Week 5 : Theory Solutions

Solution 1(a) Let $|\psi\rangle = U|\phi\rangle$, where U is an unitary operation.

$$\begin{aligned}
\langle\psi|\psi\rangle &= \langle U\phi \mid U\phi\rangle \\
&= \langle\phi \mid U^\dagger U\phi\rangle \\
&= \langle\phi \mid \phi\rangle \\
\text{U is unitary} &\implies U^\dagger U = I
\end{aligned}$$

Hence we see that the norm of $|\phi\rangle$ is preserved by unitary operations.
Solution 1(b) Let $|\psi_1\rangle = U|\phi_1\rangle$ and $|\psi_2\rangle = U|\phi_2\rangle$, where U is an unitary operation. To prove just show that $\langle\psi_1|\psi_2\rangle = \langle\phi_1|\phi_2\rangle$.
Solution 2 $\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Using the characteristic equation we have

$$|A - \lambda I| = 0$$

Therefore eigenvalues of σ_x are

$$\begin{aligned}
\left|\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right| = \lambda^2 - 1 = 0 \\
\implies \lambda = \{1, -1\}
\end{aligned}$$

Now, to find the eigenvector of σ_x we use the eigenvalue equation $\sigma_x|\psi\rangle = \lambda|\psi\rangle$. Substituting $\lambda = 1$, we get

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= 1 \cdot \begin{bmatrix} x \\ y \end{bmatrix} \\
\implies \begin{bmatrix} y \\ x \end{bmatrix} &= \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}$$

Hence the normalized eigenvector corresponding to $\lambda = 1$ is $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, the normalized eigenvector corresponding to $\lambda = -1$ is $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Now we perform eigendecomposition of σ_x to get

$$\begin{aligned}
\sigma_x &= V\Lambda V^{-1} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}
\end{aligned}$$

Therefore $e^{i\sigma_x} = Ve^{i\Lambda}V^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{i} & 0 \\ 0 & e^{i(-1)} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. The other two unitaries can be calculated similarly.

Solution 3 Note

- $A^\dagger = -A$ for Skew-Hermitian matrices.
- $AA^\dagger = I$ for Unitary matrices.

Skew-Hermitian matrices are diagonalizable. Therefore, $A = V\Lambda V^{-1}$ and $e^{iA} = Ve^{i\Lambda}V^{-1}$.
Thus $e^{iA}e^{iA^\dagger} = Ve^{i\Lambda}V^{-1}Ve^{i\Lambda^\dagger}V^{-1} = Ve^{i\Lambda}e^{i\Lambda^\dagger}V^{-1} = V \begin{bmatrix} e^{i\lambda_1}e^{-i\lambda_1} & & & \\ & e^{i\lambda_2}e^{-i\lambda_2} & & \\ & & . & \\ & & & . \end{bmatrix} V^{-1} =$
$VIV^{-1} = I$.
Hence e^{iA} is unitary.

Solution 4

$$U = e^{-iHt} = e^{-i\tilde{H}t/100} = e^{-i\tilde{H}\tilde{t}}$$

where $\tilde{t} = t/100$. Hence we we can implement U a 100 times faster with $\tilde{H}$ than with H.

Solution 5 The state is

$$\sum_{j=1}^{N} \beta_j |u_j\rangle^I \left(\sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle + \frac{C}{\lambda_j} |1\rangle \right)^S$$

Solution 6

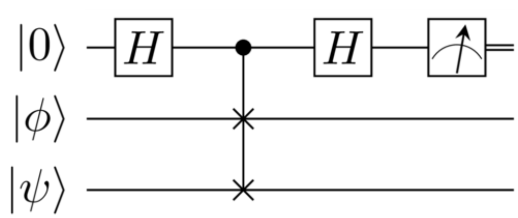

Figure 1: The SWAP test using a Fredkin's gate.[1]

Solution 6(a) The input state is

$$|0\rangle \otimes |\phi\rangle \otimes |\phi\rangle$$

After the first Hadamard gate we have,

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\phi\rangle \otimes |\phi\rangle$$

[1]Image Credit: Vtomole [CC BY-SA 4.0(https://creativecommons.org/licenses/by-sa/4.0)]

After the SWAP gate we have no change since the second and third qubits are identical.

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\phi\rangle \otimes |\phi\rangle$$

After the second Hadamard we have,

$$|0\rangle \otimes |\phi\rangle \otimes |\phi\rangle$$

Therefore we measure $|0\rangle$ with probability 1.
Solution 6(b) Since the input states are orthogonal, we have $\langle\psi|\phi\rangle = 0$.
The input state is

$$|0\rangle \otimes |\psi\rangle \otimes |\phi\rangle$$

After the first Hadamard gate we have,

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |\psi\rangle \otimes |\phi\rangle$$

After the SWAP gate we have,

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\psi\rangle \otimes |\phi\rangle + |1\rangle \otimes |\phi\rangle \otimes |\psi\rangle\right)$$

After the second Hadamard we have,

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |\psi\rangle \otimes |\phi\rangle + \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes |\phi\rangle \otimes |\psi\rangle\right]$$

$$\implies |\Phi\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle\right) \otimes |\psi\rangle \otimes |\phi\rangle + \frac{1}{2}\left(|0\rangle - |1\rangle\right) \otimes |\phi\rangle \otimes |\psi\rangle$$

Now probability of obtaining $|0\rangle$ is

$$\langle\Phi|0\rangle\langle 0| \otimes I \otimes I \otimes |\Phi\rangle$$
$$\implies \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

The intermediate steps are left as an exercise.

Solution 7 Refer [1] for such an algorithm.
Algorithm
Input: C, the initial number of steps.

1. Run eigenvalue estimation for $\hat{C} = C$ number of steps.
2. For an eigenvalue λ_i, calculate the estimate $\hat{\lambda}_i$ and the error estimate $\hat{\epsilon}$.
    - If $\hat{\epsilon} \leq O(\epsilon\hat{\lambda}_i)$, then stop.
    - Else go back to Step 1 and run for $2\hat{C}$ number of steps. This doubles the precision achieved by eigenvalue estimation.

Solution 8 As seen in the lesson, we can show that the running time is

$$O\left(\frac{1}{a^c\epsilon}\right) \times O(a\kappa) = O\left(\frac{a^{1-c}\kappa}{\epsilon}\right)$$

For this running time to give us better bounds than HHL,

$$a^{(1-c)} < \kappa$$
$$c > 1 - \frac{\log\kappa}{\log a}$$

References

[1] A. Ambainis, "Variable time amplitude amplification and a faster quantum algorithm for solving systems of linear equations," 2010.