Week 4: Fourier Based Methods

But what is a Fourier Transform?

- Classically, the Fourier transform (FT) decomposes a signal into its constituent frequencies.

- The Fourier transform is not limited to functions of time. However it is common convention to refer to the domain of the original function as the time domain.

- For example, in quantum mechanics the Fourier Transform allows us to transform between the momentum and position representations of a wave function.

Why do we need a Fourier Transform?

- This question can easily be generalized to: Why do we need transforms in general? The answer is because they make our life easier.

- The original motivation behind using Fourier Transforms is that linear operations performed in the original domain have corresponding operations in the Fourier domain, which are sometimes easier to perform.

- An example in the classical sense is as follows: If you're a sports fan, you might be horrified at the unruly audience giving away strategies to the teams on stage. However the players on stage have specialized headphones that cut off certain frequencies which make these rants inaudible to them.

What is the Quantum Fourier Transform? (Part 1)

- The Quantum Fourier Transform is the quantum analogue of the classical Discrete Fourier Transform.
- We carry out a DFT on the amplitudes of a quantum state.

$$|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i j k}{N}} |k\rangle$$

This is the effect of QFT on a basis state $|j\rangle$

What is the Quantum Fourier Transform? (Part 2)

Applying QFT on a random state would give us

$$\sum_j \alpha_j |j\rangle \to \sum_k \tilde{\alpha}_k |k\rangle$$

Here α_j are the amplitudes of the original basis states, while $\tilde{\alpha_k}$ are the DFT of the original amplitudes.

What is the Quantum Fourier Transform? (Part 3)

Since Quantum operations are unitary, we rewrite the QFT as a unitary operator in the form below.

$$\hat{F} = \frac{1}{\sqrt{N}} \sum_{j,k=0}^{N-1} e^{2\pi ijk/N} |k\rangle\langle j|$$

It can be easily verified that $\hat{F}^\dagger \hat{F} = \hat{I}$ and this does indeed carry out the transformation $|\tilde{\psi}\rangle = \hat{F}\,|\psi\rangle$.

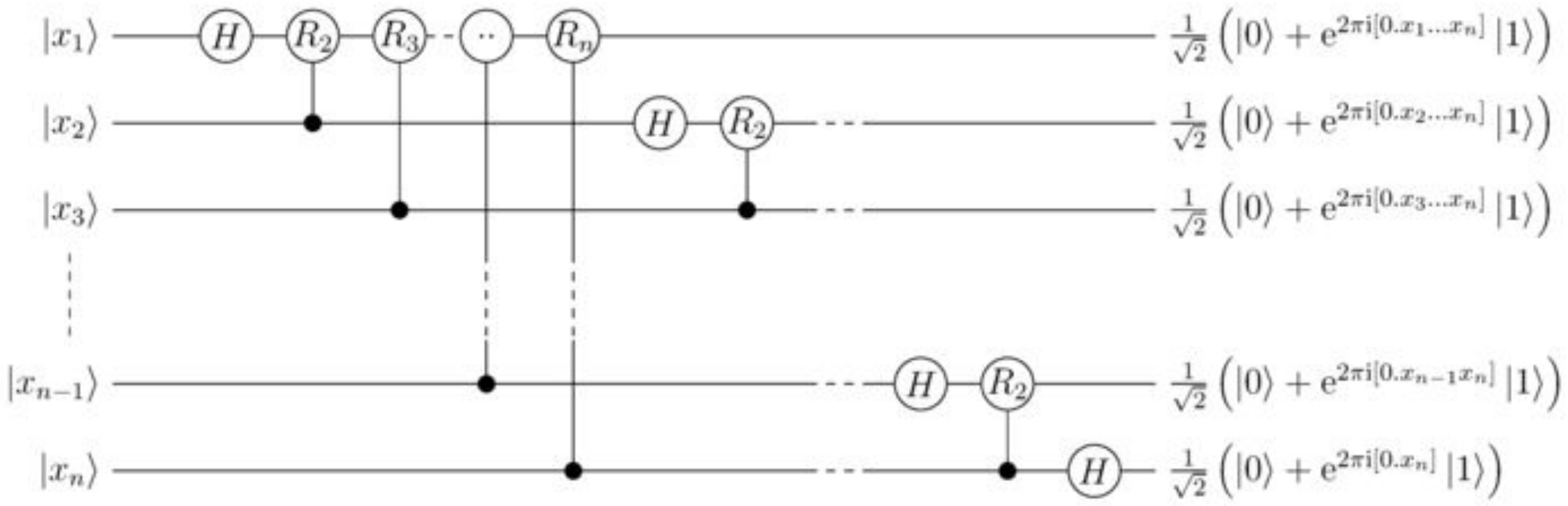
What is the Quantum Fourier Transform? (Part 4)

The QFT can be written in product representation form as follows:

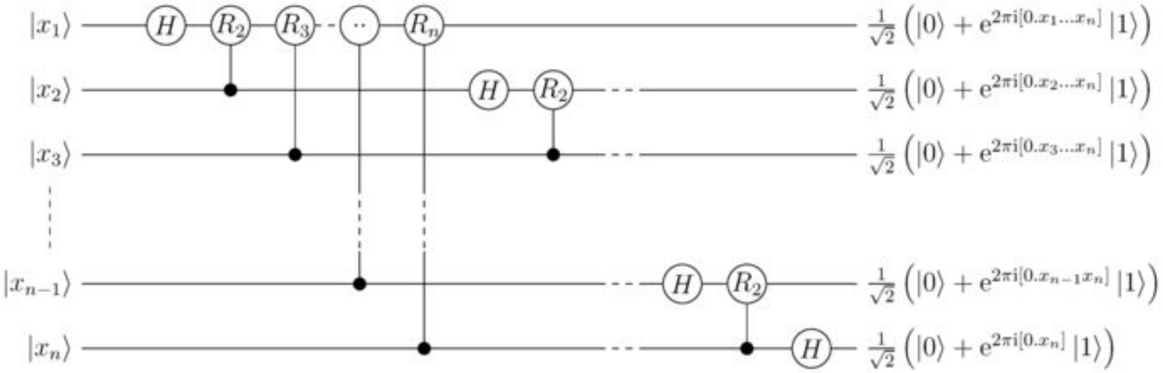
$$|j_1 \ldots j_n\rangle \rightarrow \frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle)\ldots(|0\rangle + e^{2\pi i 0.j_1 \ldots j_n}|1\rangle)}{2^{n/2}}$$

Where $|j\rangle = |j_1 j_2 \ldots j_n\rangle$.

Behold the circuit for the Quantum Fourier Transform



Behold the circuit for the Quantum Fourier Transform



$$R_k \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$$

Note that only the Hadamard gate and controlled rotation gates (shown above) are used.

Analysis of the QFT circuit

- For the first qubit we use one Hadamard gate and $n-1$ controlled rotations. Thus total n gates are used for the first qubit.

- For the second qubit we apply total $n-1$ gates (one Hadamard and $n-2$ conditional rotations).

- This analysis can be extended to the rest of the qubits, giving us a total number = $n + (n - 1) + \cdot\cdot\cdot + 1 = n(n+1)/2$

- At the end we have atmost n/2 SWAP gates, each of which is realized using 3 CNOT gates. Therefore total number of gates is O(n^2).

What is Quantum Phase Estimation?

The Quantum Phase Estimation Algorithm is an application of the Quantum Fourier Transform.

Let us have a unitary operator U with eigenvector **u** and eigenvalue λ.

Let $\lambda = \exp(-2\pi i\phi)$, where ϕ is unknown, then the goal of phase estimation is to estimate the phase ϕ.

Since λ is normalized, its modulus is 1. Then ϕ can be used to describe the eigenvalue λ completely. This is why QPE is also known as Quantum Eigen-estimation Algorithm.

The algorithm of QPE (Part 1)

- Two registers are used in the procedure, first contains t qubits, all in the state **<u>0</u>**, and the second register begins with the state **<u>u</u>** with as many qubits as necessary to store it.
    - The value of t for the first register depends on the number of digits of accuracy we want and the probability with which we want the procedure to succeed.
- Now we carry out Phase Estimation in three stages.
- First we apply the Hadamard transform to the first register. This is done by tensoring t Hadamard gates together.

(continued in the next slide)

The algorithm of QPE (Part 2)

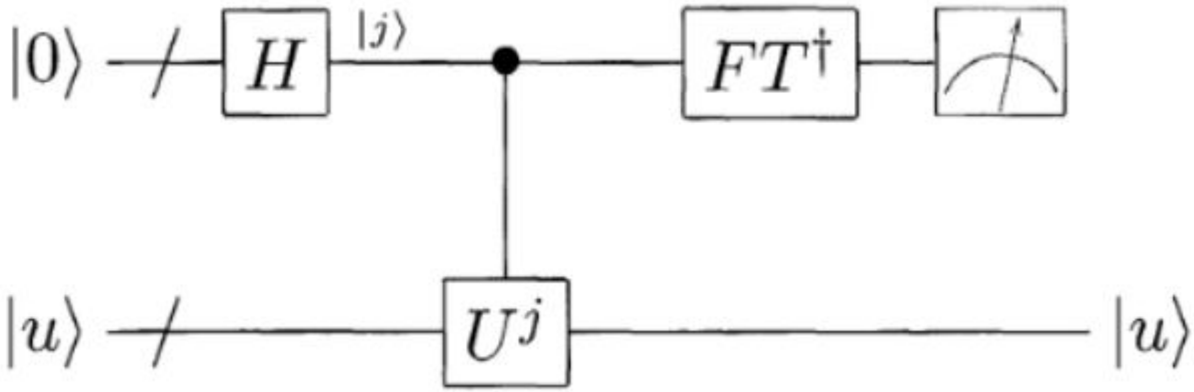
- Then we apply controlled rotation operations on the second register, where U is raised to powers of 2. The state of the first register is now

$$\frac{1}{2^{t/2}}\left(|0\rangle + e^{2\pi i 2^{t-1}\varphi}|1\rangle\right)\left(|0\rangle + e^{2\pi i 2^{t-2}\varphi}\right)\ldots\left(|0\rangle + e^{2\pi i 2^{0}\varphi}|1\rangle\right) = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{2\pi i\varphi k}|k\rangle$$

- Now we apply Inverse QFT on the first register. This is implemented by reversing the circuit for QFT. This can be done in Θ(t^2) steps.

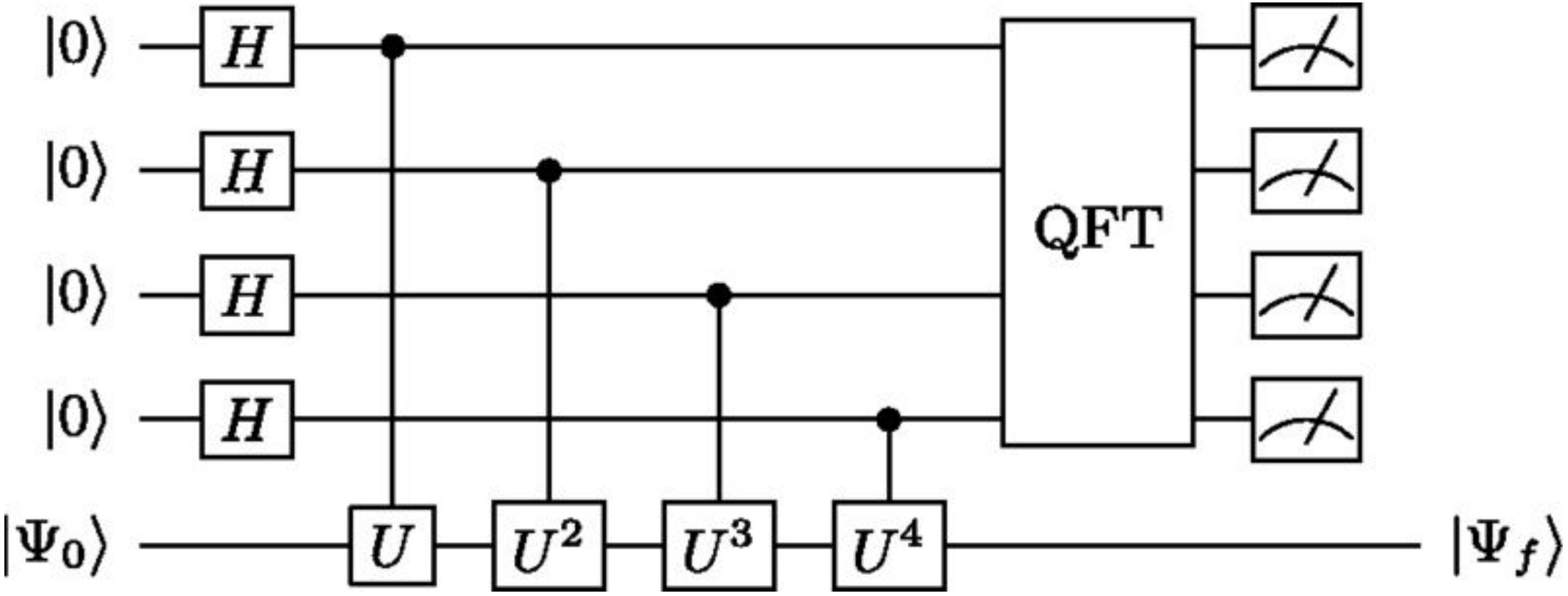
- Finally we measure the state of the first register in the computational basis, which gives us an estimate of the phase.

The circuit for QPE



A high level view of the QPE circuit

The circuit for QPE



Read Handouts for detailed description

References given in Handouts