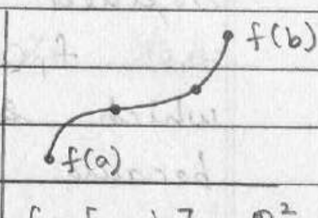
UNIT- I

Curves in the Plane & in Space

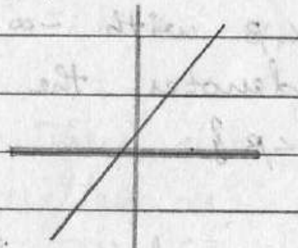
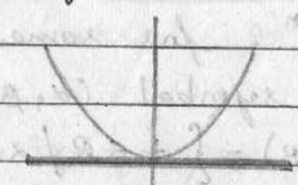
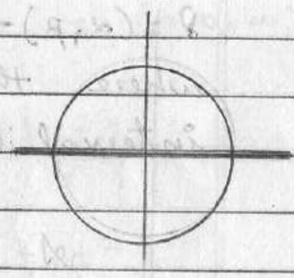
1.1) Curve

What is a Curve?



$f : [a, b] \to \mathbb{R}^2$

Examples:

i) $y - 2x = 1$	ii) $y - x^2 = 0$	iii) $x^2 + y^2 = 1$
		
"Straight line"	"Parabola"	"Circle"

In all the above examples, the curves are described through their cartesian equation $f(x,y) = c$, where f is a fn of x & y and c is a constant.

A curve can be defined to be a set of points,

$$C = \{ (x,y) \in \mathbb{R}^2 / f(x,y) = c \}$$

These egs are all curves in the plane $\mathbb{R}^2$.

The curves can also be ~~considered~~ defined in $\mathbb{R}^3$ as an example, the x-axis in $\mathbb{R}^3$ is a straight line given by

$$\{ (x, y, z) \in \mathbb{R}^3 / y = z = 0 \}$$

Generally a curve in $\mathbb{R}^3$ can be defined by a pair of equns
$$f_1(x,y,z) = c_1 \ , \ f_2(x,y,z) = c_2$$
which is called ~~linear~~ level curves because of the constants.

Definition:

A parametrised curve in $\mathbb{R}^n$ is a $\gamma : (\alpha,\beta) \to \mathbb{R}^n$, for some α,β with $-\infty \le \alpha < \beta$ where the symbol (α,β) denotes the open interval $(\alpha,\beta) = \{ t \in \mathbb{R} / \alpha < t < \beta \}$.

A parametrised curve whose image contained in a level curve C is called a parametrisation.

Example: 1

Find a parametrisation $\gamma(t)$ of the parabola $y = x^2$.

Soln:

The parametrisation can be expressed as,
$$\gamma(t) = (\gamma_1(t), \gamma_2(t)),$$
the components γ_1 and γ_2 of γ must satisfy $\gamma_2(t) = \gamma_1(t)^2 \rightarrow (*)$ $\forall\ t \in (\alpha,\beta)$ where γ is defined (yet to be ~~dep~~ decided).

When we find the values for γ_1 & γ_2 every pt on the parabola should be equal to $(\gamma_1(t), \gamma_2(t))$ for some value of $t \in (\alpha, \beta)$.

The obvious solution to (*) is to take $\gamma_1(t) = t$ and $\gamma_2(t) = t^2$.

To get every point on the parabola, we must allow t to take every real number value.

$\therefore$ We must take (α, β) to be $(-\infty, \infty)$.

Thus, the desired parametrisation can be written as

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2 \ , \ \gamma(t) = (t, t^2)$$

But this representation is not unique, we can have many representation like

$$\gamma(t) = (t^3, t^6), \ \gamma(t) = (2t, 4t^2) \text{ and so on,}$$

where the interval is $(-\infty, \infty)$.

Thus the parametrisation of a given level curves is not unique.

Example : (2)

Find a parametrisation for the circle $x^2+y^2=1$.

Soln:

To parametrize the given eqn:
if we take $x = t$,
then $y = \pm\sqrt{1-t^2}$

and if we take the parametrisation to be $\gamma(t) = (t, \sqrt{1-t^2})$, then we only obt upper half of the circle, because $\sqrt{1-t^2} \geq$

Similarly,
if we take $y = -\sqrt{1-t^2}$, then w obtain the lower half of the circle.

Thus to obtain parametrisation of the whole circle, we need fns $\gamma_1(t)$ & $\gamma_2(t)$ such that $\boxed{\gamma_1(t)^2 + \gamma_2(t)^2 = 1}$ $\forall$ $t \in (\alpha, \beta)$, a such that every pt on the circle is equal to $(\gamma_1(t), \gamma_2(t))$ for some $t \in (\alpha, \beta$

$\because$ $[\sin^2 t + \cos^2 t = 1$ $\forall$ $t]$

The obvious soln is $\boxed{\gamma_1(t) = \cos t}$, $\boxed{\gamma_2(t) = \sin t}$ for $t \in (\alpha, \beta) = (-\infty, \infty)$.

Example : (3)

Find the cartesian eqn for $\gamma(t) = (\cos^3 t, \sin^3$ (called an asteroid)

Soln:

Since $\cos^2 t + \sin^2 t = 1$ $\forall$ t, the coordinat $x = \cos^3 t$, $y = \sin^3 t$ of the pt $\gamma(t)$ satisfy

$$x^{2/3} + y^{2/3} = 1$$

This level curve coincides with the image of the map γ.

To differentiate a vector-valued fn such as $\gamma(t)$, we differentiate

Componentwise:

if $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$,

then

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \dots, \frac{d\gamma_n}{dt}\right),$$

$$\frac{d^2\gamma}{dt^2} = \left(\frac{d^2\gamma_1}{dt^2}, \frac{d^2\gamma_2}{dt^2}, \dots, \frac{d^2\gamma_n}{dt^2}\right), \text{etc.}$$

We often denote $\frac{d\gamma}{dt}$ by $\dot{\gamma}(t)$,

$\frac{d^2\gamma}{dt^2}$ by $\ddot{\gamma}(t)$, etc.

Def^n We say that γ is smooth if each of the components $\gamma_1, \gamma_2, \dots, \gamma_n$ of γ is smooth, i.e., if all the derivatives $\frac{d\gamma_i}{dt}, \frac{d^2\gamma_i}{dt^2}, \frac{d^3\gamma_i}{dt^3}, \dots$, exist, for $i = 1, 2, \dots, n$.

Tangent vector:

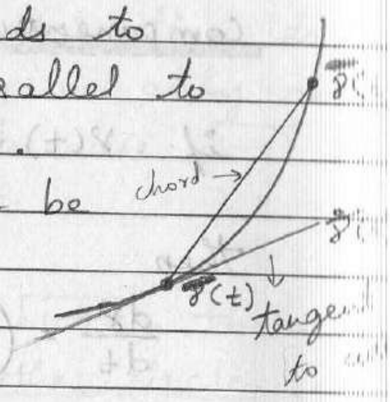
If $\gamma(t)$ is a parametrised curve, its first derivative $\frac{d\gamma}{dt}$ is called the tangent vector of γ at the point $\gamma(t)$.

Note that: The vector $\frac{\gamma(t+\delta t) - \gamma(t)}{\delta t}$ is parallel to the chord joining the two points $\gamma(t)$ and $\gamma(t+\delta t)$ of the image C of γ:

(diagram) - backside.

We expect that, as δt tends to zero, the chord becomes parallel to the tangent to C at $\gamma(t)$.

Hence, the tangent should be parallel to



$$\lim_{\delta t \to 0} \frac{\gamma(t+\delta t) - \gamma(t)}{\delta t} = \frac{d\gamma}{dt}$$

<u>Proposition 1.1</u>

If the tangent vector of a parametrised curve is constant, the image of the curve is (part of) a straight line.

<u>Proof:</u>

If $\dot{\gamma}(t) = \bar{a}$ for all t, where $\bar{a}$ is a constant vector,

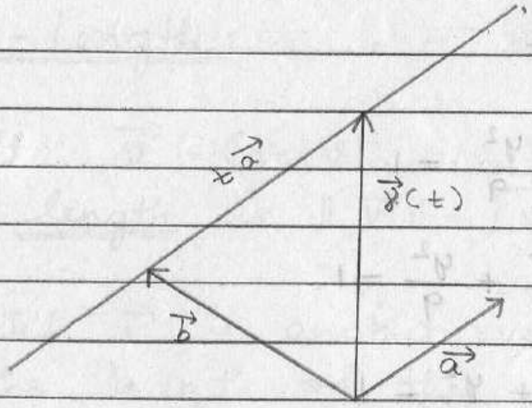
we have, <u>integrating</u> <u>componentwise</u>

$$\gamma(t) = \int \frac{d\gamma}{dt}\, dt = \int \bar{a}\, dt = t\bar{a} + \bar{b}$$

where $\bar{b}$ is another constant vector.

If $\boxed{\bar{a} \neq \bar{o}}$, this is the parametric eqn of the straight line $\parallel^{el}$ to $\bar{a}$ and passing through the point with position vector $\bar{b}$:

(diagram)



If $\boxed{\bar{a} = \bar{0}}$, the image of $\bar{\gamma}$ is a single point (namely, the pt with position vector).

Exercises:

1.1 Is $\gamma(t) = (t^2, t^4)$ a parametrisation of the parabola $y = x^2$?

Soln: Given $x = t^2, y = t^4$
Substitute these x & y values in $y = x^2$, we get,

$$t^4 = (t^2)^2$$

Hence the satisfy $y = x^2$
$\therefore \gamma(t)$ is a parametrisation of the parabola $y = x^2$.

1.2 Find parametrisation of the following level curves:

i) $y^2 - x^2 = 1$; ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Soln: i) Let $x = t$
Then $y = \sqrt{1+x^2} = \sqrt{1+t^2}$

$\Rightarrow \gamma(t) = (t, \sqrt{1+t^2})$ is the required parametrisation of $y^2 - x^2 = 1$.

ii) Take $x = 2t$

Then,

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\Rightarrow \frac{(2t)^2}{4} + \frac{y^2}{9} = 1$$

$$\Rightarrow t^2 + \frac{y^2}{9} = 1$$

$$\Rightarrow y^2 = 9(1-t^2)$$

$$\Rightarrow y = 3\sqrt{1-t^2}$$

$\therefore \gamma(t) = (2t, 3\sqrt{1-t^2})$ is the parametrisation of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ //

1.3 Find the cartesian equations of the following parametrised curves:

i) $\gamma(t) = (\cos^2 t, \sin^2 t)$;

ii) $\gamma(t) = (e^t, t^2)$.

Soln: i) Let $x = \cos^2 t$, $y = \sin^2 t$

$$\Rightarrow x + y = \cos^2 t + \sin^2 t = 1$$

$\therefore x + y = 1$ is the cartesian eqn of the curve.

ii) Let $x = e^t$, $y = t^2$

$$\Rightarrow \log x = t$$

$$\Rightarrow (\log x)^2 = t^2 \Rightarrow (\log x)^2 = y \Rightarrow \log x = y^{1/2}$$

$$\Rightarrow x = e^{y^{1/2}}$$

$\therefore x = e^{\sqrt{y}}$ is the cartesian eqn of the curve.

Arc-Length:

Defn: [Length]

If $\vec{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbb{R}^n$, its **length** is $\| \vec{v} \| = \sqrt{v_1^2 + \ldots + v_n^2}$.

If $\vec{u}$ is another vector in $\mathbb{R}^n$, $\| \vec{u} - \vec{v} \|$ is the length of the straight line segment joining the points in $\mathbb{R}^n$ with position vectors $\vec{u}$ and $\vec{v}$.

To find a formula for the length of any parametrised curve γ:

Let γ be a parametrised curve.
If ~~Let~~ δt be very small, then the part of the image C of γ between $\gamma(t)$ and $\vec{\gamma}(t+\delta t)$ is nearly a straight line, so its length is approximately

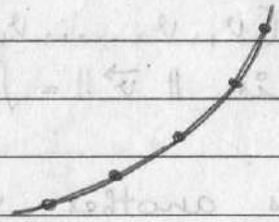
$$\| \gamma(t+\delta t) - \gamma(t) \|.$$

Since δt is very small, $\left(\frac{\gamma(t+\delta t) - \gamma(t)}{\delta t} \right)$ is nearly equal to $\dot{\gamma}(t)$, so the length is approximately

$$\| \dot{\gamma}(t) \| \delta t.$$

$$\text{Thus } \left\| \frac{\gamma(t+\delta t) - \vec{\gamma}(t)}{\delta t} \right\| \simeq \| \dot{\gamma}(t) \| \delta t \longrightarrow (*)$$

Thus we want to measure the length of part of C, we can divide it up into segments, each of which corresponds to a small increment δt) in t. Then we can calculate the length of each segment using (*), and add up the results. Letting δt tend to zero should then give the exact length.



Defn — Arc-length:

The arc-length of a curve γ starting at the point $\gamma(t_0)$ is the function $s(t)$ given by

$$\boxed{s(t) = \int_{t_0}^{t} \| \dot{\gamma}(u) \| \, du} \rightarrow (1)$$

Note: Thus $s(t_0) = 0$ and $s(t)$ is positive or negative according to whether t is larger or smaller than t_0.

If we choose a different starting point $\gamma(\tilde{t}_0)$, the resulting arc-length $\tilde{s}$ differs from s by the constant

$$\int_{t_0}^{\tilde{t}_0} \| \dot{\gamma}(u) \| \, du.$$

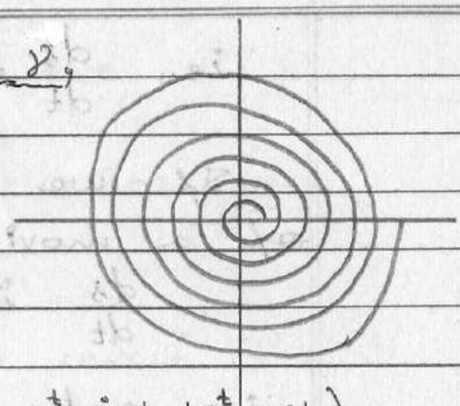
Example:

Find the arc length of the logarithmic spiral $\gamma(t) = (e^t \cos t, e^t \sin t)$.

Soln: The logarithmic spiral is given by

$$\gamma(t) = (e^t \cos t, e^t \sin t)$$

<u>To find the arc-length of γ:</u>



we will consider the initial point $t_0 = 0$ where we have $\gamma(0) = (1, 0)$

Now,

$$\dot{\gamma}(t) = (e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t)$$

$$\|\dot{\gamma}(t)\| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}$$

$$= \sqrt{e^{2t}\cos^2 t + e^{2t}\sin t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + e^{2t}\sin t \cos t}$$

$$= \sqrt{e^{2t}(2\cos^2 t + 2\sin^2 t)}$$

$$= \sqrt{2e^{2t}}$$

$$= \sqrt{2}\, e^t$$

$$\therefore \|\dot{\gamma}(u)\| = \sqrt{2}\, e^u$$

Hence $$s(t) = \int_0^t \|\dot{\gamma}(u)\|\, du$$

$$= \int_0^t \sqrt{2}\, e^u\, du$$

$$= \sqrt{2}\left[e^u\right]_0^t$$

$$= \sqrt{2}\, e^t - \sqrt{2}\, e^0$$

$$= \sqrt{2}\,(e^t - 1),$$

<u>Note:</u>

If s is the arc-length of a curve γ starting at $\gamma(t_0)$, we have

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^t \|\dot{\gamma}(u)\|\, du = \|\dot{\gamma}(t)\|$$

i.e., $\frac{ds}{dt} = \| \dot{\gamma}(t) \| \rightarrow (2)$

If we consider $\gamma(t)$ as the position of a moving point at time t, $\frac{ds}{dt}$ is the speed of the point.

i.e., rate of change of distance along the curve.

Defn - speed & unit speed:

If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$ is a parametrised curve, its speed at the point $\gamma(t)$ is $\| \dot{\gamma}(t) \|$, and γ is said to be a unit-speed if $\dot{\gamma}(t)$ is a unit vector for all $t \in (\alpha, \beta)$.

Proposition 1.2

(i) (Let $\hat{n}(t)$ be a unit vector that is a smooth function of a parameter 't'. Then, the dot product

$$\dot{\hat{n}}(t) \cdot \hat{n}(t) = 0$$

for all t, i.e. $\dot{\hat{n}}(t)$ is zero or perpendicular to $\hat{n}(t)$ for all t)

(ii) (In particular, if γ is a unit-speed curve, then $\ddot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.)

Proof:

(i) (Since $\hat{n}(t)$ is a unit vector,
we have,
$$\hat{n} \cdot \hat{n} = 1$$
Differentiating this eqn w.r.t 't',
we obtain
$$\dot{\hat{n}} \cdot \hat{n} + \hat{n} \cdot \dot{\hat{n}} = 0$$
$$2\,\dot{\hat{n}} \cdot \hat{n} = 0$$
$$\dot{\hat{n}} \cdot \hat{n} = 0$$
$$\Rightarrow \dot{\hat{n}} = 0$$
(or) $\dot{\hat{n}} \perp^r$ to $\hat{n}$) $\rightarrow$ (1)

(ii) (Since γ is a unit-speed curve by defn $\dot{\gamma}$ is a unit vector.

Replacing $\hat{n}$ with $\dot{\gamma}$ in eqn (1), we obtain
$$\ddot{\gamma} = 0$$
(or) $\ddot{\gamma} \perp^r \dot{\gamma}$)

Exercise:

1.11). Determine the arc-length of the catenary $\gamma(t) = (t, \cosh t)$ starting at the point $(0,1)$.

Soln: To find the arc length of γ,
we consider the intial point $t_0 = 0$.
Then we have, $\gamma(0) = (0, 1)$
$$\dot{\gamma}(t) = (1, \sinh t)$$
$$\|\dot{\gamma}(t)\| = \sqrt{1^2 + \sinh^2 t}$$
$$= \sqrt{\cosh^2 t} = \cosh t$$
$$\|\dot{\gamma}(u)\| = \cosh u$$

$\therefore$ The arc length is

$$s(t) = \int_0^t \|\dot{\gamma}(u)\|\, du$$

$$= \int_0^t \cosh u \, du$$

$$= \sinh u \Big]_0^t$$

$$= \sinh t$$

1.12 Show that the following curves are unit-speed:

i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$;

ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$.

Soln: i) $\dot{\gamma}(t) = \left\{\frac{1}{3}\cdot\frac{3}{2}(1+t)^{1/2}, \frac{1}{3}\frac{3}{2}(1-t)^{+1/2}(-1), \frac{1}{\sqrt{2}}\right\}$

$$= \left\{\frac{1}{2}(1+t)^{1/2}, -\frac{(1-t)^{+1/2}}{2}, \frac{1}{\sqrt{2}}\right\}$$

$$\|\dot{\gamma}(t)\| = \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}}$$

$$= \sqrt{\frac{1}{4}(1+t+1-t) + \frac{1}{2}}$$

$$= \sqrt{\frac{2}{4} + \frac{1}{2}} = \sqrt{1} = 1$$

$\therefore$ The curve $\gamma(t)$ is of unit-speed.

ii) $\dot{\gamma}(t) = \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right)$

$$\|\dot{\gamma}(t)\| = \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t}$$

$$= \sqrt{\sin^2 t + \cos^2 t} = 1$$

Reparametrisation:

Defn:

A parametrised curve $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ is a reparametrisation of a parametrised curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ (the reparametrisation map) such that the inverse map $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is also smooth and $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \ \forall \ \tilde{t} \in (\tilde{\alpha}, \tilde{\beta})$.

Note:

Since ϕ has a smooth inverse, we consider, ~~γ is a reparametrisation of $\tilde{\gamma}$~~ γ to be a reparametrization of $\tilde{\gamma}$ since we can write

$$\tilde{\gamma}(\phi^{-1}(t)) = \gamma(t) \quad \left[\because \gamma(\phi(\tilde{t})) = \gamma(\phi(\phi^{-1}(t))) = \gamma(t) \right]$$

Example 1.5

For the eqn of the ~~curve~~ circle $x^2 + y^2 = 1$, a parametrisation is given by $\gamma(t) = (\cos t, \sin t)$, reparametrize the circle.

Soln:

$\tilde{\gamma}(t) = (\sin t, \cos t)$

If $\tilde{\gamma}$ is the reparametrisation of γ, then we have to find a reparametrisation map ϕ such that

$$(\cos \phi(t), \sin \phi(t)) = (\sin t, \cos t)$$

Here instead of $\phi(\tilde{t})$, $\phi(t)$ as any have the circular fns are going to depend on the same interval $(0, 2\pi)$.

If we consider $\phi(t) = \frac{\pi}{2} - t$, then it satisfies the above eqn which is one possible solution.

Defn - [regular pt & singular pt]:

A point $\gamma(t)$ of a parametrised curve γ is called a regular point if $\dot{\gamma}(t) \neq 0$; otherwise $\gamma(t)$ is a singular point of γ. A curve is regular if all of its points are regular.

6m **Proposition 1.3**

6m

Any reparametrisation of a regular curve is regular.

Proof:

Suppose that γ & $\tilde{\gamma}$ are related as follows,

Let $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ be a reparametrisation of a parametrised curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$.

Let ϕ be a reparametrisation map given by

defn of reparametrisation

$\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ with a smooth inverse $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ such that

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \quad \forall\, \tilde{t} \in (\tilde{\alpha}, \tilde{\beta}). \to (1)$$

And therefore we have

$$t = \phi(\tilde{t}) \ \& \ \phi^{-1}(t) = \tilde{t} \to (3)$$

Now,

let us assume $\psi = \phi^{-1}$

$$③ \Longrightarrow \therefore \underbrace{\tilde{t} = \psi(t)}_{} \rightarrow ④$$

Sub ④ in ②,

$$\Rightarrow \phi\big(\overbrace{\psi(t)}^{\tilde{t}}\big) = t \rightarrow ⑤$$

Note:
$\phi(x)$
Diff 'ϕ' w.r.t 'y'
$$\frac{d\phi}{dx} \cdot \frac{dx}{dy}$$

Diff ⑤ w.r.t 't' using chain rule,

we have

$$\frac{d\phi}{d\tilde{t}} \cdot \frac{d\psi}{dt} = 1 \rightarrow ⑥$$

$$\Rightarrow \frac{d\phi}{d\tilde{t}} \neq 0$$

Diff ① w.r.t $\tilde{t}$ using chain rule,

we have

$$\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt} \cdot \frac{d\phi}{d\tilde{t}} \rightarrow ⑦$$

W.K.T,

$\frac{d\gamma}{dt} \neq 0$, since γ is regular

and by ⑥, $\frac{d\phi}{d\tilde{t}} \neq 0$

$\therefore$ By ⑦,

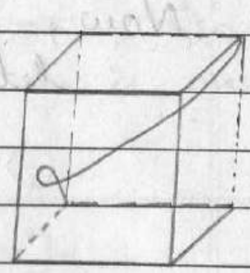
$$\frac{d\tilde{\gamma}}{d\tilde{t}} \neq 0 \Rightarrow \tilde{\gamma} \text{ is regular}$$

Problem:

Find the arc length of the twisted cubic space curve $\gamma(t) = (t, t^2, t^3)$, $-\infty < t < \infty$.

2m

Soln:



To find the arc length of γ, we consider the intial pt $t_0 = 0$

Then, we have

$$\gamma(0) = (0,0,0)$$
$$\dot{\gamma}(t) = (1, 2t, 3t^2)$$

$\because \dot{\gamma}(t) \neq 0$,
the curve is regular.

$$\|\dot{\gamma}(t)\| = \sqrt{1+4t^2+9t^4}$$
$$\|\dot{\gamma}(u)\| = \sqrt{1+4u^2+9u^4}$$

$\therefore$ The arc length is

$$s(t) = \int_0^t \|\dot{\gamma}(u)\|\, du$$
$$= \int_0^t \sqrt{1+4u^2+9u^4}\, du$$

The integral is called an elliptic integral and it can be evaluated using the familiar function.

Proposition 1.4

If $\gamma(t)$ is a regular curve, its arc-length s, starting at any point of γ, is a smooth function of it.

Proof:

W.K.T, the fn s is a differential fn of t

and $\boxed{\frac{ds}{dt} = \| \dot{\gamma}(t) \|} \rightarrow (1)$

Let us assume that γ to be a plane curve, say

$$\gamma(t) = (u(t), v(t)),$$

where u & v are smooth fns of t.

Then $\dot{\gamma}(t) = (\dot{u}(t), \dot{v}(t))$

$$\dot{\gamma} = (\dot{u}, \dot{v})$$

$$\boxed{\| \dot{\gamma}(t) \| = \sqrt{\dot{u}^2 + \dot{v}^2}} \rightarrow (2)$$

Let us define the fn $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

as $\boxed{f(u, v) = \sqrt{u^2 + v^2}} \rightarrow (3)$

From the representation of f,

W.K.T,

f is smooth on $\mathbb{R}^2 - \{(0,0)\}$.

$\therefore$ All the partial derivatives of f of all order exist and are continuous fns except at the origin $(0,0)$.

Now, Eqn (1), (2) & (3),

$$\frac{ds}{dt} = \| \dot{\gamma}(t) \|$$

$$= \sqrt{\dot{u}^2 + \dot{v}^2}$$

$$= f(\dot{u}, \dot{v}) \rightarrow (4)$$

Differentiating 's' once again w.r.t 't', we obtain

$$\frac{d^2 s}{dt^2} = \frac{\partial f}{\partial u}\ddot{u} + \frac{\partial f}{\partial v}\ddot{v}$$

where $\frac{\partial f}{\partial u} = \frac{1}{2}(u^2+v^2)^{-1/2}\, 2u$

$$= \frac{1}{2}\frac{1}{\sqrt{u^2+v^2}}\, 2u$$

$$= \frac{u}{\sqrt{u^2+v^2}}$$

and $\frac{\partial f}{\partial v} = \frac{1}{2}\frac{1}{\sqrt{u^2+v^2}}\, 2v = \frac{v}{\sqrt{u^2+v^2}}$

These are continuous except when u & v are zero. Similarly, we can find higher derivatives of s.

Proposition 1.5 γ has $\left\|\frac{d\tilde{\gamma}}{dt}\right\| = 1 \iff \frac{d\gamma}{dt} \neq 0$

20m a) 10m 6m

A parametrised curve has a unit-speed reparametrisation iff it is regular.

Proof:

Let γ be a parametrised curve with $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ and let $\tilde{\gamma}$ be its unit-speed reparametrisation with $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ where ϕ is the reparametrisation map given by $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ with the smooth inverse $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ and γ & $\tilde{\gamma}$ are connected by the relation

$$\tilde{\gamma}(\tilde{t}) = \gamma(t) \quad \longrightarrow (1)$$

Differentiation ① w.r.t '$\tilde{t}$', we have

$$\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt}\frac{dt}{d\tilde{t}}$$

i.e. $\left\|\frac{d\tilde{\gamma}}{d\tilde{t}}\right\| = \left\|\frac{d\gamma}{dt}\right\| \left\|\frac{dt}{d\tilde{t}}\right\|$

$\because$ $\tilde{\gamma}$ is of ~~unique~~ unit-speed,

we have

$$\left\|\frac{d\tilde{\gamma}}{d\tilde{t}}\right\| = 1.$$

Hence $\left\|\frac{d\gamma}{dt}\right\| \neq 0$, $\forall$ t

$\Rightarrow$ γ is a regular curve.

Conversely,

Suppose that the tangent vector $\frac{d\gamma}{dt}$ is never zero.

Since $\frac{ds}{dt} = \|\dot{\gamma}\|$ $\|\dot{\gamma}(t)\|$, we must have $\frac{ds}{dt} \neq 0$ $\forall t$

As s denotes the arc-length,

we must have $\frac{ds}{dt} > 0$ $\forall$ t.

By Proposition 1.4,

W.K.T,

s is the smooth fn of t.

By inverse fn theorem of multivariable calculus,

"See last page of this note"

W.K.T,

$s : (\alpha, \beta) \to \mathbb{R}^1$ is injective & its image is an open interval $(\tilde{\alpha}, \tilde{\beta}) \subset \mathbb{R}^1$ and that it has a smooth inverse $s^{-1} : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$.

In From prop: 1.3 → $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$

$\therefore\ t = \phi(\tilde{t})$ & $\boxed{\phi^{-1}(t) = \tilde{t}}$

$\tilde{\gamma}(\tilde{t}) = \gamma(t)$

$\Rightarrow \tilde{\gamma}(s) = \gamma(t)$

$\boxed{\phi = s^{-1}}$ We can consider the map s^{-1} to be one parametrisation map ϕ.

Then eqn (1) (from defn *) can be written as

$$\tilde{\gamma}(s) = \gamma(t) \rightarrow (3)$$

$\dot{s}(t) = \|\dot{\gamma}(t)\|$

$\frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\|$

Diff. w.r.t 't',

$$\frac{d\tilde{\gamma}}{ds}\,\frac{ds}{dt} = \frac{d\gamma}{dt}$$

$\|\ \|$ mod is +ve, so we can remove

$$\left\|\frac{d\tilde{\gamma}}{ds}\right\| \left|\frac{ds}{dt}\right| = \left\|\frac{d\gamma}{dt}\right\|$$

ie. $\left\|\frac{d\tilde{\gamma}}{ds}\right\| \frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\|$ as $\frac{ds}{dt} > 0 \quad \forall\ t$

↓ by the defn of arc-length

ie. $\left\|\frac{d\tilde{\gamma}}{ds}\right\| \frac{ds}{dt} = \frac{ds}{dt} \Rightarrow \left\|\frac{d\tilde{\gamma}}{ds}\right\| = 1$

$\Rightarrow \tilde{\gamma}$ is of unit speed.

Thus when γ is a regular curve, its reparametrisation is a unit-speed.

Corollary: 1.1

6m

Let γ be a regular curve & let $\tilde{\gamma}$ be a unit-speed reparametrisation of γ:

$$\tilde{\gamma}(u(t)) = \gamma(t) \quad \text{for all } t,$$

where u is a smooth fn of t. Then, if s is the arc-length of γ (starting at any point), we have $\boxed{u = \pm s + c} \rightarrow (*)$ where c is a constant.

Conversely, if u is gn by (*) for some value of c and with either sign, then $\tilde{\gamma}$ is a unit-speed reparametrisation of γ.

Proof:

Let γ be a regular curve with unit-speed reparametrisation $\tilde{\gamma}$ with a relation

$$\tilde{\gamma}(u(t)) = \gamma(t) \quad \forall\ t$$

Then

$$\frac{d\tilde{\gamma}}{du}\frac{du}{dt} = \frac{d\gamma}{dt} \ \ \& \ \ \left\|\frac{d\tilde{\gamma}}{du}\right\| \left|\frac{du}{dt}\right| = \left\|\frac{d\gamma}{dt}\right\|$$

As in the proposition 1.5, ← $\boxed{\gamma \text{ has } \left\|\frac{d\tilde{\gamma}}{dt}\right\| = 1 \Rightarrow \frac{d\gamma}{dt} \neq 0}$
we have

$$\left\|\frac{d\tilde{\gamma}}{du}\right\| = 1$$

As $\tilde{\gamma}$ is a unit-speed,

$$\left|\frac{du}{dt}\right| = \left\|\frac{d\gamma}{dt}\right\|$$

i.e. $\dfrac{du}{dt} = \pm\left\|\dfrac{d\gamma}{dt}\right\|$

$$\frac{du}{dt} = \pm\frac{ds}{dt}, \quad \left[\text{by defn of arc-length i.e. } \frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\|\right]$$

Integrating,

$\boxed{u = \pm s + C}$, for some constant C.

Conversely,

if u is given by this equation, we can retrace our steps and prove that $\tilde{\gamma}$ is a unit-speed.

Example :

Find the unit-speed reparametrisation for the logarithmic spiral
$$\gamma(t) = (e^t \cos t, e^t \sin t)$$

Soln:

The logarithmic spiral is gn by
$$\gamma(t) = (e^t \cos t, e^t \sin t)$$

To find the arc-length of γ, we will consider the initial point $t_0 = 0$ where we have $\gamma(0) = (1, 0)$

Now,
$$\dot{\gamma}(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t)$$

$$\|\dot{\gamma}(t)\| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}$$

$$= \sqrt{e^{2t}(2\cos^2 t + 2\sin^2 t)}$$

$$= \sqrt{2e^{2t}}$$

$$\|\dot{\gamma}(t)\| = \sqrt{2}\, e^t \quad \left[\Rightarrow \|\dot{\gamma}(t)\|^2 = 2e^{2t}\right]$$

$$\therefore \|\dot{\gamma}(u)\| = \sqrt{2}\, e^u \left[\Rightarrow \|\dot{\gamma}(u)\|^2 = 2e^{2u}\right]$$

This is never zero, so γ is regular.

$\therefore$ The arc-length of γ starting at $(1,0)$ is

$$s(t) = \int_0^t \|\dot{\gamma}(u)\|\, du$$

$$\Rightarrow s = \int_0^t \sqrt{2}\, e^u \, du$$

$$= \sqrt{2}\,\left[e^u\right]_0^t$$

$$= \sqrt{2}\,e^t - \sqrt{2}\,e^0$$

$$s = \sqrt{2}\,(e^t - 1)$$

Hence, $\Rightarrow \dfrac{s}{\sqrt{2}} = e^t - 1$

$$\Rightarrow \frac{s}{\sqrt{2}} + 1 = e^t$$

$$\Rightarrow t = \log\left(\frac{s}{\sqrt{2}} + 1\right)$$

Thus a unit-speed reparametrisation of γ is given by

$$\tilde{\gamma}(s) = \left(e^{\log\left(\frac{s}{\sqrt{2}}+1\right)} \cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right),\ e^{\log\left(\frac{s}{\sqrt{2}}+1\right)} \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right)$$

$$= \left\{\left(\frac{s}{\sqrt{2}}+1\right)\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right),\ \left(\frac{s}{\sqrt{2}}+1\right)\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right\}$$

$$\dot{\tilde{\gamma}}(s) = \left(\cancel{\left(\frac{s}{\sqrt{2}}+1\right)}\left(-\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right)\frac{1}{\cancel{\left(\frac{s}{\sqrt{2}}+1\right)}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right),\right.$$

$$\left.\cancel{\left(\frac{s}{\sqrt{2}}+1\right)}\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\frac{1}{\cancel{\left(\frac{s}{\sqrt{2}}+1\right)}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right)$$

$$\therefore \dot{\tilde{\gamma}}(s) = \left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) - \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right],\right.$$

$$\left.\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]$$

$$\|\dot{\gamma}(s)\| = \sqrt{\left(\frac{s^2}{2}+1+\frac{2s}{\sqrt{2}}\right)\sin^2\left(\log\frac{s}{\sqrt{2}}+1\right)\frac{1}{\frac{s^2}{2}+1+\frac{2s}{\sqrt{2}}}\cdot\frac{1}{2}}$$

Consider

$$\left[\left(\frac{s}{\sqrt{2}}+1\right)\left(-\sin\left(\log\frac{s}{\sqrt{2}}+1\right)\right)\frac{1}{\frac{s}{\sqrt{2}}+1}\,\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos\left(\log\frac{s}{\sqrt{2}}+1\right)\right]^2$$

$$=\left(\frac{s^2}{2}+1+\frac{2s}{\sqrt{2}}\right)\left[\sin^2\left(\log\frac{s}{\sqrt{2}}+1\right)\right]\frac{1}{\frac{s^2}{2}+1+\frac{2s}{\sqrt{2}}}\,\frac{1}{2}$$

$$+\frac{1}{2}\left[\cos^2\left(\log\frac{s}{\sqrt{2}}+1\right)\right]$$

$$+2\left(\frac{s}{\sqrt{2}}+1\right)\left[-\sin\left(\log\frac{s}{\sqrt{2}}+1\right)\right]\frac{1}{\frac{s}{\sqrt{2}}+1}\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\left[\cos\left(\log\frac{s}{\sqrt{2}}+1\right)\right]$$

Consider

$$\left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)-\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]^2$$

$$=\frac{1}{2}\left[\cos^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)+\sin^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)-2\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]$$

Consider

$$\left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)+\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]^2$$

$$=\frac{1}{2}\left[\cos^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)+\sin^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)+2\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]$$

$$\therefore \left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) - \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]^2$$

$$+ \left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]^2$$

$$= \frac{1}{2}\left[\cos^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]$$

$$+ \frac{1}{2}\left[\cos^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right) + \sin^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]$$

$$= \frac{\cancel{2}}{\cancel{2}}\left[\cos^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin^2\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]$$

$$= 1$$

Hence,

$$\|\dot{\gamma}(s)\| = \sqrt{\left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) - \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]^2 + \left[\frac{1}{\sqrt{2}}\left[\cos\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right) + \sin\left(\log\left(\frac{s}{\sqrt{2}}+1\right)\right)\right]\right]^2}$$

$$= \sqrt{1}$$

$$= 1.$$

1. Which of the following curves are regular?

i) $\gamma(t) = (t, t^2)$ ii) $\gamma(t) = (t^3, t^6)$

iii) $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $-\infty < t < \infty$;

iv) $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $0 < t < \pi/2$;

v) $\gamma(t) = (t, \cosh t)$ for $-\infty < t < \infty$.

Soln: i) $\gamma(t) = (t, t^2)$

This is a parametrisation of the parabola $y = x^2$.

Then $\dot{\gamma}(t) = (1, 2t)$

$\because \dot{\gamma}(t) \neq 0$,

the curve is regular.

ii) $\gamma(t) = (t^3, t^6)$

This is also a parametrisation of the parabola $y = x^2$.

Then $\dot{\gamma}(t) = (3t^2, 6t^5)$

$\because \dot{\gamma}(t) = 0$ when $t = 0$,

so the curve γ is not regular.

iii) $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $t \in (-\infty, \infty)$.

Then $\gamma(t) = \left(\frac{1+\cos 2t}{2}, \frac{1-\cos 2t}{2} \right)$

$\dot{\gamma}(t) = \left(-\frac{\cancel{2}\sin 2t}{\cancel{2}}, \frac{\cancel{2}\sin 2t}{\cancel{2}} \right) = (-\sin 2t, \sin 2t)$

$= \sin 2t \, (-1, 1)$

$\because \dot{\gamma}(t) = 0$,

so γ is not regular.

iv) $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $t \in (0, \pi/2)$

$\because \dot{\gamma}(t) \neq 0$ for $0 < t < \pi/2$, so γ is regular.

v) $\gamma(t) = (t, \cosh t)$ for $t \in (-\infty, \infty)$

Then $\dot{\gamma}(t) = (1, \sinh t)$

$\because \dot{\gamma}(t) \neq 0$, so γ is regular.

How much does a curve curve?

1.2) Curvature:

Remark:

1) Curvature is actually a measure to calculate how much a curve is curved.
2) The curvature should be unchanged when the curve is reparametrised.
3) Curvature of a straight line must be zero.
4) Curvature of a large circles should be small when compared to small circles.
5) $\ddot{\gamma}$ is zero then γ represents a straight line, we may define the curvature to be $\|\ddot{\gamma}\|$.

Definition:

If γ is a unit-speed curve with parameter s, its curvature $K(s)$ at the point $\gamma(s)$ is defined to be $\|\ddot{\gamma}(s)\|$.

---

1. Find the curvature of a circle with centre at (x_0, y_0) and radius R.

Soln:

Let us consider the parametrisation of the circle to be

$$\gamma(s) = \left(x_0 + R\cos\frac{s}{R},\ y_0 + R\sin\frac{s}{R}\right)$$

$\because (x-x_0)^2 + (y-y_0)^2 = R^2,$

$x - x_0 = R\cos\theta \Rightarrow x = x_0 + R\cos\frac{s}{R}$

$y - y_0 = R\sin\theta \Rightarrow y = y_0 + R\sin\frac{s}{R}$

We have

$$\gamma(s) = \left(x_0 + R\cos\frac{s}{R},\ y_0 + R\sin\frac{s}{R}\right)$$

$$\Rightarrow \dot{\gamma}(s) = \left(-\frac{R}{R}\sin\frac{s}{R},\ \frac{R}{R}\cos\frac{s}{R}\right)$$

$$= \left(-\sin\frac{s}{R},\ \cos\frac{s}{R}\right)$$

$$\therefore \|\dot{\gamma}(s)\| = \sqrt{\sin^2\frac{s}{R} + \cos^2\frac{s}{R}} = 1$$

$$\Rightarrow \gamma(s) \text{ is of unit-speed.}$$

$$\text{Hence } \ddot{\gamma}(s) = \left(-\frac{1}{R}\cos\frac{s}{R},\ -\frac{1}{R}\sin\frac{s}{R}\right)$$

$$\|\ddot{\gamma}(s)\| = \sqrt{\frac{1}{R^2}\cos^2\frac{s}{R} + \frac{1}{R^2}\sin\frac{s}{R}}$$

$$= \sqrt{\frac{1}{R^2}} = \frac{1}{R}$$

so the curvature of the circle is inversely proportional to its radius.

Note:

If $\gamma(s)$ is a unit-speed curve, the only unit-speed reparametrisations of γ are of the form $\gamma(u)$, where $u = \pm s + c$ & c is a constant.

Then, by the chain rule, we have

$$\frac{d\gamma}{ds} = \frac{d\gamma}{du}\frac{du}{ds}$$

$$= \frac{d\gamma}{du}(\pm 1) \quad \left[\because u = \pm s + c,\ \frac{du}{ds} = \pm 1\right]$$

$$= \pm\frac{d\gamma}{du}$$

Once again differentiating w.r.t 's',

$$\frac{d^2\gamma}{ds^2} = \pm \frac{d}{ds}\left(\frac{d\gamma}{du}\right)$$

$$= \pm \frac{d}{du}\left(\frac{d\gamma}{du}\right)\frac{du}{ds}$$

$$= \pm \frac{d}{du}\left(\frac{d\gamma}{du}\right)(\pm 1)$$

$$= (\pm)^2 \frac{d^2\gamma}{du^2}$$

$$= \frac{d^2\gamma}{du^2}$$

This shows that the curvature of the curve calculated using the unit-speed parameter s is the same as that calculated using the unit-speed parameter u.

Proposition 2.1

a) 20m

2am a) 10m

8m

Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then, its curvature is

$$\kappa = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

where the $\times$ indicates the vector (or cross) product and the dot denotes $\frac{d}{dt}$.

Proof:

Let $\tilde{\gamma}$ be a unit-speed reparametrisation of γ, and let us denote $\frac{d}{ds}$ by a dash. Then, by the chain rule,

$$\tilde{\gamma}(s) = \gamma(t)$$

$$\frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt}\frac{dt}{ds}$$

i.e. $\tilde{\gamma}' = \dfrac{\dot{\gamma}}{ds/dt}$

i.e. $\tilde{\gamma}' \dfrac{ds}{dt} = \dot{\gamma}$

By definition, $\to \kappa(s) = \|\ddot{\gamma}(s)\|$

$$\kappa = \|\tilde{\gamma}''\| = \left\| \frac{d}{ds}(\tilde{\gamma}') \right\| \qquad \frac{\frac{d\gamma}{dt}}{ds/dt} \qquad \frac{d\gamma}{dt} \times \frac{dt}{ds}$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \qquad = \left\| \frac{d}{ds}\left\{ \frac{\dot{\gamma}}{ds/dt} \right\} \right\| \to \times \,\&\, \div \text{ by } dt$$

$$= \left\{ \frac{\frac{ds}{dt}\ddot{\gamma} - \dot{\gamma}\frac{d^2s}{dt^2}}{\left(\frac{ds}{dt}\right)^2} \right\} = \left\| \frac{d}{dt}\left\{ \frac{\dot{\gamma}}{ds/dt} \right\} \frac{dt}{ds} \right\|$$

$$= \left\| \frac{\frac{ds}{dt}\cdot\ddot{\gamma} - \dot{\gamma}\cdot\frac{d^2s}{dt^2}}{(ds/dt)^3} \right\| \to (1)$$

W.K.T,

$$\frac{ds}{dt} = \|\dot{\gamma}\|$$

$$\left(\frac{ds}{dt}\right)^2 = \|\dot{\gamma}\|^2 = \dot{\gamma}\cdot\dot{\gamma}$$

Diff w.r.t 't'

$$\Rightarrow \cancel{2}\frac{ds}{dt}\frac{d^2s}{dt^2} = \dot{\gamma}\,\ddot{\gamma} + \ddot{\gamma}\cdot\dot{\gamma}$$

$$= \cancel{2}\,\dot{\gamma}\,\ddot{\gamma}$$

$$(1) \times \frac{ds/dt}{ds/dt}$$

$$\text{i.e. } \kappa = \left\| \frac{\left(\frac{ds}{dt}\right)^2 \ddot{\gamma} - \dot{\gamma}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)}{\left(ds/dt\right)^4} \right\|$$

$$= \left\| \frac{(\dot{\gamma}\cdot\dot{\gamma})\ddot{\gamma} - \dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})}{\|\dot{\gamma}\|^4} \right\| \longrightarrow (2)$$

$$\cancel{k = \frac{\|\dot{\gamma}\| \, \|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^4}}$$ ~~{by vector triple product identity}~~

$$\cancel{= \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}}$$

$\frac{dt}{ds}$

Using the vector triple product identity,

$$a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$$

(where $a, b, c \in \mathbb{R}^3$), we get

$a = \dot{\gamma}$, $b = \ddot{\gamma}$, $c = \dot{\gamma}$

$$\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma}) = \ddot{\gamma}(\dot{\gamma} \cdot \dot{\gamma}) - \dot{\gamma}(\dot{\gamma} \cdot \ddot{\gamma}) \longrightarrow (3)$$

Further, $\dot{\gamma}$ & $\ddot{\gamma} \times \dot{\gamma}$ are $\perp$ vectors, so

$$\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\| = \|\dot{\gamma}\| \, \|\ddot{\gamma} \times \dot{\gamma}\| \longrightarrow (4)$$

From (3) & (4), we can write (2) as follows:

$$\kappa = \left\| \frac{\ddot{\gamma}(\dot{\gamma} \cdot \dot{\gamma}) - \dot{\gamma}(\dot{\gamma} \cdot \ddot{\gamma})}{\|\dot{\gamma}\|^4} \right\|$$

$$= \frac{\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\|}{\|\dot{\gamma}\|^4}$$

$$= \frac{\cancel{\|\dot{\gamma}\|} \, \|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^{\cancel{4}\,3}}$$

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

Example: 2.1

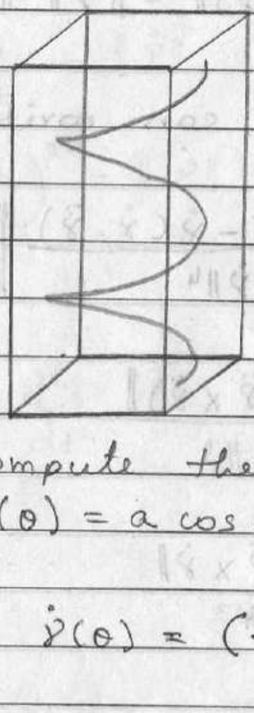
Qn A circular helix with axis the z-axis is a curve of the form

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta), \quad -\infty < \theta < \infty,$$

where a & b are constants.

Then find the curvature of the helix and prove that it is a constant and also prove that the curvature is the same as that of a circle when $b=0$ $(a \neq 0)$. Find out the curvature when $a=0$ $(b \neq 0)$ and prove that the curve becomes the straight line.

Soln:



Let us compute the curvature of the helix,

~~Let~~ Gn that $\gamma(\theta) = a\cos\theta, a\sin\theta, b\theta$

Then $\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b)$

$$\therefore \|\dot{\gamma}(\theta)\| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + b^2}$$
$$= \sqrt{a^2+b^2}$$

$\because \dot{\gamma}(\theta) \neq 0$,

so γ is regular.

Hence,

$$\ddot{\gamma}(\theta) = (-a\cos\theta, -a\sin\theta, 0)$$

Now, consider

$$\ddot{\gamma} \times \dot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\cos\theta & -a\sin\theta & 0 \\ -a\sin\theta & a\cos\theta & b \end{vmatrix}$$

$$= \hat{i}\left[-ab\sin\theta - 0\right] - \hat{j}\left[-ab\cos\theta - 0\right] + \hat{k}\left[(-a\cos\theta)(a\cos\theta) - (a\sin\theta)(a\sin\theta)\right]$$

$$= -ab\sin\theta\,\hat{i} + ab\cos\theta\,\hat{j} - (a^2\cos^2\theta + a^2\sin^2\theta)\,\hat{k}$$

$$(\ddot{\gamma} \times \dot{\gamma})(\theta) = (-ab\sin\theta, ab\cos\theta, -a^2)$$

$$\|\ddot{\gamma} \times \dot{\gamma}\| = \sqrt{a^2b^2\sin^2\theta + a^2b^2\cos^2\theta + a^4}$$

$$= \sqrt{a^2b^2 + a^4} = a \cdot \sqrt{a^2+b^2}$$

$\therefore$ The curvature is

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

$$= \frac{a(a^2+b^2)^{1/2}}{(a^2+b^2)^{3/2}} \qquad \frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1$$

$$= \frac{a}{a^2+b^2}$$

Thus, the curvature of the helix is constant.

When $b=0, (a\neq 0)$ helix is simply a circle in

$K = \frac{a}{a^2} = \frac{1}{a}$ (xy-plane)

$\Rightarrow$ the curvature of circle.

When $a=0, (b\neq 0)$ the image of helix is

$K = \frac{0}{0+b^2} = 0$ just the (z-axis)

$\Rightarrow$ the curve becomes straight line.

Problem:

1. Compute the curvature of the following curves:

i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right);$

ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right);$

v) go on ~~iii)~~ $\gamma(t) = (t, \cosh t);$

iv) $\gamma(t) = (\cos^3 t, \sin^3 t)$

Soln:

i) $\gamma(t) = \left\{\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right\}$

$\dot{\gamma}(t) = \left\{\frac{3}{2}\cdot\frac{1}{3}(1+t)^{3/2-1}, \frac{1}{3}\frac{3}{2}(1-t)^{3/2-1}(-1), \frac{1}{\sqrt{2}}\right\}$

$= \left\{\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right\}$

$$\|\dot{\gamma}(t)\| = \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}}$$

$$= \sqrt{\frac{1}{4}(1+t+1-t) + \frac{1}{2}}$$

$$= \sqrt{\frac{2}{4} + \frac{1}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{\frac{2}{2}} = \sqrt{1}$$

$$= 1$$

$$\ddot{\gamma}(t) = \left\{ \frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0 \right\}$$

$$= \left\{ \frac{1}{4}\frac{1}{\sqrt{1+t}}, \frac{1}{4}\frac{1}{\sqrt{1-t}}, 0 \right\}$$

$$\|\ddot{\gamma}(t)\| = \sqrt{\frac{1}{16}\frac{1}{1+t} + \frac{1}{16}\frac{1}{1-t}}$$

$$= \sqrt{\frac{1}{16}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)}$$

$$= \sqrt{\frac{1}{16}\frac{1-t+1+t}{(1+t)(1-t)}}$$

$$= \sqrt{\frac{1}{16}\frac{2}{(1-t^2)}}$$

$$= \frac{1}{\sqrt{8(1-t^2)}}$$

$\because$ γ is unit-sppeed,

so $\kappa = \|\ddot{\gamma}\| = \dfrac{1}{\sqrt{8(1-t^2)}}$

ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, \frac{-3}{5}\cos t\right)$

$$\dot{\gamma}(t) = \left\{ -\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t \right\}$$

$$\|\dot{\gamma}(t)\| = \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t}$$

$$= \sqrt{\sin^2 t\left(\frac{16}{25} + \frac{9}{25}\right) + \cos^2 t}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= \sqrt{1} = 1$$

Then, $\ddot{\gamma}(t) = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right)$

$$\|\ddot{\gamma}(t)\| = \sqrt{\frac{16}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t}$$

$$= \sqrt{\cos^2 t\left(\frac{16}{25} + \frac{9}{25}\right) + \sin^2 t}$$

$$= \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1}$$

$$= 1$$

$\because$ γ is unit-speed,

so $\kappa = \|\ddot{\gamma}(t)\| = 1$

iii) $\gamma(t) = (t, \cosh t)$

b)

20m

$\dot{\gamma}(t) = (1, \sinh t)$	$\ddot{\gamma}(t) = (0, \cosh t)$
$\|\dot{\gamma}(t)\| = \sqrt{1+\sinh^2 t}$	$\|\ddot{\gamma}(t)\| = \sqrt{\cosh^2 t}$
$= \sqrt{\cosh^2 t}$	$= \cosh t$
$= \cosh t$	

$$\ddot{\gamma} \times \dot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \cosh t & 0 \\ 1 & \sinh t & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-\cosh t)$$

$$= \hat{k}(-\cosh t)$$

$$\|\ddot{\gamma} \times \dot{\gamma}\| = \sqrt{\cos^2 ht}$$

$$= \cos ht$$

$\therefore$ The curvature is

$$K = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

$$= \frac{\cosh t}{\cos h^3 t} = \frac{1}{\cos h^2 t} = \sec h^2 t$$

$$= \sec h^2 t$$

iv) $\gamma(t) = (\cos^3 t, \sin^3 t)$

$$\dot{\gamma}(t) = 3\cos^2 t(-\sin t), 3\sin^2 t \cos t$$

$$\|\dot{\gamma}(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t}$$

$$= \sqrt{9\cos^2 t \sin^2 t}$$

$$= 3\cos t \sin t$$

$$\ddot{\gamma}(t) = \left[6\cos t(-\sin t)(-\sin t) + (3\cos^2 t)(-\cos t),\right.$$
$$\left. 6\sin t(\cos t)\cos t + 3\sin^2 t(-\sin t)\right]$$

$$= (6\sin^2 t\cos t - 3\cos^3 t,\ 6\sin t\cos^2 t - 3\sin^3 t)$$

$$\ddot{\gamma} \times \dot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6\sin^2 t\cos t - 3\cos^3 t & 6\sin t\cos^2 t - 3\sin^3 t & 0 \\ -3\sin t\cos^2 t & 3\sin^2 t\cos t & 0 \end{vmatrix}$$

$$= \hat{k}\left[(18\sin^4 t\cos^2 t - 9\sin^2 t\cos^4 t) + (18\sin^2 t\cos^4 t - 9\sin^4 t\cos^2 t)\right]$$

$$= \hat{k}\left[18\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t) - 9\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)\right]$$

$$= \hat{k}\left[18\sin^2 t\cos^2 t - 9\sin^2 t\cos^2 t\right]$$

$$= \hat{k}(9\sin^2 t\cos^2 t)$$

$$\|\ddot{\gamma} \times \dot{\gamma}\| = \sqrt{81\sin^4 t\cos^4 t}$$

$$= 9\sin^2 t\cos^2 t$$

$$\therefore K = \frac{9\sin^2 t\cos^2 t}{(3\cos t\sin t)^3}$$

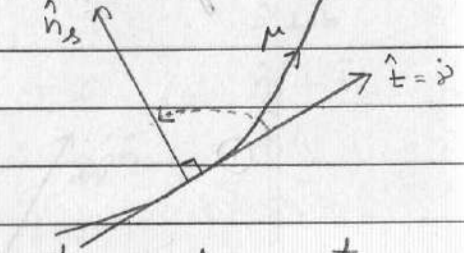
$$= \frac{\cancel{9}\,\cancel{\sin^2 t}\,\cancel{\cos^2 t}}{\cancel{27}_3\cos^{\cancel{3}} t\sin^{\cancel{3}} t}$$

$$= \frac{1}{3\cos t\sin t}$$

$$= \frac{1}{3|\cos t\sin t|}$$

Plane Curves:

Tangent & Normal:



Suppose that $\gamma(s)$ is a unit-speed curve in $\mathbb{R}^2$.

Let us denote $\frac{d}{ds}$ by a dot, and

let $\hat{t} = \dot{\gamma} \rightarrow (1)$ be the tangent vector of γ. Since γ is of unit-speed, the tangent vector $\hat{t}$ is the unit-vector.

Then there is a unit-vector $\perp r$ to t which we will denote by $\hat{n}_s$ called the <u>signed</u> <u>unit</u> <u>normal</u> of γ, which is to be the unit vector obtained by rotating $\hat{t}$ anti-clockwise by $\pi/2$.

By proposition 1.2,

$$\dot{\hat{t}} = \ddot{\gamma} \xrightarrow{(2)} \text{ is } \perp r \text{ to } \hat{t}, \text{ and hence}$$

parallel to $\hat{n}_s$. Thus there exist a number $k_s \ni \ddot{\gamma} = k_s n_s \rightarrow (3)$ where k_s is a scalar called the <u>signed curvature</u> of γ.

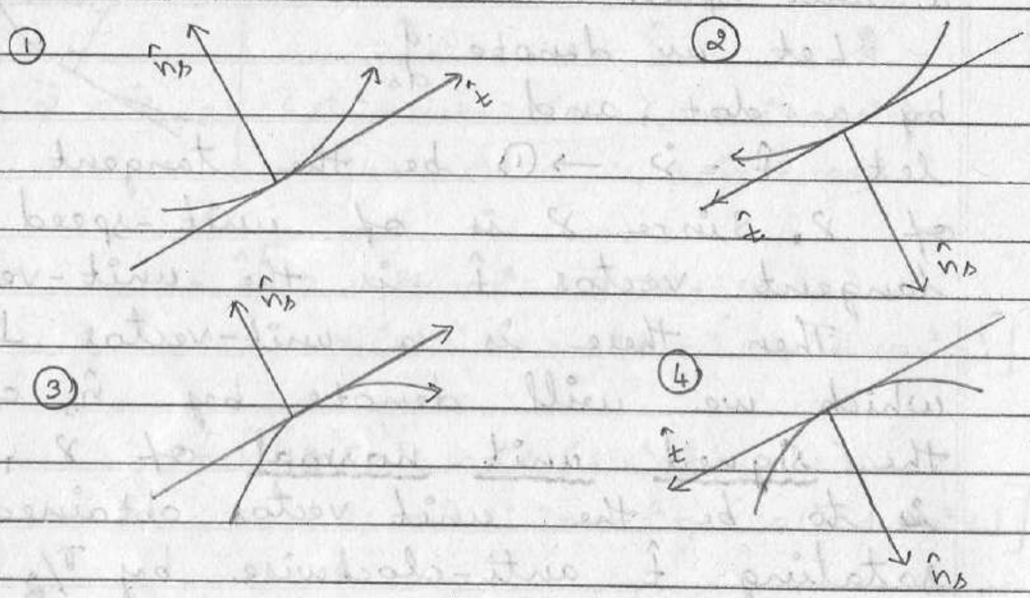
W.K.T,

$$\begin{aligned} K &= \| \ddot{\gamma} \| \\ &= \| K_s \hat{n}_s \| \\ &= | K_s | \end{aligned} \quad \left[\because \| \hat{n}_s \| = 1 \text{ as } \hat{n}_s \text{ is a unit vector} \right]$$

so the curvature of γ is absolute value of its signed curvature.

Note:

The following diagrams show how the sign of the signed curvature is determined.



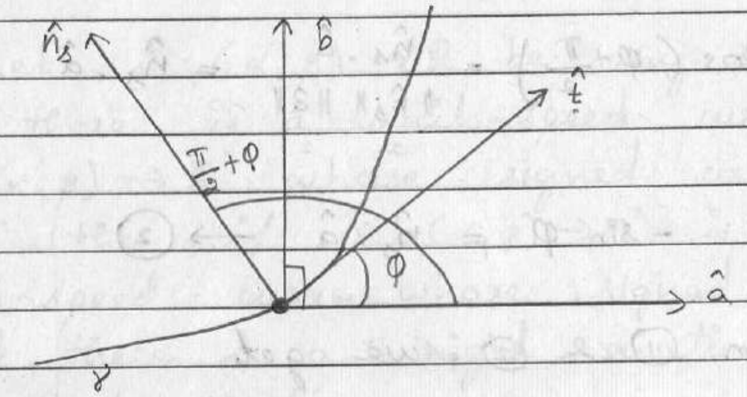
Proposition 2.2

Let $\gamma(s)$ be a unit-speed plane curve, and let $\varphi(s)$ be the angle through which a fixed unit vector must be rotated anti-clockwise to bring it into coincidence with the unit tangent vector $\hat{t}$ of γ. Then

$$\kappa_s = \frac{d\varphi}{ds}$$

where κ_s $\cancel{\kappa_s}$ is the signed curvature of γ.

Proof :



$\hat{a} \perp \hat{b}$

$\hat{n}_s \perp \hat{t}$

Let $\hat{a}$ be the fixed unit vector given in the statement which makes an angle φ with the tangent vector $\hat{t}$.

Let $\hat{b}$ be another unit vector obtained by rotating $\hat{a}$ anti-clockwise by degree $\frac{\pi}{2}$.

Then we can represent $\hat{t}$ using $\hat{a}$ & $\hat{b}$ as follows

$$\hat{t} = \hat{a} \cos\varphi + \hat{b} \sin\varphi$$

Diff. w.r.t s & denoting $\frac{d}{ds}$ by a dot,

We have, $\dot{\hat{t}} = -\hat{a} \sin\varphi \frac{d\varphi}{ds} + \hat{b} \cos\varphi \frac{d\varphi}{ds}$

Consider the dot product

$$\dot{\hat{t}} \cdot \hat{a} = -\hat{a}\cdot\hat{a} \sin\varphi \frac{d\varphi}{ds} + \hat{b}\cdot\hat{a} \cos\varphi \frac{d\varphi}{ds}$$

$$\ddot{\gamma} \cdot \hat{a} = -\sin\varphi \frac{d\varphi}{ds}$$

$$\therefore K_s \hat{n}_s \cdot \hat{a} = -\sin\varphi \frac{d\varphi}{ds} \longrightarrow (1)$$

But we know that from the diagram,

$$\cos\left(\varphi+\frac{\pi}{2}\right) = \frac{\hat{n}_s \cdot \hat{a}}{\underset{1}{\|\hat{n}_s\|}\ \underset{1}{\|\hat{a}\|}} = \hat{n}_s \cdot \hat{a}$$

ie) $-\sin\varphi = \hat{n}_s \cdot \hat{a} \longrightarrow (2)$

From (1) & (2), we get

$$\kappa_s = \frac{d\varphi}{ds}$$

Remark:

W.K.T,

a rigid motion of R^2 is a map $M : R^2 \to R^2$ of the form

$$M = T_a \circ R_\theta ,$$

where R_θ is an anti-clockwise rotation by an angle θ about the origin,

$$R_\theta(x, y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta),$$

and T_a is the translation by the vector $\hat{a}$,

$$T_a(v) = v + \hat{a} ,$$

for any vectors (x, y) and $v \in R^2$.

Theorem 2.1

Let $k : (\alpha, \beta) \to \mathbb{R}$ be any smooth fn. Then, there is a unit-speed curve $\gamma : (\alpha,\beta) \to \mathbb{R}^2$ whose signed curvature is k.
Further, if $\tilde{\gamma} : (\alpha,\beta) \to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, there is a rigid motion M of $\mathbb{R}^2$ such that
$$\tilde{\gamma}(s) = M(\gamma(s)) \quad \forall \ s \in (\alpha,\beta)$$

Proof:

First part

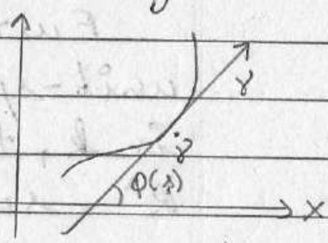
First let us assume that k is a smooth function $\ni k : (\alpha,\beta) \to \mathbb{R}'$.
Let us fix $s_0 \in (\alpha,\beta)$ and let us define, for any $s \in (\alpha,\beta)$, the smooth fn φ as follows:
$$\varphi(s) = \int_{s_0}^{s} k(u)\, du \longrightarrow ①$$

With the help of the fn that we have defined as φ, we will define a curve γ as follows:
$$\gamma(s) = \left(\int_{s_0}^{s} \cos\varphi(t)\, dt, \int_{s_0}^{s} \sin\varphi(t)\, dt \right) \longrightarrow ②$$

Diff. γ w.r.t s, we obtain
$$\dot{\gamma}(s) = (\cos\varphi(s), \sin\varphi(s)) \longrightarrow ③$$
$$\| \dot{\gamma}(s) \| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

Thus, γ is unit-speed.

From ③,
we see that the tangent vector of the curve γ makes an angle $\varphi(s)$ with the x-axis.



by proposition 2.2,

$\because \gamma$ is a unit-speed curve, we obtain the curvature as the differential of the angle φ.

ie, from ①,
its signed curvature is

$$\frac{d\varphi}{ds} = \frac{d}{ds}\int_{s_0}^{s} k(u)\, du = k(s)$$

$$\therefore \frac{d\varphi}{ds} = k(s) \longrightarrow ④$$

Thus, if k is a smooth function, then we could find a curve γ which is the unit-speed with curvature k.

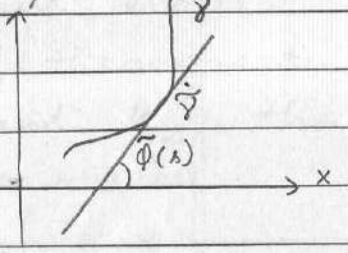
Second part Further if $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ is any other unit-speed curve with curvature as k, we can write from proposition 2.2,

$$\frac{d\tilde{\varphi}}{ds} = k(s) \longrightarrow ⑤$$

and as $\tilde{\gamma}$ is of unit-speed, we can write as in (3),

$$\dot{\tilde{\gamma}} = \left(\cos\tilde{\varphi}(s), \sin\tilde{\varphi}(s) \right) \rightarrow (6)$$

Thus, we have taken $\tilde{\varphi}(s)$ be the angle between the x-axis and the unit tangent vector $\dot{\tilde{\gamma}}$ of $\tilde{\gamma}$.



From (5),

we can obtain

$$\tilde{\varphi}(s) = \int_{s_0}^{s} k(u)\, du + \tilde{\varphi}(s_0)$$

ie, $$\tilde{\varphi}(s) = \int_{s_0}^{s} k(u)\, du + \theta \quad \text{where} \quad \theta = \tilde{\varphi}(s_0)$$
$\rightarrow (7)$

Similarly,

From (6),

we can obtain

$$\tilde{\gamma}(s) = \left(\int_{s_0}^{s} \cos\tilde{\varphi}(t)\, dt, \int_{s_0}^{s} \sin\tilde{\varphi}(t)\, dt \right) + \tilde{\gamma}(s_0)$$

$$\tilde{\gamma}(s) = \left(\int_{s_0}^{s} \cos\tilde{\varphi}(t)\, dt, \int_{s_0}^{s} \sin\tilde{\varphi}(t)\, dt \right) + \vec{a} \rightarrow (8)$$

where $\vec{a} = \tilde{\gamma}(s_0)$ is some constant vector.

Combining (1) & (7), we have

$$\tilde{\varphi}(s) = \varphi(s) + \theta \rightarrow (*)$$

Sub. ⊛ in (8), we have

$$\tilde{\gamma}(s) = \left(\int_{s_0}^{s} \cos(\varphi(t)+\theta)\,dt, \int_{s_0}^{s} \sin(\varphi(t)+\theta)\,dt \right) + \vec{a}$$

$$= T_a \left(\int_{s_0}^{s} \cos(\varphi(t)+\theta)\,dt, \int_{s_0}^{s} \sin(\varphi(t)+\theta)\,dt \right)$$

$$= T_a \left(\int_{s_0}^{s} [\cos\varphi(t)\cos\theta - \sin\varphi(t)\sin\theta]\,dt, \right.$$

$$\left. \int_{s_0}^{s} [\sin\varphi(t)\cos\theta + \cos\varphi(t)\sin\theta]\,dt \right)$$

$$= T_a \left(\cos\theta \int_{s_0}^{s} \cos\varphi(t)\,dt - \sin\theta \int_{s_0}^{s} \sin\varphi(t)\,dt, \right.$$

$$\left. \cos\theta \int_{s_0}^{s} \sin\varphi(t)\,dt + \sin\theta \int_{s_0}^{s} \cos\varphi(t)\,dt \right)$$

$$= T_a \circ R_\theta \left(\int_{s_0}^{s} \cos\varphi(t)\,dt, \int_{s_0}^{s} \sin\varphi(t)\,dt \right)$$

$$\tilde{\gamma}(s) = T_a \circ R_\theta (\gamma(s)) \quad [\text{by } (2)]$$

ie, $\tilde{\gamma}(s) = M(\gamma(s))$

Remark:

Any regular plane curve whose curvature is a positive constant is part of a circle.

Proof:

If we take κ to be the curvature of the curve γ and κ_s to be its curvature,

then $\kappa_s = \pm \kappa$

If we could have $K_s = K$ at some points of the curve and $K_s = -K$ at others, then K_s cannot be a continuous function of 's', so intermediate value theorem tells us that, if K_s takes both the value K and the value $-K$, it must take all the values between K & $-K$.

Thus if K_s is a constant, either $K_s = K$ throughout (or) $K_s = -K$ throughout. In either case for a constant K_s, the curve traced is a circle.

Remark:

The curvature of a circle with radius R is $\frac{1}{R}$ can be obtained from the angle $\varphi(s)$ that the tangent makes with the x-axis.

Proof:

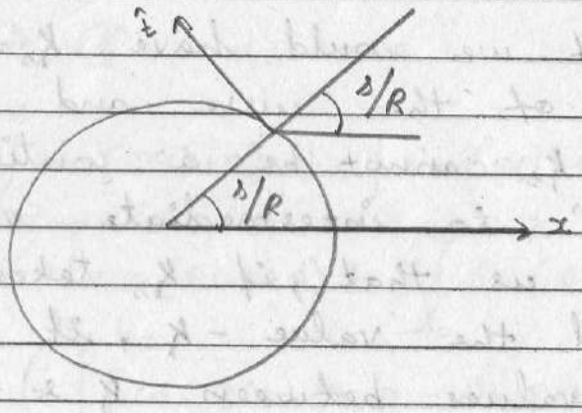
Let $\gamma(s)$ be the unit-speed parametrization of the circle with the centre at the origin and the radius R then

$$\gamma(s) = \left(R \cos \frac{s}{R}, R \sin \frac{s}{R}\right)$$

$$\text{and } \quad t = \dot{\gamma}(s) = \left(-\sin \frac{s}{R}, \cos \frac{s}{R}\right)$$

$$\text{ie, } \quad \varphi(s) = \left(\frac{\pi}{2} + \frac{s}{R}\right)$$

$$\text{and } \quad \frac{d\varphi}{ds} = \frac{1}{R}$$



$|||^{ly}$,

$\tilde{\gamma}(s) = \left(R\cos\frac{s}{R}, -R\sin\frac{s}{R}\right)$ is another parametrization of the circle with the centre at the origin and radius R.

$$\dot{\tilde{\gamma}}(s) = \left(-\sin\frac{s}{R}, -\cos\frac{s}{R}\right)$$

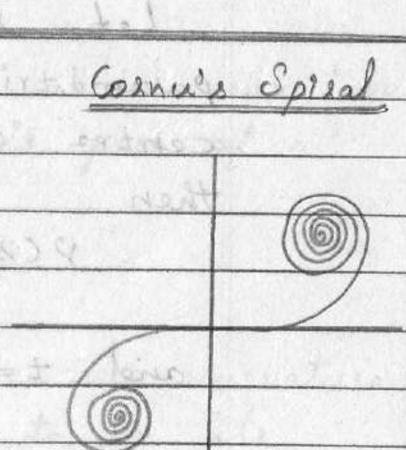
$$= \left(\cos\left(\frac{\pi}{2} - \frac{s}{R}\right), \sin\left(\frac{\pi}{2} - \frac{s}{R}\right)\right)$$

$$\tilde{\varphi}(s) = \left(\frac{\pi}{2} - \frac{s}{R}\right)$$

and $\dfrac{d\tilde{\varphi}}{ds} = -\dfrac{1}{R}$

Remark:

Cornu's Spiral



For eg thm 2.1

Let us take the signed curvature $K_s(s) = s$ and $s_0 = 0$.

Then, $\varphi(s) = \int_0^s u\,du$

$$= \frac{u^2}{2}\Big]_0^s = \frac{s^2}{2}$$

$$\therefore \gamma(s) = \left(\int_0^s \cos\left(\frac{t^2}{2}\right)dt, \int_0^s \sin\left(\frac{t^2}{2}\right)dt\right)$$

Example:

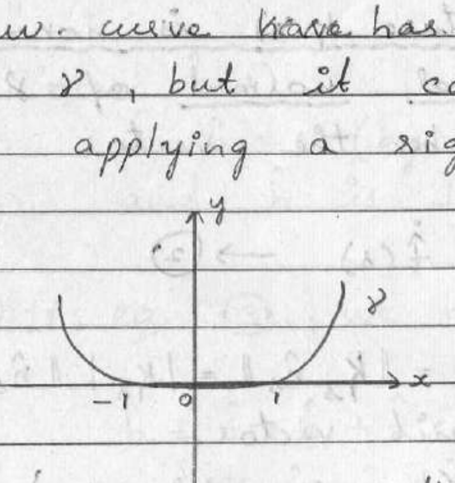
Prove that in the theorem 2.1, if we replace 'signed curvature' by 'curvature', the first part of the statement (if & only if) might hold but not (may not) the second part.

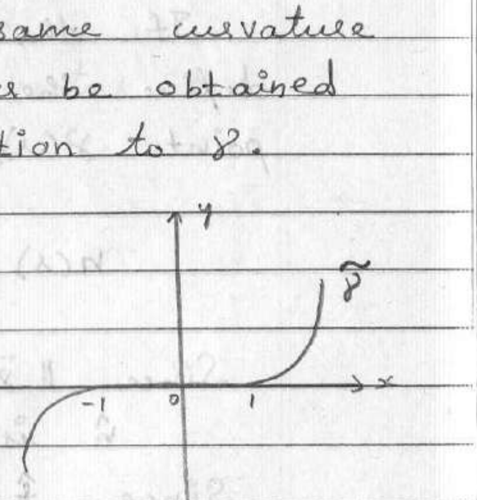
Proof:

1st part: In theorem 2.1, if we replace 'signed curvature' by 'curvature' & assume that $k \geq 0$, then γ can be chosen to have signed curvature k which will be same as the curvature k.

However, for the second part, we can consider the curve γ that coincide with the x-axis for $-1 \leq x \leq 1$ and otherwise it is above the x-axis. ~~axu~~

Let us now reflect part of the curve with $x \leq 0$ in the x-axis. The new curve ~~have~~ has the same curvature as γ, but it can never be obtained by applying a rigid motion to γ.





$$K_0 = K$$

$$\gamma(s) \neq \gamma(\tilde{\gamma}(s))$$

Space Curves:

A plane curve is essentially determined by its curvature but this is no longer true for space curves.

For example,

a circle of radius one in the xy-plane and a circular helix with $a=b=\frac{1}{2}$ both have curvature one everywhere, but it is obviously that through any rigid motion we can't obtain one with the other. Hence we need to define another type of curvature for space curves, called the torsion and any space curve is determined by the curvature & torsion together (up to a rigid motion).

Torsion Derivative:

Let $\gamma(s)$ be a unit-speed curve in R^3, and let $\hat{t} = \dot{\gamma}$ be its unit tangent vector. $\to (1)$

If the curvature $\kappa(s)$ is non-zero, we define the principal normal of γ at the point $\gamma(s)$ to be the vector

$$n(s) = \frac{1}{\kappa(s)} \dot{\hat{t}}(s) \longrightarrow (2)$$

Since $\| \ddot{\gamma} \| = \| \dot{\hat{t}} \| = \| \kappa_s \hat{n}_s \| = |\kappa_s| \, \| \hat{n}_s \| = \kappa$,
$\hat{n}$ is a unit vector.

Since $\hat{t} \cdot \dot{\hat{t}} = 0$ (by proposition 1.2),
$\hat{t}$ & $\hat{n}$ are actually $\perp r$ unit vectors.

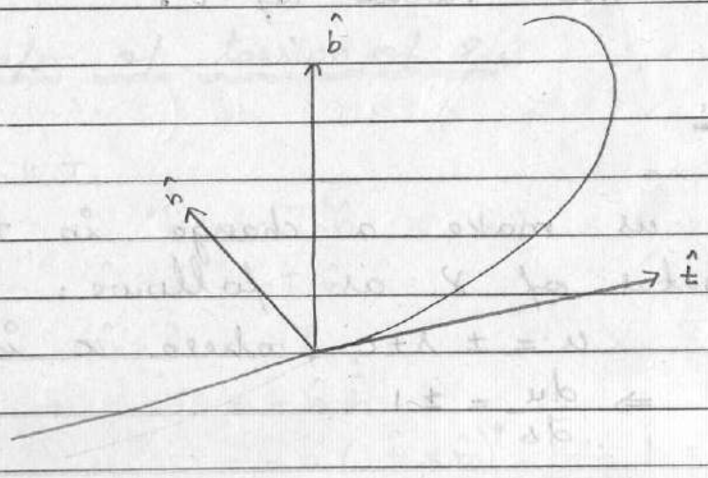
There is a third unit vector which is $\perp^r$ to both $\hat{t}$ and $\hat{n}$ for any space curve at any point, we call it as the <u>binomial</u> vector, $\hat{b}(s)$, of γ at the point $\gamma(s)$ and we can write it as follows:

$$\hat{b} = \hat{t} \times \hat{n} \longrightarrow (3)$$

Thus, the vectors $\{\hat{t}, \hat{n}, \hat{b}\}$ is an orthonormal basis of R^3, and is right-handed.

ie,

$$\hat{b} = \hat{t} \times \hat{n}, \quad \hat{n} = \hat{b} \times \hat{t}, \quad \hat{t} = \hat{n} \times b.$$



Since $\hat{b}(s)$ is a unit vector $\forall s$,

$$\dot{\hat{b}} \cdot \hat{b} = 0 \text{ (by prop. 1.2)}$$

and $\dot{\hat{b}}$ is $\perp^r$ to $\hat{b}$.

Diff. eqn (3), we have

$$\dot{\hat{b}} = \dot{\hat{t}} \times \hat{n} + \hat{t} \times \dot{\hat{n}}$$

$$= \kappa_s \underbrace{\hat{n} \times \hat{n}}_{0} + \hat{t} \times \dot{\hat{n}}$$

ie, $\dot{\hat{b}} = \hat{t} \times \dot{\hat{n}} \longrightarrow (4)$

Since $\dot{\hat{b}}$ is $\perp r$ to both $\hat{b}$ & $\hat{t}$,
$\dot{\hat{b}}$ must be $\parallel^{el}$ to $\hat{n}$ and hence we can write

$$\boxed{\dot{\hat{b}} = -\tau . \hat{n}} \rightarrow (5)$$

for some scalar τ, which is called the torsion of γ (the -ve sign is purely for convention).

Remark:

Show that a change of parameter doesn't effect the value of τ.

Proof:

Let us make a change in the unit-speed parameter of γ as follows:

$u = \pm s + c$, where c is a constant.

$$\Rightarrow \frac{du}{ds} = \pm 1$$

$$\frac{d\gamma}{ds} = \frac{d\gamma}{du}\frac{du}{ds} = \pm \frac{d\gamma}{du} \quad \& \quad \frac{d^2\gamma}{ds^2} = \frac{d^2\gamma}{du^2}$$

ie) the effect of the change of parameter is as follows:

?

a) 20m

Proposition 2.3

20m Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then, denoting $\frac{d}{dt}$ by a dot, its torsion is given by

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

Proof:

Let us first assume that γ is of

(1) unit-speed.

By defn of torsion of γ,

W.K.T,

$$\dot{\hat{b}} = -\tau . \hat{n}$$

$$\hat{n} . \dot{\hat{b}} = -\tau . \hat{n} . \hat{n}$$

$$\hat{n} . \dot{\hat{b}} = -\tau$$

$$\Rightarrow \tau = -\hat{n}\dot{\hat{b}}$$

$$= -\hat{n}\,(\hat{t} \times \hat{n})^{\bullet} \quad \left[\because \hat{b} = \hat{t} \times \hat{n}\right]$$

$$= -\hat{n}\left[\dot{\hat{t}} \times \hat{n} + \hat{t} \times \dot{\hat{n}}\right]$$

$$= -\hat{n}\left[\underbrace{\kappa \hat{n} \times \hat{n}}_{0} + \hat{t} \times \dot{\hat{n}}\right] \quad \left[\because \hat{n} = \frac{1}{\kappa}\dot{\hat{t}}\right]$$

$$(\dot{\ }) = \frac{d}{dt}(\ )$$

$$= -\hat{n}\,(\hat{t} \times \dot{\hat{n}})$$

$$= -\frac{1}{\kappa}\dot{\hat{t}}\left(\hat{t} \times \left(\frac{1}{\kappa}\dot{\hat{t}}\right)^{\bullet}\right) \leftarrow (\text{sub value of } \hat{n})$$

$$= -\frac{1}{\kappa}\dot{\hat{t}}\left\{\hat{t} \times \left[\frac{1}{\kappa}\ddot{\hat{t}} - \frac{\dot{\kappa}}{\kappa^2}\dot{\hat{t}}\right]\right\}$$

W.K.T, $\hat{t} = \dot{\gamma}$, $\dot{\hat{t}} = \ddot{\gamma}$, $\ddot{\hat{t}} = \dddot{\gamma}$

$$\Rightarrow \tau = -\frac{1}{\kappa}\ddot{\gamma}\left\{\dot{\gamma} \times \left[\frac{1}{\kappa}\dddot{\gamma} - \frac{\dot{\kappa}}{\kappa^2}\ddot{\gamma}\right]\right\}$$

$$\tau = -\frac{1}{\kappa}\ddot{\gamma}\left[\frac{1}{\kappa}\dot{\gamma}\times\dddot{\gamma} - \frac{\dot{\kappa}}{\kappa^2}\dot{\gamma}\times\ddot{\gamma}\right]$$

$$= -\frac{1}{\kappa^2}\ddot{\gamma}(\dot{\gamma}\times\dddot{\gamma}) + \frac{\dot{\kappa}}{\kappa^3}\underbrace{\ddot{\gamma}.(\dot{\gamma}\times\ddot{\gamma})}_{0}$$

$$= -\frac{1}{\kappa^2}\ddot{\gamma}(\dot{\gamma}\times\dddot{\gamma})$$

Since $\ddot{\gamma}.(\dot{\gamma}\times\ddot{\gamma}) = 0$ & $\ddot{\gamma}.(\dot{\gamma}\times\dddot{\gamma}) = -\dddot{\gamma}.(\dot{\gamma}\times\ddot{\gamma})$,
we have,

$$\tau = \frac{1}{\kappa^2}\dddot{\gamma}.(\dot{\gamma}\times\ddot{\gamma}) \rightarrow (1)$$

Since γ is of unit-speed,

$\dot{\gamma}$ & $\ddot{\gamma}$ are $\perp r$ (By proposition 1.2).

So ~~Hence~~ $\underbrace{\frac{\sin 90}{1}} = \frac{\|\dot{\gamma}\times\ddot{\gamma}\|}{\underbrace{\|\dot{\gamma}\|}_{1}\|\ddot{\gamma}\|}$ If γ is a unit-speed curve, then $\ddot{\gamma}$ is zero (or) $\perp r$ to $\dot{\gamma}$.

$$\Rightarrow \|\ddot{\gamma}\| = \|\dot{\gamma}\times\ddot{\gamma}\|$$

$\Rightarrow$

If $[\kappa = \|\dot{\gamma}\times\ddot{\gamma}\|]$,
hence $\kappa = \|\ddot{\gamma}\|$ for unit-speed curve.

Eqn (1) becomes,

$$\tau = \frac{(\dot{\gamma}\times\ddot{\gamma}).\dddot{\gamma}}{\|\dot{\gamma}\times\ddot{\gamma}\|^2}$$

—— × —— × ——

(2) In the general case,
let s be arc-length along γ
and denote $\frac{d}{ds}$ by a dash.

Then,

$$\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\,\frac{ds}{dt}$$

$$\dot{\gamma} = \gamma'\,\frac{ds}{dt}$$

$$\therefore\; \ddot{\gamma} = \frac{d}{dt}\left(\gamma'\,\frac{ds}{dt}\right)$$

$$= \frac{d}{dt}(\gamma')\,\frac{ds}{dt} + \gamma'\,\frac{d}{dt}\left(\frac{ds}{dt}\right)$$

$$= \frac{d}{ds}(\gamma')\,\frac{ds}{dt}\cdot\boxed{\frac{ds}{dt}} + \gamma'\,\frac{d^2s}{dt^2}$$

$$\ddot{\gamma} = \gamma''\left(\frac{ds}{dt}\right)^2 + \gamma'\,\frac{d^2s}{dt^2}$$

IIIly,

$$\dddot{\gamma} = \frac{d}{dt}\left\{\gamma'''\left(\frac{ds}{dt}\right)^2\right\} + \frac{d}{dt}\left\{\gamma'\,\frac{d^2s}{dt^2}\right\}$$

$$= \frac{d}{dt}(\gamma'')\left(\frac{ds}{dt}\right)^2 + \gamma''\,\frac{d}{dt}\left(\frac{ds}{dt}\right)^2$$

$$+ \frac{d}{dt}(\gamma')\,\frac{d^2s}{dt^2} + \gamma'\,\frac{d}{dt}\left(\frac{d^2s}{dt^2}\right)$$

$$= \gamma'''\left(\frac{ds}{dt}\right)^3 + \gamma''\,2\,\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}$$

$$+ \gamma''\,\frac{ds}{dt}\,\frac{d^2s}{dt^2} + \gamma'\,\frac{d^3s}{dt^3}$$

$$= \gamma'''\left(\frac{ds}{dt}\right)^3 + 3\,\gamma''\,\frac{ds}{dt}\,\frac{d^2s}{dt^2} + \gamma'\,\frac{d^3s}{dt^3}$$

Now consider,

$$\dot{\gamma}\times\ddot{\gamma} = \gamma'\left(\frac{ds}{dt}\right)\times\left\{\gamma''\left(\frac{ds}{dt}\right)^2 + \gamma'\left(\frac{d^2s}{dt^2}\right)\right\}.$$

$$\dot{\gamma} \times \ddot{\gamma} = \gamma' \times \gamma'' \left(\frac{ds}{dt}\right)^3 + \gamma' \times \gamma' \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)$$

$$= \gamma' \times \gamma'' \left(\frac{ds}{dt}\right)^3$$

And

$$(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma} = (\gamma' \times \gamma'')\left(\frac{ds}{dt}\right)^3 \left\{ \left(\frac{ds}{dt}\right)^3 \gamma''' + 3\frac{ds}{dt}\frac{d^2s}{dt^2}\gamma'' + \frac{d^3s}{dt^3}\gamma' \right\}$$

$$= (\gamma' \times \gamma'')\gamma''' \left(\frac{ds}{dt}\right)^6 + \underbrace{(\gamma' \times \gamma'')\gamma''}_{0} 3\frac{ds}{dt}\frac{d^2s}{dt^2}$$

$$+ \underbrace{(\gamma'' \times \gamma'')}_{0} \gamma' \frac{d^3s}{dt^3}$$

$$= (\gamma' \times \gamma'')\gamma''' \left(\frac{ds}{dt}\right)^6$$

Thus the torsion eqn is

$$\frac{(\dot{\gamma} \times \ddot{\gamma})\dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{(\gamma' \times \gamma'')\gamma''' \left(\frac{ds}{dt}\right)^6}{\|\gamma' \times \gamma''\|^2 \left[\left(\frac{ds}{dt}\right)^3\right]^2} = \frac{(\gamma' \times \gamma'')\gamma'''}{\|\gamma' \times \gamma''\|^2}$$

Example:

6m Compute the torsion of the circular helix
$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$

Soln:

$$\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b)$$
$$\ddot{\gamma}(\theta) = (-a\cos\theta, -a\sin\theta)$$

$$\dot{\gamma} \times \ddot{\gamma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & b \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= \hat{i}(0 + ab\sin\theta) - \hat{j}(0 + ab\cos\theta) + \hat{k}(a^2\sin^2\theta + a^2\cos^2\theta)$$

$$= ab\sin\theta\,\hat{i} - ab\cos\theta\,\hat{j} + a^2 k^2$$

$$\Rightarrow (\dot{\gamma} \times \ddot{\gamma})(\theta) = (ab\sin\theta, -ab\cos\theta, a^2)$$

$$\|\dot{\gamma} \times \ddot{\gamma}\| = \sqrt{(ab)^2\sin^2\theta + (ab)^2\cos^2\theta + a^4}$$

$$= \sqrt{a^2b^2 + a^2}$$

$$= \sqrt{a^2(a^2+b^2)}$$

$$\therefore \|\dot{\gamma} \times \ddot{\gamma}\|^2 = a^2(a^2+b^2)$$

Now,

$$\dddot{\gamma} = (a\sin\theta, -a\cos\theta)$$

$$(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma} = (ab\sin\theta\,\hat{i} - ab\cos\theta\,\hat{j} + a^2\hat{k}) \cdot (a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j})$$

$$= a^2b\sin^2\theta + a^2b\cos^2\theta$$

$$= a^2 b$$

Hence
$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

$$= \frac{a^2 b}{a^2(a^2+b^2)}$$

$$= \frac{b}{a^2+b^2}$$

Note: If $b = 0$ then $\tau = 0$, in which case the helix is just a circle in the xy-plane.

Proposition 2.4

Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature (so that the torsion τ of γ is defined). Then, the image of γ is contained in a plane $\Leftrightarrow$ τ is zero at every point of the curve.

Proof:

Let us assume that γ is of unit-speed; Otherwise we can reparametrise γ to be of unit-speed, since reparametrisation doesn't affect the nature of the curve (or) value of the torsion.

Also, let us take 's' to be the parameter of γ & denote $\frac{d}{ds}$ by a dot.

Let us first assume that γ is contained in a plane & let us write the eqn of the plane that contains γ as

$$\vec{r} \cdot \vec{a} = d \longrightarrow (1)$$

where r - p.v of an arbitrary pt of $\mathbb{R}^3$ &
d - constant scalar &
a - constant vector.

We can assume that 'a' to be a unit vector.

Since the plane represented by ① contains γ,
γ must satisfy eqn ①

ie, $\gamma . \vec{a} = d$

Diff. w.r.t 's', we have

$\dot{\gamma} . \vec{a} + \gamma . \dot{\vec{a}} = 0$ $\because \vec{a}$ is a constant ve

$\dot{\gamma} . \vec{a} = 0$ $\left[\because \vec{a} \text{ is a constant vector}, \dot{\vec{a}} = 0 \right]$

$\vec{t} . \vec{a} = 0$ $[\because \vec{t} = \dot{\gamma}]$ → ②

$\Rightarrow \hat{t} \perp r \ \vec{a}$

$\dot{\hat{t}} . \vec{a} = 0$

$K \hat{n} . \vec{a} = 0$ $\left[\because \hat{n} = \frac{1}{K} \dot{\hat{t}} \right]$

$\Rightarrow \hat{n} . \vec{a} = 0$ $[\because K \neq 0]$ → ③

$\Rightarrow \hat{n} \perp r \ \vec{a}$

From ② & ③,
it follows that $\hat{b} = \hat{t} \times \hat{n}$ $\parallel^{el}$ to a.

Since a & b are both unit vectors, & b is a smooth fn of s,
we must have $\hat{b} = \vec{a} \ \forall s$ or $\hat{b} = -\vec{a} \ \forall s$.

$\Rightarrow$ either $\hat{b} = \vec{a}$ and $\hat{b} = -\vec{a}$ ~~as~~ is a smooth function of s.

In either case, b is a constant vector.

But then, $\dot{b} = 0$

$$\Rightarrow -\tau \cdot n = 0$$

$$\Rightarrow \tau = 0 \quad [\because n \text{ is unit}, n \neq 0]$$

$\Rightarrow$ If γ is contained in a plane then $\tau = 0$.

Conversely,

~~Let~~ us suppose that $\tau = 0$ everywhere. Then we will have $\dot{b} = 0$ and that could imply b is a constant vector.

$\boxed{\dot{b} = -\tau n = 0}$

1st part of the proof.

The above discussion suggests that γ should be contained in a plane

$$\vec{\gamma} \cdot b = \text{constant}.$$

Let us therefore consider,

$$\frac{d}{ds}(\gamma \cdot b) = \dot{\gamma} \cdot b + \gamma \cdot \dot{b} \rightarrow 0$$

$$= \vec{t} \cdot \vec{b}$$

$$= 0 \quad [\because t \perp b]$$

$$\Rightarrow \gamma \cdot b = \text{constant}$$

This means that γ is indeed contained in the plane $\vec{\gamma} \cdot b = \text{const}$.

20m a) / 10m / 20m

Frenet-Serret Eqns [(or) Serret-Frenet Eqns]:

Theorem 2.2

Let γ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then,

$$\dot{t} = \kappa n$$
$$\dot{n} = -\kappa t + \tau b \qquad (t, n, b)$$
$$\dot{b} = -\tau n$$

Proof:

Let $\gamma(s)$ be a unit-speed curve in $\mathbb{R}^3$ and let $t = \dot{\gamma}$ be its unit tangent vector. If the curvature κ is non-zero, we can define the principal normal γ at the point $\gamma(s)$ as

$$n = \frac{1}{\kappa}\dot{t}$$

Thus $\dot{t} = \kappa n \rightarrow (1)$

~~Since $\kappa = \|\ddot{\gamma}\| = \|\dot{t}\|$~~

Since $t \cdot \dot{t} = 0$, we have t & n to be $\perp r$ unit vectors and let us define the binomial vector of γ at the point $\gamma(s)$ as

$$b = t \times n$$

Thus the set $\{t, n, b\}$ forms an orthonormal basis of $\mathbb{R}^3$ and

$$\dot{b} = \dot{t} \times n + t \times \dot{n}$$
$$= t \times \dot{n}$$

[This shows that $\dot{b} \perp t$]

$\because \dot{b} \perp b$,
it must be $\parallel^{el}$ to n.

Hence we define,

$$\dot{b} = -\tau n \rightarrow (2)$$

where τ is the torsion.

$$\therefore n = b \times t$$
$$\dot{n} = (\dot{b} \times t) + (b \times \dot{t})$$

Sub. (1) & (2),

$$\dot{n} = (-\tau n \times t) + (b \times \kappa n)$$
$$= -\tau(-b) + \kappa(-t)$$
$$= -\kappa t + \tau b$$

Note:

From Frenet - Serret Eqn,
we can write the coefficient matrix,

$$A = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

which expresses $\dot{\hat{t}}, \dot{\hat{n}}$ & $\dot{\hat{b}}$ in terms of $\hat{t}, \hat{n}$ & $\hat{b}$ and is a skew-symmetric ie., $A = -A^T$

Since A is skew-symmetric, it has a property that for some fixed parameter value $s_0(s)$, the vectors $\hat{t}(s_0), \hat{n}(s_0)$ & $\hat{b}(s_0)$ are orthonormal unit vectors.

Hence the vectors $\hat{t}(s), \hat{n}(s)$ & $\hat{b}(s)$ will be orthonormal unit vectors for all values of s, since $\hat{t}, \hat{n}$ & $\hat{b}$ are smooth fns of s with the ordinary diff. equs given by Frenet-Serret Equs.

8m 6m

Proposition 2.5

Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then, γ is (part of) a circle.

Proof:

By Proposition 2.4, Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature (so that the torsion τ of γ is defined). Then, the image of γ is contained in a plane $\Leftrightarrow \tau = 0$ at every pt of the curve

W.K.T, γ is a plane curve contained in a plane $\perp r$ to $\hat{b}$, the binormal (is const).

Let us consider,

$$\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\hat{n}\right) = \dot{\gamma} + \frac{\dot{\hat{n}}}{\kappa} - \frac{\dot{\kappa}}{\kappa^2}\hat{n}$$

$\left[\because \text{curvature is constant, } \kappa = \text{const} \Rightarrow \dot{\kappa} = 0\right]$

$$= \hat{t} + \frac{1}{\kappa}(-\kappa t + \tau\hat{b})$$

$$= \hat{t} - \frac{\kappa}{\kappa}\hat{t} \quad \left[\because \text{torsion is zero, } \tau = 0\right]$$

$$= \hat{t} - \hat{t} = 0$$

$$\Rightarrow \gamma + \frac{1}{k}\hat{n} = \vec{a} \text{ (const)}$$

i.e., $\gamma - \vec{a} = -\frac{1}{k}\vec{n}$

(or) $\|\gamma - \vec{a}\| = \|-\frac{1}{k}\vec{n}\|$

$= \left|-\frac{1}{k}\right| \|\vec{n}\|$

$= \frac{1}{k}$ ($\because \hat{n}$ is a unit vector)

$= \frac{1}{k}$

The above expression shows that γ is a ~~curve~~ circle with centre at $\vec{a}$ and radius $\frac{1}{k}$ i.e., $k = \frac{1}{R}$.

Theorem 2.3:

20m) Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two unit-speed curves in R^3 with the same curvature $k(s) > 0$ and the same torsion $\tau(s)$ for all s. Then, there is a rigid motion M of R^3

$$\ni \tilde{\gamma}(s) = M(\gamma(s)) \text{ for all } s.$$

Further, if k & t are smooth fns with $k > 0$ everywhere, there is a unit-speed curve in R^3 whose curvature is k and whose torsion is t.

Proof:

Let us first assume that $\gamma(s)$ & $\tilde{\gamma}(s)$ to be two unit-speed curves in $\mathbb{R}^3$ with same curvature $\kappa(s) > 0$ and same torsion $\tau(s)$ $\forall$ s.

Let $\hat{t}$, $\hat{n}$ & $\hat{b}$ be the tangent vector, principal normal and binormal of γ and let $\tilde{t}$, $\tilde{n}$ & $\tilde{b}$ be those of $\tilde{\gamma}$.

Let s_0 be a fixed value of the parameter s.

Since the set $\{t(s_0), n(s_0), b(s_0)\}$ and $\{\tilde{t}(s_0), \tilde{n}(s_0), \tilde{b}(s_0)\}$ are both right-handed orthonormal bases of $\mathbb{R}^3$, there is a rigid motion of $\mathbb{R}^3$ that takes $t(s_0)$, $n(s_0)$ & $b(s_0)$ to $\tilde{t}(s_0)$, $\tilde{n}(s_0)$ & $\tilde{b}(s_0)$, respectively.

Further it ~~has~~ takes $\gamma(s_0)$ to $\tilde{\gamma}(s_0)$, therefore we can assume that,

$$\left.\begin{aligned} t(s_0) &= \tilde{t}(s_0) \\ n(s_0) &= \tilde{n}(s_0) \\ b(s_0) &= \tilde{b}(s_0) \\ \gamma(s_0) &= \tilde{\gamma}(s_0) \end{aligned}\right\} \quad (1)$$

Let us now consider a fn,

$$A(s) = t \cdot \tilde{t} + n \cdot \tilde{n} + b \cdot \tilde{b} \longrightarrow (2)$$

which is a smooth fn of the parameter s.

Let us evaluate at the point s_0

$$A(s_0) = t(s_0).\tilde{t}(s_0) + n(s_0).\tilde{n}(s_0) + b(s_0).\tilde{b}(s_0)$$
$$= t(s_0).t(s_0) + n(s_0).n(s_0) + b(s_0).b(s_0) \text{ (by ①)}$$
$$= 1 + 1 + 1 \quad (\because t, n \,\&\, b \text{ are unit vectors})$$
$$= 3 \longrightarrow ③$$

Otherwise for any s,

$$A(s) \le 1 + 1 + 1$$
$$\le 3 \longrightarrow ④$$

[Since $t.\tilde{t} \le 1$, $n.\tilde{n} \le 1$ & $b.\tilde{b} \le 1 \longrightarrow$ (4a)
as $t, n \,\&\, b$ are $\tilde{t}, \tilde{n} \,\&\, \tilde{b}$ are unit vectors]

Diff. A w.r.t 's',

$$\dot{A}(s) = \dot{t}.\tilde{t} + \dot{n}.\tilde{n} + n.\dot{\tilde{n}} + \dot{b}.\tilde{b} + b.\dot{\tilde{b}} + t.\dot{\tilde{t}}$$

$$= \kappa n.\tilde{t} + t.\kappa\tilde{n} + (-\kappa t + \tau b)\tilde{n} + n.(-\kappa\tilde{t} + \tau\tilde{b}) + (-\tau n.\tilde{b}) + b.(-\tau\tilde{n})$$

1st part of the proof showing A is constant using Thm 2.2

[Since κ & τ are curvature and torsion of both γ & $\tilde{\gamma}$]

$$= 0$$

$\Rightarrow A$ is a constant

$\Rightarrow A = 3$ as A is a smooth fn and at s_0 $A = 3$ & so $A = 3$ everywhere.

$\Rightarrow t = \tilde{t}$, $n = \tilde{n}$ & $b = \tilde{b}$

Since (4a) $\Rightarrow t.\tilde{t} = 1$, $n.\tilde{n} = 1$ & $b.\tilde{b} = 1$ is possible only if they are equal.

Since $t = \tilde{t}$,

we have $\dot{\gamma} = \dot{\tilde{\gamma}}$

$\Rightarrow \dot{\gamma} - \dot{\tilde{\gamma}} = 0$

$\Rightarrow \gamma - \tilde{\gamma} = \text{constant}$

From eqn ①, we say that the constant must be zero.

Thus $\gamma = \tilde{\gamma} \quad \forall s$.

2nd part of the theorem

Let us assume that k and t to be smooth fns with $k > 0$ everywhere.

Let us consider the following ordinary differential eqns,

$$\dot{T} = kN \longrightarrow ⑤$$
$$\dot{N} = -kT + tB \longrightarrow ⑥$$
$$\dot{B} = -tN \longrightarrow ⑦$$

By (*),

the above ordinary diff. eqn has unique soln. $T(s), N(s), B(s) \ni T(s_0), N(s_0), B(s_0)$ are standard orthonormal unit vectors.

Because the matrix

$$\begin{bmatrix} 0 & k & 0 \\ -k & 0 & t \\ 0 & -t & 0 \end{bmatrix}$$ obtained from the eqns

expresses $\dot{T}, \dot{N}, \dot{B}$ interms of T, N, B is a skew-symmetric matrix.

Using this unit vectors let us define γ as follows

$$\gamma(s) = \int_{s_0}^{s} T(u)\,du$$

Thus $\dot{\gamma} = T$, ie, T is the tangent of the curve γ.

Since T is a unit vector,

γ is of unit-speed.

Since eqn (5) $\Rightarrow \dot{T} = kN$ and N is a unit vector, we have

k is the curvature of γ and N is its principal normal.

Since B is a unit vector $\perp$ to T & N, we must have

B to be $\|^{el}$ to $T \times N$

ie, we can write $B = \lambda(s)\, T \times N$

for some smooth fn λ with parameter s and λ can take value either $+1$ or -1.

But then we have $\hat{i}, \hat{j}, \hat{k}$ to be standard orthonormal unit vectors, hence $\mathbf{K} = \hat{i} \times \hat{j}$, hence λ must be $+1$ throughout.

Thus $B = T \times N$ & B is the binomial of γ; from eqn (7) we see that t is the torsion of γ. Thus if k & t are smooth fns with $k > 0$ everywhere. We can find a curve γ which is of unit-speed with curvature k and torsion t.

Exercise :

2.14 Calculate κ, τ, t, n & b for each of the following curves, and verify that the Frenet - Serret eqns are satisfied:

i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}} \right)$

ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t \right)$

Soln:

i) $\dot{\gamma}(t) = \left\{ \frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}} \right\}$

$\|\dot{\gamma}(t)\| = 1 \Rightarrow \gamma$ is of unit-speed

$$\dot{\gamma} = \hat{t} = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}} \right)$$

($\hat{t}$ is a unit vector, so γ is unit-speed)

$$\ddot{\gamma}(t) = \left\{ \frac{1}{4}(1+t)^{-1/2}, +\frac{1}{4}(1-t)^{-1/2}, 0 \right\} = \dot{t}$$

So, $\kappa = \|\ddot{\gamma}\|$ [$\because \gamma$ is of unit-speed]

$$= \sqrt{\frac{1}{16}(1+t)^{-1} + \frac{1}{16}(1-t)^{-1}}$$

$$= \sqrt{\frac{1}{16}\left(\frac{1}{1+t} + \frac{1}{1-t} \right)}$$

$$= \sqrt{\frac{1}{16}\left(\frac{2}{1-t^2} \right)}$$

$$= \frac{1}{\sqrt{8(1-t^2)}}$$

W.K.T,

$$\hat{n} = \frac{1}{K}\dot{\hat{t}}$$

$$\frac{1}{K} = \sqrt{8(1-t^2)}$$

$$= \sqrt{8(1+t)(1-t)}$$

$$= 2\sqrt{2}(1+t)^{1/2}(1-t)^{1/2}$$

$$\hat{n} = \left\{\frac{2\sqrt{2}}{4}(1-t)^{1/2}, \frac{2\sqrt{2}}{4}(1+t)^{1/2}, 0\right\}$$

$$= \left\{\frac{1}{\sqrt{2}}(1-t)^{1/2}, \frac{1}{\sqrt{2}}(1+t)^{1/2}, 0\right\}$$

$$\dot{\hat{n}} = \left\{-\frac{1}{2\sqrt{2}}(1-t)^{-1/2}, \frac{1}{2\sqrt{2}}(1+t)^{-1/2}, 0\right\}$$

$$\hat{b} = \hat{t}\times\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2}(1+t)^{1/2} & -\frac{1}{2}(1-t)^{1/2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(1-t)^{1/2} & \frac{1}{\sqrt{2}}(1+t)^{1/2} & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\frac{1}{2}(1+t)^{1/2}\right) - \hat{j}\left(-\frac{1}{2}(1-t)^{1/2}\right)$$

$$+ k\left(\frac{1}{2\sqrt{2}}(1+t) + \frac{1}{2\sqrt{2}}(1-t)\right)$$

$$= -\hat{i}\left(-\frac{1}{2}(1+t)^{1/2}\right) + \hat{j}\left(\frac{1}{2}(1-t)^{1/2}\right)$$

$$+ \hat{k}\left(\frac{1}{2\sqrt{2}}(1+t+1-t)\right)$$

$$\hat{b} = \left\{-\frac{1}{2}(1+t)^{1/2}, \frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right\}$$

Since γ is of unit-speed,

$$\dot{b} = -\tau n$$

$$\left(-\frac{1}{4}(1+t)^{-1/2}, -\frac{1}{4}(1-t)^{-1/2}, 0\right)$$

$$= \left(\frac{-\tau}{\sqrt{2}}(1-t)^{1/2}, \frac{-\tau}{\sqrt{2}}(1+t)^{1/2}, 0\right)$$

$$\therefore \frac{\tau}{\sqrt{2}}(1-t)^{1/2} = \frac{1}{4}(1+t)^{-1/2}$$

$$\tau = \frac{\sqrt{2}}{4}\frac{(1+t)^{-1/2}}{(1-t)^{1/2}}$$

$$= \frac{1}{2\sqrt{2}}\frac{(1+t)^{-1/2}(1+t)^{1/2}}{(1-t)^{\frac{1}{2}}(1+t)^{1/2}}$$

$$= \frac{1}{2\sqrt{2}}\frac{1}{[(1-t)(1+t)]^{1/2}}$$

$$= \frac{1}{2\sqrt{2}}\frac{1}{(1-t^2)^{1/2}}$$

$$= \frac{1}{\sqrt{8(1-t^2)}}$$

To verify Frenet-Serret eqn, we check

$$\dot{n} = -\kappa t + \tau b \qquad \sqrt{8(1-t^2)} = \sqrt{8}(1+t)^{1/2}(1-t)^{1/2}$$

$$R.H.S = -\kappa t + \tau b \qquad = 2\sqrt{2} \quad '' \quad ''$$

Consider R.H.S

To S.T R.H.S $\Rightarrow$ L.H.S

$$= -\frac{1}{\sqrt{8(1-t^2)}}\left\{\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right\}$$

$$+ \frac{1}{\sqrt{8(1-t^2)}}\left\{-\frac{1}{2}(1+t)^{1/2}, \frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right\}$$

$$= \left\{\frac{1}{4\sqrt{2}}\frac{(1+t)^{1/2}}{(1+t)^{1/2}(1-t)^{1/2}}, \frac{1}{4\sqrt{2}}\frac{(1-t)^{1/2}}{(1+t)^{1/2}(1-t)^{1/2}}, -\frac{1}{4}\cdot\frac{1}{(1-t^2)^{1/2}}\right\}$$

$$+ \left\{\frac{-1}{4\sqrt{2}}\frac{(1+t)^{1/2}}{(1+t)^{1/2}(1-t)^{1/2}}, \frac{1}{4\sqrt{2}}\frac{(1-t)^{1/2}}{(1+t)^{1/2}(1-t)^{1/2}}, \frac{1}{4}\frac{1}{(1-t^2)^{1/2}}\right\}$$

$$= \left\{ \frac{-1}{4\sqrt{2}} \cdot \frac{1}{(1-t)^{1/2}}, \frac{1}{4\sqrt{2}} \frac{1}{(1+t)^{1/2}}, \frac{-1}{4} \frac{1}{(1-t^2)^{1/2}} \right\}$$

$$+ \left\{ \frac{-1}{4\sqrt{2}} \frac{1}{(1-t)^{1/2}}, \frac{1}{4\sqrt{2}} \frac{1}{(1+t)^{1/2}}, \frac{1}{4} \frac{1}{(1-t^2)^{1/2}} \right\}$$

$$= \left\{ \frac{-1}{2\sqrt{2}} (1-t)^{-1/2}, \frac{1}{2\sqrt{2}} (1+t)^{-1/2}, 0 \right\}$$

$$= \dot{n}$$

ii) $\gamma(t) = \left(\frac{4}{5} \cos t, 1 - \sin t, \frac{-3}{5} \cos t \right)$

$\dot{\gamma}(t) = \left(-\frac{4}{5} \sin t, -\cos t, \frac{3}{5} \sin t \right)$

$\| \dot{\gamma}(t) \| = 1$

$\dot{\gamma} = t = \left(\frac{-4}{5} \sin t, -\cos t, \frac{3}{5} \sin t \right)$ is a unit vector.

so γ is of unit-speed.

$\ddot{\gamma} = \left(-\frac{4}{5} \cos t, \sin t, \frac{3}{5} \cos t \right) = \dot{t}$

Since γ is of unit-speed

$\kappa = \| \ddot{\gamma} \|$

$$= \sqrt{\frac{16}{25} \cos^2 t + \sin^2 t + \frac{9}{25} \cos^2 t}$$

$$= \sqrt{\cos^2 t + \sin^2 t}$$

$$= 1$$

$$n = \frac{1}{K}\dot{t} = \frac{1}{1}\dot{t} = \dot{t}$$

$$\hat{n} = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right)$$

$$\dot{n} = \left(\frac{4}{5}\sin t, \cos t, -\frac{3}{5}\sin t\right)$$

$$\hat{b} = \hat{t} \times \hat{n}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{4}{5}\sin t & -\cos t & \frac{3}{5}\sin t \\ -\frac{4}{5}\cos t & \sin t & -\frac{3}{5}\cos t \end{vmatrix}$$

$$= \hat{i}\left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t\right) - \hat{j}\left(-\frac{12}{15}\sin t\cos t + \frac{12}{5}\sin t\cos t\right)$$
$$+ \hat{k}\left(-\frac{4}{5}\sin^2 t - \frac{4}{5}\cos^2 t\right)$$

$$= \hat{i}\left(-\frac{3}{5}\right) + \hat{j}(0) + \hat{k}\left(-\frac{4}{5}\right)$$

$$= \left\{-\frac{3}{5}, 0, -\frac{4}{5}\right\}$$

$$\dot{b} = \hat{0}$$

$$\tau = 0 \quad \left[\because \gamma \text{ is of unit-speed, } \dot{b} = -\tau\hat{n}\right]$$

To verify:

$$\dot{n} = -\kappa t + \tau b$$

Consider the R.H.S.,

$$-\kappa t + \tau b = -\left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right) + 0 \quad [\because \tau = 0]$$

$$\therefore -\kappa t + \tau b = \left(\frac{4}{5}\sin t,\ \cos t,\ \frac{-3}{5}\sin t\right)$$

$$= \dot{n}$$

$$= L.H.S$$

Hence $\dot{n} = -\kappa t + \tau b$

Thus, Frenet-Serret eqns are verified.

Unit - II

Surfaces in three dimensions

2.1) Surface: What is a surface?

A surface is a subset of $\mathbb{R}^3$ that looks like a piece of $\mathbb{R}^2$ in the vicinity of any given point, like the surface of the Earth, which is spherical, appears to be a flat plane to an observer on the surface.

Defn: [Open]

A subset U of $\mathbb{R}^n$ is called open if, whenever 'a' is a point in U, there is a positive number $\varepsilon \ni$ every point $u \in \mathbb{R}^n$ within a distance ε of 'a' is also in U:

i.e., $a \in U$ and $\|u - a\| < \varepsilon \Rightarrow u \in U$.