UNIT- IV

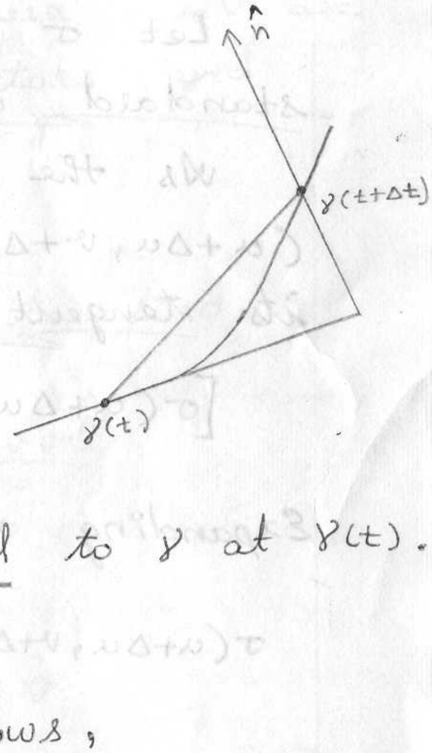
Curvature of surfaces

We introduce a measure called <u>second fundamental form</u> that can calculate how 'curved' a surface is. It can be seen that a surface patch (curvature of it) is determined up to a rigid motion of $\mathbb{R}^3$ by its first and second fundamental form.

Sec. 4.1 : Second Fundamental Form

Derivation:

Let γ be a <u>unit-speed curve</u> in $\mathbb{R}^2$.



As the parameter 't' of γ changes to $t+\Delta t$, the curve moves away from its <u>tangent line</u> at $\gamma(t)$ by a distance

$$(\gamma(t+\Delta t) - \gamma(t)) \cdot \hat{n} \longrightarrow (1)$$

where $\hat{n}$ is the <u>principal normal</u> to γ at $\gamma(t)$.

By Taylor's theorem,

we can expand γ as follows,

$$\gamma(t+\Delta t) = \gamma(t) + \dot{\gamma}(t)\,\Delta t + \frac{1}{2}\ddot{\gamma}(t)\,(\Delta t)^2 + \text{remainder terms},$$

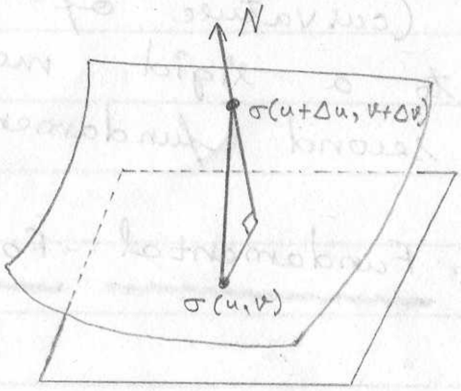
where the rest of the remainder will vanish when we let $\Delta t \to 0$ after we divide by $(\Delta t)^2$.

$$\gamma(t+\Delta t) - \gamma(t) = \dot{\gamma}(t)\,\Delta t + \frac{1}{2}\ddot{\gamma}(t)\,\Delta t^2 + \text{remainder t}$$

$$\left[\gamma(t+\Delta t) - \gamma(t)\right]\cdot\hat{n} = \dot{\gamma}\cdot\hat{n}\,\Delta t + \frac{1}{2}\ddot{\gamma}\cdot\hat{n}\,\Delta t^2 + \text{remainder}$$

$$= \underbrace{\hat{t}\cdot\hat{n}}_{0}\,\Delta t + \frac{1}{2}\kappa\,\underbrace{\hat{n}\cdot\hat{n}}_{1}\,\Delta t^2 + \text{remainder}$$

$$= \frac{1}{2}\kappa\,\Delta t^2 + \text{remainder}$$



Let σ be a surface patch in $\mathbb{R}^3$ with standard unit normal $\hat{N}$.

As the parameters (u,v) of σ change to $(u+\Delta u, v+\Delta v)$, the surface moves away from its tangent plane at $\sigma(u,v)$ by a distance

$$\left[\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)\right]\cdot\hat{N}.$$

Expanding σ by Taylor's theorem for two variables,

$$\sigma(u+\Delta u, v+\Delta v) = \sigma(u,v) + \sigma_u\Delta u + \sigma_v\Delta v + \frac{1}{2}\left(\sigma_{uu}(\Delta u)^2 + 2\sigma_{uv}\Delta u\Delta v + \sigma_{vv}(\Delta v)^2\right) + \text{remainder terms}$$

$$\left[\sigma(u+\Delta u, v+\Delta v) - \sigma(u,v)\right]\cdot\hat{N} = \underbrace{\sigma_u\cdot\hat{N}}_{0}\,\Delta u + \underbrace{\sigma_v\cdot\hat{N}}_{0}\,\Delta v + \frac{1}{2}\left(\sigma_{uu}\cdot\hat{N}(\Delta u)^2 + 2\sigma_{uv}\cdot\hat{N}\,\Delta u\Delta v + \sigma_{vv}\cdot\hat{N}(\Delta v)^2\right) + \text{remainder terms}$$

$$= \frac{1}{2}\left\{ L(\Delta u)^2 + 2M\,\Delta u\,\Delta v + N(\Delta v)^2 \right\}$$

+ remainder terms

where $L = \sigma_{uu} \cdot \hat{N}$,

$M = \sigma_{uv} \cdot \hat{N}$ &

$N = \sigma_{vv} \cdot \hat{N}$

The SFF of σ given by,

$L\,du^2 + 2M\,du\,dv + N\,dv^2$ //

Eg. 6.1

Find the 2^{nd} fundamental form of the plane, $\boxed{\sigma(u,v) = \bar{a} + u\bar{p} + v\bar{q}}$ where u & v are scalars, p & q are ⊥r unit vectors and $\bar{a}$ is a constant vector.

Soln:

To calculate SFF we need to find,

$$L = \underbrace{\sigma_{uu}}_{0} \cdot \underbrace{\hat{N}}_{1} \; ; \; M = \underbrace{\sigma_{uv}}_{0} \cdot \underbrace{\hat{N}}_{1} \; ; \; N = \underbrace{\sigma_{vv}}_{0} \cdot \underbrace{\hat{N}}_{1}$$

where $\hat{N} = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

$\sigma_u = p$; $\sigma_{uu} = 0$ so $\hat{N} = \dfrac{p \times q}{\|p \times q\|}$

$\sigma_u = p$; $\sigma_{uv} = 0$ $= \dfrac{1}{\|1\|}$

$\sigma_v = q$; $\sigma_{vv} = 0$ $= 1$

$\therefore L = M = N = 0$

Hence SFF is 0 //

Example : 6.2

Find the $\underline{FFF}$ & $\underline{SFF}$ of the surface patch σ of the surface of revolution given by

$$\boxed{\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))}$$

here $f(u) > 0$, $\forall$ value and the profile curve from $u \mapsto (f(u), 0, g(u))$ is of unit-speed.

ie) $\dot{f}^2 + \dot{g}^2 = 1$

Solution:

FFF

$$\sigma_u = (f'(u)\cos v, f'(u)\sin v, g'(u))$$

$$\sigma_v = (-f(u)\sin v, f(u)\cos v, 0)$$

$$E = \|\sigma_u\|^2$$

$$= f'(u)^2\cos^2 v + f'(u)^2\sin^2 v + g'(u)^2$$

$$= f'(u)^2 + g'(u)^2 \quad [\because \text{gn that } \dot{f}^2 + \dot{g}^2 = 1]$$

$$= 1$$

$$F = \sigma_u \cdot \sigma_v$$

$$= -f(u)f'(v)\underbrace{\cos v \sin v} + f(u)f'(u)\underbrace{\sin v \cos v} + 0$$

$$= 0$$

$$G = \|\sigma_v\|^2$$

$$= f(u)^2\sin^2 v + f(u)^2\cos^2 v$$

$$= f(u)^2$$

$\therefore$ The FFF is

$$ds^2 = (f'(u)^2 + g'(u)^2)du^2 + f(u)^2 dv^2$$

$$\Rightarrow ds^2 = du^2 + f^2 dv^2$$

SFF

$$\sigma_{uu} = (f''(u)\cos v,\ f''(u)\sin v,\ g''(u))$$

$$\sigma_{vv} = (-f(u)\cos v,\ -f(u)\sin v,\ 0)$$

$$\sigma_{uv} = (-f'(u)\sin v,\ f'(u)\cos v,\ 0)$$

$$\hat{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \qquad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f'(u)\cos v & f'(u)\sin v & g'(u) \\ -f(u)\sin v & f(u)\cos v & 0 \end{vmatrix}$$

$$\Rightarrow \hat{i}\,(0 - g'(u)f(u)\cos v) - \hat{j}\,(g'(u)f(u)\sin v) + \hat{k}\,\underbrace{(f'(u)f(u)\cos^2 v + f'(u)f(u)\sin^2 v)}_{f'(u)f(u)}$$

$$\sigma_u \times \sigma_v = (-g'(u)f(u)\cos v,\ -g'(u)f(u)\sin v,\ f'(u)f(u))$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{g'(u)^2 f(u)^2\cos^2 v + g'(u)^2 f(u)^2 \sin^2 v + f'(u)^2 f(u)^2}$$

$$= \sqrt{g'(u)^2 f(u)^2 + f'(u)^2 f(u)^2}$$

$$\sqrt{(g'(u)^2 + f'(u)^2)f(u)^2}$$

$$= \sqrt{f(u)^{\cancel{2}}} \qquad \Rightarrow \sqrt{1(f(u)^2)}$$

$$= f(u)$$

$$\therefore \hat{N} = (-\dot{g}\cos v,\ -\dot{g}\sin v,\ \dot{f})$$

i.e., $\hat{N} = \dfrac{(-g'(u)\cancel{f(u)}\cos v,\ -g'(u)\cancel{f(u)}\sin v,\ f'(u)\cancel{f(u)})}{\cancel{f(u)}}$

$\sigma_{uu}\cdot\hat{N}$

$$L = (-\ddot{f}\dot{g}\cos^2 v - \ddot{f}\dot{g}\sin^2 v + \dot{f}\ddot{g})$$

$$= \dot{f}\ddot{g} - \ddot{f}\dot{g}$$

$\sigma_{vv}\cdot\hat{N}$

$$N = (f\dot{g}\cos^2 v + f\dot{g}\sin^2 v + 0)$$

$$= f\dot{g}$$

$\sigma_{uv}\cdot\hat{N}$

$$M = \dot{f}\dot{g}\cos v\sin v - \dot{f}\dot{g}\cos v\sin v$$

$$= 0$$

$\therefore$ SFF is $(\dot{f}\ddot{g} - \ddot{f}\dot{g})\,du^2 + f\dot{g}\,dv^2$ //

Special cases:

1) When the surface is a unit sphere,

$$\sigma(u,v) = (\overset{f(u)}{\underline{\cos u}}\cos v,\ \overset{f(u)}{\underline{\cos u}}\sin v,\ \overset{g(u)}{\underline{\sin u}})$$

2) When the surface is a circular cylinder of unit radius,

$$\sigma(u,v) = (\overset{f(u)}{1}\cos v,\ \overset{f(u)}{1}\sin v,\ \overset{g(u)}{u})$$

~~Exercise:~~

~~1. Find SFF of the elliptic paraboloid,~~

Soln:

1) Here $f(u) = \cos u \ ; \ g(u) = \sin u$

$\dot{f} = -\sin u \ ; \ \dot{g} = \cos u$

$\ddot{f} = -\cos u \ ; \ \ddot{g} = -\sin u$

$\therefore$ SFF is $\left[(-\sin u)(-\sin u) - (-\cos u)(\cos u)\right] du^2 + (\cos u)(\cos u) d$

$\therefore$ SFF $= (\sin^2 u + \cos^2 u)\, du^2 + \cos^2 u\, dv^2$

$= du^2 + \cos^2 u\, dv^2$,

2) Here $f(u) = 1 \ ; \ g(u) = u$

$\dot{f} = 0 \ ; \ \dot{g} = 1$

$\ddot{f} = 0 \ ; \ \ddot{g} = 0$

$\therefore$ SFF is dv^2,

Exercise:

1. Find SFF of the elliptic paraboloid,

$$\sigma(u, v) = (\overset{u}{\cancel{x}}, v, u^2+v^2)$$

Soln:

$$\sigma(u, v) = (u, v, u^2+v^2)$$

$$\sigma_u = (1, 0, 2u) \quad ; \quad \sigma_v = (0, 1, 2v)$$

$$\sigma_{uu} = (0, 0, 2) \quad ; \quad \sigma_{vv} = (0, 0, 2)$$

$$\sigma_{uv} = (0, 0, 0)$$

$$\therefore \quad \hat{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

$$\sigma_u \times \sigma_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$

$$= \hat{i}(0-2u) - \hat{j}(2v-0) + \hat{k}(1-0)$$

$$= -2u\hat{i} - 2v\hat{j} + \hat{k}$$

$$\sigma_u \times \sigma_v = (-2u, -2v, 1)$$

$$\|\sigma_u \times \sigma_v\| = \sqrt{4u^2+4v^2+1}$$

$$\Rightarrow \hat{N} = \frac{(-2u, -2v, 1)}{\sqrt{4u^2+4v^2+1}} = \lambda(-2u, -2v, 1)$$

where $\boxed{\lambda = \frac{1}{\sqrt{4u^2+4v^2+1}} = (4u^2+4v^2+1)^{-1/2}}$

W.K.T,

$$L = \sigma_{uu} \cdot \hat{N}$$

$$M = \sigma_{uv} \cdot \hat{N}$$

$$N = \sigma_{vv} \cdot \hat{N}$$

Then

$$L = (0,0,2) \cdot \lambda(-2u,-2v,1)$$

$$= 0+0+2\lambda$$

$$= 2\lambda$$

$$M = (0,0,0) \cdot \lambda(-2u,-2v,1)$$

$$= 0$$

$$N = (0,0,2) \cdot \lambda(-2u,-2v,1)$$

$$= 2\lambda$$

$\Rightarrow$ SFF is $2\lambda du^2 + 2(0)du\,dv + 2\lambda dv^2$

$\Rightarrow$ SFF $= 2\lambda du^2 + 2\lambda dv^2 = 2\lambda(du^2+dv^2)$ //

Sec : 4.2

The Curvature of Curves on a Surface:

Let $\gamma(t) = \sigma(u(t), v(t))$ & let γ be a unit-speed curve & let σ be a surface patch.

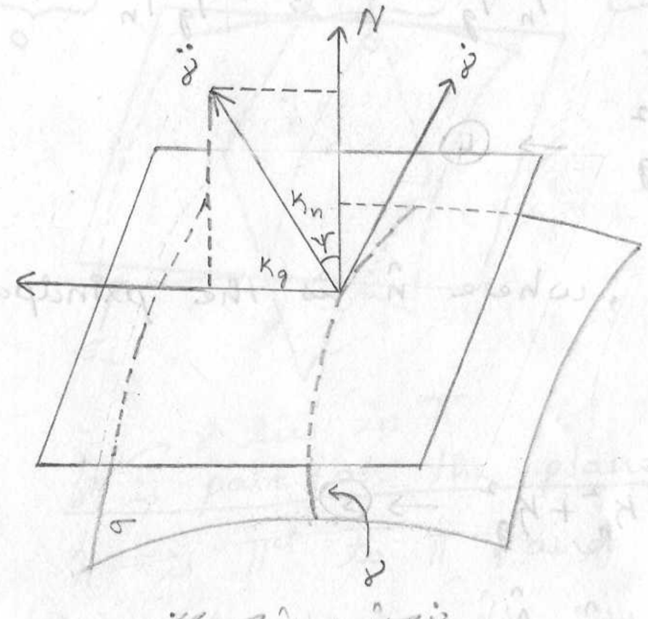
Then $\dot{\gamma}$ is a unit vector and a tangent vector to σ.

Therefore, $\dot{\gamma}$ is $\perp r$ to the standard unit normal $\hat{N}$ of σ & $\hat{N} \times \dot{\gamma}$ is $\perp r$ to both $\dot{\gamma}$ and $\hat{N}$.

So, $\dot{\gamma}, \hat{N}, \hat{N} \times \dot{\gamma}$ are mutually $\perp r$ unit vectors.

Since γ is unit-speed, $\ddot{\gamma}$ is $\perp r$ to $\dot{\gamma}$, and hence is a linear combination of $\hat{N}$ and $\hat{N} \times \dot{\gamma}$:

$$\ddot{\gamma} = K_n \hat{N} + K_g \hat{N} \times \dot{\gamma} \longrightarrow (1)$$



$\cancel{\ddot{\gamma} = K_n \hat{N} + K_g \hat{N} \times \dot{\gamma}}$

Let the scalars K_n & K_g are called the normal curvature and the geodesic curvature of γ, respectively.

Since $\hat{N}$ & $\hat{N} \times \dot{\gamma}$ are $\perp r$ unit vectors, from (1) we can write

$$\ddot{\gamma} \cdot \hat{N} = K_n \underbrace{\hat{N} \cdot \hat{N}}_{1} + K_g \underbrace{\hat{N} \times \dot{\gamma} \cdot \hat{N}}_{0}$$

$i \cdot j = 0$

$i \cdot i = 1$

$i \times i = 0$

$i \times j = 1$

$$\Rightarrow K_n = \ddot{\gamma} \cdot \hat{N} \longrightarrow (2)$$

Also,

$$\ddot{\gamma}\cdot\hat{N}\times\dot{\gamma} = K_n \underbrace{\hat{N}\cdot\hat{N}\times\dot{\gamma}}_{0} + K_g \underbrace{\hat{N}\times\dot{\gamma}\cdot\hat{N}\times\dot{\gamma}}_{1}$$

$$\Rightarrow \quad K_g = \ddot{\gamma}\cdot\hat{N}\times\dot{\gamma} \longrightarrow ③$$

Again,

$$\|\ddot{\gamma}\|^2 = \ddot{\gamma}\cdot\ddot{\gamma}$$

$$= (K_n\hat{N} + K_g\hat{N}\times\dot{\gamma})\cdot(K_n\hat{N} + K_g\cdot\hat{N}\times\dot{\gamma})$$

$$= K_n^2 \underbrace{\hat{N}\cdot\hat{N}}_{1} + K_n K_g \underbrace{\hat{N}\cdot\hat{N}\times\dot{\gamma}}_{0} + K_g K_n \underbrace{\hat{N}\times\dot{\gamma}\cdot\dot{\gamma}}_{0} + K_g^2 \underbrace{\hat{N}\times\dot{\gamma}\cdot\hat{N}\times\dot{\gamma}}_{1}$$

$$\Rightarrow \|\ddot{\gamma}\|^2 = K_n^2 + K_g^2 \longrightarrow ④$$

But $\ddot{\gamma} = K\hat{n}$, where $\hat{n}$ is the principal normal (P) of γ.

$$\|\ddot{\gamma}\| = K$$

$$\therefore ④ \Rightarrow K^2 = K_n^2 + K_g^2 \longrightarrow ⑤$$

$$② \Rightarrow K_n = K(\hat{n}\cdot\hat{N})$$

But from the above diagram,

$$\cos\psi = \frac{\hat{n}\cdot\hat{N}}{\|\hat{n}\|\,\|\hat{N}\|}$$ where ψ is the angle b/w $\hat{n}$ & $\hat{N}$

$= \hat{n}\cdot\hat{N}$ as $\hat{n}$ & $\hat{N}$ are unit vectors $\hookrightarrow (*)$

sub $(*)$ in ②,

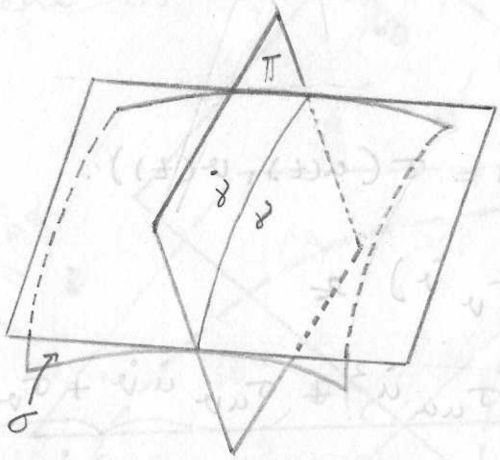
$$\therefore \Rightarrow K_n = K\cos\psi \longrightarrow ⑥$$

⑥ in ⑤ $\Rightarrow \kappa^2 = \kappa^2 \cos^2\psi + \kappa_g^2$

$$\Rightarrow \kappa_g = \pm \kappa \sin\psi \rightarrow ⑦$$

Let us consider a special case where γ is a normal section of the surface.

i.e., γ is the intersection of the surface with a plane Π, which is $\perp$r to the tangent plane of the surface at every point of γ.



Since γ is part of the plane Π ($\rightarrow$ γ lies in Π), the principal normal $\hat{n}$ is $\|^{el}$ to Π, and since Π is $\perp$r to the tangent plane, $\hat{N}$ is also $\|^{el}$ to Π.

Since $\hat{n}$ & $\hat{N}$ are both $\perp$r to $\dot{\gamma}$, and since $\dot{\gamma}$ is $\|^{el}$ to Π,

$\hat{n}$ & $\hat{N}$ must be $\|^{el}$ to each other.

Therefore, the angle b/w $\hat{n}$ & $\hat{N}$ is either 0 or π.

i.e., $\psi = 0$ or π

From eqns ⑥ & ⑦, we deduce that

$$\kappa_n = \pm \kappa \quad \& \quad \kappa_g = 0$$

Sec. 4.3

The Normal and Principal Curvatures:

Proposition : 6.1

If $\gamma(t) = \sigma(u(t), v(t))$ is a unit-speed curve on a surface patch σ, its normal curvature is given by

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,$$

where $L\,du^2 + 2M\,du\,dv + N\,dv^2$ is the SFF of σ.

Proof:

We have $\gamma(t) = \sigma(u(t), v(t))$.

$\therefore\ \dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v}$ &

doubt

$$\ddot{\gamma} = \sigma_u \ddot{u} + \sigma_{uu}\dot{u}^2 + \underbrace{\sigma_{uv}\dot{u}\dot{v} + \sigma_{vu}\dot{u}\dot{v}}_{2\sigma_{uv}\dot{u}\dot{v}} + \sigma_v \ddot{v} + \sigma_{vv}$$

W.K.T,

$$\kappa_n = \ddot{\gamma} \cdot \hat{N}$$

$$= (\sigma_{uu} \cdot \hat{N})\dot{u}^2 + 2(\sigma_{uv} \cdot \hat{N})\dot{u}\dot{v} + \overbrace{(\sigma_u \cdot \hat{N})\ddot{u}}^{0}$$

$$+ (\sigma_{vv} \cdot \hat{N})\dot{v}^2 + \underbrace{(\sigma_v \cdot \hat{N})\ddot{v}}_{0}$$

$$= L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

how?

where $L = \sigma_{uu} \cdot \hat{N}$

$M = \sigma_{uv} \cdot \hat{N}$

$N = \sigma_{vv} \cdot \hat{N}$

Required discussion to prove the proof.

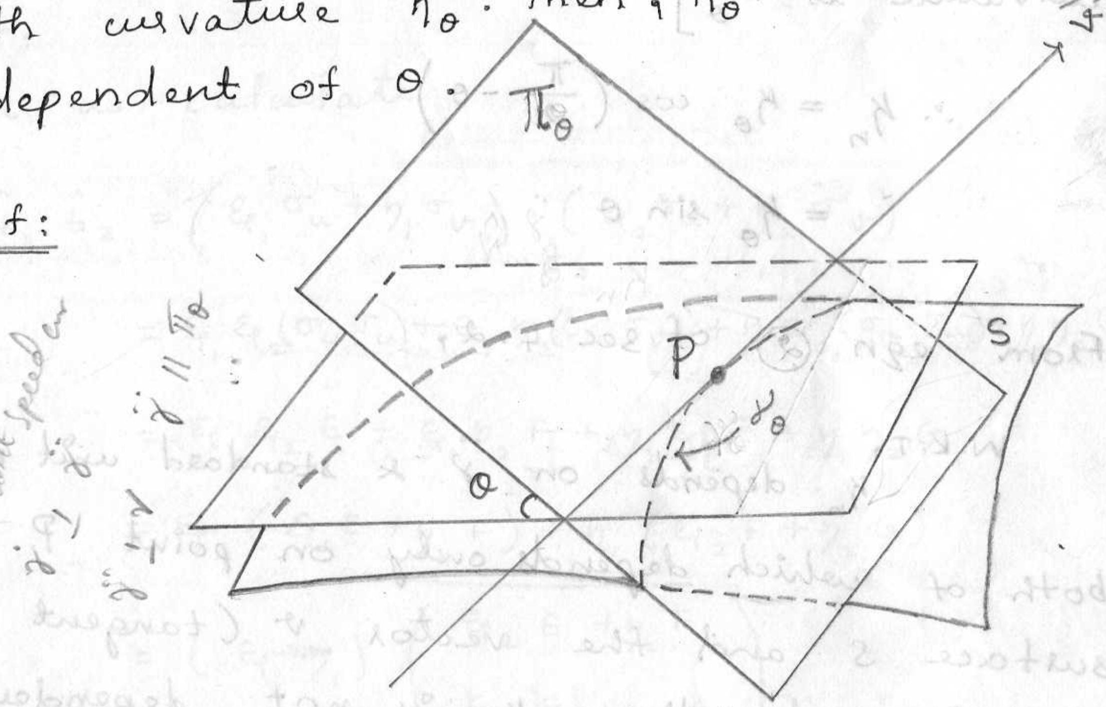
Now, we prove the thm.

Proposition : 6.2

Meusnier's Theorem

Let P be a point on a surface S and let v be a unit tangent vector to S at P. Let Π_θ be the plane containing the line through P ||el to v and making an angle θ with the tangent plane to S at P. Suppose that Π_θ intersects S in a curve with curvature K_θ. Then, $K_\theta \sin\theta$ is independent of θ.

Proof:



Assume that γ_θ is a unit-speed parametrisation of the curve of intersection of Π_θ & S, the surface.

Then, at P on the surface, we have the tangent vector $\dot{\gamma}_\theta$. Like given in the statement let us take $\dot{\gamma}_\theta$ to be either $\pm\vec{v}$.

Now, that $\ddot{\gamma}_\theta$ is ⊥r to $\dot{\gamma}_\theta$ and hence is ⊥r to the vector $\vec{v}$ and hence is ||el to the plane Π_θ.

From the discussion on sec ~~6.2~~ 4.2,

we have $\psi = \frac{\pi}{2} - \theta$ is the angle b/w $\ddot{\gamma}$ and the standard unit normal $\hat{N}$ at the point P on the surface S.

From eqn. (6) of sec ~~6.2~~ 4.2,

$$k_n = k_\theta \cos \psi$$

[from the statement it is given that the curvature is k_θ]

$$\therefore k_n = k_\theta \cos\left(\frac{\pi}{2} - \theta\right)$$

$$= k_\theta \sin \theta$$

$k_n = \ddot{\gamma} \cdot \hat{N}$

From eqn. (2) of sec 4.2,

W.K.T, k_n depends on $\ddot{\gamma}$ & standard unit normal, both of which depends only on point P in the surface S and the vector $\underline{v}$ (tangent vector at P) and thus k_n is not dependent on θ

Notations:

The FFF of the surface σ is given by,

$$E du^2 + 2F du dv + G dv^2$$

The SFF of σ is given by

$$L du^2 + 2M du dv + N dv^2$$

Let us take $\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$; $\mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

Let two tangent vectors at some point of σ be

$$\hat{t}_1 = \xi_1 \sigma_u + \eta_1 \sigma_v$$

$$\hat{t}_2 = \xi_2 \sigma_u + \eta_2 \sigma_v$$

Let $T_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}$; $T_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$

Now,

let us calculate,

$$\hat{t}_1 \cdot \hat{t}_2 = (\xi_1 \sigma_u + \eta_1 \sigma_v) \cdot (\xi_2 \sigma_u + \eta_2 \sigma_v)$$

$$= \xi_1 \xi_2 (\sigma_u \cdot \sigma_u) + \xi_1 \eta_2 (\sigma_u \cdot \sigma_v) + \eta_1 \xi_2 (\sigma_v \cdot \sigma_u) + \eta_1 \eta_2 (\sigma_v \cdot \sigma_v)$$

$$= \xi_1 \xi_2 E + \xi_1 \eta_2 F + \eta_1 \xi_2 F + \eta_1 \eta_2 G$$

$$= \xi_1 (\xi_2 E + \eta_2 F) + \eta_1 (\xi_2 F + \eta_2 G)$$

$$= (\xi_1\ \eta_1) \begin{pmatrix} \xi_2 E + \eta_2 F \\ \xi_2 F + \eta_2 G \end{pmatrix}$$

$$= (\xi_1\ \eta_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

$$= T_1^t \mathcal{F}_I T_2 \longrightarrow (*)$$

On the other hand,

Consider the tangent vector $\dot{\gamma} = \dot{u} \sigma_u + \dot{v} \sigma_v$ and let T be the matrix $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$

Then from proposition 6.1, we have

$$K_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$$

$$k_n = L\dot{u}\dot{u} + M\dot{u}\dot{v} + M\dot{u}\dot{v} + N\dot{v}\dot{v}$$

$$= \dot{u}\,(L\dot{u} + M\dot{v}) + \dot{v}\,(M\dot{u} + N\dot{v})$$

$$= (\dot{u} \;\; \dot{v}) \begin{pmatrix} L\dot{u} + M\dot{v} \\ M\dot{u} + N\dot{v} \end{pmatrix}$$

$$= (\dot{u} \;\; \dot{v}) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$$

$$= T^t \, \mathcal{F}_{II} \, T \longrightarrow (**)$$

Defn : Principal curvatures

The principal curvatures of a surface patch are the roots of the equation

$$\det(\mathcal{F}_{II} - k\,\mathcal{F}_{I}) = 0 \longrightarrow (1)$$

Note: (1)

$$(1) \Rightarrow \begin{pmatrix} L & M \\ M & N \end{pmatrix} - k \begin{pmatrix} E & F \\ F & G \end{pmatrix} = 0$$

ie.,
$$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = 0 \longrightarrow (1a)$$

(1) & (1a) are similar

$\therefore$ (1a) is a quadratic eqn for k, there are two roots.

Note : ②

Again eqn ① $\Rightarrow \det(I\,\mathcal{F}_{II} - \kappa\,\mathcal{F}_I) = 0$

ie) $\det(\mathcal{F}_I\,\mathcal{F}_I^{-1}\,\mathcal{F}_{II} - \kappa\,\mathcal{F}_I) = 0$

ie) $\det\left\{\mathcal{F}_I(\mathcal{F}_I^{-1}\,\mathcal{F}_{II} - \kappa)\right\} = 0$

ie) $\det(\mathcal{F}_I^{-1}\,\mathcal{F}_{II} - \kappa) = 0$

The principal curvatures are the eigen values of $\mathcal{F}_I^{-1}\,\mathcal{F}_{II}$.

Defn: Principal vector

If a vector $T = \begin{pmatrix}\xi \\ \eta\end{pmatrix}$ satisfies the eqn, $(\mathcal{F}_{II} - \kappa\,\mathcal{F}_I)T = 0$, → ②

the corresponding tangent vector $\hat{t} = \xi\sigma_u + \eta\sigma_v$ to the surface patch $\sigma(u, v)$ is called a principal vector corresponding to the principal curvature κ.

Proposition : 6.3

16 m

Let κ_1 & κ_2 be the principal curvatures at a point P of a surface patch σ. Then,

i) κ_1 & κ_2 are real numbers;

ii) if $\kappa_1 = \kappa_2 = \kappa$, say, then $\mathcal{F}_{II} = \kappa\,\mathcal{F}_I$ and (hence) every tangent vector to σ at P is a principal vector;

iii) if $\kappa_1 \neq \kappa_2$, then any two (non-zero) principal vectors $\hat{t}_1$ & $\hat{t}_2$ corresponding to κ_1 & κ_2, ~~are~~ respectively, are perpendicular.

In case (ii), P is called an <u>umbilic</u>.

<u>Proof:</u>

i) Let κ_1 & κ_2 be the principal curvatures at the point P of the surface patch σ then to prove case (i), let us assume $\hat{t}_1$ & $\hat{t}_2$ be any two <u>~~⊥~~ unit tangent vectors to</u> the <u>surface at</u> P.

Then we can write for some scalars, ξ_1, η_1 & ξ_2, η_2, the tangent vectors as

$$\hat{t}_1 = \xi_1 \sigma_u + \eta_1 \sigma_v \; ; \; \hat{t}_2 = \xi_2 \sigma_u + \eta_2 \sigma_v .$$

And let $T_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}$; $T_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$.

Now,

let us consider $A = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}$ and calculate $A^t \mathcal{F}_I A$,

$$A^t \mathcal{F}_I A = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \begin{pmatrix} E\xi_1 + F\eta_1 & E\xi_2 + F\eta_2 \\ F\xi_1 + G\eta_1 & F\xi_2 + G\eta_2 \end{pmatrix}$$

$$= \begin{bmatrix} \xi_1(E\xi_1+F\eta_1)+\eta_1(F\xi_1+G\eta_1) & \xi_1(E\xi_2+F\eta_2)+\eta_1(F\xi_2+G\eta_2) \\ \xi_2(E\xi_1+F\eta_1)+\eta_2(F\xi_1+G\eta_1) & \xi_2(E\xi_2+F\eta_2)+\eta_2(F\xi_2+G\eta_2) \end{bmatrix} \rightarrow (3)$$

Now,

$$\hat{t}_1 \cdot \hat{t}_1 = (\xi_1 \sigma_u + \eta_1 \sigma_v) \cdot (\xi_1 \sigma_u + \eta_1 \sigma_v)$$

$$= \xi_1 \xi_1 (\sigma_u \cdot \sigma_u) + \xi_1 \eta_1 (\sigma_u \cdot \sigma_v) + \eta_1 \xi_1 (\sigma_v \cdot \sigma_u) + \eta_1 \eta_2 (\sigma_v \cdot \sigma_v)$$

$$= \xi_1 \xi_1 E + \xi_1 \eta_1 F + \eta_1 \xi_1 F + G \eta_1 \eta_1$$

$$= \xi_1 (E\xi_1 + F\eta_1) + \eta_1 (F\xi_1 + G\eta_1) \longrightarrow (a)$$

$$\hat{t}_2 \cdot \hat{t}_2 = (\xi_2 \sigma_u + \eta_2 \sigma_v) \cdot (\xi_2 \sigma_u + \eta_2 \sigma_v)$$

$$= \xi_2 \xi_2 (\sigma_u \cdot \sigma_u) + \xi_2 \eta_2 (\sigma_u \cdot \sigma_v) + \eta_2 \xi_2 (\sigma_v \cdot \sigma_u) + \eta_2 \eta_2 (\sigma_v \cdot \sigma_v)$$

$$= \xi_2 \xi_2 E + \xi_2 \eta_2 F + \eta_2 \xi_2 F + G \eta_2 \eta_2$$

$$= \xi_2 (E\xi_2 + F\eta_2) + \eta_2 (F\xi_2 + G\eta_2) \longrightarrow (b)$$

Also,

$$\hat{t}_1 \cdot \hat{t}_2 = (\xi_1 \sigma_u + \eta_1 \sigma_v) \cdot (\xi_2 \sigma_u + \eta_2 \sigma_v)$$

$$= \xi_1 \xi_2 (\sigma_u \cdot \sigma_u) + \xi_1 \eta_2 (\sigma_u \cdot \sigma_v) + \xi_2 \eta_1 (\sigma_v \cdot \sigma_u) + \eta_1 \eta_2 (\sigma_v \cdot \sigma_v)$$

$$= \xi_1 \xi_2 E + \xi_1 \eta_2 F + \xi_2 \eta_1 F + \eta_1 \eta_2 G$$

$$= \xi_1 (E\xi_2 + F\eta_2) + \eta_1 (F\xi_2 + \eta_2 G) \longrightarrow (c)$$

Substituting (a), (b) & (c) in (3),

$$A^t \mathcal{F}_I A = \begin{pmatrix} \hat{t}_1 \cdot \hat{t}_1 & \underbrace{\hat{t}_1 \cdot \hat{t}_2}_{0} \\ \underbrace{\hat{t}_2 \cdot \hat{t}_1}_{0} & \hat{t}_2 \cdot \hat{t}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left[\because \hat{t}_1 \perp^r \hat{t}_2 \text{ \& } \hat{t}_1 \text{ \& } \hat{t}_2 \text{ are unit vectors} \right]$$

$$= I \longrightarrow (4)$$

Let us consider the matrix $\mathcal{G}_{II}$ to denote $A^t \mathcal{F}_{II} A$.

Then $\mathcal{G}_{II}$ is a symmetric matrix, because $A^t = A$

$$\mathcal{G}_{II}^t = (A^t \mathcal{F}_{II} A)^t = (A^t \mathcal{F}_{II}^t (A^t)^t)$$

$$= A^t \mathcal{F}_{II} A = \mathcal{G}_{II}$$

Since $\mathcal{F}_{II}^t = \begin{pmatrix} L & M \\ M & N \end{pmatrix}^t = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \mathcal{F}_{II}$

From linear algebra,

w.k.t symmetric matrices can be diagonalized with orthogonal matrices.

For diagonalizing the symmetric matrix $\mathcal{G}_{II}$ let us consider a orthogonal matrix B.

$\ni B^t g_{II} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \rightarrow (5)$ $\left\{ \begin{array}{l} \because B \text{ is orthogonal,} \\ BB^t = I \Rightarrow B^{-1} = B^t \end{array} \right\}$

where λ_1 & λ_2 are real numbers.

Let $C = AB$ and consider,

$$C^t \mathcal{F}_I C = (AB)^t \mathcal{F}_I (AB)$$

[if A is invertible $\Leftrightarrow \det A \neq 0$]

$$= B^t (A^t \mathcal{F}_I A) B$$

$$= B^t I B \quad (\text{by } (4))$$

$$= B^t B$$

$$= I \longrightarrow (6)$$

$|A| = 0$ (singular)

$|A| \neq 0$ (non-singular)

Consider,

$$C^t \mathcal{F}_{II} C = (AB)^t \mathcal{F}_{II} (AB)$$

$$= B^t (A^t \mathcal{F}_{II} A) B$$

$$= B^t g_{II} B$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \rightarrow (7) \quad (\text{by } (5))$$

W.K.T,

both A & B are invertible, hence the product is also invertible. Hence C is invertible and hence non-singular.

Eqn ① of principal $\overset{\text{curvatures}}{\cancel{\text{vector}}}$ gives

$$\det(\mathcal{F}_{II} - k\mathcal{F}_I) = 0$$

$$\Rightarrow \det(c^t)\det(\mathcal{F}_{II} - k\mathcal{F}_I)\det(c) = 0$$

$$\Rightarrow \det(c^t(\mathcal{F}_{II} - k\mathcal{F}_I)c) = 0$$ [where $\det(c^t)$ & $\det(c)$ are non-zero.

$$\Rightarrow \det(c^t\mathcal{F}_{II}c - c^t k\mathcal{F}_I c) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} - k\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad [\text{by } ⑥ \ \& \ ⑦]$$

$$\Rightarrow \begin{vmatrix} \lambda_1 - k & 0 \\ 0 & \lambda_2 - k \end{vmatrix} = 0$$

$$\Rightarrow (\lambda_1 - k)(\lambda_2 - k) = 0$$

$$\Rightarrow k = \lambda_1, \lambda_2$$

$\Rightarrow$ k is <u>real</u>, as λ_1, λ_2 are real principal curvatures.

ii) Let us suppose that the principal curvatures are equal to k.

ie) let us assume that $k_1 = k_2 = k$

$$⑥ \Rightarrow c^t\mathcal{F}_{I}c = I$$

$$⑦ \Rightarrow c^t\mathcal{F}_{II}c = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad \because k = k_1 = k_2$$

$$= k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

ie) $c^t\mathcal{F}_{II}c = kI$

Consider,

$$\underbrace{c^t \mathcal{F}_{II}\, c}_{} - k \underbrace{c^t \mathcal{F}_{I}\, c}_{} = kI - kI = 0$$

ie., $c^t (\mathcal{F}_{II} - k \mathcal{F}_{I}) c = 0$

$$(\mathcal{F}_{II} - k \mathcal{F}_{I}) = 0 \quad [\because c^t \,\&\, c \text{ are invertible}]$$

$\hookrightarrow$ (8)

Consider $\hat{t}$ to be any tangent vector such that $\boxed{\hat{t} = \xi\, \sigma_u + \eta\, \sigma_v}$ & corresponding co-eff matrix $T = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ then by defn of principal vector and from eqn (8), we can write,

$$(\mathcal{F}_{II} - k \mathcal{F}_{I}) T = 0$$

Hence, any tangent vector to σ at a point P will be a principal vector, which proves case (ii).

iii) Let us assume that $\hat{t}_1$ and $\hat{t}_2$ are 2 tangent vectors given by,

$$\hat{t}_1 = \xi_1 \sigma_u + \eta_1 \sigma_v$$

$$\hat{t}_2 = \xi_2 \sigma_u + \eta_2 \sigma_v$$

with $T_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}$ & $T_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$

Then $\hat{t}_1 \cdot \hat{t}_2 = T_1^{t} \mathcal{F}_{I} T_2$ [by (*)]

Let us take K_1 & K_2 to be the principal curvatures corresponding to the tangent vectors $\hat{t}_1$ & $\hat{t}_2$ and assume that $K_1 \neq K_2$

From the defn of principal vector, by eqn (2), we have

$$(\mathcal{F}_{II} - K\,\mathcal{F}_I)\,T = 0$$

Applying for the tangent vectors $\hat{t}_1$ & $\hat{t}_2$ we have

$$(\mathcal{F}_{II} - K_1 \mathcal{F}_I)\,T_1 = 0 \quad \& \quad (\mathcal{F}_{II} - K_2 \mathcal{F}_I)\,T_2 = 0$$

$$\Rightarrow \mathcal{F}_{II} T_1 = K_1 \mathcal{F}_I T_1 \quad \& \quad \mathcal{F}_{II} T_2 = K\,\mathcal{F}_I T_2$$

ie) $T_2^t \mathcal{F}_{II} T_1 = K_1 \underbrace{T_2^t \mathcal{F}_I T_1} \;\&\; T_1^t \mathcal{F}_{II} T_2 = K\underbrace{T_1^t \mathcal{F}_I T_2}$

ie) $T_2^t \mathcal{F}_{II} T_1 = K_1 (\hat{t}_1 \cdot \hat{t}_2) \;\&\; T_1^t \mathcal{F}_{II} T_2 = K_2 (\hat{t}_1 \cdot \hat{t}_2)$ $\rightarrow$ (9)

The matrix $T_2^t \mathcal{F}_{II} T_1$ is a 1×1 matrix and hence it will be equal to its transpose.

$$T_2^t \mathcal{F}_{II} T_1 = (T_2^t \mathcal{F}_{II} T_1)^t = T_1^t \mathcal{F}_{II}^t (T_2^t)^t$$
$$= T_1^t \mathcal{F}_{II} T_2$$

$$\Rightarrow \boxed{T_2^t \mathcal{F}_{II} T_1 = T_1^t \mathcal{F}_{II} T_2} \rightarrow (f)$$

By (f), ~~RHS~~ of (9) same

Therefore from (9),

LHS is the same, hence equating

the RHS,

$$K_1 (\hat{t}_1 \cdot \hat{t}_2) = K_2 (\hat{t}_1 \cdot \hat{t}_2)$$

$$\Rightarrow (K_1 - K_2)(\hat{t}_1 \cdot \hat{t}_2) = 0$$

Since $K_1 \neq K_2 \Rightarrow K_1 - K_2 \neq 0$

hence $\hat{t}_1 \cdot \hat{t}_2 = 0$

$$\Rightarrow \hat{t}_1 \perp r \ \hat{t}_2$$

which proves case (iii).

Example 6.3

Prove that for a sphere, the principal curvatures are equal to each other at every point and are constant.

sphere $\rightarrow \sigma(\theta, \phi) = (\cos\theta \cos\phi, \cos\theta \sin\phi, \sin\theta)$

Soln:

The FFF is $E = 1$; $F = 0$; $G = \cos^2\theta$

The SFF is $L = 1$; $M = 0$; $N = \cos^2\theta$

$\therefore$ The principal curvatures are the roots of the eqn, $\det(\mathcal{F}_{II} - K\mathcal{F}_I) = 0$

$$\mathcal{F}_{II} = \begin{vmatrix} L & M \\ M & N \end{vmatrix}$$

ie) $$\begin{vmatrix} L-KE & M-KF \\ M-KF & N-KG \end{vmatrix} = 0 \qquad \mathcal{F}_I = \begin{vmatrix} E & F \\ F & G \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1-\kappa & 0 \\ 0 & \cos^2\theta\,(1-\kappa) \end{vmatrix} = 0$$

$$\Rightarrow \cos^2\theta\,(1-\kappa)(1-\kappa) = 0.$$

$$\Rightarrow \kappa = 1, 1$$

Hence the principal curvatures are the same and are constant.

Example : 6.4

We consider the circular cylinder of radius one and axis the z-axis, parametised in the usual way:

$\sigma_u = (0, 0, 1)$

$\sigma_v = (-\sin v, \cos v, 0)$

$$\boxed{\sigma(u, v) = (\cos v, \sin v, u)} \rightarrow (*)$$

We found in Eg: 5.3 that

FFF of (*) $E = 1,\ F = 0,\ G = 1,$

and in eg: 6.2 that

SFF of (*) $L = 0,\ M = 0,\ N = 1.$

So the principal curvatures are the roots of

$$\begin{vmatrix} 0-\kappa & 0 \\ 0 & 1-\kappa \end{vmatrix} = 0,$$

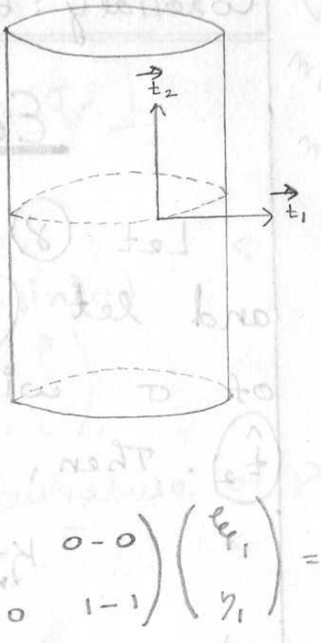
$$\Rightarrow \kappa(\kappa - 1) = 0$$

$$\Rightarrow \kappa = 0 \text{ (or) } 1$$

To find the principal vectors $\hat{t}_1$ and $\vec{t}_2$,

recall that $t_i = \xi_i \sigma_u + \eta_i \sigma_v$,

where $T_i = \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}$ satisfies



$$(\mathcal{F}_{II} - \kappa_i \mathcal{F}_I) T_i = 0$$

i.e. $$\begin{pmatrix} L-\kappa_i E & M-\kappa_i F \\ M-\kappa_i F & N-\kappa_i G \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = 0$$

$$\rightarrow \begin{pmatrix} 0-1 & 0-0 \\ 0-0 & 1-1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = 0$$

For $\boxed{\kappa_1 (=1)}$, we get

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = 0$$

$$\Rightarrow \boxed{\xi_1 = 0}$$

So T_1 is a multiple of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and hence $\hat{t}_1$ is a multiple of $0\sigma_u + 1\sigma_v = \sigma_v = (-\sin v, \cos v, 0)$.

$\|^{ly}$,

For $\boxed{\kappa_2 (=0)}$,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = 0$$

$$\Rightarrow \boxed{\eta_2 = 0}$$

So T_2 is a multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and hence $\hat{t}_2$ is a multiple of $\sigma_u = (0, 0, 1)$,

i.e. $1\sigma_u + 0\sigma_v = \sigma_u = (0, 0, 1)$.

✓ Corollary : 6.1

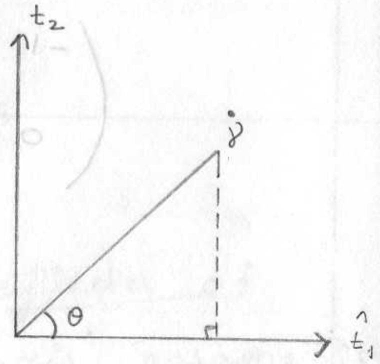
20m
8m

Euler's Theorem

Let γ be a curve on a surface patch σ, and let κ_1 & κ_2 be the principal curvatures of σ with non-zero principal vectors $\hat{t}_1$ & $\hat{t}_2$. Then, the normal curvature of γ is

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta ,$$

where θ is the angle between $\dot{\gamma}$ and $\hat{t}_1$.



Proof:

We can assume that γ is unit-speed.

Let $\hat{t}$ be the tangent vector of γ, and let

$$\hat{t} = \xi \sigma_u + \eta \sigma_v , \; T = \begin{pmatrix} \xi \\ \eta \end{pmatrix} .$$

Case (i) :

i) Suppose first that $\kappa_1 = \kappa_2 = \kappa$, say.

By proposition 6.3 (ii),

"If $\kappa_1 = \kappa_2 = \kappa$, say, then $\mathcal{F}_{II} = \kappa \mathcal{F}_I$ and (hence) every tangent vector to σ at P is a principal vector." (ie. $\hat{t}_1 \cdot \hat{t}_2 = T_1^t \mathcal{F}_I T_2$)

we have ;

the normal curvature of γ is

$$\kappa_n = T^t \mathcal{F}_{II} T \quad (\text{from prop. } 6.2 \; (**))$$

$$\Rightarrow K_n = K \underbrace{T^t \mathcal{F}_I T}$$ (by prop 6.3 (ii))

$$= K \underbrace{\hat{t}\cdot\hat{t}}_{1} \quad \left[\because \hat{t}_1 \cdot \hat{t}_2 = T_1^t \mathcal{F}_I T_2\right]$$

$$K_n = K \rightarrow (1)$$

Since $\left(K_1 \cos^2\theta + K_2 \sin^2\theta = K(\cos^2\theta + \sin^2\theta) = K\right.$,

From (1), we have, the normal curvature of γ is

$$K_n = K_1 \cos^2\theta + K_2 \sin^2\theta .$$

case ii) Assume now that $K_1 \neq K_2$.

By proposition 6.3 (iii),

"If $K_1 \neq K_2$, then any two (non-zero) principal vectors $\hat{t}_1$ and $\hat{t}_2$ corresponding to K_1 and K_2, respectively, are perpendicular."

we have

$\hat{t}_1$ and $\hat{t}_2$ are perpendicular.

We assume that $\hat{t}_1$ and $\hat{t}_2$ are unit vectors.

Let $\hat{t}_1 = \xi_1 \sigma_u + \eta_1 \sigma_v$, $T_1 = \begin{pmatrix}\xi_1\\ \eta_1\end{pmatrix}$

$\hat{t}_2 = \xi_2 \sigma_u + \eta_2 \sigma_v$, $T_2 = \begin{pmatrix}\xi_2\\ \eta_2\end{pmatrix}$

[ie. $\hat{t}_i = \xi_i \sigma_u + \eta_i \sigma_v$, $T_i = \begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix}$, for $i = 1, 2$.]

Now

$$\hat{\gamma} = \cos\theta\ \hat{t}_1 + \sin\theta\ \hat{t}_2$$

$$\Rightarrow \xi\sigma_u + \eta\sigma_v = \cos\theta(\xi_1\sigma_u + \eta_1\sigma_v) + \sin\theta(\xi_2\sigma_u + \eta_2\sigma_v)$$

$$\Rightarrow \begin{pmatrix}\xi \\ \eta\end{pmatrix} = \cos\theta\begin{pmatrix}\xi_1 \\ \eta_1\end{pmatrix} + \sin\theta\begin{pmatrix}\xi_2 \\ \eta_2\end{pmatrix}$$

$$\Rightarrow T = \cos\theta\ T_1 + \sin\theta\ T_2$$

Hence, by case (i),

the normal curvature of γ is

$$K_n = T^t \mathcal{F}_{II} T$$

$$= (\cos\theta\ T_1^t + \sin\theta\ T_2^t)\ \mathcal{F}_{II}(\cos\theta T_1 + \sin\theta T_2)$$

$$= \cos^2\theta\ T_1^t\ \mathcal{F}_{II} T_1 + \cos\theta\ \sin\theta\ T_1^t T_2 \mathcal{F}_{II}$$
$$+ \sin\theta\cos\theta(T_2^t\ \mathcal{F}_{II}\ T_1)$$
$$+ \sin^2\theta\ T_2^t\ \mathcal{F}_{II}\ T_2$$

$$= \cos^2\theta(T_1^t\ \mathcal{F}_{II}\ T_1) + \cos\theta\ \sin\theta\ (T_1^t \mathcal{F}_{II} T_2 + T_2^t \mathcal{F}_{II} T_1)$$
$$+ \sin^2\theta\ (T_2^t\ \mathcal{F}_{II}\ T_2) \rightarrow (2)$$

By defn of principal vector and

by eqn, $\hat{t}_1 \cdot \hat{t}_2 = T_1^t\ \mathcal{F}_I\ T_2$,

$$T_i^t\ \mathcal{F}_{II}\ T_j = K_i\ \overbrace{T_i^t\ \mathcal{F}_I\ T_j}^{\hat{t}_i \cdot \hat{t}_j}$$

$$\left\{\begin{array}{ll}\hat{t}_1 \cdot \hat{t}_2 = 0, & i \neq j \\ \hat{t}_1 \cdot \hat{t}_1 = 1, & i = j\end{array}\right.$$

$$= \begin{cases} K, & \text{if } i = j \\ 0, & \text{otherwise}\end{cases} \longrightarrow (3)$$

sub (3) in (2), we get

$$T_1^t \mathcal{F}_{II} T_1 = K_1 \underbrace{T_1^t \mathcal{F}_{I} T_1}$$

$$= K_1 \underset{\| \, 1}{(\hat{t}_1 \cdot \hat{t}_1)}$$

$$= K_1 \rightarrow (a)$$

$$T_1^t \mathcal{F}_{II} T_2 = 0 \xrightarrow{(b)} ; \text{ also } T_2^t \mathcal{F}_{II} T_1 = 0 \rightarrow (c)$$

$$T_2^t \mathcal{F}_{II} T_2 = K_2 \underbrace{T_2^t \mathcal{F}_{I} T_2}$$

$$= K_2 (\hat{t}_2 \cdot \hat{t}_2) = 1$$

$$= K_2 \rightarrow (d)$$

$$\cancel{K_n = K_n \cos^2\theta + K_2 \sin^2\theta}$$

Sub (a), (b), (c) & (d) in (2),

we get

$$K_n = \cos^2\theta \, K_1 + \cos\theta \sin\theta (0+0) + \sin^2\theta \, K_2$$

$$= \cos^2\theta \, K_1 + \sin^2\theta \, K_2$$

Hence the proof.

Corollary : 6.2

The principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Moreover, the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.

Proof:

i) If the principal curvatures K_1 & K_2 are different, we might as well suppose that $K_1 > K_2$.

Let K_n be the normal curvature of a curve γ on the surface.

Then,

$$K_n = K_1 \cos^2\theta + K_2 \sin^2\theta$$

$$= K_1 (1-\sin^2\theta) + K_2 \sin^2\theta$$

$$= K_1 - K_1 \sin^2\theta + K_2 \sin^2\theta$$

$$= K_1 - (K_1 + K_2) \sin^2\theta$$

check

$K_n = K_1 \cos^2\theta + K_2 (1-\cos^2\theta)$

$= K_1 \cos^2\theta + K_2 - K_2 \cos^2\theta$

$= K_2 \mp (K_1 + K_2) \cos^2\theta$

$\Rightarrow K_n \geq K_2$

Since $K_n = K_1 - (K_1 + K_2) \sin^2\theta$,

it is clear that

$K_n \leq K_1$ with equality $\Leftrightarrow \theta = 0$ or π.

ie. $\Leftrightarrow$ the tangent vector $\dot{\gamma}$ of γ is parallel to the principal vector $\hat{t}_1$.

III'y,

One shows that $K_n \geq K_2$ with equality $\Leftrightarrow \dot{\gamma}$ is parallel to $\hat{t}_2$.

ii) If $K_1 = K_2$,

~~the normal curvature of every curve is equal to K_1~~

by Euler's theorem, $\rightarrow K_n = K_1 \cos^2\theta + K_1 \sin^2\theta$

$K_n = K_1 (\cos^2\theta + \sin^2\theta)$

the normal curvature of every curve is equal to K_1 and

by proposition 6.3(ii),

every tangent vector to the surface is a principal vector.

Proposition 6.4:

Let $\hat{N}$ be the standard unit normal of a surface patch $\sigma(u,v)$.

Then $\hat{N}_u = a\sigma_u + b\sigma_v$, $\hat{N}_v = c\sigma_u + d\sigma_v$,

where $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}$.

The matrix $\mathcal{F}_I^{-1}\mathcal{F}_{II}$ is called the Weingarten matrix of the surface patch σ, and is denoted by $\mathcal{W}$.

Proof:

Since $\hat{N}$ is a unit vector,

W.K.T,

$\hat{N}_u$ & $\hat{N}_v$ are $\perp$r to $\hat{N}$.

Hence $\hat{N}_u$ & $\hat{N}_v$ are in the tangent plane to σ, and hence are linear combinations of σ_u & σ_v.

So scalars a, b, c & d satisfying the eqn

$\hat{N}_u = a\sigma_u + b\sigma_v$ & $\hat{N}_v = c\sigma_u + d\sigma_v$ exist.

$\hookrightarrow$ 1 $\qquad\qquad$ $\hookrightarrow$ 2

To calculate eqn 1 & 2,

i) $\boxed{\hat{N}\cdot\sigma_u = 0}$

Differentiating w.r.t 'u', we get

$$\hat{N}_u\cdot\sigma_u + \hat{N}\cdot\sigma_{uu} = 0$$

$$\Rightarrow \hat{N}_u\cdot\sigma_u = -\hat{N}\cdot\sigma_{uu} \rightarrow (3)$$

~~Since~~

W.K.T,

the SFF is $L\,du^2 + 2M\,du\,dv + N\,dv^2$

where
$$\left.\begin{aligned} L &= \sigma_{uu}\cdot\hat{N} \\ M &= \sigma_{uv}\cdot\hat{N} \\ N &= \sigma_{vv}\cdot\hat{N} \end{aligned}\right\} (4)$$

From (4),

(3) $\Rightarrow$ $\hat{N}_u\cdot\sigma_u = -L$

Similarly,

ii) $\boxed{\hat{N}\cdot\sigma_v = 0}$

Differentiating w.r.t 'v', we get

$$\hat{N}_v\cdot\sigma_v + \hat{N}\cdot\sigma_{vv} = 0$$

$$\Rightarrow \hat{N}_v\cdot\sigma_v = -\hat{N}\cdot\sigma_{vv}$$

by (4), $\Rightarrow \hat{N}_v\cdot\sigma_v = -N$

Also,

iii) $\boxed{\hat{N} \cdot \sigma_v = 0}$

Diff. w.r.t 'u', we get

$$\hat{N}_u \cdot \sigma_v + \hat{N} \cdot \sigma_{uv} = 0$$

$$\Rightarrow \hat{N}_u \cdot \sigma_v = -\hat{N} \cdot \sigma_{uv}$$

$$\Rightarrow \hat{N}_u \cdot \sigma_v = -M \quad [\text{by } (4')]$$

Now,

taking the <u>dot product</u> of the eqn (1') with σ_u & σ_v.

Thus $\boxed{\hat{N}_u \cdot \sigma_u} = (a\sigma_u + b\sigma_v) \cdot \sigma_u$

i)(a) $$= a\, \sigma_u \cdot \sigma_u + b\, \sigma_v \cdot \sigma_u \rightarrow (5')$$

W.K.T,

the FFF is $E\,du^2 + 2F\,du\,dv + G\,dv^2$

$$\text{where} \quad \left.\begin{aligned} E &= \|\sigma_u\|^2 \\ F &= \sigma_u \cdot \sigma_v \\ G &= \|\sigma_v\|^2 \end{aligned}\right\} (6')$$

by (6') & (4'),

$$(5') \Rightarrow \cancel{\hat{N}_u \cdot \sigma_u = \alpha E} \; -L = aE + bF$$

and $\boxed{\hat{N}_u \cdot \sigma_v} = (a\sigma_u + b\sigma_v) \cdot \sigma_v$

(ii)(a) $$= a\, \sigma_u \cdot \sigma_v + b\, \sigma_v \cdot \sigma_v$$

$$\Rightarrow -M = aF + bG \quad [\text{by } (6') \,\&\, (4')]$$

Taking the dot product of the eqn (2) with σ_u & σ_v,

Thus

$$\boxed{\hat{N}_v \cdot \sigma_u} = (c\,\sigma_u + d\,\sigma_v)\cdot\sigma_u$$

(iii)(a)

$$= c\,\sigma_u\cdot\sigma_u + d\,\sigma_v\cdot\sigma_u$$

$$\Rightarrow \quad -M = cE + dF \quad [\text{by } (4) \,\&\, (6)]$$

$$\boxed{\hat{N}_v \cdot \sigma_v} = (c\,\sigma_u + d\,\sigma_v)\cdot\sigma_v$$

(iii)(b)

$$= c\,\sigma_u\cdot\sigma_v + d\,\sigma_v\cdot\sigma_v$$

$$\Rightarrow \quad -N = cF + dG$$

$\therefore$ We have four scalar equations,

$$-L = aE + bF, \quad -M = aF + bG$$

$$-M = cE + dF, \quad -N = cF + dG.$$

These four scalar eqns are equivalent to the single matrix eqn

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\therefore \quad -\mathcal{F}_{II} = \mathcal{F}_{I}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\mathcal{F}_{I}^{-1}\,\mathcal{F}_{II}$$

Unit - V

5.2) The Pseudosphere

$\sigma(u,v) = \left(f(u)\cos v,\ f(u)\sin v,\ g(u) \right)$

Case (ii):

$K = 1$

$K = \frac{-\ddot{f}}{f}$

$1 = \frac{-\ddot{f}}{f}$

$\Rightarrow \ddot{f} + f = 0$

$f(u) = a\cos(u+b)$

$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^n$

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$

$\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$

$\phi^{-1} : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$

$\Rightarrow \tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$

where $\tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$

$\tilde{U} = U + b$

$\tilde{V} = V$

$f(u) = a\cos v$

$\dot{f}^2 + \dot{g}^2 = 1$

$\dot{g}^2 = 1 - \dot{f}^2$

$\dot{g}^2 = 1 - a^2\sin^2 u$

$\dot{g}(u) = \sqrt{1 - a^2\sin^2 u}$

$g(u) = \int \sqrt{1 - a^2\sin^2 u}\, du$

$a = 0$ or $a = \pm 1$

$a = 1$

$\Rightarrow g(u) = \sin u,$

$f(u) = \cos u$

case (iii)

$K = -1$

$K = \frac{-\ddot{f}}{f}$

$-1 = \frac{-\ddot{f}}{f} \Rightarrow \ddot{f} + f = 0$

$f(u) = ae^{u} + be^{-u}$

$= e^{u}$ $\boxed{a=1}$ $\boxed{b=0}$

W.K.T, $\dot{f}^2 + \dot{g}^2 = 1$

$\dot{g}^2 = 1 - \dot{f}^2$

$\dot{g}(u) = \sqrt{1 - e^{2u}}$

$g(u) = \int \sqrt{1 - e^{2u}}\, du$

$\boxed{u \leq 0}$

7.3) Flat Surfaces:

Divya

Example:

$$\sigma(u,v) = \gamma(u) + v\vec{\delta}(u)$$

$$\sigma_u = \dot{\gamma} + v\dot{\vec{\delta}}$$

$$\sigma_v = \vec{\delta}$$

$$\sigma_{uv} = \dot{\vec{\delta}}$$

$$\sigma_{vv} = 0$$

$$K = \frac{-(\dot{\vec{\delta}} \cdot \hat{N})^2}{EG - F^2} \leq 0$$

$$\hat{N} = \frac{(\sigma_u \times \sigma_v)}{\|\sigma_u \times \sigma_v\|}$$

$$K = 0 \iff \dot{\vec{\delta}} \cdot \hat{N} = 0$$

ie) $\dot{\vec{\delta}} \cdot (\sigma_u \times \sigma_v) = 0$

$$\sigma_u \times \sigma_v = (\dot{\gamma} + v\dot{\vec{\delta}}) \times \vec{\delta}$$

$$= \dot{\gamma} \times \vec{\delta} + v\dot{\vec{\delta}} \times \vec{\delta}$$

$$\vec{\delta} \cdot (\dot{\gamma} \cdot \vec{\delta}) = 0$$

$$\dot{\vec{\delta}} \cdot (\sigma_u \times \sigma_v) = \dot{\delta}(\dot{\gamma} \times \vec{\delta})^{\vec{a} \cdot \vec{b}} + \dot{\vec{\delta}}(v\dot{\vec{\delta}} \times \vec{\delta})$$

$$K = 0 \iff \dot{\delta}(\dot{\gamma} \times \vec{\delta}) = 0$$

$\Rightarrow K = 0 \iff \dot{\vec{\delta}}, \vec{\delta}, \dot{\gamma}$ are linearly dependent

* S_1 & S_2
  $f : S_1 \to S_2$
  Gau. cu. $P = f(P)$
* result, $K = \dfrac{LN - M^2}{EG - F^2}$
* $[\hat{e}', \hat{e}'']$
* $\{\hat{e}', \hat{e}'', \hat{N}\}$
  $\downarrow$
  assume - right-handed
  ie, $\hat{N} = \hat{e}' \times \hat{e}''$
* express PD of e' & e'' w.r.t u & v - $\{\hat{e}', \hat{e}'', \hat{N}\}$

Gauss Remarkable Theorem

Theorem : 10.1

The gaussian curvature of a surface is preserved by isometries.

Proof:

If S_1 and S_2 are two surfaces, and if $\boxed{f : S_1 \to S_2}$ is an isometry between them, then for any point P in S_1, the gaussian curvature of S_1 at P is equal to that of S_2 at $f(P)$.

In proving the theorem,

by the result,

"If $f : S_1 \to S_2$ is an isometry $\Leftrightarrow$ the surface patch σ on S_1 and $f \circ \sigma$ on S_2 have the same first fundamental form",

it is enough to prove that they have the same gaussian curvature. This is given by the formula,

$$K = \frac{LN - M^2}{EG - F^2}$$

which depends on the coefficients L, N & M of the second fundamental form as well as the coefficients E, F & G of the first fundamental form.

To prove the theorem,

we shall make use of a smooth orthonormal basis $\{\hat{e}', \hat{e}''\}$ of the tangent plane at each point of the surface patch, where "smooth" means that $\hat{e}'$ & $\hat{e}''$ are smooth functions of the surface parameters (u, v).

Then,

$\{\hat{e}', \hat{e}'', \hat{N}\}$ is an orthonormal basis of $\mathbb{R}^3$ ($\hat{N}$, being the standard unit normal of σ), and we shall assume that it is right-handed . ie., $\hat{N} = \hat{e}' \times \hat{e}''$

We can express the partial derivatives of $\hat{e}'$ and $\hat{e}''$ with respect to u and v in terms of the orthonormal basis $\{\hat{e}', \hat{e}'', \hat{N}\}$.

Since both partial derivatives of $\hat{e}'$ are perpendicular to $\hat{e}'$,

the $\hat{e}'$ components of $\hat{e}'_u$ and $\hat{e}'_v$ are zero (and similarly for $\hat{e}''$).

Thus,
$$\left.\begin{aligned} \hat{e}'_u &= \alpha\hat{e}'' + \lambda'\hat{N} \\ \hat{e}'_v &= \beta\hat{e}'' + \mu'\hat{N} \\ \hat{e}''_u &= -\alpha\hat{e}' + \lambda''\hat{N} \\ \hat{e}''_v &= -\beta\hat{e}' + \mu''\hat{N} \end{aligned}\right\} \quad (1)$$

$\lambda', \mu', \lambda'', \mu''$ (which may depend

Further to prove the theorem, we need the following lemma.

Lemma : 10.1

With the above notation in ①, we have

$$\hat{e}'_u \cdot \hat{e}''_v - \hat{e}''_u \cdot \hat{e}'_v = \lambda'\mu'' - \lambda''\mu' \longrightarrow ②$$

$$= \alpha_v - \beta_u \longrightarrow ③$$

$$= \frac{LN - M^2}{(EG - F^2)^{1/2}} \longrightarrow ④$$

Proof:

Eqn ② follows immediately from eqn ①, since $\hat{e}', \hat{e}''$ and $\hat{N}$ are perpendicular unit vectors.

ie. $\hat{e}'_u \cdot \hat{e}''_v = (\alpha \hat{e}'' + \lambda' \hat{N}) \cdot (-\beta \hat{e}' + \mu'' \hat{N})$

$$= \cancel{\lambda'} (-\alpha\beta)(\hat{e}'' \cdot \hat{e}') + \alpha\mu''(\hat{e}'' \cdot \hat{N})$$

$$+ -\lambda'\beta(\hat{N} \cdot \hat{e}') + \lambda'\mu''(\hat{N} \cdot \hat{N})$$

$$= (-\alpha\beta)(0) + \alpha\mu''(0) + -\lambda'\beta(0) + \lambda'\mu''(1)$$

$$= \lambda'\mu''$$

$$\hat{e}''_u \cdot \hat{e}'_v = (-\alpha\hat{e}' + \lambda''\hat{N}) \cdot (\beta\hat{e}'' + \mu'\hat{N})$$

$$= \lambda''\mu'$$

$$\therefore \hat{e}'_u \cdot \hat{e}''_v - \hat{e}''_u \cdot \hat{e}'_v = \lambda'\mu'' - \lambda''\mu'$$

Now, to prove eqn ③,

we compute

$$\alpha_v - \beta_u = \frac{\partial}{\partial u}(\hat{e}' \cdot \hat{e}''_v) - \frac{\partial}{\partial v}(\hat{e}' \cdot \hat{e}''_u) \quad \text{, (by ①)}$$

doubt how

$$= \hat{e}'_u \cdot \hat{e}''_v + \cancel{\hat{e}' \cdot \hat{e}''_{uv}} - \hat{e}'_v \cdot \hat{e}''_u - \cancel{\hat{e}' \cdot \hat{e}''_{uv}}$$

$$= \hat{e}'_u \cdot \hat{e}''_v - \hat{e}''_u \cdot \hat{e}'_v$$

This proves eqn ③.

Now, to prove eqn ④,

we use the formula,

$$\hat{N}_u \times \hat{N}_v = K\,(\sigma_u \times \sigma_v) \longrightarrow (*)$$

i.e. $\hat{N}_u \times \hat{N}_v = (a\sigma_u + b\sigma_v) \times (c\sigma_u + d\sigma_v)$

$$= ac\,(\sigma_u \times \sigma_u) + ad\,(\sigma_u \times \sigma_v)$$

$$+ bc\,(\sigma_v \times \sigma_u) + bd\,(\sigma_v \times \sigma_v)$$

$$= (ad - bc)(\sigma_u \times \sigma_v)$$

doubt

$$= \det(\mathcal{F}_{I}^{-1}\,\mathcal{F}_{II})\,(\sigma_u \times \sigma_v)$$

$$= \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_{I}}(\sigma_u \times \sigma_v)$$

$$= \frac{\begin{vmatrix} L & M \\ M & N \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}(\sigma_u \times \sigma_v)$$

$$= \frac{LN - M^2}{EG - F^2} (\sigma_u \times \sigma_v) \longrightarrow (**)$$

$$= K (\sigma_u \times \sigma_v)$$

where $K = \dfrac{LN - M^2}{EG - F^2}$.

Combining $(*)$ with the formulas,

$$\hat{N} = \frac{\sigma_u \times \sigma_v}{\| \sigma_u \times \sigma_v \|} \quad \text{and} \quad \| \sigma_u \times \sigma_v \| = (EG - F^2)^{1/2}$$

we get,

$$\hat{N} \cdot \| \sigma_u \times \sigma_v \| = \sigma_u \times \sigma_v$$

$$\Rightarrow \hat{N} (EG - F^2)^{1/2} = \sigma_u \times \sigma_v \longrightarrow (***)$$

From $\cancel{(*)}$, $\cancel{(**)}$ $(***)$,

we can write eqn $(**)$ as

$$\hat{N}_u \times \hat{N}_v = (LN - M^2)(EG - F^2)^{-1} (EG - F^2)^{1/2} \; \hat{N}$$

$$= (LN - M^2)(EG - F^2)^{-1/2} \; \hat{N}$$

$$= \frac{LN - M^2}{(EG - F^2)^{1/2}} \; \hat{N}$$

And hence,

$$(\hat{N}_u \times \hat{N}_v) \cdot \hat{N} = \frac{LN - M^2}{(EG - F^2)^{1/2}} (\hat{N} \cdot \hat{N})$$

$$= \frac{LN - M^2}{(EG - F^2)^{1/2}} \longrightarrow (5)$$

Since $\hat{N} = \hat{e}' \times \hat{e}''$, $(\bar{a} \times \bar{b}) \cdot (\bar{c} \times d) = (\bar{a} \cdot \bar{c})(b \cdot d) - (a \cdot d)(b \cdot c)$

we get

$$(\hat{N}_u \times \hat{N}_v) \cdot \hat{N} = (\hat{N}_u \times \hat{N}_v) \cdot (\hat{e}' \times \hat{e}'')$$

$$= (\hat{N}_u \cdot \hat{e}')(\hat{N}_v \cdot \hat{e}'') - (\hat{N}_u \cdot \hat{e}'')(\hat{N}_v \cdot \hat{e}')$$

$$= (\hat{N} \cdot \hat{e}'_u)(\hat{N} \cdot \hat{e}''_v) - (\hat{N} \cdot \hat{e}''_u)(\hat{N} \cdot \hat{e}'_v)$$

$$= \lambda' \mu'' - \lambda'' \mu' \quad [\text{by } (1)] \longrightarrow (6)$$

wherein passing from the second line to the third, we used the eqns,

$$\hat{N}_u \cdot \hat{e}' = -\hat{N} \cdot \hat{e}'_u \quad ; \quad \hat{N}_u \cdot \hat{e}'' = -\hat{N} \cdot \hat{e}''_u ,$$

$$\hat{N}_v \cdot \hat{e}' = -\hat{N} \cdot \hat{e}'_v \quad ; \quad \hat{N}_v \cdot \hat{e}'' = -\hat{N} \cdot \hat{e}''_v .$$

which follows by differentiating $\hat{N} \cdot \hat{e}' = 0 = \hat{N} \cdot \hat{e}''$ with respect to u and v.

Putting eqns (5) & (6) together shows that the R.H.S of (2) and (4) are equal. Since (2) has already been established, this

With this, now we can prove the theorem.

Combining eqns ③ & ④, we get

$$\alpha_v - \beta_u = \frac{LN - M^2}{(EG - F^2)^{1/2}} = \frac{LN - M^2}{(EG - F^2)} (EG - F^2)^{1/2}$$

$$\text{W.K.T,} \quad K = \frac{LN - M^2}{(EG - F^2)}$$

Combining this, we can write

$$K = \frac{\alpha_v - \beta_u}{(EG - F^2)^{1/2}} \longrightarrow ⑦$$

So to prove the theorem, it suffices to show that,

for a suitable choice of $\{\hat{e}', \hat{e}''\}$,

the scalars α and β depend only on E, F and G.

Now,

we shall construct $[\hat{e}', \hat{e}'']$ by applying Gram-schmidt process to the basis $\{\sigma_u, \sigma_v\}$ of the tangent plane and will then show that they have the desired property.

So, we first define,

$$\hat{e}' = \frac{\sigma_u}{\|\sigma_u\|} \Rightarrow \hat{e}' = \varepsilon\, \sigma_u \quad \text{where } \varepsilon = E^{-1/2}$$

$$\left[\because E = \|\sigma_u\|^2\right]$$

Now, we look for a vector

$\hat{e}'' = \gamma \sigma_u + \delta \sigma_v$, for some scalars γ, δ

$\ni$ $\hat{e}''$ is a unit vector $\perp r$ to $\hat{e}'$.

These conditions give,

$$\hat{e}'' . \hat{e}' = 0 \quad \text{and} \quad \| \hat{e}'' \|^2 = 1$$

ie., $\hat{e}'' . \hat{e}' = 0$

$$\Rightarrow (\gamma \sigma_u + \delta \sigma_v) . (\varepsilon \sigma_u) = 0$$

$$\Rightarrow \gamma \varepsilon (\sigma_u . \sigma_u) + \delta \varepsilon (\sigma_v . \sigma_u) = 0$$

$$\Rightarrow \gamma (E^{-1/2})(E) + \delta (E^{-1/2})(F) = 0$$

$$\Rightarrow E^{-1/2} (\gamma E + \delta F) = 0 \longrightarrow 8 \text{ (i)}$$

$$[\because E = \sigma_u . \sigma_u \ \& \ F = \sigma_u . \sigma_v]$$

and $\| \hat{e}'' \|^2 = 1$

$$(\gamma \sigma_u + \delta \sigma_v) . (\gamma \sigma_u + \delta \sigma_v) = 1$$

$$\Rightarrow \gamma^2 (\sigma_u . \sigma_u) + \gamma \delta (\sigma_u . \sigma_v) + \delta \gamma (\sigma_v . \sigma_u) + \delta^2 (\sigma_v . \sigma_v) = 1$$

$$\Rightarrow \gamma^2 E + 2 \gamma \delta F + \delta^2 G = 1 \longrightarrow 8 \text{ (ii)}$$

$$[\because G = \sigma_v . \sigma_v]$$

Eqn 8(i) gives

$$E^{-1/2} (\gamma E + \delta F) = 0 \Rightarrow \gamma E + \delta F = 0$$

$$\Rightarrow \gamma = \frac{-\delta F}{E}$$

Substituting this in (8)(ii), we get

$$\left(\frac{-\delta F}{E}\right)^2 E + 2\left(\frac{-\delta F}{E}\right)\delta F + \delta^2 G = 1$$

$$\Rightarrow \frac{\delta^2 F^2}{E^{\cancel{2}}}\cancel{E} - 2\frac{\delta^2 F^2}{E} + \delta^2 G = 1$$

$$\Rightarrow \delta^2\left(\frac{F^2}{E} - 2\frac{F^2}{E} + G\right) = 1$$

$$\Rightarrow \delta^2\left(\frac{F^2 - 2F^2}{E} + G\right) = 1$$

$$\Rightarrow \delta^2\left(-\frac{F^2}{E} + G\right) = 1$$

$$\Rightarrow \delta^2\left(\frac{EG - F^2}{E}\right) = 1$$

$$\Rightarrow \delta^2 = \frac{E}{EG - F^2}$$

$$\therefore \Rightarrow \delta = \frac{E^{1/2}}{(EG - F^2)^{1/2}} \longrightarrow (9)(i)$$

Already we have $\gamma = \frac{-\delta F}{E}$

Now, substituting the value of δ (9)(i), we get

$$\gamma = \frac{-F}{E}\left(\frac{E^{1/2}}{(EG - F^2)^{1/2}}\right)$$

$$= \frac{-F}{E^{+1/2}(EG - F^2)^{1/2}}$$

$$\therefore \gamma = \frac{-F E^{-1/2}}{(EG - F^2)^{1/2}} \text{ and } \varepsilon = E^{-1/2} \rightarrow (9)(ii)$$

We could change the sign of δ and hence also that of γ, but it could make no difference in the end.

Thus $\hat{e}' = \varepsilon\,\sigma_u$ and $\hat{e}'' = \gamma\,\sigma_u + \delta\,\sigma_v \rightarrow$ (10)

where γ, δ and ε depend only on E, F & G.

To show that

α & β also depends only on E, F & G.

We shall now compute α, β.

$$\alpha = \hat{e}'_u \cdot \hat{e}'' \quad [\text{by } (1)]$$

$$= (\varepsilon_u \sigma_u + \varepsilon\,\sigma_{uu}) \cdot (\gamma\,\sigma_u + \delta\,\sigma_v) \quad [\text{by } (10)]$$

$$= \overbrace{\varepsilon_u \gamma (\sigma_u \cdot \sigma_u) + \varepsilon_u \delta (\sigma_u \cdot \sigma_v)}^{(\varepsilon_u \sigma_u)\cdot \gamma \sigma_u + (\varepsilon_u \sigma_u)\cdot \delta \sigma_v} + \varepsilon\gamma(\sigma_{uu} \cdot \sigma_u) + \varepsilon\delta(\sigma_{uu} \cdot \sigma_v)$$

$$= (\varepsilon_u\,\sigma_u)\cdot(\gamma\cdot\sigma_u + \delta\,\sigma_v) + \varepsilon\gamma(\sigma_{uu}\cdot\sigma_u) + \varepsilon\delta(\sigma_{uu}\cdot\sigma_v)$$

× & ÷ by ε on b.s

$$= \frac{\varepsilon}{\varepsilon}(\varepsilon_u \sigma_u)\cdot(\gamma\sigma_u + \delta\sigma_v) + \frac{\varepsilon}{\cancel{\varepsilon}}\cdot\cancel{\varepsilon}\gamma(\sigma_{uu}\cdot\sigma_u) + \frac{\varepsilon}{\cancel{\varepsilon}}\cdot\cancel{\varepsilon}\delta(\sigma_{uu}\cdot\sigma_v)$$

$$= \frac{\varepsilon_u}{\varepsilon}\,\hat{e}'\cdot\hat{e}'' + \varepsilon\gamma(\sigma_{uu}\cdot\sigma_u) + \varepsilon\delta(\sigma_{uu}\cdot\sigma_v)$$

$$= \cancel{\frac{\varepsilon_u}{\varepsilon}}\,\varepsilon\gamma\left(\tfrac{1}{2}(\sigma_u\cdot\sigma_u)_u\right) + \varepsilon\delta\left[(\sigma_u\cdot\sigma_v)_u - \sigma_u\cdot\sigma_{uv}\right]$$

(ii) (P) $\leftarrow$

$$= \frac{\varepsilon\gamma}{2}\underbrace{(\sigma_u\cdot\sigma_u)_u}_{E_u} + \varepsilon\delta\Big[\underbrace{(\sigma_u\cdot\sigma_v)_u}_{F_u} - \tfrac{1}{2}\overbrace{(\sigma_u\cdot\sigma_u)_v}^{E_v}\Big]$$

$[\because\ \hat{e}'\cdot\hat{e}'' = 0]$

$$\Big[\text{since} \quad (\sigma_u\cdot\sigma_u)_u = \sigma_{uu}\cdot\sigma_u + \sigma_u\cdot\sigma_{uu}$$

$$= 2\,\sigma_{uu}\cdot\sigma_u$$

$$\Rightarrow \quad \sigma_{uu}\cdot\sigma_u = \frac{1}{2}(\sigma_u\cdot\sigma_u)_u$$

$$\text{and} \quad (\sigma_u\cdot\sigma_v)_u = \sigma_{uu}\cdot\sigma_v + \sigma_u\cdot\sigma_{uv}$$

$$\Rightarrow \quad \sigma_{uu}\cdot\sigma_v = (\sigma_u\cdot\sigma_v)_u - \sigma_u\cdot\sigma_{uv}$$

$$\text{\& also} \quad (\sigma_u\cdot\sigma_u)_v = \sigma_{uv}\cdot\sigma_u + \sigma_u\cdot\sigma_{uv} = 2\,\sigma_{uv}\cdot\sigma_u$$

$$\Rightarrow \quad \sigma_{uv}\cdot\sigma_u = \frac{1}{2}(\sigma_u\cdot\sigma_u)_v\Big]$$

$$\Rightarrow \alpha = \frac{1}{2}\,\varepsilon\gamma E_u + \varepsilon\delta\left(F_u - \frac{1}{2}E_v\right) \rightarrow (11) \quad \cancel{[\because \hat{e}'\cdot\hat{e}''=0]}$$

which does indeed depend only on E, F & G (because the same is true for γ, δ & ε).

And finally,

$$\beta = \hat{e}'_v\cdot\hat{e}''$$

$$= (\varepsilon_v\sigma_u + \varepsilon\sigma_{uv})\cdot(\gamma\sigma_u + \delta\sigma_v)$$

$$= \frac{\varepsilon_v}{\varepsilon}\,\hat{e}'\cdot\hat{e}'' + \varepsilon\gamma\,\sigma_{uv}\cdot\sigma_u + \varepsilon\delta\,\sigma_{uv}\cdot\sigma_v$$

$$= \tfrac{1}{2}\,\varepsilon\gamma E_v + \tfrac{1}{2}\,\varepsilon\delta G_u \longrightarrow (12)$$

which also depends only on E, F & G.

This completes the proof of Gauss's theorem//.

Manifolds

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Proposition : 2.1.4

The Implicit Function Theorem

If (a,b) is a point in the domain of a C^∞ function $f : \mathbb{R}^{p+q} \equiv \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ such that $f(a,b) = 0$, $\det[f^i_{,j}(a,b)] \neq 0$ $(i,j = 1,\dots,p)$

then $\exists$ a nghd V of a in $\mathbb{R}^p$ &
a nghd W of b in $\mathbb{R}^q$
$\ni$ if $w \in W$ $\exists$ a unique point $v \in V$ for which $f(v,w) = 0$.

The function $\mathbb{R}^q \to \mathbb{R}^p$ defined on W by $w \mapsto v$ is a C^∞ function.

Proof:

To prove:

i) existence of nghds of a & b in $\mathbb{R}^{p+q}$

ii) existence of a C^∞ function $g : \mathbb{R}^q \to \mathbb{R}^p$

iii) Uniqueness of g

From hypothesis,

W.K.T,

$f : A \to \mathbb{R}^p$ where A is an open subset of $\mathbb{R}^{p+q}$ and (a,b) is a point in A where $a \in \mathbb{R}^p$ & $b \in \mathbb{R}^q$.

Now,

we construct a function F to which we can apply the Inverse function theorem,

define $F : A \subset \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$ as

$$F(x,y) = (f(x,y), y) \quad , \; x \in \mathbb{R}^p \;\&\; y \in \mathbb{R}^q$$

Then we can write it as $F = (f, I)$ where I is the identity function defined by $I(y) = y$

$\therefore$ The Jacobian matrix of F on A is

$$J_F(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}(f) & \frac{\partial}{\partial y}(f) \\ \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \end{bmatrix} = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$$

Claim :

$$\det\left[F^i_{\cdot j}(x,y)\right] \neq 0$$

ie.,

$$\left| \begin{array}{cccc|ccc} \multicolumn{4}{c|}{} & \multicolumn{3}{c}{a_i} \\ F^1_{\cdot 1} & F^1_{\cdot 2} & \cdots & F^1_{\cdot p} & F^1_{\cdot p+1} & \cdots & F^1_{\cdot p+q} \\ \vdots & & D & & & & \\ F^p_{\cdot 1} & F^p_{\cdot 2} & \cdots & F^p_{\cdot p} & F^p_{\cdot p+1} & \cdots & F^p_{\cdot p+q} \\ \hline F^{p+1}_{\cdot 1} & F^{p+1}_{\cdot 2} & \cdots & F^{p+1}_{\cdot p} & F^{p+1}_{\cdot p+1} & \cdots & F^{p+1}_{\cdot p+q} \\ \vdots & & 0 & & & I & \\ F^{p+q}_{\cdot 1} & F^{p+q}_{\cdot 2} & \cdots & F^{p+q}_{\cdot p} & F^{p+q}_{\cdot p+1} & \cdots & F^{p+q}_{\cdot p+q} \end{array} \right| \neq 0 \longrightarrow (1)$$

Consider the LHS of Jacobian matrix in (1),

Put $\cap$ of 1^{st} row & $p+j^{th}$ columns $= a_1$
$\cap$ of 2^{nd} row & $p+j^{th}$ columns $= a_2$
$\vdots$
$\cap$ of p^{th} row & $p+j^{th}$ columns $= a_p$

$\cap$ p rows & $p+j$ columns $= a_1, a_2, a_p$

and
$\cap$ of 1^{st} column & $p+j^{th}$ rows $= 0$
$\cap$ of 2^{nd} column & $p+j^{th}$ rows $= 0$
$\vdots$
$\cap$ of p^{th} column & $p+j^{th}$ rows $= 0$

Also let us take

i) $\cap$ of $p+j^{th}$ rows & coloumns as I

ii) J_F of $p \times p$ matrix as D

$$\Rightarrow \quad J_F(x,y) = \begin{bmatrix} \boxed{D} & \begin{matrix} a_1 \\ \vdots \\ a_p \end{matrix} \\ 0 \; 0 \cdots 0 & I \end{bmatrix} \rightarrow (2)$$

Result :

"Let A be an n by n matrix

Let b denote its entry in row i & column j.

a) If all the entries in row i other than b vanish

then $\quad \det A = b(-1)^{i+j} \det A_{ij} \rightarrow (3)$

b) The same eqn holds if all the entries in column j other than the entry b vanish."

By using this result,

we compare ~~with~~ this result with (2),

we have

* The matrix A as $J_F(x,y)$

* ~~b~~ Entry b as I

Eqn (3) $\Rightarrow \det A = b(-1)^{i+j} \det A_{ij} \neq 0$

$$\Rightarrow \det \left[F^i_{;j}(x,y) \right] \neq 0$$

W.K.T,
Since $F = (f, I)$

since $f \in C^\infty$ & $I \in C^\infty$ because I is the identity function,

$\Rightarrow F \in C^\infty$ on A.

Now, we can apply inverse function theorem,

"If z is a point in the domain of a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}^n \ni \det J_f z \neq 0$,

$\exists$ a nghd V of z

$\ni f_{|V}$ is a diffeomorphism."

So,
by applying inverse fn thm, we say that

$\exists$ an nghd $V \times W$ of (a, b) in $\mathbb{R}^{p+q}$

$\ni F_{|V \times W}$ is a diffeomorphim $\to$ W.k.T,

$\ni$ i) $F : V \times W \to U \subset \mathbb{R}^{p+q}$ is 1-1. F is 1-1 & F, F^{-1} are C^∞ fn

ii) $F^{-1} = G : U \to V \times W$ is a C^∞ function.

Since $F(x, y) = (f(x, y), y)$,

we have $(x, y) = F^{-1}(f(x, y), y)$

$= G(f(x, y), y)$

G preserves the p-coordinates as F does.

So we can write G in the form

$G(z, y) = (h(z, y), y)$ for $z \in \mathbb{R}^p$, $y \in \mathbb{R}^q$

where $h : U \to \mathbb{R}^p \in C^\infty$ fn.

Let B be a connected nghd of b in $\mathbb{R}^q$.
Choosen small enough that $O \times B$ is contained in U

Now,
we will prove the existence of c^∞ function

$$g : B \to \mathbb{R}^p$$

If $w \in W$ then $(0, w) \in U$

so $G(0, w) = (h(0, w), w)$

$\Rightarrow F^{-1}(0, w) = (h(0, w), w)$

$\Rightarrow (0, w) = F(h(0, w), w)$

$\Rightarrow (0, w) = (f(h(0, w), w), w)$

$\Rightarrow 0 = f(h(0, w), w)$

We set $g(w) = h(0, w)$ for $w \in B$

Then B satisfies the equation

$\boxed{f(g(w), w) = 0} \to$ * for which $g(w) \in \mathbb{R}^p$

Let $v \in V \subset \mathbb{R}^p \Rightarrow v \in \mathbb{R}^p$
$(v, w) \in \mathbb{R}^{p+q}$

$G(0, w) = (h(0, w), w) = (g(w), w)$

$\Rightarrow G(0, w) = (g(w), w)$ say $v = g(w) \to$ ** when $v \in \mathbb{R}^p$

$\Rightarrow v = g(w) \to$ **

By **, we can write * as follows.

$$f(v, w) = 0$$

$\Rightarrow \exists$ a c^∞ function $g : \mathbb{R}^q \to \mathbb{R}^p$

To prove :

Uniqueness of g

It remains to prove that there is only one such function g.

Suppose $g_0|_B : \mathbb{R}^q \to \mathbb{R}^p$ be continuous function satisfying the necessary conditions of g, then we would have

$$f(g(w), w) = f(g_0(w), w)$$

$$\Rightarrow (g(w), w) = (g_0(w), w) \quad [\because f \text{ is } 1\text{-}1]$$

$$\Rightarrow g(w) = g_0(w) \quad \forall\ w \in W$$

$\Rightarrow$ the c^∞ function g is unique.

Hence the proof,,