Week 4 : Theory Solutions

Solution 1 We know that the Hadamard Gate is the QFT for $N = 2$. Accordingly, the second root of unity is $\omega = exp(\frac{2i\pi}{2}) = e^{i\pi} = -1$. For $N = 2$, we have two basis vectors $|0\rangle$ and $|1\rangle$. Therefore for $|0\rangle$ we have

$$
\begin{aligned}
|0\rangle \xrightarrow{QFT} & \frac{1}{\sqrt{2}} \sum_{y=0}^{1} \omega_2^{-0y} |y\rangle \\
&= \frac{1}{\sqrt{2}} \sum_{y=0}^{1} |y\rangle \\
&= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)
\end{aligned}
$$

and for $|1\rangle$ we have

$$
\begin{aligned}
|1\rangle \xrightarrow{QFT} & \frac{1}{\sqrt{2}} \sum_{y=0}^{1} \omega_2^{-1y} |y\rangle \\
&= \frac{1}{\sqrt{2}} (-1^{-1\times 0} |0\rangle) + \frac{1}{\sqrt{2}} (-1^{-1\times 1} |1\rangle) \\
&= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)
\end{aligned}
$$

Hence the required Fourier basis states are $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. These are sometimes known as the $|+\rangle$ and $|-\rangle$ states.

Solution 2 We have the two basic equations $A|\psi\rangle = \lambda|\psi\rangle$ and $AA^* = I$. Now,

$$
\begin{aligned}
A|\psi\rangle &= \lambda|\psi\rangle \\
|\psi\rangle^* A^* &= \lambda^* |\psi\rangle^* \\
\implies |\psi\rangle^* A^* A |\psi\rangle &= \lambda^* |\psi\rangle^* \lambda |\psi\rangle \\
\implies |\psi\rangle^* I |\psi\rangle &= \lambda^* \lambda |\psi\rangle^* |\psi\rangle \\
\implies |\psi\rangle^* |\psi\rangle &= \lambda^* \lambda |\psi\rangle^* |\psi\rangle \\
\implies ||\psi\rangle|^2 &= |\lambda|^2 ||\psi\rangle|^2 \\
\implies |\lambda|^2 &= 1 \\
\implies |\lambda| &= 1
\end{aligned}
$$

Solution 3 We know that the MS qubit has the form

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.y_n}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i y_n/2}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\pi i y_n}|1\rangle\right)$$

Since $e^{\pi i} = -1$, depending on the value of y_n (bit can be 0 or 1) we have the phase of $|1\rangle$ as $+1$ or -1. Hence we get a form of $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. This is nothing but a Hadamard transformation.

Solution 4

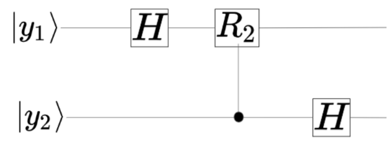

Figure 1: QFT circuit for 2 qubits

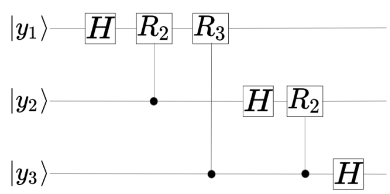

Figure 2: QFT circuit for 3 qubits

Solution 5 When $2^n\phi \neq a$, $\beta \neq 0$.

$$\Pr[a] = \frac{1}{N^2}\left|\sum_{k=0}^{N-1} e^{(2\pi i k\beta)}\right|^2$$

Now, using geometric series sum we get

$$= \frac{1}{N^2}\left|\frac{1 - e^{(2\pi i N\beta)}}{1 - e^{(2\pi i\beta)}}\right|^2$$

Now we use the observation that $\left|1 - e^{2ix}\right|^2 = 4\left|\sin x\right|^2$, and substitute this in our formula to get

$$
\begin{aligned}
&= \frac{1}{N^2}\left|\frac{\sin \pi N\beta}{\sin \pi\beta}\right|^2 \\
&\geq \frac{1}{N^2}\left|\frac{\sin \pi N\beta}{\pi\beta}\right|^2 \\
&\geq \frac{1}{N^2}\left|\frac{2N\beta}{\pi\beta}\right|^2 \\
&\geq \frac{4}{\pi^2}
\end{aligned}
$$

Second step is because $|\sin x| \leq |x|, \quad \forall |x| \leq \frac{1}{2\pi N}$. Third step is because $|\sin \pi N\beta| \geq |2N\beta|\,, \ \forall \quad |\beta| \leq \frac{1}{2N}$.

Solution 6

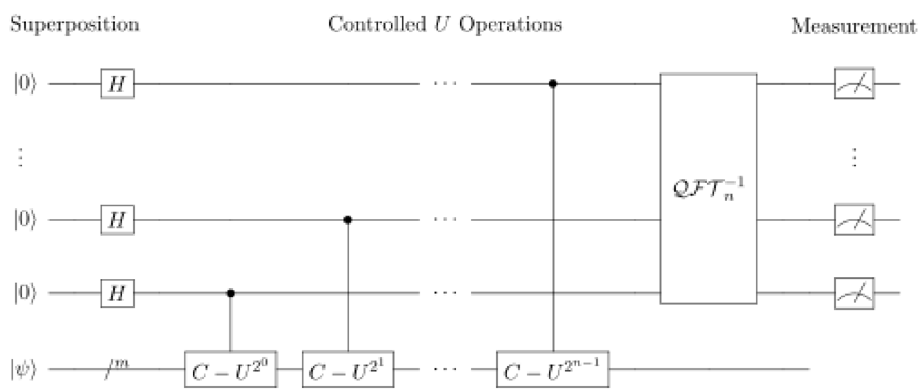

Figure 3: Circuit for Quantum Phase Estimation [1]

[1]Omrika [CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0)]