ASSIGNMENT SIX
Due Friday 4.5.24, noon, King 205a

Readings:

Please read over the definitions on this assignment sheet, my notes, and your notes.

Note: You will be graded on the correctness of your solutions and on how easy they are to read and understand. You will receive full credit for a solution to a problem if and only if the grader believes that a typical student in the class would understand how to do the problem after having read your solution.

DEFINITIONS (CARDINALITY)

Definition: If the members of a set A and the members of a set B can be put into a one-to-one correspondence (i.e., can be paired up), then A and B have the same **cardinality** and we write $|A| = |B|$. If, in addition, the set $B = \{1, 2, \ldots n\}$ for some $n \in \mathbb{N}$, then we write $|A| = n$.

Definition: A set S is **finite** if $|S| = n$ for some $n \in \mathbb{N}$.

Definition: A set S is **countably infinite** if $|S| = |\mathbb{N}|$.

Definition: A set S is **countable** if S is finite or countably infinite.

Definition: A set S is **uncountable** (or **uncountably infinite**) if S is not countable.

DEFINITIONS (FUNCTIONS)

Definition: f is a **function from** A **to** B (written $f : A \to B$) if for each $a \in A$ there exists a unique $b \in B$ such that $f(a) = b$.

Definition: $f : A \to B$ is a **surjection** if for each $b \in B$ there exists an $a \in A$ such that $f(a) = b$.

Definition: $f : A \to B$ is an **injection** if $\forall a_1 \in A : \forall a_2 \in A : a_1 \neq a_2 \to f(a_1) \neq f(a_2)$ (f maps distinct elements of A to distinct elements of B).

Definition: $f : A \to B$ is a **bijection** if it is both an injection and a surjection.

Definition: $|A| = |B|$ if there exists a bijection from A to B.

Definition: $|A| > |B|$ if there exists a surjection from A to B but there does not exist a bijection from A to B.

EXERCISES

1. Let S be the set of all one-element and two-element subsets of the nonnegative integers. Is S countable? Explain.

2. Consider the set of functions from the set $\{0, 1\}$ to $\mathbb{N}$. Is this set countable? Explain.

3. Consider the set of functions from $\mathbb{N}$ to the set $\{0, 1\}$. Is this set countable? Explain.

4. Determine which of the following functions are surjective and which of them are injective, where $\mathbb{R}$ is the set of real numbers, and $\mathbb{N}$ is the set of nonnegative integers.

a. $f : \mathbb{N} \to \mathbb{N}$ given by the rule

$$f(n) = \text{the number of distinct prime factors of } n.$$

b. $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by the rule

$$g((x, y)) = (-y, x).$$

c. $h : H \to \mathbb{N}$ given by the rule

$$h(x) = \text{the number of hairs on } x\text{'s head}$$

where H is the set of all living people. Assume that $|H| \geq 7,800,000,000$ and that no one has more than $150,000$ hairs on their head.

5. Recall that $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbf{Z}$ denotes the set of integers.

a. Find a function from $\mathbb{N}$ to $\mathbf{Z}$ that is injective but not surjective.

b. Find a function from $\mathbb{N}$ to $\mathbf{Z}$ that is surjective but not injective.

c. Find a bijection from $\mathbb{N}$ to $\mathbf{Z}$.