Week 8 : Theory Solutions

Question 1

The polynomial $p(x) = (x_1 + x_2 + x_3)/3$ represents the function in question.

Question 2

The given function is $f(x) = x_1 \cdot x_2$. Now we can see that for $x = 11$ and $B_1 = 1, B_2 = 2$, $f(x) \neq f(x^B)$. So since we have 2 disjoint subset if indices such that f is sensitive on x, the block sensitivity $bs(f) = 2$.

Question 3

The output string f_s of the function f is a 2^n bit Boolean string. If there is no solution then the $f(x) = 0$ for all $x \in \{0,1\}^n$ and $f_s = 0^{\otimes n}$. If there is exactly one solution then $f(x) = 1$ for just one $x \in \{0,1\}^n$ and $f(x) = 0$ else and $wt(f_s) = 1$ where wt() is the Hamming weight. Let $N = 2^n$. Let X be the set containing only $0^{\otimes N}$ and let Y be the set containing all N-bit strings of hamming weight 1. Let us define a relation $R = X \times Y$. Then we get $m = 2^N$ and $m' = 1$. Now, for any $i \in \{0, ..., N-1\}$, we can see that there is exactly one string $y \in Y$ such that $x_i = 0 \neq y_i$ and $(x, y) \in R$. So $l = 1$. Since we have exactly one string in X, we can have at most 1 string $x \in X$ for any $y \in Y$ such that $x_i = 0 \neq y_i$ and $(x, y) \in R$. Hence $l' = 1$. So, the minimum number of queries required to decide if $wt(f_s) = 0$ or $wt(f_s) = 1$ is $\Omega\left(\sqrt{\frac{m \cdot m'}{l \cdot l'}}\right) = \Omega\left(\sqrt{2^N}\right)$.

Question 4

The approach to this question is very similar to the previous question. Let $N = 2^n$. Let X be the set containing only the string $0^{\otimes N}$ and let Y is the set containing all N-bit strings of Hamming weight k. Define the relation $R = X \times Y$. It is obvious that $m = \binom{N}{k}$ and $m' = 1$. Now, for each any $i \in \{0, ..., N-1\}$, we see that there are exactly $\binom{2^N - 1}{k-1}$ strings $y \in Y$ such that $x_i = 0 \neq y_i$ where $x = 0^{\otimes N}$ and $(x, y) \in R$. Again since we have exactly 1 string in X, we have $l' = 1$. Therefore, the minimum number of queries required to decide if $wt(f_s) = 0$ or $wt(f_s) = k$ is $\Omega\left(\sqrt{\frac{m \cdot m'}{l \cdot l'}}\right) = \Omega\left(\sqrt{2^N/k}\right)$.