To prove :

Uniqueness of g

It remains to prove that there is only one such function g.

Suppose $g_0|_B : \mathbb{R}^q \to \mathbb{R}^p$ be continuous function satisfying the necessary conditions of g, then we would have

$$f(g(w), w) = f(g_0(w), w)$$

$$\Rightarrow (g(w), w) = (g_0(w), w) \quad [\because f \text{ is } 1\text{-}1]$$

$$\Rightarrow \quad g(w) = g_0(w) \quad \forall\ w \in W$$

$\Rightarrow$ the C^∞ function g is unique.

Hence the proof.

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DIFFERENTIAL GEOMETRY

5.1) GAUSSIAN AND MEAN CURVATURES

Now we introduce 2 new measures of the curvature of a surface called Gaussian and mean curvatures. They have greater geometrical significance.

Definition:

Let κ_1 & κ_2 be the principle curvatures of a surface patch. Then, the Gaussian curvature of the surface patch is

kappa

$$\boxed{K = \kappa_1 \kappa_2}$$

and its mean curvature is

$$\boxed{H = \frac{1}{2}(\kappa_1 + \kappa_2)}$$

Examples:

For a plane surface, $K = 0.0$ $\Rightarrow K = 0$ and $H = \frac{1}{2}(0+0)$ $\Rightarrow H = 0$

For a cylinder, $K = \frac{1}{r}.0$ $\Rightarrow K = 0$ and $H = \frac{1}{2}(\frac{1}{r}+0)$ $\Rightarrow H = \frac{1}{2r}$, r is the radius of the circle.

For a sphere, $K = \frac{1}{r}\cdot\frac{1}{r}$ $\Rightarrow K = \frac{1}{r^2}$ and $H = \frac{1}{2}\left(\frac{1}{r}+\frac{1}{r}\right)$ $\Rightarrow H = \frac{1}{r}$

For a torus, K is positive on the outer surface, negative on the inner surface and zero on the top and bottom.

Preposition 7.1

Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms

$Edu^2 + 2Fdudv + Gdv^2$ & $Ldu^2 + 2Mdudv + Ndv^2$

respectively. Then

(i) $K = \dfrac{LN - M^2}{EG - F^2}$,

(ii) $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)}$

(iii) The principal curvatures are $H \pm \sqrt{H^2 - K}$

Proof: of Principle curvatures:

By defn, the principal curvatures of a surface patch are the roots of the equation,

$$\det(\mathcal{F}_{II} - \kappa \mathcal{F}_I) = 0$$

(ie)

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0$$

$$(L - \kappa E)(N - \kappa G) - (M - \kappa F)^2 = 0$$

$$LN - \kappa LG - \kappa NE + \kappa^2 EG - M^2 - \kappa^2 F^2 + 2\kappa MF = 0$$

$$\underbrace{(EG - F^2)}_{a}\kappa^2 - \underbrace{(LG - 2MF + NE)}_{b}\kappa + \underbrace{LN - M^2}_{c} = 0 \quad - (*)$$

This is a quadratic equation in κ.

(We know in a quadratic equation $a\kappa^2 + b\kappa + c = 0$, the sum of the roots is $-b/a$ and the product of the roots is c/a). So

i) $K = \kappa_1 \kappa_2 = \text{product of roots} = \dfrac{LN - M^2}{EG - F^2}$ — (1)

$-b/a$

ii) $H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}$ (sum of roots) $= \dfrac{LG - 2MF + NE}{2(EG - F^2)}$ — (2)

iii) By the definition H and K, κ_1 & κ_2 are the roots of $\boxed{\kappa^2 - 2H\kappa + K = 0}$ (sub (1) & (2) in (*))

(ie) the principal curvatures are $\dfrac{+2H \pm \sqrt{4H^2 - 4K}}{2}$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+2H \pm 2\sqrt{H^2 - K}}{2}$$

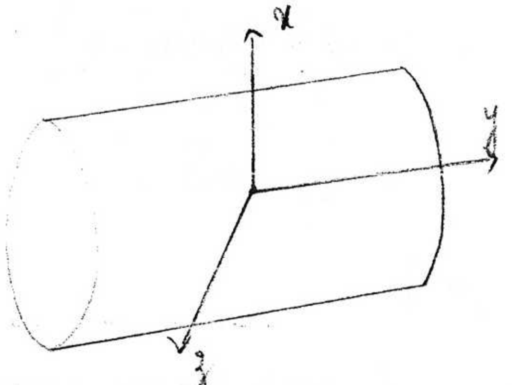
$$= \frac{+\cancel{2}(H \pm \sqrt{H^2 - K})}{\cancel{2}}$$

$$= H \pm \sqrt{H^2 - K}$$

Hence the proof.

* If one of the principal curvature is zero ; $K = K_1 K_2 = 0$, the Gaussian curvature is zero and the surface is said to have a parabolic point.

Most surfaces will contain regions of positive Gaussian curvature (elliptic points) and regions of negative Gaussian curvature separated by a curve of points with zero Gaussian curvature called a parabolic line.



Zero Curvature

Eg: 7.3

Eg: Ruled Surface

Result: "Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms $E\,du^2 + 2F\,du\,dv + G\,dv^2$ and $L\,du^2 + 2M\,du\,dv + N\,dv^2$ respectively. Then,

(i) $K = \dfrac{LN - M^2}{EG - F^2}$ (ii) Mean curvature, $H = \dfrac{K_1 + K_2}{2}$

(iii) the principal curvatures are $H \pm \sqrt{H^2 - K}$ "

A ruled surface is a surface that is a union of straight lines, called the rulings of the surface. Suppose C is a curve in $\mathbb{R}^3$ that meets each of these lines. Any point P of the surface lies on one of the given straight lines which intersects C at Q (say). If γ is a parametrisation of C with $\gamma(u) = Q$ and if $\vec{\delta}(u) \neq \vec{0}$ in the direction of the line passing through $\gamma(u)$, P has position vector of the form $\sigma(u,v) = \gamma(u) + v\vec{\delta}(u)$ for some scalar v.

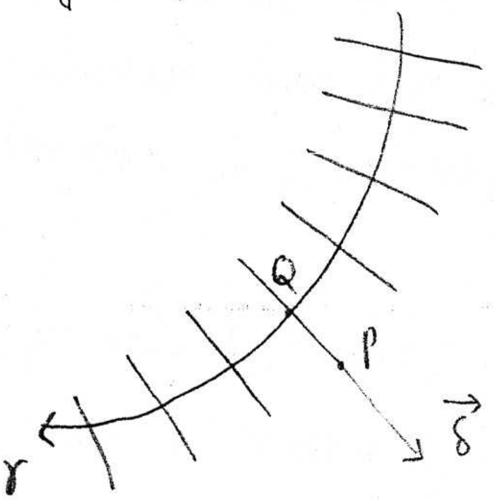
σ is regular if σ_u and σ_v are linearly independent.

$\sigma_u = \dot{\gamma} + v\dot{\vec{\delta}}$ $\qquad \sigma_v = \delta(u)$

$\sigma_{uu} = \ddot{\gamma} + v\ddot{\vec{\delta}}(u)$

$\sigma_{uv} = \dot{\vec{\delta}}$

$\sigma_{vv} = 0$



Unit normal, $\hat{N} = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

$L = \sigma_{uu} \cdot \hat{N} \qquad M = \sigma_{uv} \cdot \hat{N} \qquad N = \sigma_{vv} \cdot \hat{N} = 0$

$E = \|\sigma_u\|^2 \qquad F = \sigma_u \cdot \sigma_v \qquad G = \|\sigma_v\|^2$

Gaussian Curvature, $K = \dfrac{LN - M^2}{EG - F^2} = \dfrac{-M^2}{EG - F^2}$ (using result)

Consider, $EG - F^2 = \|\sigma_u\|^2 \|\sigma_v\|^2 - (\sigma_u \cdot \sigma_v)^2$ [using identity, $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d})$]

$= \|\sigma_u \times \sigma_v\|^2 \geq 0$

$\therefore K = \dfrac{-M^2}{EG - F^2} \leq 0$ i.e, Gaussian curvature of a ruled surface is negative

Surface of revolution.

A surface of revolution is the surface obtained by rotating a plane curve, called the profile curve, around a straight line; in a plane.

Eg 7.2

Consider a patch σ on a surface of revolution

$$\sigma(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u))$$

where we can assume that $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$ everywhere (a dot denoting d/du)

Then

$\sigma_u = (\dot{f}\cos v,\ \dot{f}\sin v,\ \dot{g})$

$\sigma_v = (-f\sin v,\ f\cos v,\ 0)$

$$\therefore E = \|\sigma_u\|^2$$
$$= \dot{f}^2\cos^2 v + \dot{f}^2\sin^2 v + \dot{g}^2$$
$$= \dot{f}^2 + \dot{g}^2$$
$$= 1$$

$$F = \sigma_u \cdot \sigma_v$$
$$= -\dot{f}f\cos v\sin v + \dot{f}f\cos v\sin v$$
$$= 0$$

$$G = \|\sigma_v\|^2$$
$$= f^2\sin^2 v + f^2\cos^2 v$$
$$= f^2(\sin^2 v + \cos^2 v)$$
$$= f^2$$

$$\hat{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

$$\sigma_u \times \sigma_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{f}\cos v & \dot{f}\sin v & \dot{g} \\ -f\sin v & f\cos v & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 - \dot{g} f \cos v) - \hat{j}\,(0 + \dot{g} f \sin v) + \hat{k}\,(\dot{f} f \cos^2 v + \dot{f} f \sin^2 v)$$

$$= -\dot{g} f \cos v\,\hat{i} - \dot{g} f \sin v\,\hat{j} + \dot{f} f\,\hat{k}$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{\dot{g}^2 f^2 \cos^2 v + \dot{g}^2 f^2 \sin^2 v + f^2 \dot{f}^2}$$

$$= \sqrt{\dot{g}^2 f^2 (\cos^2 v + \sin^2 v) + f^2 \dot{f}^2}$$

$$= \sqrt{\dot{g}^2 f^2 + f^2 \dot{f}^2}$$

$$= \sqrt{f^2 (\dot{g}^2 + \dot{f}^2)}$$

$$= \sqrt{f^2}$$

$$= f$$

$$\hat{N} = (-\dot{g}\cos v,\ -\dot{g}\sin v,\ \dot{f})$$

$$L = \sigma_{uu}\,\hat{N} \qquad M = \sigma_{uv} \cdot \hat{N} \qquad N = \sigma_{vv} \cdot \hat{N}$$

$$\sigma_{uu} = (\ddot{f}\cos v,\ f\sin v,\ \ddot{g})$$

$$\sigma_{vv} = (-f\cos v,\ -f\sin v,\ 0)$$

$$\sigma_{uv} = (-\dot{f}\sin v,\ \dot{f}\cos v,\ 0)$$

$$L = \sigma_{uu} \cdot \hat{N}$$
$$= (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}) \cdot (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f})$$
$$= -\ddot{f}\dot{g}\cos^2 v - \ddot{f}\dot{g}\sin^2 v + \dot{f}\ddot{g}$$
$$= -\ddot{f}\dot{g}(\cos^2 v + \sin^2 v) + \dot{f}\ddot{g}$$
$$= \dot{f}\ddot{g} - \ddot{f}\dot{g}$$

$$M = \sigma_{uv} \cdot \hat{N}$$
$$= (-\dot{f}\sin v, \dot{f}\cos v, 0) \cdot (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f})$$
$$= \dot{f}\dot{g}\sin v\cos v - \dot{f}\dot{g}\cos v\sin v$$
$$= 0$$

$$N = \sigma_{vv} \cdot \hat{N}$$
$$= (-f\cos v, -f\sin v, 0) \cdot (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f})$$
$$= f\dot{g}\cos^2 v + f\dot{g}\sin^2 v$$
$$= f\dot{g}(\cos^2 v + \sin^2 v)$$
$$= f\dot{g}$$

By proposition 7.1 i)

Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms $E\,du^2 + 2F\,du\,dv + G\,dv^2$ and $L\,du^2 + 2M\,du\,dv + N\,dv^2$ respectively Then,

i) $$K = \frac{LN - M^2}{EG - F^2}$$

where K is the gaussian curvature

Now,
$$E = 1, \quad F = 0 \quad G = f^2$$
$$L = \dot{f}\ddot{g} - \ddot{f}\dot{g}, \quad M = 0 \quad N = f\dot{g}$$

$$= \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2} \quad \text{---} \quad (1)$$

We can further simplify this by noting that $\dot{f}^2 + \dot{g}^2 = 1$ and by differentiating w.r.t 'u'

we have,
$$2\dot{f}\ddot{f} + 2\dot{g}\ddot{g} = 0$$
$$\Rightarrow \dot{g}\ddot{g} = -\dot{f}\ddot{f}$$

Consider
$$(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g} = \dot{f}\ddot{g}\dot{g} - \ddot{f}\dot{g}^2$$
$$= -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2$$
$$= -\ddot{f}(\dot{f}^2 + \dot{g}^2)$$
$$= -\ddot{f}$$

Substituting in (1).
$$K = \frac{-\ddot{f}f}{f^2} = -\frac{\ddot{f}}{f}$$

Eg 7.1

Special cases i) When the surface is a unit sphere we can take $f(u) = \cos u$, $g(u) = \sin u$

$\therefore \sigma(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$

Here $\dot{f} = -\sin u \qquad \dot{g} = \cos u$

$\ddot{f} = -\cos u \qquad \ddot{g} = -\sin u$

$\therefore$ Second fundamental form is given by

$$(\dot{f}\ddot{g} - \ddot{f}\dot{g})\,du^2 + f\dot{g}\,dv^2$$

$$= (\sin^2 u + \cos^2 u)\,du^2 + \cos^2 u\,dv^2$$

$$= du^2 + \cos^2 u\,dv^2$$

$\therefore L = 1, M = 0, N = \cos^2 u$

By definition, the principal curvatures are the roots of the equation $\det(\mathcal{F}_{II} - k\mathcal{F}_{I}) = 0$

(ie) $$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = 0$$

$$= \begin{vmatrix} 1-k & 0 \\ 0 & \cos^2 u(1-k) \end{vmatrix} = 0$$

$$= \cos^2 u(1-k)(1-k) = 0$$

$$\Rightarrow k = 1, 1$$

If we take $k_1 = 1$ and $k_2 = 1$,

The Gaussian curvature $K = k_1 k_2 = 1$ and

The mean curvature $H = \frac{1}{2}(k_1+k_2) = \frac{1}{2}(1+1) = \frac{2}{2} = 1$

ii) When the surface is a circular cylinder of unit radius. we take $f(u)=1$ and $g(u)=u$

$$\sigma(u,v) = (\cos v, \sin v, u)$$

$$\sigma_u(u,v) = (0,0,1)$$

$$\sigma_v = (-\sin v, +\cos v, 0)$$

$E = \|\sigma_u\|^2 = 1$ $\quad F = \sigma_u \cdot \sigma_v = 0$ $\quad G = \|\sigma_v\|^2 = \sin^2 v + \cos^2 v = 1$

$f(u) = 1$ $\quad g(u) = u$

$\dot{f} = 0$ $\quad \dot{g} = 1$

$\ddot{f} = 0$ $\quad \ddot{g} = 0$

S.F.F $\Rightarrow dv^2$

where $L=0$ $M=0$ $N=1$

The principal curvatures are the roots of the eqn

$\det(\mathcal{F}_{II} - k\mathcal{F}_I) = 0$

$$\begin{vmatrix} L-kE & M-kF \\ M-kF & N-kG \end{vmatrix} = 0$$

$$\begin{vmatrix} 0-k & 0 \\ 0 & 1-k \end{vmatrix} = 0$$

$\therefore -k(1-k) = 0 \Rightarrow k(k-1) = 0$

$k = 0$ or 1

If we take $k_1 = 0$ and $k_2 = 1$ or vice versa

Then the gaussian curvature is $K = k_1 k_2 = 0$ and

The mean curvature is $H = \frac{1}{2}(k_1+k_2) = \frac{1}{2}(0+1) = \frac{1}{2}$.

5.2) Pseudosphere

The pseudosphere

We have seen so far some surfaces of zero and constant +ve curvature. For an example of a surface with constant -ve gaussian curvature, however, we have to construct a new surface. To this end, we examine again the surface of revolution

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

WKT its gaussian curvature is

$$K = -\frac{\ddot{f}}{f} \quad — (2)$$

suppose first that $K=0$ everywhere. Then eqn (2) gives $\ddot{f}=0$, so $f(u) = au+b$ for some constants a & b. Since $\dot{f}^2+\dot{g}^2=1$, we get

$\dot{g} = \pm\sqrt{1-a^2}$ (so we must have $|a|\le 1$)

& hence $g(u) = \pm\sqrt{1-a^2}\,u+c$, where c is another constant. By applying a translation along the z-axis we can assume that $c=0$, and by applying a rotation about the x-axis (say) we can assume that the sign is +. This gives the ruled surface

$$\sigma(u,v) = (b\cos v, b\sin v, 0) + u(a\cos v, a\sin v, \sqrt{1-a^2})$$

If $a=0$ this is a circular cylinder, if $|a|=1$ it is the xy-plane, and if $0<|a|<1$ it is part of a cone (to see this, put $\bar{u} = au+b$)

Now suppose that $K=1$ everywhere (Any surface with constant +ve gaussian curvature can be reduced to this case by applying a dilation of $\mathbb{R}^3$). Then eqn. (2) becomes

$$\ddot{f} + f = 0$$

which has the general solution

$$f(u) = a\cos(u+b), \text{ where } a \text{ \& } b \text{ are constant}$$

We can assume that $b=0$ by performing a reparametrisation $\acute{u} = u+b$, $\acute{v} = v$. Then up to a change of sign and adding a constant,

$$g(u) = \int \sqrt{1-a^2\sin^2 u}\, du$$

This integral cannot be evaluated in terms of 'elementary functions unless $a=0$ or ± 1. The case $a=0$ does not give a surface, so we consider the case $a=1$ (the case $a=-1$ can be reduced to this by rotating the surface by π around the z-axis). Then,

$$f(u) = \cos u, \quad g(u) = \sin u$$ and we have the unit sphere

Suppose finally that $k=-1$. The general solution of eqn. (2) is then

$$f(u) = ae^{u} + be^{-u},$$

where a & b are arbitrary constants. For most values of a & b we cannot express g in terms of elementary functions, so we consider only the case $a=1$ & $b=0$. Then $f(u) = e^{u}$ & we can take

$$g(u) = \int \sqrt{1-e^{2u}}\, du \quad \text{——— (3)}$$

Note that we must have $u \leq 0$ for the integral in eqn(3) to make sense, since otherwise $1-e^{2u}$ would be $-ve$.

The integral in (3) can be evaluated by putting $v = e^{u}$. Then

$v = e^{u}$
$dv = e^{u} du$

$$\int \sqrt{1-e^{2u}}\, du = \int \sqrt{1-v^2}\, \frac{dv}{v}$$

$$= \int \frac{\sqrt{1-v^2}}{v}\, dv$$

$$= \int \frac{\sqrt{1-v^2}\, dv}{v\sqrt{1-v^2}\sqrt{1-v^2}}$$

$$= \int \left(\frac{1-v^2}{v}\right) \frac{dv}{\sqrt{1-v^2}}$$

$$= \int \left(\frac{1}{v} - v\right) \frac{dv}{\sqrt{1-v^2}}$$

$$= \sqrt{1-v^2} + \int \frac{dv}{v\sqrt{1-v^2}}$$

Put $\omega = v^{-1}$ in the last integral

$w = \frac{1}{v}$
$dw = \frac{-1}{v^2} dv$

$$\Rightarrow \int \sqrt{1-e^{2u}}\, du = \sqrt{1-v^2} + \int \frac{-v^2\, d\omega}{v\sqrt{1-\frac{1}{\omega^2}}}$$

$$= \sqrt{1-v^2} - \int \frac{v^2 d\omega}{v\, \frac{\sqrt{\omega^2-1}}{\omega}}$$

$$= \sqrt{1-v^2} - \int \frac{v^2\, d\omega}{v^2\sqrt{\omega^2-1}}$$

$$= \sqrt{1-v^2} - \int \frac{d\omega}{\sqrt{\omega^2-1}}$$

$$= \sqrt{1-v^2} - \cosh^{-1} w$$

$$= \sqrt{1-v^2} - \cosh^{-1}\left(\frac{1}{v}\right)$$

$$\int \sqrt{1-e^{2u}}\,du = \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})$$

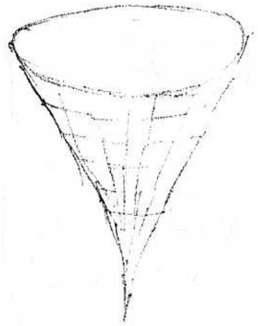
we have omitted the arbitrary constant, but we can take it to be zero suitably translating the surface in the direction of the z-axis.

$$\Rightarrow \quad f(u) = e^{u} \quad , \quad g(u) = \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})$$

~~putting~~ $x = f(u)$, $z = g(u)$, we see that the profile curve in the xz-plane has eqn.

$$\boxed{z = \sqrt{1-x^2} - \cosh^{-1}\left(\frac{1}{x}\right)} \quad \text{--- (4)}$$

Rotating this curve around the z-axis thus gives a surface called the pseudosphere, which has gaussian curvature -1 everywhere. Note that since $u \le 0$, $x = e^{u}$ is restricted to the range $0 < x \le 1$.



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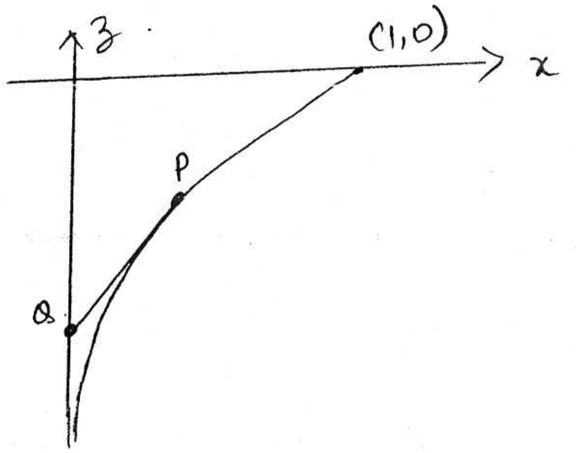
The curve defined by eqn. (4) is called the <u>tractrix</u> & it has an interesting geometric property.

Consider the tangent line at a point P of its graph and suppose that it intersects the z-axis at the point Q. Let us compute the distance PQ.



Suppose that P is a point (x_0, z_0). Either by a direct calculation or by inspecting the calculation of the integral (3), we finds that

$$\frac{dz}{dx} = \frac{\sqrt{1-x^2}}{x}$$

Hence the tangent line at P has equation

$$z - z_0 = \frac{\sqrt{1-x_0^2}}{x_0}(x - x_0)$$

This meets the z-axis at the point $(0, z_1)$,

where
$$z_1 - z_0 = \frac{\sqrt{1-x_0^2}}{x_0}(0 - x_0) = -\sqrt{1-x_0^2}.$$

Hence the distance PQ is given by
$$(PQ)^2 = x_0^2 + (z_1 - z_0)^2 = x_0^2 + 1 - x_0^2$$
$$= 1.$$

i.e the distance PQ is constant & equal to one.

Proposition 7.3:

Let P be a point of a flat surface S, and assume that P is not an umbilic. Then, there is a patch of S containing P that is a ruled surface.

Proof: Consider a patch $\sigma: U \to \mathbb{R}^3$ containing P as in proposition 7.2 which states that,

"Let P be a point of a surface S, and suppose that P is not an umbilic. Then, there is a surface patch $\sigma(u,v)$ of S containing P whose first and second fundamental forms are

$$E\,du^2 + G\,dv^2 \quad \text{and} \quad L\,du^2 + N\,dv^2, \text{ respectively,}$$

for some smooth functions E, G, L and N."

Say, $P = \sigma(u_0, v_0)$.

By proposition 7.1, which states that, "Let $\sigma(u,v)$ be a surface patch with first and second fundamental forms $E\,du^2 + 2F\,du\,dv + G\,dv^2$ and $L\,du^2 + 2M\,du\,dv + N\,dv^2$, respectively

Then, (i) $K = \dfrac{LN - M^2}{EG - F^2}$

(ii) $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)}$

(iii) the principal curvatures are $H \pm \sqrt{H^2 - K}$."

Thus, the gaussian curvature $K = \frac{LN}{EG}$.

Since the gaussian curvature is zero everywhere, either $L=0$ or $N=0$ at each point of U, and since P is not an umbilic L and N are not both zero.

Suppose that $L(u_0, v_0) \neq 0$. Then $L(u,v) \neq 0$ for (u,v) in some open subset of U containing (u_0, v_0).

We shall prove that the parameter curves $u=$ constant are straight lines. Such a curve can be parametrised by

$v \mapsto \sigma(u_0, v)$ where u_0 is the constant value of u.

A unit tangent vector to this curve is $\hat{t} = \frac{\sigma_v}{G^{1/2}}$,

so by proposition 1.1 which states that,

"If the tangent vector of parametrised curve is constant, the image of the curve is (part of) a straight line."

we need to prove that $\hat{t}_v = 0$.

By proposition 6.4, which states that,

"Let $\hat{N}$ be the standard unit normal of a surface patch $\sigma(u,v)$. Then, $N_u = a\sigma_u + b\sigma_v$, $N_v = c\sigma_u + d\sigma_v$,

where $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}$. The matrix $\mathcal{F}_I^{-1}\mathcal{F}_{II}$ is called the Weingarten matrix of the surface patch σ, and is denoted by $\mathcal{W}$."

the derivatives of the unit normal are as follows:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_I^{-1}\mathcal{F}_{II} = -\frac{1}{EG}\begin{pmatrix} G & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & N \end{pmatrix} = -\begin{pmatrix} 1/E & 0 \\ 0 & 1/G \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & N \end{pmatrix}$$

$$= -\begin{pmatrix} L/E & 0 \\ 0 & N/G \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\begin{pmatrix} L/E & 0 \\ 0 & N/G \end{pmatrix}$$

$$N_u = a\sigma_u + b\sigma_v = -\frac{L}{E}\sigma_u + 0\sigma_v$$

$$\Rightarrow N_u = -E^{-1}L\sigma_u.$$

$$N_v = c\sigma_u + d\sigma_v = 0\sigma_u - \frac{N}{G}\sigma_v = 0 \quad (\text{since } N = 0)$$

$$\Rightarrow N_v = 0.$$

Hence, $t_v \cdot \sigma_u = t_v \cdot (-EL^{-1}N_u) = -EL^{-1}t_v \cdot N_u$

Now, $\hat{t} \cdot N_u = 0$ (since t and N_u are $\perp^r$ to each other) and

$N_{uv} = 0$

We know that $\hat{t} \cdot N_u = 0$

$$\therefore (\hat{t} \cdot N_u)' = 0$$

$$t_v \cdot N_u + t \cdot N_{uv} = 0$$

$$\Rightarrow t_v \cdot N_u = -t \cdot N_{uv} = 0$$

Hence $t_v \cdot \sigma_u = -EL^{-1}t_v \cdot N_u = 0$

Next, $t_v \cdot t = 0$. Since t is a unit vector by proposition 1.2, which states that, " Let $\hat{n}(t)$ be a unit vector that is a smooth function of a parameter t. Then, the dot product $\dot{\hat{n}}(t) \cdot \hat{n}(t) = 0 \ \forall t$, ie. $\dot{n}(t)$ is zero or perpendicular to $\hat{n}(t)$ $\forall t$.

In particular, if γ is a unit-speed curve, then $\dot{\gamma}$ is zero or perpendicular to $\ddot{\gamma}$"

Hence $t_v \cdot \sigma_v = 0$

We know that $t \cdot N = 0$

$$(t \cdot N)' = 0$$

$$t_v \cdot N + t \cdot N_v = 0$$

$$t_v \cdot N = -t \cdot N_v = 0$$

Since the vectors σ_u, σ_v and N form a basis of $\mathbb{R}^3$, we have proved that $t_v = 0$.

Hence the proof.

To Describe the structure of flat ruled surfaces

no need

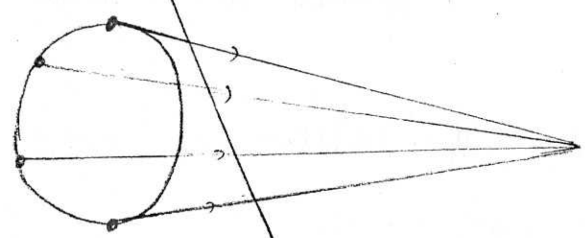
Ruled Surface :

A Surface S is said to be ruled if through every point of S there is a straight line that lies on S.

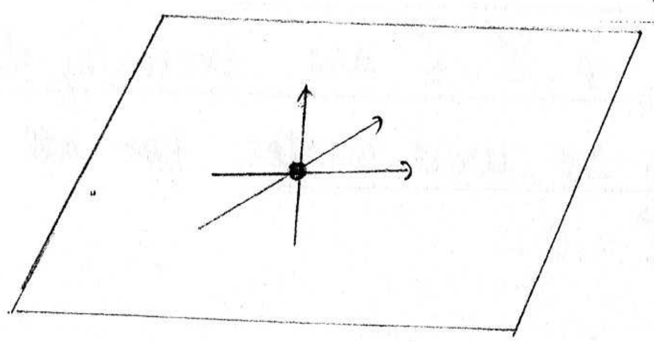
So a ruled surface can be described as a set of points swept by a moving straight line.

For eg :

(i) A cone is formed by keeping one point of a line fixed while moving another point along a circle.



(ii) Plane is the only surface which contains atleast 3 distinct lines through each of its points.



The parametrisation of the ruled surface is given by $\sigma(u,v) = \gamma(u) + v\vec{\delta}(u) \longrightarrow (1)$

where $\sigma : U \longrightarrow \mathbb{R}^3$ ×

$\left[U \subseteq \mathbb{R}^2 \text{ and } (u,v) \in U.\right]$

γ is the curve ~~that~~ describes the surface

$\vec{\delta}$ is the non-zero vector in the direction of the line passing through the point $\gamma(u)$. ×

Our task is to describe the structure of flat ruled surfaces. pg. 156

The parametrisation of ruled surface is given by (1)

$\sigma(u,v) = \gamma(u) + v\vec{\delta}(u)$

$\Rightarrow \sigma_u = \dot{\gamma} + v\dot{\vec{\delta}}$ ×

$\sigma_v = \vec{\delta}$. (where dot denotes $\frac{d}{du}$)

The Gaussian curvature is zero for σ

iff $\dot{\vec{\delta}} \cdot (\sigma_u \times \sigma_v) = 0$.

Since $\sigma_u \times \sigma_v = \dot{\gamma} \times \vec{\delta} + v\dot{\vec{\delta}} \times \vec{\delta}$

and $\dot{\vec{\delta}} \cdot (\dot{\vec{\delta}} \times \vec{\delta}) = 0$

Thus $\underline{K = 0 \text{ iff } \dot{\vec{\delta}} \cdot (\dot{\gamma} \times \vec{\delta}) = 0}$

Thus $\underline{K = 0 \text{ iff } \dot{\gamma}, \vec{\delta}, \dot{\vec{\delta}} \text{ are linearly dependent.}}$

Assuming $\vec{\delta}(u)$ to be unit vector for all values of 'u'. Then $\vec{\delta} \cdot \dot{\vec{\delta}} = 0$.

Suppose first that $\dot{\vec{\delta}}(u) = 0$ for all values of 'u'. Then, $\vec{\delta}$ is a constant vector + σ is generalised cylinder.

Suppose now that $\dot{\vec{\delta}} \neq 0$. Then $\vec{\delta}$ + $\dot{\vec{\delta}}$ are linearly independent as they're non-zero and perpendicular. So $\dot{r}, \vec{\delta}, \dot{\vec{\delta}}$ which are linearly dependent becomes

$$\dot{r}(u) = f(u)\,\vec{\delta}(u) + g(u)\,\dot{\vec{\delta}}(u) \quad — (2)$$

where 'f' + 'g' are smooth functions.

Assuming $f = \dot{g}$ everywhere.

$\therefore$ (2) $\Rightarrow$ $\dot{r} = (g\vec{\delta})^{\cdot}$

$\Rightarrow$ $r = g\vec{\delta} + \vec{a}$ where $\vec{a}$ is a constant vector

$\therefore$ (1) $\Rightarrow$ $\sigma(u,v) = (g\vec{\delta} + \vec{a}) + v\vec{\delta}$

$\Rightarrow$ $\boxed{\sigma(u,v) = \vec{a} + (v + g(u))\,\vec{\delta}(u)}$ — (3)

Putting $\tilde{u} = u$ and $\tilde{v} = v + g(u)$

$\therefore$ (3) $\Rightarrow$ $\boxed{\sigma(u,v) = \vec{a} + \tilde{v}\,\vec{\delta}(\tilde{u})}$ — (4)

(4) is the reparametrisation of a generalised cone.

Generalised cone :

A ruled surface is called a generalised cone if it can be parametrised as

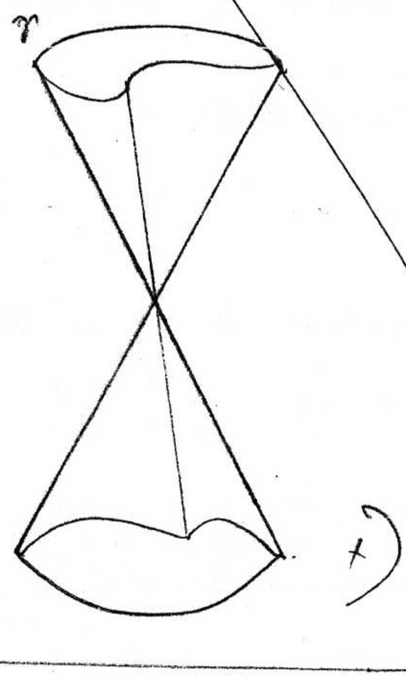
$$\sigma(u,v) = \vec{p} + v\, r(u)$$

where $\vec{p}$ = vertex

Also $r \times \dot{r} \neq 0$

In ④, $\vec{\delta}$ and $\dot{\vec{\delta}}$ are linearly independent

$$\Rightarrow \tilde{v}\, \vec{\delta} \times \dot{\vec{\delta}} \neq 0$$



suppose $\dot{\vec{\delta}} \neq 0$ and $f - \dot{g} \neq 0$ (i.e $f \neq \dot{g}$)

Define $\tilde{r}(u) = r(u) - g(u)\,\vec{\delta}(u)$ for all values of 'u'. — ⑤

and $\tilde{v} = \dfrac{v + g(u)}{f(u) - \dot{g}(u)}$ — ⑥

⑤ $\Rightarrow$ $r(u) = \tilde{r}(u) + g(u)\,\vec{\delta}(u)$ — ⑦

for all values of u

(6) $\Rightarrow$ $v = \tilde{v} f(u) - \tilde{v}\dot{g}(u) - g(u)$ — (8)

$\therefore$ (1) $\Rightarrow$ $\sigma(u,v) = \tilde{\gamma}(u) + \cancel{g(u)\vec{\delta}(u)}$

$+ \vec{\delta}(u)\left[\tilde{v} f(u) - \tilde{v}\dot{g}(u) - \cancel{g(u)}\right]$

$\Rightarrow$ $\sigma(u,v) = \tilde{\gamma}(u) + \tilde{v}\,\vec{\delta}(u)[f(u) - \dot{g}(u)]$ — (9).

From (7) $\dot{\gamma}(u) = \dot{\tilde{\gamma}}(u) + \dot{g}(u)\vec{\delta}(u) + g(u)\overset{\circ}{\vec{\delta}}(u)$. — (10)

Substitute (2) in the above eq^n (10)

$\Rightarrow$ $f(u)\vec{\delta}(u) + \cancel{g(u)\dot{\vec{\delta}}(u)} = \dot{\tilde{\gamma}}(u) + \dot{g}(u)\vec{\delta}(u)$

$+ \cancel{g(u)\dot{\vec{\delta}}(u)}$

$\Rightarrow$ $\dot{\tilde{\gamma}}(u) = [f(u) - \dot{g}(u)]\,\delta(u)$ — (11)

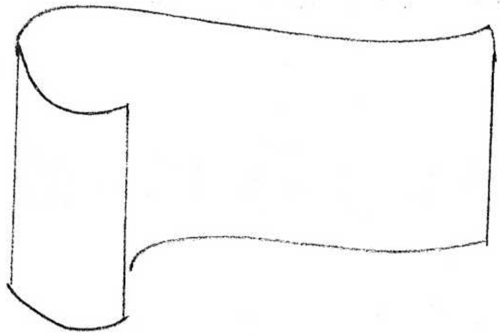
Use (11) in (9)

$\Rightarrow$ $\sigma(u,v) = \tilde{\gamma}(u) + \tilde{v}\,\dot{\tilde{\gamma}}(u)$ — (12)

(12) $\Rightarrow$ σ is a reparametrisation of part of the tangent devolapable of $\tilde{\gamma}$.

(Tangent devolapable : It is the union of tangent lines to a curve.)

Depending on certain conditions on $\vec{\delta}$, f and g we have shown that parts of the surface corresponding to certain subsets of U are parts of generalised cylinder, generalised cone and Tangent Devolapable. None of these conditions have to be necessarily satisfied. Its not true that the whole surface must be either of the 3 types. Since flat surfaces of different types can be joined together to make a smooth surface.



In general a flat surface is a patchwork consisting of pieces of generalised cylinders, generalised cones and Tangent Devolapables joined together along segments of straight lines.

— × —

$$\gamma(t) = \vec{a}\,t + \vec{b}\text{, where } \vec{a} \text{ and } \vec{b} \text{ are constant vectors.}$$

$$\dot{\gamma}(t) = \vec{a}$$

$$\ddot{\gamma} = 0.$$

i.e, A curve is a straight line if it has zero acceleration everywhere.

8(b) 5.4) Geodesics defn & Basic Properties

DEFINITION:

A curve γ on a surface S is called a <u>geodesic</u>, if $\ddot{\gamma}(t)$ is zero or perpendicular to the surface at the point $\gamma(t)$, i.e, parallel to its unit normal for all values of the parameter t.

EXAMPLE:

A particle moving on the surface and subject to no forces except a force acting perpendicular to the surface that keeps the particle on the surface, would move along a geodesic. This is because of Newton's second law of motion which says that the force on the particle is parallel to its acceleration $\ddot{\gamma}$, which would therefore be perpendicular to the surface.

PROPOSITION : 8.1 :

Any geodesic has constant speed.

PROOF:

Let $\gamma(t)$ be a geodesic on a surface S. Then, denoting $\frac{d}{dt}$ by a dot,

$$\|\dot{\gamma}\|^2 = \dot{\gamma}.\dot{\gamma}$$

$$\frac{d}{dt}\|\dot{\gamma}\|^2 = \dot{\gamma}.\ddot{\gamma} + \ddot{\gamma}.\dot{\gamma}$$

$$= 2\ddot{\gamma}.\dot{\gamma}$$

Since, γ is a geodesic, $\ddot{\gamma}$ is perpendicular to the tangent plane and is therefore perpendicular to the tangent vector $\dot{\gamma}$.

$$\Rightarrow \ddot{\gamma}.\dot{\gamma} = 0$$

This shows that, $\frac{d}{dt}\|\dot{\gamma}\|^2 = 0$

Integrating we get

$$\|\dot{\gamma}\| = \text{constant}$$

$\therefore$ Any geodesic has constant speed.

— x —

5.5) Geodesics Equation

Definition of Geodesics:

A curve γ on a surface S is called a geodesic if $\ddot{\gamma}(t)$ is zero or perpendicular to the surface at the point $\gamma(t)$, (ie) parallel to its unit normal, for all values of the parameter t.

Theorem 8.1: (State & Prove Geodesic eqns)

A curve γ on a surface S is a geodesic iff, for any part $\boxed{\gamma(t) = \sigma(u(t), v(t))}$ of γ contained in a surface patch σ of S, the following two equations are satisfied:

$$\left.\begin{aligned} \frac{d}{dt}(E\dot{u}+F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2), \\ \frac{d}{dt}(F\dot{u}+G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2), \end{aligned}\right\} \text{——} \; ①$$

where $Edu^2 + 2F du\,dv + G dv^2$ is the first fundamental form of σ. The differential equations ① are called the geodesic equations.

Proof:

First let us consider γ to be a geodesic, iff $\ddot{\gamma}$ is perpendicular to σ_u and σ_v, Since $\{\sigma_u, \sigma_v\}$ is a basis of the tangent plane of σ.

(ie) $\ddot{\gamma}\cdot\sigma_u = 0$ and $\ddot{\gamma}\cdot\sigma_v = 0$.

$\Rightarrow \left(\frac{d}{dt}\dot{\gamma}\right)\cdot\sigma_u = 0$ and $\left(\frac{d}{dt}(\dot{\gamma})\right)\cdot\sigma_v = 0$.

Here $\gamma(t) = \sigma(u(t), v(t))$ $\quad (\because t \to (u,v))$

$\Rightarrow \dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$

Hence we have,

$$\left(\frac{d}{dt}(\dot{u}\sigma_u+\dot{v}\sigma_v)\right)\cdot\sigma_u=0 \quad \text{and} \quad \left(\frac{d}{dt}(\dot{u}\sigma_u+\dot{v}\sigma_v)\right)\cdot\sigma_v=0 \qquad \text{--- (2)}$$

Now we have to show that these 2 equations are equivalent to the two geodesic equations.

Now consider the left-hand side of the 1st equation in (2),

$$\left(\frac{d}{dt}(\dot{u}\sigma_u+\dot{v}\sigma_v)\right)\cdot\sigma_u=\frac{d}{dt}\Big((\dot{u}\sigma_u+\dot{v}\sigma_v)\cdot\sigma_u\Big)-(\dot{u}\sigma_u+\dot{v}\sigma_v)\cdot\frac{d\sigma_u}{dt}$$

$$\left[\because \frac{d}{dt}(u\cdot v)=\frac{du}{dt}\cdot v+u\cdot\frac{dv}{dt}\right.$$

$$\left.\Rightarrow \frac{du}{dt}\cdot v=\frac{d}{dt}(u\cdot v)-u\cdot\frac{dv}{dt}\right]$$

$$\therefore \left(\frac{d}{dt}(\dot{u}\sigma_u+\dot{v}\sigma_v)\right)\cdot\sigma_u$$

$$=\frac{d}{dt}\Big(\dot{u}\underbrace{(\sigma_u\cdot\sigma_u)}_{E}+\dot{v}\underbrace{(\sigma_u\cdot\sigma_v)}_{F}\Big)-(\dot{u}\sigma_u+\dot{v}\sigma_v)\cdot\left(\frac{\partial\sigma_u}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\sigma_u}{\partial v}\frac{\partial v}{\partial t}\right)$$

$$=\frac{d}{dt}(E\dot{u}+F\dot{v})-(\dot{u}\sigma_u+\dot{v}\sigma_v)\cdot(\sigma_{uu}\dot{u}+\sigma_{uv}\dot{v})$$

$$=\frac{d}{dt}(E\dot{u}+F\dot{v})-\Big[\big(\dot{u}^2(\sigma_u\cdot\sigma_{uu})\big)+\dot{u}\dot{v}(\sigma_u\cdot\sigma_{uv})+\dot{u}\dot{v}(\sigma_v\cdot\sigma_{uu})+\dot{v}^2(\sigma_v\cdot\sigma_{uv})\Big]$$

where $E=\sigma_u\cdot\sigma_u$

$F=\sigma_u\cdot\sigma_v$

$$=\frac{d}{dt}(E\dot{u}+F\dot{v})-\Big[\dot{u}^2\underbrace{(\sigma_u\cdot\sigma_{uu})}_{A}+\dot{u}\dot{v}\underbrace{(\sigma_u\cdot\sigma_{uv}+\sigma_v\cdot\sigma_{uu})}_{B}+\dot{v}^2\overbrace{(\sigma_v\cdot\sigma_{uv})}^{C}\Big] \qquad \text{--- (3)}$$

Now,

$$E_u=(\sigma_u\cdot\sigma_u)_u=\sigma_{uu}\cdot\sigma_u+\sigma_u\cdot\sigma_{uu}=2(\sigma_u\cdot\sigma_{uu})$$

$$\Rightarrow \boxed{\sigma_u\cdot\sigma_{uu}=\tfrac{1}{2}E_u} \qquad \text{--- (4)}$$

A

$$F_u = (\sigma_u\cdot\sigma_v)_u = \sigma_{uu}\cdot\sigma_v + \sigma_u\cdot\sigma_{uv}$$

B

$$\Rightarrow \boxed{\sigma_u\cdot\sigma_{uv} + \sigma_v\cdot\sigma_{uu} = F_u} \quad \text{— (5)}$$

$$G_u = (\sigma_v\cdot\sigma_v)_u = \sigma_{uv}\cdot\sigma_v + \sigma_v\cdot\sigma_{uv} = 2(\sigma_v\cdot\sigma_{uv})$$

where $G = \sigma_v\cdot\sigma_v$

C

$$\Rightarrow \boxed{\sigma_v\cdot\sigma_{uv} = \tfrac{1}{2} G_u} \quad \text{— (6)}$$

Substituting (4), (5) & (6) in (3), we get

$$\left(\frac{d}{dt}(\dot u\sigma_u + \dot v\sigma_v)\right)\cdot\sigma_u = \frac{d}{dt}(E\dot u + F\dot v) - \left[\dot u^2\left(\tfrac{1}{2}E_u\right) + \dot u\dot v(F_u) + \dot v^2\left(\tfrac{1}{2}G_u\right)\right]$$

$$\left(\frac{d}{dt}(\dot u\sigma_u + \dot v\sigma_v)\right)\cdot\sigma_u = \frac{d}{dt}(E\dot u + F\dot v) - \frac{1}{2}\left(E_u\dot u^2 + 2F_u\dot u\dot v + G_u\dot v^2\right).$$

This shows that the first equation in (2) is equivalent to the first geodesic equation in (1).

Now consider the left-hand side of 2nd equation in (2),

$$\left(\frac{d}{dt}(\dot u\sigma_u + \dot v\sigma_v)\right)\cdot\sigma_v = \frac{d}{dt}\left((\dot u\sigma_u + \dot v\sigma_v)\cdot\sigma_v\right) - (\dot u\sigma_u + \dot v\sigma_v)\cdot\frac{d\sigma_v}{dt}$$

$$= \frac{d}{dt}\left(\dot u(\sigma_u\cdot\sigma_v) + \dot v(\sigma_v\cdot\sigma_v)\right) - (\dot u\sigma_u + \dot v\sigma_v)\cdot\left(\frac{\partial\sigma_v}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\sigma_v}{\partial v}\frac{\partial v}{\partial t}\right)$$

$$= \frac{d}{dt}(F\dot u + G\dot v) - (\dot u\sigma_u + \dot v\sigma_v)\cdot(\sigma_{uv}\dot u + \sigma_{vv}\dot v).$$

$$= \frac{d}{dt}(F\dot u + G\dot v) - \left[\dot u^2(\sigma_u\cdot\sigma_{uv}) + \dot u\dot v(\sigma_u\cdot\sigma_{vv}) + \dot u\dot v(\sigma_v\cdot\sigma_{uv}) + \dot v^2(\sigma_v\cdot\sigma_{vv})\right]$$

where $F = \sigma_u\cdot\sigma_v$

$G = \sigma_v\cdot\sigma_v$

$$\left(\frac{d}{dt}(\dot u\sigma_u + \dot v\sigma_v)\right)\cdot\sigma_v = \frac{d}{dt}(F\dot u + G\dot v) - \Big[\dot u^2\overbrace{(\sigma_u\cdot\sigma_{uv})}^{A(i)} + \dot u\dot v\underbrace{(\sigma_u\sigma_{vv} + \sigma_v\cdot\sigma_{uv})}_{B(i)} + \overbrace{\dot v^2(\sigma_v\cdot\sigma_{vv})}^{C(i)}\Big] \quad \text{— (7)}$$

Now,

$$E_v = (\sigma_u \cdot \sigma_u)_v = \sigma_{uv} \cdot \sigma_u + \sigma_u \cdot \sigma_{uv} = 2(\sigma_u \cdot \sigma_{uv})$$

A(i) $$\Rightarrow \boxed{\sigma_u \cdot \sigma_{uv} = \tfrac{1}{2}(E_v)} \quad \text{—— (8)}$$

$$F_v = (\sigma_u \cdot \sigma_v)_v = \sigma_{uv} \cdot \sigma_v + \sigma_u \cdot \sigma_{vv}$$

$$\Rightarrow \boxed{\sigma_u \cdot \sigma_{vv} + \sigma_v \cdot \sigma_{uv} = F_v} \quad \text{—— (9)}$$

B(i)

$$G_v = (\sigma_v \cdot \sigma_v)_v = \sigma_{vv} \cdot \sigma_v + \sigma_v \cdot \sigma_{vv} = 2(\sigma_v \cdot \sigma_{vv}).$$

C(i) $$\Rightarrow \boxed{\sigma_v \cdot \sigma_{vv} = \tfrac{1}{2}(G_v)}. \quad \text{—— (10)}.$$

Substituting (8), (9) & (10) in (7), we get,

$$\left(\frac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v)\right) \cdot \sigma_v = \frac{d}{dt}(F\dot{u} + G\dot{v}) - \left[\dot{u}^2\left(\tfrac{1}{2}E_v\right) + \dot{u}\dot{v}(F_v) + \dot{v}^2\left(\tfrac{1}{2}G_v\right)\right]$$

$$\therefore \left(\frac{d}{dt}(\dot{u}\sigma_u + \dot{v}\sigma_v)\right) \cdot \sigma_v = \frac{d}{dt}(F\dot{u} + G\dot{v}) - \tfrac{1}{2}\left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right).$$

This shows that the second equation in (2) is equivalent to the second geodesic equation in (1).

Hence the proof.

COROLLARY 8.2

An isometry between two surfaces takes the geodesics of one surface to the geodesics of the other.

PROOF

Let S_1 and S_2 be the 2 surfaces, let $f: S_1 \to S_2$ be the isometry and let γ be ~~the~~ a geodesic in S_1. Let $\sigma(u,v)$ be a patch in S_1 and suppose that the part of the geodesic lying in this patch is given by

$$\gamma(t) = \sigma(u(t), v(t))$$

Then u and v satify the geodesic equations

$$\left.\begin{aligned} \frac{d}{dt}(E\dot{u} + F\dot{v}) &= \frac{1}{2}\left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right) \\ \frac{d}{dt}(F\dot{u} + G\dot{v}) &= \frac{1}{2}\left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2\right) \end{aligned}\right\} \text{— } (1)$$

where E, F, G are the co-efficients of the first fundamental form of σ. By theorem 5.1 " A diffeomorphism $f: S_1 \to S_2$ is isometry iff for any surface patch σ_1 of S_1 the patch σ_1 and $(f \circ \sigma_1)$ of S_1 and S_2 respectively have the same first fundamental form".

$f \circ \sigma$ is a patch of S_2 with the same first fundamental form of σ. Hence, by Theorem 8.1

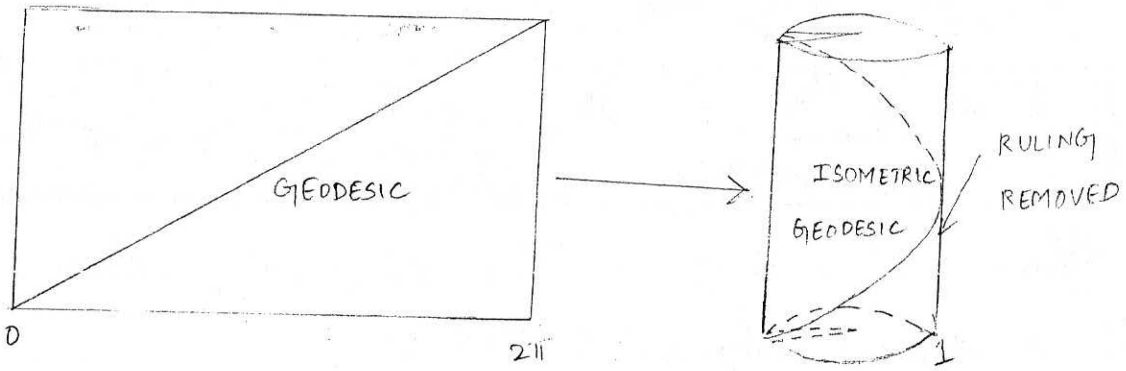
A curve γ on a surface S is a geodesic iff for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ contained in a surface patch σ of S, then eqn (1) is satisfied, where $E du^2 + 2F du dv + G dv^2$ is the first fundamental form of σ

The differential equations (1) are called the geodesic equations".

we have $t \mapsto f(\sigma(u(t), v(t)))$ ~~is~~ being a geodesic on S_2, In other words $f \circ \gamma$ is a geodesic.

Example 8.7:

Let S_1 be the infinite strip in the xy-plane given by $0 < x < 2\pi$. and S_2 be the circular cylinder $x^2+y^2=1$ with the ruling $x=1$, $y=0$ removed. On the circular cylinder we know that the circles obtained by intersecting the cylinder with planes parallel to the xy-plane are geodesics. We also know that the straight lines on the cylinder parallel to the z-axis are geodesics. However, there are certainly not the only geodesics, for there is one geodesic of each of the two types passing through each point of the cylinder.



To find the missing geodesics, we recall that the cylinder is isometric to the plane. In fact, the isometry takes the point $(u, v, 0)$ of the xy-plane to the point $(\cos u, \sin u, v)$ of the cylinder. By corollary 8.2 this map takes the geodesics on the plane (ie straight lines) to the geodesics on the cylinder and vice versa.

$K = \frac{\cdots}{(EG-F^2)^{1/2}}$ — (a)

scalars, $\gamma, \delta \ni \hat{e}'' \perp \hat{e}'$ $K = \frac{\alpha_v - \beta_u}{(EG-F^2)^{1/2}}$

$\Rightarrow \quad E^{-1/2}(\gamma E + \delta F) = 0 \quad \& \quad \gamma^2 E + 2\gamma\delta F + \delta^2 G = 1$

$\Rightarrow \quad \delta^2\left(\frac{F^2}{E} - 2\frac{F^2}{E} + G\right) = 1$

$$\therefore \boxed{\delta = \frac{E^{1/2}}{(EG-F^2)^{1/2}}, \quad \gamma = \frac{-FE^{-1/2}}{(EG-F^2)^{1/2}}, \quad \epsilon = E^{-1/2}} \longrightarrow (b)$$

Thus $\hat{e}' = \epsilon\sigma_u$ & $\hat{e}'' = \gamma\sigma_u + \delta\sigma_v$; where γ, δ, ϵ depend only on E, F & G.

Computing α and β.

$$\alpha = \hat{e}_u' \cdot \hat{e}_u''$$
$$= (e_u\sigma_u + \epsilon\sigma_{uu})\cdot(\gamma\sigma_u + \delta\sigma_v)$$
$$= \frac{e_u}{\epsilon}e'\cdot e'' + \frac{1}{2}\epsilon\gamma(\sigma_u\cdot\sigma_u)_u + \epsilon\delta((\sigma_u\cdot\sigma_v)_u - \sigma_u\cdot\sigma_{uv})$$

$$\boxed{\alpha = \frac{1}{2}\epsilon\gamma E_u + \epsilon\delta\left(F_u - \frac{1}{2}E_v\right)} \longrightarrow (c) \quad (\because \hat{e}'\cdot\hat{e}'' = 0)$$

which depend only on E, F, & G.

$$\beta = \hat{e}_v' \cdot \hat{e}''$$

$$\boxed{\beta = \frac{1}{2}\epsilon\gamma E_v + \frac{1}{2}\epsilon\delta G_u} \longrightarrow (d)$$

which depend only on E, F and G

Proof of Cor: 10:2 Qn-pg: 234

(i) if $F = 0$,

eqn (b) gives, $\gamma = 0, \quad \delta = G^{-1/2}, \quad \epsilon = E^{-1/2}$

$$(c) \Rightarrow, \quad \alpha = \frac{1}{2\sqrt{E}}(0)\cdot E_u + \frac{1}{\sqrt{E}\sqrt{G}}\left[0 - \frac{1}{2}E_v\right]$$

$$\alpha = \frac{-1}{2\sqrt{EG}}(E_v) \longrightarrow (1)$$

$$(d) \Rightarrow, \quad \beta = \frac{1}{2\sqrt{E}}(0) + \frac{G_u}{2\sqrt{E}\sqrt{G}} = \frac{1}{2\sqrt{EG}}(G_u) \rightarrow (2)$$

From ①, ② & ⓐ, we have,

$$\boxed{K = \frac{-1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial v}\left[\frac{E_v}{\sqrt{EG}}\right] + \frac{\partial}{\partial u}\left[\frac{G_u}{\sqrt{EG}}\right]\right\}} \rightarrow \text{(i)}$$

Thus (i)

(ii), if $F=0$ and further $E=1$,

Substituting $E=1$, in (i)

$$K = \frac{-1}{2\sqrt{G}}\left\{\frac{\partial}{\partial v}[0] + \frac{\partial}{\partial u}\left[\frac{G_u}{\sqrt{G}}\right]\right\}$$

$$= \frac{-1}{2\sqrt{G}}\left\{\frac{G_{uu}\sqrt{G} - \frac{G_u^2}{2\sqrt{G}}}{G}\right\} \longrightarrow (3)$$

Consider $\frac{-1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2} = \frac{-1}{\sqrt{G}}\cdot\frac{\partial}{\partial u}\left(\frac{\partial\sqrt{G}}{\partial u}\right)$

$$= \frac{-1}{\sqrt{G}}\cdot\frac{\partial}{\partial u}\left(\frac{G_u}{2\sqrt{G}}\right)$$

$$= \frac{-1}{2\sqrt{G}}\cdot\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{G}}\right)$$

$$= \frac{-1}{2\sqrt{G}}\left\{\frac{G_{uu}\sqrt{G} - \frac{G_u^2}{2\sqrt{G}}}{G}\right\} \rightarrow ④$$

RHS ③ & ④ are the same. So equating the LHS,

Hence we have $\boxed{K = \frac{-1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}}$, the required result. Hence (ii).

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Let us see an example.

Theorem 10.2

Any point of a surface of constant gaussian curvature is contained in a patch that is isometric to part of a plane, a sphere or a pseudosphere.

Proof:-

Let P be a point of a surface S with constant gaussian curvature K. By applying a dilation of $\mathbb{R}^3$; we need only consider the cases $K = 0, 1$ and -1.

We take a geodesic patch $\sigma(u,v)$ with $\sigma(0,0) = P$. Now by Proposition 8.7 which states that

"If there is an open subset U of $\mathbb{R}^2$ containing $(0,0)$ such that $\sigma: U \to \mathbb{R}^3$ is an allowable surface patch for S. Then the FFF of σ is

$$du^2 + G(u,v)\,dv^2$$

where G is a smooth function on U such that $G(0,v) = 1$, $G_u(0,v) = 0$, whenever $(0,v) \in U$."

and writing $g = \sqrt{G}$; the FFF is

$$du^2 + g(u,v)^2 dv^2 \quad \text{———} \quad ①$$

with $g(0,v) = 1$, $g_u(0,v) = 0$ ——— ②

By Corollary 10.2 (ii) : If $E=1$ and $F=0$; we have

$$\frac{\partial^2 g}{\partial u^2} + \mathcal{K} g \equiv 0 \quad \text{———} \quad ③$$

Now consider the cases $\mathcal{K} = 0, 1$ and -1.

Case (i)

If $\mathcal{K}=0$; then ③ becomes $\frac{\partial^2 g}{\partial u^2} = 0 \Rightarrow g = \alpha u + \beta$

where α, β are smooth functions of v only.

Now by the boundary conditions from ② , we will get

$$g(0,v) = \alpha \cdot 0 + \beta = 1 \Rightarrow \beta = 1$$

$$g_u(0,v) = \alpha = 0 \Rightarrow \alpha = 0$$

So ,

$$g = 0 \cdot u + 1 \Rightarrow g = 1$$

So FFF ie, ① becomes;

$du^2 + 1^2 dv^2$ ie, $\boxed{du^2 + dv^2}$; which is same as the FFF of usual parametrisation of a plane. So it shows that σ is isometric to the part of the plane.

Case (ii)

If $K=1$; then ③ becomes $\frac{\partial^2 g}{\partial u^2}+g=0$. By the auxillary equation of differential equation; we have;

$$m^2+1=0 \Rightarrow m^2=-1$$

$$m=\pm\sqrt{-1} \quad \text{i.e.,} \quad m=\pm i$$

So; $g=\alpha\cos u+\beta\sin u$; where α, and β depend only on v.

Now apply the boundary conditions ②; we will get;

$$g(0,v)=\alpha=1 \Rightarrow \alpha=1$$

$$g_u(u,v)=-\alpha\sin u+\beta\cos u \Rightarrow g_u(0,v)=\beta=0 \Rightarrow \beta=0$$

So; $g=\cos u+0 \Rightarrow g=\cos u$.

So FFF of σ is

$$\boxed{du^2+\cos^2 u\, dv^2}$$

, which is same as the FFF of a unitsphere with u and v as latitude and longitude.

Case (iii)

If $K=-1$; then ③ becomes $\frac{\partial^2 g}{\partial u^2}-g=0$. By the auxillary equation of differential equation; we have

$$m^2+1=0 \Rightarrow m^2=1$$

$$\therefore \quad m=\pm 1$$

So; $g=\alpha e^{u}+\beta e^{-u}$

Now by boundary conditions;

$$g(0,v) = \alpha+\beta=1$$

To find; $g_u(0,v) \Rightarrow \alpha e^{u} - \beta e^{-u} = g_u$

$$\therefore g_u(0,v) = \alpha-\beta=0 \Rightarrow \alpha=\beta$$

So $\alpha = 1/2 = \beta \quad (\because \alpha+\beta=1)$

So $g = \dfrac{e^{u}+e^{-u}}{2} = \cosh u$

$\therefore$ FFF will be of the form;

$$\boxed{du^2 + \cosh^2 u \, dv^2}$$

We have not encountered this FFF before. So let us reparametrise σ by defining $V = e^{v}\tanh u$ and $W = e^{v}\operatorname{sech} u$.

So; inordered to find $\tilde{E}, \tilde{F}, \tilde{G}$; we use the formula is Exercise 5.4 which says;

"Let a surface $\tilde{\sigma}(\tilde{u},\tilde{v})$ be a reparametrisation of a surface $\sigma(u,v)$ and $\tilde{E}\,d\tilde{u}^2 + 2\tilde{F}\,d\tilde{u}\,d\tilde{v} + \tilde{G}\,d\tilde{v}^2$ and $E\,du^2 + 2F\,du\,dv + G\,dv^2$ be their FFF. Let

$$J = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

be the jacobian matrix of the reparametrisation map $(\tilde{u},\tilde{v}) \mapsto (u,v)$ and let J^t denotes the transpose of J. Then;

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J$$ "

So, Here $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{pmatrix}$

$$J = \begin{pmatrix} \dfrac{1}{e^v \operatorname{sech}^2 u} & \dfrac{-1}{e^v \tanh u \operatorname{sech} u} \\ \dfrac{1}{e^v \tanh u} & \dfrac{1}{e^v \operatorname{sech} u} \end{pmatrix}; \quad J^t = \begin{pmatrix} \dfrac{1}{e^v \operatorname{sech}^2 u} & \dfrac{1}{e^v \tanh u} \\ \dfrac{-1}{e^v \tanh u \operatorname{sech} u} & \dfrac{1}{e^v \operatorname{sech} u} \end{pmatrix}$$

So, $J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \dfrac{1}{e^v \operatorname{sech}^2 u} & \dfrac{1}{e^v \tanh u} \\ \dfrac{-1}{e^v \tanh u \operatorname{sech} u} & \dfrac{1}{e^v \operatorname{sech} u} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{pmatrix}$

$$= \begin{pmatrix} \dfrac{1}{e^v \operatorname{sech}^2 u} & \dfrac{\cosh^2 u}{e^v \tanh u} \\ \dfrac{-1}{e^v \tanh u \operatorname{sech} u} & \dfrac{\cosh^2 u}{e^v \operatorname{sech} u} \end{pmatrix}$$

So, $J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J = \begin{pmatrix} \dfrac{1}{e^v \operatorname{sech}^2 u} & \dfrac{\cosh^2 u}{e^v \tanh u} \\ \dfrac{-1}{e^v \tanh u \operatorname{sech} u} & \dfrac{\cosh^2 u}{e^v \operatorname{sech} u} \end{pmatrix} \begin{pmatrix} \dfrac{1}{e^v \operatorname{sech}^2 u} & \dfrac{-1}{e^v \tanh u \operatorname{sech} u} \\ \dfrac{1}{e^v \tanh u} & \dfrac{1}{e^v \operatorname{sech} u} \end{pmatrix}$

$$= \begin{pmatrix} \dfrac{1}{(e^v \operatorname{sech}^2 u)^2} + \dfrac{\cosh^2 u}{(e^v \tanh u)^2} & \dfrac{-1}{e^{2v} \operatorname{sech}^3 u \tanh u} + \dfrac{\cosh^2 u}{e^{2v} \tanh u \operatorname{sech} u} \\ \dfrac{-1}{e^{2v} \tanh u \operatorname{sech}^3 u} + \dfrac{\cosh^2 u}{e^{2v} \tanh u \operatorname{sech} u} & \dfrac{1}{e^{2v} \tanh^2 u \operatorname{sech}^2 u} + \dfrac{\cosh^2 u}{(e^{2v} \operatorname{sech}^2 u)} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{(e^v \operatorname{sech} u)^2} \left[\dfrac{1}{\operatorname{sech}^2 u} + \dfrac{1}{\tanh^2 u} \right] & 0 \\ 0 & \dfrac{1}{(e^v \operatorname{sech} u)^2} \left[\dfrac{1}{\operatorname{sech}^2 u} + \dfrac{1}{\tanh^2 u} \right] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{W^2}\left[\frac{1}{\operatorname{sech}^2 u \tanh^2 u}\right] & 0 \\ 0 & \frac{1}{W^2}\left[\frac{1}{\operatorname{sech}^2 u \tanh^2 u}\right] \end{pmatrix}$$

$\therefore$ FFF will be of the form;

$$\frac{1}{W^2 \operatorname{sech}^2 u \tanh^2 u}\left[dv^2 + dW^2\right] = \frac{(dV^2 + dW^2)\, e^{4v}}{W^2\,(W^2V^2)}$$

Comparing with the example 8.8;

" To determine the geodesics on the pseudosphere

$\sigma(u,v) = \left(e^u \cos v,\ e^u \sin v,\ \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})\right)$

We found FFF is $du^2 + e^{2u} dv^2$.

It is convinent to reparametrise by setting $w = e^{-u}$

$\tilde{\sigma}(v,w) = \left(\frac{1}{w}\cos v,\ \frac{1}{w}\sin v,\ \sqrt{1-\frac{1}{w^2}} - \cosh^{-1}(w)\right)$ and its FFF

is, $\dfrac{dv^2 + dw^2}{w^2}$

which is the FFF of pseudosphere.

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Prop. 7.~~1~~

Thm. 8.1

Thm : 10.1

Thm : 10.2