Week 1 – Solutions

1. Compute conjugate transpose and verify.
2. Say $\rho = \sum_i p_i \langle \psi_i | \psi_i \rangle$ is a density matrix. Then,

$$\mathrm{tr}(\rho) = \sum_i p_i \, \mathrm{tr}(\langle \psi_i | \psi_i \rangle) = \sum_i p_i = 1.$$

   Say $|\varphi\rangle$ is any vector. Then,

$$\langle \varphi | \rho | \varphi \rangle = \sum_i p_i \langle \varphi | \psi_i \rangle \langle \psi_i | \varphi \rangle = p_i | \langle \varphi | \psi_i \rangle |^2 \geq 0.$$

3. Compute the conjugate transpose of each of the matrices to verify if they are unitary. Compute the value $AB - BA$ for each A, B in the Pauli X, Y and Z matrices to verify if they commute.
4. Compare $|+\rangle$ and $|-\rangle$ with $\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$ and find values of θ and φ.
5. Say we have to find relative phase of $|\psi\rangle$. Measure it using basis vectors $|+\rangle$ and $|-\rangle$ instead of the computational basis, $|0\rangle$ and $|1\rangle$. We get equations of the form,

$$\begin{aligned}\psi &= \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle, \\ &= \frac{1}{2}(|+\rangle + |-\rangle) + \frac{e^{i\theta}}{2}(|+\rangle + |-\rangle), \\ &= e^{i\frac{\theta}{2}}\big(\cos\frac{\theta}{2}|+\rangle - i\sin\frac{\theta}{2}|-\rangle\big).\end{aligned}$$

   Here, we can find the value of $\cos\frac{\theta}{2}$, since it is the probability with which we get $|+\rangle$. We can use this to find the relative phase.