Week 2 : Theory Solutions

Question 1

The quantum gate given by the matrix $\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ transforms the computational basis states into the right and left basis states.

The quantum gate given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ transforms the Hadamard basis states into the right and left basis states.

Question 2

a.

$$\begin{aligned} HXH &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z \end{aligned}$$

b.

$$\begin{aligned} HYH &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -i & i \\ i & i \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & i \end{bmatrix} = -Y \end{aligned}$$

c.

$$
\begin{aligned}
HZH &= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = X
\end{aligned}
$$

Question 3

In matrix representation, we can write the given set of gates as

$$
\begin{aligned}
(H\otimes H)CNOT_{1,2}(H\otimes H) &= \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 0 & 0 & 4\\ 0 & 0 & 4 & 0\\ 0 & 4 & 0 & 0\end{bmatrix}\\
&= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} = CNOT_{2,1}
\end{aligned}
$$

Question 4

The circuit that swaps the state of the two qubits is given by the Figure 1.

Question 5

The Toffoli gate cannot be constructed using the CNOT gates. This is because the matrix representation of Toffoli gate cannot be written as a tensor

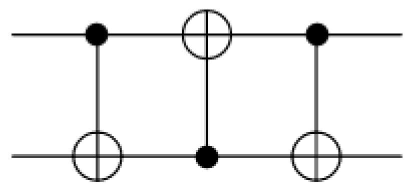

Figure 1

product of just CNOT gates. However, Toffoli gates can be decomposed into CNOT gates and Clifford gates.

Question 6

The circuit for implementing a 5-control Toffoli gate is given in Figure 2.

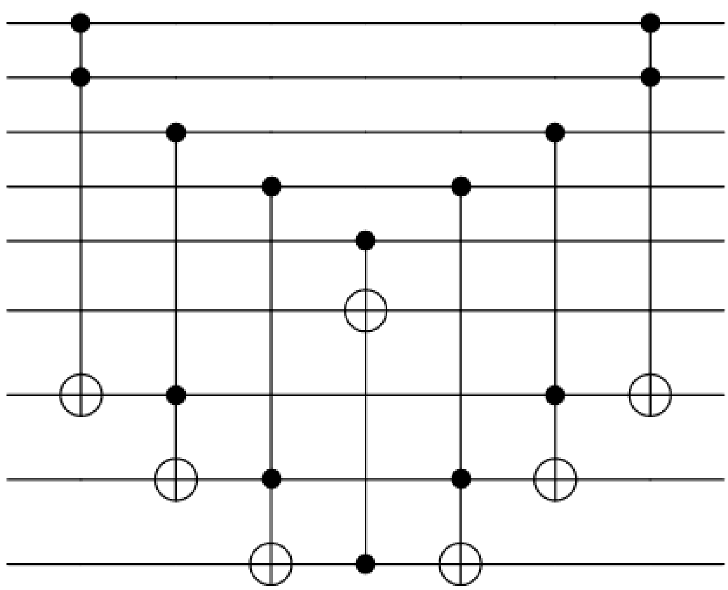

Figure 2

Question 7

The initial configuration of the system can be given as

$$|initial\rangle = |\psi\rangle \tfrac{(|00\rangle+|11\rangle)}{\sqrt{2}}.$$

The evolution of the state of the system with the implementation of each layer of gates is given as

$$\begin{aligned}
|inital\rangle &= (\alpha\,|0\rangle + \beta\,|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&\xrightarrow{CNOT_{1,2}\otimes I} \frac{1}{\sqrt{2}}(\alpha\,|0\rangle\,|00\rangle + \alpha\,|0\rangle\,|11\rangle + \beta\,|1\rangle\,|10\rangle + \beta\,|1\rangle\,|01\rangle) \\
&\xrightarrow{I\otimes H\otimes I} \frac{1}{2}(\alpha(|0\rangle + |1\rangle)\,|00\rangle + \alpha(|0\rangle + |1\rangle)\,|11\rangle + \beta(|0\rangle - |1\rangle)\,|10\rangle + \beta(|0\rangle - |1\rangle)\,|01\rangle) \\
&= \frac{1}{2}\Big(\,|00\rangle\,\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big) + |01\rangle\,\big(\beta\,|0\rangle + \alpha\,|1\rangle\,\big) + |10\rangle\,\big(\alpha\,|0\rangle - \beta\,|1\rangle\,\big) + |11\rangle\,\big(\beta\,|0\rangle - \alpha\,|1\rangle\,\big)\Big)
\end{aligned}$$