = - \int_{\Delta V} \nabla \rho \vec{q} \, dV$$

$$\text{ie.,} \int_{\Delta V} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} \right) dV = 0 \rightarrow (5)$$

Since ΔV is an arbitrary volume element in a fluid flowing for (5) to hold the integrand should vanish,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0} \rightarrow (f)$$

(f) gives the most general form of eqn of continuity at any pt of a fluid flow which is free from sources and sinks.

Corollary (1)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0 \rightarrow (f)$$

(f) can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \vec{q} + \rho (\nabla \cdot \vec{q}) = 0 \longrightarrow (fA)$$

Using the differential operator,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{q}\cdot\nabla \quad \text{operating on } \rho$$

we have

$$\frac{d\rho}{dt} = \left[\frac{\partial}{\partial t} + (\nabla\cdot\vec{q})\right]\rho$$

ie) $$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \rho(\nabla\cdot\vec{q}) \Rightarrow \frac{\partial\rho}{\partial t} = \frac{d\rho}{dt} - \rho(\nabla\cdot\vec{q})$$

ie) $$\frac{\partial\rho}{\partial t} + \nabla\rho\cdot\vec{q} + \rho(\nabla\cdot\vec{q}) = 0 \text{ (ffA) becomes}$$

$$\frac{d\rho}{dt} + \nabla\rho\cdot\vec{q} + \cancel{\rho(\nabla\cdot\vec{q})} - \cancel{\rho(\nabla\cdot\vec{q})}$$

$$\boxed{\frac{d\rho}{dt} + \rho(\nabla\cdot\vec{q}) = 0} \longrightarrow (ff)$$

(ff) is another form of eqn of continuity.

ρ is the density of the fluid $\neq 0$, therefore (ff) can be written as

$$\boxed{\frac{1}{\rho}\frac{d\rho}{dt} + \nabla\cdot\vec{q} = 0 \Leftrightarrow \frac{d}{dt}(\log\rho) + \nabla\cdot\vec{q} = 0} \rightarrow (ffA)$$

This is also another form of equation of continuity.

Corollary (2)

$\longrightarrow$ If the flow is steady, ie, the pattern of flow does not change with time and hence at any region of the locality of flow all the local derivatives or partial derivatives becomes zero.

ie, $\dfrac{\partial \rho}{\partial t} = 0$

Proof

$$\underbrace{\frac{\partial \rho}{\partial t}}_{0} + \nabla \cdot \rho \vec{q} = 0 \rightarrow (f)$$

(f) becomes $\nabla \cdot \rho \vec{q} = 0$

$\rho \neq 0 \Rightarrow \nabla \cdot \vec{q} = 0$ is a eqn of continuity.

$\longrightarrow$ If the fluid is incompressible, ρ is a constant through out the fluid at any instant and hence the total variation of ρ w.r.t time t is zero.

ie) $\dfrac{d\rho}{dt} = 0$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \vec{q}) = 0 \rightarrow (ff)$$

Hence from (ff),

$$\rho(\nabla \cdot \vec{q}) = 0$$

$$\rho \neq 0 \Rightarrow \nabla \cdot \vec{q} = 0 \rightarrow (fff)$$

→ Further if the flow is irrotational, then we have

$(\nabla \times \vec{q}) = 0$ which yields a scalar point fn ϕ : $\vec{q} = -\nabla \Phi$

∴ (Eqn of continuity of an incompressible fluid with an irrotational flow) is given by,

$$0 = \nabla \cdot \vec{q} = \nabla \cdot (-\nabla \Phi) = -\nabla^2 \Phi$$

$$\Rightarrow \nabla^2 \Phi = 0$$

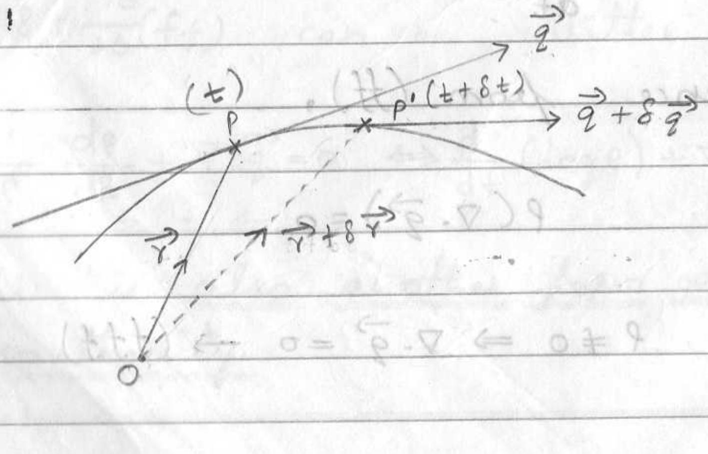
Φ is called the "velocity potential".

ie) velocity potential satisfies "Laplace equation."

Qn

1.5) Acceleration of a fluid element at any time 't':

6m



Let P be the position of fluid elt at any time t in a flowing fluid moving with velocity $\vec{q}$.

Let P' be the position of a neighbouring point to P at time $(t+\delta t)$ moving with fluid velocity $\boxed{(\vec{q}+\delta\vec{q})}$

$\therefore$ Variation in velocity at P is

$$\vec{q}+\delta\vec{q}-\vec{q}=\delta\vec{q}$$

Let O be an origin fixed in the fluid such that position vector $\vec{OP}=\vec{r}$ and velocity vector $\vec{q}=\frac{d\vec{r}}{dt}$ at P at time t.

Since $\vec{OP'}=\vec{r}+\delta\vec{r}$ at $t'=t+\delta t$,

$$\text{velocity}=\frac{\vec{r}+\delta\vec{r}-\vec{r}}{(t+\delta t)-t}=\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}=\frac{d\vec{r}}{dt}\longrightarrow (1)$$

This will exist because the fluid flow is continuous.

Average rate of change of velocity at P will be variation of velocity at P w.r.t time t

$$=\frac{\delta\vec{q}}{\delta t}$$

Acceleration of the fluid particle at P at time t.

$$= \lim_{\delta t \to 0} \frac{\delta \vec{q}}{\delta t} = \frac{d\vec{q}}{dt} = \frac{d}{dt}(\vec{q})$$

existence guaranteed, since the flow is continuous.

Denoted $\vec{f}$,

$\vec{f} = \vec{f}(x, y, z, t)$ is a vector pt fn and is a measure of acceleration of fluid elt at P at time t.

From (1),

Suppose the orthogonal curvilinear system under consideration is cartesian, then

$$\overline{OP} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\bar{q} = u\hat{i} + v\hat{j} + w\hat{k} \text{ (say)}$$

$$\therefore \bar{f} = \frac{d\bar{q}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k}$$

$\dfrac{du}{dt}$, $\dfrac{dv}{dt}$ & $\dfrac{dw}{dt}$ are called <u>acceleration components</u> of $\bar{f}$ in x-increasing, y-increasing & z-increasing directions, respectively along ox, oy & oz axis.

→ From the <u>particle rate of change operator</u>,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{q}\cdot\nabla) \quad \text{operating on } \bar{q},$$

we have

Tensor form

$$\bar{f} = \frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} \longrightarrow ②$$

$$\bar{q}\cdot\nabla = (u\hat{i}+v\hat{j}+w\hat{k})\cdot\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)$$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$(\bar{q}\cdot\nabla)\bar{q} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u\hat{i}+v\hat{j}+w\hat{k})$$

$$\left.\begin{aligned} &= \left(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u\right)\hat{i} + \\ &\quad \left(u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v\right)\hat{j} + \\ &\quad \left(u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w\right)\hat{k}. \end{aligned}\right\} \longrightarrow ③$$

and $\dfrac{\partial \bar{q}}{\partial t} = \dfrac{\partial}{\partial t}\left[u\hat{i} + v\hat{j} + w\hat{k}\right]$

$$= \frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k} \rightarrow (4)$$

and $\dfrac{d\bar{q}}{dt} = \dfrac{d}{dt}(u\hat{i} + v\hat{j} + w\hat{k})$

$$= \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} \longrightarrow (5)$$

Substituting (3), (4) and (5) in (2) & comparing $\hat{i}, \hat{j}, \hat{k}$ components, we have

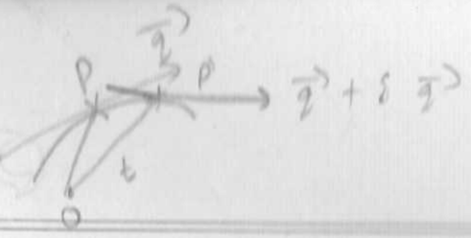
$$\left.\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\
\frac{dw}{dt} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}
\end{aligned}\right\} \rightarrow (\mathrm{I})$$

Vectorially, we represent the 3-coordinate axes as

$$x_i : (x_1, x_2, x_3) \text{ and } u_i : (u, v, w)$$

Thus the tensor representation of (I) is

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j u_{i,j} \longrightarrow (\mathrm{IA})$$



Acceleration in vector form:

Imp

Consider $\dfrac{d\bar{q}}{dt} = \dfrac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} \longrightarrow (2)$

We shall make use of two vector identities,

$$\boxed{\begin{aligned} &\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C} \\ &\Rightarrow (\bar{A}\cdot\bar{B})\bar{C} = (\bar{A}\cdot\bar{C})\bar{B} - \bar{A}\times(\bar{B}\times\bar{C}) \longrightarrow (*) \end{aligned}}$$

From (2),

$$(\bar{q}\cdot\nabla)\bar{q} = \bar{q}\left[\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\bar{q}\right]$$

$$= (\bar{q}\cdot\hat{i})\frac{\partial \bar{q}}{\partial x} + (\bar{q}\cdot\hat{j})\frac{\partial \bar{q}}{\partial y} + (\bar{q}\cdot\hat{k})\frac{\partial \bar{q}}{\partial z} \longrightarrow (6)$$

Consider,

$A = \bar{q}$; $B = \hat{i}$; $C = \dfrac{\partial \bar{q}}{\partial x}$

$$(\bar{q}\cdot\hat{i})\frac{\partial}{\partial x}\bar{q} = \left[\bar{q}\cdot\frac{\partial \bar{q}}{\partial x}\right]\hat{i} - \bar{q}\times\left(\hat{i}\times\frac{\partial \bar{q}}{\partial x}\right)$$

$$(\bar{q}\cdot\hat{j})\frac{\partial}{\partial y}\bar{q} = \left[\bar{q}\cdot\frac{\partial \bar{q}}{\partial y}\right]\hat{j} - \bar{q}\times\left(\hat{j}\times\frac{\partial \bar{q}}{\partial y}\right)$$

$$(\bar{q}\cdot\hat{k})\frac{\partial}{\partial z}\bar{q} = \left[\bar{q}\cdot\frac{\partial \bar{q}}{\partial z}\right]\hat{k} - \bar{q}\times\left(\hat{k}\times\frac{\partial \bar{q}}{\partial z}\right)$$

Sub. back in ⑥, we have

$$(\bar{q}\cdot\nabla)\bar{q} = (\bar{q}\cdot\hat{i})\frac{\partial}{\partial x}\bar{q} + (\bar{q}\cdot\hat{j})\frac{\partial\bar{q}}{\partial y} + (\bar{q}\cdot\hat{k})\frac{\partial}{\partial z}\bar{q}$$

$$= \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial x}\right)\hat{i} + \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial y}\right)\hat{j} + \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial z}\right)\hat{k}$$

$$- \left[\left(\bar{q}\times\left(\hat{i}\times\frac{\partial\bar{q}}{\partial x}\right)\right) + \left(\bar{q}\times\left(\hat{j}\times\frac{\partial\bar{q}}{\partial y}\right)\right) + \left(\bar{q}\times\left(\hat{k}\times\frac{\partial\bar{q}}{\partial z}\right)\right)\right] \quad \text{⑦}$$

We know,

$$\bar{q}^2 = \bar{q}\cdot\bar{q}$$

$$\frac{\partial}{\partial x}(\bar{q}^2) = \bar{q}\cdot\frac{\partial\bar{q}}{\partial x} + \frac{\partial\bar{q}}{\partial x}\cdot\bar{q} = 2\,\bar{q}\cdot\frac{\partial\bar{q}}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{2}\bar{q}^2\right) = \bar{q}\cdot\frac{\partial\bar{q}}{\partial x}$$

|||ly,

$$\frac{\partial}{\partial y}\left(\frac{1}{2}\bar{q}^2\right) = \bar{q}\cdot\frac{\partial\bar{q}}{\partial y}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{2}\bar{q}^2\right) = \bar{q}\cdot\frac{\partial\bar{q}}{\partial z}$$

$$\therefore \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial x}\right)\hat{i} + \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial y}\right)\hat{j} + \left(\bar{q}\cdot\frac{\partial\bar{q}}{\partial z}\right)\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\bar{q}^2\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{2}\bar{q}^2\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{2}\bar{q}^2\right)\hat{k}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{2}\bar{q}^2\right).$$

$$= \nabla\left(\frac{1}{2}\bar{q}^2\right) \longrightarrow (8)$$

Sub. (7) & (8) in (2), we have

$$\frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q} \times (\nabla \times \bar{q})$$

[Since,

$$\nabla \times \bar{q} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \bar{q}$$

$$= \left(\hat{i} \times \frac{\partial}{\partial x}\bar{q}\right) + \left(\hat{j} \times \frac{\partial}{\partial y}\bar{q}\right) + \left(\hat{k} \times \frac{\partial}{\partial z}\bar{q}\right)$$

$$\therefore \left(\bar{q} \times \left(\hat{i} \times \frac{\partial \bar{q}}{\partial x}\right)\right) + \left(\bar{q} \times \left(\hat{j} \times \frac{\partial \bar{q}}{\partial y}\right)\right) + \left(\bar{q} \times \left(\hat{k} \times \frac{\partial \bar{q}}{\partial z}\right)\right)$$

$$= \bar{q} \times \left[\left(\hat{i} \times \frac{\partial \bar{q}}{\partial x}\right) + \left(\hat{j} \times \frac{\partial \bar{q}}{\partial y}\right) + \left(\hat{k} \times \frac{\partial \bar{q}}{\partial z}\right)\right]$$

$$= \bar{q} \times (\nabla \times \bar{q}) \;]$$

$\times \xRightarrow{\text{denote}} \wedge$

Qn $\quad \therefore \bar{f} = \dfrac{d\bar{q}}{dt} = \dfrac{\partial \bar{q}}{\partial t} + \nabla\left(\dfrac{1}{2}\bar{q}^2\right) - \bar{q} \times (\nabla \times \bar{q}) \longrightarrow$ (IB)

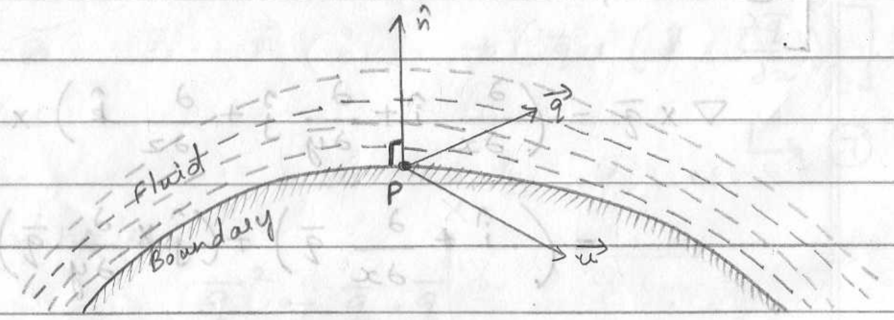
is <u>another</u> <u>vector</u> <u>form</u> <u>of</u> <u>acceleration</u>.

Suppose the flow is <u>potential</u>, then

$$\nabla \times \bar{q} = 0 \quad \text{yielding} \quad \bar{f} = \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right)$$

And $\boxed{\bar{q} = -\nabla \phi,}$ where ϕ is the scalar potential defining the potential flow (IB) is useful than (2) to work any general orthogonal curvilinear co-ordinate system.

2m Conditions at a Rigid Boundary:



→ 2m (In a fluid flowing, the line indicates rigid boundary and let (P) be an arbitrary point on the boundary. Then the fluid velocity is $\vec{q}$ and as the fluid flows the boundary moves with the $\vec{u}$.

If $\hat{n}$ represents the unit normal direction at P, then since there is no relative normal velocity at P b/w the boundary and the fluid,

we must have the two normal components to be equal,

$$\boxed{\vec{q}\cdot\hat{n} = \vec{u}\cdot\hat{n}} \longrightarrow (*) \quad \left[\because \vec{q}\cdot\hat{n} + \vec{u}\cdot(-\hat{n}) = 0\right]$$

In the case where fluid is of **invisid type**. (*) is the only condition.

For **viscous fluids**, there is no slip and the tangential components are also equal.

→ Suppose the boundary was at rest ($\vec{u}=0$),

$$\boxed{\vec{q}\cdot\hat{n}=0} \rightarrow (**)$$

ie., Normal flux is zero at every pt on the boundary.

So in a nutshell, fluid flow pass the rigid boundary as for its boundary conditions, **(*)** holding if boundary is in **motion** and **(**)** holds if boundary is in **rest**.