Unit - 1

Velocity, Relative Velocity & Angular Velocity

Fundamental quantities in mechanics are length, time and mass.

To measure these quantities there are 3 systems of units using - MKS, CGS, FPS

$\underbrace{\text{MKS, CGS}}_{\text{Metric System (French system)}}$, $\underbrace{\text{FPS}}_{\text{British System}}$

Definitions

Mass:

The material (or) the matter out of which the body is made, it is known as its mass.

Unit of mass:

*It is in MKS system is 1 kg and it is the mass of a piece of metal arbitarily chosen and preserved in paris.

*It is in FPS system is 1 pound & it is the mass of a piece of a metal preserved in london.

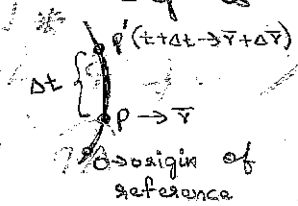
Displacement:

When a particle moves from a pt to another pt the particle is said to undergo a displacement. i.e, It is just a change of position. When a particle moves from a pt P to a pt P', the respective displacement denoted by the vector $\overline{PP'}$

Velocity:

The rate of change of position is called velocity of a particle. Velocity is denoted by $\overline{V}$.

$$\overline{V} = \frac{d\overline{r}}{dt}$$



Let P be the position of the moving particle at any instant with respect to 'O', the origin of reference, $\overline{OP} = \overline{r}$ (position vector of P.

Let P' be the position of the particle at the instant $(t+\Delta t)$ with the position vector $(\overline{r}+\Delta\overline{r})$.

Displacement at Δt time,

$$\overline{PP'} = \overline{OP'} - \overline{OP}$$
$$= (\overline{r}+\Delta\overline{r}) - \overline{r}$$
$$= \Delta\overline{r}$$

The time, rate of displacement of particle,

$$\lim_{\Delta t \to 0} \frac{\overline{PP'}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} = \frac{d\bar{r}}{dt} = \dot{\bar{r}} \text{ (or) } = \bar{V}$$

$\therefore$ The velocity of the particle is gn by

$$\bar{v} = \frac{d\bar{r}}{dt}$$

Note :

1) $\bar{r} + \Delta\bar{r}$ can also be denoted by $\bar{r}(t+\Delta t)$ and $\bar{r}$ by $\bar{r}(t)$

2) Let use dot to represent differentiation with respect to time. So, $\bar{v} = \dot{\bar{r}}$

Direction of Velocity:

* Along the tangent at P to the path described by the particle.

* Measure of the arcual distance from a fixed pt A.

* Let the arcual distance of P denoted by

$\widehat{AP}$ be s & $\widehat{AP'}$ be $s+\Delta s$

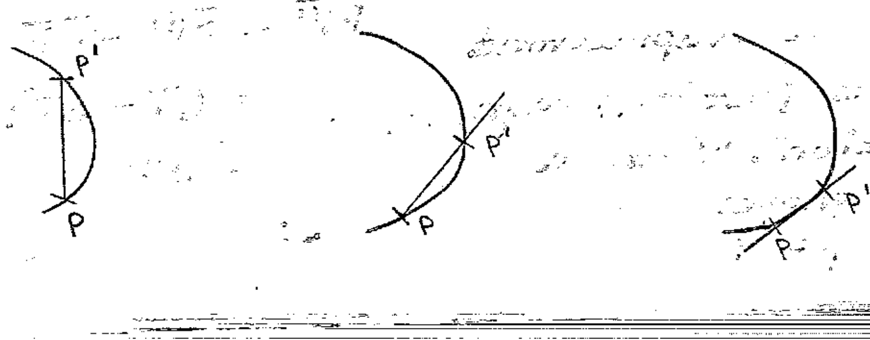
$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt} \rightarrow ①$$

Consider,

$$\frac{d\bar{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\bar{r}}{\Delta s} = \lim_{\Delta s \to 0} \frac{\overline{PP'}}{\Delta s}$$

$$\frac{d\bar{r}}{ds} = \lim_{\Delta s \to 0} \frac{PP'}{\Delta s} \widehat{PP'} \rightarrow ②$$

* When P and P' are very close to each other the chordial distance PP' and the arcual distance PP' are almost the same.



So, $\lim_{\Delta s \to 0} \frac{PP'}{\Delta s} = 1 \rightarrow (3)$

When $\Delta t \to 0$, $\Delta s \to 0$, the pt P' goes closer and closer to P and in the limiting position the chord PP' becomes the tangent at P

i.e) $\lim_{\Delta s \to 0} \hat{PP'} = \hat{T} \rightarrow (4)$

$\hat{T}$ = The unit tangent vector at P.

~~eqn. (2) becomes~~ $\bar{v} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt} =$

$(2) \Rightarrow \frac{d\bar{r}}{ds} = 1 \cdot \hat{T}$

$(1) \Rightarrow \bar{v} = \hat{T} \cdot \frac{ds}{dt} = \dot{s} \cdot \hat{T}$

* The mag. of the velocity is $\dot{s}$ and dir. of the velocity is along to tgt direction.
* The mag. of the velocity is called as speed. It is denoted by $\dot{s}$.

Unit of velocity:

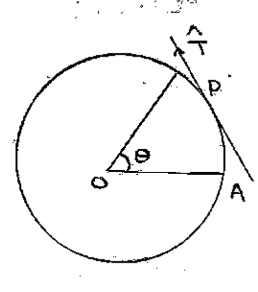
* It is easily understood as length/time.
* It is in S system.

MKS system : 1 m/sec
CGS system : 1 cm/sec
FPS system : 1 foot/sec

5M

Velocity of a particle describing a circle:

P → particle describing a circle
O → centre of the circle
a → radius of the circle
A → fixed pt on the circle
& also P → position of moving particle at time t.



$A\hat{O}P = \theta$

$s = \overset{\frown}{AP} = a\theta$ = radius x angle subtended

$\Rightarrow \dot{s} = \underbrace{a\dot{\theta}}_{\text{Mag. (or) Speed}}$

$\bar{v} = \dot{s}\hat{T} = a\dot{\theta}\hat{T}$ = velocity of the particle when the particle is a circle

$\dot{\theta} \rightarrow$ angular velocity

Resultant velocity:-

If a particle has two velocities $\overline{V_1}$ & $\overline{V_2}$ then $\overline{V_1}+\overline{V_2}$ is said to be the resultant velocity of the particle.

Book work: (1)

To find the mag. and dir. of the resultant of two velocities $\overline{V_1}$ & $\overline{V_2}$.

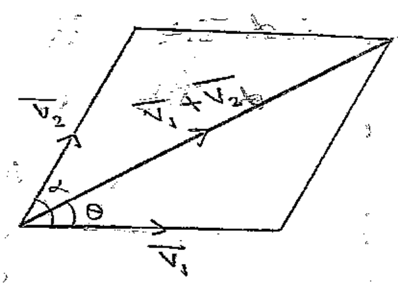
* Let the given two velocities inclined at an angle α. ($\alpha \to$ angle b/w $\overline{V_1}$ & $\overline{V_2}$)
* Let the resultant velocity make an angle θ the velocity first velocity $\overline{V_1}$. ($\theta \to$ angle b/w $\overline{V}$ & $\overline{V_1}$)
* Denote the resultant velocity by

$\overline{V} = \overline{V_1}+\overline{V_2}$

Magnitude of resultant velocity:

$$\overline{V} = \overline{V_1}+\overline{V_2}$$

$$|\overline{V}|^2 = |\overline{V_1}+\overline{V_2}|^2 = V^2$$



$$\overline{V}\cdot\overline{V} = (\overline{V_1}+\overline{V_2})\cdot(\overline{V_1}+\overline{V_2})$$

$$= \overline{V_1}\cdot\overline{V_1}+\overline{V_1}\cdot\overline{V_2}+\overline{V_2}\cdot\overline{V_1}+\overline{V_2}\cdot\overline{V_2}$$

$$= V_1^2+2\overline{V_1}\cdot\overline{V_2}+V_2^2$$

$$V^2 = V_1^2+V_2^2+2V_1V_2\cos\alpha$$

Direction of resultant velocity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{|\overline{V_1}\times\overline{V}|}{\overline{V_1}\cdot\overline{V}} = \frac{|\overline{V_1}\times(\overline{V_1}+\overline{V_2})|}{\overline{V_1}\cdot(\overline{V_1}+\overline{V_2})}$$

$(\overline{V_1}\times\overline{V_1})+(\overline{V_1}\times\overline{V_2})$

$\overline{0}$, $\overline{0} = \overline{0}$, $\overline{0}\times\overline{0}$

$|\overline{a}\times\overline{b}| = ab\sin\theta$

$\sin\theta = \frac{|\overline{a}\times\overline{b}|}{ab}$

$\overline{a}\cdot\overline{b} = ab\cos\theta$

$\cos\theta = \frac{\overline{a}\cdot\overline{b}}{ab}$

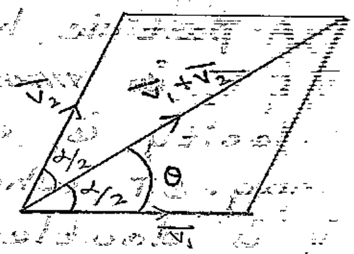
$$\tan\theta = \frac{\cancel{V_1}V_2\sin\alpha}{\cancel{V_1}(V_1+V_2\cos\alpha)}$$

$$\tan\theta = \frac{V_2\sin\alpha}{V_1+V_2\cos\alpha}$$

Case (i):

If $\overline{V_1}$ & $\overline{V_2}$ are of equal mag. to find the resultant (i.e) $|\overline{V_1}| = |\overline{V_2}|$

$\Rightarrow \boxed{V_1 = V_2}$



$$V^2 = V_1^2 + V_1^2 + 2V_1^2 \cos\alpha$$
$$= 2V_1^2 + 2V_1^2 \cos\alpha$$
$$= 2V_1^2 (1 + \cos\alpha)$$
$$= 2V_1^2 (2\cos^2 \alpha/2)$$
$$V^2 = 4V_1^2 \cos^2 \alpha/2$$
$$V = 2V_1 \cos \alpha/2$$

$\sin 2A = 2 \sin A \cos A$

$\sin A = 2 \sin A/2 \cos A/2$

$$\tan\theta = \frac{V_1 \sin\alpha}{V_1 + V_1 \cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{\cancel{2}\sin \alpha/2 \cancel{\cos \alpha/2}}{\cancel{2}\cos^{\cancel{2}} \alpha/2}$$

$$\tan\theta = \frac{\sin \alpha/2}{\cos \alpha/2} = \tan \alpha/2$$

$$\boxed{\theta = \alpha/2}$$

i.e) The resultant bisects the angle b/w velocities if the magnitudes of the velocities are equal.

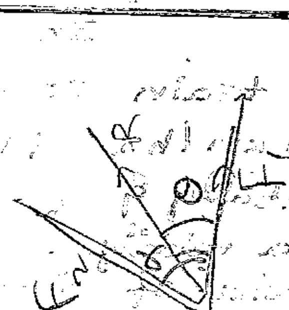
Case ii) Let the two velocities be at right angles. Take the direction as $\hat{i}$ and $\hat{j}$ axes.

$\bar{V}_1 = V_1 \hat{i}$, $\bar{V}_2 = V_2 \hat{j}$

$$V = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos 90^\circ}$$
$$= \sqrt{V_1^2 + V_2^2}$$

$$\tan\theta = \frac{V_2 \sin\alpha}{V_1 + V_2\cos\alpha} = \frac{V_2 \sin 90^\circ}{V_1 + V_2 \cos 90^\circ} = \frac{V_2}{V_1}$$

$$\tan\theta = \frac{j^{th} \text{ component}}{i^{th} \text{ component}}$$



1) A particle has two velocities $\bar{V}_1$ & $\bar{V}_2$. ~~is~~ Its resultant velocity is equal to $\bar{V}_1$ in mag. S.T when the velocity $\bar{V}_1$ is doubled, the new resultant is ⊥r to $\bar{V}_2$.

$\bar{V}_1$ & $\bar{V}_2$ – two velocities

$$|\bar{V}_1+\bar{V}_2| = |\bar{V}_1|$$

$$|\bar{V}_1+\bar{V}_2|^2 = |\bar{V}_1|^2$$

$$(\bar{V}_1+\bar{V}_2)\cdot(\bar{V}_1+\bar{V}_2) = \bar{V}_1\cdot\bar{V}_1$$

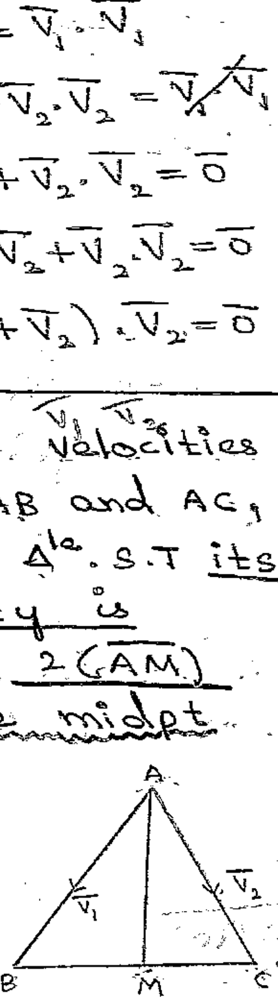
$$\cancel{\bar{V}_1\cdot\bar{V}_1}+\bar{V}_1\cdot\bar{V}_2+\bar{V}_2\cdot\bar{V}_1+\bar{V}_2\cdot\bar{V}_2 = \cancel{\bar{V}_1\cdot\bar{V}_1}$$

$$\bar{V}_1\cdot\bar{V}_2+\bar{V}_2\cdot\bar{V}_1+\bar{V}_2\cdot\bar{V}_2 = \bar{0}$$

$$2\bar{V}_1\cdot\bar{V}_2+\bar{V}_2\cdot\bar{V}_2 = \bar{0}$$

$$(2\bar{V}_1+\bar{V}_2)\cdot\bar{V}_2 = \bar{0}$$

2) A pt possesses velocities ($\bar{V}_1$, $\bar{V}_2$) represented by AB and AC, two sides of a Δ^{le}. S.T its resultant velocity is represented by $2(\overline{AM})$ where M is the midpt of BC.



$\bar{V}_1 = \overline{AB}$, $\bar{V}_2 = \overline{AC}$

$\bar{V}_1+\bar{V}_2 = 2\overline{AM}$

*M – midpt of BC

$\Rightarrow \overline{BM} = \overline{MC}$

$\overline{AB}+\overline{BM} = \overline{AM} \rightarrow ①$

$\overline{AC}+\overline{CM} = \overline{AM} \rightarrow ②$

$① + ② \Rightarrow \overline{AB}+\overline{AC}+\overline{BM}+\overline{CM} = 2\overline{AM}$

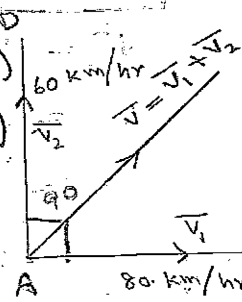
$\bar{V}_1+\bar{V}_2+\cancel{\overline{BM}}-\cancel{\overline{MC}} = 2\overline{AM}$

$\therefore \bar{V}_1+\bar{V}_2 = 2\overline{AM}$

3) A man seated in a train whose velocity is 80 km/hr, throws a ball horizontally & ⊥r to the train with a velocity of 60 km/hr. Find the velocity of the ball immediately after the throw.

$\overline{AB}$ – velocity (train)

$\overline{AD}$ – velocity (ball)



Mag. of resultant velocity,

$$\Rightarrow V = \sqrt{V_1^2+V_2^2+2V_1V_2\cos\alpha}$$

$$= \sqrt{80^2+60^2}$$

$$= 100 \text{ km/hr}$$

$\sin 90° = 1$

$\cos 90° = 0$

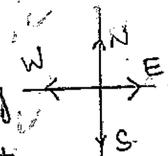
$$\Rightarrow \tan\theta = \frac{V_2}{V_1} = \frac{6\cancel{0}}{8\cancel{0}} = \frac{3}{4}$$

dir. of resultant velocity,

$$\theta = \tan^{-1}(3/4)$$

∴ The ball moves – { angle $\tan^{-1}(3/4)$; speed 100 km/hr }

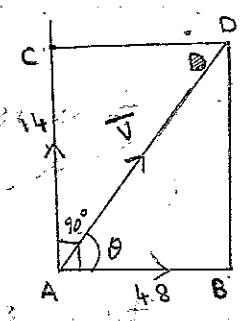
4) A boat (without water current) which can steam in still water with a velocity of 48 km/hr is steaming with its bow pointed due east, when it is carried by a current which flows northward with a speed of 14 km/hr. Find the actual distance it would travel in 12 minutes.



$\overline{AB} \rightarrow$ (east) steaming velocity

$\overline{AC} \rightarrow$ (north) water current

Resultant speed of the boat,



$$V = \sqrt{V_1^2+V_2^2}$$

$$= \sqrt{48^2+14^2}$$

$$= \sqrt{2304+196}$$

$$= \sqrt{2500} = 50 \text{ km/hr}$$

$\tan\theta = \frac{14^7}{48_{24}} \Rightarrow \theta = \tan^{-1}(7/24)$

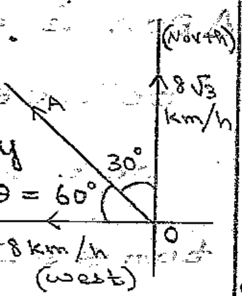
In 1 hr it moves through 50 km, in 12 minutes

1 hr = 50 km

$60 \text{ min} = 50 \text{ km} \Rightarrow 1 \text{ min} = \frac{50}{60}$ km

$$\Rightarrow 12 \text{ min} = \frac{5\cancel{0}}{\cancel{60}} \times \cancel{12}^{2} = 10 \text{ km}$$

5) A ship is steaming north at $8\sqrt{3}$ km/hr and a man walks across its deck in a direction due west at 8 km/hr. Find its resultant velocity in space.



$\overline{OA}$ – dir of the resultant velocity

∴ resultant velocity,

$V = \sqrt{V_1^2 + V_2^2}$

$= \sqrt{8^2 + (8\sqrt{3})^2} = \sqrt{64 \times 3 + 64}$

$= \sqrt{64 \times 4} = 8 \times 2 = 16$ km/h

$\tan\theta = \dfrac{j^{th} \text{ component}}{i^{th} \text{ component}}$

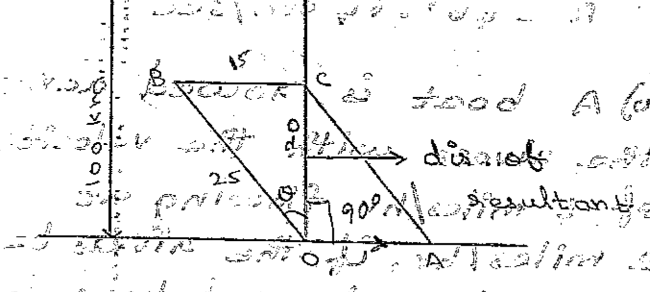
$\tan\theta = \dfrac{8\sqrt{3}}{8} = \sqrt{3}$

$\theta = \tan^{-1}(\sqrt{3})$

$\theta = 60^\circ$

∴ The dir of the man is 30° west of north (i.e.) from north to west.

6) A man can swim ⊥r across a steam of breadth 100 km in 4 min, when there is no current and in 5 min when there is a downward current. Find the velocity of the current.



Breadth – 100 mts

taken time – 4 mins

Speed = $\dfrac{\text{distance travelled}}{\text{time taken}}$

$= \dfrac{100}{4} = 25$ m/mt

Let water current be v m/mt

It is downward (i.e) horizontal

→ Taking 5 min to swim ⊥r

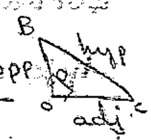
∴ Speed (OC) $= \dfrac{100}{5} = 20$ m/mt

• He starts swimming in a direction OB but by pulling of water current he moves along OC

dir of resultant velocity,

ΔOCB is a right angled Δ^{le},

$BC = OA = \sqrt{25^2 - 20^2}$



$= \sqrt{625 - 400}$

$= 15$ m/mt

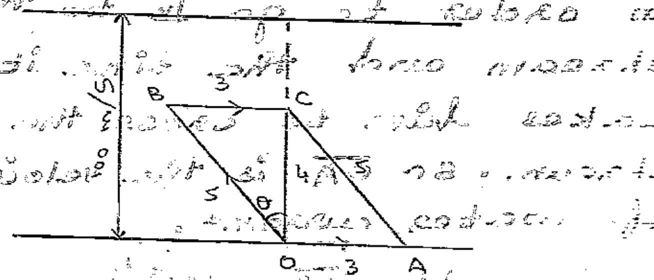
$\tan\theta = \dfrac{15}{20} = \dfrac{3}{4} = \dfrac{opp}{adj}$

$\theta = \tan^{-1}(3/4)$

∴ Man has to choose the dir. making an angle $90^\circ + \theta$ with horizontal dir, so he finally move along ⊥r direction.

→ dir $= 90^\circ + \tan^{-1}(3/4)$

7) A stream is running at 3 km/h and its breadth 10 mts. If a man can row a boat at 5 km/h in still water. Find the dir in which he must row in order to go straight across the stream and S.T the time he takes to do so in 90° sec.



$\overline{OA}$ → velocity of water current

$\overline{OA} = 3$ km/h

$\overline{OB}$ → velocity of man

$\overline{OB} = 5$ km/h

A man wants to row his boat ⊥r to water current

$\overline{OC}$ → dir. of resultant velocity

$\overline{OC} = \sqrt{5^2 - 3^2} = 4$ km/hr

= mag of resultant

In ΔOCB,
$\tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}(3/4)$

• man must choose a dir making an obtuse angle given by $90° + \tan^{-1}(3/4)$ with the dir. of water current.

→ mag of resultant = 4 km/hr

4000mts in 1hr, 100 mts ?

4 km = 1 hr

4000 m = 1 hr

$1m = \frac{1}{4000}$ hr

$100m = \frac{1}{4000} \times 100$ hr

$= \frac{1}{40}$ hr

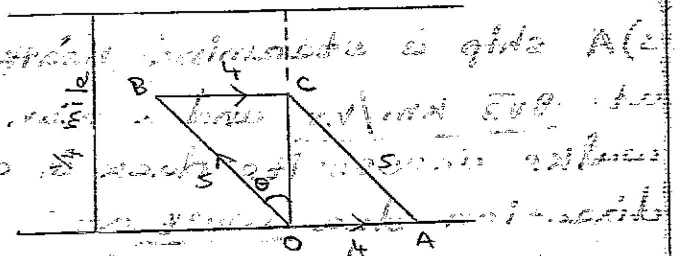
$= \frac{1}{40} \times 60$ min $= \frac{3}{2}$ min

$= \frac{3}{2} \times 60$ sec $= 90$ sec

8) A stream is running at a speed of 4 mile/hr. Its breadth is one quarter of a mile. A man can row a boat at a speed of 5 ml/h in still water. Find the dir in which he must row in order to go ⊥r to the stream and the time it takes ~~him~~ to cross the stream. So $\overline{OA}$ is the velocity of water current.

$\overline{OA} = 4$ml/h, $\overline{OB} = 5$ml/h

he wants to go ⊥r the dir. of $\overline{OC}$.



ΔBOC is a right angled Δ^{le}

$OC^2 = CB^2 - BC^2 = \sqrt{25-16} = \sqrt{9}$

$OC = 3$ ml/h

$\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}(4/3)$

• man moves along ⊥r dir.

3 mile = 1 hr

1 mile $= \frac{1}{3}$ hr

$\frac{1}{4}$ mile $= \frac{1}{3} \times \frac{1}{4}$ hr $= \frac{1}{12}$ hr

$= \frac{1}{12} \times 60$ min $= 5$ mins

∴ He takes 5 mins to across the stream. The dir has to choose makes an obtuse angle the dir of water current given by (along OB) $\tan^{-1}(4/3) + 90°$.

9) Find the resultant velocity of 2 velocities 240 cm/sec and 300 cm/sec inclined at 60° to each other.

V_1 & V_2 → 2 velocities → α

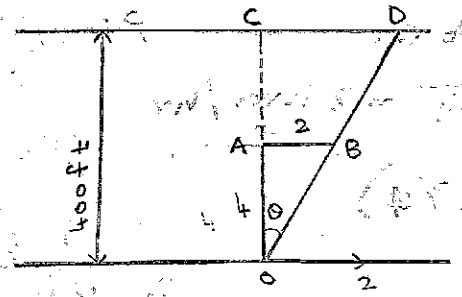
$R^2 = V_1^2 + V_2^2 + 2V_1V_2\cos\alpha$

→ $V_1 = 240$ cm/sec $\alpha = 60°$

$V_2 = 300$ cm/sec $\cos 60° = 1/2$

∴ $R = 468.64$ cm/sec

10) A boat is rowed across the rivers with the velocity of 4 miles/hr flowing at 2 miles/hr, if the rivers being 400ft broad. Find how far down will the boat reach the opposite bank.



To find CD:

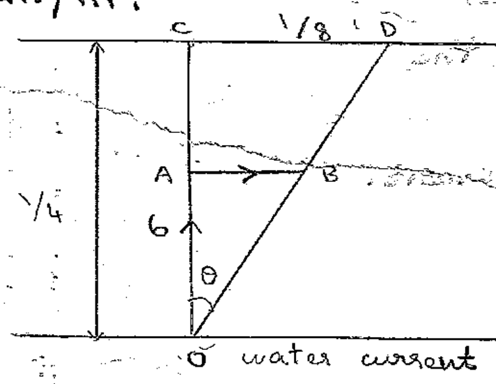
$\Delta OAB, \Delta OCD \rightarrow$ similar Δ^{les}

$$\tan\theta = \frac{AB}{OA} = \frac{CD}{OC}$$

$$\frac{2}{4} = \frac{CD}{400}$$

$$CD = 200$$

11) A man keep his boat at right angles to the current and rows across a stream $\frac{1}{4}$ km broad. He finds that he reaches the opposite bank $\frac{1}{8}$ km below the point opposite to the starting pt. Find the speed of the current if the speed of the boat due to rowing along is 6 km/hr.



Here the man starts to move in to the dir. of water current, $\overline{OA}$ is the dir of the boat to start with. But pulled by water current, he is made to move along $\overline{OB}$ which is the direction of the resultant velocity of the velocity of boat and velocity of water current. The man in the boat reaches the opposite bank at D.

The distance of D from C which is opposite to the starting pt O is given to be $\frac{1}{8}$ km.

OC is the actual breadth of the stream, $OC = \frac{1}{4}$ km.

In the $\Delta ABO, \Delta CDO$ which are similar:

$$\tan\theta = \frac{AB}{OA} = \frac{CD}{OC}$$

$$\frac{AB}{6} = \frac{1/8}{1/4} = \frac{6}{2} = 3$$

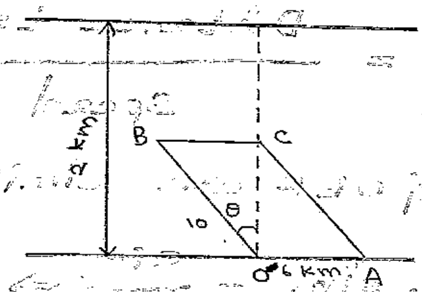
$\therefore$ The speed of water current is 3 km/hr.

12) A boat capable of moving in still water with a speed of 10 km/hr across a river, 2 km broad, flowing with a speed of 6 km/hr. Find

i) the time of crossing by the shortest route.

ii) the minimum time of crossing.

i) the time of crossing by the shortest route.



$\overline{OA}$ represents the velocity of water current.

$\overline{OB}$ " the velocity of boat.

ΔOCB is right angled Δ^{le} at C. $OB = 10 = AC$, $= 6 =$

$OC^2 = 10^2 - 6^2 = 64 \Rightarrow OC = \sqrt{64} = 8$ km/hr

$\tan\theta = \frac{6}{8} = \frac{3}{4} \Rightarrow \theta = \tan^{-1}(3/4)$

The mag of the resultant velocity is 8 km/hr

Man $\rightarrow$ moves $\rightarrow$ direction $\rightarrow 90° + \theta$ (obtuse angle)

$\rightarrow$ horizontal direction $\rightarrow \perp r$

$\rightarrow$ — — — —

8 km $\rightarrow$ 1 hr

1 km $\rightarrow \frac{1}{8}$ hr

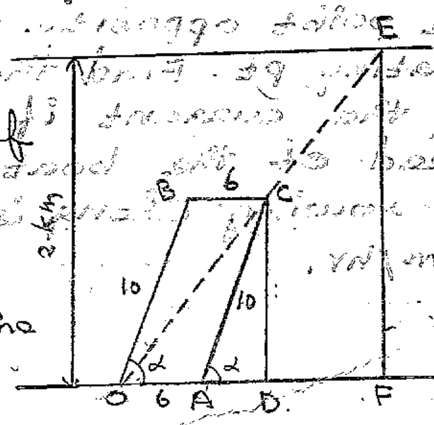
2 km $\rightarrow \frac{2}{8}$ hr

$\Rightarrow \frac{1}{4} \times 60$ min $= 15$ min

$\therefore$ The time taken to cross the river by the shortest route is 15 min. 13

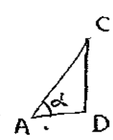
ii) the minimum time of crossing

The man steers his boat along OB ~~oft~~ which makes an angle 'α' with the direction of water current. pulled by the water current the boat moves along OC, the boat reaches the opp. bank at the pt E.



$\rightarrow$ Draw CD & EF $\perp r$ to water current.

Distance travelled $= OE$



$$\text{Speed} = \frac{\text{Distance travelled}}{\text{time taken}}$$

$$\text{time} = \frac{\text{Distance travelled}}{\text{Speed}} = \frac{OE}{OC}$$

ΔOCD & OEF are similar Δ^{les}, $\frac{OE}{OC} = \frac{EF}{CD} \Rightarrow$

ΔACD, $\sin\alpha = \frac{CD}{AC} \Rightarrow CD = AC(\sin\alpha)$

$$\therefore \frac{EF}{CD} = \frac{2}{AC(\sin\alpha)}$$

$\Rightarrow$ time $= \dfrac{EF}{CD} = \dfrac{2}{10\sin\alpha}$ $[\because AC = 10]$

* Time is minimum, when the denominator is maximum.

* The maximum value of $\sin\alpha = 1$

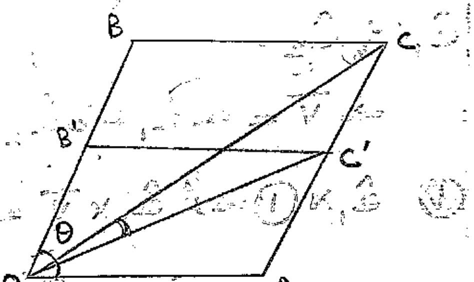
So, the minimum time $= \frac{2}{10}$ hr $= \frac{1}{5} \times 60 = 12$ mins

(i.e), the minimum time of crossing is 12 mins.

---

13. A particle has two velocities of equal mag. ~~include~~ inclined to each other at an angle θ. If one of them is halved, the angle between the other & the original resultant velocity is bisected by the new resultant. S.T. $\theta = 120°$

$\overline{OA}$ & $\overline{OB} \to$ 2 velocities or equal magnitudes.



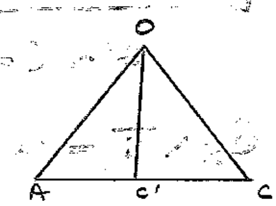
$OA = OB$

$\overline{OC} \to$ resultant & it bisects

ie) $\angle BOC = \angle COA$

* One of the velocities is halved. Let it be $\overline{OB'}$.
* $\overline{OC'}$ is the new resultant velocity of $\overline{OB'}$ & $\overline{OA}$.
* It is gn $\overline{OC'}$ bisects the $A\hat{O}C$.

$\therefore OC'$ bisects $\angle AOC$



$$\frac{OA}{AC'} = \frac{OC}{CC'}$$

C' is the midpt of AC,

$\Rightarrow AC' = CC'$

$\dfrac{OA}{OC} = \dfrac{AC'}{CC'} = 1 \Rightarrow OA = OC$ & also $OB = AC \Rightarrow OA = OB = AC$

$\therefore \triangle OAC \to$ equilateral $\triangle^{le}$

$\angle AOC = \angle OCA = \angle CAO = 60°$

in complete

$\angle AOC' = \angle C'OC = 30°$

H.W. 2 sums

$\angle AOC = \angle COB = 60°$

$\Rightarrow \theta = \angle AOB = \angle AOC + \angle COB = 60° + 60° = 120°$

$\therefore \theta = 120°$

Resolution of a velocity into its components:-

Definition:

* Given two velocities $\overline{V_1}$ & $\overline{V_2}$, we have $\overline{V_1}+\overline{V_2}$, the resultant of them;
* Conversely given $\overline{V_1}+\overline{V_2}$, the vectors $\overline{V_1}$ & $\overline{V_2}$ are said to be the components of $\overline{V_1}+\overline{V_2}$.

Book work: (2)

To resolve a velocity $\overline{V}$ into components in two given directions.

Proof: $\hat{e}_1$ & $\hat{e}_2 \rightarrow$ unit vectors in the given direction.

Let them make angle α & β with $\overline{V}$.

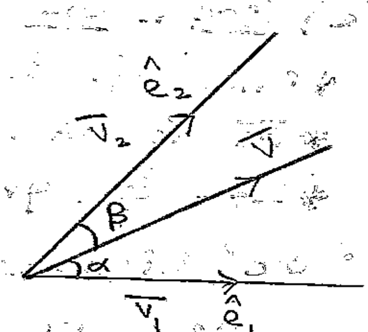
Expressing $\overline{V}$ as a linear combination of $\hat{e}_1$ & $\hat{e}_2$.

$$\Rightarrow \boxed{\overline{V} = a\hat{e}_1 + b\hat{e}_2} \rightarrow (1)$$

$$\hat{e}_1 \times (1) \Rightarrow \hat{e}_1 \times \overline{V} = a(\hat{e}_1 \times \hat{e}_1) + b(\hat{e}_1 \times \hat{e}_2)$$

$$V \sin\alpha\, \hat{n} = b \sin(\alpha+\beta)\, \hat{n}$$

Here $\hat{n}$ is $\perp r$ to both $\hat{e}_1$ & $\hat{e}_2$



$$b = \frac{V \sin\alpha}{\sin(\alpha+\beta)}$$

$|||^{ly}$ $\hat{e}_2 \times (1) \Rightarrow \hat{e}_2 \times \overline{V} = a(\hat{e}_2 \times \hat{e}_1) + b(\hat{e}_2 \times \hat{e}_2)$

$$V \sin\beta(-\hat{n}) = a \sin(\alpha+\beta)(-\hat{n})$$

$$a = \frac{V \sin\beta}{\sin(\alpha+\beta)}$$

$$\therefore \overline{V} = \frac{V \sin\beta}{\sin(\alpha+\beta)}\hat{e}_1 + \frac{V \sin\alpha}{\sin(\alpha+\beta)}\hat{e}_2$$

Component of a ~~vector~~ velocity in a given direction:

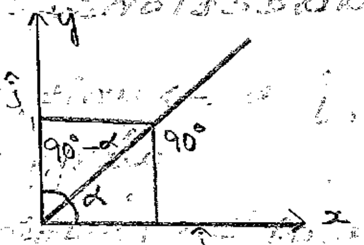
Defin

* Let $\overline{V} \rightarrow$ given velocity
* Let $\hat{e} \rightarrow$ unit vector along which we require the component.
* Then $\overline{V}\cdot\hat{e}$ is called the component of $\overline{V}$ along $\hat{e}$.

Book work: (3)

To express the velocity $\bar{v}$ in terms of its components in two $\perp$ directions.

*Let $\hat{i}$ & $\hat{j}$ → unit vectors along 2 $\perp$ directions.



$$\boxed{\bar{v} = (\bar{v}\cdot\hat{i})\hat{i} + (\bar{v}\cdot\hat{j})\hat{j}}$$

*Let $\bar{v}$ makes angle α, $(90^\circ - \alpha)$ with $\hat{i}$ & $\hat{j}$

$\Rightarrow \bar{v}\cdot\hat{i} = v\cdot 1\cdot\cos\alpha$, $\bar{v}\cdot\hat{j} = v\cdot 1\cdot\cos(90^\circ - \alpha) = v\sin\alpha$

$\therefore \bar{v} = v\cos\alpha\,\hat{i} + v\sin\alpha\,\hat{j}$

* $v\cos\alpha$ is the component of $\bar{v}$ along $\hat{i}$ direction
* $v\sin\alpha$ is the component of $\bar{v}$ along $\hat{j}$ direction

Definition of Acceleration:

Acceleration of a particle is the time rate of change of its velocity (i.e) If $\bar{v}$ is its velocity at time t, then its acceleration $\bar{a}$ at time t is $\frac{dv}{dt}$, $\bar{a} = \frac{d\bar{v}}{dt}$

Unit of acceleration:

In M.K.S system, it is 1 metre/sec^2 (or) 1m/sec^2

In C.G.S system, it is 1 cm/sec^2

In F.P.S system, it is 1 ft/sec^2

Coplanar motion:-

When a particle moves in a plane, its motion is said to be coplanar.

To find the components of velocity and acceleration along 3 sets of $\perp$ direction:

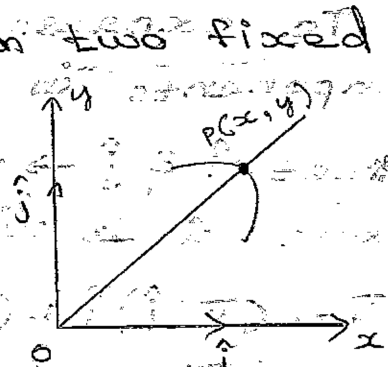
i) Two fixed $\perp$ directions, say x & y axes.

ii) The directions tangential and normal to the path of the particle.

iii) The radial and transverse directions.

Book work: 4 2M

To find the components in two fixed ⊥r directions.



* $\hat{i}$ & $\hat{j}$ be the → unit vectors along x & y axes.
* P(x, y) be the → position of the moving particle at any instant t.
* Then the position of 'P' w.r.t 'O' is,

$$\boxed{\overline{r} = \overline{OP} = x\hat{i} + y\hat{j}} \rightarrow (1)$$

Velocity $\Rightarrow \overline{V} = \dfrac{d}{dt}(\overline{r}) = \dfrac{d}{dt}(x\hat{i} + y\hat{j})$

$$= \dot{x}\hat{i} + \dot{y}\hat{j} \rightarrow (2)$$

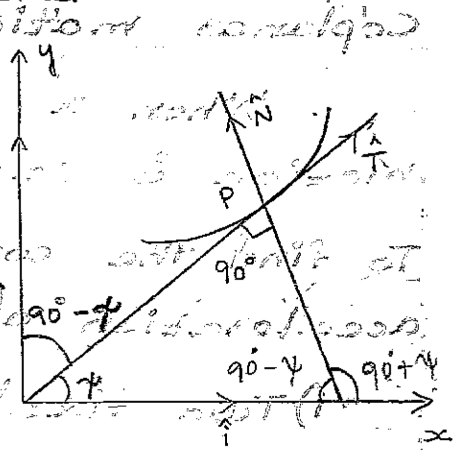
Acceleration $\Rightarrow \overline{a} = \dfrac{d}{dt}(\overline{V}) = \dfrac{d}{dt}(\dot{x}\hat{i} + \dot{y}\hat{j})$

$$= \ddot{x}\hat{i} + \ddot{y}\hat{j} \rightarrow (3)$$

* the Velocity $\xrightarrow{\text{component along the}}$ x-axis $= \dot{x}$ / y-axis $= \dot{y}$
* the acceleration ⟶ x-axis $= \ddot{x}$ / y-axis $= \ddot{y}$
* Here dot stands for diff. w.r.t time factor.

Book work: 5 5M

To find the components of the velocity and acceleration along tangential and normal directions.



* We have already proved that,

$$\overline{V} = \dot{s}\overline{T} \rightarrow (1)$$

* Velocity $\xrightarrow{\text{is only along the}}$ tangential direction,
* Velocity component along the tang. dir is $\rightarrow \dot{s}$; along the normal dir is $\rightarrow 0$
* Now we find the acceleration components.
* $\hat{T}$ & $\hat{N}$ be the → unit vectors along the tang & normal dir.
* As p moves along a curve, the direction of the tangent vector changes from instant to instant.

$$\Rightarrow \bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(\dot{s}\hat{T}) \rightarrow ②$$

$$\bar{a} = \ddot{s}\hat{T} + \dot{s}\frac{d}{dt}(\hat{T}) \rightarrow ③$$

* The unit tangent vector $\hat{T}$ makes an angle 'ψ' with the x-axis.

$$\hat{T} = \cos\psi\,\hat{i} + \sin\psi\,\hat{j} \rightarrow ④$$

$$\frac{d\hat{T}}{dt} = -\sin\psi\frac{d\psi}{dt}\hat{i} + \cos\psi\frac{d\psi}{dt}\hat{j} \rightarrow ⑤$$

$$\frac{d\hat{T}}{dt} = (-\sin\psi\,\hat{i} + \cos\psi\,\hat{j})\frac{d\psi}{dt} \rightarrow ⑥$$

* Now, we find the unit vector $\hat{N}$ along the normal direction.

* $\hat{N} \xrightarrow{\text{makes an angle}} 90° + \psi$ with the x-axis.

$$\hat{N} = \cos(90° + \psi)\hat{i} + \sin(90° + \psi)\hat{j}$$

$$\hat{N} = -\sin\psi\,\hat{i} + \cos\psi\,\hat{j} \rightarrow ⑦$$

Sub ⑦ in ⑥ $\Rightarrow \frac{d\hat{T}}{dt} = \hat{N}\frac{d\psi}{dt} \rightarrow ⑧$

Sub ⑧ in ③ $\Rightarrow \bar{a} = \ddot{s}\hat{T} + \dot{s}\hat{N}\frac{d\psi}{dt} \rightarrow ⑨$

We have to simplify $\frac{d\psi}{dt}$

chain rule

$$\frac{d\psi}{dt} = \frac{d\psi}{ds}\frac{ds}{dt} = \dot{s}\frac{d\psi}{ds}$$

* W.k.T the rate at which ψ changes w.r.t the actual distance, s is the curvature of the curve.

i.e, $\frac{d\psi}{ds}$ = curvature $= \frac{1}{\rho}$ [where ρ is the radius of curvature]

$$\frac{d\psi}{dt} = \frac{\dot{s}}{\rho} \rightarrow ⑩$$

$\bar{V} = \dot{s}\hat{T}$

$\bar{V} = V\hat{V}$

$\dot{s} = V \Rightarrow \dot{s}^2 = V^2$

$\ddot{s} = \frac{dv}{dt}$

Sub $\bar{a} = \ddot{s}\hat{T} + \frac{\dot{s}^2}{\rho}\hat{N} \rightarrow ⑪$

* Acceleration component along the normal $= \frac{\dot{s}^2}{\rho}$

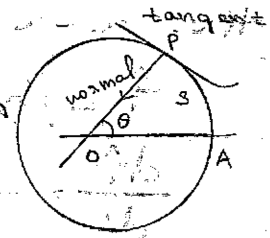
along the tangent $= \ddot{s}$

$$\therefore \bar{a} = \frac{dv}{dt}\hat{T} + \frac{V^2}{\rho}\hat{N}$$

Book work : 6

The path of the moving particle is a circle. To find the components of velocity and acceleration along the tangential and normal direction. Measure the arcual distance from the fixed pt A.

$\hat{AP} = s$, $A\hat{O}P = \theta$ $(s = b\theta,\ \dot{s} = b\dot{\theta},\ \ddot{s} = b\ddot{\theta})$



* $\bar{a} \xrightarrow{\text{denotes}}$ acceleration a & mag. of acceleration
* So we say 'b' for radius

$$\boxed{\bar{v} = \dot{s}\hat{T}} = b\dot{\theta}\hat{T} \rightarrow (1)$$

$$\boxed{\bar{a} = \ddot{s}\hat{T} + \frac{\dot{s}^2}{b}\hat{N}} \rightarrow (2)$$

$$= b\ddot{\theta}\hat{T} + \frac{b^2\dot{\theta}^2}{b}\hat{N}$$

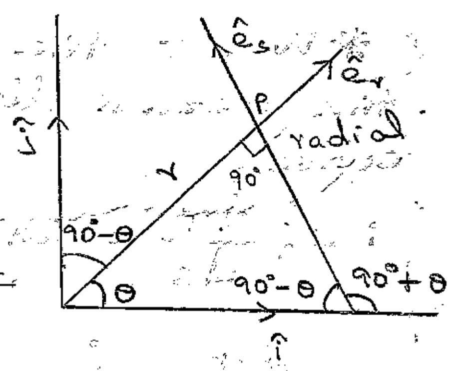
$$\bar{a} = b\ddot{\theta}\hat{T} + b\dot{\theta}^2\hat{N} \rightarrow (3)$$

* Component of velocity along the tangent $= b\dot{\theta}$
 normal $= 0$

* Component of acceleration along the normal $= b\ddot{\theta}$
 inward normal $= b\dot{\theta}^2$

Book work : 7

To find the components of velocity & acceleration of a particle in the radial & transverse directions.



* P → position of moving particle at any instant t.
* Choose origin as pole.
* $\hat{i}$ & $\hat{j}$ → unit vectors along x & y axes.
* $\hat{e}_r$ & $\hat{e}_s$ → unit vectors along radial & transverse directions.
* P changes its position, $\hat{e}_r$ & $\hat{e}_s$ also change direction

$$\Rightarrow \bar{v} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) \rightarrow (1) \qquad \left[\because OP = r,\ \hat{OP} = \hat{e}_r\right] \qquad \dot{\theta} = \frac{d\theta}{dt}$$

$$\bar{v} = \dot{r}\hat{e}_r + r\frac{d}{dt}(\hat{e}_r) \rightarrow (2)$$

To get $\frac{d}{dt}(\hat{e}_r) \Rightarrow \hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \rightarrow (3)$

$\therefore \frac{d}{dt}(\hat{e}_r) = -\sin\theta \frac{d\theta}{dt}\hat{i} + \cos\theta \frac{d\theta}{dt}\hat{j} \rightarrow (4)$

$= (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,\dot{\theta} \rightarrow (5) \quad \left[\because \dot{\theta} = \frac{d\theta}{dt}\right]$

* $\hat{e}_s \xrightarrow{\text{unit vector along transverse direction}}$ makes an angle $(\theta + 90^\circ) \rightarrow x$-axis

$\Rightarrow \hat{e}_s = \cos(90^\circ + \theta)\,\hat{i} + \sin(90^\circ + \theta)\,\hat{j} \rightarrow (6)$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \rightarrow (7)$

Sub (7) in (5), $\frac{d}{dt}\hat{e}_r = \dot{\theta}\,\hat{e}_s \rightarrow (8)$

Sub (8) in (2), $\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s \rightarrow (9)$

To find acceleration vector:

$\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s$

$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s)$

$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_s) \rightarrow (10)$

consider $\frac{d}{dt}(r\dot{\theta}\hat{e}_s) = \dot{r}\dot{\theta}\hat{e}_s + r\ddot{\theta}\hat{e}_s + r\dot{\theta}\frac{d}{dt}(\hat{e}_s) \rightarrow (11)$

To get $\frac{d}{dt}(\hat{e}_s) \Rightarrow \hat{e}_s = \cos(90^\circ + \theta)\,\hat{i} + \sin(90^\circ + \theta)\,\hat{j}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\Rightarrow \frac{d}{dt}\hat{e}_s = -\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j}$

$= -\left[\cos\theta\frac{d\theta}{dt}\hat{i} + \sin\theta\frac{d\theta}{dt}\hat{j}\right]$

$= -\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]\frac{d\theta}{dt}$

$\frac{d}{dt}\hat{e}_s = -\dot{\theta}\hat{e}_r \rightarrow (12)$

$\bar{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_s + \dot{r}\dot{\theta}\hat{e}_s + r\ddot{\theta}\hat{e}_s + r\dot{\theta}(-\dot{\theta})\hat{e}_r$

$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_s \rightarrow (13)$

* Velocity component along the → radial direction $= \dot{r}$
  → transverse " $= r\dot{\theta}$

* Acceleration component along the → radial " $= \ddot{r} - r\dot{\theta}^2$
  → transverse " $= \dot{r}\dot{\theta} + r\ddot{\theta}$

we simplify $2\dot{r}\dot{\theta}+r\ddot{\theta}$

consider $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(2r\dot{r}\dot{\theta}+r^2\ddot{\theta})$

$= 2\dot{r}\dot{\theta}+r\ddot{\theta} \rightarrow (14)$

using (14) in (13),

$\bar{a} = (\ddot{r}-r\dot{\theta}^2)\hat{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{e}_s \rightarrow (15)$

∴ The acceleration component along the transverse direction is $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

Book work : (8) *transverse & radial*

The path is a circle. $r = b$

$\dot{r} = 0 \quad \ddot{r} = 0$

$\boxed{\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s}$

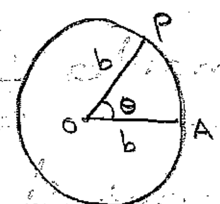
$\left[r = OP = \text{radius} = \text{a constant} \right]$

$\bar{v} = (0)\hat{e}_r + b\dot{\theta}(\hat{e}_s) \rightarrow (1)$

$\angle AOP = \theta$

$\Rightarrow \bar{a} = (\ddot{r}-r\dot{\theta}^2)\hat{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{e}_s$

$= (0-b\dot{\theta}^2)\hat{e}_r + \frac{1}{b}(b^2\ddot{\theta})\hat{e}_s$



$= -b\dot{\theta}^2\hat{e}_r + b\ddot{\theta}\hat{e}_s$

$\frac{d}{dt}\angle AOP = \dot{\theta}$ ↳ angular velocity

* $\dot{\theta}$ is known as the angular velocity

where $\angle AOP = \theta \Rightarrow \frac{d}{dt}\angle AOP = \dot{\theta}$ as shown in diagram

4. If the angular velocity of a point (particle) moving on a plane curve is a constant about a fixed point (origin), S.T its transverse acceleration is proportional to its radial velocity.

2M (f)

Radial velocity $= \left\{ \begin{array}{l} \text{component of velocity} \\ \text{along the radial dir.} \end{array} \right\} = \dot{r} \rightarrow (1)$

Transverse acceleration $= \boxed{2\dot{r}\dot{\theta}+r\ddot{\theta}}$

$\boxed{\begin{array}{l} \bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s \\ \dot{\theta} = c \text{ (given)} \end{array}}$

$= 2\dot{r}c + r(0)$

$= 2c\dot{r}$

$\Rightarrow$ T.A $\propto$ R.V

15. A particle moves so that the radial and transverse components of its velocity are ar & $b\theta$. ST the radial & transverse components of its acceleration are

$$a^2 r - \frac{b^2\theta^2}{r}, \quad ab\theta + \frac{b^2\theta}{r}$$

$$\bar{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s, \quad \bar{a} = (\ddot{r} + r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_s$$

Here $\dot{r} = ar \rightarrow (1)$, $r\dot{\theta} = b\theta \rightarrow (2)$

To get radial acceleration

$$\ddot{r} - r\dot{\theta}^2 = \frac{d}{dt}(ar) - r\left(\frac{b\theta}{r}\right)^2$$

$$= a\dot{r} - \frac{b^2\theta^2}{r} = a(ar) - \frac{b^2\theta^2}{r} = a^2 r - \frac{b^2\theta^2}{r}$$

To get transverse acceleration.

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 2(ar)\left(\frac{b\theta}{r}\right) + r\frac{d}{dt}(\dot{\theta}) \rightarrow (3)$$

Consider,

$$r\frac{d}{dt}(\dot{\theta}) = r\frac{d}{dt}\left(\frac{b\theta}{r}\right) = \frac{r(rb\dot{\theta} - \dot{r}b\theta)}{r^2} = \frac{rb\dot{\theta} - \dot{r}b\theta}{r}$$

$$= \frac{rb\dot{\theta} - ar(b\theta)}{r} = \dot{\theta}b - ab\theta$$

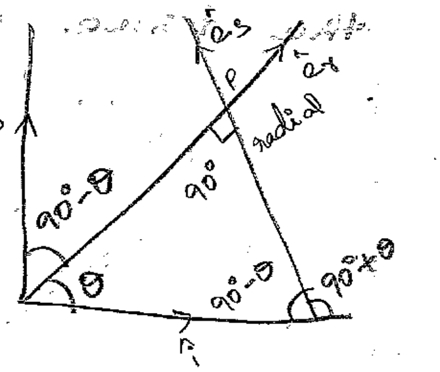
$$\boxed{r\frac{d}{dt}(\dot{\theta}) = \dot{\theta}b - ab\theta} \rightarrow (4)$$

Sub (4) in (3),

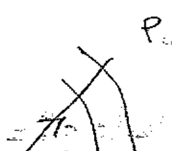
$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 2ab\theta + b\dot{\theta} - ab\theta$$

$$= ab\theta + b\left(\frac{b\theta}{r}\right)$$

$$= ab\theta + \frac{b^2\theta}{r}$$



Definition of Relative velocity:



* A & P $\rightarrow$ two moving points.
* $\overline{AP}$ $\rightarrow$ pv of 'P' w.r.t 'A'
* $\frac{d}{dt}(\overline{AP})$ $\rightarrow$ velocity of P relative to A.
* O $\rightarrow$ origin of reference.

Relative velocity of P w.r.to A

$$\frac{d}{dt}(\overline{AP}) = \frac{d}{dt}(\overline{OP}-\overline{OA}) = \frac{d}{dt}(\overline{OP}) - \frac{d}{dt}(\overline{OA})$$

$$= \text{velocity of } p - \text{velocity of } A$$

$$= \overline{V}_p - \overline{V}_A \longrightarrow (1)$$

i.e., $\frac{d}{dt}(\overline{AP})$ = True velocity of p − velocity of A → (2)

= True velocity of p + (−velocity of A)

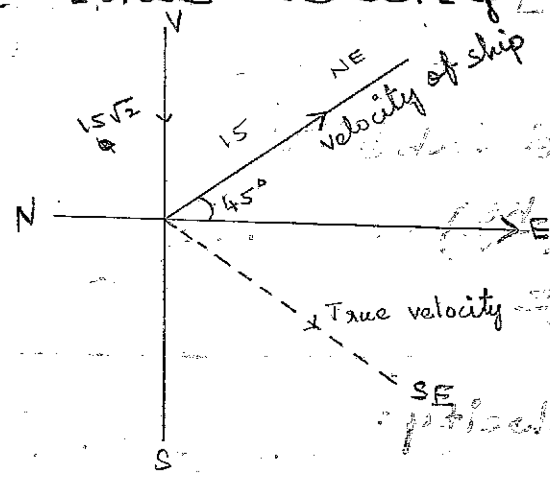
= True velocity of p + reversed velocity of A → (3)

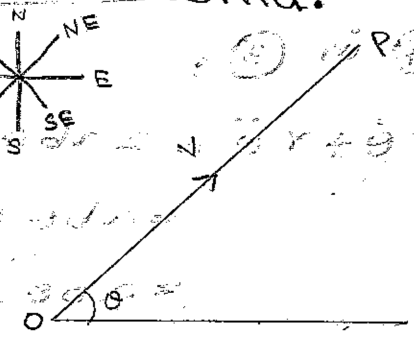
∴ the relative velocity of p w.r.t A is the resultant velocity of p & the reversed velocity of A.

<u>Note</u> :- (Appears → denote the dir of relative velocity)

When we consider the relative velocity of P w.r.t A, through A is also in motion, we treat it to be at rest) & assume that P is moving with the relative velocity $= \overline{V}_P - \overline{V}_A$ which is the resultant velocity of velocity of p & the reversed velocity of A.

16. A <u>ship</u> sails north-east at 15 km/hr and to a passanger on ~~board~~ ship, the <u>wind appears</u> to blow from north with a velocity of $15\sqrt{2}$ km/hr. Find the true velocity of the wind.





$\overline{OP} = OP\hat{OP} = r(\cos\theta\, \hat{i} + \sin\theta\, \hat{j})$

P → wind / A → ship

Relative velocity of <u>P</u> w.r.t <u>A</u>

= True velocity of P − ~~true~~ velocity of A

Relative velocity of wind w.r.t ship

= True velocity of wind − ~~true~~ velocity of ship

$\propto$ R.V of P w.r.t A $= -15\sqrt{2}\,\hat{j}$

$\propto$ True velocity of P $= v_1\hat{i} + v_2\hat{j}$

$\propto$ Velocity of A (ship) $= 15(\cos 45\,\hat{i} + \sin 45\,\hat{j})$

$$= 15\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$\therefore\ -15\sqrt{2}\,\hat{j} = (v_1\hat{i} + v_2\hat{j}) - 15\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$-15\sqrt{2}\,\hat{j} = v_1\hat{i} + v_2\hat{j} - 15\frac{1}{\sqrt{2}}\hat{i} - 15\frac{1}{\sqrt{2}}\hat{j}$$

$$-15\sqrt{2}\,\hat{j} = \left(v_1 - \frac{15}{\sqrt{2}}\right)\hat{i} + \left(v_2 - \frac{15}{\sqrt{2}}\right)\hat{j}$$

$$\Rightarrow v_1 - \frac{15}{\sqrt{2}} = 0, \quad v_2 - \frac{15}{\sqrt{2}} = -15\sqrt{2}$$

$$\boxed{v_1 = \frac{15}{\sqrt{2}}}, \qquad v_2 = -15\sqrt{2} + \frac{15}{\sqrt{2}}$$

$$\boxed{v_2 = \frac{-15}{\sqrt{2}}}$$ now doubt

$\therefore$ True velocity of wind $= v_1\hat{i} + v_2\hat{j}$

$$= \frac{15}{\sqrt{2}}\hat{i} - \frac{15}{\sqrt{2}}\hat{j}$$

$\propto$ Mag. of T.V of wind $= \sqrt{\left(\frac{15}{\sqrt{2}}\right)^2 + \left(\frac{-15}{\sqrt{2}}\right)^2}$

$$= \sqrt{15^2\left(\frac{1}{2} + \frac{1}{2}\right)}$$

$$= 15\ \text{km/hr}$$

* $\hat{j}$ th component $\to$ −ve & $\hat{i}$ th component $\to$ +ve, the dir. is in the 4th quadrant.
* i.e, dir $\to$ towards the South East (SE).

---

17. To a man walking along a level road at 5 km/hr. The rain appears to beat into his face at 8 km/hr at an angle 60° with the vertical. Find the direction and the mag of the true velocity of the ~~rain~~ rain.

Relative velocity of the rain (P) w.r.t man (A)

= True velocity of rain (P) − velocity of man (A)

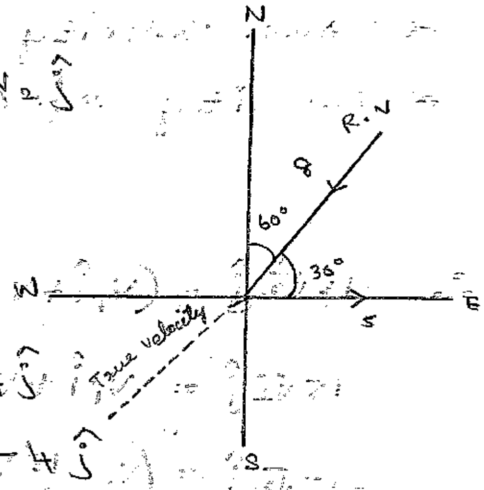
$-(\cos 30°\,\hat{i} + \sin 30°\,\hat{j})$

$$-8\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) = (v_1\hat{i} + v_2\hat{j}) - 5\hat{i}$$

$$-4\sqrt{3}\,\hat{i} - 4\hat{j} = (v_1 - 5)\hat{i} + v_2\hat{j}$$

$\boxed{v_2 = -4}$, $v_1 - 5 = -4\sqrt{3}$

$\boxed{v_1 = -4\sqrt{3} + 5}$



∴ T.V of wind $= (5 - 4\sqrt{3})\hat{i} - 4\hat{j}$

$= -(4\sqrt{3} - 5)\hat{i} - 4\hat{j}$

The mag of T.V of rain $= \sqrt{(5 - 4\sqrt{3})^2 + 4^2}$

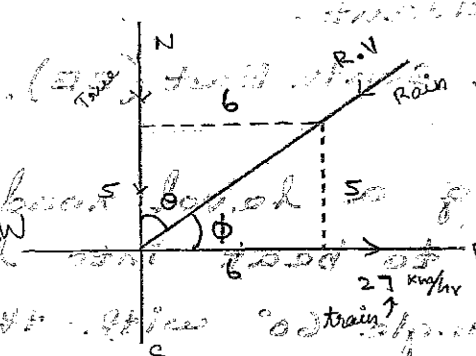
$= 4.44$ km/hr

The dir. is inclined at an angle θ with the x-axis is given by:-

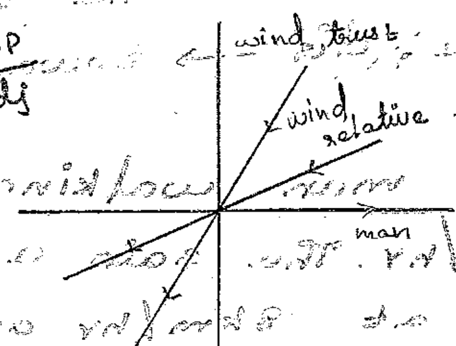
$$\tan\theta = \frac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \frac{-4}{-(4\sqrt{3}-5)} = \frac{4}{4\sqrt{3}-5}$$

As Both i^{th} & j^{th} comp → −ve, the dir is towards the 3rd quadrant.

18. A train travels at the rate of 27 km/hr. Rain which is falling vertically, appears to a man in the train to make an angle of $\tan^{-1}(6/5)$ with the vertical. Find the mag of the True velocity of the rain.



$\tan\theta = \frac{\text{opp}}{\text{adj}}$



Relative velocity of the rain w.r.t train

= True velocity of rain − velocity of train

$= -v\hat{j} - 27\hat{i}$

↳ $-\hat{j}$ (as it is falling vertically)

$v_1\hat{i} + v_2\hat{j} = 0 - v\hat{j}$ (let us take $-v\hat{j}$)

* angle made by the dir. $\to$ R.V of rain with the x-axis.

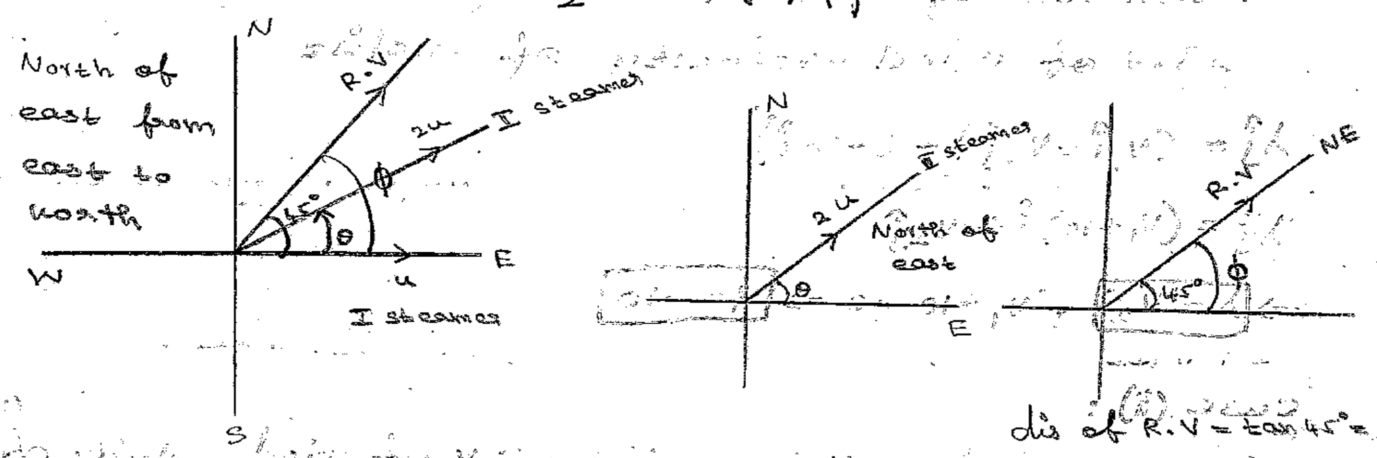
$$\tan\phi = \frac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \frac{-v}{-27}$$

$$\frac{5}{6} = \frac{v}{27} \quad \left[\because \tan^{-1}(6/5) = \theta \Rightarrow \tan\theta = \frac{6}{5} = \frac{opp}{adj}\right]$$

$$\frac{5}{\cancel{6}_2} \times \cancel{27}^{9} = v \Rightarrow v = \frac{5}{2}\times 9 = \frac{45}{2}$$

$\therefore$ The mag of the T.V of the rain $= \frac{45}{2}$ km/hr.

19. A steamer is travelling due east at the rate of u km/hr. A second steamer is travelling at 2u km/hr in a dir. of θ north of east and appears to be travelling north east to a passanger on the first steamer. PT $\theta = \frac{1}{2}\sin^{-1}(3/4)$



R.V of P w.r.t A = T.V of P − velocity of A

R.V of II steamer w.r.t I steamer

= T.V of II steamer − velocity of I steamer

$= 2u(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) - u\,\hat{i}$

$= (2\cos\theta - 1)u\,\hat{i} + 2u\sin\theta\,\hat{j}$

R.V of II w.r.t I makes an angle 45° with the x-axis

$$\underset{\substack{\downarrow\\ \tan 45^\circ}}{\tan\phi} = \frac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \frac{2u\sin\theta}{(2\cos\theta - 1)u}$$

$$1 = \frac{2\cancel{u}\sin\theta}{(2\cos\theta - 1)\cancel{u}} \Rightarrow 2\cos\theta - 1 = 2\sin\theta$$

$\Rightarrow 2(\cos\theta - \sin\theta) = 1 \Rightarrow \cos\theta - \sin\theta = \frac{1}{2}$

$\Rightarrow (\cos\theta - \sin\theta)^2 = \frac{1}{4} \Rightarrow \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta = \frac{1}{4}$

$\Rightarrow 1 - 2\sin\theta\cos\theta = \frac{1}{4} \Rightarrow -2\sin\theta\cos\theta = \frac{1}{4} - 1$

$\Rightarrow -2\sin\theta\cos\theta = \frac{-3}{4} \Rightarrow \sin 2\theta = \frac{3}{4} \Rightarrow 2\theta = \sin^{-1}(\frac{3}{4})$

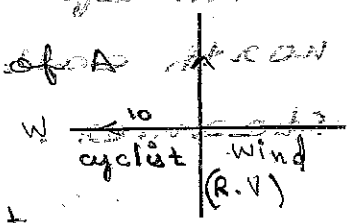
$\therefore \theta = \frac{1}{2}\sin^{-1}(3/4)$

---

20. To a cyclist riding due west at 10 km/hr, the wind appears (i) to blow from south. When he doubles his speed, it (wind) appears (ii) him to blow from south-west. ST the speed of the wind is $10\sqrt{2}$ km/hr and it is from south east.

Case (i):

$\Rightarrow$ R.V of P w.r.t A = T.V of P − T.V of A



Relative velocity of wind w.r.t cyclist

= T.V of wind − velocity of cyclist

$\lambda\hat{j} = (v_1\hat{i} + v_2\hat{j}) - (-10\hat{i})$

(using λ for case (i), μ for case (ii) for differentiation)

$\lambda\hat{j} = (v_1 + 10)\hat{i} + v_2\hat{j}$

equating coefficients,

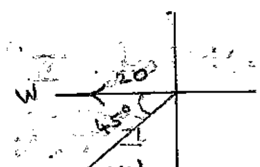
$\Rightarrow \boxed{\lambda = v_2}$, $v_1 + 10 = 0 \Rightarrow \boxed{v_1 = -10}$

case (ii):

R.V of wind w.r.t cyclist = T.V of wind − velocity of cyclist

$\mu(\cos 45\hat{i} + \sin 45\hat{j}) = (v_1\hat{i} + v_2\hat{j}) - (-20\hat{i})$

$\mu\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) = (v_1 + 20)\hat{i} + v_2\hat{j}$



$\frac{\mu}{\sqrt{2}}\hat{i} + \frac{\mu}{\sqrt{2}}\hat{j} = (v_1 + 20)\hat{i} + v_2\hat{j}$

$\Rightarrow v_1 + 20 = \frac{\mu}{\sqrt{2}} \Rightarrow -10 + 20 = \frac{\mu}{\sqrt{2}} \Rightarrow 10 = \frac{\mu}{\sqrt{2}}$ $[\because v_1 = -10]$

$\Rightarrow v_2 = \frac{\mu}{\sqrt{2}} = 10$ $\left[\because \frac{\mu}{\sqrt{2}} = 10\right] \Rightarrow \boxed{v_2 = 10}$

$\therefore v_1 = -10, v_2 = 10$

$\Rightarrow$ T.V of wind $= v_1\hat{i} + v_2\hat{j}$

$= -10\hat{i} + 10\hat{j}$

$\Rightarrow$ Mag of T.V of wind $= \sqrt{(-10)^2+(10)^2}$

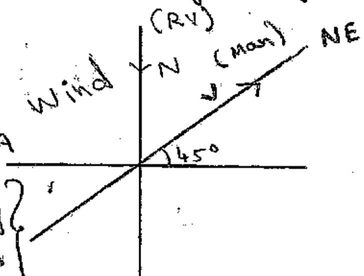
$= 10\sqrt{2}$

$\Rightarrow \tan\theta = \dfrac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \dfrac{10}{-10} = -1$

* i^{th} comp $\to$ -ve & j^{th} comp $\to$ +ve.
* It is in 2^{nd} quadrant
* Dir of T.V from southeast.

---

21. To a man travelling towards Northeast the wind appears to come from north. But when he doubled his speed, the wind appears him to come from a ~~distanc~~ direction inclined at an angle $\theta = \cot^{-1}(2)$ east of north. Find the true velocity of the wind.



Case (i):

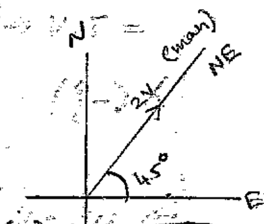
R.V of P w.r.t A = T.V of P - velocity of A

R.V of wind w.r.t Man $= \begin{Bmatrix}\text{T.V of} & - & \text{Velocity} \\ \text{wind} & & \text{of man}\end{Bmatrix}$

$\lambda(-\hat{j}) = (\omega_1\hat{i} + \omega_2\hat{j}) - v(\cos 45\hat{i} + \sin 45\hat{j})$

Equating, we get $\Rightarrow \omega_1 - \dfrac{v}{\sqrt{2}} = 0 \Rightarrow \boxed{\omega_1 = \dfrac{v}{\sqrt{2}}}$

$\boxed{\omega_2 - \dfrac{v}{\sqrt{2}} = -\lambda}$



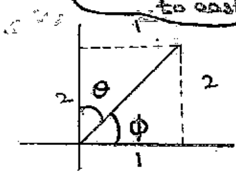
$\tan\phi = \dfrac{2}{1} = \dfrac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \dfrac{\omega_2\sqrt{2} - 2v}{\omega_1\sqrt{2} - 2v}$

Case (ii): dir of RV $= \theta = \cot^{-1}(2)$

$\cot\theta = 2/1$

$\tan\theta = 1/2$

East of North $\to$ from north to east



R.V. of wind w.r.t to man

= T.V. of wind - velocity of man

$\cancel{-\mu\left[\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}\right]} = [W_1\hat{i} + W_2\hat{j}] - 2v\left[\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right]$

$\omega_1\hat{i} - \frac{2v\hat{i}}{\sqrt{2}} + \omega_2\hat{j} - \frac{2v}{\sqrt{2}}\hat{j}$

$\cancel{-\frac{\mu}{\sqrt{5}}\hat{i} - \frac{2\mu}{\sqrt{5}}\hat{j}} =$

sub ω_1 value, we get it.

$\Rightarrow \cancel{-\dfrac{\mu}{\sqrt{5}} = W_1 - \dfrac{2v}{\sqrt{2}}} \Rightarrow \cancel{-\dfrac{\mu}{\sqrt{5}} = \dfrac{v}{\sqrt{2}} - \dfrac{2v}{\sqrt{2}}} \Rightarrow \boxed{\cancel{\dfrac{+\mu}{\sqrt{5}} = \dfrac{+v}{\sqrt{2}}}}$

$\Rightarrow \cancel{-\dfrac{2\mu}{\sqrt{5}} = \omega_2 - \dfrac{2v}{\sqrt{2}}} \Rightarrow \cancel{\dfrac{-2v}{\sqrt{2}} = \omega_2 - \dfrac{2v}{\sqrt{2}}} \Rightarrow \boxed{\cancel{\omega_2 = 0}}$

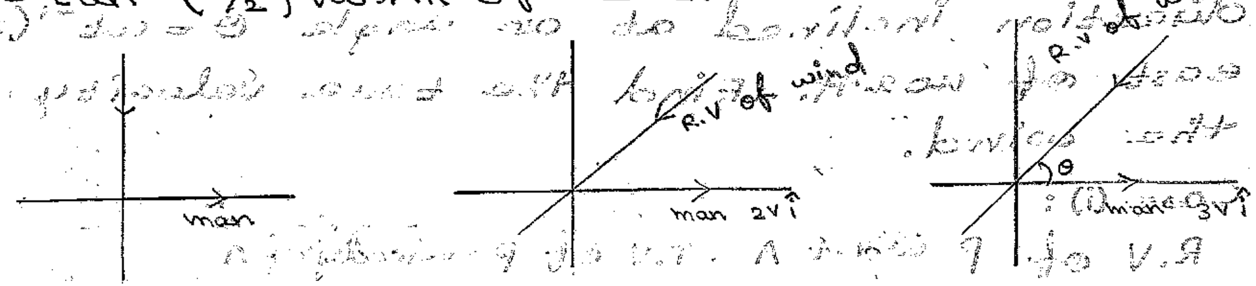
* True velocity of wind $= \omega_1\hat{i} + \omega_2\hat{j}$

$$= \frac{V}{\sqrt{2}}\hat{i} + 0 = \frac{V}{\sqrt{2}}\hat{i}$$

* mag of T.V of wind $= \sqrt{\frac{V^2}{(\sqrt{2})^2}} = \frac{V}{\sqrt{2}}$

* Dir. of T.V of wind = Eastern direction

22. A (10M) person travelling eastward finds the wind to blow from north. On doubling his speed, he finds it to come from N.E. ST if he trebles his speed the wind would appear to him to come from a direction making an angle $\theta = \tan^{-1}(1/2)$ north of east.



case (i) case (ii) case (iii)

Case (i):

R.V of P (wind) w.r.t A (person)

$= $ T.V of P − V of A

$\lambda(-\hat{j}) = (\omega_1\hat{i} + \omega_2\hat{j}) - V(\hat{i}) \Rightarrow$ ①

$\Rightarrow \omega_1 - V = 0$

$\omega_1 = V$

$\Rightarrow \omega_2 = -\lambda$

Case (ii):

R.V of P w.r.t A

$= $ T.V of P − V of A

$-\mu(\cos 45\hat{i} + \sin 45\hat{j}) = (\omega_1\hat{i} + \omega\hat{j}) - 2V\hat{i}$

$-\mu\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) = (\omega_1 - 2V)\hat{i} + \omega_2\hat{j}$

$\Rightarrow \frac{-\mu}{\sqrt{2}} = \omega_1 - 2V$

$\Rightarrow \frac{-\mu}{\sqrt{2}} = \omega_2$

$\omega_1 - 2V = \omega_2$

$V - 2V = \omega_2$

$-V = \omega_2$

Case (iii):

R.V of P w.r.t A

$= $ T.V of P − V of A

$-\eta(\cos\theta\hat{i} + \sin\theta\hat{j}) = (\omega_1\hat{i} + \omega_2\hat{j}) - 3V\hat{i}$

$\Rightarrow -\eta(\cos\theta) = \omega_1 - 3V$

$\Rightarrow -\eta\sin\theta = \omega_2$

Sub value of ω_1 &

$\omega_1 - 2V = \omega_2$

$V - 2V = \omega_2$

$\omega_2 = -V$

Sub it

$-\eta\cos\theta = \omega_1 - 3V = -$

$\tan\theta = \frac{-\eta\sin\theta}{-\eta\cos\theta} = \frac{\omega_2}{\omega_1 - }$

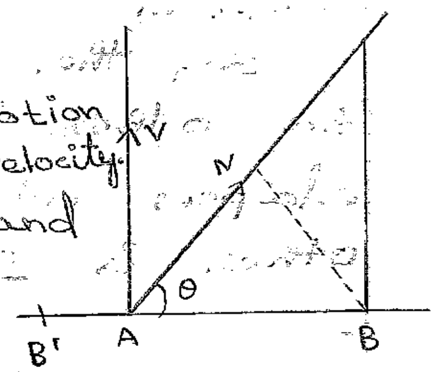
$$\tan\theta = \frac{\cancel{x}}{\cancel{2x}} \Rightarrow \tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}(1/2)$$

* Angle made by the dir. of RV of wind w.r.t x-axis.

* Direction makes an angle $\theta = \tan^{-1}(1/2)$ north of east (from east through an angle θ towards the northern direction.

~~Definition of Angular velocity~~

23. At a instant, if the distance b/w two moving points A and B is 'a', if v is their resultant velocity and if u and v are the component of v along and ⊥r to AB. ST the distance when they are nearest to each other is $\frac{av}{V}$ and the time elapses when they are nearest to each other is $\frac{au}{V^2}$



* When two particles are in motion only, we can consider the relative velocity.

* If one particle is at rest and other alone moves with speed of relative velocity along the direction of relative velocity from A

* Let the positions of the particles be at A and B, where AB = a

* The first particle moves along AP and the second particle stays at B.

* Let $\hat{i}, \hat{j}$ be the unit vectors along AB and ⊥r to AB', where AB' denotes the reversed velocity of B.

* Let AP makes an angle θ with the x-axis.

∡ R.V whose mag is V is given by

$$\cos\theta\,\hat{i} + \sin\theta\,\hat{j} = v\hat{i} + v\hat{j}$$

$\left(\cos\theta = \frac{u}{V} , \sin\theta = \frac{v}{V} \right.$

Draw BN $\perp$ to AP.

where A is at N, the two particles are nearest each other because the shortest distance b/w a line and a pt. in the $\perp$ distance and a moves along the line AP..

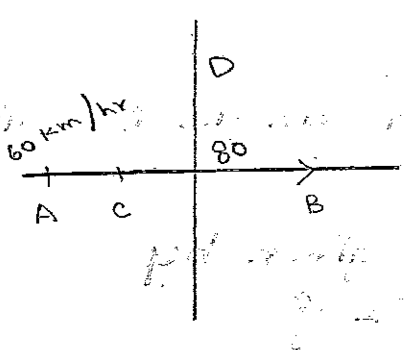
$$\frac{BN}{AB} = \sin\theta , BN = a \sin\theta = \frac{av}{V}$$

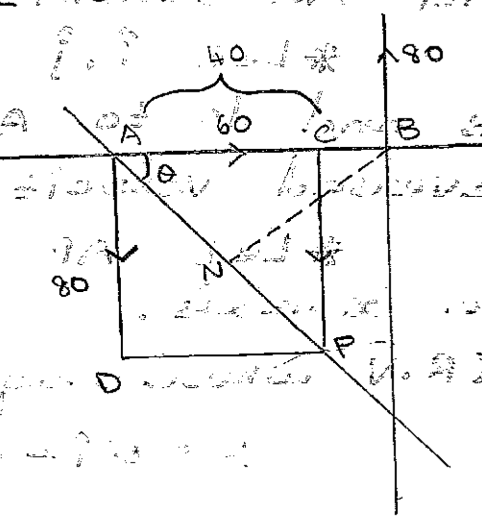
$\therefore$ The time taken to reach N by the first particle,

$$= \frac{\text{distance to be travelled}}{\text{speed}} = \frac{AN}{\text{mag. of R.V}} = \frac{a\cos\theta}{V} = \frac{au}{V^2}$$

So, the distance when they are nearest to other is $\frac{av}{V}$ and the time that elapses when they are nearest to each other is $\frac{au}{V^2}$.

24. The cars A and B are moving due east and due north at 60 km/hr and 80 km/hr respectively. At noon A is west of B at a distance 40 km, when are the cars nearest to each other and what is the distance b/w the other.





* $\overline{AC} \xrightarrow{\text{represents}}$ the velocity of A
* $\overline{AD} \xrightarrow{\text{"}}$ reversed velocity of B
* $\overline{AP} \longrightarrow$ resultant velocity

$\therefore \overline{AP}$ is the dis. of the relative velocity

* A moves along AP, Draw BN $\perp$r to AP.
* N $\xrightarrow{\text{be the}}$ position of the cars A when A & B are nearest to each other.
* The nearest distance b/w them = BN

$$\frac{BN}{AB} = \sin\theta \Rightarrow BN = AB\sin\theta = 40\sin\theta \quad [\because AB = 40$$

In ΔAPC, $AP^2 = AC^2 + CP^2$

$$AP^2 = 60^2 + 80^2 = 100^2$$

$$AP = 100$$

$$\sin\theta = \frac{80}{100} = \frac{4}{5} \quad \cos\theta = \frac{60}{100} = \frac{3}{5}$$

$$\therefore BN = 40 \times \frac{4}{5} = 32 \text{ km}$$

The minimum distance b/w the cars = 32 km

* The time taken by A to move from A to N

$$t = \frac{\text{distance travelled}}{\text{speed}} = \frac{AN}{\text{R.speed}} = \frac{AB\cos\theta}{100}$$

$$t = \frac{40}{100} \times \frac{3}{5} = \frac{6}{25} \text{ hr}$$

The cars are nearest after $6/25$ hr from 12 noon

$$\frac{6}{25} \text{ hr} = \frac{6}{25} \times 60 \text{ min} = \frac{72}{5} = 14.4 \text{ min}$$

1 min $\rightarrow$ 60 sec

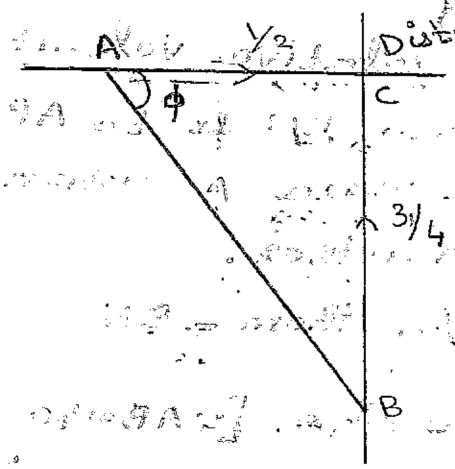
4 min $\rightarrow \frac{60 \times 4}{1} = 240$ seconds

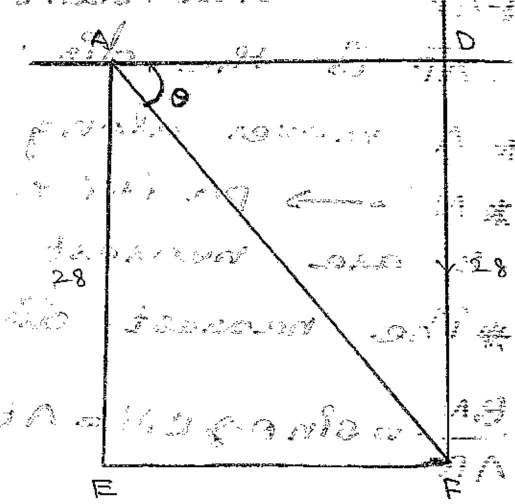
$\therefore$ They are nearest at 12 hr. 14 m. 240 seconds.

---

25. Two motor cars A, B are travelling along st. roads at right angles to each other with velocities of 21 km/hr and 28 km/hr towards C, the pt at which the roads cross.

If $AC = \frac{1}{2}$ km, $BC = \frac{3}{4}$ km, Find the S.D b/w them during the subsequent motion.





$\therefore AC = \frac{1}{2}$ km, $BC = \frac{3}{4}$ km (represents velocities)

$\Rightarrow AB^2 = AC^2 + BC^2 = \frac{1}{4} + \frac{9}{16} = \frac{4+9}{16} = \frac{13}{16} \Rightarrow AB = \frac{\sqrt{13}}{4}$

$$\sin\phi = \frac{3/4}{\sqrt{13}/4} = \frac{3}{\sqrt{13}}, \cos\phi = \frac{1/2}{\sqrt{13}/4} = \frac{2}{\sqrt{13}}$$

* $\overline{AD} \rightarrow$ the velocity of A $\Rightarrow |\overline{AD}| = 21$ km/h
* $\overline{AE} \rightarrow$ the reversed velocity of B $\Rightarrow |\overline{AE}| = 28$ km/h
* $\overline{AF} \rightarrow$ the R.V. of A w.r.t. B

$$AF^2 = AD^2 + AE^2 = 21^2 + 28^2$$

$$AF = \sqrt{(3\times 7)^2 + (4\times 7)^2}$$

$$= \sqrt{(9+16)7^2} = 5\times 7 = 35 \text{ km}$$

$|\overline{AF}| = 35$ km/hr is the mag. of relative veloci

$$\Rightarrow \sin\theta = \frac{28}{35} = \frac{4}{5}, \cos\theta = \frac{21}{35} = \frac{3}{5}$$

$$\sin\phi > \sin\theta$$

$$\frac{3}{\sqrt{13}} > \frac{4}{5}$$

Squaring,

$$\frac{9}{13} > \frac{16}{25}$$

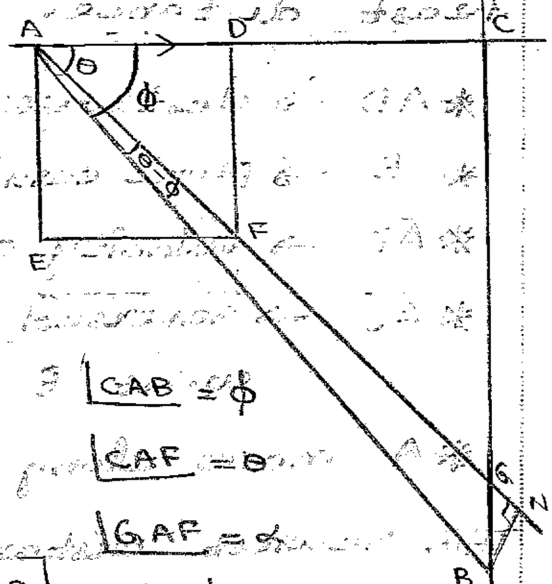
$$9\times 25 > 16\times 13$$

$225 > 208$ which is valid

As $\sin\phi > \sin\theta \Rightarrow \phi > \theta$

Combining the velocities & distances (remembering $\phi > \theta$ (combining fig 1 & fig 2 to draw)

* $A\phi \rightarrow$ dir. of R.V of A
* Draw BN $\perp$ Aϕ
* N $\rightarrow$ position of A when it is nearest to B.



$\rightarrow$ BN = S.D b/w A & B

$\angle CAB = \phi$

$\angle CAF = \theta$

$\angle GAF = \alpha$

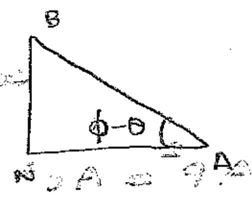
$\alpha = \phi - \theta$

$$\frac{BN}{AN} = \sin\alpha \Rightarrow BN = AB\sin\alpha$$

$$= \frac{\sqrt{13}}{4}\sin(\phi - \theta)$$

$$BN = \frac{\sqrt{13}}{4}\left[\sin\phi\cos\theta - \cos\phi\sin\theta\right]$$

$$= \frac{\sqrt{13}}{4}\left[\frac{3}{\sqrt{13}} \times \frac{3}{5} - \frac{2}{\sqrt{13}} \times \frac{4}{5}\right]$$



$$= \frac{\sqrt{13}}{4}\left[\frac{9}{5\sqrt{13}} - \frac{8}{5\sqrt{13}}\right]$$

$$= \frac{\sqrt{13}}{4} \times \frac{1}{5\sqrt{13}} = \frac{1}{20}\text{ km} = 0.05\text{ km}$$

missed b/w them = 0.05 km

The time taken when they are nearest is asked we proceed as follows

$$t = \frac{\text{distance travelled}}{\text{Relative speed}} = \frac{AN}{35} = \frac{AB\cos\alpha}{35}$$

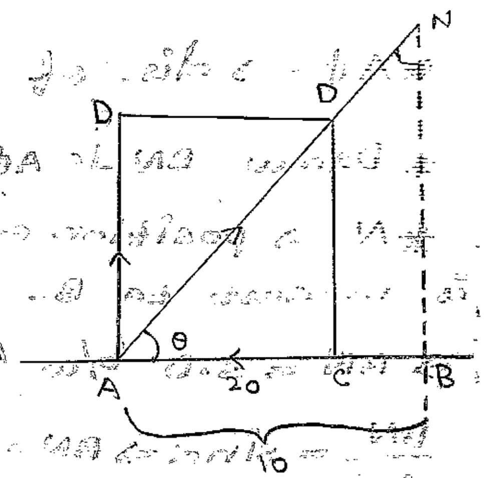
$$= \frac{\sqrt{13}}{4 \times 35}\cos(\phi - \theta) = \frac{\sqrt{13}}{140}\left[\cos\phi\cos\theta + \sin\phi\sin\theta\right]$$

$$= \frac{\sqrt{13}}{140}\left[\frac{2}{\sqrt{13}} \times \frac{3}{5} + \frac{3}{\sqrt{13}} \times \frac{4}{5}\right] = \frac{\sqrt{13}}{140}\left[\frac{6}{5\sqrt{13}} + \frac{12}{5\sqrt{13}}\right]$$

$$= \frac{\sqrt{13}}{140} \times \frac{18}{5\sqrt{13}} = \frac{9}{350}\text{ hr}$$

26 A destroyer streaming north at the rate of 15 km/hr observes a plane carries due east of itself at a distance of 10 km. If the batter is steaming due west at the rate of

20 km/hr after what time they are at the least distance from each and what is the least distance.



* AD $\xrightarrow{\text{be}}$ destroyer
* B $\xrightarrow{\text{be}}$ plane carrier
* $\overline{AD}$ → velocity of A
* $\overline{AC}$ → reversed velocity of A w.r.t B
* A moves along AP, Draw BN ⊥r AP

The nearest distance b/w them = BN

In ΔAPC, $\sin\theta = \frac{15}{25} = \frac{3}{5}$

$$\cos\theta = \frac{20}{25} = \frac{4}{5}$$

$$AP^2 = AC^2 + CP^2 = 20^2 + 15^2 = 400 + 225 = 625$$

$$AP = 25$$

$\therefore BN = AB\sin\theta = 10 \times 3/5 = 6\text{ km}$

* The minimum distance b/w A and B → 6 km
* The time taken by A to move A to N,

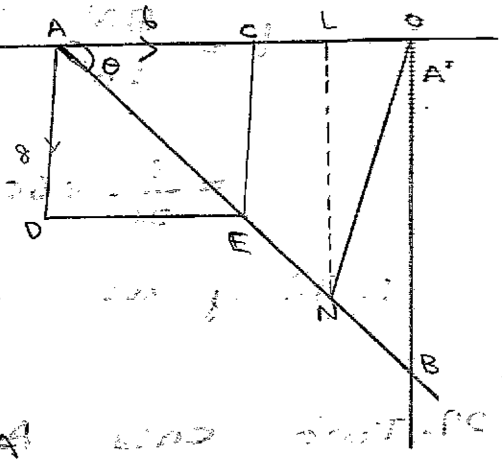
$$t = \frac{\text{distance travelled}}{\text{speed}} = \frac{AN}{\text{relative speed}}$$

$$= \frac{AB\cos\theta}{25} = \frac{10 \times 4/5}{25} = \frac{8}{25}\text{ hr}$$

∴ They are nearest at $8/25$ hrs.

27\. Two men A and A′ walk with velocities 6 km/hr and 8 km/hr along two st. roads which across at right angles at O. When A′ is at O, A is at a distance 100 m from O & walk towards O. ST they be nearest together, when A has walked 36 metres.

* $\overline{AO}$ & $\overline{BO}$ → dir. of velocities of A & A'.



* ST → directions are ⊥r when A' is at O.

AA' = AO = 100 metres

* $\overline{AD}$ → reversed velocity of A'
* $\overline{AE}$ → dir. of R.V of A' w.r.t A'
* A moves along the line AE
* Draw AN ⊥r AE

The distance AN, $\frac{AN}{OA} = \cos\theta \Rightarrow AN = OA\cos\theta = 100 \times \frac{3}{5}$

$= 60$ metres

* Draw NL ⊥r AO

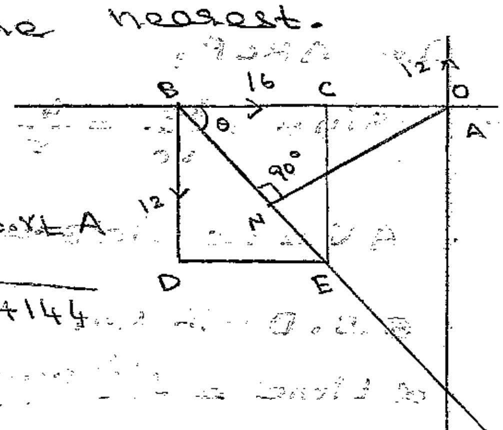
The distance AL, In ΔALN ⇒ $\frac{AL}{AN} = \cos\theta \Rightarrow AL = AN\cos\theta$

$= 60 \times 3/5$

$= 36$ m.

∴ They be nearest together to each other when A has walked 36 metres.

28. The courses of two ships A and B are towards north and east & their speeds are 12 and 16 knots respectively. At noon A is east of B and 10 ~~naut~~ nautical miles away. Find the time ~~along~~ when they are nearest.



* $\overline{BD}$ → reversed velocity of A.
* $\overline{BC}$ → velocity of B
* $\overline{BE}$ → relative velocity of B w.r.t A.

Mag. of the R.V $= \sqrt{16^2+12^2} = \sqrt{256+144}$

$= 20$ knots

* Draw AN ⊥r BE.
* N corresponds to the position when they are nearest.

Time taken to reach N is

$$t = \frac{\text{distance travelled}}{\text{R. speed}}$$

Note:
The unit for distance travelled in the sea is nautical mile.
* 1 nautical mile is travelled in 1 hr, we say the speed is 1 knot.

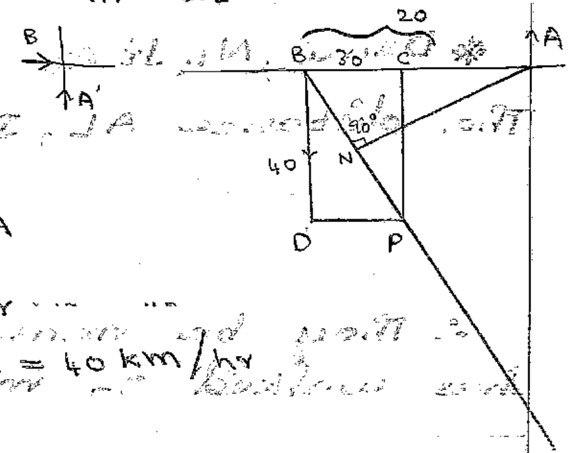
$$t = \frac{BN}{20} = \frac{OB\cos\theta}{20} = \frac{10 \times 16/20}{20} = \frac{8}{20}\ hr$$

$$= \frac{8}{20} \times 60\ min = 24\ min$$

∴ They are nearest 24 min after 12 o'clock.

---

29. Two cars A and B are moving due north and due east at 40 km/hr and 30 km/hr. At noon B is west of A at a distance of 20 km. When are the cars close to each other and what is the distance b/w them at the time.



$BA = 20\ km$

R.V of B w.r.t A = T.V of B

velocity of A

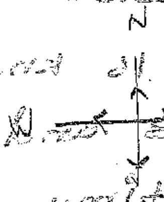
* $\overline{BC}$ → velocity of B = 30 km/hr
* $\overline{BD}$ → reversed velocity of A = 40 km/hr

* Draw AN ⊥ BP

In ΔABN,

$\sin\theta = \frac{AN}{AB} \Rightarrow AN = AB\sin\theta = 20\sin\theta$

$\cos\theta = \frac{BN}{AB} \Rightarrow BN = AB\cos\theta = 20\cos\theta$



In ΔBCP,

$\sin\theta = \frac{40}{50} = \frac{4}{5}$, $\cos\theta = \frac{30}{50} = \frac{3}{5}$

$AN = 20\sin\theta = 20 \times 4/5 = 16\ km$

S.D = 16 km

$$time = \frac{distance\ travelled}{speed} = \frac{BN}{BP}$$

$$t = \frac{20 \times 3/5}{50} = 12/50$$

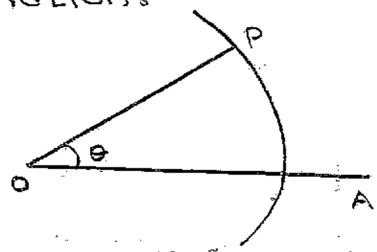
$$t = 6/25\ hrs$$

3/12/14
Tuesday

Unit - II:

1) **Definition of Angular velocity:**

* $P \rightarrow$ particle having a coplanar motion.
* $O \rightarrow$ fixed pt
* $OA \rightarrow$ fixed line in the plane of motion.



* Then the time-rate of change of the angle $A\hat{O}P$ is called the angular velocity of the particle about 'O'.

i.e) $A\hat{O}P = \theta$

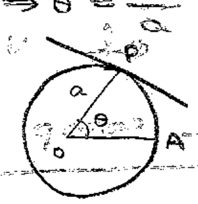
$\frac{d}{dt}(A\hat{O}P) = \frac{d\theta}{dt} = \dot{\theta}$ is the angular velocity of one particle P about O.

$\Rightarrow$ The unit $\xrightarrow{\text{will}}$ one radian / second

2) If the motion of the particle is a circle, the linear velocity $\rightarrow v$ and the angular velocity $\rightarrow \theta$ then W.K.T, $\dot{s} = a\dot{\theta}$ ($\because \hat{AP} = s = a\theta$ = radius × angle substended) $\dot{s} = a\dot{\theta}$

$\Rightarrow \vec{v} = \dot{s}\hat{T} = a\dot{\theta}\hat{T} = v\hat{v} \Rightarrow v = a\dot{\theta} \Rightarrow \dot{\theta} = \frac{v}{a} = \frac{\text{linear velocity}}{\text{radius}}$

formula, $\dot{\theta} = \frac{v}{a} = \frac{\text{linear velocity}}{\text{radius}}$



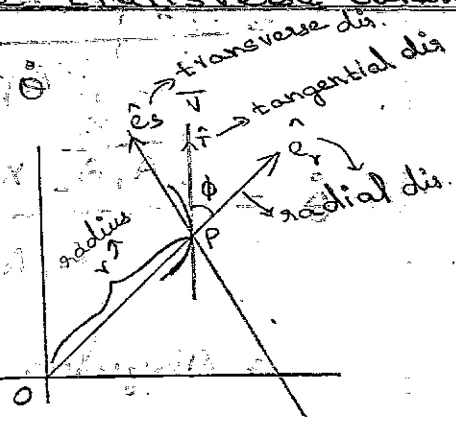
3) **Component of velocity v along the transverse direction**

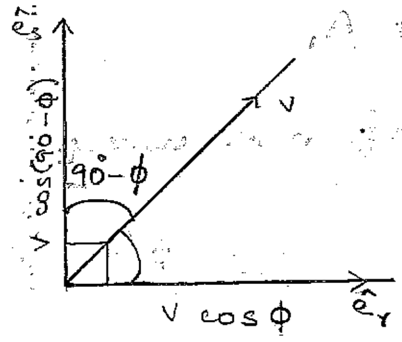
comp. of v along trans. dir. $= r\dot{\theta}$

$OP = r$

$v \rightarrow$ velocity at P

$\phi \rightarrow$ angle b/w OP & tgt at P





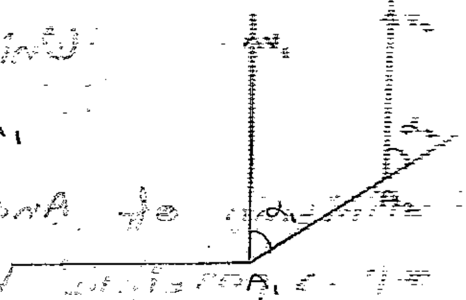
$\dot{\theta} = \frac{r\dot{\theta}}{r} = \frac{\text{the comp of velocity} \perp \text{r OP}}{OP}$ (Trans. dir)

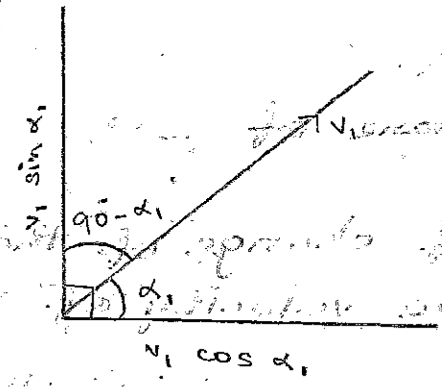
$\dot{\theta} = \frac{v\cos(90-\phi)}{r} = \frac{v\sin\phi}{r} = \frac{rv\sin\phi}{r^2}$

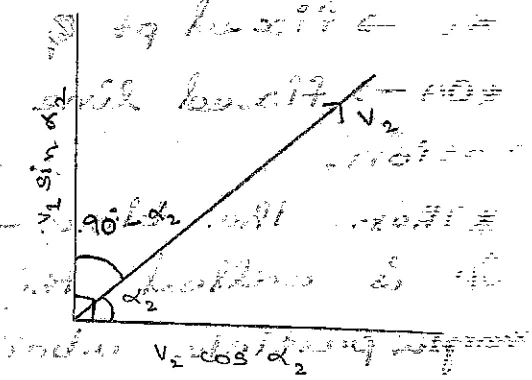
Angular Speed $= \dot{\theta} = \frac{|\vec{r} \times \vec{v}|}{r^2}$

Relative angular velocity :

* Velocity of A_2 relative to A_1 is $\overline{V_2}-\overline{V_1}$







* A_1 and A_2 → 2 particles → moving in plane
* $\overline{V_1}$ & $\overline{V_2}$ → velocities
* $\overline{V_1}$ makes an angle α_1 with line A_1A_2
* $\overline{V_2}$ makes an angle α_2 with line A_1A_2
* $\overline{V_1} \longrightarrow 90-\alpha_1$ with dir. $\perp^r$ to A_1A_2
* $\overline{V_2} \longrightarrow 90-\alpha_2$ with dir. $\perp^r$ to A_1A_2
* Comp of velocity V_1 $\perp^r$ to $A_1A_2 = V_1 \sin\alpha_1$
* Comp of velocity V_2 $\perp^r$ to $A_1A_2 = V_2 \sin\alpha_2$

→ $\dot{\theta} = \dfrac{\text{the comp of velocity } \perp^r A_1A_2}{A_1A_2}$

$$= \frac{v_2 \sin\alpha_2 - v_1 \sin\alpha_1}{A_1A_2}$$

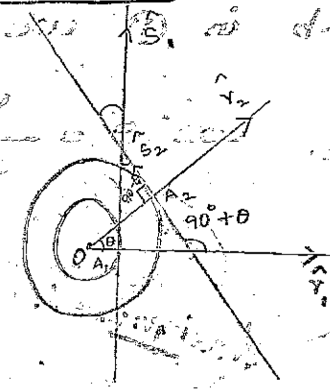
$$\boxed{\dot{\theta} = \frac{\left|\overline{A_1A_2} \times (\overline{V_2}-\overline{V_1})\right|}{A_1A_2^{\,2}}}$$

= Angular speed of A_2 about A_1

= Relative angular velocity of one w.r.t another.

Book work : 1

Two particles A_1, A_2 describe concentric circles of radii a_1, a_2 and centre O with speed v_1 and v_2. When the relative angular velocity of one particle about the other vanishes.



* $O \longrightarrow$ common centre.
* Particle $A_1 \longrightarrow$ circle with radius a_1 & the linear velocity $\overline{v_1}$.
* Particle $A_2 \longrightarrow$ circle with radius a_2 & the linear velocity $\overline{v_2}$.
* $\hat{r}_1$ & $\hat{r}_2 \longrightarrow$ unit vectors along the ~~tangents~~ A_1 & A_2 radii.
(i.e) along $\overline{OA_1}$ & $\overline{OA_2}$ $(\overline{OA_1} = a_1\hat{r}_1, \overline{OA_2} = a_2\hat{r}_2)$
* $\hat{s}_1$ & $\hat{s}_2 \longrightarrow$ unit vectors along the tangents A_1 & A_2, velocity is along the tgt. $(\overline{v_1} = v_1\hat{s}_1, \overline{v_2} = v_2\hat{s}_2)$
  pt
* Velocity is along the tangential direction

$$\angle A_1OA_2 = \theta$$

Angle b/w

$\hat{r}_1$ and $\hat{s}_1$ is $90°$

$\hat{r}_2$ and $\hat{s}_2$ is $90°$

$\hat{r}_1$ and $\hat{s}_2$ is $90°+\theta$

$\hat{r}_2$ and $\hat{s}_1$ is $90°-\theta$

$$\text{Gn :- } \dot{\theta} = \frac{|\overline{A_1A_2} \times (\overline{v_2} - \overline{v_1})|}{A_1A_2^2}$$

[Relative A.V of one particle about the other vanishes]

$$\Rightarrow |\overline{A_1A_2} \times (\overline{v_2} - \overline{v_1})| = \overline{0}$$

$$\overline{A_1A_2} \times (\overline{v_2} - \overline{v_1}) = \overline{0}$$

$$(\overline{OA_2} - \overline{OA_1}) \times (\overline{v_2} - \overline{v_1}) = \overline{0}$$

$$(a_2\hat{r}_2 - a_1\hat{r}_1) \times (v_2\hat{s}_2 - v_1\hat{s}_1) = \overline{0} \longrightarrow (1)$$

$|\bar{a}\times\bar{b}| = ab\sin\theta$

$\bar{a}\times\bar{b} = ab\sin\theta\,\hat{n}$

$\hat{e}_1\times\hat{e}_2 = \sin\theta\,\hat{n}$

$|\hat{e}_1\times e_2| = \sin\theta$

$\bar{r}_1\times\hat{s}_1 = \sin 90^\circ\,\hat{n} = \hat{n}$

$\bar{r}_2\times\hat{s}_2 = \sin 90^\circ\,\hat{n} = \hat{n}$

$\bar{r}_1\times\hat{s}_2 = \sin(90^\circ+\theta)\,\hat{n} = \cos\theta\,\hat{n}$

$\hat{r}_2\times\hat{s}_1 = \sin(90^\circ-\theta)\,\hat{n} = \cos\theta\,\hat{n}$

$(a_2\hat{r}_2 - a_1\hat{r}_1)\times(v_2\hat{s}_2 - v_1\hat{s}_1) = \bar{0} \rightarrow (1)$

$a_2v_2(\hat{r}_2\times\hat{s}_2) - a_2v_1(\hat{r}_2\times\hat{s}_1) - a_1v_2(\hat{r}_1\times\hat{s}_2) + a_1v_1(\hat{r}_1\times\hat{s}_1) = \bar{0}$

$a_2v_2\hat{n} - a_2v_1\cos\theta\,\hat{n} - a_1v_2\cos\theta\,\hat{n} + a_1v_1\hat{n} = \bar{0}$

$\hat{n}[a_2v_2 + a_1v_1 - \cos\theta(a_1v_2 + a_2v_1)] = \bar{0}$

$a_2v_2 + a_1v_1 - \cos\theta(a_1v_2 + a_2v_1) = 0$

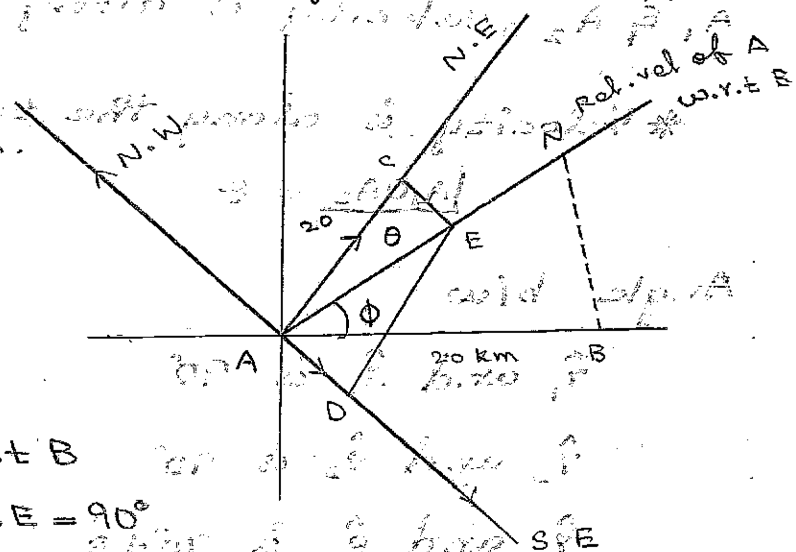
sub in (1) we get

$$\cos\theta = \frac{a_1v_1 + a_2v_2}{a_1v_2 + a_2v_1}$$

Assignment

Unit - I (continuation)

30. At a given instant one steamer is 20 km west of another. The first travels north-east 20 km/hr and the second north-west at the rate of 16 km/hr. Find the distance of nearest approach between them.



* $\overline{AC}$ → velocity of A.

→ $\overline{AC} = 20$ km/hr

* $\overline{AD}$ → reverse the velocity of B.

→ $\overline{AD} = 16$ km/hr

* $\overline{AE}$ → R.V of A w.r.t B
* Angle b/w N.E & S.E = 90°
* Draw BN ⊥r to the dir. of R.V

BN is the reqd distance of nearest approach b/w A & B → $\frac{BN}{AB} = \sin\phi$

$$BN = AB\sin\phi$$

$$BN = 20\sin\phi \rightarrow (1)$$

To find φ:

$$\phi = C\hat{A}B - C\hat{A}E = 45^\circ - \theta$$

$$\sin\phi = \sin(45^\circ - \theta)$$

$$\sin\phi = \sin 45^\circ\cos\theta - \cos 45^\circ\sin\theta \rightarrow (2)$$

In ΔCAE,

$$AE = \sqrt{20^2+16^2} = \sqrt{400+256} = \sqrt{656}$$

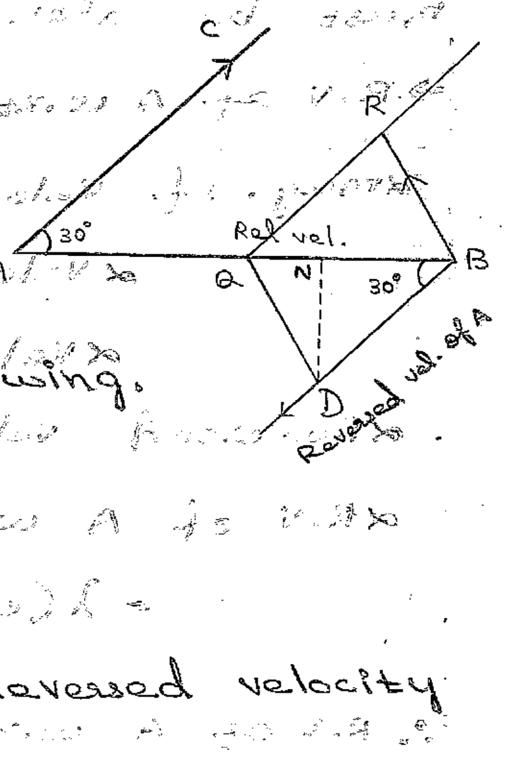
$$\cos\theta = \frac{20}{\sqrt{656}},\ \sin\theta = \frac{16}{\sqrt{656}}$$

Sub. in (2),

$$\sin\phi = \frac{1}{\sqrt{2}}\frac{20}{\sqrt{656}} - \frac{1}{\sqrt{2}}\frac{16}{\sqrt{656}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{656}}(20-16) = \frac{4}{\sqrt{2}\sqrt{656}}$$

$$\Rightarrow BN = 20 \times \frac{4}{\sqrt{2}\sqrt{656}} = \frac{20\times 4}{\sqrt{2}\sqrt{656}} = \frac{20\times 2}{2\sqrt{82}} = \frac{20}{\sqrt{82}}\text{ km}$$

31. A ship A streaming in the direction 30° north of east at 30 knots sees another ship B at 20 nautical miles in the east. Find the minimum speed of B so that it may intercept at A. [1 knot is the speed of 1 nautical mile/hr]



* Reduce A to be at rest
* As B has to meet A, the dir. of R.V of B w.r.t A must be along BA. Let it be $\overline{BQ}$
* If $\overline{BD}$ is the reversed velocity of A then we have the following.

$|\overline{BD}|$ = mag. of velocity of A

= 30 knots

* $\overline{AC}$ is the velocity of A.
* At B, $\overline{BD}$ represents the reversed velocity of A.
* $\overline{BR}$ must be velocity of B
* Complete the parallelogram with $\overline{BD}$ & $\overline{BR}$ as adjacent sides, $\overline{BQ}$ becomes R.V of B w.r.t A.
* The minimum velocity of B is required.
* Minimum of $|\overline{DQ}|$ corresponds to the $\perp$r distance for $|\overline{DQ}|$ is minimum only $\overline{DQ} \perp$r $\overline{BA}$

Draw DN $\perp^r$ BR

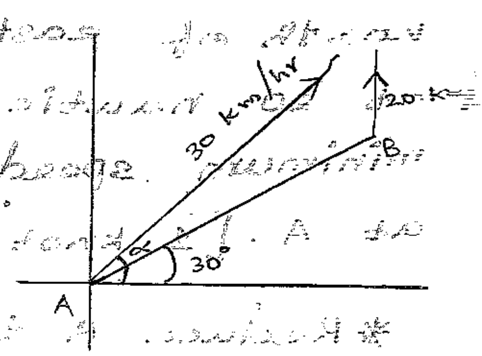
$$\frac{DN}{BD} = \sin 30^\circ$$

$$DN = BD \sin 30^\circ = 30 \times \frac{1}{2} = 15 \text{ knots}$$

∴ The minimum speed of B so that B may intercept A is 15 knots.

---

32. A cruiser (A) which can steam at 30 knots receives a report that an enemy vessel (B) steaming due north at 20 knots is at 29 nautical miles away in a direction 30° N of East. S.T the cruiser can overtake the vessel in almost 2 hrs.



* If A has to overtake B, then A has to move along AB.
i.e) dir. of R.V of A w.r.t B must be along AB.

→ R.V of A w.r.t B = True velocity of A − velocity of B

mag. of velocity of A = 30 knots

Velocity of A $= 30(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}) \to (1)$

Velocity of B $= 20\,\hat{j}$

Reversed velocity of B $= -20\,\hat{j} \to (2)$

R.V of A w.r.t B where λ is the mag of R.V

$= \lambda(\cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j})$

∴ R.V of A w.r.t B = T.V of A − V of B

$$\lambda(\cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j}) = 30(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}) - 20\,\hat{j}$$

$$= (30\cos\alpha)\,\hat{i} + (30\sin\alpha - 20)\,\hat{j} \to (3)$$

→ As A has to overtake B, it has to cover a distance of 29 miles in time t with the speed of relative velocity along AB.

Distance travelled = speed × time

$$\left[30\cos\alpha\,\hat{i} + (30\sin\alpha - 20)\hat{j}\right]t = 29\cos 30^\circ\,\hat{i} + 29\sin 30^\circ\,\hat{j}$$

- $(30\cos\alpha)t = 29\cos 30^\circ = \frac{29\sqrt{3}}{2} \rightarrow (4)$
- $(30\sin\alpha - 20)t = \frac{29}{2}$

$$(30\sin\alpha)t = \frac{29}{2} + 20t \rightarrow (5)$$

Squaring & adding (4) & (5)

$$900t^2(\cos^2\alpha + \sin^2\alpha) = 29^2 \times \frac{3}{4} + \frac{29^2}{4} + 2 \times \frac{29}{2} \times 20t + 400t^2$$

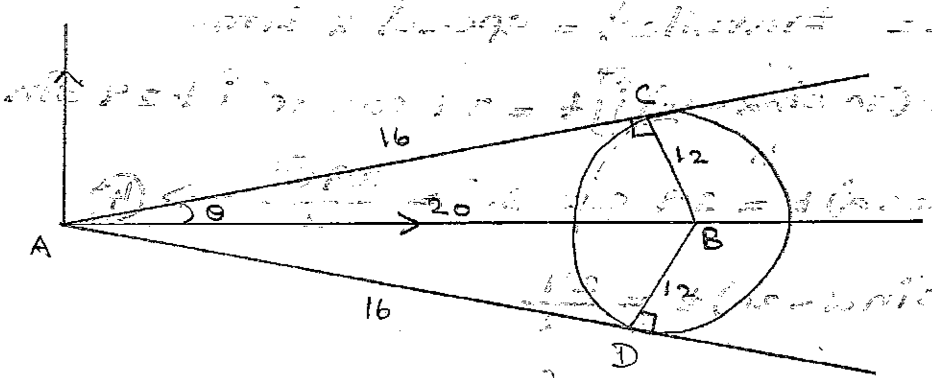
$$900t^2 = \boxed{29^2 + 29 \times 20t + 400t^2}$$

$$500t^2 = 580t + 841$$

$$500t^2 - 580t - 841 = 0$$

33. A ship A observes another ship B, 20 km due east of A. B is steaming north at 24 km/hr. After 24 minutes if the ships are found to be nearest to each other at a distance of 12 km, find the velocity of A.

The path of the Relative motion of A will be a st. line at a ⊥r distance 12 km from B. So, the path will touch the circle with centre at B and radius 12. So, there are 2 paths which are tangential to the circle.



$\Rightarrow AB = 20\ km$

$\Rightarrow AC$ & AD are two paths

$AC = \sqrt{AB^2 - CB^2} = \sqrt{20^2 - 12^2} = \sqrt{400-144} = 16$

$\boxed{AC = AD = 16\ km}$

$\cos\theta = \frac{16}{20} = \frac{4}{5}$, $\sin\theta = \frac{12}{20} = \frac{3}{5}$

We have to find the velocity of A.

Case (i): (A moves along AC)

* Actual velocity be $u_1\hat{i} + v_1\hat{j}$
* Actual velocity of $B = 24\hat{j}$
* mag. of R.V of $A = v_1$
* dir. makes an angle θ with x-axis

R.V of A w.r.t B = T.V of A − vel. of B

$$v_1 \hat{AC} = (u_1\hat{i} + v_1\hat{j}) - 24\hat{j}$$

$$v_1(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = (u_1\hat{i} + v_1\hat{j}) - 24\hat{j}$$

$$v_1\left(\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right) = u_1\hat{i} + (v_1 - 24)\hat{j}$$

• $t = \frac{AC}{v_1} = \frac{\text{distance}}{\text{speed}}$ [∵ dis. AC travelled with the speed of R.V]

time = 24 min (i.e) $\frac{24}{60}$ hrs

$\Rightarrow \frac{24}{60} = \frac{16}{v_1}$

$\frac{2}{5} = \frac{16}{v_1}$

$\boxed{v_1 = 40}$

Sub. the value of v_1,

$$40\left(\frac{4}{5}\hat{i}+\frac{3}{5}\hat{j}\right) = u_1\hat{i} + (v_1-24)\hat{j}$$

$$32\hat{i}+24\hat{j} = u_1\hat{i}+(v_1-24)\hat{j}$$

$\boxed{u_1 = 32}$, $v_1 - 24 = 24 \Rightarrow \boxed{v_1 = 48}$

$\therefore$ Velocity of $A = 32\hat{i}+48\hat{j}$

Its mag $= \sqrt{32^2+48^2} = \sqrt{16\times2\times16\times2+16\times3+16\times3}$

$= 16\sqrt{4+9} = 16\sqrt{13}$

- $\tan\theta = \dfrac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \dfrac{48}{32} = \dfrac{3}{2}$

$\therefore \tan^{-1}(3/2)$ north of east.

Case (ii) :- (AD be Relative path of A)

* T.V be $u_2\hat{i}+v_2\hat{j}$

$\rightarrow \hat{AD} = \cos(-\theta)\hat{i}+\sin(-\theta)\hat{j}$ (line AD makes an angle θ

$= \cos\theta\,\hat{i}-\sin\theta\,\hat{j}$ b/o below the x-axis)

* $V_2 \rightarrow$ mag of R.V.

then $u_2\hat{i}+(v_2-24)\hat{j} = V_2(\cos\theta\,\hat{i}-\sin\theta\,\hat{j})$

$$= 40\left(\frac{4}{5}\hat{i}-\frac{3}{5}\hat{j}\right)$$

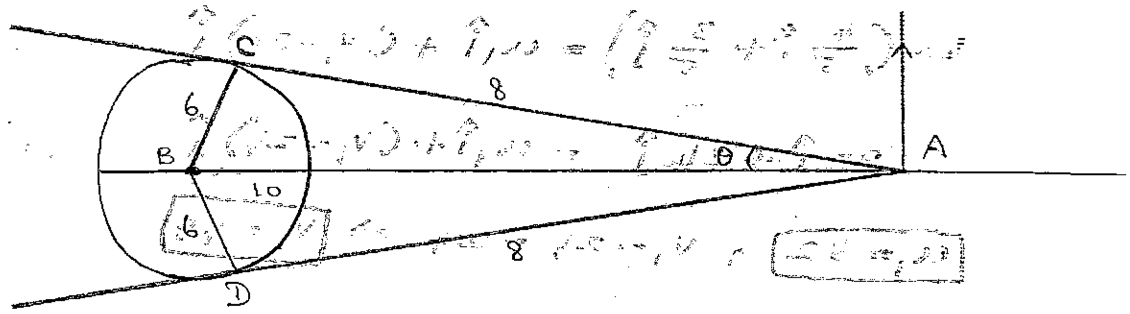
$$= 32\hat{i}-24\hat{j}$$

$\boxed{u_2 = 32}$, $v_2 - 24 = -24 \Rightarrow \boxed{v_2 = 0}$

$\therefore$ The velocity of $A = 32\hat{i}$

$= 32$ km/hr eastward

34. A ship A observes another ship B, 10 km due ~~north~~ east of A. B is steaming north at 12 km/hr. After 24 minutes ~~the ships~~ are found to be the nearest at a distance 6 km apart. Find the velocity of A.



The path of the relative motion of A will be a st. line at a distance 6 km from B. So, the path will touch the circle with centre at B and radius 6. So, there are 2 such paths which are tangential to the circle.

$\Rightarrow AB = 10$ km.

$\Rightarrow$ AC & AD are 2 paths

$$AC = \sqrt{AB^2 - CD^2} = \sqrt{10^2 - 6^2} = \sqrt{100-36} = 8$$

$$AC = AD = 8 \text{ km}$$

$$\cos\theta = \frac{8}{10} = \frac{4}{5}, \quad \sin\theta = \frac{6}{10} = \frac{3}{5}$$

We have to find the velocity of A.

Case (i):

R.V of A w.r.t B = T.V of A − vel. of B

$$v_1 \hat{AC} = (u_1\hat{i} + v_1\hat{j}) - 12\hat{j}$$

$$v_1(\cos\theta\hat{i} + \sin\theta\hat{j}) = u_1\hat{i} + (v_1 - 12)\hat{j}$$

$$v_1\left(\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right) = u_1\hat{i} + (v_1 - 12)\hat{j}$$

$$\text{time} = \frac{AC}{v_1}$$

$$\frac{24}{60} = \frac{8}{v_1} \quad \left[\because t = 24 \text{ min} \Rightarrow t = \frac{24}{60} \text{ hr}\right]$$

$$\frac{2}{5} = \frac{8}{v_1}$$

$$\boxed{v_1 = 20}$$

Sub the value, $20\left(\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right) = u_1\hat{i} + (v_1 - 12)\hat{j}$

$$16\hat{i}+12\hat{j} = u_1\hat{i}+(v_1-12)\hat{j}$$

$$u_1 = 16,\ v_1-12 = 12 \Rightarrow v_1 = 24$$

$\therefore$ the velocity of $A = 16\hat{i}+24\hat{j}$

$\Rightarrow$ Its mag. $= \sqrt{16^2+24^2} = \sqrt{4\times4\times4\times4+6\times4\times6\times4}$

$$= 4\sqrt{16+36} = 4\sqrt{52}$$

$\Rightarrow \tan\theta = \dfrac{j^{th}\text{ comp}}{i^{th}\text{ comp}} = \dfrac{24}{16} = \dfrac{12}{8} = \dfrac{3}{2}$

$\therefore \tan^{-1}(3/2)$ north of east

Case (ii):

$$T.V = u_2\hat{i}+v_2\hat{j}$$

$\Rightarrow$ unit velocity along AD is $\hat{AD} = \cos(-\theta)\hat{i}+\sin(-\theta)\hat{j}$

$= \cos\theta\hat{i}-\sin\theta\hat{j}$

R.V. of A w.r.t B = T.V. of A − Vel. of B

$$v_2(\cos\theta\,\hat{i}-\sin\theta\,\hat{j}) = (u_2\hat{i}+v_2\hat{j}) - 12\hat{j}$$

$$20\left(\frac{4}{5}\hat{i}-\frac{3}{5}\hat{j}\right) = u_2\hat{i}+(v_2-12)\hat{j}$$

- $u_2 = 20\cdot\dfrac{4}{5} = 16$ & $v_2-12 = -20\cdot\dfrac{3}{5} = -12$

$\boxed{u_2 = 16}$ $\boxed{v_2 = 0}$

$\therefore$ The velocity of $A = 16\hat{i}$

$= 16$ km/hr eastward.

10/12/14 Unit − II (continuation)

Note:

When the relative angular velocity of one particle w.r.t another does not vanish then.

$$\dot{\theta} = \frac{|\overline{A_1A_2}\times(\overline{v_2}-\overline{v_1})|}{A_1A_2^2} \neq 0$$

$$\dot{\theta} = \frac{|(\overline{OA_2}-\overline{OA_1})\times(\overline{v_2}-\overline{v_1})|}{A_1A_2^2}$$

$$\dot{\theta} = \frac{|(a_2\hat{r}_2-a_1\hat{r}_1)\times(v_2\hat{s}_2-v_1\hat{s}_1)|}{A_1A_2^2}$$

$$\dot{\theta} = \frac{|a_2 v_2 (\hat{r}_2 \times \hat{s}_2) - a_2 v_1 (\hat{r}_2 \times \hat{s}_1) - a_1 v_2 (\hat{r}_1 \times \hat{s}_2) + a_1 v_1 (\hat{r}_1 \times \hat{s}_1)|}{A_1 A_2^2}$$

$$\dot{\theta} = \frac{a_2 v_2 |\hat{r}_2 \times \hat{s}_2| - a_2 v_1 |\hat{r}_2 \times \hat{s}_1| - a_1 v_2 |\hat{r}_1 \times \hat{s}_2| + a_1 v_1 |\hat{r}_1 \times \hat{s}_1|}{A_1 A_2^2}$$

$$\dot{\theta} = \frac{a_2 v_2 \sin 90^\circ - a_2 v_1 \cos\theta - a_1 v_2 \cos\theta + a_1 v_1 \sin 90^\circ}{A_1 A_2^2}$$

$$\boxed{\dot{\theta} = \frac{(a_1 v_1 + a_2 v_2) - (a_2 v_1 + a_1 v_2)\cos\theta}{A_1 A_2^2}}$$

11. Two planets describe nearly circles radii a_1, a_2 round the sun as centre with speed varying inversely as the square roots of the radii. S.T their relative angular velocity vanishes when the angle b/w the radii to these planets is

$$\cos^{-1}\left(\frac{\sqrt{a_1 a_2}}{a_1 - \sqrt{a_1 a_2} + a_2}\right)$$

1. Figure

2. To prove :→ $\cos A_1\hat{O}A_2 = \frac{\sqrt{a_1 a_2}}{(a_1 - \sqrt{a_1 a_2} + a_2)}$

3. Let $A_1\hat{O}A_2 = \theta$

4. The Relative angular velocity of A_2 w.r.t A_1

$$\boxed{\dot{\theta} = \frac{|\overline{A_1 A_2} \times (\overline{v_2} - \overline{v_1})|}{A_1 A_2^2}}$$

5. Gn that Relative A.V vanishes

$$\Rightarrow |\overline{A_1 A_2} \times (\overline{v_2} - \overline{v_1})| = \overline{0}$$

$$\Rightarrow \boxed{\cos\theta = \frac{a_1 v_1 + a_2 v_2}{a_1 v_2 + a_2 v_1}} \longrightarrow (1)$$

6. Let a_1 & a_2 → radii of 2 circles
& v_1 & v_2 → velocities

7. Gn : $v_1 \propto \frac{1}{\sqrt{a_1}}$, & $v_2 \propto \frac{1}{\sqrt{a_2}}$

⇒ $\boxed{v_1 = \frac{k}{\sqrt{a_1}}}$, $\boxed{v_2 = \frac{k}{\sqrt{a_2}}}$

8. Sub in ①,

$$\cos\theta = \frac{a_1 \cdot \frac{k}{\sqrt{a_1}} + a_2 \cdot \frac{k}{\sqrt{a_2}}}{a_1 \cdot \frac{k}{\sqrt{a_2}} + a_2 \cdot \frac{k}{\sqrt{a_1}}}$$

$$= \frac{\sqrt{a_1} + \sqrt{a_2}}{\frac{a_1}{\sqrt{a_2}} + \frac{a_2}{\sqrt{a_1}}}$$

$$= \frac{(\sqrt{a_1} + \sqrt{a_2})\sqrt{a_1 a_2}}{a_1\sqrt{a_1} + a_2\sqrt{a_2}}$$

$$= \frac{(\sqrt{a_1} + \sqrt{a_2})\sqrt{a_1 a_2}}{a_1^{3/2} + a_2^{3/2}}$$

$a_1^{3/2} + a_2^{3/2} = (\sqrt{a_1})^3 + (\sqrt{a_1})^3$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$$= \frac{(\sqrt{a_1} + \sqrt{a_2})\sqrt{a_1 a_2}}{(\sqrt{a_1})^3 + (\sqrt{a_2})^3}$$

$\sqrt{a_1}^3 + \sqrt{a_2}^3 = (\sqrt{a_1} + \sqrt{a_2})(\sqrt{a_1}^2 - \sqrt{a_1}\sqrt{a_2} + \sqrt{a_2}^2)$

$$= \frac{\cancel{(\sqrt{a_1} + \sqrt{a_2})}\sqrt{a_1 a_2}}{(\sqrt{a_1} + \sqrt{a_2})(a_1 - \sqrt{a_1 a_2} + a_2)}$$

$$= \frac{\sqrt{a_1 a_2}}{(a_1 - \sqrt{a_1 a_2} + a_2)}$$

$$\theta = \cos^{-1}\left[\frac{\sqrt{a_1 a_2}}{(a_1 - \sqrt{a_1 a_2} + a_2)}\right]$$

2. Points A_1 and A_2 describe concentric circles of radii a_1 and a_2 with speeds varying inversely as the radii, S.T their relative velocity is $\parallel^{el}$ $\xrightarrow{\text{vanishes}}$ to the line A_1A_2 when the angle b/w the radii through A_1 and A_2 is $\cos^{-1}\left(\frac{2a_1a_2}{a_1^2+a_2^2}\right)$ 5

● 1 ..... 6 → same as previous problem.

7. Gn: $V_1 \propto \frac{1}{a_1}$, $V_2 \propto \frac{1}{a_2}$

$\Rightarrow V_1 = \frac{k}{a_1}$, $V_2 = \frac{k}{a_2}$

5) Given R. Motion is $\parallel^{el}$ to the line A_1A_2
⇒ R.A.V vanishes
⇒ same as previous but in 5th point

8. Sub in ①,

$$\cos\theta = \frac{\frac{a_1k}{a_1}+\frac{a_2k}{a_2}}{\frac{a_1k}{a_2}+\frac{a_2k}{a_1}} = \frac{2k}{\left(\frac{a_1^2k+a_2^2k}{a_1a_2}\right)} = \frac{2k}{\frac{k(a_1^2+a_2^2)}{a_1a_2}}$$

$$\cos\theta = \frac{2a_1a_2}{a_1^2+a_2^2}$$

$$\theta = \cos^{-1}\left(\frac{2a_1a_2}{a_1^2+a_2^2}\right)$$

3. Two particles A_1, A_2 describe concentric circles of radii a_1, a_2 with angular speeds ω_1, ω_2 about the common centre O. ST when the R.A.V of one particle about other vanishes then $\cos A_1\hat{O}A_2 = \frac{a_1^2\omega_1+a_2^2\omega_2}{a_1a_2(\omega_1+\omega_2)}$

● 1 ..... 6 → same as before one.

7. W.K.T, $\dot{\theta} = \frac{v}{a} \Rightarrow v = a\dot{\theta}$

⇒ linear velocity = radius × ang.

$\therefore V_1 = a_1\dot{\theta}_1 = a_1\omega_1$,

$V_2 = a_2\dot{\theta}_2 = a_2\omega_2$

$\omega_1, \omega_2 \rightarrow$ angular velocities

8. Sub in ①,

$$\cos\theta = \frac{a_1^2\omega_1 + a_2^2\omega_2}{a_1a_2\omega_2 + a_2a_1\omega_1}$$

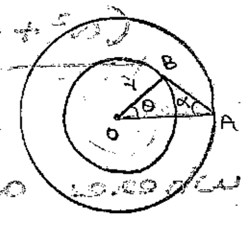
$$\cos\theta = \frac{a_1^2\omega + a_2^2\omega_2}{a_1a_2(\omega_1+\omega_2)}$$

$$\cos A_1\hat{O}A_2 = \frac{a_1^2\omega_1 + a_2^2\omega_2}{a_1a_2(\omega_1+\omega_2)}$$

4. Two ~~partic~~ points A_1 and A_2 describe concentric circles

4. Two points A and B move with speeds v and 2v along two concentric circles with centre at O and radii 2r & r respectively. If $\angle OAB = \alpha$. ST $\cot\alpha = 2$, when their R. Motion is along AB.

1. Figure

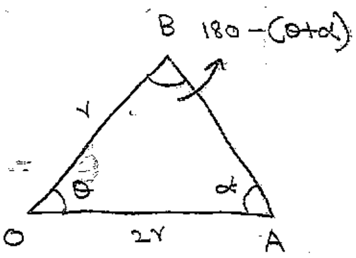


2. To prove $\cot\alpha = 2$
3. Let $\angle AOB = \theta$
4. Same in previous
5. Gn that R. Motion is along AB

$\Rightarrow$ R.A.V vanishes

$\Rightarrow |\overline{A_1A_2} \times (\overline{V_2} - \overline{V_1})| = \overline{0}$

$\Rightarrow \cos\theta = \dfrac{a_1v_1 + a_2v_2}{a_1v_2 + a_2v_1} \longrightarrow$ ①



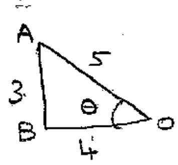
6. Same as previous
7. Gn : $a_1 = 2r$, $a_2 = r$

$v_1 = v$, $v_2 = 2v$

8. Sub in ①,

$\sin\theta = \dfrac{opp}{adj}$

$$\cos\theta = \frac{(2r)v + r(2v)}{(2r)(2v) + rv} = \frac{4rv}{5rv} = \frac{4}{5}, \ \sin\theta = \frac{3}{5}$$



From ΔOAB, Apply sine formula

$$\frac{OB}{\sin\alpha} = \frac{OA}{\sin[180^\circ - (\theta+\alpha)]} \rightarrow \frac{r}{\sin\alpha} = \frac{2r}{\sin(\theta+\alpha)}$$

$$\sin(\theta+\alpha) = 2\sin\alpha$$

$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = 2\sin\alpha$$

$\div \sin\alpha \Rightarrow \sin\theta\cot\alpha + \cos\theta = 2$

$$\frac{3}{5}\cot\alpha + \frac{4}{5} = 2$$

$$\frac{3}{5}\cot\alpha = 2 - \frac{4}{5} = \frac{6}{5}$$

$$\cot\alpha = \frac{6}{5} \times \frac{5}{3} = 2$$

$$\boxed{\cot\alpha = 2}$$

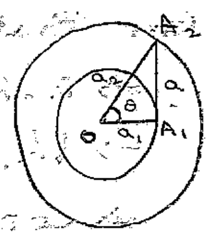
5M 19. Two points A_1 and A_2 describe concentric circles of radii a_1 and a_2 with A. Velocities ω_1 and ω_2 respectively. S.T the R.A.V of one about the other is

$$\frac{(a^2+a_1^2-a_2^2)\omega_1 + (a^2+a_2^2-a_1^2)\omega_2}{2a^2}$$

where a is the distance b/w them.

The R.A.V of A_2 w.r.t A_1

$$\dot\theta = \frac{|\overline{A_1A_2} \times (\overline{V_2}-\overline{V_1})|}{A_1A_2}$$



$$\dot\theta = \frac{|(\overline{OA_2}-\overline{OA_1}) \times (\overline{V_2}-\overline{V_1})|}{A_1A_2^2}$$

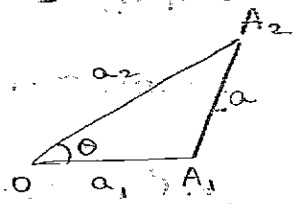
$$\Rightarrow \dot\theta = \frac{|(a_2\hat{r}_2 - a_1\hat{r}_1) \times (v_2\hat{s}_2 - v_1\hat{s}_1)|}{a^2}$$

$$= \frac{|a_2v_2(\hat{r}_2\times\hat{s}_2) + (a_1v_1)(\hat{r}_1\times\hat{s}_1) - a_1v_2(\hat{r}_1\times\hat{s}_2) - a_2v_1(\hat{r}_2\times\hat{s}_1)|}{a^2}$$

$$= \frac{|(a_2v_2+a_1v_1)\hat{k} - a_1v_2\sin(90^\circ+\theta)\hat{k} - a_2v_1\sin(90^\circ-\theta)\hat{k}|}{a^2}$$

$$= \frac{(a_1v_1+a_2v_2) - (a_1v_2+a_2v_1)\dfrac{(a_1^2+a_2^2-a^2)}{2a_1a_2}}{a^2}$$

Apply cosine formula to the ΔOA_1A_2



$$a^2 = a_1^2 + a_2^2 - 2a_1a_2\cos\theta$$

$$\cos\theta = \frac{a_1^2 + a_2^2 - a^2}{2a_1a_2}$$

$$\therefore \dot{\theta} = \frac{(a_1v_1 + a_2v_2) - (a_1v_2 + a_2v_1)\cos\theta}{a^2}$$

Sub $v_1 = a_1\omega_1$, $v_2 = a_2\omega_2$

$$\dot{\theta} = \frac{(a_1a_1\omega_1 + a_2a_2\omega_2) - (a_1a_2\omega_2 + a_2a_1\omega_1)\dfrac{(a_1^2 + a_2^2 - a^2)}{2a_1a_2}}{a^2}$$

$$= \frac{(a_1^2\omega_1 + a_2^2\omega_2) - (\omega_2 + \omega_1)\cancel{(a_1a_2)}\dfrac{(a_1^2 + a_2^2 - a^2)}{2\cancel{(a_1a_2)}}}{a^2}$$

$$= \frac{a_1^2\omega_1 + a_2^2\omega_2 - (\omega_2 + \omega_1)(a_1^2 + a_2^2 - a^2)/2}{a^2}$$

$$= \frac{{}_1\cancel{2}a_1^2\omega_1 + {}_1\cancel{2}a_2^2\omega_2 - a_1^2\omega_2 - a_2^2\omega_1 + a^2\omega_2 + a^2\omega_1 - \cancel{a_2^2\omega_2} - \cancel{a_1^2\omega_1}}{2a^2}$$

$$= \frac{a_1^2\omega_1 + a_2^2\omega_2 - a_1^2\omega_2 - a_2^2\omega_1 + a^2\omega_2 + a^2\omega_1}{2a^2}$$

$$= \frac{(a_1^2 + a^2 - a_2^2)\omega_1 + (a^2 + a_2^2 - a_1^2)\omega_2}{2a^2}$$

$\therefore$ R.A.V. of A_2 w.r.t A_1 is

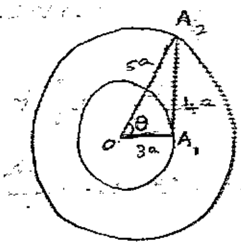
$$\frac{(a^2 + a_1^2 - a_2^2)\omega_1 + (a^2 + a_2^2 - a_1^2)\omega_2}{2a^2}$$

6. Two points are describing concentric circles of radii $3a$ and $5a$ in the same sense with angular velocities ω and ω'. S.T. the angular velocity of the line joining them when its length is $4a$ is equal to ω'.

Let A.V of A_1A_2 denoted by λ,

$$\lambda = \frac{(a_1v_1 + a_2v_2) - (a_1v_2 + a_2v_1)\cos\theta}{A_1A_2^2}$$

ΔOA_1A_2 is right angle at A_1

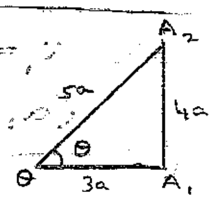


$$\cos\theta = \frac{3a}{5a} = \frac{3}{5}$$

$V_1 = $ (radius) (angular velocity)

$V_1 = 3a\omega$, $V_2 = 5a\omega'$

$$\therefore \lambda = \frac{3a \times 3a\omega + 5a \times 5a\omega' - (3a \times 5a\omega' + 5a \times 3a\omega) \times 3/5}{16a^2}$$



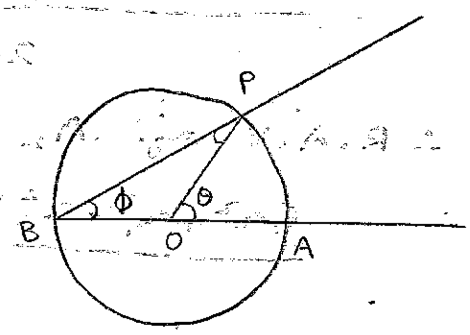
$$= \frac{9a^2\omega + 25a^2\omega' - 9a^2\omega' - 9a^2\omega}{16a^2}$$

$$= \frac{16a^2\omega'}{16a^2} = \omega'$$

Hence the angular velocity of the line joining them when the length is $4a$ is ω'

7. A point moves along a circle. ST at any instant its angular velocity about the centre is twice its angular velocity about ~~the centre~~ any point on the circumference.

*For angular velocity, we must have a fixed point and a fixed line.



*O $\rightarrow$ centre of the circle.

*AOB $\rightarrow$ fixed diameter.

*P $\rightarrow$ moving point

The A.V of P w.r.t to the centre is

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d}{dt}(A\hat{O}P) \rightarrow (1)$$

The A.V of P w.r.t a pt (fixed) on the circumference is

$$\phi = \frac{d\phi}{dt} = \frac{d}{dt}(A\hat{B}P) \rightarrow (2)$$

In ΔOBP, $OB = OP = $ radius

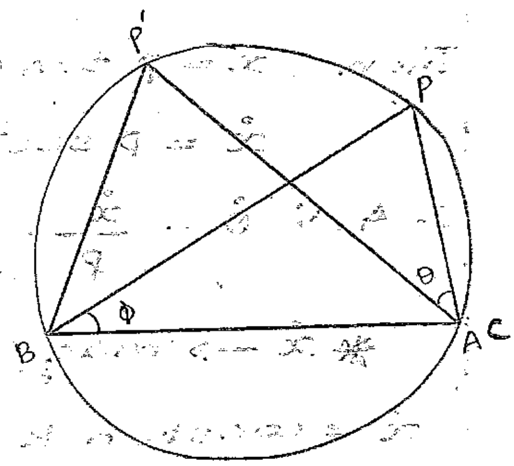
$\underline{\angle OBP} = \underline{\angle OPB} = \phi$

$$\angle\theta = \angle 2\phi$$

$$\dot{\theta} = 2\dot{\phi}$$

$\therefore$ A.V about the centre is twice the A.V about any pt on the circumference.

8. If a pt moves so that as angular velocity about two fixed pts are the same, PT it describes a circle.



* A & B $\rightarrow$ fixed pts.
* P $\rightarrow$ moving pt,

produce AB to C.

$\angle CBP = \phi$, $\angle CAP = \theta$

The A.V of P w.r.t A and B,

$$\frac{d}{dt}(CAP) = \frac{d\theta}{dt} \rightarrow (1)$$

$$\frac{d}{dt}(CBP) = \frac{d\phi}{dt} \rightarrow (2)$$

They are equal

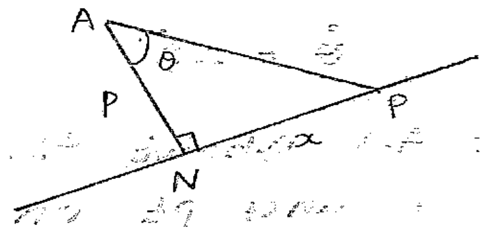
$$\frac{d\phi}{dt} = \frac{d\theta}{dt} \rightarrow \frac{d}{dt}(\theta - \phi) = 0 \rightarrow (\theta - \phi) = A\hat{P}B = k \text{ is a}$$

constant angle for various positions of the moving point P.

* The angles in the same segment, the angles subtended by a chord at any pt on the circumference of a circle will be the same.
* $A\hat{P}B = k$ we conclude P describes a circle with AB as a chord.

9. S.T the angular velocity about a fixed pt A of a particle P moving uniformly in a st. line, varies inversely as the square of the distance of the pt from the fixed point.

* A → fixed pt.



* A moves uniformly along the st. line.

* Draw AN ⊥r to the line

* N → fixed pt as it is the foot of the ⊥r from the fixed pt A on the fixed line

* AN = p, Measure the distance of P from N.

* $P\hat{A}N = \theta$

Then, $x = p \tan\theta \rightarrow (1)$

$$\dot{x} = p \sec^2\theta \, \dot{\theta}$$

∴ A.V $\dot{\theta} = \frac{\dot{x}}{p} \cdot \frac{1}{\sec^2\theta} \rightarrow (2)$

* $\dot{x}$ → mag of the velocity of P.

$\dot{x}$ = constan k

$$\Rightarrow \dot{\theta} = \frac{k}{p}\cos^2\theta = \frac{k}{p}\left(\frac{p}{AP}\right)^2 = kp \frac{1}{AP^2} \rightarrow (3)$$

∴ AV is inversely proportional to the square of the distance of P from the fixed point.

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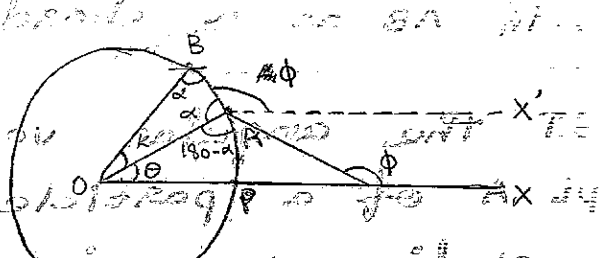
10. Two particles A and B describe a circle of radius a in the same sense with the same speed u. S.T the R.A.V of each w.r.t the other is $\frac{u}{a}$.

* O → centre



* OX → fixed line

* A & B → moving pts

* As they move with the same speed in the same sense (i.e) in the same dir, the distance b/w A & B will be the same.

i.e) AB is a constant. So, $\angle AOB = k$.

$\angle OAB = \alpha$, As $OA = OB$, α is constant

$\Rightarrow \cos 180^\circ - k = 2\alpha$

AX' be a line $\parallel^{el}$ to the fixed line OX. So AX' is also a fixed line.

$\angle X'AB = \phi$, $\angle XPA = \phi$

From ΔOAP,

$\phi = \theta + O\hat{A}P = \theta + 180^\circ - \alpha \rightarrow (1)$

$\dot{\phi} = \dot{\theta}$; $\dot{\theta}$ is the A.V of A moving on a circle about the centre.

linear velocity u = radius x A.V

$$u = a\dot{\theta}$$

$$\frac{u}{a} = \dot{\theta} = \dot{\phi}$$

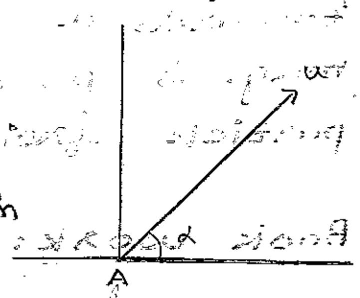
But $\dot{\phi}$ is the A.V of B w.r.t A

$\angle \phi = X'\hat{A}B$

Hence the R.A.V of B w.r.t A is $\dot{\phi} = \dot{\theta} = \frac{u}{a}$

---

11. The line joining two pts A and B is of constant length a and the velocities of A and B are in direction which make angles α and β respectively with AB. P.T the A.V of the AB about A is $\frac{u \sin(\alpha - \beta)}{a \cos \beta}$ where u is the velocity of A

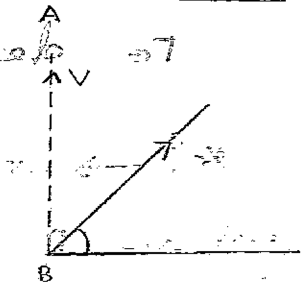


* u & $V \rightarrow$ velocities of A & B.

* They make angles α & β with AB respectively.

Components of u and V along AB:

$u \cos \alpha$, $V \cos \beta$



$\perp^r$ to AB : $u \sin\alpha$, $V \sin\beta$

* As AB is of constant length, the component of velocity of B relative to A along AB is zero.

$$V \cos\beta - u \cos\alpha = 0$$

$$V = \frac{u \cos\alpha}{\cos\beta}$$

$\therefore$ A.V of AB abt A be ω

$$\Rightarrow \omega = \frac{\text{comp of A.R.V of B } \perp^r \text{ AB}}{\text{Length of AB}}$$

$$= \frac{V \sin\beta - u \sin\alpha}{a}$$

$$= \frac{\left(\frac{u\cos\alpha}{\cos\beta}\right)\sin\beta - u\sin\alpha}{a}$$

$$= \frac{u\cos\alpha \sin\beta - u \sin\alpha \cos\beta}{a \cos\beta}$$

$$= \frac{u \sin(\beta - \alpha)}{a \cos\beta}$$

Definition of Simple Harmonic Motion:

The motion of a particle along a st. line with an acceleration which is always towards a fixed pt on the st. line and whose mag. is proportional to the distance of the particle from the fixed pt is called a SHM

Book work:

To derive the eqn of motion for a SHM

* $\hat{i} \rightarrow$ unit vector along the dir. $\overline{OP}$ where P is the position of the particle whic

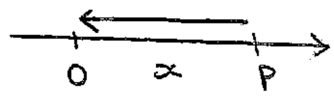
is in SHM at time t.

$\overline{r} = \overrightarrow{OP} \Rightarrow \overline{r} = x\hat{i} \Rightarrow \dot{\overline{r}} = \dot{x}\hat{i}$ ($OP.\hat{OP}$, displacement) $\dot{\overline{r}} = \dot{x}\hat{i} \rightarrow$ velocity

$\ddot{\overline{r}} = \ddot{x}\hat{i} \rightarrow$ acceleration

W.K.T Acceleration is proportional to the distance of particle from O

* Mag. of the acceleration $= n^2 x$
* Dⁿ of deceleration is towards O $\boxed{\text{ie}, (-\hat{i})}$



$\therefore \ddot{\overline{r}} = -n^2 x\hat{i} \longrightarrow ③$

Equating ② & ③, $\ddot{x}\hat{i} = n^2 x\hat{i}$

$\boxed{\ddot{x} = -n^2 x} \rightarrow ④ \rightarrow$ eqn of SHM

Book Work:

In the SHM whose eqn is $\ddot{x} = -n^2 x$. find

1) x in 't' 2) $\dot{x}$ in 't' 3) $\dot{x}$ in 'x'

To prove: $\boxed{v^2 = n^2(a^2 - x^2)}$

Eqn of SHM is $\ddot{x} = -n^2 x$

$$\ddot{x} + n^2 x = 0$$

$$\frac{d^2x}{dt^2} + n^2 x = 0 \Rightarrow (D^2 + n^2)x = 0$$

The accalery eqn is $m^2 + n^2 = 0$

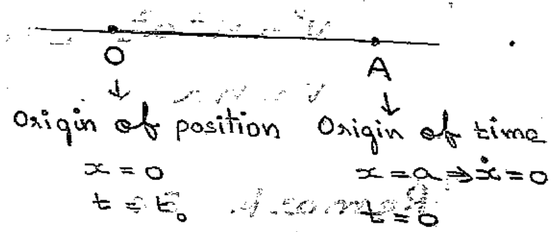
$$m^2 = -n^2$$

$$m = \pm ni$$

Real part $(\alpha) = 0$, Imaginary part $(\beta) = n$

$(D^2 + n^2)y = 0 \Rightarrow m = \alpha \pm i\beta \rightsquigarrow y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

$y \rightarrow x \qquad x \rightarrow t$



$\therefore x = A\cos nt + B\sin nt \rightarrow ①$

$\dot{x} = -An\sin nt + Bn\cos nt \rightarrow ②$

Sub $x = a, t = 0$ in ①,

$a = A\cos n0 + B\sin n0 \Rightarrow a = A$

Sub $x = a, \dot{x} = 0, t = 0$ in ②

$0 = -An\sin n0 + Bn\cos n0 \Rightarrow B = 0$

Sub $A = a, B = 0$ in ① & ②,

$x = a\cos nt \rightarrow ③$

$\dot{x} = -an\sin nt \rightarrow ④$

From ③ $\Rightarrow \frac{x}{a} = \cos nt \rightarrow$ ⑤

From ④ $\Rightarrow \frac{\dot{x}}{-an} = \sin nt \rightarrow$ ⑥

$⑤^2 + ⑥^2 \Rightarrow \frac{x^2}{a^2} + \frac{\dot{x}^2}{a^2 n^2} = 1$ ← $(\cos^2 nt + \sin^2 nt = 1)$

$$\frac{x^2 n^2 + \dot{x}^2}{a^2 n^2} = 1$$

$$\dot{x}^2 = a^2 n^2 - x^2 n^2$$

$$\dot{x}^2 = n^2(a^2 - x^2)$$

↓ velocity

$$V^2 = n^2(a^2 - x^2) \quad [\because \dot{x} = V]$$

$x \rightarrow$ displacement
$\dot{x} \rightarrow$ velocity
$\ddot{x} \rightarrow$ acceleration

Remark 1:

*Here the origin of position is O the particle starts its motion from the point A.

*A is the origin of time.

O ↓ origin of position = $x = 0$ & $t = t_0$

A ↓ origin of time $x = a$ $\Rightarrow \dot{x} = 0$ ↓ Particle starts its motion from here.

Remark 2:

* At $x = 0$

↳ Speed is maximum.

↳ Magnitude of velocity is ~~minimum~~ maximum.

$$V^2 = n^2(a^2 + x^2)$$

$$V^2 = n^2 a^2$$

$V = na \rightarrow$ mag. of maximum speed is $v = na$.

Remark 3:

O — A

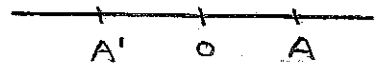
* At $x = a \Rightarrow \dot{x} = 0$

↳ Magnitude of acceleration is maximum.

Mag. of acc is $\ddot{x} = n^2 x$	$V^2 = n^2(a^2 - x^2)$
$\ddot{x} = n^2(0)$	$V^2 = n^2(a^2 - a^2)$
$\ddot{x} = 0$	$V^2 = 0 \Rightarrow V = 0$
acceleration is zero	Velocity is zero

- The acc is maximum at the pt A & it is attained at A an extreme position which is also known as origin of time.

Amplitude:



If A is the extreme position or the origin of the time and O is the origin position. The distance between O & A is known as the amplitude of the SHM and it is denoted by 'a'. That is the maximum displacement of a particle is called amplitude.

$$\ddot{x} = n^2 x$$

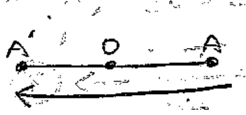
* At 'O', acceleration = 0 but velocity is constant and is the maximum.

Motion of particle having SHM:

The particle reaching 'O' cannot be brought to rest as it has a velocity; so it moves beyond O reaching at pt A' when $\overline{OA'}$ = a length of the particle is reduced to be at rest.

The velocity at A' is zero. But the particle has an acceleration at the pt A' which is towards 'O' so, it starts moving toward 'O' as the velocity at 'O' is maximum it goes beyond 'O' and reaches A. It cannot beyond A but it starts towards 'O'. So if a particle has a SHM the particle moves between A to A' and A' to A (i.e) ~~&~~ it between A and A'.

Vibration:

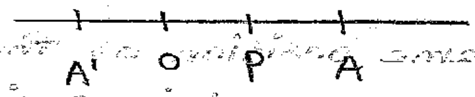


The motion of the particle from extremity A to the other extremity A' of its path is called as vibration.

Oscillation:

One complete motion of the particle from one extremity A to the other extremity A' and back to the same extremity A is called an oscillation. It can also be defined as follows:

Motion of particle from P to A' then A' to A and at last A

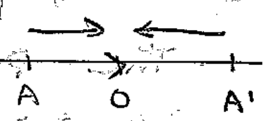


One complete motion of the particle from a pt P on its path to one extremity of its path A' then to the other extremity A and back to the pt is called as oscillation.

Period (T):-

The time taken for 1 oscillation is called period. It is denoted by T.

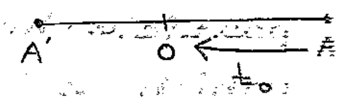
To prove: $\boxed{T = \frac{2\pi}{n}}$



* $t_0 \rightarrow$ time taken to move from A to O.
* The time taken to move from O to A', A' to O, O to A all are equal to 't_0' due to symmetry.

$\therefore$ total time taken for one oscillation is $4t_0$.

$$T = 4t_0 \rightarrow (1)$$



Consider $x = a\cos nt \rightarrow (2)$

At A, $x = a$, $t = 0$, $\dot{x} = 0$

$\therefore x = a\cos nt$ (at A)

At O, $t = t_0$ & $x = 0$ sub in (2),

$\rightarrow 0 = a\cos nt_0$

$\rightarrow \cos nt_0 = 0$

$$nt_0 = \cos^{-1}(0) = \pi/2$$

$$t_0 = \frac{\pi}{2n} \rightarrow (3)$$

sub (3) in (1),

$$T = 4\frac{\pi}{2n} = \frac{2\pi}{n}$$

Frequency:

The no. of oscillations made per second is defined as the frequency. It is denoted by 'f'

Relation b/w T and f :

* T & f are reciprocals of each other

T sec → 1 oscillation

1 sec → $\frac{1}{T}$ oscillation = f

$$\boxed{f = \frac{1}{T} \Rightarrow T = \frac{1}{f}}$$

Other equation of SHM :

i) $v^2 = n^2(a^2 - x^2)$

ii) $x = a \cos nt$

iii) $x = A \cos nt + B \sin nt$

iv) $x = a \cos(nt + E)$

v) $x = a \sin(nt + E)$

nt + E → phase of SHM

nt + E , if t = 0

↓

E → Epoch of SHM

displacement → x

magi of velocity → $\dot{x}$

magi of acceleration → $\ddot{x}$

i) $v^2 = n^2(a^2 - x^2)$

If $\dot{x}^2 = n^2(a^2 - x^2)$ then it represents SHM

differentiate both sides w.r.t 't'.

$v^2 = \dot{x}^2 = n^2(a^2 - x^2)$

$2\dot{x}\ddot{x} = n^2(-2\dot{x}x)$

$$\boxed{\ddot{x} = -n^2 x}$$

ii) $x = a \cos nt$

$\dot{x} = -an \sin nt$

$\ddot{x} = -an^2 \cos nt$

$\ddot{x} = -n^2 [a \cos nt]$

$$\boxed{\ddot{x} = -n^2 x}$$

iii) $x = A \cos nt + B \sin nt$

$\dot{x} = -An \sin nt + B \sin nt$

$\ddot{x} = -An^2 \cos nt - Bn^2 \sin nt$

$$\boxed{\ddot{x} = -n^2 x}$$

iv) $x = a \cos(nt + E)$

$\dot{x} = -a \sin(nt + E)n$

$\ddot{x} = -an^2 \cos(nt + E)$

$$\boxed{\ddot{x} = -n^2 x}$$

When the particle extreme the SHM starts its motion from the pt A the extreme position the displacement is given by

$x = a \cos(nt + E) \rightarrow (1)$

In (1),

When $t = 0 \Rightarrow x = a \cos t$

When $\cos t = 1 \Rightarrow x = a$

[Particle is at A]

When $\cos t > 1$, $x < a$

[Particle is at P differ from A]

v) $x = a \sin(nt + E)$

$\dot{x} = a \cos(nt + E)n$

$\ddot{x} = -a \sin(nt + E)n^2$

$$\boxed{\ddot{x} = -an^2 \sin(nt + E) = -n^2 x}$$

1. P.T in a SHM if f is the acceleration, v is the velocity at any moment and T is the period time then $f^2T^2 + 4\pi^2 v^2$ is a constant.

Gn that $f^2T^2 + 4\pi^2 v^2 \rightarrow$ (1)

$f = \underbrace{n^2}_{\text{constant}} x$ = mag. of acceleration, $f \propto x$

$v^2 = n^2(a^2 - x^2)$, $T = \frac{2\pi}{n}$

Sub in (1) $\Rightarrow = n^4 x^2 \cdot \frac{4\pi^2}{n^2} + 4\pi^2 [n^2(a^2 - x^2)]$

$= \cancel{4\pi^2 n^2 x^2} + 4\pi^2 n^2 a^2 - \cancel{4\pi^2 n^2 x^2}$

$= 4\pi^2 n^2 a^2$ = a constant

---

2. A body moving with a SHM has an amplitude 'a' and a period T. S.T the velocity V at a distance x from the mean position is gn by $V^2T^2 = 4\pi^2(a^2 - x^2)$

Gn that $V^2T^2 = 4\pi^2(a^2 - x^2)$

$\Rightarrow$ LHS $= V^2T^2 \rightarrow$ (1)

$\boxed{V^2 = n^2(a^2 - x^2)} \rightarrow$ SHM

$\boxed{T = \frac{2\pi}{n}} \Rightarrow T^2 = \frac{4\pi^2}{n^2}$

Sub in (1), $V^2T^2 = \cancel{n^2}(a^2 - x^2) \cdot \frac{4\pi^2}{\cancel{n^2}}$

$= 4\pi^2(a^2 - x^2)$ = RHS

---

3. The displacement x of a particle moving along a st. line given by $\boxed{x = A\cos nt + B\sin nt}$ where a, b is n constant. S.T its motion is SHM $x = A\cos nt + B\sin nt$. If A is 3, B = 4, n = 2 find period $\left(\frac{2\pi}{n}\right)$, amplitude (a), maximum velocity (na) and maximum acceleration (n^2a).

$x = A\cos nt + B\sin nt$

$x = 3\cos 2t + 4\sin 2t \rightarrow$ (1)

$V = \dot{x} = -6\sin 2t + 8\cos 2t \rightarrow$ (2)

$\frac{\dot{x}}{2} = \frac{V}{2} = -3\sin 2t + 4\cos 2t \rightarrow$ (3)

$①^2 + ③^2 \Rightarrow x^2 + \frac{v^2}{4} = (3\cos 2t + 4\sin 2t)^2 + (-3\sin 2t + 4\cos 2t)^2$

$x^2 + \frac{v^2}{4} = 9\cos^2 2t + 16\sin^2 2t + \cancel{24\cos 2t \sin 2t} + 9\sin^2 2t + 16\cos^2 2t - \cancel{24\cos 2t \sin 2t}$

$= 9(\cos^2 2t + \sin^2 2t) + 16(\sin^2 2t + \cos^2 2t)$

$= 9 + 16$

$x^2 + \frac{v^2}{4} = 25 \Rightarrow \frac{v^2}{4} = 25 - x^2 \Rightarrow v^2 = 4(25 - x^2) \Rightarrow v^2 = 2^2(5^2 - x^2) \rightarrow ④$

Comparing ④ with $\boxed{v^2 = n^2(a^2 - x^2)}$

$\therefore n = 2, \; a = 5$

$\Rightarrow$ Its motion is SHM.

*Maximum amplitude $= a = 5$

*Maximum velocity $= na = 2 \times 5 = 10$

*Maximum acceleration $= n^2 a = 4 \times 5 = 20$

*Time period $\Rightarrow T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

4. The displacement x of a particle moving along a st. line is given by $x = a\cos nt + b\sin nt$. S.T the motion is simple harmonic motion with amplitude $\boxed{\sqrt{a^2+b^2}}$ & period $\boxed{2\pi/n}$.

$x = a\cos nt + b\sin nt \rightarrow ①$

$v = \dot{x} = -an\sin nt + bn\cos nt$

$\Rightarrow \frac{\dot{x}}{n} = \frac{v}{n} = -a\sin nt + b\cos nt \rightarrow ②$

$①^2 + ②^2 \Rightarrow x^2 + \frac{v^2}{n^2} = (a\cos nt + b\sin nt)^2 + (-a\sin nt + b\cos nt)^2$

$x^2 + \frac{v^2}{n^2} = a^2 + b^2 \Rightarrow \frac{v^2}{n^2} = (a^2 + b^2) - x^2$

$\Rightarrow v^2 = n^2\left[(a^2 + b^2) - x^2\right] = n^2\left[\left(\sqrt{a^2+b^2}\right)^2 - x^2\right] \rightarrow ③$

Compare ③ with $\boxed{v^2 = n^2(a^2 - x^2)}$

$\therefore a = \sqrt{a^2+b^2}, \; n = n, \; x = x$

$\Rightarrow$ Its motion is SHM

*amplitude $= a = \sqrt{a^2+b^2}$

* period $\Rightarrow T = \frac{2\pi}{n}$

5. If the ~~amplitude~~ distance x of a pt moving on a st. line measured from a fixed pt on it and its velocity v are connected by relation, $\boxed{4v^2 = (25 - x^2)}$. S.T the motion SHM. Find the period and amplitude of the motion.

Gn that $4v^2 = 25 - x^2$ $\qquad v^2 = n^2(a^2 - x^2)$

$$v^2 = \frac{25 - x^2}{4} = \dot{x}^2$$

$$v^2 = \frac{1}{4}(25 - x^2) = \left(\frac{1}{2}\right)^2 (5^2 - x^2)$$

It is in the form of $v^2 = n^2(a^2 - x^2)$

Hence it represent SHM.

$\therefore n = 1/2$, $a = 5$

* amplitude $= a = 5$

* period $\Rightarrow T = \frac{2\pi}{n} = \frac{2\pi}{1/2} = 4\pi$

At A, $x = a$ & $t = 0$ & $v = 0$

$\Rightarrow 4v^2 = 25 - x^2$

$$0 = 25 - x^2$$

$$x^2 = 25 \Rightarrow a^2 = 25$$

$$\boxed{a = 5}$$

---

6. A particle moves in a st. line of its velocity v whose distance x from a fixed pt in the line. If $\boxed{v^2 = \alpha - \beta x^2}$. S.T the motion is SHM & determine period and amplitude. Here α & β are constants.

Gn that $\boxed{v^2 = \alpha - \beta x^2}$

$$\dot{x}^2 = v^2 = \beta\left(\alpha/\beta - x^2\right) = (\sqrt{\beta})^2\left(\left(\frac{\sqrt{\alpha}}{\sqrt{\beta}}\right)^2 - x^2\right) \quad (1)$$

Comparing (1) with $\boxed{v^2 = n^2(a^2 - x^2)}$

$\therefore n^2 = (\sqrt{\beta})^2 \Rightarrow n = \sqrt{\beta}$, $a = \frac{\sqrt{\alpha}}{\sqrt{\beta}}$

* amplitude $= a = \sqrt{\frac{\alpha}{\beta}}$

* period $\Rightarrow T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\beta}}$

At A, $x=a$ & $t=0$ & $v=0$

$\Rightarrow v^2 = \alpha - \beta x^2$

$0 = \alpha - \beta a^2$

$\beta a^2 = \alpha \Rightarrow a^2 = \alpha/\beta \Rightarrow \boxed{a = \sqrt{\frac{\alpha}{\beta}}}$

---

7. The velocity of a particle moving in a st. line at a distance x from a fixed pt in the line is given by $\boxed{v = k\sqrt{a^2 - x^2}}$ where 'k' & 'a' are constant S.T the motion is SHM. Find the amplitude & period time.

Gn that $v = k\sqrt{a^2 - x^2}$

$\dot{x}^2 = v^2 = k^2(a^2 - x^2)$

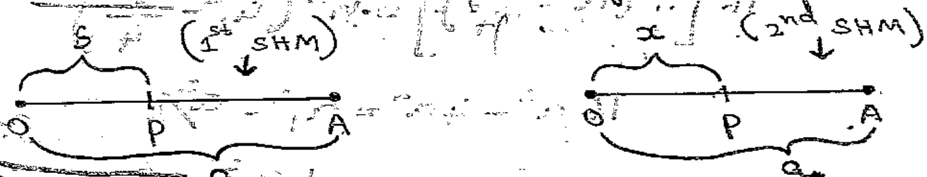
It is in the form $\dot{x}^2 = n^2(a^2 - x^2)$

Hence it is SHM, here $n = k$, $a = a$, $x = x$

* amplitude $= a$
* period time $\Rightarrow T = \frac{2\pi}{n} = \frac{2\pi}{k}$

---

8. A (*) 5M particle is moving in a st. line with the SHM of amplitude at 'a' distance 's' from the centre of motion the particle receives a blow in the direction of motion with instaneously doubles the velocity. Find the amplitude.



$(2V)^2$ $\boxed{V^2 = n^2(a^2 - x^2)}$

* When the particle is at P, where op=s it receives a blow which doubles the speed and so the amplitude of the particle is also change to 'a_1'.

* We have to find the new amplitude a_1.

With respect to the first SHM,

$v^2 = n^2(a^2 - s^2) \longrightarrow (1)$ $[\because v^2 = n^2(a^2 - x^2)]$

W.r.t the new SHM

$(2V)^2$

$4V^2 = n^2(a_1^2 - s^2) \longrightarrow (2)$

$① \times 4 \Rightarrow 4v^2 = 4n^2(a^2 - s^2) \rightarrow ③$

$② = ③ \Rightarrow \cancel{n^2}(a_1^2 - s^2) = 4\cancel{n^2}(a^2 - s^2)$

$$\boxed{a_1^2 = 4a^2 - 4s^2 + s^2}$$

$$a_1^2 = 4a^2 - 3s^2$$

$$\boxed{a_1 = \sqrt{4a^2 - 3s^2}}$$

9. A particle is executing a SHM with O as the mean position, 'a' is the amplitude. When it is at a distance $a/2$ from O its velocity is ~~quads~~ quatrupled (4 times) by a blows. S.T the new amplitude is $\frac{7a}{2}$

By $v^2 = n^2(a^2 - x^2)$,

$\Rightarrow v^2 = n^2\left(a^2 - \frac{a^2}{4}\right) \rightarrow ①$

$\overbrace{O \quad P}^{a/2}$

*Let the new amplitude be a_1 and the new velocity be '4V',

$\Rightarrow (4v)^2 = n^2\left(a_1^2 - \frac{a^2}{4}\right)$

$16v^2 = n^2\left(a_1^2 - \frac{a^2}{4}\right) \rightarrow ②$

sub the value v^2 in ② from ①

$$16\left[\cancel{n^2}\left(a^2 - \frac{a^2}{4}\right)\right] = \cancel{n^2}\left(a_1^2 - \frac{a^2}{4}\right)$$

$$16a^2 - 4a^2 = a_1^2 - a^2/4$$

$$12a^2 = \frac{4a_1^2 - a^2}{4}$$

$$48a^2 = 4a_1^2 - a^2 \Rightarrow 4a_1^2 = 49a^2$$

$$\Rightarrow a_1^2 = \frac{49a^2}{4} \Rightarrow a_1 = \sqrt{\frac{49a^2}{4}} = \sqrt{\frac{(7)^2 a^2}{(2)^2}}$$

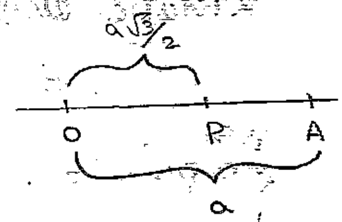
$$\therefore \boxed{a_1 = \frac{7a}{2}}$$

10. A particle is executing a SHM with O as the mean position $2\pi/n$ as the period & 'a' as the amplitude. When it is at a distance $\frac{a\sqrt{3}}{2}$, fro[m] it receives a blow which increases its velocit[y]

by na. S.T the new amplitude is $a\sqrt{3}$

$v^2 = n^2(a^2 - x^2)$ suitably,

$$\Rightarrow v^2 = n^2\left(a^2 - \frac{a^2(\sqrt{3})^2}{(2)^2}\right) = n^2\left(a^2 - \frac{a^2 \cdot 3}{4}\right)$$



$$v^2 = n^2\left[\frac{4a^2 - 3a^2}{4}\right] = n^2 a^2/4$$

$$\boxed{V = na/2}$$

*Let the new amplitude be a_1 and the new velocity is $(v + na)$

$$\underbrace{(v+na)^2}_{\text{value}} = n^2\left(a_1^2 - \frac{3a^2}{4}\right)$$

$$\left(\frac{na}{2} + na\right)^2 = n^2\left(a_1^2 - \frac{3a^2}{4}\right) \Rightarrow \left(\frac{3na}{2}\right)^2 = n^2 a_1^2 - \frac{3n^2a^2}{4}$$

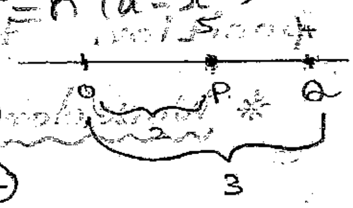
$$\Rightarrow \frac{9n^2a^2}{4} + \frac{3n^2a^2}{4} = n^2 a_1^2 \Rightarrow 3a^2 = a_1^2 \Rightarrow \boxed{a_1 = a\sqrt{3}}$$

---

11. A particle is moving in a st.line with SHM. Its velocity has values 5ft/sec, 4ft/sec when its distance from the mean positions are 2ft and 3ft respectively. Find the length of its path & period of its motion.

$\boxed{2a}$

$v = 5,\ x = 2$ / $v = 4,\ x = 3$ $\quad v^2 = n^2(a^2 - x^2)$



$$\Rightarrow v^2 = n^2(a^2 - x^2) \Rightarrow \boxed{5^2 = n^2(a^2 - 2^2)} \rightarrow (1)$$

$$\Rightarrow 4^2 = n^2(a^2 - 3^2) \rightarrow (2)$$

$$\boxed{16 = n^2(a^2 - 9)} \rightarrow (3)$$

$$\frac{(1)}{(3)} \Rightarrow \frac{25}{16} = \frac{\cancel{n^2}(a^2 - 2^2)}{\cancel{n^2}(a^2 - 3^2)} = \frac{a^2 - 4}{a^2 - 9} \Rightarrow 25(a^2 - 9) = 16(a^2 - 4)$$

$$\Rightarrow 25a^2 - 225 = 16a^2 - 64 \Rightarrow 9a^2 = 161 \Rightarrow a^2 = \frac{161}{9} \Rightarrow \boxed{a = \frac{\sqrt{161}}{3}}$$

Sub $a^2 = \frac{161}{9}$ in (3),

$$\Rightarrow 16 = n^2\left(\frac{161}{9} - 9\right) = n^2\left(\frac{161 - 81}{9}\right) = n^2\left(\frac{80}{9}\right)$$

$$\Rightarrow \frac{16 \times 9}{80} = n^2 \Rightarrow n^2 = \frac{144}{80} = \frac{9}{5} \Rightarrow \boxed{n = \frac{3}{\sqrt{5}}}$$

Length of the path $= 2a = \frac{2\sqrt{161}}{3}$

Period, $T = \frac{2\pi}{n} = \frac{2\pi}{3/\sqrt{5}}$

$$\boxed{T = \frac{2\sqrt{5}\pi}{3}}$$

Kinetic energy & Potential energy.

Energy in a particle is measured by its capacity to do work. Mechanical energy is divided into two parts.

1) Kinetic energy (K.E)

2) Potential energy (P.E)

Kinetic Energy:

Energy possessed by particle interms of its motion $= \frac{1}{2}mv^2$

Potential Energy:

↳ Energy possessed by particle when particles moves from present position to standard position (or)

↳ P.E is measured interms of the workdone by particle by moving it from present position to standard position.

* Work done : F × distance moved by the particle along the direction of force.

* Force : mass × acceleration = ma

* ST in SHM , K.E + P.E = constant

$$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\left[n^2(a^2-x^2)\right]$$

$$= \frac{1}{2}mn^2(a^2-x^2) \longrightarrow (1)$$

$$P.E = \int_x^0 (\text{force})\,dx = \int_x^0 m\ddot{x}\,dx = \int_x^0 m(-n^2x)\,dx$$

$$= \int_0^x mn^2x\,dx = \frac{mn^2x^2}{2} \longrightarrow (2)$$

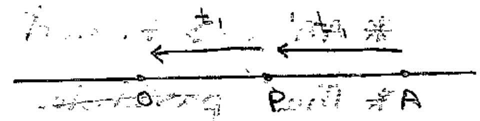
$$(1)+(2) \Rightarrow K.E + P.E = \frac{1}{2}mn^2(a^2-x^2) + \frac{1}{2}mn^2x^2$$

$$= \frac{1}{2}mn^2a^2$$

$$= \text{constant}$$

12. A particle is executing a SHM of period with O as the mean position. The particle passes through a point P with velocity V in the direction of OP. S.T the time which lapses before it ~~position~~ return to P is

$$\frac{T}{\pi} \tan^{-1}\left(\frac{VT}{2\pi(OP)}\right)$$



$\cos\theta = \frac{adja}{hypo}$

To prove:

$$2t_1 = \frac{T}{\pi} \tan^{-1}\left(\frac{VT}{2\pi(OP)}\right)$$

Time which lapses before its return to P

Let $OP = b$, $x = a\cos nt$ is in SHM $\rightarrow$ (1)

At P, $x = b$ & $t = t_1$

Sub in (1),

$\Rightarrow b = a\cos nt \Rightarrow \cos nt_1 = \frac{b}{a}$

From right angle $\triangle^{le}$,

$$\tan nt_1 = \frac{\sqrt{a^2-b^2}}{b} \rightarrow (2)$$

$\Rightarrow T = \frac{2\pi}{n} \Rightarrow \boxed{n = \frac{2\pi}{T}} \rightarrow (3)$

$\Rightarrow$ As the particle is in the SHM

$V^2 = n^2(a^2 - b^2) \Rightarrow \frac{V^2}{n^2} = a^2 - b^2 \Rightarrow \boxed{\frac{V}{n} = \sqrt{a^2-b^2}} \rightarrow (4)$

Sub (3) & (4) in (2)

$$\tan \frac{2\pi}{T} t_1 = \frac{VT}{2\pi(OP)}$$

$$2t_1 = \frac{T}{\pi} \tan^{-1}\left(\frac{VT}{2\pi(OP)}\right)$$

Book work:

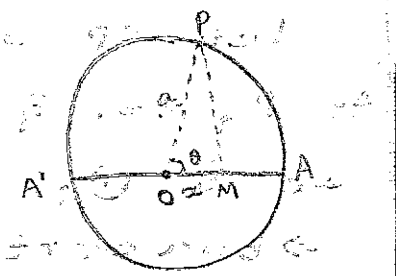
Projection of a particle having a uniform circular motion.

Gn: Particle moves along a circle with a uniform speed.

T.P : Motion of its projection on circular motion along fixed diameter is SHM.

Proof:

* Let a particle form a circle
* Constant angular velocity $\rightarrow \omega \rightarrow$ (1)
* $AA' \rightarrow$ fixed diameter
* The particle starts its motion from a point A, at any instant 't'.
* Let it be at a pt. 'p' $\Rightarrow \angle AOP = \theta$
* Draw PM $\perp$ AA'
* M $\longrightarrow$ foot of the $\perp$
* Then M $\longrightarrow$ projection of P
* OM $= x$ & OP $= a =$ radius



$\cos\theta = \frac{x}{a}$ [fig]

* $x = a\cos\theta \longrightarrow$ (2)

$\dot{x} = -a\sin\theta\,\dot{\theta}$

$\ddot{x} = -a\cos\theta\,\dot{\theta}^2 = -\dot{\theta}^2\, \underset{x}{\underbrace{a\cos\theta}}$

From (1) & (2) $\rightarrow \ddot{x} = -\omega^2 x$

This is the eqn of SHM.

$\rightarrow$ Amplitude $= a$ & Period $\left(\frac{2\pi}{\omega} = T\right)$

* Projection M $\longrightarrow$ A to A' : P $\longrightarrow$ uppersemicircle
* Projection M $\longrightarrow$ A' to A : P $\longrightarrow$ lowersemicircle

13. If initially the particle is project from with the velocity v away from the pt. O where OA = a, such that it executes a SHM. Find x the amplitude and the velocity.

$x = c\cos nt + D\sin nt \rightarrow$ SHM $\longrightarrow$ (1)

$\dot{x} = -cn\sin nt + Dn\cos nt \longrightarrow$ (2)

At A: $\dot{x} = V,\ x = a,\ t = 0$

sub in (1) $\Rightarrow a = C + D.0 \Rightarrow \boxed{a = C}$

sub in (2) $\Rightarrow V = Dn \Rightarrow \boxed{D = \frac{V}{n}}$

sub value of C & D,

(1) $\Rightarrow x = a\cos nt + \frac{V}{n}\sin nt \rightarrow$ (3)

↳ Displacement

(2) $\Rightarrow \dot{x} = -an\sin nt + \frac{V}{n}n\cos nt \rightarrow$ (4)

↳ Velocity

At O: $\dot{x} = v$

Sub in (4) $\Rightarrow \frac{v}{n} = -a\sin nt + \frac{V}{n}\cos nt \rightarrow$ (5)

$(3)^2 + (5)^2 \Rightarrow x^2 + \frac{v^2}{n^2} = a^2 + \frac{V^2}{n^2}$

$$n^2x^2 + v^2 = a^2n^2 + V^2$$

$$v^2 = n^2(a^2 - x^2) + V^2 \longrightarrow (6)$$

At A':

Amplitude be a corresponding to the extreme position of the SHM.

$x = a',\ v = 0$

sub in (6) $\Rightarrow 0 = (a^2 - a'^2)n^2 + V^2$

$$0 = a^2n^2 - a'^2n^2 + V^2$$

$$a'^2n^2 = V^2 + a^2n^2$$

$$a'^2 = \frac{V^2 + a^2n^2}{n^2} = a^2 + \frac{V^2}{n^2}$$

$$a' = \sqrt{a^2 + \frac{V^2}{n^2}}$$

$\therefore$ The new amplitude $a' = \sqrt{a^2 + \frac{V^2}{n^2}}$

---

14. A particle executing SHM in a st. line has velocities 8, 7, 4 at 3 pts distant one foot from each other. Find the period.

Let the distances of the points from O be $x, x+1, x+2$ respectively.

$$v^2 = n^2(a^2 - x^2) \longrightarrow (1)$$

$$64 = n^2(a^2 - x^2) \longrightarrow (2)$$

$$49 = n^2(a^2 - (x+1)^2) \longrightarrow (3)$$

$$16 = n^2(a^2 - (x+2)^2) \longrightarrow (4)$$

$$(2) - (3) \Rightarrow 64 - 49 = \cancel{n^2a^2} - \cancel{n^2x^2} - \cancel{n^2a^2} + \cancel{n^2x^2} + n^2 2x + n^2$$

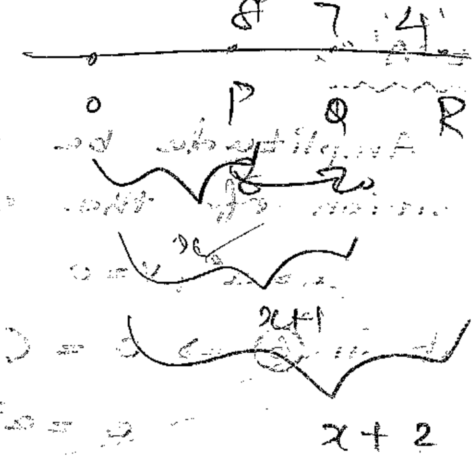
$$15 = n^2 2x + n^2$$

$$15 = n^2(2x+1) \longrightarrow (5)$$

$$(3) - (4) \Rightarrow 49 - 16 = \cancel{n^2a^2} - \cancel{n^2x^2} - 2n^2x - n^2 - \cancel{n^2a^2} + \cancel{n^2x^2} + 4n^2 + 4xn^2$$

$$33 = 2xn^2 + 3n^2$$

$$33 = n^2(2x+3) \longrightarrow (6)$$



$$\frac{(5)}{(6)} \Rightarrow \frac{15}{33} = \frac{\cancel{n^2}(2x+1)}{\cancel{n^2}(2x+3)} = \frac{2x+1}{2x+3}$$

$$\Rightarrow \frac{15}{33} - \frac{2x+1}{2x+3} = 0$$

$$\frac{15(2x+3) - 33(2x+1)}{33(2x+3)} = 0$$

$$\frac{30x + 45 - 66x - 33}{66x + 99} = 0$$

$$-36x + 12 = 0 \Rightarrow -36x = -12$$

$$\Rightarrow x = \frac{12}{36} = \frac{1}{3}$$

Sub x in (2) & (4),

$$64 = n^2(a^2 - 1/9) \longrightarrow (7)$$

$$16 = n^2\left[a^2 - \left(\frac{1}{3} + 2\right)^2\right] = n^2\left[a^2 - \frac{1}{9} - 4 - \frac{4}{3}\right]$$

$$16 = n^2\left[a^2 - \frac{49}{9}\right] \longrightarrow (8)$$

$$\frac{⑦}{⑧} \Rightarrow \frac{64}{16} = \frac{n^2(9a^2-1)}{n^2(9a^2-49)} \Rightarrow 4 = \frac{9a^2-1}{9a^2-49}$$

$$\Rightarrow 4(9a^2-49) = 9a^2-1 \Rightarrow 36a^2-196-9a^2+1=0$$

$$\Rightarrow 27a^2 = 195 \Rightarrow a^2 = \frac{195}{27} \Rightarrow \boxed{a^2 = \frac{65}{9}}$$

Sub a^2 in ②,

$$64 = n^2\left[\frac{65}{9} - \frac{1}{9}\right] = n^2\left[\frac{64}{9}\right]$$

$$\frac{64\times 9}{64} = n^2 \Rightarrow n^2 = 9 \Rightarrow \boxed{n=3}$$

$$\boxed{T = \frac{2\pi}{n} = \frac{2\pi}{3}}$$

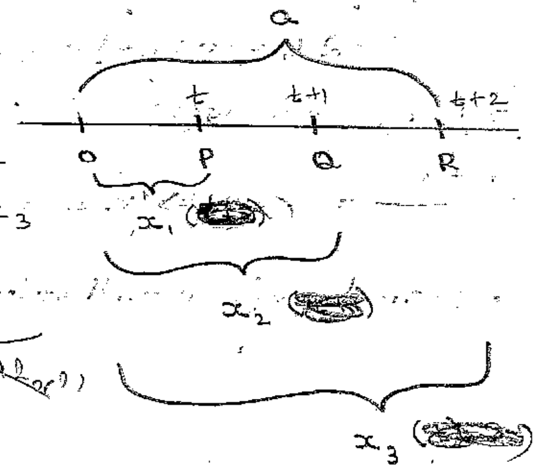
15. A particle is moving with a SHM and while moving from the mean position to one extreme position its distance at 3 consecutive seconds are $x_1, x_2, x_3 \ldots$ S.T the period is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_2}{2x_2}\right)}$ ①

(or)

A particle moving with SHM from an extremity of the path towards the centre is observed to be at distances x_1, x_2, x_3 from the centre at the ends of 3 consecutive sec. S.T the time of a complete oscillation is $T = \dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$. ②

Method 1: ②

Let the distances of the particle at the ends of time $t, t+1, t+2$ seconds be x_1, x_2, x_3 respectively from the centre.



we have

$$\boxed{x_1 = a\cos nt} \rightarrow ①$$

$$x_2 = a\cos n(t+1) \rightarrow ②$$

$$x_3 = a\cos n(t+2) \rightarrow ③$$

$$\Rightarrow x_1 + x_3 = a\left[\cos(nt+2n) + \cos nt\right], \quad 2x_2 = 2a\cos(nt+n)$$

$$\frac{x_1+x_3}{2x_2} = \frac{\cancel{a}\left[\cos(nt+2n)+\cos nt\right]}{2\cancel{a}\cos(nt+n)}$$

$\cos C = \cos(nt+2n)$

$\cos D = \cos nt$

$$= \frac{\cancel{2}\cos\left(\frac{\cancel{2}nt+\cancel{2}n}{\cancel{2}}\right)\cos\left(\frac{\cancel{nt}+\cancel{2}n-\cancel{nt}}{\cancel{2}}\right)}{\cancel{2}\cos(nt+n)}$$

$$\left[\because \cos C+\cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

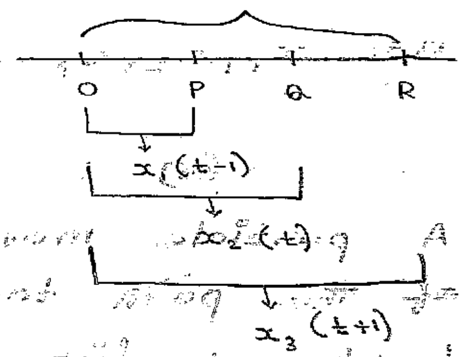
$$\frac{x_1+x_3}{2x_2} = \frac{\cancel{\cos(nt+n)}\cos n}{\cancel{\cos(nt+n)}}$$

$$\frac{x_1+x_3}{2x_2} = \cos n \Rightarrow n = \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$$

$$\therefore T = \frac{2\pi}{n} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$$

Method 2:

We take the consecutive seconds as $t-1,\ t,\ t+1$



we write $x = a\cos nt$ suitably

$x_1 = a\cos n(t-1) \rightarrow (1)$

$x_2 = a\cos n(t) \rightarrow (2)$

$x_3 = a\cos n(t+1) \rightarrow (3)$

Consider x_1+x_3,

$x_1+x_3 = a\left[\cos(nt-n)+\cos(nt+n)\right]$

$= a\left[\cos nt\cdot\cos n + \sin \cancel{nt}\cdot\sin n + \cos nt\cdot\cos n - \sin \cancel{nt}\cdot\sin n\right]$

$= 2(a\cos nt)\cos n$

$x_1+x_3 = 2x_2\cos n$

$\frac{x_1+x_3}{2x_2} = \cos n \Rightarrow n = \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$

$\therefore$ period of oscillation is $\frac{2\pi}{n} = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$

16. The maximum velocity of a particle moving in a SHM is 2ft/sec and its period is $\frac{1}{5}$ sec. P.T the amplitude is $\frac{1}{5\pi}$ feet.

Maximum velocity $= na = 2 \rightarrow (1)$

$$T = \frac{2\pi}{n} = \frac{1}{5} \Rightarrow n = 10\pi \rightarrow (2)$$

Sub (2) in (1) $\Rightarrow a = \frac{2}{n} = \frac{2}{10\pi} = \frac{1}{5\pi}$

$\therefore$ The amplitude is $\frac{1}{5\pi}$ feet.

---

17. A particle executing SHM makes 100 complete oscillation per minute and its maximum speed is 15 ft/sec. What is the length of its path and maximum acceleration.

α $\boxed{\text{Frequency} = \text{no. of oscillations made per second}}$

In 60 sec = 100 oscillation

$$1 \text{ sec} = \frac{100}{60} = \frac{5}{3} \text{ oscillation.}$$

α $f = \frac{10}{6} = \frac{5}{3}$, $T = \frac{1}{f} = \frac{3}{5}$, $na = 15$ (maxi. speed)

$$T = \frac{3}{5} \qquad a = \frac{15 \times 3}{10\pi} = \frac{9}{2\pi}$$

$$\frac{2\pi}{n} = \frac{3}{5} \Rightarrow \boxed{n = \frac{10\pi}{3}}, \quad \boxed{a = \frac{9}{2\pi}} \text{ ft}$$

i) Length of the path $= 2a = \frac{9}{\pi}$ ft

ii) Maximum acceleration $= n^2 a = \frac{\cancel{100}^{50} \pi^{\cancel{2}}}{\cancel{9}} \times \frac{\cancel{9}}{\cancel{2\pi}}$

$$= 50 \text{ ft/sec}^2 = 50\pi$$

Book work:

S.T the resultant motion of two SHM of same period along two ⊥r lines is along an ellipse.

The particle is given two SHM. One along the x-axis and the other along the y-axis. The periods are the same.

Let the particle starts from the extreme position which is also along the x-axis.

$\therefore \boxed{x = a\cos nt} \longrightarrow (1)$

For the 2nd SHM which is along the y-axis the point A is not the extreme point.

$\therefore \boxed{y = b\cos(nt+\varepsilon)} \longrightarrow (2)$ $[\cos(A+B) = \cos A\cos B - \sin A\sin B]$

From (1), $\frac{x}{a} = \cos nt \longrightarrow (3) \Rightarrow \cos^2 nt = \frac{x^2}{a^2}$

$\Rightarrow \cos^2 nt + \sin^2 nt = 1 \Rightarrow \frac{x^2}{a^2} + \sin^2 nt = 1 \Rightarrow \sin^2 nt = 1 -$

$\Rightarrow \sin nt = \pm\sqrt{1 - x^2/a^2} \longrightarrow (4)$

Expand the right side of (2),

$$y = b[\cos nt\cos\varepsilon - \underline{\sin nt}\sin\varepsilon]$$

Sub (3) & (4),

$$y = b\left[\frac{x}{a}\cos\varepsilon \pm \sqrt{1-\frac{x^2}{a^2}}\sin\varepsilon\right] \longrightarrow (5)$$

$$\frac{y}{b} - \frac{x}{a}\cos\varepsilon = \pm\left[\sqrt{1-\frac{x^2}{a^2}}\sin\varepsilon\right]$$

Squaring on both sides,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2}\cos^2\varepsilon - \frac{2xy}{ab}\cos\varepsilon = \left(1-\frac{x^2}{a^2}\right)\sin^2\varepsilon$$

$$\frac{y^2}{b^2} + \frac{x^2}{a^2}\cos^2\varepsilon - \frac{2xy}{ab}\cos\varepsilon = \sin^2\varepsilon - \frac{x^2}{a^2}\sin^2\varepsilon$$

$$\frac{y^2}{b^2} + \frac{x^2}{a^2}\cos^2\varepsilon + \frac{x^2}{a^2}\sin^2\varepsilon - \frac{2xy}{ab}\cos\varepsilon = \sin^2\varepsilon$$

$$\frac{y^2}{b^2} + \frac{x^2}{a^2}(\underline{\cos^2\varepsilon + \sin^2\varepsilon}) - \frac{2xy}{ab}\cos\varepsilon = \sin^2\varepsilon$$

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\varepsilon = \sin^2\varepsilon$$

$$1/a^2\, x^2 + 1/b^2\, y^2 - 2/ab\; xy\cos\varepsilon = \sin^2\varepsilon \longrightarrow (6)$$

$\therefore$ (6) is the eqn of the curve described by the particle which has two SHM along $\perp r$ lines.

To find the shape of the curve:

We consider $(h^2 - ab)$

$$ax^2 + by^2 + 2hxy = \text{constant}$$

Here $2h \rightarrow$ coefficient of xy

$a \rightarrow$ coefficient of x

$b \rightarrow$ coefficient of y

$h^2 - ab < 0 \rightarrow$ ellipse
$h^2 - ab > 0 \Rightarrow$ hyperbola
$h^2 - ab = 0 \rightarrow$ parabola

From (6) ,

$\Rightarrow 2h = \frac{-2}{ab} \cos \varepsilon \Rightarrow h = \frac{-1}{ab} \cos \varepsilon$

$x^2 = 1/a^2$, $y^2 = 1/b^2$

$$\left[\because \left(\frac{1}{a^2}\right)x^2 + \left(\frac{1}{b^2}\right)y^2 - \left(\frac{2}{ab} \cos \varepsilon\right) xy = \text{constant}\right]$$

So, $h^2 - ab = \frac{1}{a^2b^2} \cos^2 \varepsilon - \frac{1}{a^2b^2} = \frac{1}{a^2b^2}(\cos^2 \varepsilon - 1)$

As $(h^2 - ab < 0)$, the path is an <u>ellipse</u>.

Note: 2m

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos \varepsilon = \sin^2 \varepsilon$$

If $\varepsilon = \pi$, sub in the above eqn we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

$$\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

Hence, the eqn become two incident lines.

State Hooke's law:

$$\text{Tension} = \lambda \left[\frac{\text{extension (or) compression}}{\text{natural length}}\right]$$

Book work:

S.T the resultant of 2 SHM of the same period along the same st. line is also a SHM with the same period.

* The general form of the displacem- is taken.

* Let a_1 & a_2 $\longrightarrow$ amplitude

* Let ε_1 & ε_2 $\longrightarrow$ epoch of the SHM

$\therefore x_1 = a_1 \cos(nt + \varepsilon_1) \longrightarrow (1)$ $\boxed{x = a\cos(nt+\varepsilon}$

$x_2 = a_2 \cos(nt + \varepsilon_2) \longrightarrow (2)$

(1) + (2), $[\cos(A+B) = \cos A \cos B - \sin A \sin B]$

$x_1 + x_2 = a_1[\cos(nt + \varepsilon_1)] + a_2[\cos(nt + \varepsilon_2)]$

$= a_1(\cos nt \cos\varepsilon_1 - \sin nt \sin\varepsilon_1) + a_2(\cos nt \cos$
$- \sin nt \sin$

$= a_1 \cos nt \cos\varepsilon_1 + a_2 \cos nt \cos\varepsilon_2 - a_1 \sin nt \sin$
$- a_2 \sin nt$

$= \underbrace{(a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2)}_{a\cos\varepsilon} \cos nt - \underbrace{(a_1 \sin\varepsilon_1 + a}_{a\sin\varepsilon}$ $\sin nt$

$= a\cos\varepsilon \cos nt - a\sin\varepsilon \sin nt$

$= a(\cos\varepsilon \cos nt - \sin\varepsilon \sin nt)$

$\boxed{x_1 + x_2 = a\cos(nt + \varepsilon)}$

This is the eqn of an SHM.

$\propto$ amplitude of a SHM $= a$

$\propto$ epoch of the SHM $= \varepsilon$

$\Rightarrow$ we now get a & ε,

$a\cos\varepsilon = a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2$

$a\sin\varepsilon = a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2$ ← doubt

$\therefore a = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2)}$

$$\boxed{\tan \mathcal{E} = \frac{a \sin \mathcal{E}}{a \cos \mathcal{E}}} = \frac{a_1 \sin \mathcal{E}_1 + a_2 \sin \mathcal{E}_2}{a_1 \cos \mathcal{E}_1 + a_2 \cos \mathcal{E}_2}$$

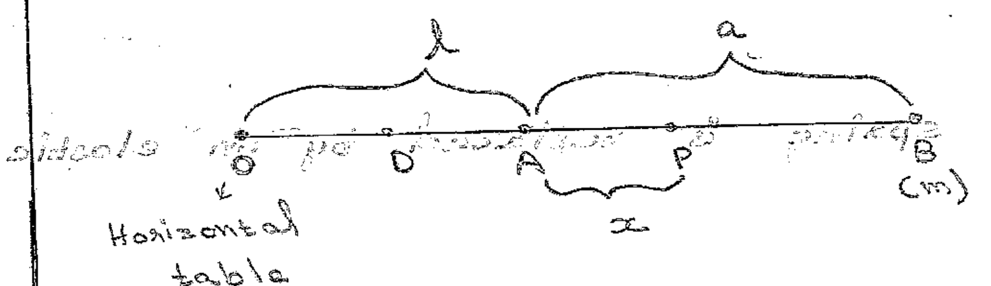
$$\therefore \mathcal{E} = \tan^{-1}\left[\frac{a_1 \sin \mathcal{E}_1 + a_2 \sin \mathcal{E}_2}{a_1 \cos \mathcal{E}_1 + a_2 \cos \mathcal{E}_2}\right]$$

$\therefore$ The resultant motion is also be SHM.

Hence proved.

Bookwork : one end of spring of length l is fixed to a fixed pt O.

One end of a SHM along the horizontal line spring of length λ is fixed to fixed pt 'o' on a smooth horizontal table and a heavy particle of mass 'm' is attach to the other end. if the particle is pulled through a distance 'a', $a < \lambda$ and then let go. Find its motion.



*Let OA $\rightarrow$ elastic spring kept on horizontal table.

i.e) $OA = l$

*O $\rightarrow$ fixed point

*At the other end, A attach a mass 'm' pulled through a distance 'a' & then let go.

*As there is extension, there is a tension

*P $\rightarrow$ position of the particle

i.e) $AP = x$

Applying Newton's II law,

$$\boxed{ma = F \Rightarrow m\ddot{x} = T} \rightarrow (1)$$

$F = T$

$a = \ddot{x}$

By Hooke's law, $\boxed{T = \frac{\lambda x}{l}}$ ~~(H)~~ (2)

As it is towards 'O', there is (-) sign $\rightarrow T = \frac{-\lambda x}{l}$

Sub ② in ⑬;

$$m\ddot{x} = \frac{-\lambda x}{l}$$

$$\ddot{x} = \frac{-\lambda x}{lm}$$

$$\boxed{\ddot{x} = -n^2 x}$$

This eqn represents SHM.

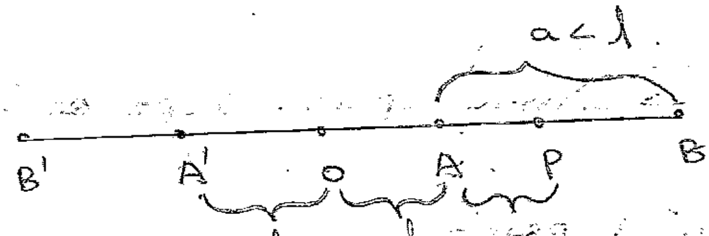
- From B to A there is an extension.
- From A to D there is a compression
- then B to D there is an extension.

Here $n^2 = \frac{\lambda}{ml} \Rightarrow \boxed{n = \sqrt{\frac{\lambda}{ml}}}$

- $\Rightarrow x = a\cos nt = a\cos\sqrt{\frac{\lambda}{ml}}\,t$
- $\Rightarrow v^2 = n^2(a^2 - x^2) = \frac{\lambda}{ml}(a^2 - x^2)$
- $\Rightarrow T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{ml}{\lambda}}$

Remark:

✓ If the spring is replaced by an elastic string



- OA = natural length l.
- The mass attached to the end A is pulled through a distance "a".
- As there is extension there is a T and proceeding as before we find the motion to be a SHM between B to A.
- When the mass reaches the point
- There is no extension of the string therefore no tension and so no SHM.

* But reaching the point A', the particle has the maximum velocity na. So the particle proceeds further as there is extension when it moves beyond A', there is tension.
$\therefore$ the motion is SHM b/w A' to B'.

* So the particle is found to execute the SHM b/w B to A, A' to B', B' to A' and A to B. $\Rightarrow$ Time taken $= \frac{2\pi}{\sqrt{\lambda/ml}} = 2\pi\sqrt{\frac{ml}{\lambda}}$ (previous B.W)

* It moves with a constant velocity b/w A to A' and A' to A.

$$\text{Time taken} = \frac{2 \times \text{distance travelled}}{\text{speed}} = \frac{2AA'}{na}$$

$$= \frac{2(2l)}{a}\sqrt{\frac{ml}{\lambda}}$$

$\therefore$ Total time taken for the journey of 'm' (move from B to B' and B' to B) $= \frac{2\pi}{\sqrt{\lambda/ml}} + 2\left(\frac{AA'}{na}\right)$

$$= 2\pi\sqrt{\frac{ml}{\lambda}} + \frac{4l}{a}\sqrt{\frac{ml}{\lambda}} = 2\pi\sqrt{\frac{ml}{\lambda}} + \frac{4l}{a}\sqrt{\frac{lm}{\lambda}}$$

$$= 2\sqrt{\frac{ml}{\lambda}}\left[\pi + \frac{2l}{a}\right]$$

Book work: (X) 5M

SHM along a vertical line.

A light spiral string of length 'l' and hangs vertically in the position where OA is a point of suspension. Let B be its equilibrium position. When a mass m is attached to its lower end and $AB = a$, if m is pulled vertically downwards from B to C through a distance b let go, find its motion.

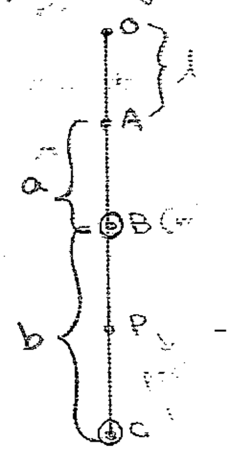
* OA is the natural length of the string.

i.e) $OA = l$

At A:

* A mass 'm' attached to the pt pulls it down to the pt B.
* Such that B is the equilibrium pos[illegible]
* So the forces acting at B must ba[illegible] each other. (forces are balanced)

i.e) the tension towards BA = the weight with opposite direction.



At B:

$$\uparrow T = \frac{\lambda a}{l} = \text{weight} \quad ; \text{weight} = mg \downarrow$$

$$\Rightarrow \frac{\lambda a}{l} = mg \longrightarrow (1)$$

B $\uparrow T$ $\downarrow mg$

* The mass m is pulled to the pt C & let

i.e) $BC = b$

* Let P $\longrightarrow$ position of mass m at any inst[illegible]

i.e) $BP = x$

* The new tension at $P = \lambda\left(\frac{\text{extension}}{\text{natural length}}\right)$

new tension $\uparrow T_1$

weight $\downarrow mg$ P $= \frac{\lambda(a+x)}{l}$

* Here this is towards O.

i.e) along the x decreasing direction.

$\therefore$ a (−) sign is attached. mg is downw[illegible] along the x increasing direction.

At P:

Newton's II law,

$\boxed{ma = F}$ $\rightarrow$ tension + weight

$$m\ddot{x} = mg - \frac{\lambda(a+x)}{l}$$

$$= mg - \frac{\lambda a}{l} - \frac{\lambda x}{l}$$

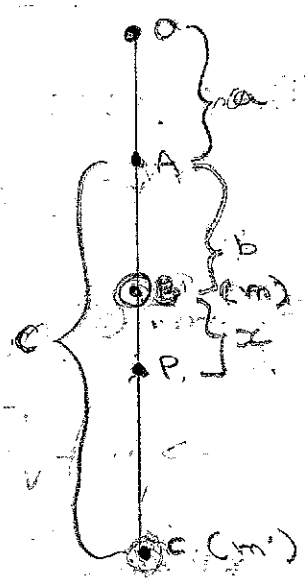
$$m\ddot{x} = \frac{\lambda a}{\lambda} - \frac{\lambda a}{\lambda} - \frac{\lambda x}{\lambda} \quad \left(\because mg = \frac{\lambda a}{\lambda}\right) (\text{by } ⑬)$$

$$m\ddot{x} = -\frac{\lambda x}{l} \Rightarrow \ddot{x} = -\left(\frac{\lambda}{ml}\right)x = -(n^2)x \rightarrow \text{SHM}$$

$$\Rightarrow n^2 = \frac{\lambda}{ml} \Rightarrow n = \sqrt{\frac{\lambda}{ml}}$$

$\therefore$ The period is $2\pi\sqrt{\frac{lm}{\lambda}}$

18. Two body of mass m and m' are attached to the lower end of an elastic string whose upperend is fixed and hangs at rest M' falls off. S.T the distance of 'm' from the upper end of the string and time T is $a+b+c\left(\cos\sqrt{g/b}\ t\right)$



At A:

*mass 'm' attached $\rightarrow$ pulled to B $\rightarrow$ equilibrium at B $\rightarrow$ force is balanced.

At B:

The tension at B is gn by

$$\uparrow \boxed{T = \frac{\lambda b}{a}} \rightarrow \text{Hookes law} = \text{weight} = mg \downarrow \rightarrow ①$$

$$\therefore mg = \frac{\lambda b}{a} \Rightarrow \frac{\lambda}{am} = \frac{g}{b} \rightarrow ②$$

At C:

The tension at C is gn by

$$\uparrow T = \frac{\lambda c}{a} = \text{weight} = m'g \downarrow$$

$$\therefore m'g = \frac{\lambda c}{a} \rightarrow ③$$

* m' fall of the mass m starts moving upwards towards B. At any instant let P be the position of mass m. i.e) PB = x

At P

* The extension at P is $b+x$.

* The new tension $= -\lambda\left(\frac{b+x}{a}\right)$

* The tension is towards O the wgh act vertically downwards.

* The x increasing direction is downwards.

The eqn of motion at P is given by

By newton's II law, $ma = F \rightarrow$ tension + weight

$$\Rightarrow m\ddot{x} = mg - \frac{\lambda(b+x)}{a} = mg - \frac{\lambda b}{a} - \frac{\lambda x}{a}$$

$$m\ddot{x} = \frac{\lambda b}{a} - \frac{\lambda b}{a} - \frac{\lambda x}{a} = -\frac{\lambda x}{a} \quad \left(\because mg = \frac{\lambda b}{a}\right)$$

$$\ddot{x} = \frac{-\lambda}{ma}x \qquad \ddot{x} = n^2 x$$

The eqn represents a SHM. SHM

$$n^2 = \frac{\lambda}{am}$$

From (2), $\frac{\lambda}{am} = \frac{g}{b}$

$$\Rightarrow n = \sqrt{\frac{\lambda}{am}} = \sqrt{\frac{g}{b}}$$

* Hence the amplitude of the ~~SHM exec~~ by the mass 'm' is given by

$$x = a \cos nt$$

$$x = c \cos nt \quad [\because a = c]$$

$$= c \cos\sqrt{\frac{g}{b}}\,t$$

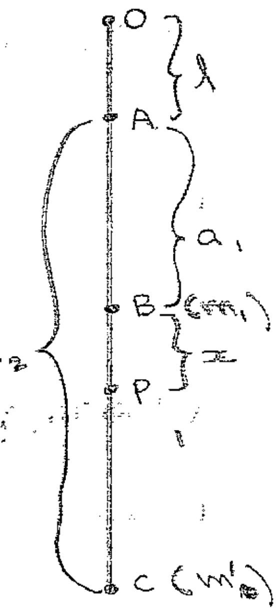
* The distance of P at any instant from O is $OP = OA + AB + BP$

$$= a + b + x$$

$$= a + b + c\cos\sqrt{g/b}\,t$$

19. If T_1 and T_2 are the periods corresponding to 2 different masses. When attached to a vertical elastic string and if a_1 and a_2 are the statical extension due to mass. S.T

$$g = \frac{4\pi^2(a_1 - a_2)}{T_1^2 - T_2^2}$$



* Let OA = natural length l

* AB = a_1 (statical extension)

<u>At B:</u>

$\uparrow$ Tension at B = T_1 = $\dfrac{\lambda(\text{extension})}{\text{original length}} = \dfrac{+\lambda a_1}{l}$

upward direction

$\downarrow$ weight $= m_1 g = \dfrac{\lambda a_1}{l} \rightarrow (1)$

(downward direction

$$\Rightarrow a_1 = \frac{m_1 g l}{\lambda} \rightarrow (2)$$

<u>At C:</u>

Tension at C $= \dfrac{\lambda(\text{extension})}{\text{original length}} = \dfrac{-\lambda a_2}{l}$

Two forces, Tension $\rightarrow \uparrow$ weight $\rightarrow \downarrow$

weight $= m_2 g = \dfrac{\lambda a_2}{l} \rightarrow (3)$

$$\Rightarrow a_2 = \frac{m_2 g l}{\lambda}$$

<u>At P</u>: extension at P $= a + x$, PB $= x$

$\uparrow$ Tension at P $= \dfrac{\lambda(\text{extension})}{\text{original length}} = \dfrac{-\lambda(a_1 + x)}{l}$

$\downarrow$ weight $= m_1 g$

Newton's II law,

$$m_1\ddot{x} = m_1 g - \lambda\left(\frac{a_1 + x}{l}\right) = m_1 g - \frac{\lambda a_1}{l} - \frac{\lambda x}{l}$$

$$m_1\ddot{x} = \frac{\lambda a_1}{l} - \frac{\lambda a_1}{l} - \frac{\lambda x}{l} = \frac{-\lambda x}{l}$$

$$\ddot{x} = \frac{-\lambda x}{m_1 l} = -n_1^2 x \rightarrow (4)$$

The eqn represents a SHM.

$$n_1^2 = \frac{\lambda}{m_1 l} \Rightarrow n_1 = \sqrt{\frac{\lambda}{m_1 l}}$$

$$\text{III}^{ly}\quad n_2^2 = \frac{\lambda}{m_2 l} \Rightarrow n_2 = \sqrt{\frac{\lambda}{m_2 l}}$$

$$\therefore T_1 = \frac{2\pi}{n_1},\ T_2 = \frac{2\pi}{n_2}$$

$$\Rightarrow T_1^2 - T_2^2 = \frac{4\pi^2}{n_1^2} - \frac{4\pi^2}{n_2^2} = 4\pi^2\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right] = 4\pi^2\left[\frac{1}{\lambda/m_1 l} - \dots\right]$$

$$= 4\pi^2\left[\frac{m_1 l}{\lambda} - \frac{m_2 l}{\lambda}\right]$$

$$= 4\pi^2 \frac{l}{\lambda}\left[m_1 - m_2\right] \longrightarrow (5)$$

$$(1) \Rightarrow m_1 g = \frac{\lambda a_1}{l} \Rightarrow a_1 = \frac{m_1 g l}{\lambda}$$

$$(3) \Rightarrow m_2 g = \frac{\lambda a_2}{l} \Rightarrow a_2 = \frac{m_2 g l}{\lambda}$$

$$4\pi^2(a_1 - a_2) = 4\pi^2\left[\frac{m_1 g l}{\lambda} - \frac{m_2 g l}{\lambda}\right]$$

$$= \frac{4\pi^2 g l}{\lambda}\left[m_1 - m_2\right] \longrightarrow (6)$$

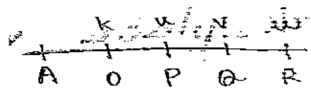
$$\frac{(6)}{(5)} \Rightarrow \frac{4\pi^2(a_1 - a_2)}{T_1^2 - T_2^2} = \frac{\frac{4\pi^2 g l}{\lambda}(m_1 - m_2)}{\frac{4\pi^2 l}{\lambda}(m_1 - m_2)}$$

$$\frac{4\pi^2(a_1 - a_2)}{T_1^2 - T_2^2} = g$$

Assignment

20. A particle describe a SHM along a st. line which A is a fixed point. If P, Q, R are p[oints] on the st. line lying on the same side o[f A] such that $AP = p$, $AQ = q$, $AR = r$ & u, v, w are [the] speeds of the particle at these points. [Show that] the periodic time T of the motion is g[iven by]

$$4\pi^2 \frac{(q-r)(r-p)(p-q)}{T^2} = u^2(q-r) + v^2(r-p) + w^2(\dots$$



* A → fixed point.

* O → mean position.

* P, Q, R lie on the same side of A.

* The distance of these pts from A are → $AP = p$, $AQ = q$, $AR = r$.

* The distance of the pts from the mean position → $AO = k$, $OP = p-k$, $OQ = q-k$, $OR = r-k$

* Velocities at P, Q, R are u, v, w.

* Amplitude → a

use $v^2 = n^2(a^2 - x^2)$ suitably

$$\rightarrow u^2 = n^2\left[a^2 - (p-k)^2\right]$$

$$v^2 = n^2\left[a^2 - (q-k)^2\right]$$

$$w^2 = n^2\left[a^2 - (r-k)^2\right]$$

Consider $\Sigma u^2(q-r)$,

$$\Sigma u^2(q-r) = n^2\Big[\left[a^2(q-r+r-p+p-q)\right] - \big[(q-r)(p-k)^2 + (r-p)(q-k)^2 + (p-q)(r-k)^2\big]\Big]$$

$$= n^2\Big[-\big[k^2(q-r+r-p+p-q) - 2k(pq-pr+qr-pq+pr-qr) + p^2(q-r) + q^2(r-p) + r^2(p-q)\big]\Big]$$

$$= -n^2\left[p^2(q-r) + q^2(r-p) + r^2(p-q)\right]$$

$$= -n^2\left[p^2q - p^2r + q^2r - q^2p + r^2p - r^2q\right] \rightarrow (1)$$

Consider,

$$(q-r)(r-p)(p-q) = (qr - r^2 - pq + pr)(p-q)$$

$$= pqr - pr^2 - p^2q + p^2r - q^2r + qr^2 + pq^2 - pqr \rightarrow (2)$$

Sub (2) in (1),

$$\Sigma u^2(q-r) = -n^2(-1)(q-r)(r-p)(p-q)$$

$$= n^2(q-r)(r-p)(p-q) \rightarrow (3)$$

Replace $n = \frac{2\pi}{T}$

$$\Sigma u^2 (q-r) = \frac{4\pi^2}{T^2} (q-r)(r-p)(p-q)$$

Hence result.

21. A body moving in a st. line OAB with SHM has zero velocity when at the pt. A & B whose distance from 'O' are a & b respectively and has velocity V when half way b/w them. S.T. the complete period is $\frac{\pi(b-a)}{V}$

Let α be the amplitude of SHM

we have, $v^2 = n^2(\alpha^2 - x^2) \rightarrow (1)$

At A, $0 = n^2(\alpha^2 - a^2)$

At B, $0 = n^2(\alpha^2 - b^2)$

$\alpha^2 = a^2 \rightarrow (2)$

$\alpha^2 = b^2 \rightarrow (3)$

$(2) + (3) \Rightarrow 2\alpha^2 = a^2 + b^2$

$$\alpha^2 = \frac{a^2+b^2}{2}$$

At the mid pt of AB,

$$V = \frac{n}{2}(b-a)$$

$$V^2 = n^2\left[\alpha^2 - \left(\frac{a+b}{2}\right)^2\right]$$

i.e) $n = \frac{2V}{b-a}$

$$\therefore V^2 = n^2\left[\frac{1}{2}(a^2+b^2) - \left(\frac{a+b}{2}\right)^2\right] = \frac{n^2}{4}(b-a)^2$$

$$V^2 = \frac{n^2}{4}(b-a)^2$$

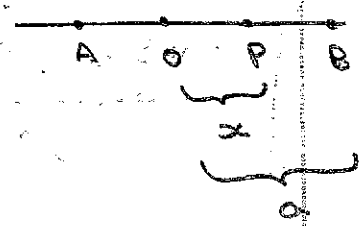
$$V = \frac{n}{2}(b-a)$$

$$\boxed{n = \frac{2V}{b-a}}$$

$$\therefore T = \frac{2\pi}{n} = \frac{2\pi}{2V/b-a} = \frac{\pi(b-a)}{V}$$

22. A particle is performing SHM b/w pts A & B. If the period of oscillation is 2π. S.T the velocity at any pt P is a mean proportional b/w AP & BP. [b is said to be the mean proportional b/w a & c if $b^2 = ac$]

$T = \frac{2\pi}{n}$, Here $T = 2\pi$



so, $n = 1$

If v is the velocity at P

$v^2 = n^2(a^2 - x^2)$

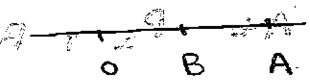
$v^2 = n^2(OB^2 - OP^2) = n^2(OB + OP)(OB - OP)$

$v^2 = 1.(AO + OP)(OB - OP)$

$v^2 = AP \cdot BP$

Hence the result.

23. S.T a particle executing SHM requires sixth of its period to move from one position of maximum displacement to one in which the displacement is half the amplitude.



* Let a be the amplitude.

* Let t_1 be the time taken from A, the position of maximum displacement.

$\Rightarrow x = a$ (which is the position of rest also)

to the position when $x = a/2$

$x = a \cos nt_1$

$a/2 = a \cos nt_1$

$1/2 = \cos nt_1$

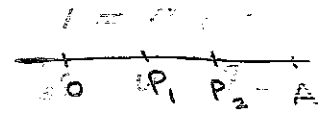
$nt_1 = \pi/3 \Rightarrow \boxed{t_1 = \frac{\pi}{3n}}$

$\therefore \boxed{T = \frac{2\pi}{n}} \Rightarrow \frac{1}{6}T = \frac{1}{6_3} \cdot \frac{2\pi}{n} \Rightarrow \frac{1}{6}T = t_1$

Hence the result,

24. A point executes SHM. ST in 2 position the velocities are u, v and the correspon acceleration are α & β. ST the distance the position, $\frac{v^2-u^2}{\alpha+\beta}$. Prove also that the amplitude of motion is

$$\left[\frac{(v^2-u^2)(\alpha^2 v^2-\beta^2 u^2)}{\alpha^2-\beta^2}\right]^{1/2}$$



Let $OA = a$, the amplitude of the S
Let P_1, P_2 be two position at whi the velocities are u & v and the accele are α and β.

$OP_1 = x_1$, $OP_2 = x_2$

$$\ddot{x} = -n^2 x$$

$$v^2 = n^2(a^2 - x^2)$$

At P_1, $\alpha = -n^2 x_1 \longrightarrow (1)$

$$u^2 = n^2(a^2 - x_1^2) \longrightarrow (2)$$

At P_2, $\beta = -n^2 x_2 \longrightarrow (3)$

$$v^2 = n^2(a^2 - x_2^2) \longrightarrow (4)$$

To [illegible] the distance b/w P_1, P_2

$P_1P_2 = x_2 - x_1$

$(4) - (2) \Rightarrow v^2 - u^2 = n^2(x_1^2 - x_2^2) \longrightarrow (5)$

$(1) + (3) \Rightarrow \alpha + \beta = -n^2(x_1 + x_2) \longrightarrow (6)$

$$\frac{(5)}{(6)} \Rightarrow \frac{v^2-u^2}{\alpha+\beta} = \frac{-(x_1+x_2)(x_1-x_2)}{x_1+x_2} = x_2 - x_1 \longrightarrow$$

cosider,

$$\alpha^2 v^2 - \beta^2 u^2 = n^4 x_1^2 \cdot n^2(a^2 - x_2^2) - n^4 x_2^2 n^2(a^2 - x_1^2)$$

$$= n^6\left[a^2x_1^2 - x_1^2x_2^2 - a^2x_2^2 + x_1^2 + x_2^2\right]$$

$$\alpha^2v^2 - \beta^2u^2 = n^6a^2(x_1^2 - x_2^2)$$

$$\therefore \left[\frac{(v^2-u^2)(\alpha^2v^2-\beta^2u^2)}{\alpha^2-\beta^2}\right]^{1/2} = \frac{\left[n^2(x_1^2-x_2^2)\cdot n^6a^2(x_1^2-x_2^2)\right]^{1/2}}{n^4(x_1^2-x_2^2)}$$

$$= \frac{n^4(x_1^2-x_2^2)a}{n^4(x_1^2-x_2^2)}$$

$$= a$$

25. A particle executing SHM has velocities v_1 & v_2. When its distance from O. The centre of oscillations are d_1, d_2 respectively. Find the amplitude, periodic time and the velocity of the particle. When its distance from O is $\frac{1}{2}(d_1+d_2)$

$$v^2 = n^2(a^2-x^2)$$

$$x = d_1,\ v = v_1,\ v_1^2 = n^2(a^2-d_1^2) \rightarrow (1)$$

$$x = d_2,\ v = v_2,\ v_2^2 = n^2(a^2-d_2^2) \rightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{v_1^2}{v_2^2} = \frac{a^2-d_1^2}{a^2-d_2^2} \Rightarrow \frac{v_1^2}{v_2^2} - \frac{a^2-d_1^2}{a^2-d_2^2} = 0$$

$$\frac{v_1^2a^2 - v_1^2d_2^2 - v_2^2a^2 + v_2^2d_1^2}{v_2^2(a^2-d_2^2)} = 0$$

$$v_1^2a^2 - v_2^2a^2 = v_1^2d_2^2 - v_2^2d_1^2$$

$$a^2 = \frac{d_2^2v_1^2 - d_1^2v_2^2}{v_1^2 - v_2^2}$$

$$a = \sqrt{\frac{d_2^2v_1^2 - d_1^2v_2^2}{v_1^2-v_2^2}}$$

From ①,

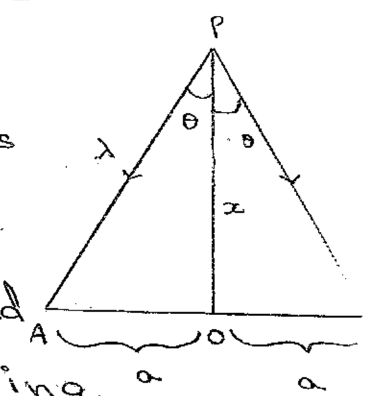
$$v_1^2 = n^2 \left[\frac{d_2^2 v_1^2 - d_1^2 v_2^2 - d_1^2}{v_1^2 - v_2^2} \right]$$

$$= n^2 \left[\frac{(d_2^2 - d_1^2) v_1^2}{v_1^2 - v_2^2} \right]$$

26. The ends of an elastic string of natu[ral] length a are fixed at pts A and B distance 2a apart on a smooth horiz[ontal] table. A particle of mass 'm' is atta[ched] to the middle pt of the string & slightly displaced along the direction [⊥] to AB. ST period of oscillation is $\pi\sqrt{\frac{2}{}}$

* The length of the elastic string is a.

* It is tied b/w two pts A & B where AB = 2a.

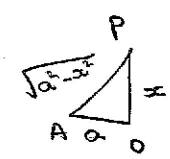


* There is a mass attached to the middle pt of the string.

* The mass is pulled in a direction [⊥] to AB through a distance x and then let go.

* The mass executes a SHM.

<u>To find the period:</u>



The new length = AP + BP = 2 (AP)

The extension = 2(AP) - (original length)

i.e) = 2 (AP) - a

$= 2\left(\sqrt{a^2-x^2}\right) - a \rightarrow (1)$

* The forces acting on the mass a[re] tension T along PA and tension T alo[ng] the PB.

* Their components $2T\cos A\hat{O}P$ act towards O along PO.

Applying the Newton's II law,

$$m\ddot{x} = \text{sum of the force} = 2T\cos A\hat{P}O$$

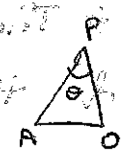
$$= -2\left[\frac{\lambda\ \text{Extension}}{\text{original}}\right]\cdot\frac{x}{\sqrt{a^2-x^2}}$$

$$= \frac{-2\lambda}{a}\left[2\sqrt{a^2-x^2}-a\right]\cdot\frac{x}{\sqrt{a^2-x^2}}$$

$$= \frac{-2\lambda}{a}\left[2-\frac{a}{\sqrt{a^2-x^2}}\right]\cdot x$$

As x is very very small x^2 is much smaller. So we get x^2 term.

Hence we have,



$$= \frac{-2\lambda}{a}\left[2-\frac{a}{a}\right]x$$

$$\cos APO = \frac{OP}{AP} = \frac{x}{\sqrt{a^2+x^2}}$$

$$= \frac{-2\lambda}{a}\left[2-1\right]x$$

$$\cos\theta = \frac{x}{\sqrt{a^2+x^2}}$$

$$m\ddot{x} = -\frac{2\lambda}{a}x$$

Comparing ① by $\ddot{x}$

$$\ddot{x} = -n^2 x$$

$$n^2 = \frac{2\lambda}{am} \Rightarrow n = \sqrt{\frac{2\lambda}{am}}$$

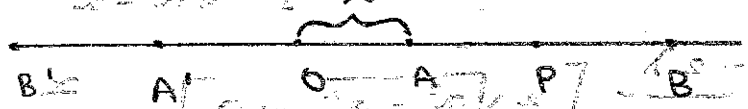
$$\therefore \ddot{x} = \frac{-2\lambda}{am}x$$

$$\text{Period } T = \frac{2\pi}{\sqrt{2\lambda}}\sqrt{am} = \pi\sqrt{\frac{2am}{\lambda}}$$

---

27. The end of a elastic string whose modulus of elasticity is λ and natural length 'a' is fixed to a pt on a smooth horizontal plane and other end is attached to a particle of mass 'm' lying on the table. The particle is pulled from the fixed pt

to a distance $2a$ and then let go. S.T. the time of a complete oscillation is $2\sqrt{\frac{am}{\lambda}}(\pi+2)$



* If the spring is replaced by an elastic string we discuss its motion as follows.

* Let OA = natural length l

* The mass attached to the end A pulled through a distance 'a'.

* A' and B' are pts such that OA' = OA, OB' = OB.

* Then the motion of the particle will be SHM, so long as the tension act otherwise it will be a free motion under no force.

* Thus the motions from B to A, A' to B' to A', A to B from a complete oscillation of the SHM and the motion from A to A' and A' to A are free motions with a constant speed.

$\therefore$ The time taken [for one full to and for motion from B to B' and Back to B]

= time taken to execute a SHM

+ time taken to describe AA' & A'A

$$= \frac{2\pi}{n} + \frac{2AA'}{\text{Max. velo}} \quad [\because OA = OA' = l]$$

$$= \frac{2\pi}{n} + \frac{2(2a)}{na} = \frac{2}{n}(\pi + 2)$$

$$= \frac{2}{\sqrt{\frac{\lambda}{ma}}}(\pi + 2)$$

Finding 'n'

By newton's law : $m\ddot{x} = T \rightarrow (1)$

By hooke's law : $T = \lambda \left(\frac{\text{extension}}{\text{original length}}\right)$,

$$T = \frac{\lambda x}{a}$$

$$T = \frac{-\lambda x}{a} \quad [\because \text{the tension acts towards 'O'}]$$

$$\Rightarrow m\ddot{x} = \frac{-\lambda x}{a}$$

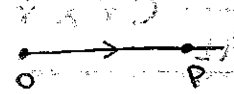
$$\ddot{x} = \frac{-\lambda}{ma} x$$

$$\ddot{x} = -n^2 x \rightarrow (2)$$

From (2) $\Rightarrow n^2 = \frac{\lambda}{am} \Rightarrow n = \sqrt{\frac{\lambda}{am}}$

Unit - V

Central Orbits



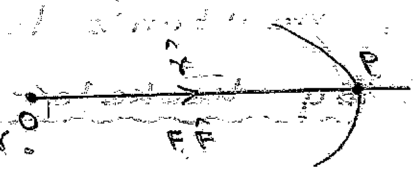
Definition:

When a particle is under the action of a force which is always either towards or away from a fixed point. The particle is said to be under the action of a central force i.e.) a central force is a force whose line of action always passes through a fixed point. The fixed point is called centre of force. The path described is called orbits.

Book work : ①

ST a central orbit is a planar curve

* Let $P \rightarrow$ position of a particle at any instant t.



* The unit vector along $OP = \hat{r}$.

* Let $\phi(r)\hat{r} \rightarrow$ force per unit mass. $\quad \bar{F} = m(\phi(r)\hat{r})$

* The C.F (central force) $\begin{cases} \text{away from 'O'} \rightarrow \hat{r} \\ \text{towards 'O'} \rightarrow -\hat{r} \end{cases}$

* If m is the mass of a particle.

By Newton's II law,

$$ma = F \Rightarrow m\bar{a} = \bar{F}$$

Mass constant

$$m\ddot{\bar{r}} = m(\phi(r)\hat{r}) \rightarrow ① \quad [\because \bar{F} = m(\phi(r)\hat{r})]$$

acceleration

$$\ddot{\bar{r}} = \phi(r)\hat{r} \rightarrow ②$$

To prove :

$$\boxed{\frac{d}{dt}(\underbrace{\bar{r}}_{\text{distance}} \times \underbrace{\dot{\bar{r}}}_{\text{velocity}}) = \bar{0}} \Rightarrow \bar{r} \times \dot{\bar{r}} = c$$

$$\Rightarrow \frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = (\dot{\bar{r}} \times \dot{\bar{r}}) + (\bar{r} \times \ddot{\bar{r}}) \rightarrow ③$$

$$= \bar{0} + (r\hat{r} \times \phi(r)\hat{r})$$

$$= r\phi(r)(\hat{r} \times \hat{r})$$

$$\frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = \bar{0} \rightarrow ④$$

To prove : $\boxed{\bar{r} \cdot \bar{c} = \bar{0}}$

$$\Rightarrow \bar{r} \times \dot{\bar{r}} = c$$

Xply by $\bar{r}$ on both sides,

$$\bar{r} \cdot (\bar{r} \times \dot{\bar{r}}) = \bar{r} \cdot c$$

$$\Rightarrow \bar{r} \cdot c = [\bar{r}\ \bar{r}\ \dot{\bar{r}}] = \bar{0}$$

[$\because$ In a scalar triple product if two vect are the same its value is zero]

$\therefore \bar{r} \cdot \bar{c} = \bar{0}$

Conclusion:

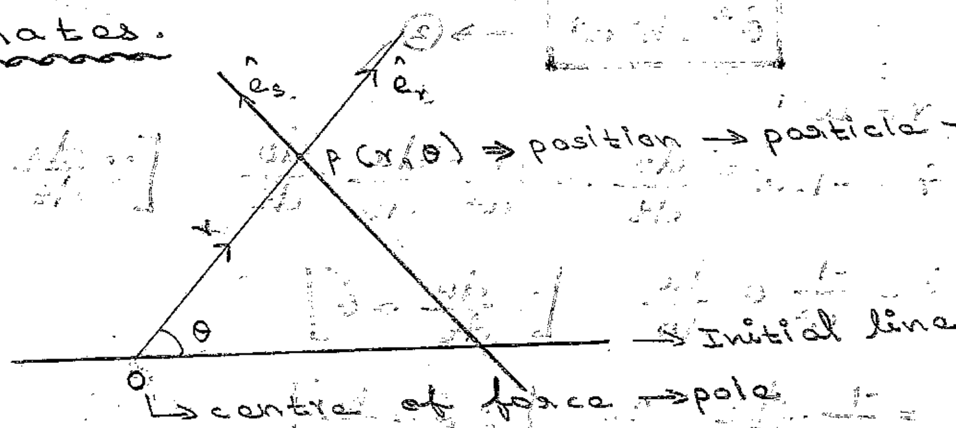
distance $\perp r$ to constant

$\boxed{\bar{r} \perp r\ \bar{c} \Rightarrow \overline{OP} \perp r\ \bar{c}}$

* $\bar{r} \rightarrow$ pv of the moving pt p for all position of p.
* $\overline{OP}$ $\perp r$ to the same vector $\bar{c}$.
* The various positions of p must be on the same plane.
* The path described by p is a planar curve.
* Hence the central orbit is a planar curve.

Book work: 2

Differential eqn of a central orbit in polar co-ordinates.



* Let O $\xrightarrow{\text{is the}}$ central force $\xrightarrow{\text{when it is}}$ pole
* Let x-axis $\rightarrow$ initial line.
* Let $p(r,\theta) \rightarrow$ position of the particle at any instant 't'.
* As the particle is under (the) action of the central force which acts along op and away from o, the force acting on the particle is given by $m[\phi(r)]\hat{r}$.
* The comp of force is along the radial direction.
* So the acceleration component along the transverse direction = 0

i.e) $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

By Newton's II law,

$$\bar{F} = m\bar{a} \Rightarrow m\bar{a} = \bar{F}$$

$$\boxed{\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_s}$$

$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{comp. of } \bar{a} \text{ along rad direction}}\hat{e}_r + \underbrace{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})}_{\text{comp. of } \bar{a} \text{ along transverse direction}}\hat{e}_s$$

$$\therefore \bar{F} = m\bar{a} \Rightarrow \cancel{m}(\phi(r)\cancel{\hat{r}}) = \cancel{m}(\ddot{r} - r\dot{\theta}^2)\cancel{\hat{r}}$$

$$\boxed{\phi(r) = \ddot{r} - r\dot{\theta}^2} \longrightarrow (1)$$

sub $r = \frac{1}{u}$

To find: $\ddot{r}$ & $r\dot{\theta}^2$

Let $r^2\dot{\theta} = h$

$$\dot{\theta} = \frac{h}{r^2}$$

formula $\dot{\theta} = hu^2$ $\left[\because r = \frac{1}{u} = u^{-1} \Rightarrow u = \frac{1}{r}\right]$

$$\boxed{\dot{\theta}^2 = h^2u^4} \longrightarrow (2)$$

$$\Rightarrow r = u^{-1}$$

$$\dot{r} = -1 \cdot u^{-2} \cdot \frac{du}{dt} = \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} \qquad \left[\because \frac{du}{dt} = \frac{du}{d\theta} \cdot \frac{d\theta}{dt}\right]$$

$$\dot{r} = \frac{-1}{u^2}\dot{\theta}\frac{du}{d\theta} \qquad \left[\because \frac{d\theta}{dt} = \dot{\theta}\right]$$

$$\frac{d}{dt}(\theta)$$

$$= \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{h}{r^2} \qquad \left[\because \dot{\theta} = \frac{h}{r^2}\right]$$

$$= \frac{-1}{\cancel{u^2}} \cdot \frac{du}{d\theta} \cdot h\cancel{u^2}$$

$$\boxed{\dot{r} = -h\frac{du}{d\theta}} \longrightarrow (3)$$

$$\Rightarrow \ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) \qquad \frac{du}{d\theta} = \frac{d}{d\theta}(u)$$

$$= -h\frac{d}{d\theta}\left(\frac{du}{dt}\right)$$

$$= -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}(hu^2)$$

$$\boxed{\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}} \longrightarrow (4)$$

$\Rightarrow r\dot{\theta}^2 = r(hu^2)^2$

$= rh^2u^4$

$$\boxed{r\dot{\theta}^2 = h^2u^3} \longrightarrow (5) \quad \left[\because r = \frac{1}{u}\right]$$

Sub. (4) & (5) in (1),

$$\phi(r) = -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3$$

$$\phi(r) = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$\therefore$ The differential eqn of the central orbit is

$$\boxed{h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\phi(r)}$$

Attractive force	Repulsive force
* When the c.F is towards ~~away from~~ O, it is attractive force.	* When the c.F is away from O, it is repulsive force.
* There is a +ve sign on the right side.	* There is a -ve sign on the right side.
$\Rightarrow \phi(r) = +F$	$\Rightarrow \phi(r) = -F$
$h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = -\phi(r) = -F$	$h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = -\phi(r) = +F$
$\frac{d^2u}{d\theta^2} + u = \frac{-F}{h^2u^2}$	$\frac{d^2u}{d\theta^2} + u = \frac{+F}{h^2u^2}$

* Force / unit mass:

$$\phi(r)\,\hat{r} = \text{force / unit mass}$$

$$\phi(r)\,\hat{r} = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)\hat{r}$$

Book work: (3)

To find the velocity of the central orbit.

$$\bar{V} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_s \longrightarrow (1)$$

Formula

To find : $\dot{r}$ and $r\dot{\theta}$

Let $r^2\dot{\theta} = h \Rightarrow \dot{\theta} = \frac{h}{r^2} = hu^2$

$\Rightarrow r = \frac{1}{u} = u^{-1}$ $\left[\because u = \frac{1}{r}\right]$

$$\dot{r} = -1 . u^{-2} . \frac{du}{dt}$$

$$= \frac{-1}{u^2} . \frac{du}{d\theta} . \frac{d\theta}{dt}$$

$$= -\frac{1}{u^2} . \dot{\theta} . \frac{du}{d\theta}$$

$$= \frac{-1}{u^2} . hu^2 . \frac{du}{d\theta}$$

$$\dot{r} = -h\frac{du}{d\theta} \rightarrow (2)$$

$$r\dot{\theta} = \frac{1}{u} hu^2 = hu \rightarrow (3)$$

sub (2) & (3) in (1)

$$\overline{V} = -h\frac{du}{d\theta}\hat{e}_r + hu\hat{e}_\theta$$

$$V^2 = \overline{V} . \overline{V} = h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2$$

$$V^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$V = h\sqrt{\left(\frac{du}{d\theta}\right)^2 + u^2}$$

Laws of a central force:

When the equation of a central orbit is given to obtain the force per unit mass and the speed of the particle at a distance r from the centre of force we have to calculate

$$-h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)\hat{r} , h\sqrt{\left(\frac{du}{d\theta}\right)^2 + u^2}$$

$$\left[\phi(r) = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \text{ \& } V^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\right]$$

Book work : (4)

Differential equation of central orbit in p-r co-ordinates.

* O → centre of force → pole

$\frac{d}{dx} x^n = n x^{n-1}$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

Let $PV = h$

$$\boxed{p^2 v^2 = h^2}$$

$$p^2 \left[\cancel{h^2}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)\right] = \cancel{h^2}$$

$$p^2 = \frac{1}{u^2 + \left(\frac{du}{d\theta}\right)^2}$$

reciprocal, $p^{-2} = \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

diff. w.r.t θ

$$-2p^{-3} \frac{dp}{d\theta} = 2u \cdot \frac{du}{d\theta} + 2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right)$$

$$\frac{-\cancel{2}}{p^3} \frac{dp}{d\theta} = \cancel{2} \frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$\frac{\cancel{-2}}{p^3} \frac{dp}{\cancel{d\theta}} = \cancel{2} \frac{du}{\cancel{d\theta}}\left(\cancel{-}\frac{\phi(r)}{h^2u^2}\right)$$

$$\therefore \boxed{-\phi(r) = h^2u^2\left[\left(\frac{d^2u}{d\theta^2} + u\right)\right]}$$

$$\frac{dp}{p^3} = \frac{\phi(r)}{h^2u^2} \cdot du$$

$\Rightarrow u = \frac{1}{r} \Rightarrow u^2 = \frac{1}{r^2}$

$du = \frac{-1}{r^2} dr$

$$\therefore \frac{dp}{p^3} = \frac{\phi(r)}{h^2/r^2} \times \frac{-1}{r^2} dr$$

$$\frac{h^2}{p^3} \cdot \frac{dp}{dr} = -\phi(r)$$

D.E for a attractive C.F (p-r coordinates):-

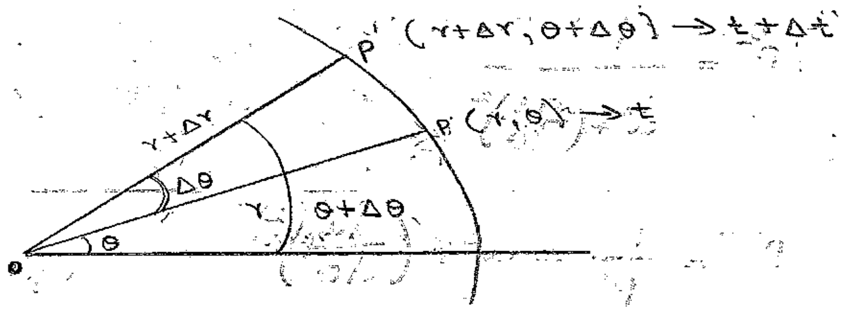
If the central force is an attractive one of magnitude F per unit mass, then $\phi(r) = -F$ and the p-r eqn of orbit is

$$\frac{h^2}{p^3} \frac{dp}{dr} = F$$

This is the differential equation of central orbit in p-r co-ordinates (or) → Pedal differential equation (or) Pedal equation of the orbit.

Book work: (5)

To prove areal velocity is constant of particle describing a central orbit.



* The particle P describes the central orbit.

* The position of the particle at time t P & at time t+Δt is P'.

* The coordinates of P & P' are (r, θ) & (r+Δr, θ+Δθ).

* The elementary area denoted by ΔA is approximately = area of the ΔOPP'.

i.e) ΔA (elementary area) $\simeq \Delta OPP'$

$$\Delta A \simeq \frac{1}{2}(OP)(OP')\sin\Delta\theta$$

area of $\Delta = \frac{1}{2}bh$ and also $= \frac{1}{2}\times$ one side $\times$ another side $\times \sin$ (including

$$\div \Delta t \Rightarrow \frac{\Delta A}{\Delta t} \simeq \frac{1}{2}r(r+\Delta r)\frac{\sin\Delta\theta}{\Delta t}$$

$$\frac{\Delta A}{\Delta t} \simeq \frac{1}{2}r(r+\Delta r)\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta t}$$

When $\Delta t \to 0$, $\Delta r \to 0$ and $\Delta\theta \to 0$

$$\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} \simeq \frac{1}{2}r^2 \lim_{\Delta\theta\to 0}\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}$$

So the areal velocity of P is

$$\underset{\Delta t \to 0}{lt} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$\underset{\Delta t \to 0}{lt} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$

$\underset{\Delta t \to 0}{lt} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta}$

$\underset{x \to 0}{lt} \frac{\sin x}{x} = 1$

Constancy of the areal velocity in central orbit:-

$$\frac{dA}{dt} = \frac{1}{2} \underline{r^2 \dot{\theta}} = \text{constant denoted by } \frac{h}{2} \quad [\because r^2\dot{\theta} = h]$$

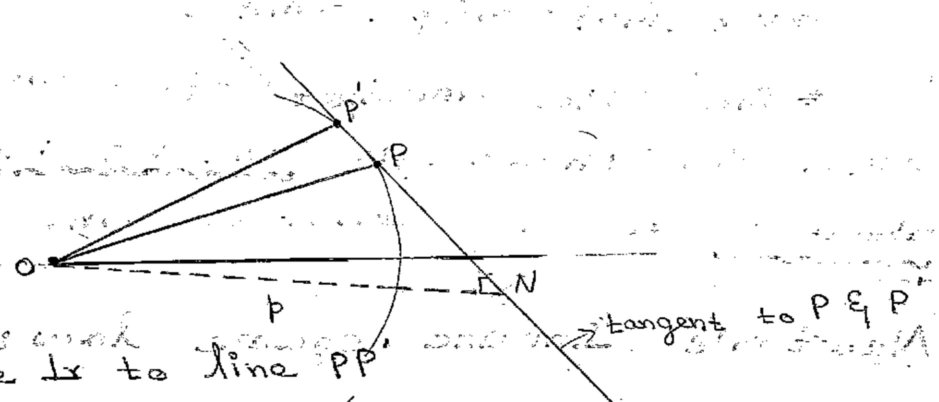
$$\frac{dA}{dt} = \frac{h}{2} = \text{a constant}$$

ie) The areal velocity in a central orbit is a constant.

Book work: 6

Alternative form for areal velocity

Another expression



* Let ON be $\perp$r to line PP'

* Here ΔA = Area of $\Delta OPP'$

$$\Delta A = \frac{1}{2} (ON)(PP') \sin 90° \rightarrow (1)$$

$\frac{(1)}{\Delta t}$ & taking the limit,

$$\Rightarrow \underset{\Delta t \to 0}{lt} \frac{\Delta A}{\Delta t} = \frac{1}{2} p \underset{\Delta t \to 0}{lt} \frac{PP'}{\Delta t} \longrightarrow (2)$$

* When $\Delta t \to 0$, the chord PP' becomes the tangent at P.

* When we take the limit, ON becomes the $\perp$r drawn from O to the tangent at P. It is denoted by p.

* $\underset{\Delta t \to 0}{lt} \frac{PP'}{\Delta t}$ is the mag. of the velocity denoted by V.

$$\lim_{\Delta t \to 0} \frac{PP'}{\Delta t} = v \longrightarrow (3)$$

Sub (3) in (2):

$$\frac{dA}{dt} = \frac{1}{2} p v = \frac{h}{2} = \text{constant}$$

$\left[\because \text{speed} = \text{mag. of velocity} = v = \dfrac{\text{distance travelled}}{\text{time taken}} \right]$

Hence we have $pv = h$

Definition:-

Constancy of moment of momentum (or) angular momentum about O:-

* The momentum of the particle is mv which ~~is along the tangent~~.

* Its momentum about O is

$$ON \times (mv) = m(pv) = mh$$

* Thus the moment of momentum about O, otherwise known as angular momentum about O, is the constant mh.

Newton's Inverse square law:-

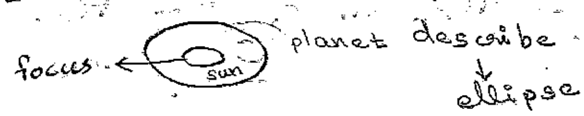
Two particles of masses m_1 & m_2 attract each other with a force whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance.

ie) $\underset{\text{Force}}{F} \propto \dfrac{\overbrace{m_1 m_2}^{\text{masses}}}{\underset{\text{distance}}{r^2}} \Rightarrow F = \dfrac{\gamma'' m_1 m_2}{r^2}$ $p_1 \rightarrow \leftarrow p_2$ m_1 attract with force m_2

Kepler's laws of planetary motion: V.V.V. (*)

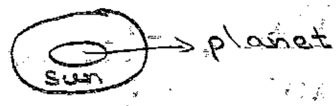
I law:

The planet describe ellipse about the sun as focus.



II law:

The radius vector drawn from the sun to a planet sweeps out equal areas in equal ~~times~~ interval of time.



~~III~~ ~~[illegible]~~ 2^{nd} law implies ⤵

* For each planet, the areal velocity $= \frac{1}{2} r^2 \dot{\theta}$

$= \frac{h}{2}$ = constant

* Transverse comp. of the force acting on the planet $= 0$
* The force acting on the planet is along the radius vector & thus it is a central force.

1^{st} law implies ⤵

$$\left.\begin{matrix}\text{Force of attraction}\\ \text{on the planet}\end{matrix}\right\} \propto \frac{1}{(\text{distance from the sun})^2}$$

III law:

The squares of the periodic times of the planets are proportional to the cubes of the semi major axis of their respective orbits.

$$T^2 \propto a^3$$

Book work : (7)

Conic as a central orbit

To find :- (i) law of force

(ii) speed of a particle

full blue

when the central orbit is a conic with the centre of force at one focus.

Proof :-

$$\boxed{\text{Polar eqn of conic : } \frac{l}{r} = 1 + e\cos\theta}$$

[$l \rightarrow$ semi lactus rectum of conic]

$u = \frac{1}{r} \rightarrow ul = 1 + e\cos\theta$

$$\Rightarrow u = \frac{1}{l} + \frac{e\cos\theta}{l} \rightarrow (1)$$

Differentiate with respect to θ

$$\frac{du}{d\theta} = \frac{-e}{l}\sin\theta \rightarrow (2)$$

$$\frac{d^2u}{d\theta^2} = \frac{-e}{l}\cos\theta \rightarrow (3)$$

$$\boxed{h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right] = -\phi(r)} \rightarrow (4)$$

Subs for $\frac{d^2u}{d\theta^2}$ & u (3) & (1) in (4) we get the above

$$\Rightarrow \phi(r) = \frac{-h^2u^2}{l}$$

$$\phi(r)\hat{e}_r = \frac{h^2u^2}{l} \times -\hat{e}_r$$

$$= \frac{h^2}{l} \times \frac{1}{r^2} \times -\hat{e}_r \quad \left[\because \frac{1}{r} = u\right]$$

$\hookrightarrow$ force $\rightarrow$ (attractive) } $\rightarrow$ force / unit mass

$$\text{mag. of force} = \frac{h^2}{l}\cdot\frac{1}{r^2}$$

$$\text{mag. of force} \propto \frac{1}{r^2}$$

i) law of force:

Inverse square law

ii) To find speed v:

$$\boxed{v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]}$$

$$= h^2\left[\frac{e^2}{l^2}\sin^2\theta + \frac{1}{l^2} + \frac{e^2\cos^2\theta}{l^2} + \frac{2e\cos\theta}{l^2}\right]$$

$$= \frac{h^2}{l^2}\left[e^2+1+2e\cos\theta\right]$$

$$= \frac{h^2}{l^2}\left[e^2+1+2\left(\frac{l-r}{r}\right)\right] \quad \left(\because \frac{l}{r} = 1+e\cos\theta,\ e\cos\theta = \frac{l-r}{r}\right)$$

$$= \frac{h^2}{l^2}\left[e^2+1+\frac{2l}{r} - \frac{2r}{r}\right]$$

$$v^2 = \frac{h^2}{l^2}\left[e^2 - 1 + \frac{2l}{r}\right]$$

$$= \frac{h^2}{l}\left[\frac{e^2-1}{l} + \frac{2}{r}\right]$$

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right] \qquad \left[\because \mu = \frac{h^2}{l}\right]$$

i) When the path is parabola,

$$e = 1 \Rightarrow v^2 = \frac{2\mu}{r}$$

ii) When the path is ellipse,

$$b^2 = a^2(1-e^2)$$

$$\Rightarrow e^2 - 1 = \frac{-b^2}{a^2}$$

$$l = \frac{b^2}{a}$$

$$\therefore v^2 = \mu\left[\frac{-b^2/a^2}{b^2/a} + \frac{2}{r}\right]$$

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

iii) When the path is hyperbola,

$$b^2 = a^2(e^2-1)$$

$$\Rightarrow e^2 - 1 = \frac{b^2}{a^2}$$

$$l = \frac{b^2}{a}$$

$$\therefore v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]$$

When the path is the branch of hyperbola, nearer to the centre of force.

Note:

When the path is the branch of hyperbola not nearer to the ~~point~~ centre of force.

Equation: $\frac{l}{r} = -1 + e\cos\theta$

force / unit mass $= \frac{h^2}{l} \cdot \frac{1}{r^2}\hat{e}_r$ (Repulsive force)

Periodic Time:

When the orbit is an ellipse, the periodic time of a particle is

$$T = \frac{\text{total area}}{\text{Areal velocity}}$$

$$= \frac{\pi ab}{\frac{1}{2} h}$$

$$= \frac{\pi ab}{\frac{1}{2}\sqrt{\mu l}} \quad \left[\because \mu = \frac{h^2}{l} \Rightarrow h = \sqrt{\mu l}\right]$$

$$= \frac{2\pi}{\sqrt{\mu}} \frac{ab\sqrt{a}}{b} \quad \left[\because l = \frac{b^2}{a}\right]$$

$$= \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$T = k a^{3/2}$$

$$\boxed{T^2 = k^2 a^3} \quad \text{v.v.}$$

Hence Kepler's third law is verified.

1. S.T force towards the pole under which a particle describes the curve $r^n = a^n \cos n\theta$ varies inversely as the $(2n+3)^{rd}$ power of the distance of the particle from the pole.

$$r^n = a^n \cos n\theta$$

$$\frac{1}{u^n} = a^n \cos n\theta$$

$$u^{-n} = a^n \cos n\theta$$

Taking log,

$$-n \log u = \log a^n + \log \cos n\theta \longrightarrow (1)$$

Differentiating (1) w.r.t θ

$$\cancel{-}\cancel{n}\frac{1}{u}\frac{du}{d\theta} = \frac{\cancel{-}1}{\cos n\theta}(\cancel{n} \sin n\theta)$$

$$\frac{1}{u}\frac{du}{d\theta} = \tan n\theta$$

$$\frac{du}{d\theta} = u \tan n\theta \longrightarrow (2)$$

Differentiating (2) w.r.t. θ,

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + un\sec^2 n\theta$$

$$\frac{d^2u}{d\theta^2} = u\tan^2 n\theta + un\sec^2 n\theta$$

$$\Rightarrow \boxed{h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)} = h^2u^2\left[u\tan^2 n\theta + un\sec^2 n\theta + u\right]$$

$$F = h^2u^3\left(\tan^2 n\theta + n\sec^2 n\theta + 1\right) \quad [\because 1+\tan^2 n\theta = \sec^2 n\theta]$$

$$= h^2u^3(n+1)\sec^2 n\theta$$

$$= h^2u^3(n+1)\frac{a^{2n}}{r^{2n}} \quad \left[\because \cos n\theta = \frac{r^n}{a^n},\ \sec n\theta = \frac{a^n}{r^n},\ \sec^2 n\theta = \frac{a^{2n}}{r^{2n}}\right]$$

$$= \frac{h^2(n+1)a^{2n}}{r^3 r^{2n}}$$

$$= \frac{h^2(n+1)a^{2n}}{r^{2n+3}}$$

$\therefore$ The force varies inversely as the $(2n+3)^{rd}$ power of r.

(2.) V.V. 2M If a particle describes a curve $r = e^\theta$ under a central force at the pole, ST the force varies inversely as the cube of the distance of the particle from the pole.

$$r = e^\theta \Rightarrow u = e^{-\theta}$$

$$\frac{du}{d\theta} = -e^{-\theta} \Rightarrow \frac{d^2u}{d\theta^2} = e^{-\theta} = u$$

$$\therefore \boxed{h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right]} = h^2u^2(u+u)$$

$$= 2h^2u^3$$

$$= \frac{2h^2}{r^3}$$

3. If the eqn of the orbit is $r = ae^{\theta\cot\alpha}$,
H.W find the law of force. (✗) 2M

$$r = ae^{\theta\cot\alpha} \Rightarrow u^{-1} = ae^{\theta\cot\alpha}$$

$$u = ae^{-\theta\cot\alpha} \longrightarrow (1)$$

Differentiating (1) w.r.t θ,

$$\frac{du}{d\theta} = -ae^{-\theta\cot\alpha}(\cot\alpha)$$

$$\frac{d^2u}{d\theta^2} = -ae^{-\theta\cot\alpha}(-\cot\alpha)(\cot\alpha)$$

$$= ae^{-\theta\cot\alpha}\cot^2\alpha$$

$$\frac{d^2u}{d\theta^2} = u\cot^2\alpha$$

$$\therefore \boxed{h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right]} = h^2u^2(u\cot^2\alpha + u)$$

$$= h^2u^3\operatorname{cosec}^2\alpha$$

$$= \frac{h^2\operatorname{cosec}^2\alpha}{r^3}$$

$\therefore$ The force is inversely proportional to the cube of the distance.

H.W
4. If $r^n = A\cos n\theta - B\sin n\theta$ find the law of force proceed as follows.

$$A = a^n\cos\phi,\ B = a^n\sin\phi,\ A^2+B^2 = a^{2n},\ \tan\phi = \frac{B}{A}$$

$$\Rightarrow r^n = a^n\cos\phi\cos n\theta - a^n\sin\phi\sin n\theta$$

$$u^{-n} = a^n\cos(n\theta+\phi) \quad [\because \cos(A+B) = \cos A\cos B - \sin A\sin B]$$

Taking log,

$$-n\log u = n\log a + \log\cos(n\theta+\phi)$$

$$\frac{1}{\cos(n\theta+\phi)} = \frac{a^n}{u^{-n}}$$

Differentiating w.r.t θ,

$$-\cancel{n}\frac{1}{u}\frac{du}{d\theta} = -\cancel{n}\frac{1}{\cos(n\theta+\phi)}\sin(n\theta+\phi)$$

$\sec(n\theta+\phi) =$

$$\frac{1}{u}\frac{du}{d\theta} = \tan(n\theta+\phi)$$

$\sec^2(n\theta+\phi)$

(2)

$$\frac{du}{d\theta} = a\tan(n\theta+\phi)$$

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan(n\theta+\phi) + un\sec^2(n\theta+\phi)$$

$$\therefore h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right] = h^2u^3\left[\tan^2(n\theta+\phi)+n\sec^2(n\theta+\phi)+1\right]$$

$$= h^2u^3(n+1)\sec^2(n\theta+\phi)$$

$1+\tan^2\theta = \sec^2\theta$

sub ② in

$$= \frac{h^2u^3(n+1)a^{2n}}{r^{2n}}$$

$$= \frac{h^2(n+1)a^{2n}}{r^{2n}\,r^3}$$

$$= \frac{h^2a^{2n}(n+1)}{r^{2n+3}}$$

5. The position vector of a particle at time t is $\bar{r} = \bar{A}\cos nt + \bar{B}\sin nt$, where $\bar{A}, \bar{B}$ are constant vector and 'n' constant. Find the acceleration & the nature of force.

$F = m\ddot{\bar{r}}$ $\quad \bar{a}, \bar{b}, n \rightarrow$ constant

$$\bar{r} = \bar{a}\cos nt + \bar{b}\sin nt$$
$$\dot{\bar{r}} = -n\bar{a}\sin nt + n\bar{b}\cos nt$$
$$\ddot{\bar{r}} = -n^2\bar{A}\cos nt - n^2\bar{B}\sin nt$$
$$= -n^2(\bar{A}\cos nt + \bar{B}\sin nt)$$
$$\ddot{\bar{r}} = -n^2\bar{r}$$

$F = m\ddot{\bar{r}}$
$= m(-n^2\bar{r})$
$= mn^2(-\bar{r})$

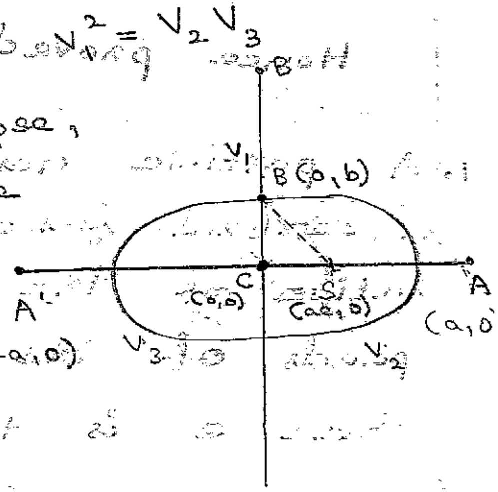
Force $\rightarrow$ towards O

$\rightarrow$ attractive central force

6. V.V. (*) 5M A particle describes an ellipse. Orbit is under a central force towards one focus S. If V_1 is the speed at the end B of the minor axis and V_2, V_3 the speeds at the ends of A, A' of the major axis. S.T $V_1^2 = V_2V_3$

* As the orbit is an ellipse, the velocity at any pt on the orbit is given by

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \longrightarrow ①$$



* Here the distance r is measured from which is the centre of force.

* We find the distances of A, A' and B from S.

$\therefore SA = CA - CS = a - ae = a(1-e) \rightarrow (2)$

$SA' = CS + CA' = a + ae = a(1+e) \rightarrow (3)$

$B = (0, b)$; $S = (ae, 0)$

$\rightarrow x_2, y_2$; $\rightarrow x_1, y_1$

$SB = \sqrt{(0-ae)^2 + (b-0)^2} = \sqrt{a^2e^2 + b^2}$

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{a^2e^2 + a^2(1-e^2)}$

$= a \longrightarrow (4)$

* The velocities at B, A, A' are v_1, v_2, v_3 respectively. $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

$$v_1^2 = \mu\left(\frac{2}{SB} - \frac{1}{a}\right) = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a} \rightarrow (5)$$

$$v_2^2 = \mu\left(\frac{2}{SA} - \frac{1}{a}\right) = \mu\left(\frac{2}{a(1-e)} - \frac{1}{a}\right) = \mu\left[\frac{2-1+e}{a(1-e)}\right]$$

$$v_2^2 = \frac{\mu}{a}\left[\frac{1+e}{1-e}\right] \longrightarrow (6)$$

$$v_3^2 = \mu\left(\frac{2}{SA'} - \frac{1}{a}\right) = \mu\left(\frac{2}{a(1+e)} - \frac{1}{a}\right) = \frac{\mu}{a}\left(\frac{1-e}{1+e}\right) \rightarrow (7)$$

Consider,

$$v_2^2 v_3^2 = \frac{\mu^2}{a^2}\left(\frac{1+e}{1-e}\right) \times \left(\frac{1-e}{1+e}\right) = \frac{\mu^2}{a^2} = v_1^4$$

$\therefore v_1^4 = v_2^2 v_3^2$

$\Rightarrow v_1^2 = v_2 v_3$

Hence proved.

combine question To find i) $v_1^2 = v_2 v_3$ ii) minimum : maximum speed = ?

---

7. A particle describes an elliptic orbit under a central force towards one focus S. the ratio of the minimum and maximum speeds of the particle is $(1-e) : (1+e)$ where e is the eccentricity.

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

At A':

* The distance of the particle from the focus is maximum.

* When r is maximum, the speed is minimum.

$\Rightarrow$ Minimum speed is $v_3^2 = \dfrac{\mu}{a}\dfrac{(1-e)}{(1+e)} \rightarrow (1)$

At A:

* The distance of the particle from the focus is minimum.

* When r is minimum, the speed is maximum.

$\Rightarrow$ Maximum speed is $v_2^2 = \dfrac{\mu}{a}\dfrac{(1+e)}{(1-e)} \rightarrow (2)$

From (1) & (2),

The ratio of minimum speed and maximum speed

$$= \frac{\sqrt{\mu/a}\sqrt{(1-e)/(1+e)}}{\sqrt{\mu/a}\sqrt{(1+e)/(1-e)}} = \frac{1-e}{1+e}$$

Hence proved.

8. A particle describes an ellipse under a law $\dfrac{\mu}{(\text{distance})^2}$ towards the focus. If it was projected with a velocity v from a point and distance r from the centre of force. ST the periodic time is $\dfrac{2\pi}{\sqrt{\mu}}\left(\dfrac{2}{r}-\dfrac{v^2}{\mu}\right)^{-3/2}$

As the central orbit is an ellipse, the velocity is given by

$$v^2 = \mu\left(\frac{2}{r}-\frac{1}{a}\right) \quad \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}$$

$\Rightarrow$ Rewriting $\dfrac{\mu}{a} = \dfrac{2\mu}{r} - v^2 \Rightarrow a = \left(\dfrac{2}{r}-\dfrac{v^2}{\mu}\right)^{-1}$

The periodic time is gn by,

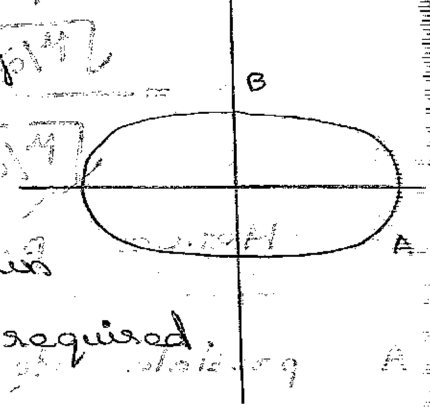
$$\boxed{T = \frac{2\pi}{\sqrt{\mu}}\, a^{3/2}}$$

$$T^2 = \frac{(2\pi)^2}{\mu}\, a^3 = \frac{4\pi^2}{\mu}\left(\frac{2}{r} - \frac{V^2}{\mu}\right)$$

$$\therefore\ T = \frac{2\pi}{\sqrt{\mu}}\left(\frac{2}{r} - \frac{V^2}{\mu}\right)^{-3/2}$$

---

9. A planet describing an ellipse about the sun as focus. ST its velocity away from the sun is greater when the radius vector to the planet is at right angles to the major axis of the path and that it is $\frac{2\pi a e}{T\sqrt{1-e^2}}$ where $2a$ is the major axis of the path, e is the eccentricity and T is the periodic time.



* The velocity vector is gn by,

$$\boxed{\bar{V} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_s} \rightarrow (1)$$

* Velocity away from the sun is along $\hat{e}_r$, maximum $\dot{r}$ is required.

$$\boxed{\frac{l}{r} = 1 + e\cos\theta \longrightarrow (2)}$$

Differentiating (1),

$$\frac{-l}{r^2}\dot{r} = -e\sin\theta\,\dot{\theta}$$

$$\dot{r} = \frac{e}{l}(r^2\dot{\theta})\sin\theta \longrightarrow (3)$$

$$\dot{r} = \frac{e}{l}(h)\sin\theta \quad \left[\because r^2\dot{\theta} = \text{constant for the central orbit}\right]$$

* $\dot{r}$ is maximum, when $\sin\theta$ is maximum $= 1$

$$\max \dot{r} = \frac{eh}{l} \longrightarrow (4)$$

$$\max \dot{r} = \frac{e}{l}\sqrt{\mu l} \quad \left[\because \mu = \frac{h^2}{l} \Rightarrow \sqrt{\mu l} = h\right]$$

$$\max \dot{r} = e\sqrt{\frac{\mu}{l}} \rightarrow (5)$$

Periodic Time when it is an ellipse

$$\max \dot{r} = \frac{e}{\sqrt{l}} \frac{2\pi}{T} a^{3/2} \rightarrow (6) \quad \left[\because T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \Rightarrow \sqrt{\mu} = \frac{2\pi}{T} a^{3/2}\right]$$

$$\Rightarrow l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

$$\sqrt{l} = \sqrt{a(1-e^2)} \rightarrow (7)$$

Sub (7) in (6)

$$\therefore \max \dot{r} = \frac{2\pi e}{T} \frac{a^{3/2}}{\sqrt{a}\sqrt{1-e^2}}$$

$$= \frac{2\pi e a}{T\sqrt{1-e^2}}$$

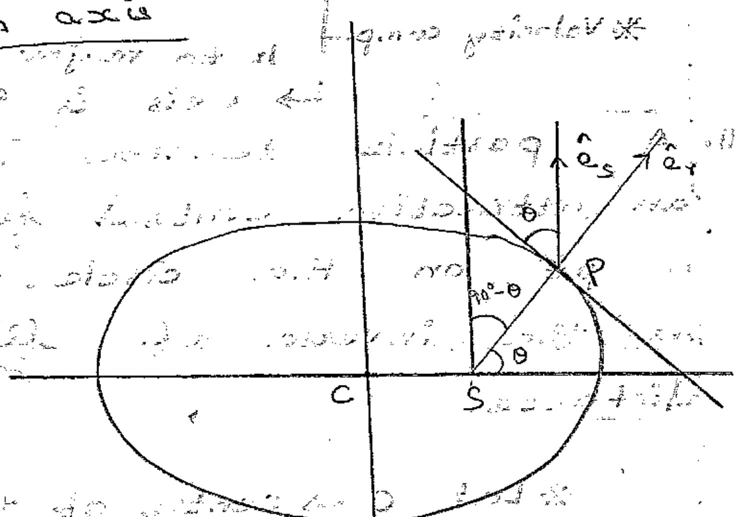
---

10. ST the velocity of a particle moving in an ellipse about the centre at a focus is compounded

i) $\frac{\mu}{h}$ $\perp r$ to the Radius Vector

ii) $\frac{\mu}{h}$ $\perp r$ to the major axis

$$\boxed{\bar{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s}$$



$$\frac{l}{r} = 1 + e\cos\theta$$

$$-\frac{l}{r^2}\dot{r} = -e\sin\theta\,\dot{\theta}$$

$$\dot{r} = \frac{e}{l}(r^2\dot{\theta})\sin\theta$$

$$\dot{r} = \frac{eh}{l}\sin\theta \rightarrow (1)$$

$$\boxed{\text{consider } r\dot{\theta} = \frac{r^2\dot{\theta}}{r} = \frac{h}{r}}$$

$$r\dot{\theta} = \frac{h(1+e\cos\theta)}{l} \rightarrow (2) \quad \left[\because \frac{1}{r} = \frac{1+e\cos\theta}{l}\right]$$

$$\Rightarrow \frac{h^2}{l} = \mu \Rightarrow \frac{h}{l} = \frac{\mu}{h} \rightarrow (3)$$

sub ③ in ②,

$$r\dot{\theta} = \frac{\mu}{h}(1+e\cos\theta)$$

$$r\dot{\theta} = \frac{\mu}{h} + \frac{e\mu}{h}\cos\theta \rightarrow ④$$

$$\bar{V} = \frac{eh}{l}\sin\theta\,\hat{e}_r + \left(\frac{\mu}{h} + \frac{e\mu}{h}\cos\theta\right)\hat{e}_s$$

$$= \frac{eh}{l}\sin\theta\,\hat{e}_r + \frac{e\mu}{h}\cos\theta\,\hat{e}_s + \frac{\mu}{h}\hat{e}_s$$

$$= \frac{e\mu}{h}(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_s) + \frac{\mu}{h}\hat{e}_s \rightarrow ⑤$$

* y-axis is $\perp$r to the major axis, the unit vector being $\hat{j}$, it makes an angle $(90°-\theta)$ with $\hat{e}_r$ and an angle θ with $\hat{e}_s$ where $\hat{e}_r, \hat{e}_s$ are $\perp$r.

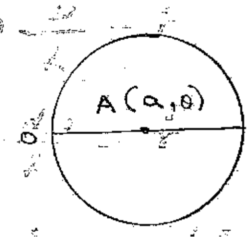
$$\hat{j} = \cos(90°-\theta)\hat{e}_r + \cos\theta\,\hat{e}_s = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_s \rightarrow ⑥$$

sub ⑥ in ⑤ $\Rightarrow \bar{V} = \frac{e\mu}{h}\hat{j} + \frac{\mu}{h}\hat{e}_s \rightarrow ⑦$

* Velocity comp —
  - $\perp$r to radius vector is along $\hat{e}_s$ is $\frac{\mu}{h}$.
  - $\perp$r to major axis is along minor axis is $e\mu/h$.

11. A particle describes a circular orbit under an attractive central force directed towards a pt on the circle. ST the force varies as the inverse of fifth power of the distance.

* Let O → centre of the force in the circumference of the circle.



* A(a, 0) → centre.

The eqn of the circle is

$$\boxed{(x-a)^2 + (y-0)^2 = a^2}$$

$$x^2 + y^2 - 2ax + a^2 = a^2$$

$$x^2 + y^2 = 2ax$$

$$r^2 = 2a(r\cos\theta)$$

* $r = 2a\cos\theta$ is the eqn of the given circle in polar coordinates.

$$\frac{1}{u} = 2a\cos\theta$$

Diff. of $\sec\theta\tan\theta = \sec\theta(\sec^2\theta) + \tan\theta\sec\theta\tan\theta$
$= \sec^3\theta + \tan^2\theta\sec\theta$

$$u = \frac{1}{2a\cos\theta} = \frac{\sec\theta}{2a}$$

$$\frac{du}{d\theta} = \frac{\sec\theta\tan\theta}{2a} \Rightarrow \frac{d^2u}{d\theta^2} = \frac{1}{2a}\left[\sec^3\theta + \sec\theta\tan^2\theta\right]$$

Consider,

$$h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = \frac{h^2u^2}{2a}\left[\sec^3\theta + \sec\theta\tan^2\theta + \sec\theta\right]$$

$$= \frac{h^2u^2\sec\theta}{2a}\left[\sec^2\theta + \tan^2\theta + 1\right]$$

$$= \frac{h^2u^2}{2a}\sec\theta\left[2\sec^2\theta\right] \quad \left[\because 1+\tan^2\theta = \sec^2\theta\right]$$

$$= \frac{2h^2u^2\sec^3\theta}{2a}$$

$$= \frac{h^2u^2(2au)^3}{a} = \frac{8a^3u^2u^3h^2}{a}$$

$$= 8a^2h^2u^5$$

$$\therefore F = \frac{8a^2h^2}{r^5}$$

$\therefore$ F varies inversely as the fifth power of the distance.

~~This ... ...~~

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12. The eccentricity of the earth's orbit round the sun is $\frac{1}{60}$. ST the earth's distance from the sun exceeds the length of semi major axis of the orbit during about 2 days more than half year.

$S = (ae, 0)$, $B = (0, b)$

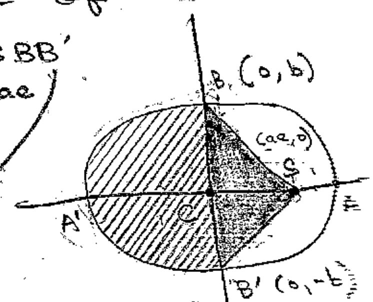
$$SB = \sqrt{a^2e^2+b^2}$$

$$a = \sqrt{a^2e^2+a^2(1-e^2)} \quad [\because SB = a = SB']$$

* At any pt on the boundary b/w B and B' to the distance of the particle from the sun is a.

* The time taken to describe the area SBA'B'S is required.

Area of SBA'B'S = area of $\left.\underset{\frac{1}{2}\text{ year}}{BA'B'B}\right\}$ + $\left\{\begin{array}{l}\text{area of} \\ \Delta SBB' \\ = \frac{1}{2}\times 2b\times ae\end{array}\right.$



Time taken to describe area BA'B'B

= area of the ellipse

= ½ year

To find the time taken to describe area ΔSBB'

$$\Delta SBB' = \frac{1}{2}(BB')(CS)$$

$$= \frac{1}{2}(2b)(ae)$$

$\pi ab = 365$ days, $abe = ?$ days

$$\Rightarrow \frac{abe}{\pi ab} = \frac{?}{365} \Rightarrow \frac{365e}{\pi} = ?$$

$$\Rightarrow 365\times 7/22 \times 1/6 = 1.94 = 2 \text{ days}$$

To describe the entire ellipse whose area $\pi ab = 365$ d

$$aeb = \frac{365}{\pi}\times\frac{1}{60} = \frac{365\times 7}{22\times 60} = 1.94 = 2 \text{ days}$$

∴ The required time is 2 days more than half a year.

Hence proved.

Book work: 8

Converse of law of force and velocity

$\frac{l}{r} = 1+e\cos\theta$: To find the orbit when the force is attractive central force varying inversely as the square of distance.

* Choose the centre of focus as the pole.

The differential eqn of the orbit is

$$\boxed{h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right]=\phi(r)}$$ [∵ attractive force]

W.K.T $\phi(r)=\frac{\mu}{r^2}$

$\mu \rightarrow$ constant of proportionality

$r^2 \rightarrow$ square of distance

$$\therefore h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right]=\left[\frac{\mu}{r^2}\right] \rightarrow (1)$$

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2} \rightarrow (2)$$

$$\left(\frac{d^2}{d\theta^2}+1\right)u=\frac{\mu}{h^2}e^{0\theta}$$

Take $D=\frac{d}{d\theta}$, $\Rightarrow (D^2+1)u=\frac{\mu}{h^2} \rightarrow (3)$

$m=\pm i$, $\alpha=0$, $\beta=1$

$C.F = e^{0\theta}[c_1\cos\theta+c_2\sin\theta]$

$= c_1\cos\theta+c_2\sin\theta$

$= A\cos(\theta+B)$

how means,

$C.F = A\cos(\theta+B)$

$= A(\cos\theta\cos B - \sin\theta\sin B)$

$=(A\cos B)\cos\theta + (-A\sin B)\sin\theta$

$= c_1\cos\theta + c_2\sin\theta$

The auxilary eqn $m^2+1=0 \Rightarrow m=\pm i$, the roots are $-i, i$

$C.F = A\cos(\theta+B)$

$$P.I = \frac{1}{D^2+1}\left(\frac{\mu}{h^2}\right)e^{\alpha x} = \frac{\mu}{h^2}$$

$\frac{1}{0+1}\left(\mu/h^2\right)$

So, the general soln of diff. eqn is

$$u=\frac{\mu}{h^2}+A\cos(\theta+B)$$

$$\frac{1}{r}=\frac{\mu}{h^2}\left[1+\frac{h^2A}{\mu}\cos(\theta+B)\right]$$

$$\frac{h^2}{\mu r}=1+\frac{h^2A}{\mu}\cos(\theta+B) \rightarrow (4)$$

Take $\frac{h^2}{\mu}=l$, $e=\frac{h^2A}{\mu}$

$(4) \Rightarrow \frac{l}{r}=1+e\cos(\theta+B)$, which is a conic.

To find the nature of conic:

Multiply (2) by $\frac{2du}{d\theta}$ & integrate w.r.t θ

$$(2) \Rightarrow \frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}$$

(2) $\Rightarrow h^2\left[\dfrac{d^2u}{d\theta^2}+u\right]=\mu$

(2) $\times \dfrac{2du}{d\theta} \Rightarrow h^2\left[\dfrac{2du}{d\theta}\,\dfrac{d^2u}{d\theta^2}+\dfrac{2du}{d\theta}\,u\right]=\mu\,\dfrac{2du}{d\theta}$

integral, $h^2\displaystyle\int\left(\frac{2du}{d\theta}\,\frac{d^2u}{d\theta^2}+2u\frac{du}{d\theta}\right)d\theta = 2\mu\int\frac{du}{d\theta}\,d\theta \rightarrow$ (5)

$$h^2\int\frac{d}{d\theta}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]d\theta = 2\mu\int du$$

$$h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\mu u+c \rightarrow (6)$$

$$V^2=\frac{2\mu}{r}+c \rightarrow (7)$$

$\Rightarrow u = A\cos(\theta+B)+\mu/h^2 \rightarrow$ (8)

$\dfrac{du}{d\theta} = -A\sin(\theta+B)$

$\left(\dfrac{du}{d\theta}\right)^2 = A^2\sin^2(\theta+B) \longrightarrow$ (9)

Sub (8) & (9) in (6),

$$h^2\left[A^2\sin^2(\theta+B)+A^2\cos^2(\theta+B)+\frac{\mu^2}{h^4}+\frac{2\mu}{h^2}A\cos(\theta+B)\right]=2\mu u+c$$

$$h^2\left[A^2+\frac{2\mu}{h^2}A\cos(\theta+B)+\frac{\mu^2}{h^4}\right]=2\mu u+c$$

$$h^2A^2+2\mu A\cos(\theta+B)+\frac{\mu^2}{h^2}=2\mu u+c$$

$$h^2A^2+2\mu\left(u-\frac{\mu}{h^2}\right)+\frac{\mu^2}{h^2}=2\mu u+c$$

$u=\dfrac{\mu}{h^2}+A\cos(\theta+B)$

$u-\dfrac{\mu}{h^2}=A\cos(\theta+B)$

$$h^2A^2+2\cancel{\mu u}-\frac{2\mu^2}{h^2}+\frac{\mu^2}{h^2}-2\cancel{\mu u}-c=0$$

$$h^2A^2-\frac{\mu^2}{h^2}-c=0$$

$\Rightarrow h^2A^2=\dfrac{\mu^2}{h^2}+c \longrightarrow$ (10)

$$h^2A^2=\frac{\mu^2+h^2c}{h^2}$$

$$h^4A^2 = \mu^2 + h^2c \Rightarrow h^4A^2 = \mu^2\left[1+\frac{h^2}{\mu^2}c\right]$$

$$\frac{h^4A^2}{\mu^2} = 1+\frac{h^2}{\mu^2}c = e^2 \quad \left[\because e = \frac{h^2A}{\mu} \Rightarrow e^2 = \frac{h^4A^2}{\mu^2} = 1+\frac{h^2c}{\mu^2}\right]$$

The central orbit is an ellipse if

$$1+\frac{h^2}{\mu^2}c < 1 \quad \text{i.e) if } c<0$$

The central orbit is a hyperbola if

$$1+\frac{h^2}{\mu^2}c > 1 \quad \text{i.e) } c>0$$

The central orbit is a parabola if

$$1+\frac{h^2}{\mu^2}c = 1 \quad \text{i.e) } c=0$$

using these in (7), we get

$$v^2-\frac{2\mu}{r}<0, \quad v^2-\frac{2\mu}{r}=0, \quad v^2-\frac{2\mu}{r}>0$$

$$\text{i.e) } v<\sqrt{\frac{2\mu}{r}}, \quad v=\sqrt{\frac{2\mu}{r}}, \quad v>\sqrt{\frac{2\mu}{r}}$$

For an ellipse, parabola, hyperbola respectively.

The quantity $\sqrt{\frac{2\mu}{r}}$ is called the critical speed at a distance r i.e, the nature of the conic depends on the critical speed.

Book work: (9)

To find the orbit of a particle moving under an attractive force varying the distance:

(* Let the pv of a particle of mass m, at time t be $\bar{r}$.

* If n^2r is the force per unit mass, then the eqn of the motion is

$$m\ddot{\bar{r}} = -mn^2\bar{r} \quad (\text{or}) \quad \ddot{\bar{r}} = -n^2\bar{r}$$

If $\vec{r} = x\hat{i} + y\hat{j}$ we get

$$-n^2(x\hat{i}+y\hat{j}) = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

Equating $\hat{i}$ & $\hat{j}$ component

$$\ddot{x} = -n^2 x \ , \ \ddot{y} = -n^2 y$$

* The general soln of these differential eqn is

(2) $x = A\cos nt + B\sin nt \Rightarrow A\cos nt + B\sin nt - x = 0$

$y = C\cos nt + D\sin nt \Rightarrow C\cos nt + D\sin nt - y = 0$

where the constants A, B, C, D depends upon the initial conditions.

* Solving these two eqns for cos nt, sin nt

By cross multiplication rule

$$\frac{\cos nt}{\begin{vmatrix} B & -x \\ D & -y \end{vmatrix}} = \frac{-\sin nt}{\begin{vmatrix} -x & A \\ -y & C \end{vmatrix}} = \frac{1}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}$$

$$\frac{\cos nt}{-By+Dx} = \frac{-\sin nt}{-Cx+Ay} = \frac{1}{AD-BC}$$

$$\frac{\cos nt}{Dx-By} = \frac{\sin nt}{Cx-Ay} = \frac{1}{AD-BC}$$

$$\cos nt = \frac{-By+Dx}{AD-BC} \ , \ \sin nt = \frac{-Ay+Cx}{AD-BC}$$

(3) Squaring and adding, we get the eqn of the path as

$$\cos^2 nt + \sin^2 nt = 1$$

$$\left[(Cx-Ay)^2 + (Dx-By)^2\right] = (AD-BC)^2$$

Book work: (9)

(1) $\bar{r} = x\hat{i} + y\hat{j}$

$\dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$

$\ddot{\bar{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

$= -n^2 x\hat{i} - n^2 y\hat{j}$

$= -n^2(x\hat{i} + y\hat{j})$

$= -n^2\bar{r}$

$$\boxed{\begin{aligned}\ddot{x} &= -n^2 x\\ \ddot{y} &= -n^2 y\end{aligned}}$$

(2) $\bar{x} = \bar{A}\cos nt + \bar{B}\sin nt$

$x = A\cos nt + B\sin nt$

$y = C\cos nt + D\sin nt$

$A\cos nt + B\sin nt - x = 0$

$C\cos nt + D\sin nt - y = 0$

By multiplication rule

$\cos nt$	$\sin nt$		
B	$-x$	A	B
D	$-y$	C	D

$$\frac{\cos nt}{\begin{vmatrix} B & -x \\ D & -y \end{vmatrix}} = \frac{\sin nt}{-\begin{vmatrix} -x & A \\ -y & C \end{vmatrix}} = \frac{1}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}$$

critical velocity at the ... So, the nature of the orbit depends on the critical velocity.

Proof:

The critical velocity can be seen to be the velocity that would be acquired by a particle in the same force field, in reaching the pt in question. Starting from rest at pt at infinity towards the pole is

$$\ddot{r} = \frac{-\mu}{r^2} \rightarrow (1)$$

$$\frac{\cos nt}{-By + Dx} = \frac{\sin nt}{-(-Cx + Ay)} = \frac{1}{AD - BC}$$

$$\cos nt = \frac{Dx - By}{AD - BC}, \quad \sin nt = \frac{Cx - Ay}{AD - BC}$$

(3) $\cos^2 nt + \sin^2 nt = 1$

$(Dx - By)^2 + (Cx - Ay)^2 = (AD - BC)^2$

$D^2x^2 + B^2y^2 - 2BDxy + C^2x^2 + A^2y^2 - 2ACxy = (AD - BC)^2$

$(D^2 + C^2)x^2 + (B^2 + A^2)y^2 - 2xy(BD + AC) = ''$

$ax^2 + by^2 + 2hxy =$

$= 0 \rightarrow$ parabola

$< 0 \rightarrow$ ellipse

$> 0 \rightarrow$ hyperbola

(4) $a = D^2 + C^2, \; b = B^2 + A^2, \; h = -(BD + AC)$

Consider

$h^2 - ab = (BD + AC)^2 - (D^2 + C^2)(B^2 + A^2)$

$h^2 - ab = B^2D^2 + A^2C^2 + 2ABCD$

$\quad - [D^2B^2 + D^2A^2 + C^2B^2 + C^2A^2]$

$= -[D^2A^2 + C^2B^2 - 2ABCD]$

$= -[AD - BC]^2 < 0$

$x^2($ ...

Here $a =$ ...

$b =$ ...

$h =$ ...

Consider $h^2 - ab =$ ...

$h^2 - ab = (AC +$ ...

$= -(AD -$ ...

* which being ...

represents a ...

* Further it ...

* Hence it ...

* since $(-x, -$ ...

ellipse is symm...

* So the cen...

centre of the ...

Critical velocit...

The square ...

critical velocit...

nature of the ...

critical velocity ...

Proof:

The critic...

be the veloci...

by a particl...

in reaching ...

from rest at ...

the pole is

$\ddot{r} = \frac{-\mu}{\dots}$

$$x^2(C^2+D^2)+y^2(A^2+B^2)-2xy(AC+BD)=(AD-BC)^2$$

Here a = co-efficient of x^2

b = co-efficient of y^2

h = co-efficient of $2xy = -(AC+BD)$

Consider h^2-ab,

$$h^2-ab=(AC+BD)^2-(C^2+D^2)(A^2+B^2)$$

$$=-(AD-BC)^2<0$$

* which being a second degree eqn represents a conic.
* Further it satisfies $h^2-ab<0$.
* Hence it represents an ellipse.
* Since $(-x,-y)$ satisfies the eqn, the ellipse is symmetrical about the origin.
* So the centre of force is at the centre of the ellipse.

Critical velocity:

The quantity $\sqrt{\frac{2\mu}{r}}$ is called the critical velocity at the distance r. So, the nature of the orbit depends on the critical velocity.

Proof:

The critical velocity can be seen to be the velocity that would be acquired by a particle in the same force field, in reaching the pt in question. Starting from rest at pt at infinity towards the pole is

$$\ddot{r}=\frac{-\mu}{r^2}\rightarrow (1)$$

Multiplying both sides by $2\dot{r}$ in ①;

$$① \Rightarrow 2\dot{r}\ddot{r} = -\frac{\mu}{r^2} 2\dot{r} \rightarrow ②$$

Integrating ② w.r.t t we get,

$$2\int \dot{r}\ddot{r}\, dt = -2\mu \int \frac{1}{r^2}\dot{r}\, dt$$

$$\int \frac{d}{dt}(\dot{r}^2)\, dt = 2\mu \int -\frac{1}{r^2}\dot{r}\, dt$$

$$\int \frac{d}{dt}(\dot{r}^2)\cancel{dt} = 2\mu \int \frac{d}{dt}\left(\frac{1}{r}\right)\cancel{dt}$$

$$\dot{r}^2 = \frac{2\mu}{r} + c$$

$$\left[\dot{r}^2\right]_0^v = 2\mu\left[\frac{1}{r}\right]_{-\infty}^{r}$$

$$\therefore v^2 = \frac{2\mu}{r} \Rightarrow v = \sqrt{\frac{2\mu}{r}}$$

Note:

1) Let the particle be under the action of a attractive central force, where the force satisfies the inverse law in p-r coordinates

We have; $\frac{h^2}{p^3}\frac{dp}{dr} = F$

$$\frac{h^2}{p^3}\, dp = \frac{\mu}{r^2}\, dr$$

$$\boxed{\begin{array}{l} \frac{h^2}{p^3}\frac{dp}{dr} = -\phi(r) \rightarrow \text{D.E in p-r coordinates} \\ \frac{h^2}{p^3}\frac{dp}{dr} = F \quad [\because \phi(r) = -F] \\ \frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2} \quad \left[\because F = \frac{\mu}{r^2}\right] \end{array}}$$

This box must be written in exam it includes 2 marks

Integrating, $\frac{h^2}{-2p^2} = \frac{-\mu}{r} + c$

$$\frac{h^2}{p^2} = \frac{2\mu}{r} + D$$

$$\frac{\cancel{p^2}v^2}{\cancel{p^2}} = \frac{2\mu}{r} + D \quad [\because pv = h]$$

$$v^2 = \frac{2\mu}{r} + D$$

2) The polar eqns of an ellipse, parabola and hyperbola are given by

orbit:

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 \text{ (ellipse)}$$

$$p^2 = ar \text{ (parabola)}$$

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1 \text{ (hyperbola)}$$

formula

Comparing we conclude that the central orbit is an ellipse if $D<0$, it is a parabola if $D=0$ and it is a hyperbola if $D>0$

W.K.T $\Rightarrow v^2 - \frac{2\mu}{r} = D$ $\qquad v^2 - \frac{2\mu}{r} < 0$

i) When $D<0$, we get an ellipse.

ie) $v^2 < \frac{2\mu}{r}$ (or) $v < \sqrt{\frac{2\mu}{r}}$

ii) When $D=0$, we get a parabola. $v^2 - \frac{2\mu}{r} = 0$

ie) $v^2 = \frac{2\mu}{r}$ (or) $v = \sqrt{\frac{2\mu}{r}}$

iii) When $D>0$, we get a hyperbola. $v^2 - \frac{2\mu}{r} > 0$

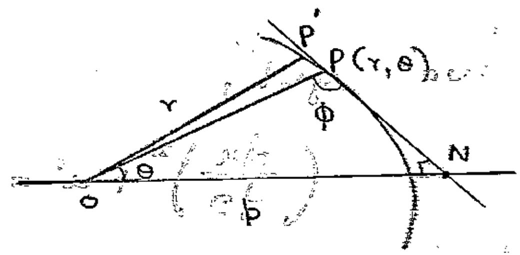
ie) $v^2 > \frac{2\mu}{r}$ (or) $v > \sqrt{\frac{2\mu}{r}}$

Apse:

An apse is a point on the central orbit, whose distance from the centre of force is either a maximum (or) a minimum [ie) r is ~~mini~~ minimum (or) maximum]. If r is the distance of the point from the central of force

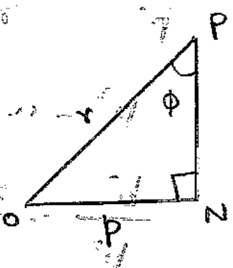
ie) u is maximum (or) minimum corresponding to an apse.

ie) $\frac{du}{d\theta} = 0$, at an apse.



* Also OP ⊥r to the tgt at P if P is an apse.

$p = r\sin\phi$ in general at an apse $\phi = 90^\circ$



$\therefore p = r$ at an apse.

For a central orbit $r^2\dot{\theta} = h$

$p = r\sin\phi$

$p = r\sin 90^\circ \Rightarrow p = r$

so, $\dot{\theta} = \dfrac{h}{r^2}$

* If r is minimum, $\dot{\theta}$ is maximum

* If r is maximum, $\dot{\theta}$ is minimum

$\therefore$ At an apse the angular velocity is either a maximum (or) a minimum.

13. A particle acted on by a central attractive force μu^3 is projected with the velocity $\sqrt{\mu}/a$ at an angle of $\frac{\pi}{4}$ with its initial distance 'a' from the centre of force. ST the path is the equiangular spiral whose eqn is $r = ae^{-\theta}$

10m

The differential eqn of the central orbit is

$$\boxed{h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = \phi(r)} \longrightarrow (1)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^3}{h^2 u^2} \longrightarrow (2)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u}{h^2} \longrightarrow (3)$$

Multiply by $\frac{2du}{d\theta}$ & integrate both sides

$$\int 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + \int 2u\frac{du}{d\theta} = \frac{\mu}{h^2}\int 2u\frac{du}{d\theta}\,d\theta \longrightarrow$$

we get,

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2}u^2 + c \longrightarrow (5)$$

When $\phi = \frac{\pi}{4}$,

$$\boxed{p = r\sin\phi}$$

$$= \frac{a}{\sqrt{2}} \quad [\because r = a]$$

The velocity at an apse

$$v = \sqrt{\frac{\mu}{a}}$$

$$pv = h$$

$$h = \frac{a}{\sqrt{2}} \cdot \frac{\sqrt{\mu}}{a} = \frac{\sqrt{\mu}}{\sqrt{2}}$$

$$\sqrt{\mu} = \sqrt{2}\, h$$

$$\mu = 2h^2$$

$$\boxed{v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]}$$

$$\frac{v^2}{h^2} = \left(\frac{du}{d\theta}\right)^2 + u^2 = 1/p^2$$

$$\left[\because pv = h,\ p = \frac{h}{v},\ \frac{v}{h} = \frac{1}{p}\right]$$

From (5) we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu u^2}{h^2} + c$$

$$\frac{1}{p^2} = \frac{\mu u^2}{h^2} + c \quad (6)$$

$$\Rightarrow \frac{2}{a^2} = \frac{2h^2u^2}{h^2} + c$$

$$\left[\because p = \frac{h}{\sqrt{\mu}/a} = \frac{ah}{\sqrt{\mu}} = \frac{ah}{\sqrt{2}\,h}\right.$$

$$p = \frac{a}{\sqrt{2}}$$

$$\left. p^2 = \frac{a^2}{2}\right]$$

$$\Rightarrow \frac{2}{a^2} = \frac{2}{r^2} + c$$

Initially r is a

$$\frac{2}{a^2} = \frac{2}{a^2} + c$$

$$c = 0$$

sub. in (5)

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2u^2$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2$$

$$\frac{du}{d\theta} = u$$

Here $u = \frac{1}{r}$, $\frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}$

$$\Rightarrow \frac{1}{r} = \frac{-1}{r^2}\frac{dr}{d\theta}$$

$$\frac{-dr}{r} = d\theta$$

Integrating,

$$\log\left(\frac{1}{r}\right) = \theta + c' \quad (7)$$

r is 'a', when $\theta = 0$

$$\log\left(\frac{1}{a}\right) = 0 + c'$$

$$c' = \log\left(\frac{1}{a}\right)$$

sub in (7)

$$\log\left(\frac{1}{r}\right) = \theta + \log\left(\frac{1}{a}\right)$$

ie) $\log\left(\frac{1}{r}\right) - \log\left(\frac{1}{a}\right) = \theta$

$$\theta = \log\left(\frac{a}{r}\right)$$

Taking exponential form,

$$\frac{a}{r} = e^{\theta}$$

$r = ae^{-\theta}$ is the eqn of central orbit.

Hence proved.

14. A particle of mass m moves under a central force $\mu\left[3au^4 - 2(a^2-b^2)u^5\right]$, $a>b$, & it is projected from an apse, at a distance $r = (a+b)$ with a velocity $v = \frac{\sqrt{\mu}}{a+b}$. ST the eqn of orbit is $\boxed{r = a + b\cos\theta}$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu\left[3au^2 - 2(a^2-b^2)u^3\right]}{h^2} \longrightarrow (1)$$

we have $pv = h$

where $p = r = (a+b)$ and $v = \frac{\sqrt{\mu}}{a+b}$

Substituting, $(a+b)\frac{\sqrt{\mu}}{a+b} = h$

$$h = \sqrt{\mu} \longrightarrow (2)$$

we have $\frac{du}{d\theta} = 0$ (at an apse)

Multiply (1) by $\frac{2du}{d\theta}$ and integrate

$$\int 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + \int 2\frac{du}{d\theta}\,u = \frac{2\mu}{h^2}\int\left[3au^2 - 2(a^2-b^2)u^3\right]\frac{du}{d\theta}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\left[\frac{3au^3}{3} - \frac{2(a^2-b^2)u^4}{4}\right] + c$$

$$= 2au^3 - a^2u^4 + b^2u^4 + c$$

$$= 2\left[au^3 - \frac{(a^2-b^2)}{2}u^4\right] + c \longrightarrow (4)$$

To get the value of c

$$\frac{du}{d\theta} = 0,\; u = \frac{1}{r} = \frac{1}{a+b}$$

we get $u^2 = 2u^3\left[a - \frac{(a^2-b^2)}{2}u\right] + c$

$$1 = 2u\left[a - \frac{(a^2-b^2)}{2r}\right] + c$$

$$1 = 2u\left[a - \frac{(a^2-b^2)}{2(a+b)}\right] + c$$

$$1 = \frac{2}{a+b}\left[\frac{2a^2+2ab-a^2+b^2}{2(a+b)}\right] + c$$

$$1 = \frac{2}{a+b}\left[\frac{(a+b)^2}{2(a+b)}\right] + c$$

$$\boxed{c = 0}$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + u^2 = 2\left[au^3 - \frac{(a^2-b^2)}{2}u^4\right]$$

$$\left(\frac{du}{d\theta}\right)^2 = 2au^3 - a^2u^4 + b^2u^4 - u^2$$

$$= u^2\left[2au - a^2u^2 + b^2u^2 - 1\right]$$

$$= u^2\left[2au - (a^2-b^2)u^2 - 1\right]$$

$$\left(\frac{du}{d\theta}\right) = u\sqrt{2au - (a^2-b^2)u^2 - 1}$$

Put $u = 1/r$

$$\frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}$$

$$\frac{-1}{r^2}\frac{dr}{d\theta} = \frac{1}{r}\sqrt{\frac{2a}{r} - \frac{(a^2-b^2)}{r^2} - 1}$$

$$\frac{-dr}{r\sqrt{\frac{2a}{r} - \frac{(a^2-b^2)}{r} - 1}} = d\theta \Rightarrow \frac{-dr}{r \cdot \frac{1}{r}\sqrt{2ar - (a^2-b^2) - r^2}} = d\theta$$

$$\frac{-dr}{\sqrt{b^2 - (r-a)^2}} = d\theta.$$

Integrating, $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta + c'$

using the initial condition $r = a+b$, when $\theta = 0$

$$\cos^{-1}\left(\frac{b}{b}\right) = 0 + c'$$

$$c' = \cos^{-1}(1)$$

$$c' = 0$$

Hence we have, $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta$

$$r - a = b\cos\theta$$

$\boxed{r = a + b\cos\theta}$ is the equation of the orbit.

Hence proved.

---

15. A particle moves with a central acceleration $\mu\left[r^{-7}\right]$ and starts from an apse at a distance 'a' with a velocity equal to the velocity which would be acquired by the particle travelling from rest at infinity to the apse. ST the eqn of its orbit is $r^2 = a^2\cos 2\theta$

For the motion along the radius vector ~~from~~ from ∞ to any point $P(r,\theta)$ on the orbit, we have

$$\boxed{\ddot{r} = \frac{-\mu}{r^7}} \rightarrow (1)$$

Let v be the velocity attained by the particle travelling from rest at ∞ to

Then, multiplying both sides of (1) by $2\dot{r}$ & integrating with respect to 't'

$$\int_0^v 2\dot{r}\ddot{r}\frac{dr}{dt} = -2\mu\int_\infty^r \frac{1}{r^7}\frac{dr}{dt}\,dt$$

$$\left(\dot{r}^2\right)_0^{r} = \left(\frac{-2\mu}{-6r^6}\right)_{\infty}^{r}$$

$$v^2 = \frac{\mu}{3r^6}$$

So, the square of the velocity v^2 of projection at the apse $r = a$, is $v^2 = \frac{\mu}{3a^6}$

Next we shall find the value of the constant h^2.

Initially at the apse $r = p = a$ & from $h = pv$

$$h^2 = p^2v^2 = a^2\left(\frac{\mu}{3a^6}\right) = \frac{\mu}{3a^4} \rightarrow (2)$$

Now the differential ~~eqn of the~~ orbit is

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \mu/r^7$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2u^2r^7} = \frac{\mu u^5}{h^2}$$

From (2), $\frac{d^2u}{d\theta^2} + u = \frac{\mu u^5 \cdot 3a^4}{\mu} = 3a^4 u^5$

Multiplying both sides by $\frac{2du}{d\theta}$ & integrate with respect to θ

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \int 2(3a^4u^5)\,du$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = a^4u^6 + c$$

But initially,

$$\frac{du}{d\theta} = 0,\ u = \frac{1}{a}$$

$$0 + \frac{1}{a^2} = a^4 \cdot \frac{1}{a^6} + c$$

$$\boxed{c = 0}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = a^4u^6$$

$$\left(\frac{du}{d\theta}\right)^2 = a^4u^6 - u^2$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2\left(a^4u^4-1\right) \longrightarrow (3)$$

$$\left(\frac{du}{d\theta}\right) = u\sqrt{a^4u^4-1}$$

Substituting in eqn (3)

$$u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = \frac{-1}{r^2}\,\frac{dr}{d\theta}$$

$$\Rightarrow \frac{-1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6} - \frac{1}{r^2}$$

$$\frac{-1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4-r^4}{r^6}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4-r^4}{r^2}$$

$$\frac{dr}{d\theta} = \frac{\sqrt{a^4-r^4}}{r}$$

$$\frac{r}{\sqrt{a^4-r^4}}\,dr = d\theta \quad (5)$$

Putting $r^2 = y$ & integrating,

$2r\,dr = dy$ $[\because r^2 = y]$

$$r\,dr = \frac{dy}{2}$$

$$\frac{1}{2}\,\frac{dy}{\sqrt{a^4-y^2}} = d\theta$$

$$\sin^{-1}\left(\frac{y}{a^2}\right) = 2\theta + c'$$

If the initial line is choosen through the apse so that the apse is $(a, 0)$ then $r = a,\ \theta = 0$ $(y = r^2 = a^2)$

This gives, $2\theta + c' = \sin^{-1}\left(\frac{a^2}{a^2}\right)$

$$c' = \pi/2$$

$$\therefore \sin^{-1}(y/a^2) = 2\theta + c \Rightarrow \frac{r^2}{a^2} = \sin\left(2\theta + \frac{\pi}{2}\right)$$

$$\Rightarrow \sin(\theta + 90^\circ) = \cos\theta$$

$$\boxed{r^2 = a^2\cos 2\theta}$$

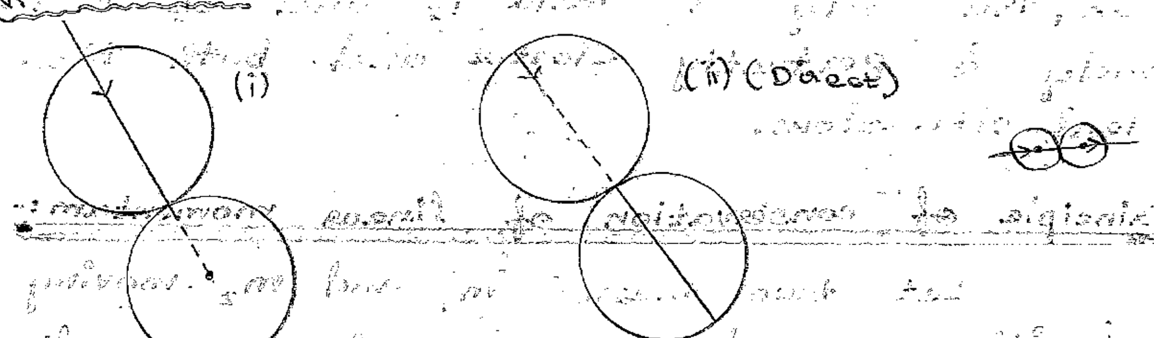
17/2/15

Unit - IV

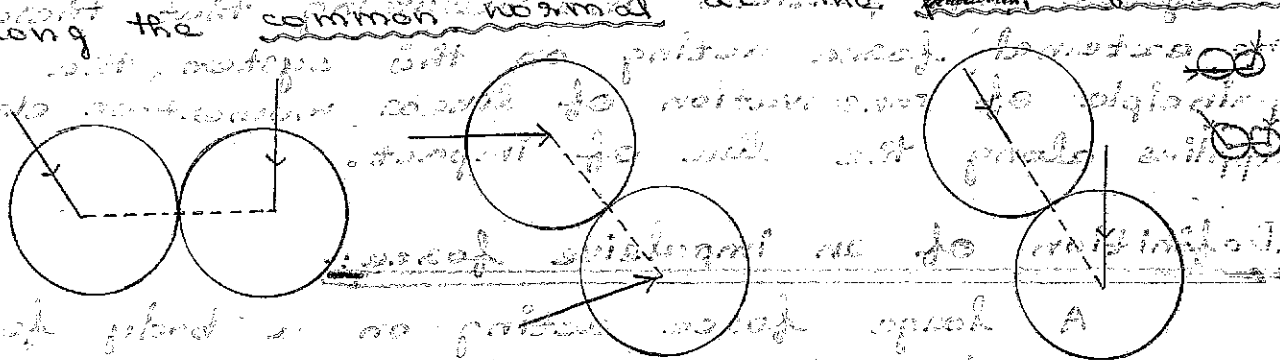
Impact

Definitions:

1) Two bodies are said to be impinge directly if the direction of motion of each is along the common normal at the point of contact.



2) Two bodies are said to impinge obliquely if the direction of motion of either or both is not along the common normal at the point of contact.



3) The common normal at the point of contact is called the line of impact.

4) When a body collides against another body or a fixed plane, it is slightly deformed. If the body tends to resume its original shape and the process rebounds, then it is called an elastic body, otherwise (if not) it is called an inelastic body.

Newton's experimental laws:-

i) When two bodies impinge directly, the relative velocity of one with respect to the other after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction.

ii) When two bodies impinge obliquely the relative velocity of one with respect to the other resolved along the common normal after impact bears a constant ratio to the relative velocity of one w.r.t the other resolved along the common normal before impact and is of opposite direction.

The constant ratio, called the coefficient of elasticity or restitution and denoted by e, depends on the material with which the body is made of and is independent of its shape, size or mass. In general, e lies between 0 and 1. If $e=0$, the body is inelastic and if $e=1$, the body is perfectly elastic and both these are ideal situations.

Principle of conservation of linear momentum:-

Let two masses m_1 and m_2 moving with velocities u_1 and u_2 respectively, impinge directly and let their velocities after impact be v_1 and v_2 respectively, considering these two masses as forming a system of particles and assuming that there is no external force acting on this system, the principle of conservation of linear momentum clearly applies along the line of impact.

Definition of an impulsive force:-

A large force acting on a body for an a body for an infinitesimally small period producing a finite change of momentum in that interval is called an impulsive force.

Example:

The force experienced by a ball due to hit by a bat or the blow of a hammer on a nail are examples of impulsive forces.

Result : 1

The effect of the action of an impulsive force is measured by the change momentum produced by the force. This change is called the impulsive of the impulsive force and it is denoted by $\overline{I}$

$$\overline{I} = m\overline{v} - m\overline{u} = m\overline{V} \quad \left[\because \overline{V} = \overline{v} - \overline{u}\right]$$

N.E.L

1) 2 bodies → I.D

u_1 u_2 $\xrightarrow{\text{velocity}}$ before impact

v_1 v_2 $\xrightarrow{\text{velocity}}$ after impact

$$\boxed{v_2 - v_1 = -e(u_2 - u_1)}$$

∴ R.V of one w.r.t the other after impact
= constant ratio (R.V of one w.r.t the other before impact)

→ opposite direction

2) 2 bodies → I.O

$u_1 \to \alpha_1$ $u_2 \to \alpha_2$

$v_1 \to \theta_1$ $v_2 \to \theta_2$

$$\boxed{v_2\cos\theta_2 - v_1\cos\theta_1 = -e(u_2\cos\alpha_2 - u_1\cos\alpha_1)}$$

∴ R.V of one w.r.t the other resolved along the common normal after impact
= constant ratio (R.V of one w.r.t the other resolved along the common normal before impact)

→ opposite direction

Constant ratio:

Definition +

In general, $0 < e < 1$

$e = 0$ → body : inelastic
$e = 1$ → body : perfectly elastic

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \to$ Ideal situations

Principle of conservation of linear momentum:

m_1 m_2
↓ ↓
u_1 u_2
I.D
v_1 v_2

form a system of particle

elasticity of the body. In general

P.C.L.M $\xrightarrow{\text{applies along}}$ line of impact

Formula :-

* $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$ $\boxed{D.I}$

* $m_1v_1 \cos\theta_1 + m_2v_2 \cos\theta_2 = m_1u_1 \cos\alpha_1 + m_2u_2 \cos\alpha_2$ $\boxed{O.I}$

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(1) S.T the work done by an impulsive force is equal to the product of the impulse and the mean of the velocities before and after the blow?

Work done by a force F acting on a particle

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

W.K.T, change in momentum $= mv - mu$

$I = m(v-u)$

$\therefore W = \frac{1}{2}m(v^2-u^2)$

$= \frac{1}{2}m(v+u)(v-u)$

$= I\frac{(u+v)}{2}$

= Impulse x Average velocity

where, Average velocity $= \frac{u+v}{2}$

= mean of the velocities before and after blow.

Result 2

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same.

By Newt

When $\overline{F} = \overline{0}$

i.e) $\overline{a} =$

i.e) $\frac{d\overline{v}}{dt}$

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Definition

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To prove: $\overline{I} = \int_0^t \overline{F}\, dt$

$$\overline{I} = m\overline{v} - m\overline{u} = m(\overline{v})_0^t = m\int d\overline{v} = m\int_0^t \left(\frac{d\overline{v}}{dt}\right) dt$$

$$I = m\int_0^t \overline{a}\, dt = \int_0^t \underbrace{(m\overline{a})}_{F} dt$$

$$\overline{I} = \int_0^t \overline{F}\, dt$$

*The unit for impulse is the same as the unit for momentum.

*In the FPS, the unit for impulse is pound foot/sec.

*In the CGS system, gm cm/sec.

*In the MKS system, kgm m/sec.

Note:
$I = mv - mu$
$\overline{I} = m\overline{v} - m\overline{u}$

Result : 2

If the applied force on a particle along a direction is zero, then the linear momentum of the particle along the direction remains the same.

By Newton's II law, we know $\overline{F} = m\overline{a}$

When $\overline{F} = \overline{0}$, $m\overline{a} = \overline{0}$

i.e) $\overline{a} = \overline{0}$

i.e) $\frac{d\overline{v}}{dt} = \overline{0}$

This implies $\overline{v}$ is a constant vector. At any instant t, $\overline{v}$ is the same. So, momentum $m\overline{v}$ remains the same i.e) momentum is conserved along a direction if there is no force acting along the direction.

Definition of Direct Impact:- Definitions ① + ②

When two bodies collide with each other, there is a ~~dis~~ distortion (rotation) taking place around the point of contact. The capacity of the bodies to regain the original shape is known as the elasticity of the body. The co-efficient of restitution denoted by 'e' measures the elasticity of the body. In general $0 < e < 1$.

* If e is 1, the body is said to be perfectly elastic.

* If e is 0, the bodies are said to be perfectly inelastic.

When two bodies collide with each other, it is only due to internal forces. So, there is no external force during a collision.

We denote the two rigid bodies as two spheres with the centres C_1 and C_2. The line C_1C_2 is called the line of centres. (If both the rigid bodies move along the line of centres before impact, the impact is called a ~~direct impact~~.) In the case of a direct impact, the bodies will move the line of centres after impact also.

Let m_1 and m_2 be the masses of the spheres, u_1 and u_2 be their velocities before impact and v_1 and v_2 their velocities after impact.

There is force acting along the line of centres during collision.

$\therefore$ The sum of the momenta before impact must be the same as the sum of the momenta after impact along the line of centres.

$$\therefore m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

This is due to the application of the principle of conservation of linear momentum.

~~**Book work:** (1)~~

~~To find the velocities, the impulse imparted and the loss in K.E due to a direct impact.~~

~~The centres of the spheres are C_1 and C_2. The masses are m_1 and m_2. The velocities before impact are u_1 and u_2. The velocities after impact are v_1 and v_2. They are along the line of centres.~~

By Newton's experimental law,

we have $V_2 - V_1 = -e(u_2 - u_1) \longrightarrow (1)$

By the principle law of conservation of linear momentum,

we have $m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \longrightarrow (2)$

$\times (1)$ by m_1

$m_1 V_2 - m_1 V_1 = -m_1 e(u_2 - u_1) \longrightarrow (3)$

$(2) + (3) \Rightarrow -m_1 v_1 + m$

Book work : (1) (*) 5M

To find the velocities, the impulse imparted and the loss in K.E due to a direct impact.

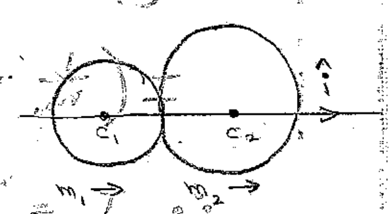
* c_1 & c_2 → centres of the spheres

* m_1 & m_2 → masses

* u_1 & u_2 → velocities before impact

* V_1 & V_2 → velocities after impact

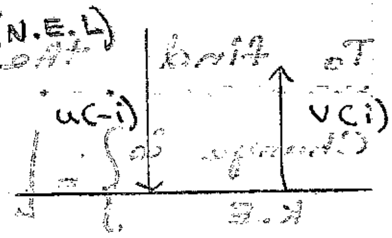
* They are along the line of centres.



By Newton's experimental law, (N.E.L)

$$\boxed{V_2 - V_1 = -e(u_2 - u_1)}$$

$$V_2 - V_1 = e(u_1 - u_2) \longrightarrow (1)$$



By principle [law] of conservation of linear momentum,

$$\boxed{m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2} \longrightarrow (2)$$

$m_1 \times (1) \Rightarrow m_1 v_2 - m_1 v_1 = m_1 e(u_1 - u_2) \longrightarrow (3)$

$(2) + (3) \Rightarrow (m_1 + m_2) v_2 = m_1 u_1 + m_2 u_2 + e m_1 (u_1 - u_2) \longrightarrow (4)$

$$V_2 = \frac{1}{(m_1 + m_2)}\left[m_1 u_1 + m_2 u_2 + e m_1 (u_1 - u_2)\right]$$

$[m_2 \times (1)] - (2) \Longrightarrow$ |||ly $V_1 = \frac{1}{(m_1 + m_2)}\left[m_1 u_1 + m_2 u_2 + e m_2 (u_2 - u_1)\right]$

To find the expression for impulse imparted to the spheres:

Take the directions $c_1 c_2$ as $\hat{i}$

The impulse imparted to the second sphere $=$ $I = mv - mu$

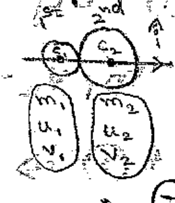
$$I(\hat{i}) = m_2 v_2 \hat{i} - m_2 u_2 \hat{i} \rightarrow (5)$$

$$\Rightarrow I = m_2 v_2 - m_2 u_2 \quad [\because \text{cancelling } \hat{i}]$$

The impulse imparted to the first sphere $=$

$$I(-\hat{i}) = m_1 v_1 \hat{i} - m_1 u_1 \hat{i}$$

$$\Rightarrow I = -m_1 v_1 + m_1 u_1 \rightarrow (6) \quad [\because \text{cancelling } (-\hat{i})]$$



$$\frac{(5)}{m_2} + \frac{(6)}{m_1} \Rightarrow \frac{I}{m_2} + \frac{I}{m_1} = v_2 - u_2 + u_1 - v_1$$

$$I\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = (v_2 - v_1) - (u_2 - u_1) \longrightarrow (7)$$

Applying Newton's experimental law,

$$\boxed{v_2 - v_1 = -e(u_2 - u_1)} \longrightarrow (8)$$

Sub (8) in (7),

$$I\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = -e(u_2 - u_1) - (u_2 - u_1) = (1+e)(u_1 - u_2)$$

$$\therefore I = \frac{m_1 m_2}{m_1 + m_2}(1+e)(u_1 - u_2) \longrightarrow (9)$$

To find the change in K.E due to collision:

$$\left.\begin{array}{l}\text{Change in}\\ \text{K.E}\end{array}\right\} = \left[\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right] - \left[\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2\right]$$

$$= \frac{1}{2}\left[m_1(v_1^2 - u_1^2) + m_2(v_2^2 - u_2^2)\right]$$

$$= \frac{1}{2}\left[m_1(v_1 - u_1)(v_1 + u_1) + m_2(v_2 - u_2)(v_2 + u_2)\right]$$

From (3) & (1)

$$\frac{1}{2}\left[m_1(v_1+u_1) \times -(u_1 - v_1) + \dots\right]$$

$$-I(v_1 + u_1)$$

$$= \frac{1}{2}\left[I(-v_1 + u_1) + I(v_2 + u_2)\right]$$

$$-v_1 - u_1 + v_2 + u_2$$

$$= \frac{I}{2}\left[(v_2 - v_1) + (u_2 - u_1)\right]$$

K.E $= \frac{1}{2} m v^2$

Change in K.E $= \frac{1}{2} m (v^2 - \dots$

We have m_1, m_2; v_1, v_2; u_1, u_2

Applying Newton's experimental law,

$$\boxed{v_2 - v_1 = -e(u_2 - u_1)}$$

$$= \frac{I}{2}\left[-e(u_2 - u_1) + (u_2 - u_1)\right]$$

$$= \frac{I}{2}(1-e)(u_2 - u_1)$$

Sub ⑨

$$\left.\begin{array}{l}\text{change} \\ \text{in K.E}\end{array}\right\} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1+e)(u_1-u_2)(u_2-u_1)(1-e)$$

$$= -\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2$$

$\therefore$ (−) sign shows, final K.E < initial K.E

So there is a loss in K.E due to impact.

Newton's experimental law for direct impact:

- If u_1 & u_2 → velocities of the two bodies along the common normal before impact.
- If v_1 & v_2 → velocities of two bodies along the common normal after impact.

Then $v_2 - v_1 = -e(u_2 - u_1)$

$\therefore$ e → relative velocity of one with respect to the other after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction.

Book work : ② 5m

Direct impact with the fixed plane

* Take the direction of velocity of rebound as $\hat{i}$.

→ Initial velocity of the particles → $-u\hat{i}$

→ Final velocity is → $v\hat{i}$

* Here the plane → fixed.

Plane $\left\{\begin{array}{l} u_1 \quad u_2 \\ v_1 \quad v_2 \end{array}\right\}$ ball

So, $u_1 = v_1 = 0$

By Newton's experimental law,

Plane : $u_1\ v_1 \to 0 \to$ fixed

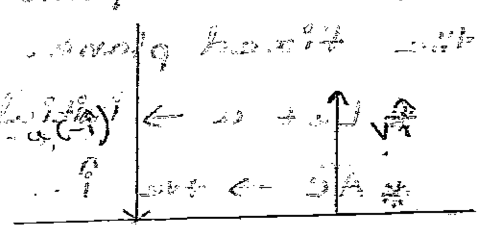
Ball : $u_2 \to -u$, $v_2 \to v$

$$v_2 - v_1 = -e(u_2 - u_1)$$

i.e) $v - 0 = -e(-u - 0)$ [∵ u is opposite in direction]

$$\boxed{v = eu}$$

* As e<1, the velocity of rebound is less than the initial velocity.



To find the impulse imparted to the sphere

$\overline{I} = mv\hat{i} - mu(-\hat{i})$ [$\because \overline{I} \to$ change in momentum] $\overline{I} = m\overline{v} - m\overline{u}$

$\overline{I} = (mv + mu)\hat{i}$

$\Rightarrow \overline{I} = m(v+u)$

$= m(eu + u)$

$= mu(1+e)$

$I = m(1+e)u$

To find the change in K.E

change in K.E = final K.E − initial K.E

$= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2}m(v^2 - u^2)$

$= \frac{1}{2}m(e^2u^2 - u^2)$

$= \frac{1}{2}mu^2(e^2 - 1)$

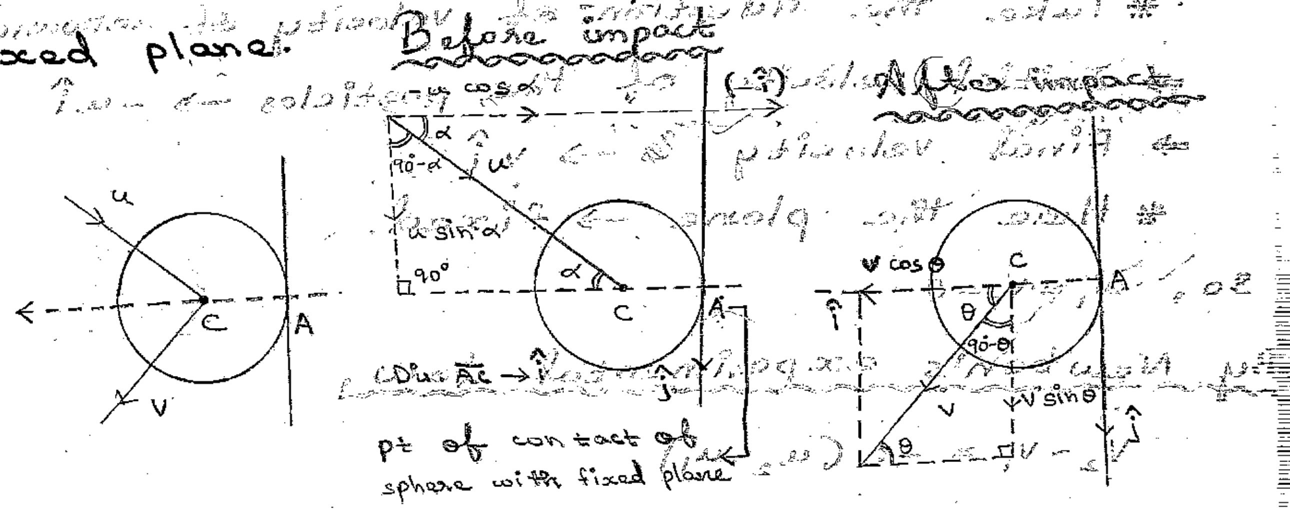
$= -\frac{1}{2}mu^2(1-e^2)$

Here, $(1-e^2) > 0$ as $e^2 < 1$ & the other terms are +ve

$\therefore$ The $(-)$ sign shows that there is a loss in K.E.

Book work: (3)

To discuss the oblique impact with a fixed plane.



* Let C → centre of the sphere.
* Let A → point of contact of the sphere with the fixed plane.
* Let u → initial velocity of the sphere.
* $\overline{AC} \to$ +ve $\hat{i}$.

* As the impact is oblique, the initial velocity u makes an angle α with the normal AC.

* After the impact the particle moves along a direction making an angle θ with the normal AC.

* So the components of the velocities along the normal AC before and after impact are $(u\cos\alpha)(-\hat{i})$ and $(v\cos\theta)\hat{i}$ [u_2 & v_2 w.r.to sphere]

By Newton's experimental law,

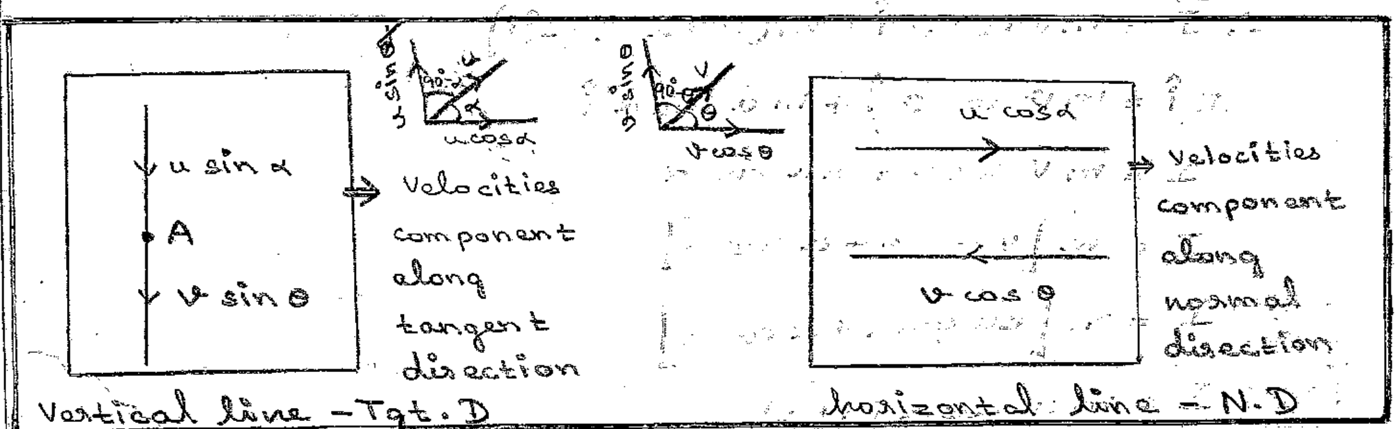
$$v_2 - v_1 = -e(u_2 - u_1) \longrightarrow \text{D.I}$$

$$v\cos\theta - 0 = -e[-u\cos\alpha - 0]$$

plane $\begin{cases} u_1 = 0 & u_2 = -u\cos\alpha \\ v_1 = 0 & v_2 = v\cos\theta \end{cases}$

$$v\cos\theta = eu\cos\alpha \longrightarrow (1)$$

$$\boxed{\begin{array}{l} u_1 = v_1 = 0 \\ u_2 = -u\cos\alpha \\ v_2 = v\cos\theta \end{array}}$$



* Component of velocities along the tangential direction ($\perp$ to AC) before and after impact is $(u\sin\alpha)\hat{j}$ & $(v\sin\theta)\hat{j}$

* Plane and Sphere $\rightarrow$ smooth

* The force of friction has to act along the tangential direction will be absent. [R → N.D, F → T.D]

* Component of velocity along the tangential direction must remain the same

So

i.e) $u\sin\alpha = v\sin\theta \longrightarrow (2)$

Squaring and adding (1) & (2)

$$v^2 = e^2u^2\cos^2\alpha + u^2\sin^2\alpha$$

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$$

$$v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$$

$\theta \rightarrow$ After impact

$\alpha \rightarrow$ Before impact

$$\Rightarrow \tan\theta = \frac{v\sin\theta}{v\cos\theta} \quad (2)$$

$$= \frac{u\sin\alpha}{eu\cos\alpha}$$

$$\tan(\text{After impact}) = \frac{\tan(\text{Before impact})}{e}$$

$$\tan\theta = \frac{\tan\alpha}{e}$$

To get impulse imparted:

* The velocity component along the $\hat{j}$ direc- remains the same.
* There is no change of momentum along $\hat{j}$ direction.
* Hence there is no impulse imparted along the $\hat{j}$ direction.
* Hence the impulse imparted is only along the $\hat{i}$ direction.

$$\therefore \bar{I} = mv\cos\theta\,\hat{i} - m(u\cos\alpha(-\hat{i}))$$

$$I\hat{i} = mv\cos\theta\,\hat{i} + mu\cos\alpha\,\hat{i}$$

$$I = mv\cos\theta + mu\cos\alpha$$

$$I = m\left[v\cos\theta + u\cos\alpha\right]$$

$$I = m\left[eu\cos\alpha + u\cos\alpha\right]$$

$$I = mu\cos\alpha(1+e)$$

Now we find the expression for change in K.E:

$$\text{Change in K.E} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$= \frac{1}{2}m\left[e^2u^2\cos^2\alpha + u^2\sin^2\alpha - u^2\right]$$

$$= \frac{1}{2}m\left[e^2u^2\cos^2\alpha - u^2(1-\sin^2\alpha)\right]$$

$$= \frac{1}{2}m\left[e^2u^2\cos^2\alpha - u^2\cos^2\alpha\right]$$

$$= \frac{1}{2}mu^2\cos^2\alpha(e^2-1)$$

$$= -\frac{1}{2}mu^2\cos^2\alpha(1-e^2) \quad [\because e<1]$$

$\therefore$ The negative sign shows that there is loss in K.E.

1. A ball of mass 8 gms moving with the velocity of 10 cm/sec impinges directly on another mass of 24 gms moving at 2 cm/sec in the same direction. If $e = 1/2$ find the velocities after impact. Also Find I and calculate the loss in K.E

I ↳ Impulse impacted

$m_1 = 8$ gms , $m_2 = 24$ gms , $e = 1/2$

$u_1 = 10$ cm/sec , $u_2 = 2$ cm/sec

$$\Rightarrow \boxed{V_1 = \frac{1}{m_1+m_2}\left[m_1u_1 + m_2u_2 + em_2(u_2-u_1)\right]}$$

$$= \frac{1}{32}\left[80+48+\frac{1}{2}\times 24(-8)\right] = \frac{1}{32}\left[80+48-96\right] = \frac{1}{32}(32)$$

$V_1 = 1$ cm/sec

$$\Rightarrow \boxed{V_2 = \frac{1}{m_1+m_2}\left[m_1u_1 + m_2u_2 + em_1(u_1-u_2)\right]}$$

$$= \frac{1}{32}\left[80+48+\frac{1}{2}\times 8(8)\right] = \frac{1}{32}\left[128+32\right] = \frac{1}{32}(160) = 5$$

$V_2 = 5$ cm/sec

$$\Rightarrow \boxed{I = \frac{m_1m_2}{m_1+m_2}(1+e)(u_1-u_2)}$$

$$= \frac{192}{32}(1+1/2)(10-2) = \frac{\cancel{192}^{24}}{\cancel{32}_{8}} \times \frac{3}{\cancel{2}} \times \cancel{8}^{4}$$

$I = 72$

$$\Rightarrow \boxed{\text{Loss in K.E} = -1/2\left(\frac{m_1m_2}{m_1+m_2}\right)(1-e^2)(u_1-u_2)^2}$$

$$= -1/2\,(192/32)(1-1/4)(64) = -1/2 \times 192 \times \frac{3}{4} \times 2$$

$$= -144$$

2. A ball of mass 'm' impinges on another of mass '2m' which is moving in the same direction as the first but with 1/7 of its velocity. If e is 3/4, ST the first ball is reduced to rest after impact. Find I, V_2 & loss in K.E

↳ $V_1 = 0$

$m_1 = m$, $m_2 = 2m$, $u_1 = u$, $u_2 = \frac{1}{7}u$, $e = \frac{3}{4}$

$$\Rightarrow V_1 = \frac{1}{3m}\left[mu + \frac{2}{7}mu + \frac{3}{4}\times 2m\left(\frac{1}{7}u - u\right)\right]$$

$$= \frac{1}{3m}\left[mu\left(1+\frac{2}{7} + \frac{3}{2}\times\frac{1}{7} - \frac{3}{2}\right)\right]$$

$$= \frac{1}{3}\left[u\left(\frac{14+4+3-21}{14}\right)\right]$$

$$\boxed{V_1 = 0}$$

$$\Rightarrow I = \frac{2m^2}{3m}\left(1+\frac{3}{4}\right)\left(u - \frac{1}{7}u\right)$$

$$= \frac{2}{3}m\left(\frac{7}{4}\right)\left(\frac{6u}{7}\right)$$

$$\boxed{I = mu}$$

$$\Rightarrow \text{Loss in K.E} = -\frac{1}{2}\left(\frac{2m^2}{3m}\right)\left(1-\frac{9}{16}\right)\left(\frac{6u}{7}\right)^2$$

$$= -\frac{m}{3}\times\frac{7}{16}\times\frac{36u^2}{49}$$

$$\boxed{\text{Loss in K.E} = \frac{-3mu^2}{28}}$$

$$\Rightarrow V_2 = \frac{1}{3m}\left[mu + \frac{2}{7}mu + \frac{3}{4}\times m\left(u - \frac{1}{7}u\right)\right]$$

$$= \frac{1}{3m}\left[mu\left(1+\frac{2}{7} + \frac{3}{4} - \frac{3}{4}\times\frac{1}{7}\right)\right]$$

$$= \frac{1}{3}\left[u\left(\right.\right.$$

$$\boxed{V_2 = \frac{9u}{14}}$$

3. A ball A impinges on an exactly equal and similar ball B lying on a horizontal plane. If the co-efficient of restitution is e. PT after impact, the velocity of B will be equal to that of A is $(1+e):(1-e)$. Find I & loss in K.E

$m_1 = m_2 = m$, $u_2 = 0$

$u_1 = u$, $u_2 = 0$, $m_1 = m_2 = m$

$$\Rightarrow V_1 = \frac{1}{2m}\left[mu + 0 + em(0-u)\right] = \frac{1}{2m}(mu - emu) = \frac{1}{2m}(1-e)mu$$

$$\boxed{V_1 = \frac{u}{2}(1-e)}$$

$$\Rightarrow V_2 = \frac{1}{2m}\left(mu + 0 + em(u-0)\right) = \frac{1}{2m}\left(mu + emu\right)$$

$$\boxed{V_2 = \frac{u}{2}(1+e)}$$

$$\therefore V_2 : V_1 = \frac{\frac{u}{2}(1+e)}{\frac{u}{2}(1-e)} = (1+e) : (1-e)$$

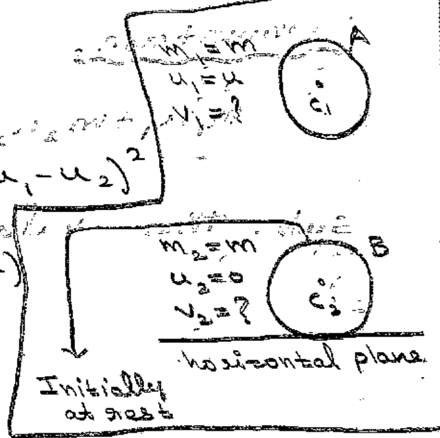
$$\Rightarrow I = \frac{m_1 m_2}{m_1+m_2}\left[(1+e)(u_1-u_2)\right] = \frac{m^2}{2m}\left[(1+e)(u)\right]$$

$$\boxed{I = \frac{mu}{2}(1+e)}$$

$$\Rightarrow \text{Loss in K.E} = -\frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(1-e^2)(u_1-u_2)^2$$

$$= -\frac{1}{2}\left(\frac{m^2}{2m}\right)(1-e^2)(u^2)$$

$$= -\frac{m}{4}(1-e^2)u^2$$



4. Hail stones are observed to strike the surface of a frozen lake in a direction making an angle of 30° with the vertical and is found to rebound at an angle of 60° (after impact). Assuming the contact to be smooth (friction is absent) find e.

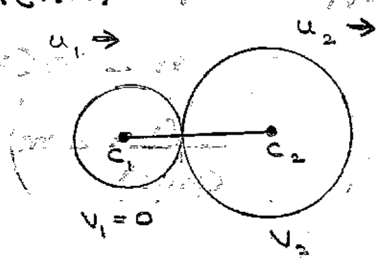
Here $\alpha = 30°$, $\theta = 60°$ ← After impact (rebound)
← Before impact (strike)

$$\Rightarrow \tan\theta = \frac{\tan\alpha}{e}$$

$$\therefore e = \frac{\tan\alpha}{\tan\theta} = \frac{\tan 30°}{\tan 60°} = \frac{1/\sqrt{3}}{\sqrt{3}} = \frac{1}{3}$$

5. A ball impinges directly on another ball of mass 'm' times its mass which is moving with $1/n$ times its velocity in the same direction. If the impact reduces the first ball to rest ($v_1 = 0$). P.T

$$e = \frac{(m+n)}{m(n-1)} \text{ and } m > \frac{n}{n-2}$$



m_1, u_1 ; $m_2 = m m_1$, $u_2 = \frac{1}{n}u_1$

$v_1 = 0$, $v_2 = ?$

* Let the mass of I ball $\rightarrow m_1$
* mass of the II ball $\rightarrow m_2 = m m_1$
* velocity of the I ball before impact $= u_1$
* velocity of the II ball before impact $\rightarrow u_2 = \frac{1}{n}$
* velocity of the I ball after impact $\rightarrow v_1 = 0$

By the law of principle of conservation of lin. momentum,

$$\boxed{m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2}$$

sub. the values $\rightarrow \cancel{m_1} u_1 + m \cancel{m_1}\left(\frac{1}{n} u_1\right) = \cancel{m_1} \cdot 0 + m \cancel{m_1} v_2$

$$u_1 + \frac{m u_1}{n} = m v_2$$

$$u_1\left(1 + \frac{m}{n}\right) = m v_2$$

$$u_1\left(\frac{n+m}{n}\right) = m v_2 \rightarrow (2)$$

By Newton's experimental law,

$$\boxed{v_2 - v_1 = -e(u_2 - u_1)}$$

$$v_2 = -e\left(\frac{1}{n} u_1 - u_1\right)$$

$$v_2 = -e\left(\frac{1}{n} - 1\right) u_1 \rightarrow (3)$$

(3) $\times m \Rightarrow m v_2 = \frac{m(n-1)}{n} u_1 e \rightarrow (4)$

Equating (2) & (5),

$$\cancel{u_1}\left(\frac{n+m}{\cancel{n}}\right) = \frac{m(n-1)}{\cancel{n}} \cancel{u_1} e \Rightarrow e = \frac{m+n}{m(n-1)}$$

To prove : $m > \frac{n}{}$

We know $\boxed{e < 1}$ in general

$\therefore \cancel{\frac{m+n}{m(n-1)} < 1} \Rightarrow \cancel{m + n < mn - m}$

$\cancel{m + m - mn < -n} \Rightarrow \cancel{2m - mn < -n}$

$\cancel{-m(n-2) < -n} \Rightarrow \cancel{m > \frac{n}{n-2}}$

~~Hence the result,~~

$m + n < mn - m$

$n < mn - m - m$

$n < mn - 2m$

$n < m(n-2)$

$\frac{n}{(n-2)} < m$

6. If two equal spheres which are perfectly elastic collide with each other, find their velocities after impact.

$\rightarrow e=1$

$\rightarrow v_1 \,\&\, v_2$

$$\Rightarrow v_1 = \frac{1}{2m}\left[mu_1 + mu_2 + m(u_2-u_1)\right] = \frac{1}{2m} m(u_1+u_2+u_2-u_1)$$

$$\boxed{v_1 = u_2} \qquad \boxed{m_1 = m_2 = m,\ e=1}$$

$$\Rightarrow v_2 = \frac{1}{2m} m\left[u_1+u_2+u_1-u_2\right] = \frac{2u_1}{2}$$

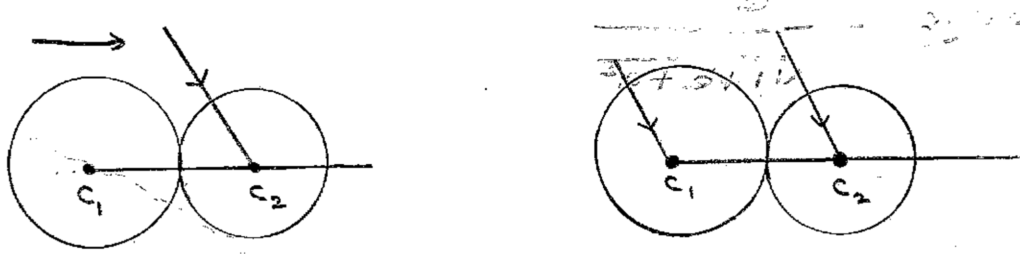
$$\boxed{v_2 = u_1}$$

$\therefore$ In this case they interchange their velocities after impact.

$$\boxed{\begin{matrix} u_1 & & u_2 \\ & \times & \\ v_1 & & v_2 \end{matrix}}$$

Definition of oblique impact:-

* Two bodies are said to be impinge obliquely if the direction of motion of either or both is not along the common normal at the point of contact before impact.



* During collision, the bodies will undergo a distortion. The capacity to regain the original shape will depend on the material out of which the bodies are made and the capacity is independent of the shape, size and mass.

* The capacity is said to be the elasticity of the body. The co-efficient of the elasticity is denoted by e.

* When $e=0 \rightarrow$ the body is perfectly inelastic.

* When $e=1 \rightarrow$ the body is perfectly elastic.

* These two situations are ideal.

* In general $0<e<1$.

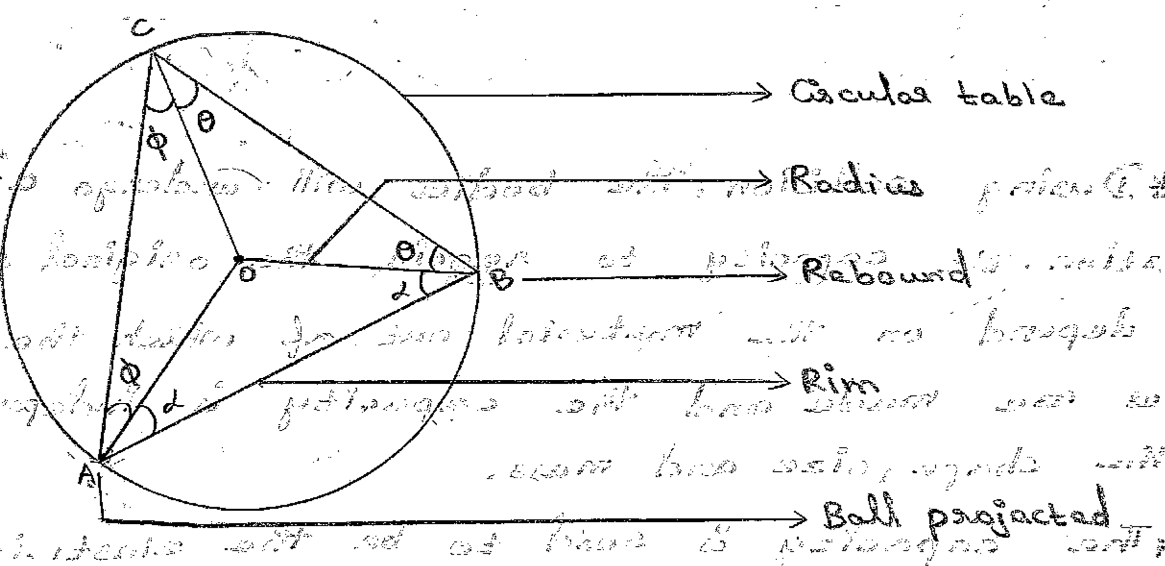
6. Hail stones are observed to strike the surface of a frozen lake in a direction making an angle of 30° with the vertical and is found to rebound at an angle of 60°. Assuming the contact to be smooth find e.

Here $\alpha = 30^\circ$, $\theta = 60^\circ$

$$\therefore \tan\theta = \frac{\tan\alpha}{e}$$

$$e = \frac{\tan\alpha}{\tan\theta} = \frac{1}{3}$$

7. A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. S.T a ball projected along the table from a point A on the rim in a direction making an angle α with the radius through A will return to the point of projection after two impacts is $\tan\alpha = \dfrac{e^{3/2}}{\sqrt{1+e+e^2}}$.



* The ball is projected from a point on the rim on a circular table.

* It strikes the rim at the point B and gets rebound.

* It strikes the rim again at C and get rebound. But it is given that it reaches the point A finally.

OA, OB & OC → Normals at A, B & C

At B :-

The velocity before impact makes an angle α with the normal and the velocity after impact makes an angle θ with the normal.

$$\therefore \tan\theta = \frac{\tan\alpha}{e} \rightarrow (1)$$

Impact at C :-

$$\tan\phi = \frac{\tan\theta}{e} \longrightarrow (2) \quad \left[\because \tan\phi = \frac{\tan\theta}{e} = \frac{\tan\alpha}{e^2}\right] \text{ By (1)}$$

From the ΔABC,

$$\alpha+\alpha+\theta+\theta+\phi+\phi = 180^\circ$$

$$2\alpha + 2\theta + 2\phi = 180^\circ$$

$$\alpha+\theta+\phi = 90^\circ$$

$$\alpha = 90^\circ - (\theta+\phi)$$

$$\tan\alpha = \tan[90^\circ - (\theta+\phi)]$$

$$= \cot(\theta+\phi)$$

$$= \frac{1}{\tan(\theta+\phi)}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$= \frac{1-\tan\theta\tan\phi}{\tan\theta+\tan\phi}$$

$$= \frac{1-\frac{\tan\alpha}{e}\cdot\frac{\tan\alpha}{e^2}}{\frac{\tan\alpha}{e}+\frac{\tan\alpha}{e^2}}$$

$$= \frac{1-\frac{\tan^2\alpha}{e^3}}{\tan\alpha\left(\frac{1}{e}+\frac{1}{e^2}\right)}$$

$$\tan^2\alpha\left(\frac{1}{e}+\frac{1}{e^2}\right) = 1-\frac{\tan^2\alpha}{e^3} \Rightarrow \tan^2\alpha\left(\frac{1}{e}+\frac{1}{e^2}\right)+\frac{\tan^2\alpha}{e^3} = 1$$

$$\Rightarrow \tan^2\alpha\left(\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right) = 1 \Rightarrow \tan^2\alpha\,(e^2+e+1) = e^3$$

$$\tan^2\alpha = \frac{e^3}{1+e+e^2}$$

$$\tan\alpha = \frac{e^{3/2}}{\sqrt{1+e+e^2}}$$ Hence showed

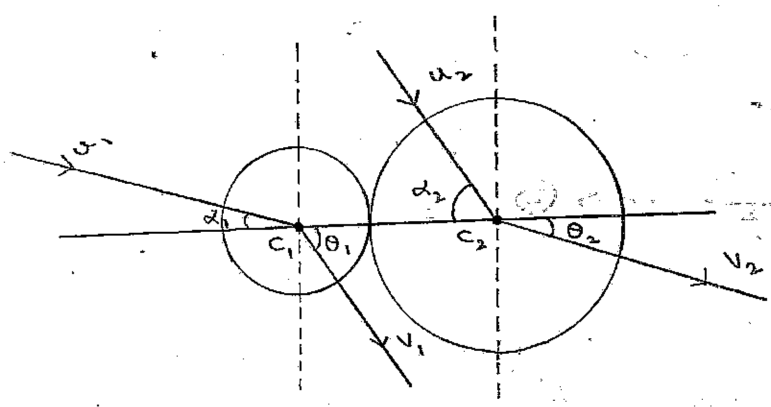
Book work : (4) V.V.V (X) 10m

To find i) Velocities

ii) Impulse imparted

iii) Change in K.E

in an oblique impact.



* u_1 & u_2 → initial velocities of the two spheres
* initial velocities makes an angle α_1 & α_2 with a line of centres.
* V_1 & V_2 → final velocities of the two spheres
* final velocities makes an angle θ_1 & θ_2 with a line of centres.

Component of velocities along N.D :-

- $u_1 \cos \alpha_1$ & $v_1 \cos \theta_1$ → 1st sphere
- $u_2 \cos \alpha_2$ & $v_2 \cos \theta_2$ → 2nd sphere

Component of velocities along T.D :-

- $u_1 \sin \alpha_1$ & $v_1 \sin \theta_1$ → 1st sphere
- $u_2 \sin \alpha_2$ & $v_2 \sin \theta_2$ → 2nd sphere

* As the two spheres are smooth, the force of friction is absent.
* The velocity components along the tangential directions before & after impact must be the sa

$$\left.\begin{aligned} u_1 \sin \alpha_1 &= v_1 \sin \theta_1 \\ u_2 \sin \alpha_2 &= V_2 \sin \theta_2 \end{aligned}\right\} \rightarrow (1)$$

By Newton's experimental law,

$$V_2 - V_1 = -e(u_2 - u_1) \rightarrow \text{D.I}$$

$$V_2 \cos\theta_2 - V_1 \cos\theta_1 = -e(V_2 \cos\alpha_2 - V_1 \cos\alpha_1) \rightarrow \text{O.I} \rightarrow (2)$$

By the law of principle of conservation of linear momentum,

$$m_1 u_1 + m_2 u_2 = m_1 V_1 + m_2 V_2 \rightarrow \text{D.I}$$

$$m_1 V_1 \cos\theta_1 + m_2 V_2 \cos\theta_2 = m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 \rightarrow \text{O.I} \rightarrow (3)$$

$$(2) \times m_1 \Rightarrow m_1 v_1 \cos\theta_2 - m_1 V_1 \cos\theta_1 = -e m_1 [u_2 \cos\alpha_2 - u_1 \cos\alpha_1] \rightarrow (4)$$

$$(3) + (4) \Rightarrow m_2 V_2 \cos\theta_2 + m_1 V_2 \cos\theta_2 = m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\theta_2 + e m_1 [u_1 \cos\alpha_1 - u \cos\alpha_2]$$

$$(m_1 + m_2)\cos\theta_2 V_2 = m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 + e m_1 [u_1 \cos\alpha_1 - u_2 \cos\alpha_2]$$

$$V_2 \cos\theta_2 = \frac{1}{m_1 + m_2}\begin{bmatrix} m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 \\ + e m_1 (u_1 \cos\alpha_1 - u_2 \cos\alpha_2) \end{bmatrix}$$

$|||^{ly}$ $$V_1 \cos\theta_1 = \frac{1}{m_1 + m_2}\begin{bmatrix} m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 \\ + e m_2 (u_2 \cos\alpha_2 - u_1 \cos\alpha_1) \end{bmatrix}$$

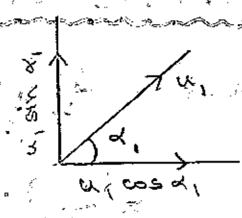
To find the impulse imparted to the spheres:-

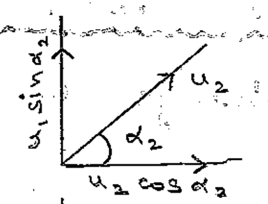
$$\overline{u_1} = u_1 \cos\alpha_1 \hat{i} + u_1 \sin\alpha_1 \hat{j}$$

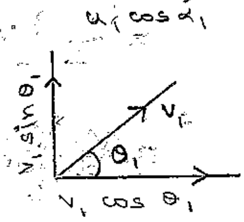
$$\overline{u_2} = u_2 \cos\alpha_2 \hat{i} + u_2 \sin\alpha_2 \hat{j}$$

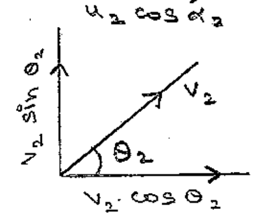
$$\overline{V_1} = V_1 \cos\theta_1 \hat{i} + V_1 \sin\theta_1 \hat{j}$$

$$\overline{V_2} = V_2 \cos\theta_2 \hat{i} + V_2 \sin\theta_2 \hat{j}$$









Impulse imparted to the 2nd sphere:-

$$\overline{I} = m_2 \overline{V_2} - m_2 \overline{u_2}$$

$$I \hat{i} = m_2 V_2 \cos\theta_2 \hat{i} - m_2 u_2 \cos\alpha_2 \hat{i}$$

$$I = m_2 [V_2 \cos\theta_2 - u_2 \cos\alpha_2]$$

$$\frac{I}{m_2} = V_2 \cos\theta_2 - u_2 \cos\alpha_2 \rightarrow (i)$$

Impulse imparted to the 1^{st} sphere:

$$\bar{I} = m_1 \bar{v}_1 - m_1 \bar{u}_1$$

$$I(-\hat{i}) = m_1 v_1 \cos\theta_1 \hat{i} - m_1 u_1 \cos\alpha_1 \hat{i}$$

$$-I = m_1 [v_1 \cos\theta_1 - u_1 \cos\alpha_1]$$

$$I = m_1 [u_1 \cos\alpha_1 - v_1 \cos\theta_1]$$

$$\frac{I}{m_1} = u_1 \cos\alpha_1 - v_1 \cos\theta_1 \longrightarrow \text{(ii)}$$

$$\text{(i)} + \text{(ii)} \Rightarrow \frac{I}{m_1} + \frac{I}{m_2} = u_1 \cos\alpha_1 - v_1 \cos\theta_1 + v_2 \cos\theta_2 - u_2 \cos\alpha_2$$

$$I\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = (v_2 \cos\theta_2 - v_1 \cos\theta_1) - (u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$$

From (2),
$$I\left(\frac{m_1 m_2}{m_1 + m_2}\right) = -e(u_2 \cos\alpha_2 - u_1 \cos\alpha_1) - (u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$$

$$I = \frac{m_1 m_2}{m_1 + m_2}\left[-e(u_1 \cos\alpha_1 - u_2 \cos\alpha_2) + (u_1 \cos\alpha_1 - u_2 \cos\alpha_2)\right]$$

$$= \frac{m_1 m_2}{m_1 + m_2}\left[e(u_1 \cos\alpha_1 - u_2 \cos\alpha_2) + (u_1 \cos\alpha_1 - u_2 \cos\alpha_2)\right]$$

$$= \frac{m_1 m_2}{m_1 + m_2}(1+e)(u_1 \cos\alpha_1 - u_2 \cos\alpha_2)$$

To find change in K.E

$$\text{Change in K.E} = \frac{1}{2}\left[(m_1 v_1^2 + m_2 v_2^2) - (m_1 u_1^2 + m_2 u_2^2)\right]$$

$$= \frac{1}{2}\left[m_1 v_1^2 \cdot 1 + m_2 v_2^2 \cdot 1 - m_1 u_1^2 \cdot 1 - m_2 u_2^2 \cdot 1\right]$$

$$= \frac{1}{2}\left[m_1 v_1^2 (\cos^2\theta_1 + \sin^2\theta_1) + m_2 v_2^2 (\cos^2\theta_2 + \sin^2\theta_2) - m_1 u_1^2 (\cos^2\alpha_1 + \sin^2\alpha_1) - m_2 u_2^2 (\cos^2\alpha_2 + \sin^2\alpha_2)\right]$$

$$= \frac{1}{2}\left[m_1 [v_1^2 \cos^2\theta_1 + v_1^2 \sin^2\theta_1 - u_1^2 \cos^2\alpha_1 - u_1^2 \sin^2\alpha_1] + m_2 [v_2^2 \cos^2\theta_2 + v_2^2 \sin^2\theta_2 - u_2^2 \cos^2\alpha_2 - u_2^2 \sin^2\alpha_2]\right]$$

$$\left.\begin{array}{l}\text{Change}\\ \text{in K.E}\end{array}\right\} = \frac{1}{2}\begin{bmatrix} m_1[(v_1\cos\theta_1+u_1\cos\alpha_1)(v_1\cos\theta_1-u_1\cos\alpha_1)+\\ (v_1\sin\theta_1+u_1\sin\alpha_1)(v_1\sin\theta_1-u_1\sin\alpha_1)]\\ +m_2[(v_2\cos\theta_2+u_2\cos\alpha_2)(v_2\cos\theta_2-u_2\cos\alpha_2)\\ +(v_2\sin\theta_2+u_2\sin\alpha_2)(v_2\sin\theta_2-u_2\sin\alpha_2)]\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} m_1(v_1\cos\theta_1+u_1\cos\alpha_1)(v_1\cos\theta_1-u_1\cos\alpha_1)\\ +m_2(v_2\cos\theta_2+u_2\cos\alpha_2)(v_2\cos\theta_2-u_2\cos\alpha_2)\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} \cancel{m_1}(v_1\cos\theta_1+u_1\cos\alpha_1)(-I/\cancel{m_1})\\ +\cancel{m_2}(v_2\cos\theta_2+u_2\cos\alpha_2)(I/\cancel{m_2})\end{bmatrix}$$

$$=\frac{I}{2}\left[(v_2\cos\theta_2-v_1\cos\theta_1)+(u_2\cos\alpha_2-u_1\cos\alpha_1)\right]$$

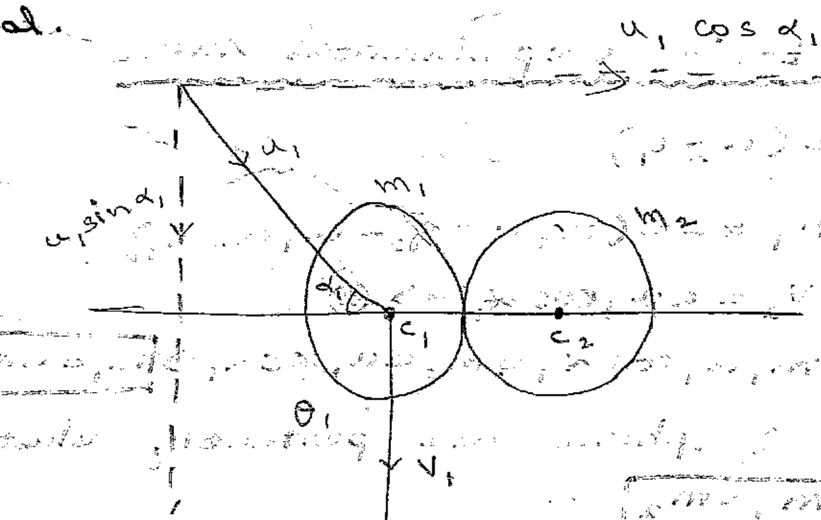
$$=\frac{I}{2}\left[-e(u_2\cos\alpha_2-u_1\cos\alpha_1)+(u_2\cos\alpha_2-u_1\cos\alpha_1)\right]$$

$$=\frac{I}{2}(1-e)(u_2\cos\alpha_2-u_1\cos\alpha_1)$$

$$=\frac{-1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1\cos\alpha_1-u_2\cos\alpha_2)^2$$

$\therefore$ -ve sign indicates that there is a loss in K.E

8. A smooth sphere of mass m_1 impinges obliquely on a smooth sphere of mass m_2 at rest. If the directions after the impact are at right angle. ST, $m_1 = m_2 e$. show also that if the spheres are perfectly elastic their masses are equal.



* 2^{nd} sphere $\rightarrow$ at rest before impact.

$u_2 = 0$, $\alpha_2 = 0$

* Two spheres are smooth.
* The tangential components of the velocit remain the same.

$V_2 \sin\theta_2 = u_2 \sin\alpha_2$

$V_2 \sin\theta_2 = 0$ $[\because u_2 = 0]$

$\Rightarrow V_2 \neq 0$; $\therefore \sin\theta_2 = 0$

$\theta_2 = 0$

* θ_2 is the angle made by the velocity after impact.

$\Rightarrow$ The second sphere after impact moves alon the line of centres.

Given:

* The direction after impact are at right angle $\Rightarrow \theta_1 = 90°$

By the principle of conservation of linear momentum, along the line of centres

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \rightarrow$ D.I

$m_1u_1\cos\alpha_1 + m_2u_2\cos\alpha_2 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2 \rightarrow$ O.I

Here $u_2 = 0$, $\alpha_2 = 0$, $\theta_2 = 0$, $\theta_1 = 90°$

$\Rightarrow m_1u_1\cos\alpha_1 = m_1v_1\cos 90° + m_2v_2\cos 0$

$m_1u_1\cos\alpha_1 = m_2v_2 \rightarrow (1)$

$\alpha_1 \rightarrow \theta_1$, $v_1 \rightarrow \theta_1$, $u_2 \Rightarrow \alpha_2$, $v_2 \Rightarrow \theta_2$

Applying Newton's experimental law,

$V_2 - V_1 = -e(u_2 - u_1) \rightarrow$ D.I

$V_2\cos\theta_2 - V_1\cos\theta_1 = -e(u_2\cos\alpha_2 - u_1\cos\alpha_1) \rightarrow$ O.I

$V_2 = eu_1\cos\alpha_1 \rightarrow (2)$ $(u_2 = \alpha_2 = \theta_2 = 0,\ \theta_1 = 90°)$

Sub (2) in (1), $m_1u_1\cos\alpha_1 = m_2eu_1\cos\alpha_1 \Rightarrow \boxed{m_1 = em_2}$

* If the 2 spheres are perfectly elastic then $e = 1$. $\therefore \boxed{m_1 = m_2}$

Hence proved

Unit - 3

15. A particle moves with a central acceleration $\mu[r^{-7}]$ and starts from an apse at a distance 'a' with a velocity equal to the velocity which would be acquired by the particle ~~remaining~~ travelling from rest at infinity to the apse. ST the eqn of its orbit is $r^2 = a^2 \cos 2\theta$

Already written as 15 Sum in 5s 3rd unit

Given : acceleration $= \mu r^{-7}$

↳ starts → apse → at a distance 'a'

Velocity → particle → rest at ∞ to apse.

To prove : The eqn of orbit is $r^2 = a^2 \cos 2\theta$

Proof :

r → displacement
$\dot{r}$ → velocity
$\ddot{r}$ → acceleration

$$\ddot{r} = -\mu r^{-7} = \frac{-\mu}{r^7} \quad [\because \text{attractive force}]$$

× by $2\dot{r}$,

$$2\dot{r}\ddot{r} = \frac{-\mu}{r^7} 2\dot{r}$$

Integrating w.r.to 't'

$$\int \cancel{2}\dot{r}\ddot{r}\, dt = -\cancel{2}\mu \int \frac{\dot{r}}{r^7}\, dt$$

$$\int \frac{d}{\cancel{dt}} (\dot{r})^2 \cancel{dt} = -2\mu \int r^{-7} \frac{dr}{\cancel{dt}} \cancel{dt}$$

$$\int = \frac{r^{n+1}}{n+1}$$

$$\text{diff} = n r^{n-1}$$

$$\left[\dot{r}^2\right]_{0 \to \text{pole}}^{v \to \text{speed}} = +\cancel{2}\mu \left[\frac{r^{-6}}{\cancel{+6}3}\right]_{\infty}^{r}$$

$$v^2 = \boxed{\frac{\mu}{3r^6}}$$

At apse :

$p = r = a$

$$\boxed{v^2 = \frac{\mu}{3a^6}}$$

W.K.T, $h = pv$

$$h^2 = p^2 v^2 = \frac{\cancel{a^2}\mu}{3a^{\cancel{6}4}} = \frac{\mu}{3a^4} \longrightarrow (1)$$

D.E in polar co-ordinates

$$h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right] = -\phi(r) = F = \frac{\mu}{r^7} \quad [\text{attractive}$$

$$\Rightarrow \frac{d^2u}{d\theta^2}+u = \frac{\mu}{h^2u^2r^7} = \frac{\mu u^{\cancel{7}5}}{h^2\cancel{u^2}} = \frac{\mu u^5}{h^2} = \frac{\cancel{\mu}u^5 3a^4}{\cancel{\mu}}$$

$$\frac{d^2u}{d\theta^2}+u = 3a^4u^5$$

$\times$ by $2.\frac{du}{d\theta}$ and integrating w.r.to θ

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}+2u\frac{du}{d\theta} = 3a^4u^5\times 2\frac{du}{d\theta}$$

$$\int\frac{d}{d\theta}\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)d\theta = 6a^4\int u^5\frac{du}{d\theta}d\theta$$

$$\left(\frac{du}{d\theta}\right)^2+u^2 = \cancel{6}a^4\times\frac{u^6}{\cancel{6}}+c$$

$$\left(\frac{du}{d\theta}\right)^2+u^2 = a^4u^6+c \longrightarrow (2)$$

To find c:

Initially $r=a$

$u = 1/r = 1/a$

$\frac{du}{d\theta} = 0$

Sub in (2) $\Rightarrow u^2 = a^4u^6+c$

$$\frac{1}{a^2} = a^4\frac{1}{a^6}+c \Rightarrow \boxed{c=0}$$

Sub in (2) $\Rightarrow \left(\frac{du}{d\theta}\right)^2+u^2 = a^4u^6$

$$\left(\frac{du}{d\theta}\right)^2 = a^4u^6-u^2 \longrightarrow (3)$$

$$= \cancel{u^2(a^4u^4-1)}$$

$$\Rightarrow \frac{du}{d\theta} = \cancel{u\sqrt{(a^4u^4-1)}}$$

$$\boxed{u = 1/r \Rightarrow \frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \longrightarrow (4)$$

Comparing (3) & (4),

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6} - \frac{1}{r^2} \longrightarrow (5)$$

~~To prove~~ $\cancel{\frac{dr}{d\theta} = \frac{\sqrt{a^4-r^4}}{r}}$

From (5) $\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4-r^4}{r^6}$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4-r^4}{r^2}$$

$$\boxed{\frac{dr}{d\theta} = \frac{\sqrt{a^4-r^4}}{r}}$$

$$\frac{r\,dr}{\sqrt{a^4-r^4}} = d\theta$$

$$\frac{r\,dr}{\sqrt{(a^2)^2-(r^2)^2}} = d\theta$$

Put $r^2 = y \Rightarrow 2r\,dr = dy$ and integrating

$$\int \frac{dy}{2\sqrt{(a^2)^2-y^2}} = \int d\theta$$

$$\frac{1}{2}\int \frac{dy}{\sqrt{(a^2)^2-y^2}} = \int d\theta$$

$$\frac{1}{2}\sin^{-1}\left(y/a^2\right) = \theta + K$$

$$\sin^{-1}\left(y/a^2\right) = 2\theta + 2K$$

$$\sin^{-1}\left(y/a^2\right) = 2\theta + c' \longrightarrow (6)$$

Find c': $\sin^{-1}\left(\cancel{a^2}/\cancel{a^2}\right) = 2\times 0 + c'$ $\left[\because y = r^2 = a^2\right]\left[\theta = 0\right]$

$\cancel{\sin^{-1}}\left(\cancel{\sin}\,\pi/2\right) = c' \Rightarrow \boxed{c' = \pi/2}$ $\sin(90^\circ+2\theta)$ $2\theta+90^\circ$

Sub c' in (6) $\Rightarrow \sin^{-1}\left(\frac{r^2}{a^2}\right) = 2\theta + \frac{\pi}{2} \Rightarrow \frac{r^2}{a^2} = \sin\left(2\theta+\pi/2\right)$

$$\boxed{r^2 = \cos 2\theta \cdot a^2}$$

Hence proved.

16. If the law of acceleration is $5\mu u^3 + 8\mu c^2 u^5$ and the particle is projected from an apse a distance 'c' with the velocity $\frac{3\sqrt{\mu}}{c}$. P.T the eqn of the orbit is $r = c\cos(2\theta/3)$

The force is $5\mu u^3 + 8\mu c^2 u^5$

$$\text{So, } h^2 u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \mu u^2(5u + 8c^2u^3) \longrightarrow (1)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{h^2 u^2}(5u + 8c^2u^3) \longrightarrow (2)$$

$\times$ by $\frac{2du}{d\theta}$,

$$2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + 2\frac{du}{d\theta}\cdot u = \frac{2du}{d\theta}\cdot\frac{\mu u^2}{h^2u^2}(5u + 8c^2u^3)$$

Integrating it,

$$\int 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + 2\frac{du}{d\theta}\,u = \frac{\mu}{h^2}\int 2\frac{du}{d\theta}(5u + 8c^2u^3)\,d\theta$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2}(5u^2 + 4c^2u^4) + c_1 \longrightarrow (3)$$

W.K.T, $pv = h$, $p = c$, $v = \frac{3\sqrt{\mu}}{c}$

$$\Rightarrow h = \frac{3\sqrt{\mu}}{c}\cdot c = 3\sqrt{\mu} \Rightarrow \sqrt{\mu} = \frac{h}{3}$$

$$\frac{\sqrt{\mu}}{h} = \frac{1}{3} \Rightarrow \frac{\mu}{h^2} = \frac{1}{9}$$

$r = c$, $u = \frac{1}{c}$, $\frac{du}{d\theta} = 0$ at an apse.

So (3) becomes,

$$1/c^2 = 1/9(5u^2 + 4c^2u^4) + c_1$$

$$\frac{1}{c^2} = \frac{1}{9}\left(\frac{5}{c^2} + \frac{4}{c^2}\right) + c_1$$

$$\frac{1}{c^2} = \frac{1}{c^2} + c_1 \Rightarrow \boxed{c_1 = 0}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{9}\left[5u^2 + 4c^2u^4\right]$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{9}\left[5u^2 + 4c^2u^4\right] - u^2 = \frac{1}{9}\left[4c^2u^4 - 4u^2\right]$$

$$= \frac{4u^2}{9}\left[c^2u^2 - 1\right]$$

$$\frac{du}{d\theta} = \frac{2u}{3}\sqrt{c^2u^2-1}$$

$$\frac{du}{u\sqrt{c^2u^2-1}} = \frac{2}{3}\,d\theta$$

Integrating,

$$\sec^{-1}(cu) = \frac{2}{3}\theta + c_2$$

Take the apse line as the initial line,

$$r = c,\ \theta = 0,\ u = \frac{1}{c}$$

$$\text{ie) } \sec^{-1}(1) = 0 + c_2 \Rightarrow c_2 = 0$$

$$\therefore \sec^{-1}(cu) = \frac{2}{3}\theta$$

$$cu = \sec\left(\frac{2}{3}\right)\theta$$

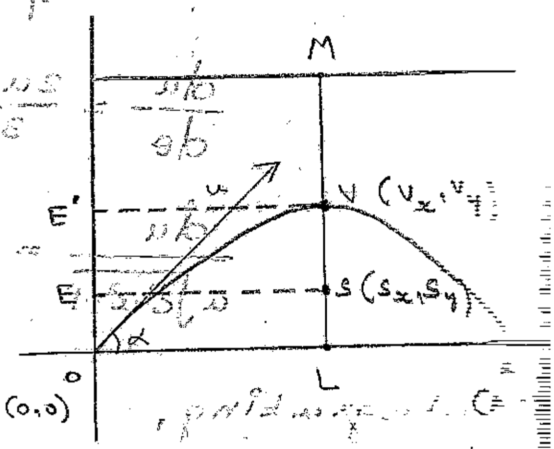
$$\frac{c}{r} = \sec\left(\frac{2}{3}\right)\theta$$

$$r = c\cos\left(\frac{2}{3}\right)\theta$$

Unit - 3

Projectile

Definition :-



* V → Vertex
* S → focus
* ML → directrix
* $y = ML = MV + VL$
  = eqn of directrix

⇒ $MV = VS$
where VS is distance b/w vertex and focus
⇒ $VL = V_y$, $SL = S_y = VL - VS$
$VE' = V_x$, $SE = S_x$
Here $V_x = S_x$

* OA → Horizontal Range
* → Distance b/w vertex and focus = $\frac{1}{4}$ (Latus rectum)

1) Projectile :-

Projectile is a particle in a direction inclined to the direction of gravity namely to the vertical.

2) Trajectory :-

The path of the projectile is called Trajectory.

3) Angle of projectile :-

The angle of projection is the angle with the direction of projection makes with the horizontal.

4) Horizontal Range :-

The distance between the point of projection and the point where the projectory meets the horizontal plane is called the horizontal angle.

5) Time of flight :-

The time of flight is the time taken to complete the horizontal range.

Book work :- N.V.V (*) 10M compulsory

The path of the projectile is a parabola.

* A path is projected from origin with a velocity of magnitude u inclined at an angle α with x-axis.
* Initial velocity vector $= u(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}) = \dot{\bar{r}}$
* A particle describe a curve on an vertical plane,
* We have to find the nature of path.
* We neglect air resistance and any other forces that may act at the particle.
* We assume that only one force that is the weight of the particle (mg) to act on it.

By Newton's II law,

$$m\bar{a} = \bar{F}$$

downward direction

$$m\ddot{\bar{r}} = mg(-\hat{j}) \rightarrow (1)$$

Integrating (1) w.r.t 't'

$$\dot{\bar{r}} = -gt\hat{j} + \bar{c}_1 \rightarrow (2)$$

When $t=0$, $\dot{\bar{r}} = \bar{c}_1$

Sub in (2),

$$\dot{\bar{r}} = -gt\hat{j} + u(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$$

$$= +u\cos\alpha\,\hat{i} + (u\sin\alpha - gt)\hat{j} \longrightarrow (3)$$

Integrating (3) w.r.t 't'

$$\bar{r} = (u\cos\alpha)t\,\hat{i} + \left[(u\sin\alpha)t - gt^2/2\right]\hat{j} + \bar{c}_2 \longrightarrow (4)$$

When $t=0$, $\bar{r} = 0$ (displacement) [$\because$ the particle is at the origin]

$\Rightarrow \bar{c}_2 = \bar{0}$ (ie, $\bar{0} = \bar{0} + \bar{0} + \bar{0} + \bar{c}_2$)

Sub in (4),

$$\bar{r} = (u\cos\alpha)t\,\hat{i} + \left[(u\sin\alpha)t - \frac{gt^2}{2}\right]\hat{j} \longrightarrow (5)$$

$$\bar{r} = x\hat{i} + y\hat{j}$$

equating i^{th} and j^{th} component,

$$\boxed{x = (u\cos\alpha)t, \quad y = (u\sin\alpha)t - \frac{1}{2}gt^2}$$

These are called the parametric equation of the path. ~~we eliminate t and get the cartesian equation for the path.~~

To find the cartesian eqn of the path:-

$\Rightarrow x = (u\cos\alpha)t$

$$t = \frac{x}{u\cos\alpha}$$

sub value of 't',

$$y = (u\sin\alpha)t - \frac{1}{2}gt^2$$

$$y = (\cancel{u}\sin\alpha)\left(\frac{x}{\cancel{u}\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\alpha}\right)$$

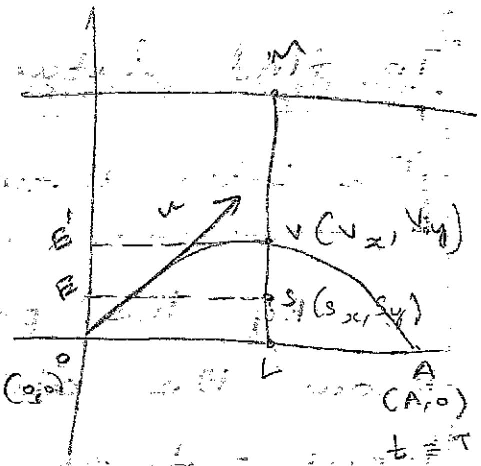
$$\boxed{y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}} \longrightarrow (6)$$

In this eqn there is only one term of degree 2 i.e, x^2 and it is perfect square

∴ eqn ⑥ represents a parabola. (ie) the path of projectile is a parabola.

<u>To find the following:</u>

i) Vertex
ii) Latus rectum
iii) Focus
iv) Eqn of directrix
v) Horizontal range
vi) Time of flight



<u>To find the vertex:</u>

From ⑥,

$$y = x\tan\alpha - \frac{g x^2}{2u^2\cos^2\alpha}$$

$$(2u^2\cos^2\alpha)\,y = (2u^2\cos^2\alpha)(x\tan\alpha) - gx^2$$

$$gx^2 - (2u^2\cos^2\alpha)(x\tan\alpha) = -(2u^2\cos^2\alpha)\,y$$

÷ by g,

$$x^2 - \frac{(2u^2\cos^{\cancel{2}}\alpha)}{g}\cdot\frac{\sin\alpha}{\cancel{\cos}\alpha}\cdot x = \frac{-(2u^2\cos^2\alpha)}{g}\cdot y$$

$$x^2 - \frac{2u^2\cos\alpha\sin\alpha\, x}{g} + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} - \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{-(2u^2\cos^2\alpha)}{g}\cdot y$$

$$\left[x - \frac{u\sin\alpha\cos\alpha}{g}\right]^2 - \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{-2u^2\cos^2\alpha}{g}\,y$$

$$\left[x - \frac{u\sin\alpha\cos\alpha}{g}\right]^2 = \frac{-2u^2\cos^2\alpha}{g}\cdot y + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

$$\left[x - \frac{u\sin\alpha\cos\alpha}{g}\right]^2 = \frac{-2u^2\cos^2\alpha}{g}\left[y - \frac{u^2\sin^2\alpha}{2g}\right]$$

$$\boxed{\begin{aligned}\text{Vertex} &= V\left(\frac{u^2\cos\alpha\sin\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)\\ &= (V_x, V_y)\end{aligned}}$$

$\therefore$ The vertex is shifted from origin $O(0,0)$ to $V(V_x, V_y)$ $\quad V_y \rightarrow$ greatest height

To find latus rectum:-

$$\boxed{\text{The Latus rectum of the parabola is } \frac{2u^2\cos^2\alpha}{g}}$$

By the property of a parabola, we know the <u>distance between</u> the vertex and focus $= \frac{1}{4}$ of the Latus rectum

$$\boxed{VS = \frac{1}{4}\,\frac{2u^2\cos^2\alpha}{g}}$$

focus (S), Vertex (V)

To find the focus:-

$$S_x = V_x = \frac{u^2\cos\alpha\sin\alpha}{g} = \frac{u^2\sin 2\alpha}{2g}$$

$$S_y = SL = VL - VS$$

$$S_y = V_y - \frac{1}{4}\,\frac{2u^2\cos^2\alpha}{g}$$

$$= \frac{u^2\sin^2\alpha}{2g} - \frac{1}{\cancel{4}_2}\,\frac{\cancel{2}u^2\cos^2\alpha}{g}$$

$$= \frac{u^2\sin^2\alpha}{2g} - \frac{u^2\cos^2\alpha}{2g}$$

$$= \frac{u^2}{2g}(\sin^2\alpha - \cos^2\alpha)$$

$$= \frac{-u^2}{2g}(\cos^2\alpha - \sin^2\alpha)$$

$$S_y = \frac{-u^2}{2g}\cos 2\alpha$$

$$\boxed{\begin{aligned} \text{Focus} &= S\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2 \cos 2\alpha}{2g}\right) \\ &= (S_x, S_y) \end{aligned}}$$

To find the eqn of directrix :-

It is a line $\parallel$ to x-axis at a height $= ML$

Its eqn is $y = ML$

$$= MV + VL$$

$$= VS + V_y$$

$$= \frac{1}{\cancel{2}2} \frac{\cancel{2} u^2 \cos^2\alpha}{g} + \frac{u^2 \sin^2\alpha}{2g}$$

$$= \frac{u^2}{2g}(\sin^2\alpha + \cos^2\alpha)$$

$$y = \frac{u^2}{2g}$$

$$\boxed{\therefore \left.\begin{array}{l}\text{The eqn of} \\ \text{the directrix}\end{array}\right\} \text{is } y = \frac{u^2}{2g}}$$

To find the Horizontal range :-

The horizontal range $= OA$

$$= 2(OL)_{V_x}$$

$$= 2 \frac{u^2 \sin\alpha \cos\alpha}{g}$$

$$= \frac{u^2 \sin 2\alpha}{g}$$

$$\boxed{\therefore \text{Horizontal range } R = \frac{u^2 \sin 2\alpha}{g}}$$

To find the time of flight :-

W.K.T, $y = (u \sin\alpha) t - \frac{1}{2} g t^2$

Let T be the time of flight.

When $t = 0$, the particle is at O.

$t = T$, the particle is at A.

$\therefore$ put $t = T$ and $y = 0$

$$\Rightarrow 0 = (u \sin\alpha)T - \frac{1}{2} g T^2$$

$$\frac{1}{2} g T^2 = u \sin\alpha\, T$$

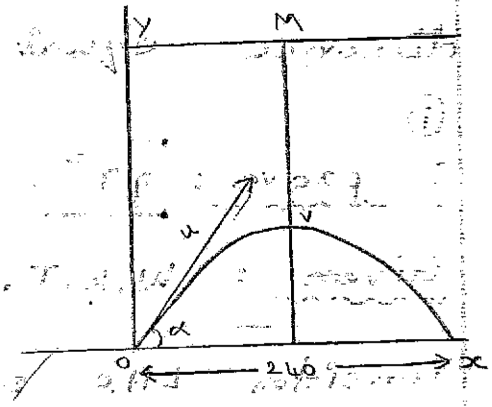
$$\frac{1}{2} g (T)^{\cancel{2}} = u \sin\alpha\, \cancel{T}$$

$$gT = 2u \sin\alpha$$

$$T = \frac{2u\sin\alpha}{g}$$

$$\boxed{\therefore \text{Time of flight is } T = \frac{2u\sin\alpha}{g}}$$

1\. S.T the particle starting with a velocity of 100 feet/sec at angle $\tan^{-1} 3/4$ to the horizontal will just clear a wall 36 feet high at a horizontal distance of 240 feet from the point of projection. Here $g = 32$



Given:

$u = 100$ ft/sec

$\alpha = \tan^{-1} 3/4 \Rightarrow \tan\alpha = 3/4$

$x = 240$ ft, $y = 36$ ft, $g = 32$

Soln:-

$$\boxed{y = x\tan\alpha - \frac{g}{2u^2\cos^2\alpha}x^2} \rightarrow \text{formula}$$

~~Sub the values,~~

$$y = x\tan\alpha - \frac{g}{2u^2}\sec^2\alpha\, x^2$$

$$y = x\tan\alpha - \frac{g}{2u^2}(1+\tan^2\alpha)\,x^2$$

Sub the values,

$$y = \cancel{240}^{60}\left(3/\cancel{4}\right) - \frac{\cancel{32}^{16}}{\cancel{2}(100)^2}\left[1+\left(\frac{3}{4}\right)^2\right](240)^2$$

$$= 180 - \frac{16}{10000}\left(\frac{16+9}{16}\right)57600$$

$$= 180 - \frac{25}{100}(576)$$

$$= 180 - 144$$

$$= 36 \text{ feet}$$

2. If T is the time of flight, R the horizontal range and α the angle of projections. S.T $gT^2 = 2R\tan\alpha$. If $\alpha = 60°$ the particle in turn of R the height of particle when it has moved through horizontal distance equal to $\frac{3R}{4}$

(i)

To prove : $\boxed{gT^2 = 2R\tan\alpha} \longrightarrow (1)$

Given : W.K.T, $T = \frac{2u\sin\alpha}{g}$, $R = \frac{u^2\sin 2\alpha}{g}$

Consider LHS of eqn (1),

$$gT^2 = \cancel{g}\left(\frac{2^2 u^2 \sin^2\alpha}{g^{\cancel{2}}}\right)$$

$$= \frac{4u^2}{g}\sin^2\alpha \longrightarrow (2)$$

Consider RHS of eqn (1),

$$2R\tan\alpha = 2 \times \frac{u^2\sin 2\alpha}{g} \times \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{2u^2 \cdot 2\sin\alpha\cos\alpha}{g} \cdot \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{4u^2}{g}\sin^2\alpha \longrightarrow (3)$$

From (2) & (3),

LHS = RHS $\Rightarrow gT^2 = 2R\tan\alpha$

(ii)

To find : the height of particle "y".

Given : $\alpha = 60°$, $x = \frac{3R}{4}$ [$\because$ Particle when it has moved through horizontal distance equal to $\frac{3R}{4}$]

Solution : $R = \frac{u^2\sin 2\alpha}{g} = \frac{u^2\sin 2(60°)}{g} = \frac{u^2\sin 120°}{g}$

$$R = \frac{\sqrt{3}}{2}\cdot\frac{u^2}{g}$$

$$\boxed{y = x \tan\alpha - \frac{g x^2}{2u^2 \cos^2\alpha}}$$

$$y = \frac{3R}{4} \times \tan 60° - \frac{g}{2u^2 \cos^2 60°} \times \left(\frac{3R}{4}\right)^2$$

$$= \frac{3R}{4} \tan 60° - \frac{g}{2u^2 (\cos 60°)^2} \frac{9R}{(4)^2} R$$

$$= \frac{3R}{4} \tan 60° - \frac{\cancel{g}}{2\cancel{u^2} (\cos 60°)^2} \frac{9R}{(4)^2} \cdot \frac{\sqrt{3}\, \cancel{u^2}}{2\cancel{g}}$$

$$= \frac{3R}{4} \cdot \sqrt{3} - \frac{1}{2\left(\frac{1}{2}\right)^2} \cdot \frac{9R}{(4)^2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{4} R - \frac{1}{2} \cdot \frac{\cancel{4}}{1} \cdot \frac{9R}{\cancel{16}\,4} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{4} R - \frac{9R}{4} \cdot \frac{\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}}{4} R \left[1 - \frac{3}{4}\right]$$

$$= \frac{3\sqrt{3}}{4} R \left(\frac{1}{4}\right)$$

$$y = \frac{3\sqrt{3}}{16} R$$

---

3. A particle projected under gravity in a vertical plane with a velocity u, at angle α to the horizontal. If the range on the horizontal be R and the greatest height attained is h. S.T $\frac{u^2}{2g} = h + \frac{R^2}{16h}$ and $\tan\alpha = \frac{4h}{R}$

To prove :

i) $\frac{u^2}{2g} = h + \frac{R^2}{16h} \rightarrow (1)$

ii) $\tan\alpha = \frac{4h}{R} \rightarrow (2)$

W.K.T, $R = \frac{u^2 \sin 2\alpha}{g}$, $h = \frac{u^2 \sin^2\alpha}{2g} = V_y$

horizontal range   greatest height

Consider RHS of ①,

$$h + \frac{R^2}{16h} = \frac{u^2 \sin^2\alpha}{2g} + \left(\frac{u^2 \sin 2\alpha}{g}\right)^2 \cdot \frac{1}{16\left(\frac{u^2 \sin^2\alpha}{2g}\right)}$$

$$= \frac{u^2 \sin^2\alpha}{2g} + \frac{u^4 \sin^2 2\alpha}{g^{\cancel{2}}} \cdot \frac{\cancel{2}g}{8\,\cancel{16}\, u^2 \sin^2\alpha}$$

$$= \frac{u^2 \sin^2\alpha}{2g} + \frac{u^{\cancel{4}2}(2 \sin\alpha \cos\alpha)^2}{g \cdot 8 \cdot \cancel{u^2} \sin^2\alpha}$$

$$= \frac{u^2 \sin^2\alpha}{2g} + \frac{u^2 \cdot \cancel{4} \cdot \cancel{\sin^2\alpha} \cos^2\alpha}{g \cdot \cancel{8}_2 \cdot \cancel{\sin^2\alpha}}$$

$$= \frac{u^2}{2g}(\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{u^2}{2g} = L.H.S$$

Consider RHS of ②,

$$\frac{4h}{R} = \overset{2}{\cancel{4}}\left(\frac{\cancel{u^2} \sin^2\alpha}{\cancel{2g}}\right) \cdot \frac{\cancel{g}}{\cancel{u^2} \sin 2\alpha}$$

$$= \frac{\cancel{2} \sin^{\cancel{2}}\alpha}{\cancel{2}\, \cancel{\sin}\alpha \cos\alpha} = \frac{\sin\alpha}{\cos\alpha}$$

$$= \tan\alpha = L.H.S$$

4. If the greatest height attained by a projectile is one quater of its range on the horizontal plane. ST the angle of projection is 45°

Given:

$$\boxed{\text{Greatest height} = \frac{1}{4}\ \text{Horizontal range}} \rightarrow ①$$

To prove: $\alpha = 45^\circ$

Solution:

W.K.T, $\underset{\text{greatest height}=v_y}{h} = \dfrac{u^2 \sin^2\alpha}{2g}$, $\underset{\text{horizontal range}}{R} = \dfrac{u^2 \sin 2\alpha}{g}$

Sub the value of 'h' & 'R' in (1),

$$\frac{\cancel{u^2} \sin^2\alpha}{\cancel{2}\cancel{g}} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{u^2} \sin 2\alpha}{\cancel{g}}$$

$$\sin^{\cancel{2}}\alpha = \frac{\cancel{2}\, \cancel{\sin\alpha} \cos\alpha}{\cancel{2}}$$

$$\frac{\sin\alpha}{\cos\alpha} = 1$$

$$\tan\alpha = 1$$

$$\alpha = \tan^{-1}(1)$$

$$\alpha = 45^\circ$$

---

5. If the range on the horizontal plane of a projectile and the greatest height above the point of projection are R and H respectively ST the velocity of projection is $\sqrt{2gH + \frac{gR^2}{8H}}$

To prove: $u = \sqrt{2gH + \dfrac{gR^2}{8H}}$

W.K.T, $R = \dfrac{u^2 \sin 2\alpha}{g} \Rightarrow \boxed{\sin 2\alpha = \dfrac{Rg}{u^2}} \rightarrow (1)$

$H = \dfrac{u^2 \sin^2\alpha}{2g} \Rightarrow \sin^2\alpha = \dfrac{2Hg}{u^2}$

$$\Rightarrow \frac{1-\cos 2\alpha}{2} = \frac{2Hg}{u^2}$$

$$\Rightarrow \boxed{\cos 2\alpha = -\left(\frac{4Hg - u^2}{u^2}\right)} \rightarrow (2)$$

From ① & ②,

$$\boxed{\sin^2 2\alpha + \cos^2 2\alpha = 1}$$

$$\left(\frac{Rg}{u^2}\right)^2 + \left[-\left(\frac{4Hg - u^2}{u^2}\right)\right]^2 = 1$$

$$\frac{R^2 g^2 + (4Hg - u^2)^2}{u^4} = 1$$

$$R^2 g^2 + 16 H^2 g^2 + \cancel{u^4} - 8u^2 Hg = \cancel{u^4}$$

$$-8u^2 Hg = -R^2 g^2 - 16 H^2 g^2$$

$$8u^2 Hg = R^2 g^2 + 16 H^2 g^2$$

$$\div 8Hg \Rightarrow u^2 = \frac{R^2 g^{\cancel{2}}}{8H\cancel{g}} + \frac{\overset{8}{\cancel{16}} H^{\cancel{2}} g^{\cancel{2}}}{\cancel{8} \cancel{H} \cancel{g}}$$

$$u^2 = \frac{R^2 g}{8H} + 2Hg .$$

$$u = \sqrt{2Hg + \frac{R^2 g}{8H}}$$

6. S.T for a given velocity of projection u and horizontal range R. There are two different direction of projection and that the respective maximum height and time of flight are given by the equation $16H^2 g - 8u^2 H + gR^2 = 0$, $T^4 g^2 - 4u^2 T^2 + 4R^2 = 0$

To prove :

i) $16 H^2 g - 8u^2 H + gR^2 = 0$

ii) $T^4 g^2 - 4u^2 T^2 + 4R^2 = 0$

Solution:-

$$\boxed{\sin^2 2\alpha + \cos^2 2\alpha = 1}$$

i) W.K.T, $R = \dfrac{u^2 \sin 2\alpha}{g}$

$$\boxed{\Rightarrow \sin 2\alpha = \frac{Rg}{u^2}}$$

$$H = \frac{u^2 \sin^2\alpha}{2g}$$

$$\boxed{\Rightarrow \cos 2\alpha = 1 - \frac{4Hg}{u^2}}$$ (refer previous problem)

Sub in ①,

$$\left(\frac{Rg}{u^2}\right)^2 + \left(1 - \frac{4Hg}{u^2}\right)^2 = 1$$

$$\frac{R^2 g^2}{u^4} + \frac{(u^2 - 4Hg)^2}{u^4} = 1$$

$$R^2 g^2 + \cancel{u^4} + 16 H^2 g^2 - 8u^2 Hg = \cancel{u^4}$$

$$\div g \Rightarrow R^2 g + 16 H^2 g - 8u^2 H = 0$$

$$\therefore 16 H^2 g - 8u^2 H + gR^2 = 0$$

ii) W.K.T, $T = \dfrac{2u \sin\alpha}{g} \Rightarrow \sin\alpha = \dfrac{gT}{2u}$

$$\Rightarrow \sin^2\alpha = \frac{g^2 T^2}{4u^2} \Rightarrow \frac{1-\cos 2\alpha}{2} = \frac{g^2 T^2}{4u^2}$$

$$\boxed{\Rightarrow \cos 2\alpha = \frac{2u^2 - T^2 g^2}{2u^2}}$$

$$R = \frac{u^2 \sin 2\alpha}{g}$$

$$\boxed{\Rightarrow \sin 2\alpha = \frac{Rg}{u^2}}$$

Sub in ①,

$$\left(\frac{Rg}{u^2}\right)^2 + \left(\frac{2u^2 - T^2 g^2}{2u^2}\right)^2 = 1$$

$$\frac{R^2 g^2}{u^4} + \frac{4u^4 + T^4 g^4 - 4u^2 T^2 g^2}{4u^4} = 1$$

$$4R^2 g^2 + 4u^4 + T^4 g^4 - 4u^2 T^2 g^2 = 4u^4$$

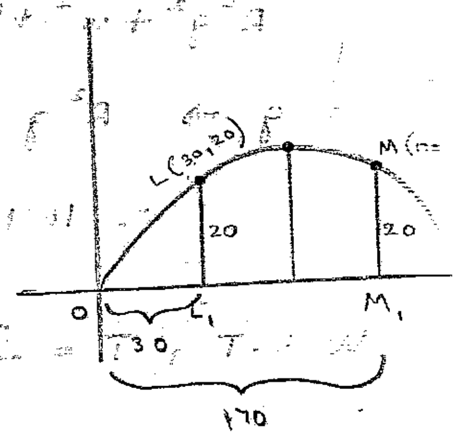
$$\div g^2 \Rightarrow 4R^2 + T^4 g^2 - 4u^2 T^2 = 0$$

$$\therefore T^4 g^2 - 4u^2 T^2 + 4R^2 = 0$$

---

7. A particle is projected so as just to graze the tops of 2 wall, each of height 30 feet and 20 feet at distance of 170 feet respectively from the point of projections. Find the angle of projection and the highest point reached in the flight.

Gn:- $OL_1 = 30$, $OM_1 = 170$, $LL_1 = MM_1 = 20$



Soln:- $$\boxed{y = x\tan\alpha - \frac{g}{2u^2\cos^2\alpha}x^2}$$

$$\Rightarrow y - x\tan\alpha = \frac{-g}{2u^2\cos^2\alpha}x^2$$

$L(30, 20)$; $20 - 30\tan\alpha = \dfrac{-g}{2u^2\cos^2\alpha}(30)^2$

$$20 - 30\tan\alpha = \frac{-g}{2u^2}\sec^2\alpha\,(30)^2 \longrightarrow (1)$$

$M(170, 20)$; $20 - 170\tan\alpha = \dfrac{-g}{2u^2\cos^2\alpha}(170)^2$

$$20 - 170\tan\alpha = \frac{-g}{2u^2}\sec^2\alpha\,(170)^2 \longrightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{20 - 30\tan\alpha}{20 - 170\tan\alpha} = \frac{-\frac{g}{2u^2}\sec^2\alpha\,(30)^2}{-\frac{g}{2u^2}\sec^2\alpha\,(170)^2} = \frac{900}{28900}$$

$$= \frac{9}{289}$$

$$\frac{\cancel{16}(2-3\tan\alpha)}{\cancel{16}(2-17\tan\alpha)} = \frac{9}{289}$$

$$289(2-3\tan\alpha) = 9(2-17\tan\alpha)$$

$$578 - 867\tan\alpha = 18 - 153\tan\alpha$$

$$578 - 18 = 867\tan\alpha - 153\tan\alpha$$

$$560 = 714\tan\alpha$$

$$\tan\alpha = \frac{560}{714} = \frac{40}{51}$$

$$(1) \Rightarrow 20 - 30\tan\alpha = \frac{-g}{2u^2}\sec^2\alpha\,(30)^2$$

$$20 - 30\tan\alpha = \frac{-g}{2u^2}(1+\tan^2\alpha)(900)$$

$$20 - 30\left(\frac{40}{51}\right) = \frac{-g}{2u^2}\left(1+\left(\frac{40}{51}\right)^2\right)900$$

$$20 - \frac{1200}{51} = \frac{-g}{2u^2}\left(1+\frac{1600}{2601}\right)900$$

$$\frac{1020-1200}{51} = \frac{-g}{2u^2}\left(\frac{2601+1600}{2601}\right)900$$

$$\frac{-\cancel{180}^{60}}{\cancel{51}_{17}} = \frac{-g}{2u^2}\left(\frac{4201}{2601}\right)900$$

$$-\frac{60}{17} = \frac{-g}{2u^2}(1.6151)900$$

$$-3.529 = \frac{-g}{2u^2}(1453.6)$$

$$\frac{2u^2}{g}\times 3.529 = 1453.6$$

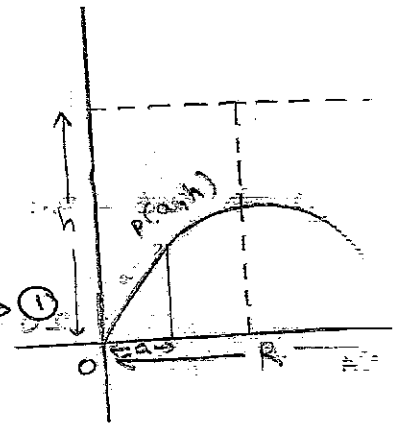
$$\frac{u^2}{g}(7.058) = 1453.6$$

$$\frac{u^2}{g} = \frac{1453.6}{7.058} = 205.9506$$

$$\div 2 \Rightarrow \frac{u^2}{2g} = 102.97$$

8. A particle is projected from given pt on ground and just clears the wall of height 'h' at a distance 'a' from the pt of projection. If the particle moves in a vertical plane $\perp^r$ to the wall & if the horizontal range is R. S.T the elevation of the projection is gn by $\boxed{\tan\alpha = \dfrac{Rh}{a(R-a)}}$

$$\boxed{y = x\tan\alpha - \frac{g}{2u^2\cos^2\alpha}x^2}$$



$$\Rightarrow y - x\tan\alpha = \frac{-g}{2u^2\cos^2\alpha}x^2$$

$$P(a,h)\ ;\ h - a\tan\alpha = \frac{-g}{2u^2\cos^2\alpha}a^2 \rightarrow (1)$$

$$A(R,0)\ ;\ 0 - R\tan\alpha = \frac{-gR^2}{2u^2\cos^2\alpha} \rightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{h - a\tan\alpha}{-R\tan\alpha} = \frac{a^2}{R^2}$$

$$R(h - a\tan\alpha) = -a^2\tan\alpha$$

$$Rh - Ra\tan\alpha = -a^2\tan\alpha$$

$$Rh = -a^2\tan\alpha + Ra\tan\alpha$$

$$Rh = \tan\alpha(Ra - a^2)$$

$$\tan\alpha = \frac{Rh}{Ra - a^2}$$

$$\tan\alpha = \frac{Rh}{a(R-a)}$$

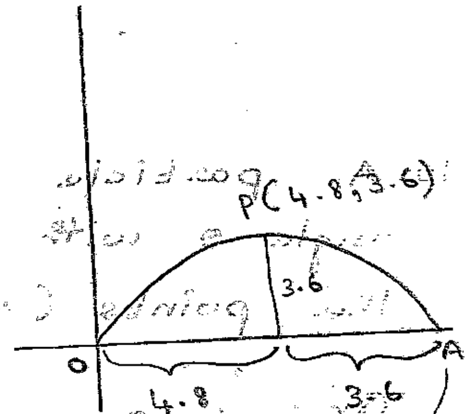
Note: Suppose, $P(a,h)$ replaced by $P(x,y)$ with $A($

$$\Rightarrow \tan\alpha = \frac{Ry}{x(R-x)}$$

9. Find the velocity and the angle of projection of a stone. S.T it make just clear a tree 3.6 meters height 4.8 mts distance & fall on the other side at a distance 3.6 from the ~~stree~~ tree. $g = 9.8$

To find:

* Velocity 'u'
* angle of projection 'α'



Solution:

$$\boxed{y - x\tan\alpha = \frac{-gx^2}{2u^2\cos^2\alpha}}$$

$P(4.8, 3.6)$; $3.6 - 4.8\tan\alpha = \dfrac{-g(4.8)^2}{2u^2\cos^2\alpha}$ → ① [(4.8+3.6), 0] (8.4, 0)

$A(8.4, 0)$; $0 - 8.4\tan\alpha = \dfrac{-g(8.4)^2}{2u^2\cos^2\alpha}$ → ②

$$\frac{①}{②} \Rightarrow \frac{3.6 - 4.8\tan\alpha}{0 - 8.4\tan\alpha} = \frac{-\cancel{g}(4.8)^2}{2\cancel{u^2}\cancel{\cos^2\alpha}} \cdot \frac{2\cancel{u^2}\cancel{\cos^2\alpha}}{-\cancel{g}(8.4)^2} = \frac{(4.8)^2}{(8.4)^2}$$

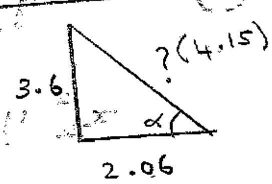
$$\frac{3.6 - 4.8\tan\alpha}{-8.4\tan\alpha} = \frac{23.04}{70.56}$$

$$3.6 = 4.8\tan\alpha - 2.743\tan\alpha$$

$$\Rightarrow 2.06\tan\alpha = 3.6$$

$$\tan\alpha = \frac{3.6}{2.06}$$

$$\alpha = \tan^{-1}\left(\frac{3.6}{2.06}\right)$$



$$\text{hyp} = \sqrt{(3.6)^2 + (2.06)^2}$$
$$= \sqrt{17.20}$$
$$= 4.15$$

$\tan\alpha = \dfrac{3.6}{2.06}$ $\dfrac{\text{Opp}}{\text{Adj}}$

$\sin\alpha = \dfrac{3.6}{4.15}$ $\dfrac{\text{Opp}}{\text{hyp}}$

$\cos\alpha = \dfrac{2.06}{4.15}$ $\dfrac{\text{Adj}}{\text{hyp}}$

$$R = \frac{u^2\sin 2\alpha}{g} \quad (H.R)$$

$A(8.4, 0)$; $8.4 = \dfrac{u^2 \cdot 2\sin\alpha\cos\alpha}{g}$

10. A particle is projected making an angle θ with the horizon, if it passes through the points (x_1, y_1) & (x_2, y_2). P.T $\tan\theta = \dfrac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)}$

Gn: $\alpha = \theta =$ angle of projection

soln: $$\boxed{y = x\tan\theta - \frac{g x^2}{2u^2\cos^2\theta}}$$

$$y - x\tan\theta = \frac{-g x^2}{2u^2\cos^2\theta}$$

(x_1, y_1); $$y_1 - x_1\tan\theta = \frac{-g x_1^2}{2u^2\cos^2\theta} \longrightarrow (1)$$

(x_2, y_2); $$y_2 - x_2\tan\theta = \frac{-g x_2^2}{2u^2\cos^2\theta} \longrightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{y_1 - x_1\tan\theta}{y_2 - x_2\tan\theta} = \frac{x_1^2}{x_2^2}$$

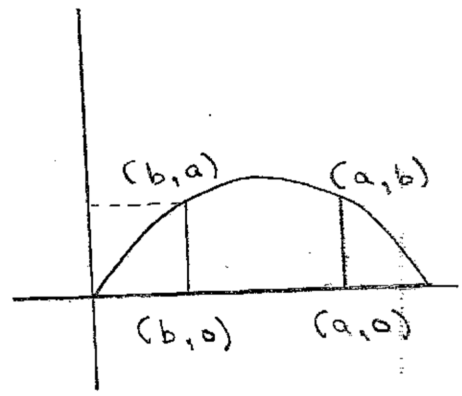
$$x_2^2 y_1 - x_2^2 x_1 \tan\theta = x_1^2 y_2 - x_1^2 x_2 \tan\theta$$

$$x_2^2 y_1 - x_1^2 y_2 = x_2^2 x_1 \tan\theta - x_1^2 x_2 \tan\theta$$

$$x_2^2 y_1 - x_1^2 y_2 = (x_2^2 x_1 - x_1^2 x_2)\tan\theta$$

$$\tan\theta = \frac{x_2^2 y_1 - x_1^2 y_2}{x_2^2 x_1 - x_1^2 x_2}$$

11. A particle is projected so as to clear two walls first of height 'a' at a distance 'b' from the point of projection and the second of height 'b' at a distance 'a' from the point of projection. S.T the range on the horizontal plane is $\frac{a^2+ab+b^2}{a+b}$ and the angle of projection exceeds $\tan^{-1}(3)$



To prove:

$$H.R = R = \frac{a^2+ab+b^2}{a+b}$$

$$\alpha > \tan^{-1}(3)$$

Solution:

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$y - x\tan\alpha = \frac{-gx^2}{2u^2\cos^2\alpha}$$

$$(b,a);\quad a - b\tan\alpha = \frac{-gb^2}{2u^2\cos^2\alpha} \rightarrow (1)$$

$$(a,b);\quad b - a\tan\alpha = \frac{-ga^2}{2u^2\cos^2\alpha} \rightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{a-b\tan\alpha}{b-a\tan\alpha} = \frac{b^2}{a^2}$$

$$a^2(a-b\tan\alpha) = b^2(b-a\tan\alpha)$$

$$a^3 - a^2b\tan\alpha = b^3 - ab^2\tan\alpha$$

$$a^3 - b^3 = a^2b\tan\alpha - ab^2\tan\alpha$$

$$a^3 - b^3 = (a^2b - ab^2)\tan\alpha$$

$$\Rightarrow \tan\alpha = \frac{a^3 - b^3}{ab(a-b)}$$

$$= \frac{\cancel{(a-b)}(a^2+ab+b^2)}{ab\cancel{(a-b)}}$$

$$= \frac{a^2+ab+b^2}{ab}$$

$$= \frac{a}{b} + \frac{b}{a} + 1$$

$$\tan\alpha = \left(\frac{a}{b} + \frac{b}{a} - 2\right) + 3$$

$$\alpha = \tan^{-1}\left[\left(\frac{a}{b} + \frac{b}{a} - 2\right) + 3\right]$$

exceeds $\tan^{-1}(3)$

If $\tan\alpha = 3$

$$\alpha = \tan^{-1}(3)$$

$$y = x \tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$y(2u^2\cos^2\alpha) = 2u^2 x\cos^2\alpha \tan\alpha - gx^2$$

$$2yu^2\cos^2\alpha = 2u^2 x \cos^2\alpha \frac{\sin\alpha}{\cos\alpha} - gx^2$$

$$2yu^2\cos^2\alpha = 2u^2 x\cos\alpha\sin\alpha - gx^2$$

(b, a) ; $2au^2\cos^2\alpha = 2u^2 b\sin\alpha\cos\alpha - gb^2 \to$ (1)

(a, b) ; $2bu^2\cos^2\alpha = 2u^2 a\sin\alpha\cos\alpha - ga^2 \to$ (2)

(1) $\times b \Rightarrow 2abu^2\cos^2\alpha = 2u^2b^2\sin\alpha\cos\alpha - gb^3 \to$ (3)

(2) $\times a \Rightarrow 2abu^2\cos^2\alpha = 2u^2a^2\sin\alpha\cos\alpha - ga^3 \to$ (4)

$$\frac{③}{④} \Rightarrow 1 = 2u^2 \sin\alpha \cos\alpha (b^2-a^2) - g(b^3-a^3)$$

$$u^2 \, 2 \sin\alpha \cos\alpha (b^2-a^2) = g(b^3-a^3)$$

$$\frac{u^2 \sin 2\alpha}{g} = \frac{b^3-a^3}{b^2-a^2}$$

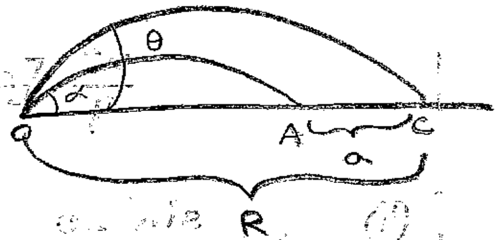
$$\therefore R = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{\cancel{(b-a)}(b+a)}$$

$$R = \frac{b^2+ab+a^2}{a+b}$$

---

12. Two projectiles are projected from the same point with the same velocity at angle α and β and aimed at an target on the horizontal plane through O. One falls 'a' feet too short and the other 'b' feet too far from the target. If θ is the correct angle of projection so as to hit the target. S.T

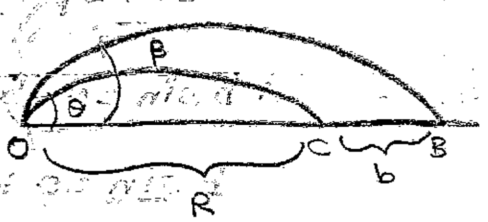
$$\theta = \frac{1}{2} \sin^{-1}\left[\frac{a \sin 2\beta + b \sin 2\alpha}{a+b}\right]$$



Given:

$A(a,0) \Rightarrow$ Range $AC = a$,

angle of projection α



$B(b,0) \Rightarrow$ Range $CB = b$,

angle of projection β

$C(R,0) \Rightarrow$ Range $OC = R$,

angle of projection θ

To prove: $\theta = \frac{1}{2} \sin^{-1}\left[\frac{a \sin 2\beta + b \sin 2\alpha}{a+b}\right]$

If $OC = R$, $R = \dfrac{u^2 \sin 2\theta}{g} \longrightarrow (1)$

If $OA = a$, $R - a = \dfrac{u^2 \sin 2\alpha}{g} \longrightarrow (2)$

If $OB = b$ $R + b = \dfrac{u^2 \sin 2\beta}{g} \longrightarrow (3)$

Sub (1) in (2)

$$\frac{u^2 \sin 2\theta}{g} - a = \frac{u^2 \sin 2\alpha}{g}$$

$$\frac{u^2 \sin 2\theta}{g} - \frac{u^2 \sin 2\alpha}{g} = a$$

$$\frac{u^2}{g}\left[\sin 2\theta - \sin 2\alpha\right] = a \longrightarrow (4)$$

Sub (1) in (3)

$$\frac{u^2 \sin 2\theta}{g} + b = \frac{u^2 \sin 2\beta}{g}$$

$$\frac{u^2 \sin 2\beta}{g} - \frac{u^2 \sin 2\theta}{g} = b$$

$$\frac{u^2}{g}\left[\sin 2\beta - \sin 2\theta\right] = b \longrightarrow (5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta} = \frac{a}{b}$$

$$b \sin 2\theta - b \sin 2\alpha = a \sin 2\beta - a \sin 2\theta$$

$$b \sin 2\theta + a \sin 2\theta = a \sin 2\beta + b \sin 2\alpha$$

$$(b+a) \sin 2\theta = a \sin 2\beta + b \sin 2\alpha$$

$$\boxed{\sin 2\theta = \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}}$$

$$\Rightarrow 2\theta = \sin^{-1}\left(\frac{a \sin 2\beta + b \sin 2\alpha}{a+b}\right)$$

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{a \sin 2\beta + b \sin 2\alpha}{a+b}\right)$$

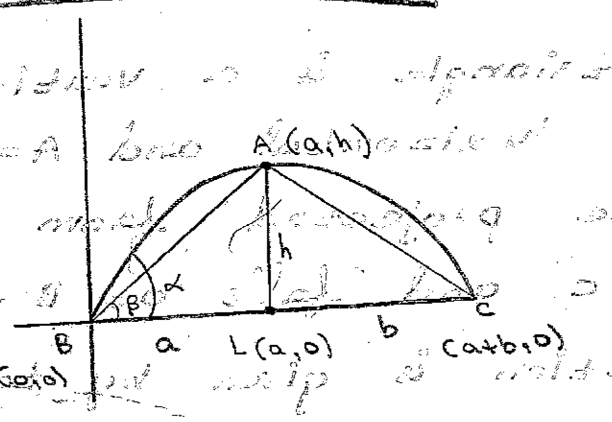
---

13. A particle is projected over a triangle from one end of its horizontal base to the vertex and followed at another end of the base. If B and C are the base angle's and α the angle of projection. S.T $\tan\alpha = \tan B + \tan C$

ii) A particle is projected from a point at an angle 30° to the horizontal. If P, Q is the horizontal range. If the angle of elevation of the particle at P at and at any instant of flight are A and B respectively.

S.T $\tan A + \tan B = \frac{1}{\sqrt{3}}$

(i)



$$\tan B = \frac{h}{a}$$

$$\tan C = \frac{h}{b}$$

$$\boxed{y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}}$$

$$y - x\tan\alpha = \frac{-gx^2}{2u^2\cos^2\alpha}$$

$$A(a,h) ;\ h - a\tan\alpha = \frac{-ga^2}{2u^2\cos^2\alpha} \longrightarrow (1)$$

$$C(a+b, 0) ;\ 0 - (a+b)\tan\alpha = \frac{-g(a+b)^2}{2u^2\cos^2\alpha} \longrightarrow (2)$$

$$\frac{①}{②} \Rightarrow \frac{h - a\tan\alpha}{(a+b)\tan\alpha} = \frac{a^2}{(a+b)^2}$$

$$h(a+b) - a(a+b)\tan\alpha = a^2 \tan\alpha$$

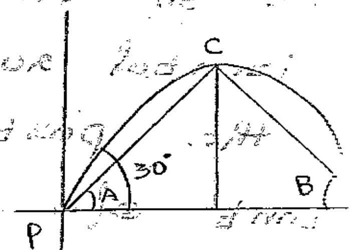
$$ha + hb - (a^2+ab)\tan\alpha = a^2\tan\alpha$$

$$ha + hb - \cancel{a^2\tan\alpha} - ab\tan\alpha = \cancel{a^2\tan\alpha}$$

$$ab\tan\alpha = ha + hb$$

$$\tan\alpha = \frac{ha+hb}{ab} = \frac{h}{b} + \frac{h}{a}$$

$$\tan\alpha = \tan A + \tan B$$



(ii) * $\alpha = 30°$

$$\tan\alpha = \tan A + \tan B$$

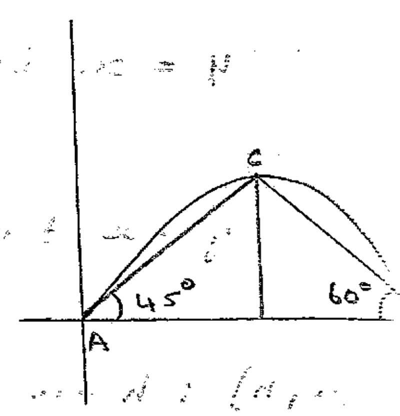
$$\Rightarrow \tan A + \tan B = \tan 30° = \frac{1}{\sqrt{3}}$$

---

14. ABC is the triangle in a vertical plane with (AB) horizontal and $A = 45°$, $B = 60°$. A particle projected from A passes through (C) and falls at (B). S.T the angle of projection is given by $\tan\theta = \sqrt{3} +$

Given: $A = 45°$, $B = 60°$

To prove: $\tan\theta = \sqrt{3} + 1$



From previous problem – Derive $\tan\alpha = \tan A + \tan B$ Here $\alpha = \theta$

$$\tan\theta = \tan A + \tan B = \tan 45° + \tan 60°$$

$$\therefore \tan\theta = 1 + \sqrt{3}$$

15. If the particles are projected from the same point in the same vertical plane to describe equal parabola. S.T the vertices of the path lie on a parabola.

The particles are projected from the same point in the same vertical plane to describe equal parabola.

$\Rightarrow u$ & α varies but the Latus rectum remains the same.

$$\text{Latus rectum} = \frac{2u^2\cos^2\alpha}{g} = k \text{ (constant)}$$

$$V = \left(\frac{u^2\cos\alpha\sin\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right) = (x, y)$$

$$\therefore ky = k\left(\frac{u^2\sin^2\alpha}{2g}\right)$$

$$= \frac{\cancel{2}u^2\cos^2\alpha}{g} \times \frac{u^2\sin^2\alpha}{\cancel{2}g}$$

$$= \frac{u^4\cos^2\alpha\sin^2\alpha}{g^2}$$

$$= \left(\frac{u^2\cos\alpha\sin\alpha}{g}\right)^2$$

$$ky = x^2$$

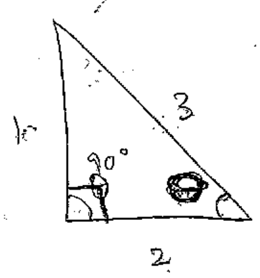
ie) $x^2 = ky$

$\hookrightarrow$ Latus rectum

$\Rightarrow$ Vertices of the path lie on the parabola extra sums :

(Like 12th sum)
1) A man who can rowed a boat at 12 km/hr in still water wishes to cross the river 3km broad flo[wing] at 5km/hr. i) Find the time of crossing by the shortest route, ii) Find the minimum time of cross[ing]

(Like 11th sum)
2) A boat is rowed with the speed of 4m/sec st. across the river flowing at 3m/s. The width of the river is ... Find how far down the river, the boat will be carri[ed]



$\tan\theta = \frac{1}{2}$

$\sin\theta = \frac{1}{3}$

$\cos\theta = \frac{2}{3}$

16. A no. of particles projected simultaneously from the same point with the same velocity in the same vertical plane but in the different direction. Find the locus of the focci of their path.

Gn:-

A no. of particles projected ....

.... but in the different direction.

Soln:-

$$S = \left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2 \cos 2\alpha}{2g} \right) = S(x, y)$$

$$x^2 + y^2 = \frac{u^4 \sin^2 2\alpha}{4g^2} + \frac{u^4 \cos^2 2\alpha}{4g^2}$$

$$= \left(\frac{u^2}{2g}\right)^2 \quad \left[\because x^2+y^2=r^2\right]$$

$$x^2 + y^2 = \left(\frac{u^2}{2g}\right)^2$$

$$(x-0)^2 + (y-0)^2 = u^2/2g$$

$\therefore$ Locus is a circle with centre at origin &

$$\text{radius} = \frac{u^2}{2g}$$

17. Three particles are projected from the same point in the same vertical plane with velocities (u, v, w) at angle of elevations (α, β, γ). P.T the focci of their path will lie on a straight line if

$$\frac{\sin 2(\beta-\gamma)}{u^2} + \frac{\sin 2(\gamma-\alpha)}{v^2} + \frac{\sin 2(\alpha-\beta)}{w^2} = 0$$

* 3 particles from the same pt in the same vertical plane with velocities (u, v, w) at angle of elevations (α, β, γ)

Focus $S = (S_x, S_y)$

$$= \left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2 \cos 2\alpha}{2g} \right)$$

$$\left(\frac{v^2 \sin 2\beta}{2g}, \frac{-v^2 \cos 2\beta}{2g} \right)$$

$$\left(\frac{w^2 \sin 2\gamma}{2g}, \frac{-w^2 \cos 2\gamma}{2g} \right)$$

* The 3 focci are collinear then

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{vmatrix} \frac{u^2 \sin 2\alpha}{2g} & \frac{-u^2 \cos 2\alpha}{2g} & 1 \\ \frac{v^2 \sin 2\beta}{2g} & \frac{-v^2 \cos 2\beta}{2g} & 1 \\ \frac{w^2 \sin 2\gamma}{2g} & \frac{-w^2 \cos 2\gamma}{2g} & 1 \end{vmatrix} = 0$$

$$\begin{array}{c} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \begin{vmatrix} \frac{u^2 \sin 2\alpha}{2g} & \frac{-u^2 \cos 2\alpha}{2g} & 1 \\ \frac{v^2 \sin 2\beta - u^2 \sin 2\alpha}{2g} & \frac{u^2 \cos 2\alpha - v^2 \cos 2\beta}{2g} & 0 \\ \frac{w^2 \sin 2\gamma - u^2 \sin 2\alpha}{2g} & \frac{u^2 \cos 2\alpha - w^2 \cos 2\gamma}{2g} & 0 \end{vmatrix} = 0$$

$$\left(\frac{v^2\sin 2\beta - u^2\sin 2\alpha}{2g}\right)\left(\frac{u^2\cos 2\alpha - w^2\cos 2\gamma}{2g}\right)$$

$$+\left(\frac{u^2\cos 2\alpha - v^2\cos 2\beta}{2g}\right)\left(\frac{w^2\sin 2\gamma - u^2\sin 2\alpha}{2g}\right) = 0$$

$$\frac{1}{2g}\Big[u^2v^2\sin 2\beta\cos 2\alpha - v^2w^2\sin 2\beta\cos 2\gamma$$
$$\cancel{- u^4\sin 2\alpha\cos 2\alpha} + u^2w^2\cos 2\gamma\sin 2\alpha$$
$$- u^2w^2\cos 2\alpha\sin 2\gamma \cancel{+ u^4\sin 2\alpha\cos 2\alpha}$$
$$+ v^2w^2\cos 2\beta\sin 2\gamma - u^2v^2\cos 2\beta\sin 2\alpha\Big] = 0$$

$$u^2v^2(\sin 2\beta\cos 2\alpha - \cos 2\beta\sin 2\alpha)$$
$$+ v^2w^2(\cos 2\beta\sin 2\gamma - \sin 2\beta\cos 2\gamma)$$
$$+ u^2w^2(\cos 2\gamma\sin 2\alpha - \cos 2\alpha\sin 2\gamma) = 0$$

$$- u^2v^2(\cos 2\beta\sin 2\alpha - \sin 2\beta\cos 2\alpha)$$
$$- v^2w^2(\sin 2\beta\cos 2\gamma - \cos 2\beta\sin 2\gamma)$$
$$- u^2w^2(\cos 2\alpha\sin 2\gamma - \cos 2\gamma\sin 2\alpha) = 0$$

$$-\Big[u^2v^2(\sin 2\alpha\cos 2\beta - \sin 2\beta\cos 2\alpha)$$
$$+ v^2w^2(\sin 2\beta\cos 2\gamma - \sin 2\gamma\cos 2\beta)$$
$$+ u^2w^2(\cos 2\alpha\sin 2\gamma - \cos 2\gamma\sin 2\alpha)\Big] = 0.$$

$$u^2v^2\sin(2\alpha - 2\beta) + v^2w^2\sin(2\beta - 2\gamma) + u^2w^2\sin(2\gamma - 2\alpha) = 0$$

÷ by $u^2v^2w^2$, we get

$$\frac{\sin 2(\alpha-\beta)}{w^2} + \frac{\sin 2(\beta-\gamma)}{u^2} + \frac{\sin 2(\gamma-\alpha)}{v^2} = 0$$

18. S.T the greatest height reached by a particle whose initial velocity is V and the angle of projection α is unaltered if V is increased by kV and α is decreased by λ were $\operatorname{cosec}\lambda = k[\cot\lambda - \cot\alpha]$

Given:

case (i):-

* Initial velocity is V
* Angle of projection is α
* greatest height is $h_1 = \dfrac{V^2\sin^2\alpha}{2g} \rightarrow (1)$

case (ii):-

* Initial velocity is kV
* Angle of projection is $\alpha-\lambda$
* greatest height is $h_2 = \dfrac{k^2V^2\sin^2(\alpha-\lambda)}{2g} \rightarrow (2)$

To prove: $h_1 = h_2$

Solution:

From given,

$$\operatorname{cosec}\lambda = k[\cot\lambda - \cot\alpha]$$

$$\frac{1}{\sin\lambda} = k\left[\frac{\cos\lambda}{\sin\lambda} - \frac{\cos\alpha}{\sin\alpha}\right]$$

$$\frac{1}{\cancel{\sin\lambda}} = k\left[\frac{\sin\alpha\cos\lambda - \cos\alpha\sin\lambda}{\cancel{\sin\lambda}\sin\alpha}\right]$$

$$\sin\alpha = k\sin(\alpha-\lambda) \rightarrow (3)$$

$$(2) \Rightarrow h_2 = \frac{k^2V^2\sin^2(\alpha-\lambda)}{2g}$$

$$h_2 = \frac{V^2(k\sin(\alpha-\lambda))^2}{2g}$$

$$h_2 = \frac{v^2 \sin^2\alpha}{2g} \quad (\text{using } ③)$$

$$h_2 = h_1 \quad (\text{using } ①)$$

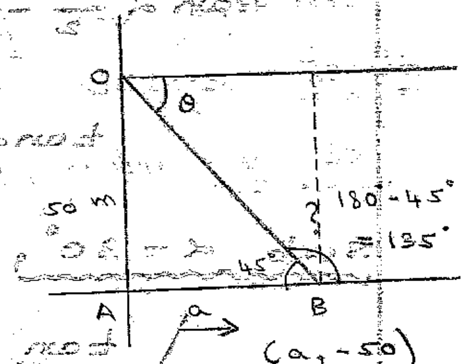
Hence proved.

---

19. A particle projected from the top 'O' of a wall 'AO' 50 m high at an angle of projection 30° above the horizontal strikes the level ground through A & B at an angle of 45°. S.T. the angle of dipression of B from O is $\tan^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)$

To show:

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)$$



Solution :-

slope $\left(\frac{dy}{dx}\right) = \tan 135° = \text{slope}$

consideration $\alpha = 30°$

- $\tan\theta = \frac{OA}{AB} = \frac{50}{a} \Rightarrow ①$

- $y = x\tan\alpha - \frac{g x^2}{2u^2\cos^2\alpha}$

$$\Rightarrow y - x\tan\alpha = \frac{-g x^2}{2u^2\cos^2\alpha} \rightarrow ②$$

At B(a, -50)

$$-50 - a\tan\alpha = \frac{-g a^2}{2u^2\cos^2\alpha} \rightarrow ③$$

Diff ② w.r.t x, $\Rightarrow \frac{dy}{dx} - \tan\alpha = \frac{-g\cancel{2}x}{\cancel{2}u^2\cos^2\alpha}$

$$-1 - \tan\alpha = \frac{-gx}{u^2\cos^2\alpha}$$

At $B(a, -50)$, $-1 - \tan\alpha = \frac{-ga}{u^2\cos^2\alpha} \longrightarrow$ ④

$\frac{a}{2} \times$ ④ $\Rightarrow -\frac{a}{2} - \frac{a}{2}\tan\alpha = \frac{-ga^2}{2u^2\cos^2\alpha} \longrightarrow$ ⑤

③ = ⑤ $\Rightarrow$

$$-50 - a\tan\alpha = -\frac{a}{2} - \frac{a}{2}\tan\alpha$$

$$\tan\alpha\,\frac{a}{2} - a\tan\alpha = -\frac{a}{2} + 50$$

$$\tan\alpha\left(\frac{a}{2}\right) = \left(\frac{a}{2} - 50\right)$$

sub $\alpha = 30^\circ$,

$$\tan 30^\circ\left(\frac{a}{2}\right) = \frac{a}{2} - 50$$

$$\frac{1}{\sqrt{3}}\ \frac{a}{2} = \frac{a}{2} - 50$$

$$\frac{a}{\sqrt{3}\cdot 2} - \frac{a}{2} = -50$$

$$\frac{a}{2}\left(\frac{1}{\sqrt{3}} - 1\right) = -50$$

$$\frac{a}{2}\left(\frac{1-\sqrt{3}}{\sqrt{3}}\right) = -50$$

$$a\left(\frac{1-\sqrt{3}}{\sqrt{3}}\right) = -100$$

$$a = \frac{-100\sqrt{3}}{1-\sqrt{3}}$$

$$\therefore a = \frac{100\sqrt{3}}{\sqrt{3}-1}$$

$$\tan\theta = \frac{s}{a}$$

$$= \frac{\frac{50}{100\sqrt{3}}}{\sqrt{3}-1}$$

$$\tan\theta = \frac{\sqrt{3}-1}{2\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)$$

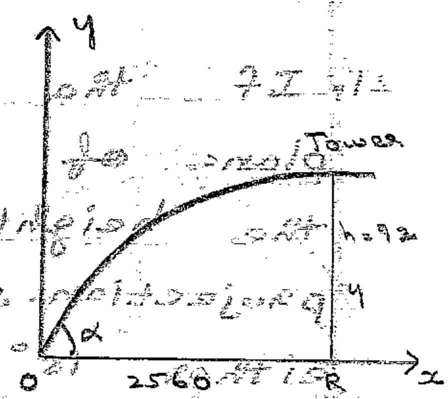
Hence proved

---

20. Find the angle of projection if the velocity is 320 ft/second to hit an object on the top of a tower 92 ft height and 2560 ft away, were $g = 32$

Eqn of projectory

$$\boxed{y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}} \rightarrow (1)$$



Here $u = 320$ ft/sec

$y = 92$ ft

$x = 2560$ ft

Sub in (1)

$$92 = 2560\tan\alpha - \frac{32(2560)^2}{2(320)^2\cos^2\alpha}$$

$$92 = 2560\tan\alpha - 1024\sec^2\alpha$$

$$92 = 2560\tan\alpha - 1024(1+\tan^2\alpha)$$

$$92 = 2560\tan\alpha - 1024 - 1024\tan^2\alpha$$

$$1024\tan^2\alpha - 2560\tan\alpha + 1116 = 0$$

$$\div 4 \Rightarrow 256\tan^2\alpha - 640\tan\alpha + 279 = 0$$

$$\tan\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{640 \pm \sqrt{640^2 - 4(256)(279)}}{2(256)}$$

$$= \frac{640 \pm \sqrt{123904}}{2(256)}$$

$$= \frac{640 \pm 352}{512}$$

$$\tan\alpha = \frac{640 + 352}{512}, \frac{640 - 352}{512}$$

$$\tan\alpha = \frac{31}{16}, \frac{9}{16}$$

$$\alpha = \tan^{-1}\left(\frac{31}{16}\right) \text{ or } \tan^{-1}\left(\frac{9}{16}\right)$$

---

21. If the range on the horizontal plane of the projectile is equal to the height due to the velocity of projection. S.T the angle of projection is either 15° or 75°.

Here the height reached is due to the projection vertically upwards so we have to use,

$$\boxed{v^2 = u^2 + 2as} \text{ suitably}$$

Here, the final velocity is zero the intial velocity is u

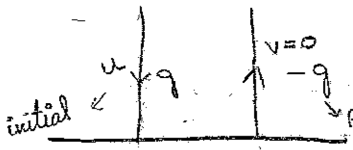
Acceleration = retardation = $-g$

Height travelled is h

$\therefore$ we have $0 = u^2 - 2gh$

$$\frac{u^2}{2g} = h$$

here, we have $\dfrac{u^2 \sin 2\alpha}{g} = \dfrac{u^2}{2g}$



$$\sin 2\alpha = 1/2$$

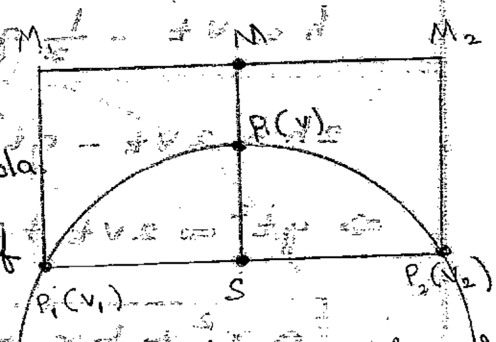
$$2\alpha = \sin^{-1}(1/2)$$

$$2\alpha = 30$$

$$\alpha = 15^\circ$$

22. If V_1, V_2 are the velocities of a projectile at the ends of the focal chord of its path and v the horizontal component of its velocity, S.T $\dfrac{1}{V_1^2} + \dfrac{1}{V_2^2} = \dfrac{1}{v^2}$

Let P_1SP_2 → focal chord. path of



- l → semi latus rectum of parabola
- $2l$ → latus rectum of parabola.
- harmonic mean H.M b/w the segment of the focal chord.

formula

$$\boxed{H.M = \frac{N}{\Sigma \frac{1}{x}}} \Rightarrow l = \frac{2}{\frac{1}{SP_1} + \frac{1}{SP_2}} \Rightarrow \frac{2}{l} = \frac{1}{SP_1} + \frac{1}{SP_2}$$

(B.W)

$$V_1^2 = 2g P_1M_1 = 2gSP_1 \qquad [\because v^2 = 2g.PM]$$

$$V_2^2 = 2g P_2M_2 = 2gSP_2$$

$$v^2 = 2g \frac{PM}{AA} = 2g \frac{SP}{SA} = 2g(1/4 \text{ latus rectum})$$

$$= 2g(1/4 \cdot 2l) = gl$$

$$\frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{2g\,SP_1} + \frac{1}{2g\,SP_2}$$

$$= \frac{1}{2g}\left(\frac{1}{SP_1} + \frac{1}{SP_2}\right)$$

$$= \frac{1}{2g} \cdot \frac{2}{\lambda}$$

$$= \frac{1}{g\lambda} = \frac{1}{V^2}$$

$$\therefore \frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{V^2}$$

---

23. A particle is projected with a velocity whose horizontal and vertical components are u and v. If P_1 and P_2 are the points on its trajectory at the same height h above the point of projection.

S.T $P_1P_2 = \frac{2u}{g}\sqrt{v^2 - 2gh}$

$$S = ut + \frac{1}{2}at^2$$

$$h = vt - \frac{1}{2}gt^2$$

$$2h = 2vt - gt^2$$

$$\Rightarrow gt^2 - 2vt + 2h = 0 \quad \text{—} \ (1)$$

$$\boxed{\begin{array}{l} ax^2 + bx + c = 0 \\ \text{sum of the roots} = -b/a \\ \text{Product of the roots} = c/a \end{array}}$$

Let t_1 & $t_2 \rightarrow$ roots of (1)

$$t_1 + t_2 \longrightarrow \frac{2v}{g}$$

$$t_1 t_2 \longrightarrow \frac{2h}{g}$$

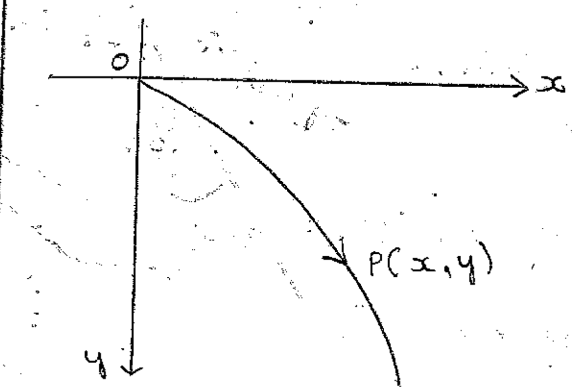
$$(t_2-t_1)^2 = t_2^2+t_1^2-2t_1t_2$$
$$t_2^2+t_1^2-2t_1t_2+2t_1t_2-2t_1t_2$$
$$=(t_1+t_2)^2-4t_1t_2$$
$$=\left(\frac{2v}{g}\right)^2-4\left(\frac{2h}{g}\right)$$
$$(t_2-t_1)^2=\frac{4}{g^2}(v^2-2hg)$$
$$t_2-t_1=\frac{2}{g}\sqrt{v^2-2gh}$$

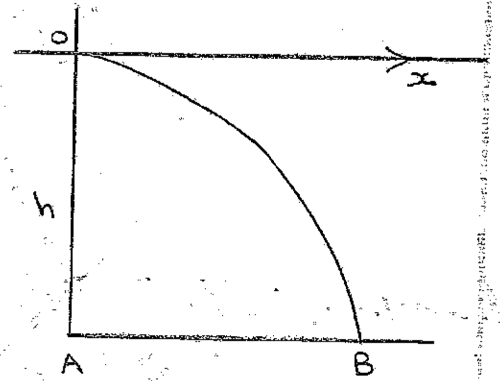
The time interval → particle → P_1 to P_2

t_2-t_1.

$$P_1P_2 = u(t_2-t_1)$$
$$=u\cdot\frac{2}{g}\sqrt{v^2-2gh}$$
$$=\frac{2u}{g}\sqrt{v^2-2gh}$$

Projectile projected horizontally:-





$x = ut \rightarrow (1) \Rightarrow t=\frac{x}{u}$

$\left[\because x=ut \Rightarrow x = uT = u\sqrt{\frac{2h}{g}} \rightarrow H.R\right]$ $\rightarrow h=\frac{1}{2}gT^2 \Rightarrow T^2=\frac{2h}{g} \Rightarrow T=\sqrt{\frac{2h}{g}}$ Time of flight

$s = ut+\frac{1}{2}at^2 \Rightarrow y=\frac{1}{2}gt^2 \rightarrow (2)$ $[\because u=0]$

$$y=\frac{1}{2}g\left(\frac{x}{u}\right)^2$$
$$y=\frac{1}{2}g\frac{x^2}{u^2}$$

$$\Rightarrow x^2 = \frac{2u^2}{g} y$$

This is a parabola → vertex at origin → bent downwards.

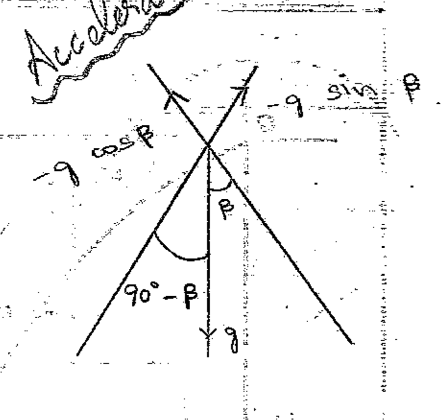
24. A particle projected with a speed u strikes at right angles through the pt of projection inclined at angle β to the horizontal. If α, T, R the angle of projection, the time of flight, range on the inclined plane.

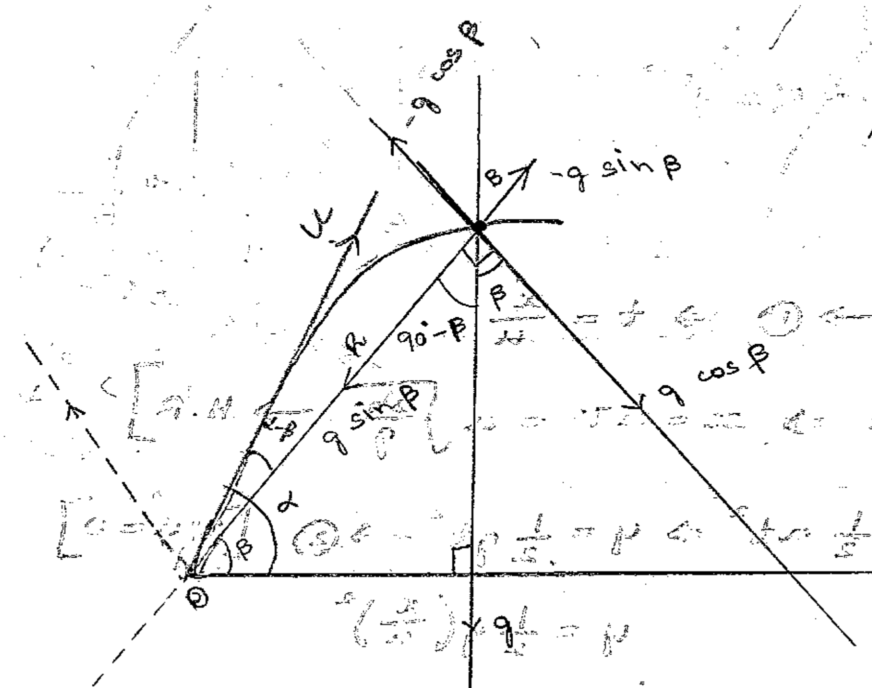
S.T i) $\cot\beta = 2\tan(\alpha - \beta)$

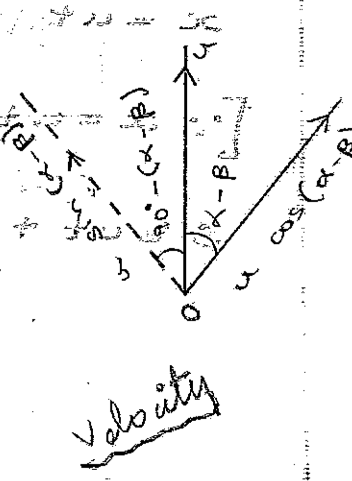
ii) $\cot\beta = \tan\alpha - 2\tan\beta$

iii) $T = \dfrac{2u}{g\sqrt{1+3\sin^2\beta}}$

iv) $R = \dfrac{2u^2\sin\beta}{g(1+3\sin^2\beta)}$







$$\left.\begin{array}{l}\text{Component of } \underline{\text{velocity}} \\ \text{along the inclined plane}\end{array}\right\} = u\cos(\alpha-\beta)$$

$$\left.\begin{array}{l}\text{Component of } \underline{\text{Velocity}} \perp \\ \text{to the inclined plane}\end{array}\right\} = u\sin(\alpha-\beta)$$

$$\left.\begin{array}{l}\text{Component of } \underline{\text{acceleration}} \\ \text{along inclined plane}\end{array}\right\} = -g\sin\beta$$

$$\left.\begin{array}{l}\text{Component of } \underline{\text{acceleration}} \\ \perp \text{ to inclined plane}\end{array}\right\} = -g\cos\beta$$

$$s = ut + \frac{1}{2}at^2$$

∴ the particle is at the point B

The $\perp$ distance travelled $\perp$ to the plane is zero.

$$0 = u\sin(\alpha-\beta)t - \frac{g\cos\beta}{2}t^2$$

$$\frac{g\cos\beta t^{\cancel{2}}}{2} = u\sin(\alpha-\beta)\cancel{t}$$

$$t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

$$v = u + at$$

The final velocity along the inclined plane is zero.

$$0 = u\cos(\alpha-\beta) - g\sin\beta t$$

$$0 = u\cos(\alpha-\beta) - \cancel{g}\sin\beta \cdot \frac{2u\sin(\alpha-\beta)}{\cancel{g}\cos\beta}$$

$$\cancel{u}\cos(\alpha-\beta) = 2\cancel{u}\sin(\alpha-\beta)\cdot\frac{\sin\beta}{\cos\beta}$$

$$\frac{\cos\beta}{\sin\beta} = 2\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

$$\cot\beta = 2\tan(\alpha-\beta)$$

ii) $$\frac{\cos\beta}{\sin\beta} = \cot\beta = 2\,\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

$$\cos\beta\cdot\cos(\alpha-\beta) = 2\sin(\alpha-\beta)\sin\beta$$

$$\cos\beta\left[\cos\alpha\cos\beta+\sin\alpha\sin\beta\right] = 2\left[\sin\alpha\cos\beta-\cos\alpha\sin\beta\right]\sin\beta$$

$$\Rightarrow \cos\alpha\cos^2\beta+\sin\alpha\sin\beta\cos\beta = 2\sin\alpha\cos\beta\sin\beta - 2\cos\alpha\sin^2\beta$$

$\div \cos\alpha\cos\beta\sin\beta$

$$\frac{\cos\alpha\cos^2\beta}{\cos\alpha\cos\beta\sin\beta}+\frac{\sin\alpha\sin\beta\cos\beta}{\cos\alpha\cos\beta\sin\beta} = \frac{2\sin\alpha\cos\beta\sin\beta}{\cos\alpha\cos\beta\sin\beta} - \frac{2\cos\alpha\sin^2\beta}{\cos\alpha\cos\beta\sin\beta}$$

$$\frac{\cos\beta}{\sin\beta}+\frac{\sin\alpha}{\cos\alpha} = 2\,\frac{\sin\alpha}{\cos\alpha} - 2\,\frac{\sin\beta}{\cos\beta}$$

$$\cot\beta+\tan\alpha = 2\tan\alpha - 2\tan\beta$$

$$\cot\beta = 2\tan\alpha - \tan\alpha - 2\tan\beta$$

$$\cot\beta = \tan\alpha - 2\tan\beta$$

iii) $$T = t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$ $$T = \frac{2u\sin\alpha}{g}$$

$$\frac{gT\cos\beta}{2} = u\sin(\alpha-\beta) \longrightarrow (1)$$

$$v = u+at \Rightarrow v = u + aT$$

$0 = u\cos(\alpha-\beta) - g\sin\beta\, T$ [along the inclined plane]

$g\sin\beta\, T = u\cos(\alpha-\beta) \rightarrow ②$

$①^2 + ②^2 \Rightarrow u^2 = \dfrac{g^2T^2}{4}\left[4\sin^2\beta + \cos^2\beta\right]$

$$u^2 = \frac{g^2T^2}{4}\left[3\sin^2\beta + \sin^2\beta + \cos^2\beta\right]$$

$$u^2 = \frac{g^2T^2}{4}\left[3\sin^2\beta + 1\right]$$

$$T^2 = \frac{4u^2}{g^2(3\sin^2\beta+1)}$$

$$T = \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

iv) $s = ut + \frac{1}{2}at^2$

$s = uT + \frac{1}{2}aT^2$

$R = u\cos(\alpha-\beta)T - \frac{1}{2}g\sin\beta\, T^2$ [along the inclined plane]

$R = g\sin\beta\, T^2 - \frac{1}{2}g\sin\beta\, T^2$ [② ⇒ $u\cos(\alpha-\beta) = g\sin\beta\, T$]

$R = \frac{1}{2}g\sin\beta\, T^2$

$$R = \frac{1}{2}g\sin\beta \cdot \frac{4u^2}{g^2(3\sin^2\beta+1)}$$

$$R = \frac{2u^2\sin\beta}{g(3\sin^2\beta+1)}$$ [H.R. of the inclined plane]

25. A particle is projected with speed u so as to strick at right angle. A plane through the point of projection inclined at an angle 30° to the horizontal S.T the range of inclined plane is $\frac{4u^2}{7g}$

$$\boxed{R = \frac{2u^2 \sin\beta}{g(1+3\sin^2\beta)}}$$ inclined plane rage

sub $\beta = 30^\circ$

$$R = \frac{2u^2 \sin 30^\circ}{g(1+3\sin^2 30^\circ)}$$

$$= \frac{\cancel{2} u^2 (1/\cancel{2})}{g(1+3(1/4))} = \frac{u^2}{g(7/4)} = \frac{4u^2}{7g}$$

$$R = \frac{4u^2}{7g}$$

26. A ball is thrown with a velocity of 96 ft/sec from the top of a tower 200 ft high. If the angle of projection is 45°. Find at what distance from the foot of the tower the ball will strike the ground. $g=32$

OA = Tower of height 200

A = point of projection

$\alpha = 45^\circ$, $u = 96$ ft/sec

Eqn of Trajectory is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$y = x\tan 45 - \frac{32x^2}{2(96)^2\cos^2 45^\circ}$$

$$y = x - \frac{\cancel{32}x^2}{\cancel{96}\cdot 96\cdot\frac{1}{2}}$$

$$y = x - \frac{x^2}{288}$$

$AL \rightarrow$ ground level

Co-ordinates of L are $(AL, -200)$

$$-200 = AL - \frac{AL^2}{288}$$

$$-57600 = 288\,AL - AL^2$$

$$AL^2 - 288\,AL - 57600 = 0$$

$$AL = \frac{288 \pm \sqrt{288^2 - 4(1)(-57600)}}{2(1)}$$

$$= \frac{288 \pm \sqrt{313344}}{2}$$

$$= \frac{288 \pm 96\sqrt{34}}{2}$$

$$AL = 144 \pm 48\sqrt{34}$$

As the co-ordinate of L is positive, (AL) is positive.

$$AL = 144 + 48\sqrt{34}$$

27. S.T the greatest height which are particle with initial velocity 'u' can reach on a vertical wall at a distance 'a' from the point of projection is $\frac{u^2}{2g} - \frac{a^2 g}{2u^2}$

If α is the angle of projection & y is the height reached on the wall, we have

$$y = a \tan\alpha - \frac{g a^2}{2u^2 \cos^2\alpha} \quad (\text{here } x = a)$$

$$\text{i.e) } y = a \tan\alpha - \frac{g a^2}{2u^2} \sec^2\alpha \rightarrow (1) \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\sec^2 x = 2\sec x \sec x \tan x = 2\sec^2 x \tan x$$

$$\frac{dy}{dx} = a \sec^2\alpha - \frac{g a^2}{\cancel{2}u^2} \cancel{2}\sec^2\alpha \tan\alpha \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{dy}{dx} = a \sec^2\alpha - \frac{g a^2}{u^2} \sec^2\alpha \tan\alpha$$

$$\frac{d^2y}{dx^2} = 2a \sec^2\alpha \tan\alpha - \frac{g a^2}{u^2}\left[\sec^4\alpha + 2\sec^2\alpha \tan^2\alpha\right]$$

For a maximum, $\frac{dy}{dx} = 0$

$$\text{i.e) } a \sec^2\alpha - \frac{g a^2}{u^2} \sec^2\alpha \tan\alpha = 0$$

$$\cancel{a}\, \cancel{\sec^2\alpha} = \frac{g a^{\cancel{2}}}{u^2} \cancel{\sec^2\alpha} \tan\alpha$$

$$\text{i.e) } \tan\alpha = \frac{u^2}{ag} \text{ we can}$$

S.T $\frac{d^2y}{dx^2}$ is negative for $\tan\alpha = \frac{u^2}{ag}$

$\therefore \tan\alpha = \frac{u^2}{ag}$ gives maximum

$$\text{Greatest height} = a \tan\alpha - \frac{g a^2}{2u^2}(1 + \tan^2\alpha)$$

$$\Rightarrow a\left(\frac{u^2}{ag}\right) - \frac{g a^2}{2u^2}\left(1 + \frac{u^4}{a^2 g^2}\right) \Rightarrow \frac{u^2}{g} - \frac{\cancel{g}\cancel{a^2}}{2u^2}\left(\frac{a^2 g^2 + u^4}{\cancel{a^2} g^{\cancel{2}}}\right)$$

$$\Rightarrow \frac{u^2}{g} - \frac{a^2 g}{2u^2} - \frac{u^2}{2g} \Rightarrow \frac{u^2}{2g} - \frac{g a^2}{2u^2}$$

28. A hill is inclined at angle of 30° to the horizontal from a point on the hill. One projectile is projected up the hill and another down the hill, both starting with the same velocity and same angle of projection of 45° with the horizontal. S.T the range of the one particle is nearly $\frac{15}{4}$ times that of the other.

$\alpha = 45^\circ$
$\beta = 30^\circ$

$$R_1 = \frac{2u^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$R_2 = \frac{2u^2 \sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$$

$$\therefore \frac{R_1}{R_2} = \frac{\cancel{2u^2}\,\cancel{\cos\alpha}\,\sin(\alpha-\beta)}{\cancel{g}\,\cancel{\cos^2\beta}} \times \frac{\cancel{g}\,\cancel{\cos^2\beta}}{\cancel{2u^2}\,\cancel{\cos\alpha}\,\sin(\alpha+\beta)}$$

$$= \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$$

$$= \frac{\sin(45^\circ-30^\circ)}{\sin(45^\circ+30^\circ)}$$

$$= \frac{\sin 45^\circ\cos 30^\circ - \cos 45^\circ \sin 30^\circ}{\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ}$$

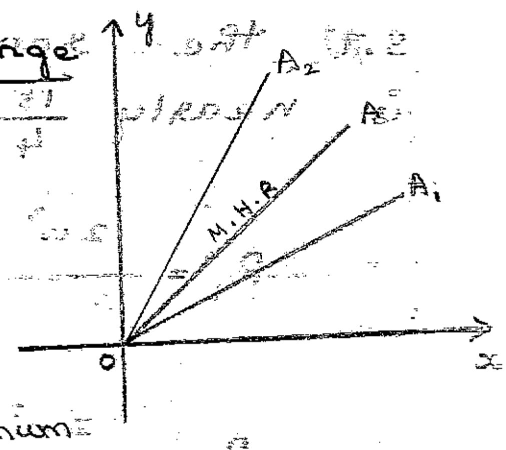
$$= \frac{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}}{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{\sqrt{3}+1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = 0.268$$

$$\frac{R_1}{R_2} = \frac{4}{15} \qquad \frac{15}{4} R_1 = R_2$$

Book work : 2

Find the direction of maximum horizontal range point there are two different direction for the same horizont[al range]. Prove also that the direction correspondin[g] to maximum horizontal range is equally inclined with 2 directions.



$$HR = \frac{u^2 \sin 2\alpha}{g}$$

$\sin 2\alpha = 1$; if HR is maximum

$\Rightarrow \sin 2\alpha = \sin \frac{\pi}{2}$

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4} = 45^\circ$$

Hence in order to get maximum HR for a given velocity u the particle must be projected at an angle 45°

$$HR = \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin(180 - 2\alpha)}{g} = \frac{u^2 \sin 2(90^\circ - \alpha)}{g}$$

Given any two velocity u, 2 different direction α and $90^\circ - \alpha$ give the same horizontal range.

* OA_1 & OA_2 $\xrightarrow{\text{correspond to}}$ 2 different directions giving the same horizontal range.

$\Rightarrow X\hat{O}A_1 = \alpha$; $X\hat{O}A_2 = 90^\circ - \alpha$

* $OA \xrightarrow{\text{correspond to}}$ maximum horizontal range

$\Rightarrow X\hat{O}A = 45^\circ$

* The angle between OA_1 and OA is equal to $A_1\hat{O}A$.

$$A_1\hat{O}A = X\hat{O}A - X\hat{O}A_1$$

$$= 45 - \alpha \rightarrow (1)$$

* The angle between OA and OA_2 is equal to $A\hat{O}A_2$

$$A\hat{O}A_2 = X\hat{O}A_2 - X\hat{O}A$$

$$= 90^\circ - \alpha - 45^\circ$$

$$= 45^\circ - \alpha \rightarrow (2)$$

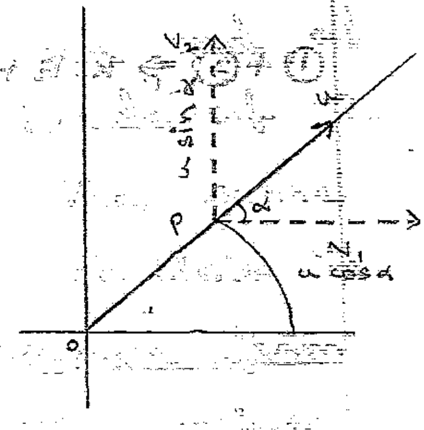
From (1) & (2),

$$A_1\hat{O}A = A\hat{O}A_2$$

The direction corresponding to maximum horizontal range is equally inclined with the two directions corresponding to the same horizontal range.

Book work : (3)

To find the velocity at time t



* $P \rightarrow$ particle at time 't'
* $V \rightarrow$ its velocity
* V_1 & $V_2 \rightarrow$ horizontal and vertical component of V

$\Rightarrow V_1 = u\cos\alpha$

Consider suitably

$$\boxed{v = u + at}$$

$\Rightarrow v_2 = u \sin\alpha - gt \rightarrow$ considering the vertical displacement

$\therefore$ Resultant velocity $= v = \sqrt{v_1^2 + v_2^2}$

$$= \sqrt{u^2 + g^2t^2 - 2ugt\sin\alpha}$$

$\therefore$ Direction $= \tan\theta = \dfrac{v_2}{v_1} = \dfrac{u\sin\alpha - gt}{u\cos\alpha}$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{u\sin\alpha - gt}{u\cos\alpha}\right)$$

Book work : 4

To verify in the case of a projectile kinetic energy + potential energy = constant

* Let $m \rightarrow$ mass of the projectile at time 't'

then $K.E = \frac{1}{2}mv^2$

$$= \frac{1}{2}m\left(u^2 + g^2t^2 - 2ugt\sin\alpha\right) \rightarrow (1)$$

* $h \rightarrow$ vertical height of the particle at time 't'

$s = ut + \frac{1}{2}at^2$ $\rightarrow$ suitable

$h = u\sin\alpha t - \frac{1}{2}gt^2$ $\rightarrow$ considering the vertical displacement

$P.E = mgh$

$$= mg\left(u\sin\alpha\, t - \frac{1}{2}gt^2\right) \rightarrow (2)$$

$(1) + (2) \Rightarrow K.E + P.E = \frac{1}{2}mu^2 + \cancel{\frac{1}{2}mg^2t^2} - \cancel{mugt\sin\alpha} + \cancel{mug\sin\alpha t} - \cancel{\frac{1}{2}mg^2t^2}$

$$= \frac{1}{2}mu^2$$

$$= \text{constant}$$

Book work: 5

Maximum horizontal range for a given velocity.

* Suppose a particle is projected with a velocity 'u' and angle of projection 'α' then H.R is

$$R = \frac{u^2 \sin 2\alpha}{g}$$

* When $\sin 2\alpha$ is maximum, it attains the maximum value.

* But the maximum value of $\sin 2\alpha$ is 1.

So, the maximum H.R $= \frac{u^2}{g}$ and angle of projection is given by

$$\sin 2\alpha = 1$$

$$2\alpha = \sin^{-1}(1) = \frac{\pi}{2} = 90^\circ$$

$$2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4} = 45^\circ$$

$$\alpha = 45^\circ$$

$$\therefore R = \frac{u^2}{g} = \text{maximum H.R}$$

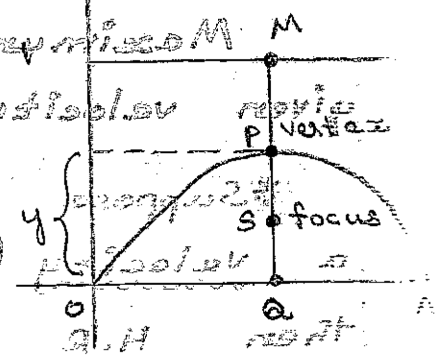
Book work: 6

Find the velocity of the projectile magnitude and direction at the end of the time t. S.T the magnitude of the velocity at any point is the same would be acquired by a particle in falling freely a vertical distance from the level of directrix to the point.

Parametric eqn of parabola

$x = (u\cos\alpha)\,t$

$y = (u\sin\alpha)\,t - \frac{1}{2}gt^2$ → (1)



$\Rightarrow \frac{dx}{dt} = \dot{x} = u\cos\alpha$

$\Rightarrow \frac{dy}{dt} = \dot{y} = u\sin\alpha - gt$

*Resultant velocity $v = \sqrt{\dot{x}^2 + \dot{y}^2}$

$v = \sqrt{u^2 + g^2t^2 - 2u\sin\alpha\, gt}$ → (2)

*$\tan\theta = \frac{j^{th}\text{ comp}}{i^{th}\text{ comp}}$

$= \frac{\dot{y}}{\dot{x}} = \frac{u\sin\alpha - gt}{u\cos\alpha}$

From (2),

$$v^2 = u^2 - 2(u\sin\alpha)gt + g^2t^2$$
$$= u^2 - 2g\left[u\sin\alpha\, t - \frac{1}{2}gt^2\right]$$
$$= u^2 - 2gy \quad (\text{from } (1))$$
$$= 2g\left[\frac{u^2}{2g} - y\right]$$

→ eqn of directrix

$$= 2g[QM - PQ]$$
$$= 2g\,PM$$
$$= 2g\,SP \quad [\because PM = SP]$$

$v^2 = 2g(\frac{1}{4}\text{ Latus rectum})$

$\downarrow u=0 \quad \uparrow v=0$

$v^2 = u^2 + 2as$ (suitably chosen)

$\Rightarrow v_1^2 = 2g\,PM$

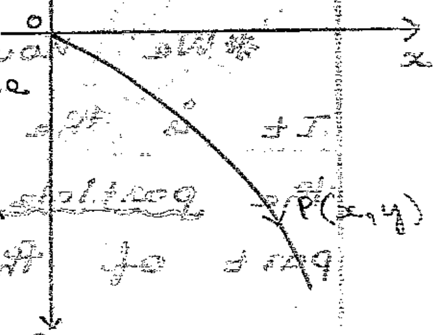
$v_1^2 = 2g\,PM = v^2 \Rightarrow \boxed{v_1 = v}$

the velocity at P of the particle falling freely from M

Book work : 7

Projectile projected horizontally

A particle is projected horizontally from a height 'h'. Find the equation of the path.



* Choose the horizontal line and verticle downward line through the point of projection.

* P(x, y) → position of the particle at time 't' [the initial velocity is horizontal and so]

* The horizontal components of the velocity is constant equal to u.

★ The distance travelled horizontally is given by

$$\boxed{x = ut} \longrightarrow (1)$$

Now consider the y coordinate of the point P,

$$s = ut + \tfrac{1}{2}at^2 \text{ (suitably)}$$

Here the initial velocity is zero

i.e) $u = 0$, $a = g$

we have $y = \frac{1}{2} g t^2 \longrightarrow (2)$

$(1) \Rightarrow t = \dfrac{x}{u}$

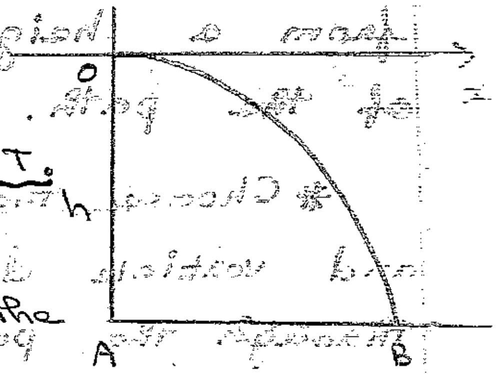
Sub in (2)

$$y = \frac{1}{2} g \frac{x^2}{u^2} = \frac{g}{2u^2} x^2 \Rightarrow x^2 = \frac{2u^2}{g} \cdot y$$

∴ This is the eqn of parabola with vertex at the origin bent downwards.

Book work : 8

'A' is the point on the ground O is a point above A such that OA = h. Find the time of flight T and the range AB.



* We now get the time T.

It is the time taken by the particle to describe the part of the parabola

$$x^2 = \frac{2u^2}{g}y$$

i.e) To describe OB corresponding to time T, y = h

The distance travelled horizontally is given by

$$x = ut \Rightarrow (1)$$

Now we get the y co-ordinates of the point P,

$$s = ut + \frac{1}{2}at^2 \text{ (suitably)}$$

Here the initially vertical velocity is 0, a = g.

we have, $y = \frac{1}{2}gt^2 \Rightarrow (2)$

sub y = h in (2), $\Rightarrow h = \frac{1}{2}gT^2$

Time of flight $T = \sqrt{\frac{2h}{g}}$

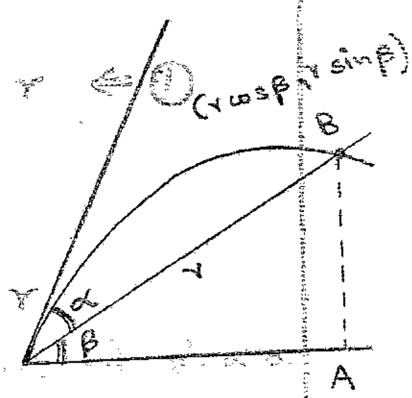
Horizontal range $= AB = uT = u\sqrt{\frac{2h}{g}}$

Range on the inclined plane:-

Find the range on the inclined plane and time of flight T.

$\alpha \rightarrow$ angle of projection

$\beta \rightarrow$ angle of inclination



$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

At the point $(r\cos\beta, r\sin\beta)$

$$r\sin\beta = r\cos\beta\tan\alpha - \frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$$

$$r\sin\beta = r\cos\beta\frac{\sin\alpha}{\cos\alpha} - \frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$$

$$\frac{g\cos^2\beta}{2u^2\cos^2\alpha}\cdot r = \frac{\cos\beta\sin\alpha}{\cos\alpha} - \sin\beta$$

$$\frac{g\cos^2\beta}{2u^2\cos^2\alpha}\cdot r = \frac{\cos\beta\sin\alpha - \sin\beta\cos\alpha}{\cos\alpha}$$

$$\frac{g\cos^2\beta}{2u^2\cos^2\alpha}\cdot r = \frac{\sin(\alpha-\beta)}{\cos\alpha}$$

$$\frac{\sin(\alpha-\beta)\cdot 2u^2\cos^2\alpha}{\cos\alpha\cdot g\cos^2\beta}$$

$$r = \frac{\sin(\alpha-\beta)\,2u^2\cos\alpha}{g\cos^2\beta} \longrightarrow (1)$$

This is the range up the inclined plane.

The range down the inclined plane is got by replacing β by $(-\beta)$

$$r = \frac{\sin(\alpha-(-\beta))\,2u^2\cos\alpha}{g\cos^2(-\beta)}$$

Range on the inclined plane:

$$r = \frac{\sin(\alpha-\beta)\, 2u^2 \cos\alpha}{g \cos^2\beta} \rightarrow (1)$$

$$(1) \Rightarrow r = \frac{\sin(\alpha-\beta)\, 2u^2 \cos\alpha}{g \cos^2\beta}$$

$$r = \frac{2 \sin(\alpha-\beta) \cos\alpha\, u^2}{g \cos^2\beta}$$

$$[\because 2 \sin A \cos B = \sin(A+B) + \sin(A-B)]$$

$$r = \frac{[\sin(\alpha-\beta+\alpha) + \sin(\cancel{\alpha}-\beta-\cancel{\alpha})]u^2}{g \cos^2\beta}$$

$$r = \frac{(\sin(2\alpha-\beta) - \sin\beta)u^2}{g \cos^2\beta}$$

$\therefore$ Range up the inclined plane is

$$r = \frac{(\sin(2\alpha-\beta) - \sin\beta)\, u^2}{g \cos^2\beta} \longrightarrow (3)$$

$\therefore$ Replace β by $-\beta$,

Range down the inclined plane is

$$r = \frac{(\sin(2\alpha+\beta) + \sin\beta)\, u^2}{g \cos^2\beta} \longrightarrow (4)$$

max range up the inclined plane $= r_1$

Range is maxi.

$\Rightarrow$ Numerator of (3) is maxi

$\Rightarrow \sin(2\alpha-\beta) = 1$

$$③ \Rightarrow r_1 = \frac{(1-\sin\beta)u^2}{g(1-\sin^2\beta)}$$

$$r_1 = \frac{\cancel{(1-\sin\beta)}\,u^2}{g(1+\sin\beta)\cancel{(1-\sin\beta)}}$$

$$r_1 = \frac{u^2}{g(1+\sin\beta)} \longrightarrow ⑤$$

Maxi range down the inclined plane is got by replacing β by $(-\beta)$ in ⑤. It is denoted by r_2

$$r_2 = \frac{u^2}{g(1-\sin\beta)}$$

consider

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{\frac{u^2}{g(1+\sin\beta)}} + \frac{1}{\frac{u^2}{g(1-\sin\beta)}}$$

$$= \frac{g(1+\sin\beta)+g(1-\sin\beta)}{u^2}$$

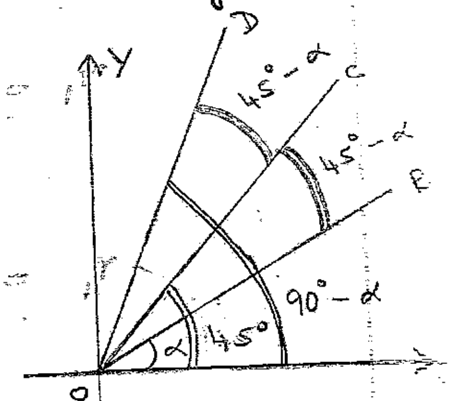
$$= \frac{g+g\cancel{\sin\beta}+g-g\cancel{\sin\beta}}{u^2}$$

$$= \frac{2g}{u^2}$$

∴ relation between maxi range up the inclined plane and maxi range down the inclined plane.

Book work : 9

Two trajectories with the given speed and the range.



Given speed $= u$

Given HR $= R = \dfrac{u^2 \sin 2\theta}{g}$

$\Rightarrow \sin 2\theta = \dfrac{Rg}{u^2}$

$\sin 2\theta = 1$ for maxi HR

$2\theta = \sin^{-1} 1$

$2\theta = \dfrac{\pi}{2}$

$\theta = \dfrac{\pi}{4} = 45^\circ$

$\Rightarrow HR = \dfrac{u^2 \sin 2\theta}{g}$

$= \dfrac{u^2 \sin (180^\circ - 2\theta)}{g}$

$= \dfrac{u^2 \sin 2(90^\circ - \theta)}{g}$

$\therefore XOA_1 = \theta$

$XOA_2 = 90^\circ - \theta$

} OA_1 & OA_2 correspond to 2 trajec giving the same HR

$\Rightarrow X\hat{O}A = 45^\circ \longrightarrow OA$ corres to max HR

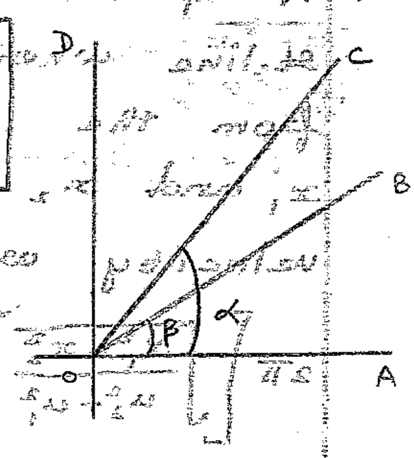
- $A_1\hat{O}A = X\hat{O}A - X\hat{O}A_1 = 45^\circ - \theta$
- $A\hat{O}A_2 = X\hat{O}A_2 - X\hat{O}A = 90^\circ - \theta - 45^\circ = 45^\circ - \theta$

$\Rightarrow$ direction corres to max HR is equally inclined with the 2 traje corres to same HR.

Book work (10)

Horizontal range up the inclined plane.

$$\boxed{r = \frac{2u^2}{g\cos^2\beta}\left[\sin(\alpha-\beta)\cos\alpha\right]}$$



previous problem

$$= \frac{u^2}{g\cos^2\beta}\left[2\sin(\alpha-\beta)\cos\alpha\right] \quad \text{(refer range on the inclined plane)}$$

$$= \frac{u^2}{g\cdot\cos^2\beta}\left[\sin(2\alpha-\beta) - \sin\beta\right]$$

HR is maxi $\Rightarrow \sin(2\alpha-\beta) = 1$

$$2\alpha - \beta = \sin^{-1} 1$$

$$2\alpha - \beta = \frac{\pi}{2}$$

$$2\alpha = 90^\circ + \beta$$

$$\alpha = \frac{90^\circ}{2} + \frac{\beta}{2}$$

$$\alpha = 45^\circ + \beta/2$$

Let $A\hat{O}B = \beta$ &

$A\hat{O}C = \alpha = 45^\circ + \beta/2$

$B\hat{O}C = A\hat{O}C - A\hat{O}B = \alpha - \beta$

$= 45^\circ + \frac{\beta}{2} - \beta$

$= 45^\circ - \frac{\beta}{2}$

$C\hat{O}D = A\hat{O}D - A\hat{O}C$

$= 90^\circ - \alpha$

$= 90^\circ - (45^\circ + \beta/2)$

$= 45^\circ - \beta/2$

$\Rightarrow B\hat{O}C = C\hat{O}D$

Simple Harmonic Motion

1. A particle is moving with the SHM in a st.line when the distance of the particle from the equilibrium position as the value x_1 and x_2. The corresponding value of velocity are u_1 and u_2. S.T the period is $2\pi\left[\sqrt{\frac{x_1^2-x_2^2}{u_2^2-u_1^2}}\right]$

A distance of the particle from the equilibrium when $x = x_1$, the velocity is

$$u_1^2 = n^2(a^2 - x_1^2) \rightarrow (1)$$

when $x = x_2$, the velocity is

$$u_2^2 = n^2(a^2 - x_2^2) \rightarrow (2)$$

sub. (1) & (2) we get,

$$u_2^2 - u_1^2 = n^2\left[a^2 - x_2^2 - a^2 + x_1^2\right]$$

$$u_2^2 - u_1^2 = n^2\left[x_1^2 - x_2^2\right]$$

$$n^2 = \frac{u_2^2 - u_1^2}{x_1^2 - x_2^2}$$

$$n = \sqrt{\frac{u_2^2 - u_1^2}{x_1^2 - x_2^2}}$$

The period of SHM is $\frac{2\pi}{n}$

$$\therefore \text{Period} = \frac{2\pi}{\sqrt{\frac{u_2^2 - u_1^2}{x_1^2 - x_2^2}}}$$

$$\therefore T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{u_2^2 - u_1^2}}$$

2. A particle is moving with the SHM has speeds v_1 and v_2 ($v_1 > v_2$) and acceleration with magnitude α_1 and α_2 at the pt A & B which lie on the same side of the mean position O. S.T i) $AB = \dfrac{v_1^2 - v_2^2}{\alpha_1 + \alpha_2}$ ii) $T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

i) $\ddot{x} = n^2 x \rightarrow$ eqn of SHM and $v^2 = n^2(a^2 - x^2)$

$\alpha_1 = n^2 x_1$, $\alpha_2 = n^2 x_2$

$$v_1^2 = n^2(a^2 - x_1^2)$$

$$v_2^2 = n^2(a^2 - x_2^2)$$

$$v_1^2 - v_2^2 = n^2(a^2 - x_1^2 - a^2 + x_2^2)$$

$$= n^2\left[x_2^2 - x_1^2\right]$$

$$= -n^2\left[x_1^2 - x_2^2\right] \longrightarrow \text{(I)}$$

$$v_1^2 - v_2^2 = -n^2(x_1 + x_2)(x_1 - x_2)$$

$$\frac{v_1^2 - v_2^2}{n^2(x_1 + x_2)} = x_2 - x_1 \Rightarrow \frac{v_1^2 - v_2^2}{n^2 x_1 + n^2 x_2} = x_2 - x_1$$

$$\frac{v_1^2 - v_2^2}{\alpha_1 + \alpha_2} = AB$$

Hence (i) proved

ii) (I) $\Rightarrow v_1^2 - v_2^2 = -n^2(x_1^2 - x_2^2)$

$$v_1^2 - v_2^2 = n^2(x_2^2 - x_1^2)$$

$$n^2 = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2} \Rightarrow n = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$$

$$\therefore T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}}$$

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

Hence (ii) proved

2. A particle is moving with the SHM has speeds v_1 and v_2 ($v_1 > v_2$) and acceleration with magnitude α_1 and α_2 at the pt A & B which lie on the same side of the mean position O. S.T i) $AB = \dfrac{v_1^2 - v_2^2}{\alpha_1 + \alpha_2}$ ii) $T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

i) $\ddot{x} = n^2 x \rightarrow$ eqn of SHM and $v^2 = n^2(a^2 - x^2)$

$\alpha_1 = n^2 x_1 \,,\; \alpha_2 = n^2 x_2$

$$v_1^2 = n^2(a^2 - x_1^2)$$

$$v_2^2 = n^2(a^2 - x_2^2)$$

$$v_1^2 - v_2^2 = n^2(a^2 - x_1^2 - a^2 + x_2^2)$$

$$= n^2\left[x_2^2 - x_1^2\right]$$

$$= -n^2\left[x_1^2 - x_2^2\right] \longrightarrow \text{(I)}$$

$$v_1^2 - v_2^2 = -n^2(x_1 + x_2)(x_1 - x_2)$$

$$\frac{v_1^2 - v_2^2}{n^2(x_1 + x_2)} = x_2 - x_1 \Rightarrow \frac{v_1^2 - v_2^2}{n^2 x_1 + n^2 x_2} = x_2 - x_1$$

$$\frac{v_1^2 - v_2^2}{\alpha_1 + \alpha_2} = AB$$

Hence (i) proved

ii) (I) $\Rightarrow v_1^2 - v_2^2 = -n^2(x_1^2 - x_2^2)$

$$v_1^2 - v_2^2 = n^2(x_2^2 - x_1^2)$$

$$n^2 = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2} \Rightarrow n = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$$

$$\therefore T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}}$$

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

Hence (ii) proved

S.A. Thara Jani

DYANAMICS (2m)

1. Mass :-

The material or the matter out of which the body is made, it is known as its mass.

Unit of mass:-

It is in MKS and FPS system is 1 kg and 1 pound and it is the mass of a piece of metal (arbitarily chosen) preserved in paris & london.

2. Displacement :-

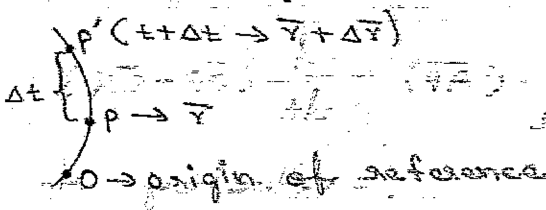
When a particle moves from a pt to another pt, the particle is said to undergo a displacement. i.e, it is just a change of position.

When a particle moves from a pt 'p' to a pt 'p'', the respective displacement denoted by the vector $\overline{PP'}$

3. Velocity :-

The rate of change of position is called velocity of a particle. Velocity is denoted by $\overline{v}$.

$$\overline{v} = \frac{d\overline{r}}{dt}$$



4. Direction of velocity :-

$$\overline{v} = \hat{T}\frac{ds}{dt} = \dot{s}\,\hat{T}$$

* The mag. of the velocity is $\dot{s}$ and dir. of the velocity is along to the tgt direction.

* The mag. of the velocity is called as speed. It is denoted by $\dot{s}$.

5. Unit of velocity:-

(m.s)

* It is easily understood as length / time

* It is in S system,

MKS system : 1 m/sec

CGS system : 1 cm/sec.

FPS system : 1 foot/sec

6. Resultant Velocity:-

If a particle has two velocities $\overline{V_1}$ & $\overline{V_2}$ then $\overline{V_1}+\overline{V_2}$ is said to be the resultant velocity of the particle.

7. Relative velocity:-

A & P → two moving points

$\overline{AP}$ → PV of 'P' w.r.t

$\frac{d}{dt}(\overline{AP})$ → velocity of P relative to A

O → origin of reference

Relative velocity of P w.r.t A

$$\frac{d}{dt}(\overline{AP}) = \frac{d}{dt}(\overline{OP}-\overline{OA}) = \frac{d}{dt}(\overline{OP}) - \frac{d}{dt}(\overline{OA})$$

$$= \text{velocity of P} - \text{velocity of A}$$

$$= \overline{V_P} - \overline{V_A}$$

i.e, $\frac{d}{dt}(\overline{AP})$ = True velocity of P − velocity of A

= T.V of P + (− velocity of A)

= T.V of P + reversed velocity of A.

∴ The relative velocity of P w.r.t A is the resultant velocity of P and the reversed velocity of A.

...eration:-

Acceleration of a particle is the time-rate of change of its velocity. (i.e) if $\bar{v}$ is its velocity at time t, then its acceleration $\bar{a}$ at time t is $\frac{d\bar{v}}{dt}$, $\boxed{\bar{a} = \frac{d\bar{v}}{dt}}$

Unit of acceleration:-

M.K.S system : $1 m/sec^2$

C.G.S system : $1 cm/sec^2$

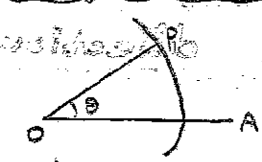
F.P.S system : $1 ft/sec^2$

9. Coplanar motion:-

When a particle moves in a plane, its motion is said to be coplanar.

10. Angular velocity:-

* P → particle having a coplanar motion
* O → fixed point
* OA → fixed line in the plane of motion
* Then the time-rate of change of the angle $A\hat{O}P$ is called the angular velocity of the particle about 'O'.

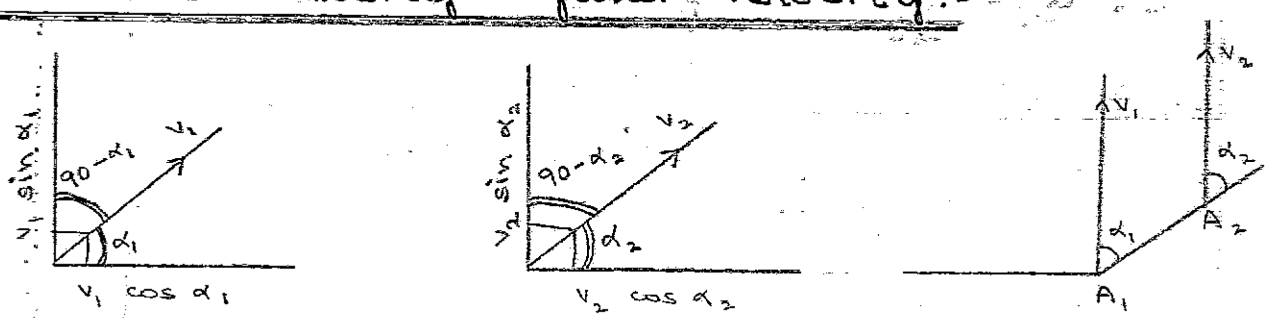


i.e) $A\hat{O}P = \theta$

$\frac{d}{dt}(A\hat{O}P) = \frac{d\theta}{dt} = \dot{\theta}$ is the angular velocity of the particle P about O.

⇒ The unit is 1 rad/sec.

11. Relative ~~velocity~~ angular velocity:-



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$$= \overline{V_P} - \overline{V_A}$$

i.e, $\frac{d}{dt}(\overline{AP})$ = True velocity of P - velocity of A

= T.V of P + (- velocity of A)

= T.V of P + reversed velocity of A.

∴ The relative velocity of P w.r.t A is the resultant velocity of P and the reversed velocity of A.

* velocity of A_2 relative to A_1 is $\overline{V_2}$
* A_1 & $A_2 \rightarrow$ 2 particles $\rightarrow$ moving in plane
* $\overline{V_1}$ & $\overline{V_2} \rightarrow$ velocities
* $\overline{V_1}$ & $\overline{V_2}$ makes an angle α_1 & α_2 with line A_1A_2
* $\overline{V_1}$ & $\overline{V_2} \longrightarrow (90-\alpha_1)$ & $(90-\alpha_2)$ with dir $\perp$ to A_1A_2
* Comp of velocity V_1 & V_2 $\perp$ to $A_1A_2 = V_1 \sin\alpha_1$

& $A_1A_2 = V_2 \sin\alpha_2$

$$\Rightarrow \dot{\theta} = \frac{\text{the comp of velocity} \perp A_1A_2}{A_1A_2}$$

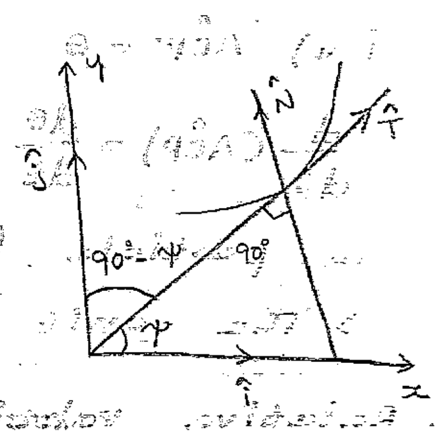
$$= \frac{\overbrace{V_2 \sin\alpha_2 - V_1 \sin\alpha_1}^{\overline{V_2}-\overline{V_1}}}{A_1A_2}$$

$$\boxed{\dot{\theta} = \frac{|\overline{A_1A_2} \times (\overline{V_2}-\overline{V_1})|}{A_1A_2^{\,2}}}$$

= Angular speed of A_2 about A_1

= Relative angular velocity of one w.r.t another.

12. **Comp. of velo & Acceleration along tgt & normal directions.**



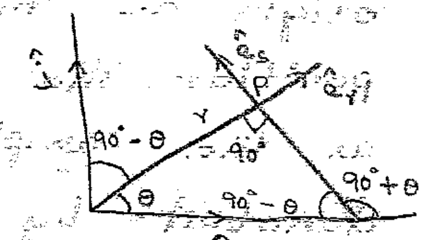
* Velocity comp: tang. dir is $\dot{s}$; nor. dir is 0
* Acceleration comp: tgt. $= \ddot{s}$; nor. $= \dfrac{\dot{s}^2}{\rho}$

$$\therefore \overline{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N}$$

Comp. of velocity & acceleration of a particle along the radial & transverse directions:-

* Velocity comp: radial dir $= \dot{r}$; trans. dir $= r\dot{\theta}$

* Acceleration comp: rad. dir $= \ddot{r} - r\dot{\theta}^2$; trans. dir $= 2\dot{r}\dot{\theta} + r\ddot{\theta}$

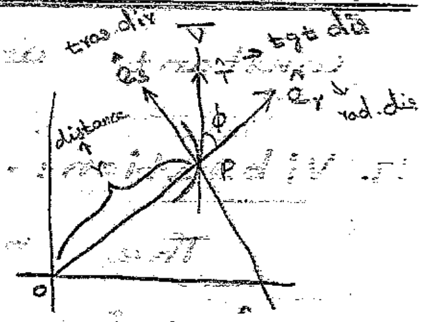


$$\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_s$$

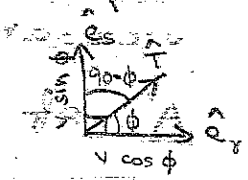
$$\therefore \bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_s$$

14. Comp of velocity v along transverse direction:

Comp of v along trans. dir $= r\dot{\theta}$



* OP = r
* V → velocity at P
* ϕ → angle b/w OP & tgt at P



$$\therefore \dot{\theta} = \frac{r\dot{\theta}}{r} = \frac{\text{comp of velo} \perp r \text{ OP}}{OP}$$

→ transverse direction.

$$\dot{\theta} = \frac{v\cos(90-\phi)}{r} = \frac{v\sin\phi}{r} = \frac{r\,v\sin\phi}{r^2}$$

$$\text{Angular speed} = \dot{\theta} = \frac{|\bar{r} \times \bar{v}|}{r^2}$$

15. Simple Harmonic Motion:-

The motion of a particle along a st. line with an acceleration which is always towards a fixed pt on the st. line and whose mag is proportional to the distance of the particle from the fixed point is called a SHM.

→ Eqn of SHM is $\boxed{\ddot{x} = -n^2 x}$

* velocity of A_2 relative to A_1 is $\overline{V_2}$
* A_1 & A_2 → 2 particles → moving in plane
* $\overline{V_1}$ & $\overline{V_2}$ → velocities
* $\overline{V_1}$ & $\overline{V_2}$ makes an angle α_1 & α_2 with line A_1A_2
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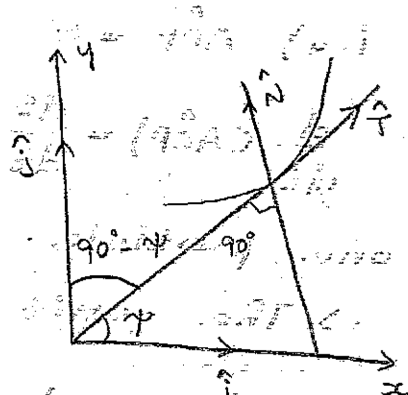
* Acceleration comp → tgt. $= \ddot{s}$; nor. $= \dfrac{\dot{s}^2}{\rho}$



$$\therefore \overline{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N}$$

16. **Amplitude :-** A' O A

If A is the extreme position or origin of the time and O is the origin position. The distance b/w O & A is known as the amplitude of the SHM and it is denoted by (a) i.e) the maxi displacement of a particle is called amplitude.

$$\boxed{\ddot{x} = n^2 x}$$

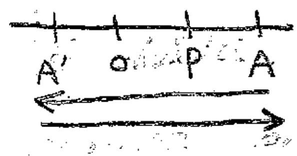
At 'O' acceleration = 0, but velocity is constant and is the maximum.

17. **Vibration :-**

The motion of the particle from extremity A to the other extremity A' of its path is called as vibration.

18. **Oscillation :-**

One complete motion of the particle from one extremity A to the other extremity A' and back to the same extremity A is called an oscillation. ~~It can also be defined as~~



(Motion of a particle from P to A' then A' to A and at last A to P.)

19. **Period (T) :-**

The time taken for 1 oscillation is called period. It is denoted by (T)

$$T = \frac{2\pi}{n}$$

Frequency:-

The no. of oscillations made per second is defined as the frequency. It is denoted by (f)

21. ## Relation b/w T & f :-

* T & f are reciprocals of each other

- T sec → 1 oscillation

1 sec → $\frac{1}{T}$ oscillation = f

$\therefore f = \frac{1}{T} \Rightarrow T = \frac{1}{f}$

22. ## K.E :-

Energy possessed by particle interms of its motion.

P.E :-

P.E is measured interms of the workdone by particle by moving it from present position to standard position.

* Workdone : F x distance moved by the particle along the dir. of force

* Force : mass x acceleration = ma

23. ## S.T in SHM, K.E + P.E = constant

$$K.E = \frac{1}{2} mv^2 = \frac{1}{2} m[n^2(a^2 - x^2)] = \frac{1}{2} mn^2(a^2 - x^2)$$

$$P.E = \int_P^0 (\text{force})\,dx = \int_x^0 m\ddot{x}\,dx = \int_x^0 m(-n^2x)\,dx = \int_0^x mn^2x\,dx = \frac{mn^2x^2}{2}$$

$$K.E + P.E = \frac{1}{2} mn^2(a^2 - x^2) + \frac{1}{2} mn^2x^2$$

$$= \frac{1}{2} mn^2a^2$$

$$= \text{constant}$$

24. Hooke's law:-

$$\text{Tension} = \lambda \left[\frac{\text{extension (or) compression}}{\text{Natural length}}\right]$$

25. Central orbits:-

When a particle is under the action of a force which is always either towards or away from a fixed point. The particle is said to be under the action of a central force i.e) a central force is a force whose line of action always passes through a fixed point. The fixed point is called centre of force. The path described is called orbits.

26. Laws of central force:-

When the equation of a central orbit is given, to obtain the force per unit mass and the speed of the particle at a distance r from the centre of force, we have to calculate

$$-h^2 u^2\left(\frac{d^2u}{d\theta^2}+u\right)\hat{r},\ h\sqrt{\left(\frac{du}{d\theta}\right)^2+u^2}$$

27. Constancy of moment of momentum (or) angular momentum about O:-

* The momentum of the particle is mv which is along the tangent.
* Its momentum about O is

$$ON \times (mv) = m(pv) = mh$$

*Thus the moment of momentum about O, otherwise known as angular momentum about O, is the constant mh.

28. **Newton's Inverse square law:-**

Two particles of masses m_1 & m_2 attract each other with a force whose magnitude is directly proportional to the product of the masses and inversely proportional to the square of the distance.

$$\text{ie)}\ F \propto \frac{m_1 m_2}{r^2} \Rightarrow F = \frac{\gamma\, m_1 m_2}{r^2}$$

29. **Kepler's laws of planetary motion:-**

I law:

The planet describe ellipse about the sun as focus.

$$\left.\begin{array}{l}\text{*Force of attraction}\\ \text{on the planet}\end{array}\right\} \propto \frac{1}{(\text{distance from the sun})^2}$$

II law:

The radius vector drawn from the sun to a planet sweeps out equal areas in equal intervals of time.

*For each planet,

The areal velocity $= \frac{1}{2} r^2 \dot{\theta} = \frac{h}{2} =$ constant

*Transverse comp. of the force acting on the planet $= 0$

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The areal velocity $= \frac{1}{2} r^2 \dot{\theta} = \frac{h}{2} =$ constant

* Transverse comp. of the force acting on the planet $= 0$

* The force acting on the planet is along the radius vector & thus it is central force.

III Law:

The squares of the periodic times of the planets are proportional to the cubes of the semi major axis of their respective orbits.

$$T^2 \propto a^3$$

30. Periodic Time:-

When the orbit is an ellipse, the periodic time of a particle is

$$T = \frac{\text{total area}}{\text{Areal velocity}} = \frac{\pi ab}{\frac{1}{2}h} = \frac{\pi ab}{\frac{1}{2}\sqrt{\mu l}} \quad \left[\because \mu = \frac{h^2}{l} \Rightarrow h = \sqrt{\mu l}\right]$$

$$T = \frac{2\pi}{\sqrt{\mu}}\frac{ab\sqrt{a}}{b} \quad \left[\because l = \frac{b^2}{a}\right] = \frac{2\pi}{\sqrt{\mu}}a^{3/2} = K a^{3/2}$$

$$\Rightarrow \boxed{T^2 = K^2 a^3}$$

Hence Kepler's third law is verified.

31. Critical velocity:-

The quantity $\sqrt{\frac{2\mu}{r}}$ is called the critical velocity at the distance r. So, the nature of the orbit depends on the critical velocity.

i.e) $V = \sqrt{\frac{2\mu}{r}}$

Projectile :-

Projectile is a particle in a direction inclined to the direction of gravity namely to the vertical.

33. Trajectory :-

The path of the projectile is called Trajectory.

34. Angle of projectile :-

The angle of projection is the ~~ang~~ angle with the ~~dist~~ direction of projection makes with the horizontal.

35. Horizontal Range :-

The distance b/w the pt of projection and the pt where the projectory meets the horizontal plane is called the horizontal angle.

36. Time of flight :-

The time of flight is the time taken to complete the horizontal range.