Unit - III

The First Fundamental Form

3.1) Lengths of curves on surfaces:

If $\gamma(t) = \sigma(u(t), v(t)) \rightarrow (1)$ is a curve in a surface patch σ, its arc length starting at a point $\gamma(t_0)$ is given by,

$$s = \int_{t_0}^{t} \|\dot{\gamma}(u)\| \, du \rightarrow (2)$$

Diff (1) w.r.t 't',

$$\dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v} \rightarrow (3)$$

$$\|\dot{\gamma}\|^2 = \dot{\gamma} \cdot \dot{\gamma}$$

$$= (\sigma_u \dot{u} + \sigma_v \dot{v}) \cdot (\sigma_u \dot{u} + \sigma_v \dot{v})$$

$$= (\sigma_u \cdot \sigma_u) \dot{u}^2 + (\sigma_u \cdot \sigma_v) \dot{u}\dot{v} + (\sigma_v \cdot \sigma_u) \dot{v}\dot{u} + (\sigma_v \cdot \sigma_v) \dot{v}^2$$

$$= \|\sigma_u\|^2 \dot{u}^2 + 2(\sigma_u \cdot \sigma_v) \dot{u}\dot{v} + \|\sigma_v\|^2 \dot{v}^2$$

$$= E \dot{u}^2 + 2F \dot{u}\dot{v} + G \dot{v}^2$$

where $E = \|\sigma_u\|^2$

$F = \sigma_u \cdot \sigma_v$

$G = \|\sigma_v\|^2$

$$\therefore s = \int_{t_0}^{t} (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}\, dt \rightarrow (f)$$

$$= \int_{t_0}^{t} \left\{ \underbrace{E\left(\frac{du}{dt}\right)^2}_{\left(\frac{du}{dt}\right)^2 dt^2 = du^2} + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \right\}^{1/2} dt$$

$$= \int \sqrt{ds^2} \rightarrow (4)$$

where $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \rightarrow (*)$ is the **first fundamental form of σ.**

If now γ is a curve on arbitrary surface S, its length can be computed by breaking γ into segments each of which lies in a surface patch and using (f) to compute the length of each segment. The first fundamental form will be changed if the surface patch is changed.

Eg: ~~V.Imp~~ 5.1

3.1) expression of FFF + this 8m

For the plane, $\sigma(u,v) = \bar{a} + u\bar{p} + v\bar{q}$ with $\bar{p}$ & $\bar{q}$ being $\perp r$ unit vectors.

We have, $\sigma_u = \bar{p}$; $\sigma_v = \bar{q}$

So, $E = \|\sigma_u\|^2 = \|\bar{p}\|^2 = \bar{p}\cdot\bar{p} = 1$; $F = \sigma_u \cdot \sigma_v = \bar{p}\cdot\bar{q} = 0$; $G = \|\sigma_v\|^2 = \|\bar{q}\|^2 = \bar{q}\cdot\bar{q} = 1$

The FFF is $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

$\therefore ds^2 = du^2 + dv^2$

Eg: ~~UNLB~~ 5.2

For the sphere in latitude-longitude coordinates $\sigma(\theta,\phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$.

We have,

$$\bar{\sigma}_\theta = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)$$

$$\bar{\sigma}_\phi = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0)$$

So, $E = \|\sigma_\theta\|^2$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta$$
$$= \sin^2\theta(\sin^2\phi + \cos^2\phi) + \cos^2\theta$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1$$

$F = \sigma_\theta \cdot \sigma_\phi$

$$= \sin\theta\cos\theta\cos\phi\sin\phi - \sin\theta\cos\theta\cos\phi\sin\phi$$
$$= 0$$

$G = \|\sigma_\phi\|^2$

$$= \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi + 0$$
$$= \cos^2\theta(\sin^2\phi + \cos^2\phi)$$
$$= \cos^2\theta$$

The FFF is $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

$$= d\theta^2 + 2(0) + \cos^2\theta\, d\phi^2$$
$$= d\theta^2 + \cos^2\theta\, d\phi^2$$

Eg : ~~UNKN~~ 5.3

We consider a generalized cylinder,

$$\sigma(u, v) = \gamma(u) + v\vec{a}$$

We assume that γ is unit-speed, $\bar{a}$ is a unit vector and γ is contained in a plane $\perp r$ to $\bar{a}$.

Then denoting $\frac{d}{du}$ by a dot,

$$\sigma_u = \dot{\gamma} \; ; \; \sigma_v = \bar{a}$$

So, $E = \|\sigma_u\|^2 \; ; \; F = \sigma_u \cdot \sigma_v \; ; \; G = \|\sigma_v\|^2$

$= \|\dot{\gamma}\|^2 \quad = \dot{\gamma}\cdot\bar{a} \quad = \|\bar{a}\|^2$

$= 1 \quad = 0 \quad = 1$

[$\because \gamma$ is of unit-speed] [$\because \gamma \perp r$ to $\bar{a}$] [$\because \bar{a}$ is a unit vector]

The FFF of σ is $du^2 + dv^2$

This is same as the FFF of the plane.

Eg: ~~UNKN~~ 5.4

We consider a generalized cone,

$$\sigma(u, v) = (1+v)\,\bar{p} - \gamma(u)$$

see eg 4.11 (compare)

First, translating the surface by $\bar{p}$, we get the surface patch, $\sigma_1 = \sigma - \bar{p}$

$= v(\bar{\gamma} - \bar{p})$

so, we replace γ by $\gamma_1 = \gamma - \bar{p}$, we get $\sigma_1 = v\gamma_1$, we may also assume $\bar{p} = \bar{o}$.

For $\bar{\sigma}$ to be a regular surface patch, γ must not pass through the origin, so we can define a new curve, $\tilde{\gamma}$ by $\tilde{\gamma}(u) = \dfrac{\gamma(u)}{\|\gamma(u)\|}$.

Setting $\tilde{u} = u$ & $\tilde{v} = \dfrac{(1+v)}{\|\gamma(u)\|}$, we get a reparametrization

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \tilde{v}\,\tilde{\gamma}(\tilde{u}) \text{ of } \bar{\sigma} \text{ with } \|\tilde{\gamma}\| = 1.$$

We can therefore assume to begin with that $\sigma(u, v) = v\,\gamma(u)$ with $\|\gamma(u)\| = 1$ for all values of u (geometrically this means that we can replace γ by the intersection of the cone with S^2).

Finally, reparametrizing again we can assume that γ is [unit-speed], σ to be regular, γ must be regular.

With these assumptions and with a dot denoting $\frac{d}{du}$, we have

$\sigma_u = v\dot{\gamma}$; $\sigma_v = \gamma$

$E = \|v\dot{\gamma}\|^2 = v^2\|\dot{\gamma}\|^2 = v^2$

$F = v\dot{\gamma}\cdot\gamma = 0$ $[\because \|\gamma\| = 1]$

$G = \|\gamma\|^2 = 1$

The FFF is $ds^2 = v^2 du^2 + dv^2$,

Exercise:

Calculate FFF

6m 1) $\sigma(u,v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$

6m 2) $\sigma(u,v) = (u-v, u+v, u^2+v^2)$

3) $\sigma(u,v) = (\cosh u, \sinh u, v)$

4) $\sigma(u,v) = (u, v, u^2+v^2)$

What kinds of sulfaces are these?

(Name the sulfaces)

Soln:

1) $\sigma(u,v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$

Name of the sulface: Quadric cone ; $x^2+z^2=y^2$

$\sigma_u = (\cosh u \sinh v, \cosh u \cosh v, \cosh u)$

$\sigma_v = (\sinh u \cosh v, \sinh u \sinh v, 0)$

$E = \|\sigma_u\|^2$

$= \cosh^2 u \sinh^2 v + \cosh^2 u \cosh^2 v + \cosh^2 u$

$= \cosh^2 u (\sinh^2 v + \cosh^2 v + 1)$

$= \cosh^2 u (\cosh^2 v - \cancel{1} + \cosh^2 v + \cancel{1})$

$= 2\cosh^2 u \cosh^2 v$

$F = \sigma_u \cdot \sigma_v$

$= \cosh u \sinh v \cosh v \sinh u + \cosh u \cosh v \sinh u \sinh v$

$= 2 \sinh u \cosh u \sinh v \cosh v$

$$G = \|\sigma_v\|^2$$
$$= \sinh^2 u \cosh^2 v + \sinh^2 u \sinh^2 v$$
$$= \sinh^2 u (\cosh^2 v + \sinh^2 v)$$
$$= \sinh^2 u \cosh 2v$$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\therefore$ The FFF is

$$ds^2 = 2\cosh^2 u \cosh^2 v \, du^2 + 4 \sinh u \cosh u \sinh v \cosh v \, du dv + \sinh^2 u \cosh 2v \, dv^2$$

2) $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$

<u>Name of the surface</u>: Paraboloid of revolution

$\sigma_u = (1, 1, 2u)$; $\sigma_v = (-1, 1, 2v)$

$E = \|\sigma_u\|^2$; $F = \sigma_u \cdot \sigma_v$; $G = \|\sigma_v\|^2$

$= 1 + 1 + 4u^2$; $= -1 + 1 + 4uv$; $= 1 + 1 + 4v^2$

$= 2 + 4u^2$; $= 4uv$; $= 2 + 4v^2$

$\therefore$ The FFF is

$$ds^2 = (2 + 4u^2) du^2 + 8uv \, du dv + (2 + 4v^2) dv^2$$

3) $\sigma(u, v) = (\cosh u, \sinh u, v)$

<u>Name of the surface</u>: Hyperbolic cylinder

$\sigma_u = (\sinh u, \cosh u, 0)$; $\sigma_v = (0, 0, 1)$

$E = \|\sigma_u\|^2 = \sinh^2 u + \cosh^2 u = \cosh 2u$

$F = \sigma_u \cdot \sigma_v = 0$

$G = \|\sigma_v\|^2 = 1$

4) $\sigma(u,v) = (u, v, u^2+v^2)$

Name of the surface: Paraboloid of revolution

$\sigma_u = (1,0,2u)$; $\sigma_v = (0,1,2v)$

$E = \|\sigma_u\|^2 = 1+4u^2$
$F = \sigma_u \cdot \sigma_v = 4uv$
$G = \|\sigma_v\|^2 = 1+4v^2$

$\therefore$ The FFF is

$$ds^2 = (1+4u^2)\,du^2 + 8uv\,du\,dv + (1+4v^2)\,dv^2$$

Sec 3.2: Isometries of surfaces:

Defn: [Isometry] 2m

If S_1 & S_2 are surfaces, a diffeomorphism $f: S_1 \to S_2$ is called an isometry if it takes curves in S_1 to curves of the same length in S_2. If an isometry $f: S_1 \to S_2$ exists, we say that S_1 & S_2 are isometric.

6m

Theorem: 5.1

20 m

A diffeomorphism $f: S_1 \to S_2$ is an isometry $\Leftrightarrow$ for any surface patch σ_1 of S_1, the patches σ_1 and $f\circ\sigma_1$ of S_1 & S_2, respectively, have the same fundamental form.

Proof:

If any surface is covered by more than one surface patch, then the curve that we are considering could be part of more than one surface patch, in that case the length of the curve can be computed as the sum of the lengths of the bits of that curve lying in each of the surface patches.

Instead we can assume that the entire surface is covered by a single surface patch and thus the length of the curve needs to be calculated for that one surface fn.

Let us assume that S_1 and S_2 are two surfaces that are covered by single surface patch σ_1 & σ_2, resp.

Let f be a diffeomorphism from S_1 to S_2, then as in prop. 4.3, we have the following relation $f \circ \sigma_1 = \sigma_2$.

Suppose first that σ_1 & σ_2 have the same 1st fundamental form which is given by,

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \rightarrow (1)$$

Prop. 4.3

Let $f: S_1 \rightarrow S_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on S_1, then $f \circ \sigma_1$ is an allowable surface patch on S_2.

If $t \mapsto (u(t), v(t))$ is any curve in U and $\gamma_1(t) = \sigma_1(u(t), v(t))$ and $\gamma_2(t) = \sigma_2(u(t), v(t))$ are the corresponding curves in S_1 & S_2, then the diffeomorphism f takes γ_1 to γ_2,

since $f(\gamma_1(t)) = f(\sigma_1(u(t), v(t)))$

$$= (f \circ \sigma_1)(u(t), v(t))$$

$$= \sigma_2(u(t), v(t))$$

$$= \gamma_2(t).$$

Since γ_1 & γ_2 have the same length as they are represented by σ_1 and σ_2, both of which have the same first fundamental form given by (1).

Since γ_1 & γ_2 have the same length,
By defn,
f is an isometry.

Conversely,

Suppose f is an isometry, assume that $t \mapsto (u(t), v(t))$ be any curve in U.

Let f take the curve $\gamma_1(t) = \sigma_1(u(t), v(t))$ to $\gamma_2(t) = \sigma_2(u(t), v(t))$, since f is isometry, γ_1 & γ_2 have the same length i.e) σ_1 & σ_2 have the same fundamental form.

Hence

$$\int_{t_0}^{t}\left(E_1\left(\frac{du}{dt}\right)^2 + 2F_1\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G_1\left(\frac{dv}{dt}\right)^2\right)^{1/2} dt$$

$$= \int_{t_0}^{t}\left(E_2\left(\frac{du}{dt}\right)^2 + 2F_2\frac{du}{dt}\frac{dv}{dt} + G_2\left(\frac{dv}{dt}\right)^2\right)^{1/2} dt$$

$\forall\ t_0, t \in (\alpha, \beta)$

where E_1, F_1 & G_1 are co-effs of FFF of σ_1 & E_2, F_2 & G_2 are co-effs of FFF of σ_2.

The above eqn implies that the integrand are the same.

ie)

$$E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2 = E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2 \rightarrow (2)$$

Let us fix a $t_0 \in (\alpha, \beta)$ and correspondingly fix

$u_0 = u(t_0)$ & $v_0 = v(t_0)$.

Let us apply eqn (2) for the following 3 choices of the curve $t \mapsto (u(t), v(t))$.

i) $u = u_0 + t - t_0,\ v = v_0$

ii) $u = u_0,\ v = v_0 + t - t_0$

iii) $u = u_0 + t - t_0,\ v = v_0 + t - t_0$

i) $\Rightarrow \dot{u}=1,\ \dot{v}=0$

Sub in (2'), $\boxed{E_1 = E_2} \rightarrow (3')$

ii) $\Rightarrow \dot{u}=0,\ \dot{v}=1$

sub in (2'), $\boxed{G_1 = G_2} \rightarrow (4')$

iii) $\Rightarrow \dot{u}=1,\ \dot{v}=1$

Sub in (2'),

$$E_1 + 2F_1 + G_1 = E_2 + 2F_2 + G_2 \rightarrow (5)$$

sub ~~in~~ (3') & (4) in (5'),

$$\cancel{E_1} + 2F_1 + \cancel{G_1} = \cancel{E_1} + 2F_2 + \cancel{G_1}$$

$$\boxed{F_1 = F_2} \rightarrow (6)$$

Hence σ_1 & σ_2 have the same first fundamental form.

Example: 5.5

If f is a diffeomorphism from S_1 to S_2 where S_1 represents part of a plane, S_2 represents a cylinder then f is an isometry.

Soln: Let S_1 be the infinite strip in the xy-plane given by $0 < x < 2\pi$ and S_2 be the circular cylinder $x^2+y^2=1$ with the ruling given by $x=1, y=0$ removed.

Let ~~us~~ consider the surface patch σ_1 to be the single surface patch the surface S_1 then $\boxed{\sigma_1(u,v) = (u, v, 0)}$ ii) $0<x<2\pi$

And let σ_2 be the single surface patch covering S_2 given by

$$\boxed{\sigma_2(u,v) = (\cos u, \sin u, v)} \text{ where } 0<u<2\pi.$$

Now, the FFF is given by the expression,

$E_1 d\bar{u}^2 + 2F_1 d\bar{u}d\bar{v} + G_1 d\bar{v}^2$ for σ_1 &

$E_2 d\bar{u}^2 + 2F_2 d\bar{u}d\bar{v} + G_2 d\bar{v}^2$ for σ_2

where $E_1 = \|\sigma_{1u}\|^2 = 1^2+0+0 = 1$ $\quad F_1 = \sigma_{1u}\cdot\sigma_{1v} = 0$

$G_1 = \|\sigma_{1v}\|^2 = 1$ $\quad \sigma_{2u} = -\sin u, \cos u, 0$

$\sigma_{2v} = 0, 0, 1$

The FFF for σ_1 is $du^2 + dv^2$

$E_2 = \|\sigma_{2u}\|^2 = \sin^2 u + \cos^2 u = 1$ $\quad F_2 = \sigma_{2u}\cdot\sigma_{2v} = 0$ $\quad G_2 = \|\sigma_{2v}\|^2 = 1$

The FFF for σ_2 is $d\bar{u}^2 + d\bar{v}^2$

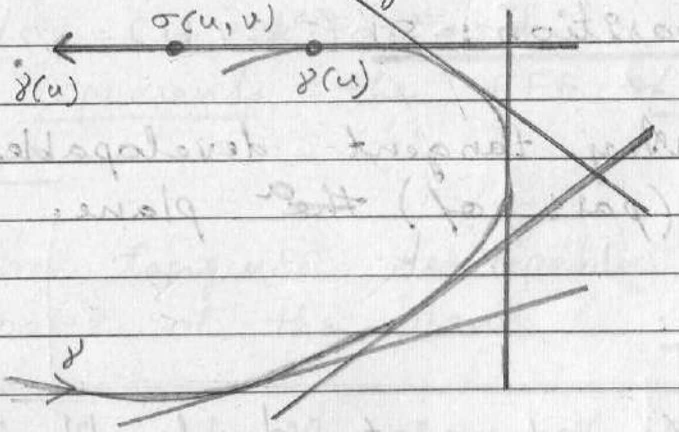
Since both FFF's are the same, by theorem 5.1, f is an isometry.

Defn : [Tangent developables]

A **tangent developable** is the union of the tangent lines to the curve in $\mathbb{R}^3$ which may be given by

$$\sigma(u, v) = \gamma(u) + v\,\dot{\gamma}(u)$$

where $\gamma(u)$ is point on the curve and $\dot{\gamma}(u)$ is the tangent at the point.



Remark :

A tangent developable,

$$\sigma(u, v) = \gamma(u) + v\,\dot{\gamma}(u)$$

where $\sigma_u = \dot{\gamma}(u) + v\,\ddot{\gamma}(u)$

$\sigma_v = \dot{\gamma}(u)$

$$\begin{aligned}
\sigma_u \times \sigma_v &= (\dot{\gamma}(u) + v\,\ddot{\gamma}(u)) \times \dot{\gamma}(u) \\
&= \underbrace{\dot{\gamma} \times \dot{\gamma}}_{=0} + v\,\ddot{\gamma} \times \dot{\gamma} \\
&= v\,\ddot{\gamma} \times \dot{\gamma} \\
&= v\,\dot{t} \times \dot{\gamma} \quad [\because \ddot{\gamma} = \dot{t} \ \&\ \dot{\gamma} = t] \\
&= v\,\dot{t} \times \hat{t} \\
&= v\,k\,\hat{n} \times \hat{t} \quad [\because \dot{t} = k\hat{n}]
\end{aligned}$$

$\Rightarrow \sigma_u \times \sigma_v \neq 0$ if $v\,k\,\hat{n} \neq 0$ ie) if $k > 0$

$j \times j = 0$
$i \times j = 1$
$j \cdot j = 1$
$i \cdot j = 0$

In this case when $\boxed{k>0}$, we have σ to be regular. It should also be noted that we cannot take v to be zero either $v>0$ or $v<0$ but never be equal to zero that means in the tangent developable formed by the curve γ, we have to remove the edge where $\boxed{v=0}$.

Proposition: 5.1

6m

Any tangent developable is isometric to (part of) ~~the~~ a plane.

Proof:

The tangent developable for the curve γ is given by $\boxed{\sigma(u,v)=\gamma(u)+v\dot{\gamma}(u)}$

Let us assume that γ is of unit speed and $k>0$.

Then, we have

$$\sigma_u = \dot{\gamma} + v\ddot{\gamma} \quad ; \quad \sigma_v = \dot{\gamma}$$

$$
\begin{aligned}
E &= \|\sigma_u\|^2 = \sigma_u \cdot \sigma_u \\
&= (\dot{\gamma} + v\ddot{\gamma}) \cdot (\dot{\gamma} + v\ddot{\gamma}) \\
&= \dot{\gamma}\dot{\gamma} + v\dot{\gamma}\ddot{\gamma} + v\dot{\gamma}\ddot{\gamma} + v^2\ddot{\gamma}\ddot{\gamma} \\
&= 1 + 2v\underbrace{\dot{\gamma}\ddot{\gamma}}_{0} + v^2\|\ddot{\gamma}\|^2 \quad \left[\because \ddot{\gamma}\cdot\ddot{\gamma} = \|\ddot{\gamma}\|^2\right] \\
&= 1 + v^2k^2 \quad \left[\because \gamma \text{ is of unit speed, } \|\ddot{\gamma}\| = k\right]
\end{aligned}
$$

$$F = \sigma_u \cdot \sigma_v$$
$$= (\dot{\gamma} + v\ddot{\gamma}) \cdot (\dot{\gamma})$$
$$= \dot{\gamma}\cdot\dot{\gamma} + v\,\underbrace{\dot{\gamma}\cdot\ddot{\gamma}}_{0}$$
$$= 1$$

$$G = \|\sigma_v\|^2 = \sigma_v \cdot \sigma_v$$
$$= \dot{\gamma}\cdot\dot{\gamma}$$
$$= 1$$

$\therefore\ ds^2 = (1+v^2k^2)\,du^2 + 2\,du\,dv + dv^2$

which represents the FFF of part of a plane.

Hence tangent developable is isometric to part of the plane.

3.3) Conformal Mappings of surfaces:

Let us suppose that 2 curves γ & $\tilde{\gamma}$ on a surface S intersect at a point P that lies in the σ patch of S, then

$$\gamma(t) = \sigma(u(t), v(t))$$
$$\& \quad \tilde{\gamma}(t) = \sigma(\tilde{u}(t), \tilde{v}(t))$$

for some smooth fns $u, v, \tilde{u}$ & $\tilde{v}$ and for some parameters t_0 & $\tilde{t}_0$ at P, we have

$$\sigma(u(t_0), v(t_0)) = \sigma(\tilde{u}(\tilde{t}_0), \tilde{v}(\tilde{t}_0)) = P$$

The **angle** θ of intersection of γ & $\tilde{\gamma}$ at p is defined to be the angle b/w the tangent vectors $\dot{\gamma}$ & $\dot{\tilde{\gamma}}$ (evaluated at $t = t_0$ & $\tilde{t} = \tilde{t}_0$, respectively).

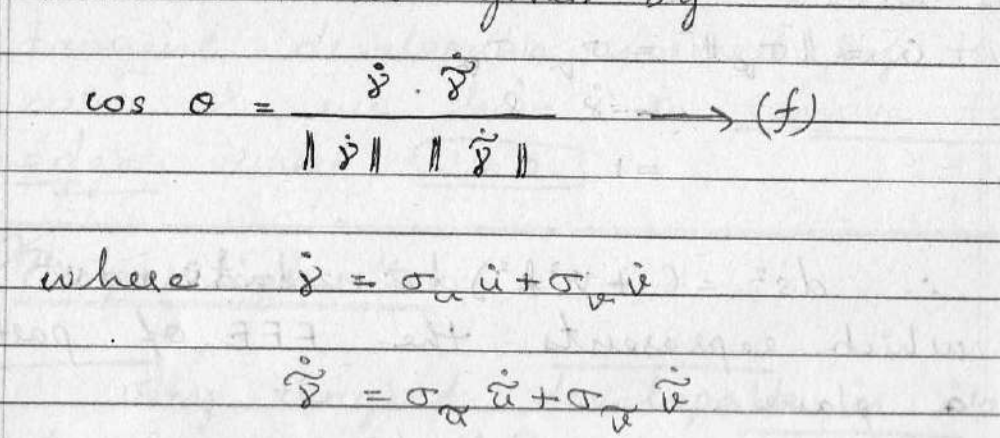
Using the dot product formula for the angle b/w vectors, we see that θ is given by

$$\cos\theta = \frac{\dot{\gamma}\cdot\dot{\tilde{\gamma}}}{\|\dot{\gamma}\|\,\|\dot{\tilde{\gamma}}\|} \longrightarrow (f)$$

where $\dot{\gamma} = \sigma_u\dot{u} + \sigma_v\dot{v}$

$$\dot{\tilde{\gamma}} = \sigma_{\tilde{u}}\dot{\tilde{u}} + \sigma_{\tilde{v}}\dot{\tilde{v}}$$

ie) $\dot{\tilde{\gamma}} = \sigma_u\dot{\tilde{u}} + \sigma_v\dot{\tilde{v}}$ $\left[\begin{array}{l} \because \text{at } P, \\ \sigma(u,v) = \sigma(\tilde{u},\tilde{v}) \\ \& \ \sigma_u = \sigma_{\tilde{u}} \end{array}\right]$



Now,

$$\dot{\gamma}\cdot\dot{\tilde{\gamma}} = (\sigma_u\dot{u} + \sigma_v\dot{v})\cdot(\sigma_u\dot{\tilde{u}} + \sigma_v\dot{\tilde{v}})$$

$$= \underbrace{(\sigma_u\cdot\sigma_u)}_{E}\dot{u}\dot{\tilde{u}} + \underbrace{(\sigma_u\sigma_v)}_{F}(\dot{u}\dot{\tilde{v}} + \dot{v}\dot{\tilde{u}}) + \underbrace{(\sigma_v\sigma_v)}_{G}\dot{v}\dot{\tilde{v}}$$

$$= E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{v}\dot{\tilde{u}}) + G\dot{v}\dot{\tilde{v}} \longrightarrow (fA)$$

$$\|\dot{\gamma}\| = (\dot{\gamma}\cdot\dot{\gamma})^{1/2}$$

$$= \left[(\sigma_u \dot{u} + \sigma_v \dot{v})\cdot(\sigma_u \dot{u} + \sigma_v \dot{v})\right]^{1/2}$$

$$= \left[(\sigma_u\cdot\sigma_u)\dot{u}^2 + 2(\sigma_u\cdot\sigma_v)\dot{u}\dot{v} + (\sigma_v\cdot\sigma_v)\dot{v}^2\right]^{1/2}$$

$$= (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2} \rightarrow (f_B)$$

Similarly,

$$(f_C)\ \|\dot{\tilde{\gamma}}\| = (E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2} \rightarrow (f_C)$$

sub $(f_A), (f_B)$ & (f_C) in (f) $\rightarrow$

$$\therefore \cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{v}\dot{\tilde{u}}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}} \hookrightarrow (f)$$

Eg: 5.6

The parameter curves on a surface patch $\sigma(u,v)$ can be parametrised as follows:

$$\gamma(t) = \sigma(a,t) \quad \& \quad \tilde{\gamma}(t) = \sigma(t,b)$$

(a & b are constant vectors)

Find the angle b/w the 2 curves.

Soln:

Since $\gamma(t) = \sigma(a,t)$,

we have

$$u(t) = a \ ; \ v(t) = t$$

Hence $\dot{u} = 0$ & $\dot{v} = 1$.

Also $\tilde{\gamma}(t) = \sigma(t, b)$

Hence $\tilde{u}(t) = t$; $\tilde{v}(t) = b$

$\dot{\tilde{u}} = 1$; $\dot{\tilde{v}} = 0$

From eqn (†),

we have the angle b/w the curves to be

$$\cos\theta = \frac{E(0) + F(0+1) + G(0)}{(E(0) + 2F(0) + G(1))^{1/2}(E + F(0) + 0)^{1/2}}$$

$$= \frac{F}{\sqrt{G}\sqrt{E}}$$

$$= \frac{F}{\sqrt{GE}}$$

(Parameter - orthogonal $\Leftrightarrow F = 0$)

→ If $F = 0$, the parameter curves are orthogonal since the angle b/w the curves is $90°$.

Defn: [Conformal]

If S_1 & S_2 are surfaces, a diffeomorphism f from S_1 to S_2 is said to be conformal if whenever f takes intersecting curves γ_1 & $\tilde{\gamma}_1$ on S_1 to curves γ_2 & $\tilde{\gamma}_2$ on S_2; the angle of intersection of γ_1 & $\tilde{\gamma}_1$ is equal to the angle of intersection of γ_2 & $\tilde{\gamma}_2$.

ie) we say f is conformal if & only if it preserves angle.

S_2 γ_2 $\tilde{\gamma}_2$ angle of

f: S_1 γ_1 & $\tilde{\gamma}_1$ angle of

Theorem : 5.2

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A diffeomorphism $f : S_1 \to S_2$ is conformal $\Leftrightarrow$ for any surface patch σ_1 on S_1, the FFF's σ_1 and $f \circ \sigma_1$ are proportional.

Proof:

Let $\boxed{\sigma_1 : U \to \mathbb{R}^3}$ be a single surface patch covering the surface S_1 and let $\boxed{\sigma_2 = f \circ \sigma_1}$ be a single surface patch covering the surface S_2.

Suppose that $f : S_1 \to S_2$ is a diffeomorphism.

First let us assume that the FFF's of σ_1 & σ_2 are proportional.

ie) let us take

$$E_2 du^2 + 2F_2 du dv + G_2 dv^2 = \lambda(E_1 du^2 + 2F_1 du dv + G_1 dv^2) \quad \text{(2)}$$

where λ is a smooth fn on U and also we take $\boxed{\lambda > 0}$ everywhere because E_1 & E_2 both are greater than zero.

Let us consider $\boxed{\gamma(t) = \sigma_1(u(t), v(t))}$ & $\boxed{\tilde{\gamma}(t) = \sigma_1(\tilde{u}(t), \tilde{v}(t))}$ to be two intersecting curves in S_1.

Then the diffeomorphism f takes γ & $\tilde{\gamma}$ to the curves $\boxed{\sigma_2(u(t), v(t))}$ & $\boxed{\sigma_2(\tilde{u}(t), \tilde{v}(t))}$ in S_2, respectively.

Let us find the angle of intersection of the curves in S_2 using eqn (1).

$$\cos\theta = \frac{E_2 \dot{u}\dot{\tilde{u}} + F_2(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G_2 \dot{v}\dot{\tilde{v}}}{(E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2)^{1/2}(E_2\dot{\tilde{u}}^2 + 2F_2\dot{\tilde{u}}\dot{\tilde{v}} + G_2\dot{\tilde{v}}^2)^{1/2}}$$

by (2),

$$E_2 du^2 + 2F_2 du dv + G_2 du^2 = \lambda E_1 du^2 + \lambda 2F_1 du dv + \lambda G_1 dv^2$$

By equating,

$$E_2 = \lambda E_1, \quad \cancel{2}F_2 = \lambda\cancel{2}F_1, \quad G_2 = \lambda G_1$$

$$\therefore \cos\theta = \frac{\lambda E_1 \dot{u}\dot{\tilde{u}} + \lambda F_1(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + \lambda G_1 \dot{v}\dot{\tilde{v}}}{(\lambda E_1\dot{u}^2 + 2\lambda F_1\dot{u}\dot{v} + \lambda G_1\dot{v}^2)^{1/2}(\lambda E_1\dot{\tilde{u}}^2 + 2\lambda F_1\dot{\tilde{u}}\dot{\tilde{v}} + \lambda G_1\dot{\tilde{v}}^2)^{1/2}}$$

$$= \frac{\cancel{\lambda}[E_1 \dot{u}\dot{\tilde{u}} + F_1(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G_1 \dot{v}\dot{\tilde{v}}]}{\cancel{\lambda^{1/2}}(E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2)^{1/2}\cancel{(\lambda)^{1/2}}(E_1\dot{\tilde{u}}^2 + 2F_1\dot{\tilde{u}}\dot{\tilde{v}} + G_1\dot{\tilde{v}}^2)^{1/2}}$$

$$= \frac{E_1 \dot{u}\dot{\tilde{u}} + F_1(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G_1 \dot{v}\dot{\tilde{v}}}{(E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2)^{1/2}(E_1\dot{\tilde{u}}^2 + 2F_1\dot{\tilde{u}}\dot{\tilde{v}} + G_1\dot{\tilde{v}}^2)^{1/2}}$$

= The angle b/w the intersecting curves in S_1.

$\Rightarrow f$ preserves angles.

Conversely,

For the converse, let us assume that f is ~~of intersection~~ conformal

ie) let us assume that the angle of intersection of curves in S_1 is equal to the angle of intersection of curves in S_2.

ie)
$$\frac{E_1 \dot{u}\dot{\tilde{u}} + F_1(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G_1 \dot{v}\dot{\tilde{v}}}{(E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2)^{1/2}(E_1\dot{\tilde{u}}^2 + 2F_1\dot{\tilde{u}}\dot{\tilde{v}} + G_1\dot{\tilde{v}}^2)^{1/2}}$$
$$= \frac{E_2 \dot{u}\dot{\tilde{u}} + F_2(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G_2 \dot{v}\dot{\tilde{v}}}{(E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2)^{1/2}(E_2\dot{\tilde{u}}^2 + 2F_1\dot{\tilde{u}}\dot{\tilde{v}} + G_2\dot{\tilde{v}}^2)^{1/2}} \quad \to (3)$$

Fix $(a,b) \in U$ and let us consider the following curves,

$\boxed{\gamma(t) = \sigma_1(a+t, b)}$, $\boxed{\tilde{\gamma}(t) = \sigma_1(a + t\cos\phi,\ b + t\sin\phi)}$ where ϕ is constant

For these curves,

$u(t) = a+t$; $v(t) = b$

$\tilde{u}(t) = a + t\cos\phi$; $\tilde{v}(t) = b + t\sin\phi$

$\dot{u} = 1$; $\dot{v} = 0$

$\dot{\tilde{u}} = \cos\phi$; $\dot{\tilde{v}} = \sin\phi$

Sub. in eqn (3),

$$\frac{E_1\cos\phi + F_1\sin\phi}{(E_1 + 2F_1(0) + G_1(0))^{1/2}(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)^{1/2}}$$
$$= \frac{E_2\cos\phi + F_1\sin\phi}{(E_2)^{1/2}(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)^{1/2}}$$

Squaring on both sides,

$$\frac{(E_1\cos\phi + F_1\sin\phi)^2}{E_1(E_1\cos^2\phi + 2F_1\sin\phi\cos\phi + G_1\sin^2\phi)} = \frac{(E_2\cos\phi + F_2\sin\phi)^2}{E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)}$$

Adding & subtracting ~~on the LHS~~ $\boxed{E_1G_1\sin^2\phi}$ **on the LHS** $\boxed{E_2G_2\sin^2\phi}$ **on the RHS**, we have

$$\frac{E_1^2\cos^2\phi + 2E_1F_1\cos\phi\sin\phi + F_1^2\sin^2\phi + E_1G_1\sin^2\phi - E_1G_1\sin^2\phi}{E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)}$$

$$= \frac{E_2^2\cos^2\phi + 2E_2F_2\cos\phi\sin\phi + F_2^2\sin^2\phi + E_2G_2\sin^2\phi - E_2G_2\sin^2\phi}{E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)}$$

$$\frac{E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi) - (E_1G_1 - F_1^2)\sin^2\phi}{E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)}$$

$$= \frac{E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi) - (E_2G_2 - F_2^2)\sin^2\phi}{E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)}$$

$$\cancel{1} - \frac{(E_1G_1 - F_1^2)\sin^2\phi}{E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)}$$

$$= \cancel{1} - \frac{(E_2G_2 - F_2^2)\sin^2\phi}{E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)}$$

$$(E_1G_1 - F_1^2)E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)$$

$$= (E_2G_2 - F_2^2)E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)$$

Let us take

$$\lambda = \frac{(E_2 G_2 - F_2^2) E_1}{(E_1 G_1 - F_1^2) E_2}$$

$$\Rightarrow E_2 \cos^2\varphi + 2F_2 \cos\varphi \sin\varphi + G_2 \sin^2\varphi$$
$$= \lambda (E_1 \cos^2\varphi + 2F_1 \cos\varphi \sin\varphi + G_1 \sin^2\varphi)$$

$$\therefore (E_2 - \lambda E_1)\cos^2\varphi + 2(F_2 - \lambda F_1)\cos\varphi \sin\varphi + (G_2 - \lambda G_1)\sin^2\varphi = 0 \rightarrow (4)$$

Let us take $\phi = 0$ and then $\phi = \pi/2$ in (4),

Sub $\phi = 0$

$$\Rightarrow (E_2 - \lambda E_1)\underbrace{\cos^2(0)}_{1} = 0$$

$\cos 0 = 1$
$\sin 0 = 0$

$$\Rightarrow E_2 = \lambda E_1 \rightarrow (5)$$

$\frac{\pi}{2} = 90°$

Sub $\phi = \pi/2$

$$\Rightarrow (G_2 - \lambda G_1)\underbrace{\sin^2(\pi/2)}_{1} = 0$$

$\sin 90° = 1$
$\cos 90° = 0$

$$\Rightarrow G_2 = \lambda G_1 \rightarrow (6)$$

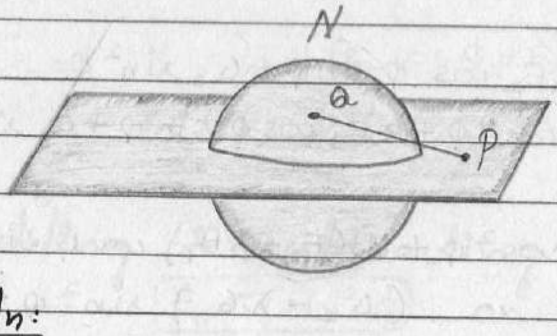
Sub (5) & (6) in (4),

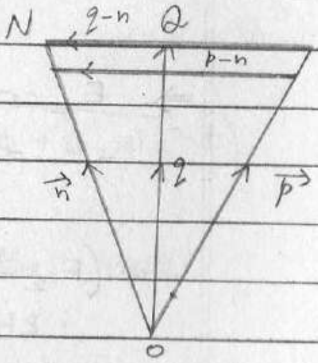
$$\Rightarrow F_2 = \lambda F_1 \rightarrow (7)$$

$\therefore$ (5), (6) & (7) proves the forms of propotional

Example: 5.7

The stereographic projection is conformal.





Soln:

Consider the unit sphere $x^2+y^2+z^2=1$.
If $\boxed{P=(u,v,0)}$ for any point on xy-plane drawn a straight line through P and the north pole $\boxed{N=(0,0,1)}$. This line intersect the sphere at point Q, say.
Every point Q of the sphere can be mapped onto a plane except for the north pole.

The vector $\overrightarrow{NQ}$ is $\|^{el}$ to the vector $\overrightarrow{NP}$,

$\therefore$ We can find a scalar, say ℓ, such that position vector $\overrightarrow{NP}$ & $\overrightarrow{NQ}$ can be related as $\vec{q}-\vec{n}=\ell(\vec{p}-\vec{n})$.

$$\therefore \vec{q}=\vec{n}+\ell(\vec{p}-\vec{n}) \quad \left[\because \ell\vec{p}-\ell\vec{n}=\ell(\vec{p}-\vec{n})\right]$$
$$=(0,0,1)+\ell\left[(u,v,0)-(0,0,1)\right]$$
$$=(0,0,1)+\ell(u,v,0)-\ell(0,0,1)$$
$$=\left(0+\ell u-(0)\ell,\ 0+\ell v-(0)\ell,\ 1+(0)\ell-(1)\ell\right)$$

$\Rightarrow$ ie.) $\vec{q}=(\ell u, \ell v, 1-\ell)$

Since $\vec{q}$ lies on the sphere which satisfy the eqn of the sphere

$$\boxed{(x^2+y^2+z^2)=1}$$

$$\Rightarrow (\rho u)^2+(\rho v)^2+(1-\rho)^2=1$$

$$\Rightarrow \rho^2u^2+\rho^2v^2+\cancel{1}-2\rho+\rho^2=\cancel{1}$$

$$\Rightarrow \rho^{\cancel{2}}(u^2+v^2+1)=2\cancel{\rho}$$

$$\Rightarrow \boxed{\rho=\frac{2}{(u^2+v^2+1)}}$$

$$\Rightarrow \vec{q}=\left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

Let us denote the R.H.S to be

$$\sigma_1(u,v)=\left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

and let us parametrise the plane as

$$\sigma_2^{(u,v)}=(u,v,0)$$

The map that takes Q to P takes $\sigma_1^{(u,v)}$ to $\sigma_2^{(u,v)}$ is called the **stereographic projection**.

We will now prove that it is conformal.

The co-efficient of FFF of σ_2 are E_2, F_2, G_2

where $E_2 = \sigma_2(u) . \sigma_2(u)$
$= (1,0,0).(1,0,0)$
$= (1,0,0) = 1$

$ds_2^2 = du^2 + dv^2$

$$\left\{\begin{aligned} E_2 &= \|\sigma_2(u)\|^2 = 1^2 + 0^2 + 0^2 = 1 \\ F_2 &= \sigma_{2u} \cdot \sigma_{2v} = (1,0,0)\cdot(0,1,0) = 0 \\ G_2 &= \|\sigma_{2v}\|^2 = 0^2 + 1^2 + 0^2 = 1 \end{aligned}\right.$$

The coefficient of FFF of σ_1 are E_1, F_1, G_1

where $E_1 = \sigma_{1u} \cdot \sigma_{1u}$

$$\sigma_{1u} = \left(\frac{(u^2+v^2+1)(2) - 2u(2u)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{(u^2+v^2+1)(2u) - (u^2+v^2-1)(2u)}{(u^2+v^2+1)^2} \right)$$

$$\cancel{2u^3} + \cancel{2uv^2} + 2u - \cancel{2u^3} - \cancel{2uv^2} + 2u = 4u$$

$$= \left(\frac{2v^2 - 2u^2 + 2}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)}, \frac{4u}{(u^2+v^2+1)^2} \right)$$

$$\sigma_{1u} \cdot \sigma_{1u} = \frac{2(v^2-u^2+1)}{(u^2+v^2+1)^2} \cdot \frac{2(v^2-u^2+1)}{(u^2+v^2+1)^2}$$

$$= \frac{4(v^4 - u^2v^2 + v^2 - u^2v^2 + u^4 - u^2 + v^2 - u^2 + 1)}{(u^2+v^2+1)^4}.$$

$$= \frac{4(v^4+u^4-2u^2v^2+2v^2-2u^2+1)}{(u^2+v^2+1)^4}$$

$$= \frac{4(u^2-v^2+1)^2}{(u^2+v^2+1)^4}$$

$$\sigma_{1v}\cdot\sigma_{1v} = \frac{16u^2v^2}{(u^2+v^2+1)^4}$$

$$\sigma_{1w}\cdot\sigma_{1w} = \frac{16u^2}{(u^2+v^2+1)^4}$$

$$\sigma_{1u}\cdot\sigma_{1u} = \frac{4v^4+4u^4-8u^2v^2+8u^2-8v^2+4+16u^2v^2+16u^2}{(u^2+v^2+1)^4}$$

$$= \frac{4v^4+4u^4+8u^2v^2+8u^2+8v^2+4}{(u^2+v^2+1)^4}$$

$$= \frac{4(u^2+v^2+1)^{\cancel{2}}}{(u^2+v^2+1)^{\cancel{4}\,2}} = \frac{4}{(u^2+v^2+1)^2}$$

$$F_1 = \sigma_{1u}\cdot\sigma_{1v}$$

$$\sigma_{1u} = \left[\frac{2(v^2-u^2+1)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right]$$

$$\sigma_{1v} = \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(u^2+v^2+1)-2v(2v)}{(u^2+v^2+1)^2}, \frac{(u^2+v^2+1)(2v)-(u^2+v^2-1)(2v)}{(u^2+v^2+1)^2}\right)$$

$$= \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(u^2-v^2+1)}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2}\right)$$

by thm 5.2,

conformal $\Leftrightarrow$ proportional

Here, $$du^2+dv^2 = \frac{4}{(u^2+v^2+1)^2}(du^2+dv^2)$$

$\Rightarrow$ proportional $\Leftrightarrow$ conformal

$$\sigma_{1u}\cdot\sigma_{1v} = \frac{-8uv(v^2-u^2+1)-8uv(u^2-v^2+1)+16uv}{(u^2+v^2+1)^4}$$

$$= \frac{\cancel{-8uv^3}+\cancel{8u^3v}-\cancel{8uv}-\cancel{8u^3v}+\cancel{8uv^3}-\cancel{8uv}+\cancel{16uv}}{(u^2+v^2+1)^4}$$

$$= 0$$

$$G_1 = \sigma_{1v}\cdot\sigma_{1v}$$

$$= \frac{-4uv(-4uv)+2(u^2-v^2+1)\cdot 2(u^2-v^2+1)+4v\cdot 4v}{(u^2+v^2+1)^4}$$

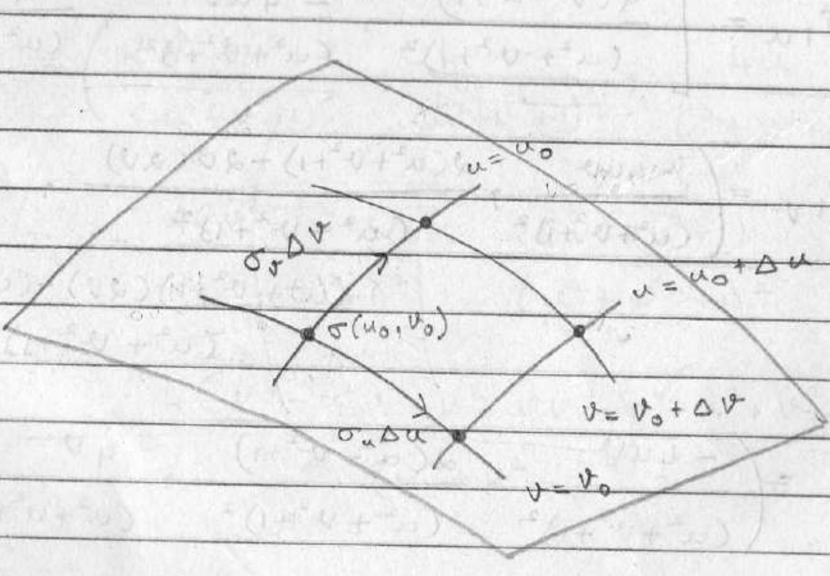
$$= \frac{16u^2v^2+4u^4+4v^4+4-4u^2v^2-4u^2v^2+4u^2-4v^2+4u^2-4v^2+16v^2}{(u^2+v^2+1)^4}$$

$$= \frac{4u^4+4v^4+4+8u^2v^2+8u^2+8v^2}{(u^2+v^2+1)^4}$$

$$= \frac{4(u^2+v^2+1)^2}{(u^2+v^2+1)^4} = \frac{4}{(u^2+v^2+1)^2}$$

$$ds_1^2 = \frac{4}{(u^2+v^2+1)^2}(du^2+dv^2)$$

3.4) Surface Area:



Suppose that $\sigma : U \to \mathbb{R}^3$ is a surface patch on a surface S. The image of σ is covered by two families of parameter curves obtained by setting $u = \text{const}$, $v = \text{const}$.

Fix a value $(u_0, v_0) \in U$ and let Δu and Δv be very small. Since the change in $\sigma(u, v)$ corresponding to a small change Δu in u is approximately $\sigma_u \Delta u$.

Similarly, corresponding to the small change Δv in v is approx. $\sigma_v \Delta v$.

The part of the surface contained by the parameter curves in the surface corresponding to $u = u_0$, $u = u_0 + \Delta u$, $v = v_0$ & $v = v_0 + \Delta v$ is almost a parallelogram in the plane with the sides given by the vectors $\sigma_u \Delta u$ and $\sigma_v \Delta v$.

Since area of parallelogram with sides $\bar{a}$ & $\bar{b}$ is given by $\|\bar{a} \times \bar{b}\|$, the area of the parallelogram on the considered surface is approx.

$$\|\sigma_u \Delta u \times \sigma_v \Delta v\| = \|\sigma_u \times \sigma_v\| \Delta u \Delta v$$

Defn: [Area]

The area $A_\sigma(R)$ of the part $\sigma(R)$ of the surface patch $\sigma: U \to \mathbb{R}^3$, corresponding to the region R contained in U is given by, ($R \subseteq U$)

$$A_\sigma(R) = \iint_R \|\sigma_u \times \sigma_v\| \, du \, dv.$$

Proposition 5.2

6m

$$\|\sigma_u \times \sigma_v\| = (EG - F^2)^{1/2}$$

Proof:

W.k.T, if $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are vectors in $\mathbb{R}^3$, then $(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (a.c)(b.d) - (a.d)(b.c)$

Applying this to $\|\sigma_u \times \sigma_v\|^2$,

$$\begin{aligned}\|\sigma_u \times \sigma_v\|^2 &= (\sigma_u \times \sigma_v)(\sigma_u \times \sigma_v) \\ &= (\sigma_u \cdot \sigma_u)(\sigma_v \cdot \sigma_v) - (\sigma_u \cdot \sigma_v)(\sigma_v \cdot \sigma_u) \\ &= EG - F^2\end{aligned}$$

$$\Rightarrow \|\sigma_u \times \sigma_v\| = (EG - F^2)^{1/2}$$

Proposition 5.3:

8m 6m

The area of the surface patch is unchanged by reparametrization.

20m

Prop 4.2 statement

Proof:

Let $\sigma : U \to \mathbb{R}^3$ be a surface patch and let $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be a reparametrisation of σ, with reparametrisation map $\Phi : \tilde{U} \to U$.

Thus, we have $\Phi(\tilde{u}, \tilde{v}) = (u, v)$ and
$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(u, v).$$

Let $\tilde{R} \subseteq \tilde{U}$ be a region and let $R = \Phi(\tilde{R}) \subseteq U$

Hence by defn of surface area, we have

$$A_\sigma(R) = \iint_R \|\sigma_u \times \sigma_v\| \, du\, dv$$

$$A_{\tilde{\sigma}}(\tilde{R}) = \iint_{\tilde{R}} \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| \, d\tilde{u}\, d\tilde{v}$$

$$= \iint_{\tilde{R}} \|\sigma_u \times \sigma_v\| \, |\det J(\Phi)| \, d\tilde{u}\, d\tilde{v} \quad (\text{by Prop. 4.2 eqn } 1)$$

$$= \iint_R \|\sigma_u \times \sigma_v\| \, du\, dv \quad \left[\text{by change of variable formula for double integral}\right]$$

the RHS of this eqn is exactly ←

$$= A_\sigma(R)$$

$\therefore$ Area of surface patch is unchanged.