Unit V

Chapter 9

Tests of Hypotheses for a Single Sample

- Hypotheses testing
- Tests on the mean of a normal distribution- *variance known*
- Tests on the mean of a normal distribution- *variance unknown*
- Tests on the variance and standard deviation of a normal distribution
- Tests on a population proportion
- Testing for goodness of fit

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Statistical inference may be divided into two major areas:

- Parameter estimation
- Hypothesis testing

Statistical hypothesis testing and confidence interval estimation of parameters are the fundamental methods used at the data analysis stage of a **comparative experiment,** in which the engineer is interested, for example, in comparing the mean of a population to a specified value.

A **statistical hypothesis** is a statement about the parameters of one or more populations.

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

For example, suppose that we are interested in the burning rate of a solid propellant used to power aircrew escape systems.

- Now burning rate is a random variable that can be described by a probability distribution.
- Suppose that our interest focuses on the **mean** burning rate (a parameter of this distribution).
- Specifically, we are interested in deciding **whether or not the mean burning rate is 50 centimeters per second**.

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Two-sided Alternative Hypothesis

H_0: $\mu = 50$ centimeters per second

H_1: $\mu \neq 50$ centimeters per second

null hypothesis

alternative hypothesis

One-sided Alternative Hypotheses

H_0: $\mu = 50$ centimeters per second

H_1: $\mu < 50$ centimeters per second

or

H_0: $\mu = 50$ centimeters per second

H_1: $\mu > 50$ centimeters per second

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Test of a Hypothesis :

- A procedure leading to a decision about the null hypothesis

•Hypotheses are statements about population or distribution under study.

•A procedure leading to a decision about a particular hypothesis

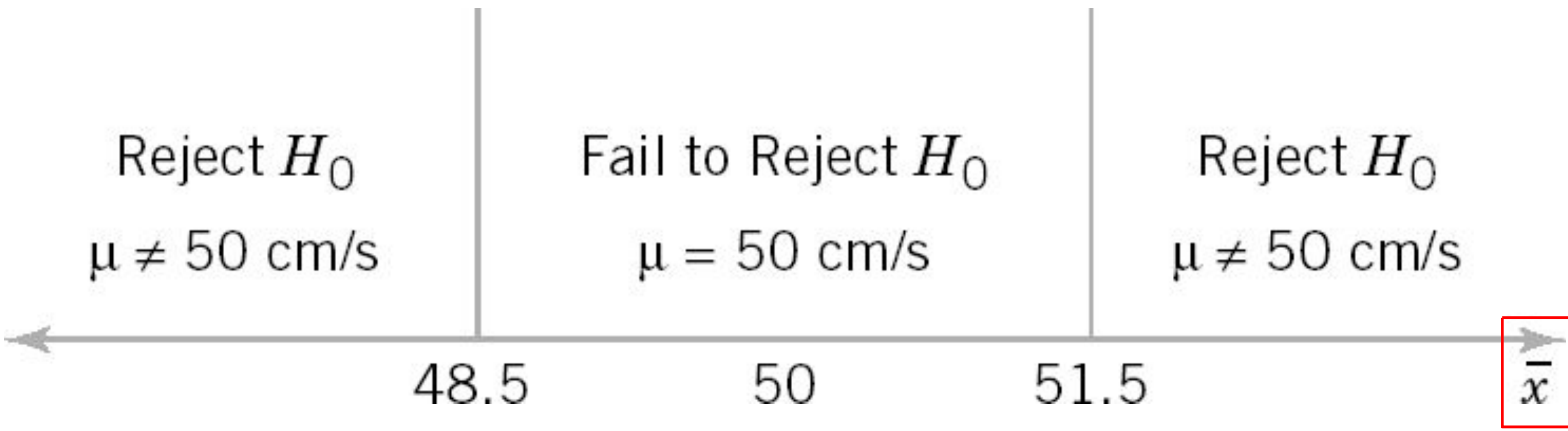
•Hypothesis-testing procedures rely on using the information in a **random sample from the population of interest**.

•If this information is *consistent* with the hypothesis, then we will conclude that the hypothesis is **true (fail to reject hypothesis)**; if this information is *inconsistent* with the hypothesis, we will conclude that the hypothesis is **false (reject hypothesis)**.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

$$H_0\colon \mu = 50 \text{ centimeters per second}$$
$$H_1\colon \mu \neq 50 \text{ centimeters per second}$$



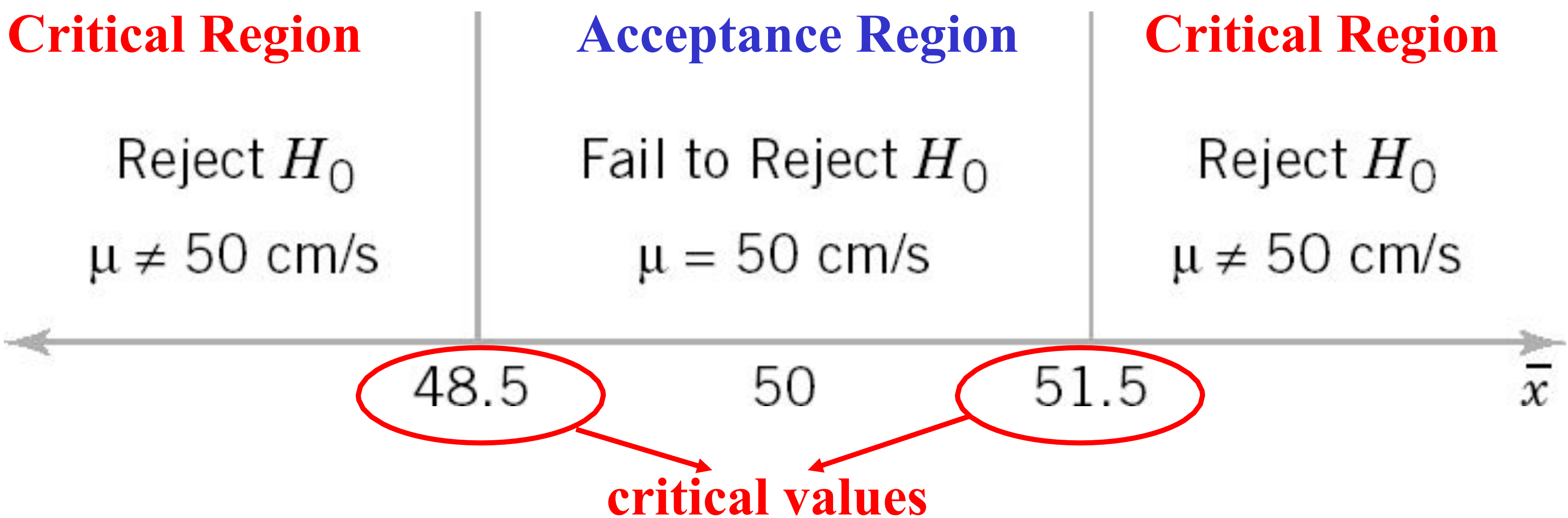
Decision criteria for testing H_0:μ = 50 cm/s versus H_1:$\mu \neq$ 50 cm/s.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

$$H_0: \mu = 50 \text{ centimeters per second}$$

$$H_1: \mu \neq 50 \text{ centimeters per second}$$



Decision criteria for testing H_0:μ = 50 cm/s versus H_1:$\mu \neq 50$ cm/s$_{8}$.

Example

n=10, μ = 50 centimeters per second

Suppose that if $48.5 \le \ddot{x} \le 51.5$, we will not reject the null hypothesis H0: $\mu = 50$, and

if either $\ddot{x} < 48\ 5.$ or $\ddot{x} > 51\ 5.$, we will reject the null hypothesis in favor of the alternative hypothesis H1: $\mu \neq 50$.

- The values of ẍ that are less than 48.5 and greater than 51.5 constitute the critical region for the test;
- All values that are in the interval $48.5 \le \ddot{x} \le 51.5$ form a region for which we will fail to reject the null hypothesis. □ By convention, this is usually called the acceptance region.
- The boundaries between the critical regions and the acceptance region are called the critical values. In our example, the critical values are 48.5 and 51.5.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Two wrong conclusions are possible:

Type I Error

Rejecting the null hypothesis H_0 when it is true is defined as a **type I error.**

Type II Error

Failing to reject the null hypothesis when it is false is defined as a **type II error.**

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Decision	H_0 Is True	H_0 Is False
Fail to reject H_0	no error	type II error
Reject H_0	type I error	no error

The probability of making a type I error is denoted by the Greek letter α.

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

Sometimes the type I error probability is called the **significance level**, or the **α-error**, or the **size** of the test.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Finding the probability of making a type I (significance level of the test)

Ex: $\sigma = 2.5$ $n=10$

Standard deviation of the sample mean:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{2.5}{\sqrt{10}} = 0.79$$

$$\alpha = P(\bar{X} < 48.5 \text{ when } \mu = 50) + P(\bar{X} > 51.5 \text{ when } \mu = 50)$$

The z-values that correspond to the critical values 48.5 and 51.5 are

$\mu_{\bar{x}}$

$$z_1 = \frac{48.5 - 50}{0.79} = -1.90 \quad \text{and} \quad z_2 = \frac{51.5 - 50}{0.79} = 1.90$$

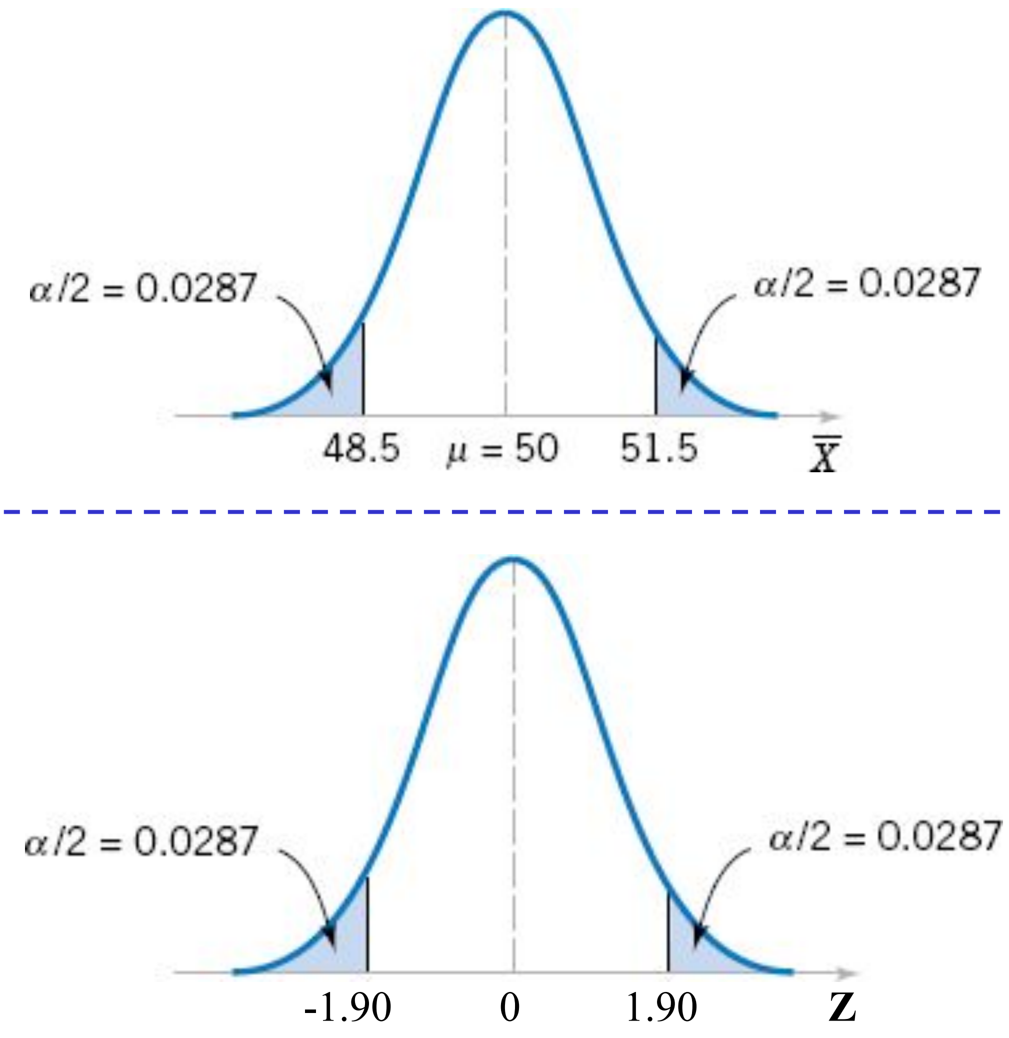
$\sigma_{\bar{X}}$

Therefore

$$\alpha = P(Z < -1.90) + P(Z > 1.90) = 0.028717 + 0.028717 = 0.057434$$

9-1 Hypothesis Testing

Critical Region for $H_0: \mu = 50 \quad H_1: \mu \neq 50$



Probability distribution of $\overline{X}$

Corresponding probability distribution of Z

9-1 Hypothesis Testing

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \quad (9\text{-}3)$$

1. How can we reduce α ?

-by widening acceptance region

Type I error probability does not depend on the true mean

(if take critical values 48 and 52, α =0.0114, Verify !)

For example, if we make the critical values 48 and 52, the value of α is

$$\alpha = P\left(z < -\frac{48-50}{0.79}\right) + P\left(z > \frac{52-50}{0.79}\right) = P(z < -2.53) + P(z > 2.53)$$

$$= 0.0057 + 0.0057 = 0.0114$$

Type I error calculation only involves the assumption that H0 is true (μ=50) and is independent of other true means like 50.5.

- by increasing sample size

We could also reduce α by increasing the sample size. If $n = 16$, $\sigma/\sqrt{n} = 2.5/\sqrt{16} = 0.625$

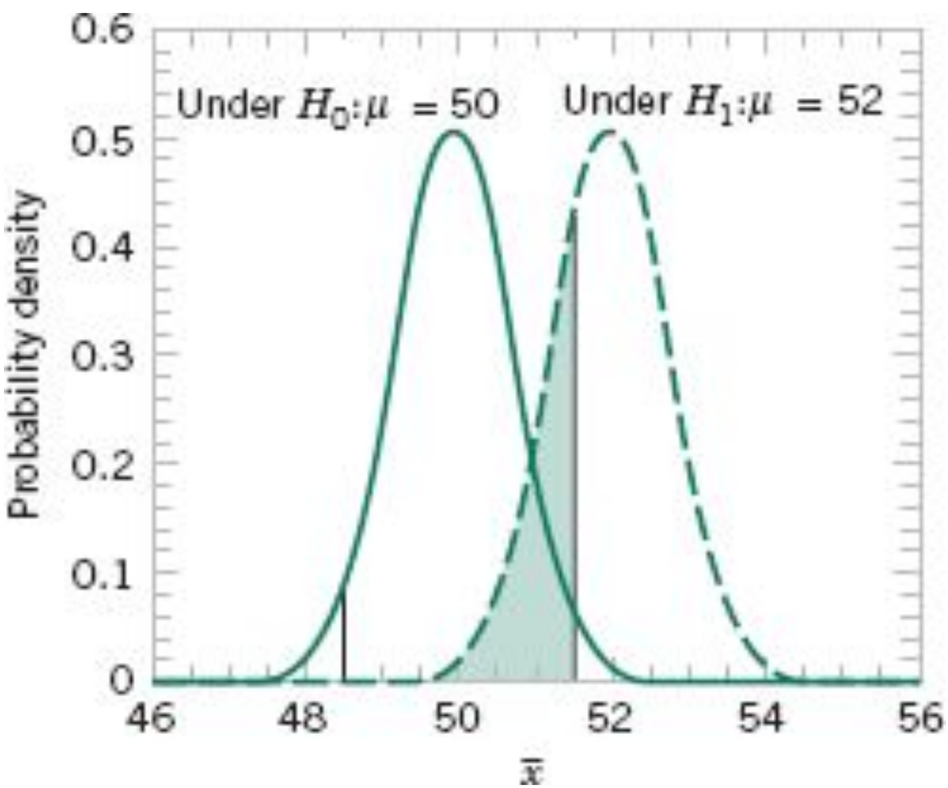
$$z_1 = \frac{48.5-50}{0.625} = -2.40 \quad \text{and} \quad z_2 = \frac{51.5-50}{0.625} = 2.40$$

Therefore,

$$\alpha = P(Z < -2.40) + P(Z > 2.40) = 0.0082 + 0.0082 = 0.0164$$

9-1 Hypothesis Testing

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}) \qquad (9\text{-}4)$$



The probability of type II error when <u>μ = 52</u> and n = 10.

9-1 Hypothesis Testing

Example to find Type II error :

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 52)$$

The z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

$$z_1 = \frac{48.5 - 52}{0.79} = -4.43 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.79} = -0.63$$

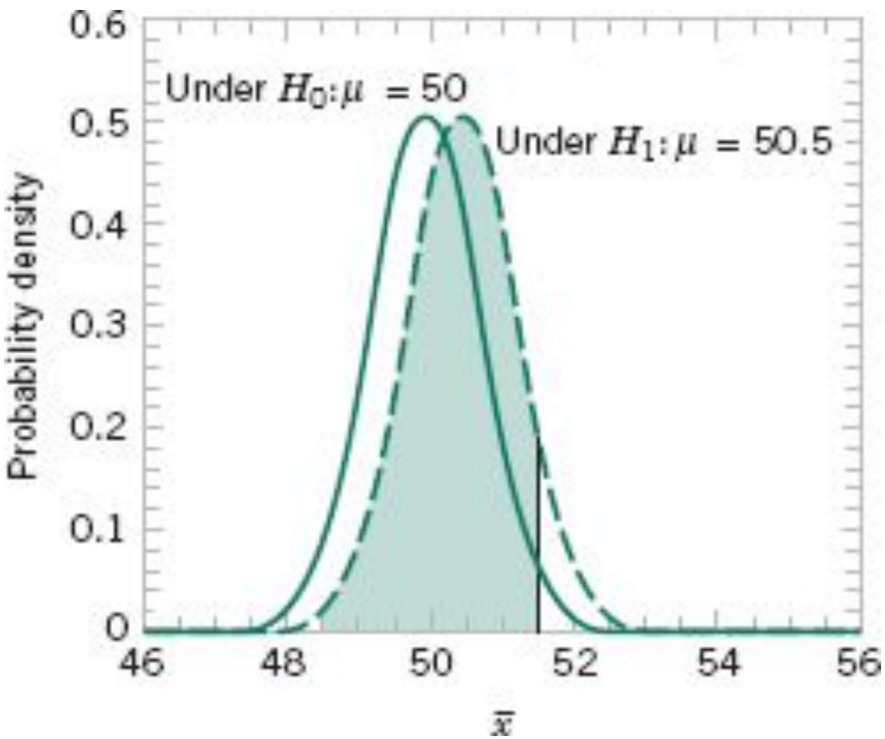
Therefore

$$\begin{aligned}\beta &= P(-4.43 \le Z \le -0.63) = P(Z \le -0.63) - P(Z \le -4.43)\\ &= 0.2643 - 0.0000 = 0.2643\end{aligned}$$

Thus, if we are testing H0 : μ = 50 against H1 : μ ≠ 50 with n = 10 and the true value of the mean is μ = 52, the probability that we will fail to reject the false null hypothesis is 0.2643.

9-1 Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 50.5)$$



The probability of type II error when $\mu = 50.5$ and $n = 10$.

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 50.5)$$

As shown in Fig. 9-4, the z-values corresponding to 48.5 and 51.5 when $\mu = 50.5$ are

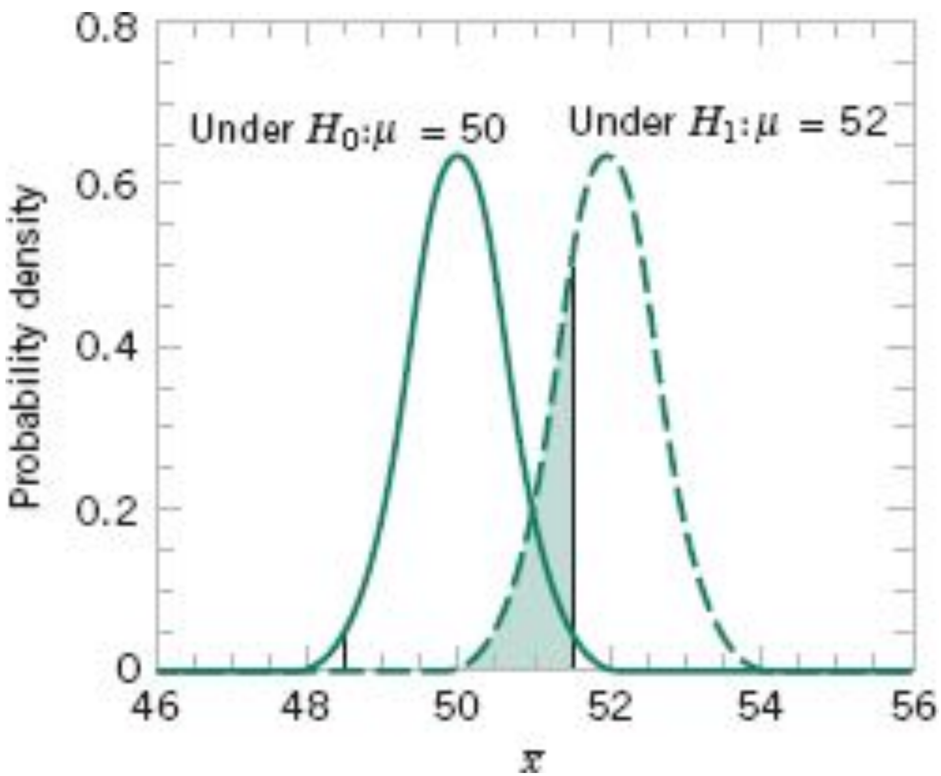
$$z_1 = \frac{48.5 - 50.5}{0.79} = -2.53 \qquad \text{and} \qquad z_2 = \frac{51.5 - 50.5}{0.79} = 1.27$$

Therefore

$$\begin{aligned}\beta &= P(-2.53 \le Z \le 1.27) = P(Z \le 1.27) - P(Z \le -2.53) \\ &= 0.8980 - 0.0057 = 0.8923\end{aligned}$$

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 52)$$



The probability of type II error when μ = 52 and n = 16.

9-1 Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 52)$$

When $n = 16$, the standard deviation of $\overline{X}$ is $\sigma/\sqrt{n} = 2.5/\sqrt{16} = 0.625$, and the z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

$$z_1 = \frac{48.5 - 52}{0.625} = -5.60 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.625} = -0.80$$

Therefore

$$\begin{aligned}\beta &= P(-5.60 \leq Z \leq -0.80) = P(Z \leq -0.80) - P(Z \leq -5.60) \\ &= 0.2119 - 0.0000 = 0.2119\end{aligned}$$

9-1 Hypothesis Testing

- Generally α is controlable when critical values are selected.
- Thus, rejection of null hypothesis H_0 is a strong conclusion.
- β is not constant but depends on the true value of the parameter and sample size.
- Accepting H_0 is a weak conclusion unless β is acceptably small.
- Prefer the terminology "**fail to reject H_0**" rather than "accept H_0"
- Fail to reject H_0
  - implies we have not found sufficient evidence to reject H_0.
  - does not necessarily mean there is a high probability that H_0 is true.
  - means more data are required to reach a strong conclusion.

9-1 Hypothesis Testing

Power

The **power** of a statistical test is the probability of rejecting the null hypothesis H_0 when the alternative hypothesis is true.

- The power is computed as 1 - β, and power can be interpreted as *the probability of correctly rejecting a false null hypothesis.* We often compare statistical tests by comparing their power properties.

- For example, consider the propellant burning rate problem when we are testing H_0 : μ = 50 cm/s against H_1 : μ not equal 50 cm/s . Suppose that the true value of the mean is μ = 52. When n = 10, we found that β = 0.2643, so the power of this test is 1 - β = 1 - 0.2643 = 0.7357 when μ = 52.

9-1 Hypothesis Testing

9-1.3 One-Sided and Two-Sided Hypotheses

Two-Sided Test:

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

One-Sided Tests:

$$H_0: \mu = \mu_0 \qquad\qquad H_0: \mu = \mu_0$$
$$\text{or}$$
$$H_1: \mu > \mu_0 \qquad\qquad H_1: \mu < \mu_0$$

9-1 Hypothesis Testing

Example 9-1

- Suppose if the propellant burning rate is less than 50 cm/s
- Want to show this with a strong conclusion **CLAIM**
- Hypotheses should be stated as

  H_0: $\mu = 50$ cm/s

  H_1: $\mu < 50$ cm/s
- Since the rejection of H_0 is always a strong conclusion, this statement of the hypotheses will produce the desired outcome if H_0 is rejected.
- Although H_0 is stated with an equal sign, it is understood to include any value of μ not specified by H_1.
- Failing to reject H_0 does not mean $\mu = 50$ cm/s exactly.
- Failing to reject H_0 means we do not have strong evidence in support of H_1.

9-1 Hypothesis Testing

The bottler wants to be sure that the bottles meet the specification on mean internal pressure or bursting strength, which for 10-ounce bottles is a minimum strength of 200 psi.

The bottler has decided to formulate the decision procedure for a specific lot of bottles as a hypothesis testing problem.

There are two possible formulations for this problem: either

$$H_0: \mu = 200 \text{ psi} \qquad \qquad H_0: \mu = 200 \text{ psi}$$
$$H_1: \mu > 200 \text{ psi} \qquad \text{or} \qquad H_1: \mu < 200 \text{ psi}$$

Which is correct? Depends on the objective of the analysis.

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

•When H_0 is rejected at a specified α level, this gives no idea about whether the computed value of the test statistic

- is just barely in the rejection region
- or it is very far into this region.

•Thus, P-value has been adopted widely in practice

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

Consider the two-sided hypothesis test for burning rate:

$H_0 : \mu = 50$ cm/s

$H_1 : \mu \neq 50$ cm/s

$n=16$, $\sigma=2.5$, Sample mean = 51.3

P-value?

Hint : to calculate p-value
Two sided : 2(1-p(Z))
One sided (upper) : 1- p(Z)
One sided (lower)=p(Z)

$$
\begin{aligned}
P\text{-value} &= 1 - P(48.7 < \bar{X} < 51.3) \\
&= 1 - P\left(\frac{48.7 - 50}{2.5/\sqrt{16}} < Z < \frac{51.3 - 50}{2.5/\sqrt{16}}\right) \\
&= 1 - P(-2.08 < Z < 2.08) \\
&= 1 - 0.962 = 0.038
\end{aligned}
$$

OR

Using the standard normal distribution table, find P(Z>2.08):
P(Z>2.08) = (1-0.9812) = 0.0188
P-value=2(0.0188)=0.038

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

Consider the one sided hypothesis test for burning rate: $H_0 : \mu = 50$ cm/s

$H_1 : \mu > 50$ cm/s

$n=16$, $\sigma=2.5$, Sample mean = 51.3

P-value? Since we know the population standard deviation, we can use the z-test for the hypothesis test. The test statistic z is given by:

Hint : to calculate p-value
Two sided : 2(1-p(Z))
One sided (upper) : 1- p(Z)
One sided (lower)=p(Z)

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

$$z = \frac{51.3 - 50}{2.5/4} = \frac{1.3}{0.625} = 2.08$$

Using standard normal tables or a calculator, we find the P-value corresponding to $z = 2.08$.

$$P(Z \geq 2.08) = 1 - P(Z \leq 2.08) \approx 1 - 0.9812 = 0.0188$$

The P-value is approximately **0.0188**.

9-1 Hypothesis Testing

9-1.5 Connection between Hypothesis Tests and Confidence Intervals

Close relation between **hypothesis tests** and **confidence intervals**

$H_0 : \mu = 50$ cm/s n=16, σ=2.5, α=0.05, $\bar{x} = 51.3$
$H_1 : \mu \neq 50$ cm/s

Critical z values are $z_{\alpha/2}=z_{0.025}=1.96$ and $-z_{0.025}=-1.96$ which corresponds to

Critical values $50 \pm 1.96 \dfrac{2.5}{\sqrt{16}} = [48.775 ; 51.225]$

$\bar{x} = 51.3$ is not in the acceptance region [48.775 ; 51.225]. So reject null hypothesis.

Confidence interval for μ at α=0.05 is $51.3 \pm 1.96 \dfrac{2.5}{\sqrt{16}}$

same conclusion !

That is $50.075 \leq \mu \leq 52.525$

μ=50 is not in the confidence interval [50.075 ; 52.525]. So reject null hypothesis.

9-1 Hypothesis Testing

9-1.5 Connection between Hypothesis Tests and Confidence Intervals

There is a close relationship between the test of a hypothesis about any parameter, say θ, and the confidence interval for θ. If $[l, u]$ is a $100(1-\alpha)\%$ confidence interval for the parameter θ, the test of size α of the hypothesis

$$H_0: \theta = \theta_0$$
$$H_1: \theta \neq \theta_0$$

will lead to rejection of H_0 if and only if θ_0 is **not** in the $100(1-\alpha)\%$ CI $[l, u]$. As an illustration, consider the escape system propellant problem with $\bar{x} = 51.3$, $\sigma = 2.5$, and $n = 16$. The null hypothesis H_0: $\mu = 50$ was rejected, using $\alpha = 0.05$. The 95% two-sided CI on μ can be calculated using Equation 8-7. This CI is $51.3 \pm 1.96(2.5/\sqrt{16})$ and this is $50.075 \leq \mu \leq 52.525$. Because the value $\mu_0 = 50$ is not included in this interval, the null hypothesis H_0: $\mu = 50$ is rejected.

9-1 Hypothesis Testing

9-1.6 General Procedure for Hypothesis Tests

1. From the problem context, identify the parameter of interest.
2. State the null hypothesis, H_0 .
3. Specify an appropriate alternative hypothesis, H_1 .
4. Choose a significance level, α.
5. Determine an appropriate test statistic.
6. State the rejection region for the statistic (based on test statistic or p-value).
7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
8. Decide whether or not H_0 should be rejected and report that in the problem context.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

We wish to test:

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

The **test statistic** is:

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \tag{9-8}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

Reject H_0 if the observed value of the test statistic z_0 is either:

$$z_0 > z_{\alpha/2} \text{ or } z_0 < -z_{\alpha/2}$$

Fail to reject H_0 if

$$-z_{\alpha/2} < z_0 < z_{\alpha/2}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

We may also develop procedures for testing hypotheses on the mean μ where the alternative hypothesis is one-sided. Suppose that we specify the hypotheses as

$$H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0 \tag{9-11}$$

In defining the critical region for this test, we observe that a negative value of the test statistic Z_0 would never lead us to conclude that H_0: $\mu = \mu_0$ is false. Therefore, we would place the critical region in the **upper tail** of the standard normal distribution and reject H_0 if the computed value of z_0 is too large. That is, we would reject H_0 if

$$z_0 > z_\alpha \tag{9-12}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean (Continued)

as shown in Figure 9-7(b). Similarly, to test

$$H_0: \mu = \mu_0$$
$$H_1: \mu < \mu_0 \qquad (9\text{-}13)$$

we would calculate the test statistic Z_0 and reject H_0 if the value of z_0 is too small. That is, the critical region is in the **lower tail** of the standard normal distribution as shown in Figure 9-7(c), and we reject H_0 if

$$z_0 < -z_\alpha \qquad (9\text{-}14)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

Aircrew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specifications require that the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is $\sigma = 2$ centimeters per second. The experimenter decides to specify a type I error probability or significance level of $\alpha = 0.05$ and selects a random sample of $n = 25$ and obtains a sample average burning rate of $\bar{x} = 51.3$ centimeters per second. What conclusions should be drawn?

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

We may solve this problem by following the eight-step procedure outlined in Section 9-1.4. This results in

1. The parameter of interest is μ, the mean burning rate.
2. H_0: $\mu = 50$ centimeters per second
3. H_1: $\mu \neq 50$ centimeters per second
4. $\alpha = 0.05$
5. The test statistic is

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

6. Reject H_0 if $z_0 > 1.96$ or if $z_0 < -1.96$. Note that this results from step 4, where we specified $\alpha = 0.05$, and so the boundaries of the critical region are at $z_{0.025} = 1.96$ and $-z_{0.025} = -1.96$.

If $\alpha = 0.05$, $\alpha/2 = 0.025$
$\Rightarrow$ CI % = 95%
$Z_{\alpha/2} = Z_{0.025} = 1.96$
Critical region $\Rightarrow$
$+Z_{0.025}$ to $-Z_{0.025}$
$\Rightarrow +1.96$ to -1.96.

7. Computations: Since $\bar{x} = 51.3$ and $\sigma = 2$,

$$z_0 = \frac{51.3 - 50}{2/\sqrt{25}} = 3.25$$

8. Conclusion: Since $z_0 = 3.25 > 1.96$, we reject H_0: $\mu = 50$ at the 0.05 level of significance. Stated more completely, we conclude that the mean burning rate differs from 50 centimeters per second, based on a sample of 25 measurements. In fact, there is strong evidence that the mean burning rate exceeds 50 centimeters per second.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean (Continued)

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic: $Z_0 = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$

Alternative hypothesis	Rejection criteria
$H_1: \mu \neq \mu_0$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu > \mu_0$	$z_0 > z_{\alpha}$
$H_1: \mu < \mu_0$	$z_0 < -z_{\alpha}$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

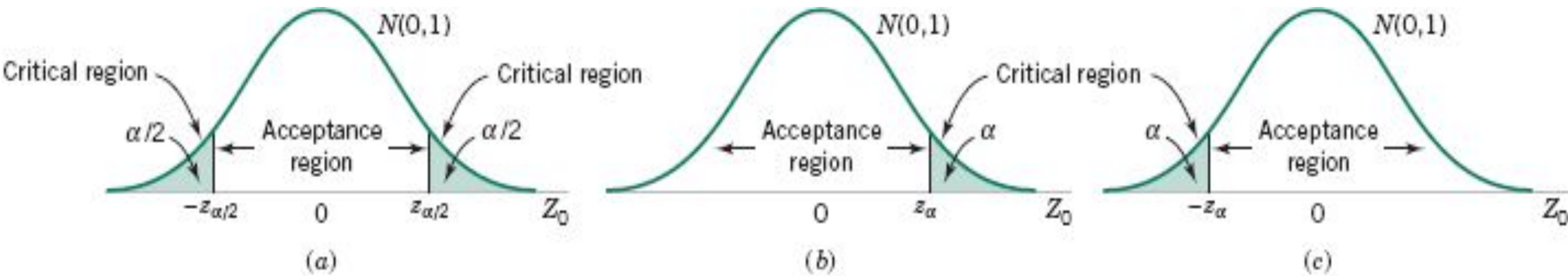


Figure 9-7 The distribution of Z_0 when H_0: $\mu = \mu_0$ is true, with critical region for (a) the two-sided alternative $H_1 : \mu \neq \mu_0$, (b) the one-sided alternative $H_1 : \mu > \mu_0$, and (c) the one-sided alternative $H_1 : \mu < \mu_0$.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

P-Values in Hypothesis Tests

The **P-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

$$P = \begin{cases} 2\left[1 - \Phi(|z_0|)\right] & \text{for a two-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for a upper-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu < \mu_0 \end{cases} \qquad (9\text{-}15)$$

P-Values in Hypothesis Tests

For the Example 9-2, we use p-value to decide….

6. Computations: Because $\bar{x} = 51.3$ and $\sigma = 2$,

$$z_0 = \frac{51.3 - 50}{2/\sqrt{25}} = 3.25$$

7. Conclusion: Because the P-value $= 2\left[1 - \Phi(3.25)\right] = 0.0012$ we reject H_0: $\mu = 50$ at the 0.05 level of significance.

Hint : to calculate p-value
Two sided : 2(1-p(|Z|))
One sided (upper) : 1- p(Z)
One sided (lower)=p(Z)

Example 1 :

9-35. For the hypothesis test $H_0: \mu = 7$ against $H_1: \mu \neq 7$ and variance known, calculate the P-value for each of the following test statistics.

(a) $z_0 = 2.05$ (b) $z_0 = -1.84$ (c) $z_0 = 0.4$

For (a) :

Step 1: Find the tail probability.

Using the standard normal distribution table or statistical software, we look up the probability for $Z>2.05$.

$P(Z>2.05) = 1-0.97981 = 0.0202$

Step 2: Calculate the P-value for the two-sided test.

$P=2\times0.0202=0.0404$

The P-value for the test statistic z0=2.05 is approximately **0.0404**.

For (b) & (c) – Do by yourself

Hint : to calculate p-value
Two sided : 2(1-p(|Z|))
One sided (upper) : 1- p(Z)
One sided (lower)=p(Z)

Example 2 :

9-36. For the hypothesis test $H_0: \mu = 10$ against $H_1: \mu > 10$ and variance known, calculate the P-value for each of the following test statistics.

(a) $z_0 = 2.05$ (b) $z_0 = -1.84$ (c) $z_0 = 0.4$

Step 1: Find the tail probability.

Using the standard normal distribution table or statistical software, we find the probability for Z>2.05.

$P(Z>2.05) \approx 1-0.9812 = 0.0202$

the P-value for the one-sided test.

P=0.0202

The P-value for the test statistic z0=2.05 is approximately **0.0202**.

Hint : to calculate p-value

Two sided : 2(1-p(|Z|))

One sided (upper) : 1- p(Z)

One sided (lower)=p(Z)

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

Consider the two-sided hypothesis

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

Suppose that the null hypothesis is false and that the true value of the mean is $\mu = \mu_0 + \delta$, say, where $\delta > 0$. The test statistic Z_0 is

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\overline{X} - (\mu_0 + \delta)}{\sigma/\sqrt{n}} + \frac{\delta\sqrt{n}}{\sigma}$$

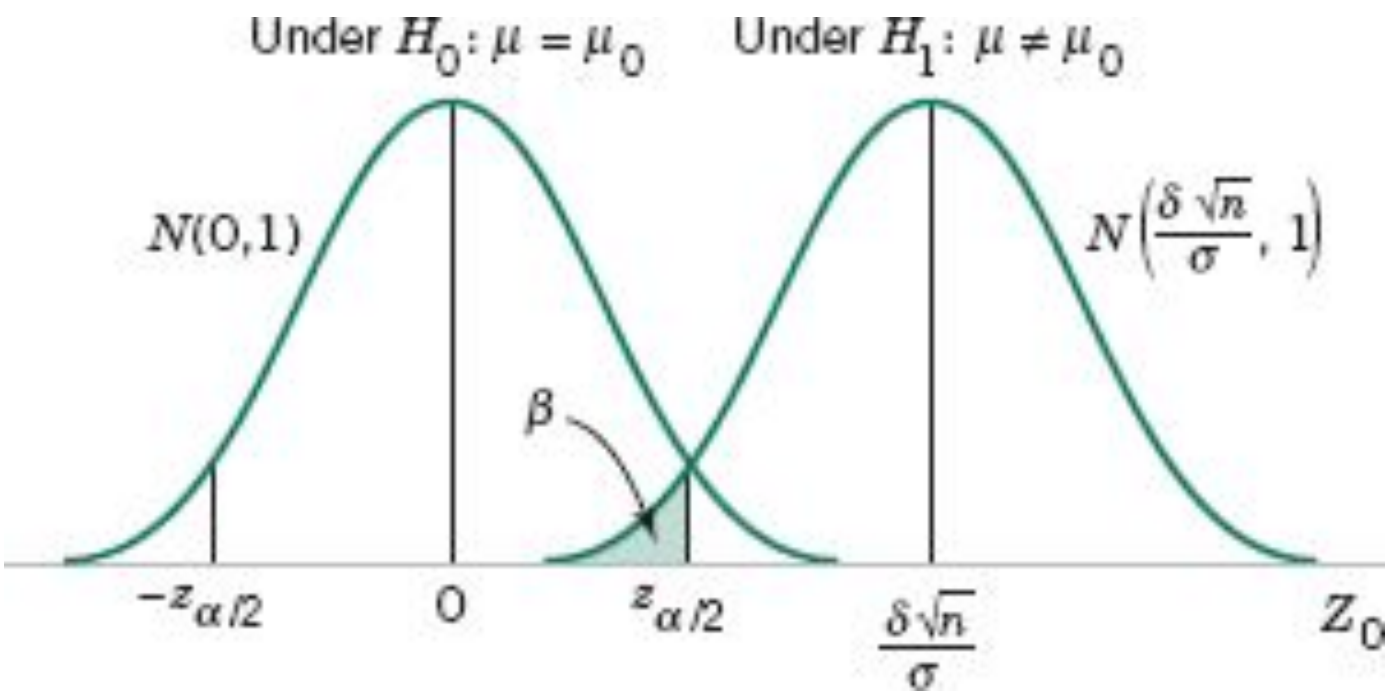
Therefore, the distribution of Z_0 when H_1 is true is

$$Z_0 \sim N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right) \qquad (9\text{-}16)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β



The distribution of Z_0 under H_0 and H_1

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \underbrace{\Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)}_{0 \quad \text{if} \quad \delta>0} \qquad (9\text{-}17)$$

Let z_β be the 100β <u>upper percentile</u> of the standard normal distr.

$\beta = \Phi(-z_\beta)$

$$-z_\beta \simeq z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Sample Size Formulas

For a two-sided alternative hypothesis:

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} \quad \text{where} \quad \delta = \mu - \mu_0 \tag{9-19}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Sample Size Formulas

For a one-sided alternative hypothesis:

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{\delta^2} \quad \text{where} \quad \delta = \mu - \mu_0 \qquad (9\text{-}20)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

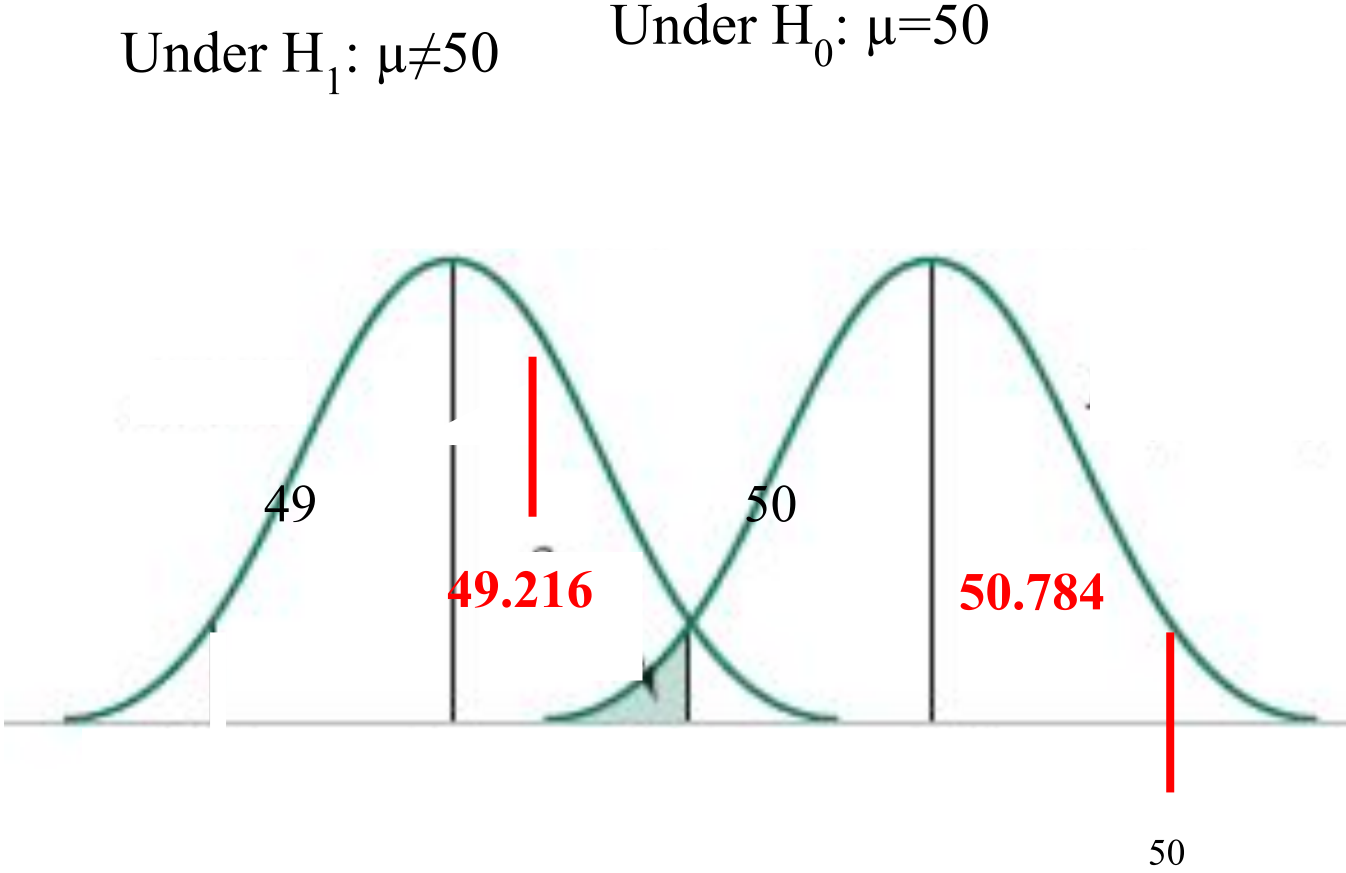
Example 9-3

Finding the Probability of Type II Error β

H_0: $\mu=50$
H_1: $\mu\neq 50$
$\sigma=2$
$\alpha=0.05$
$n=25$
If true $\mu=49$, $\beta=?$

$z_{0.025}=1.96$

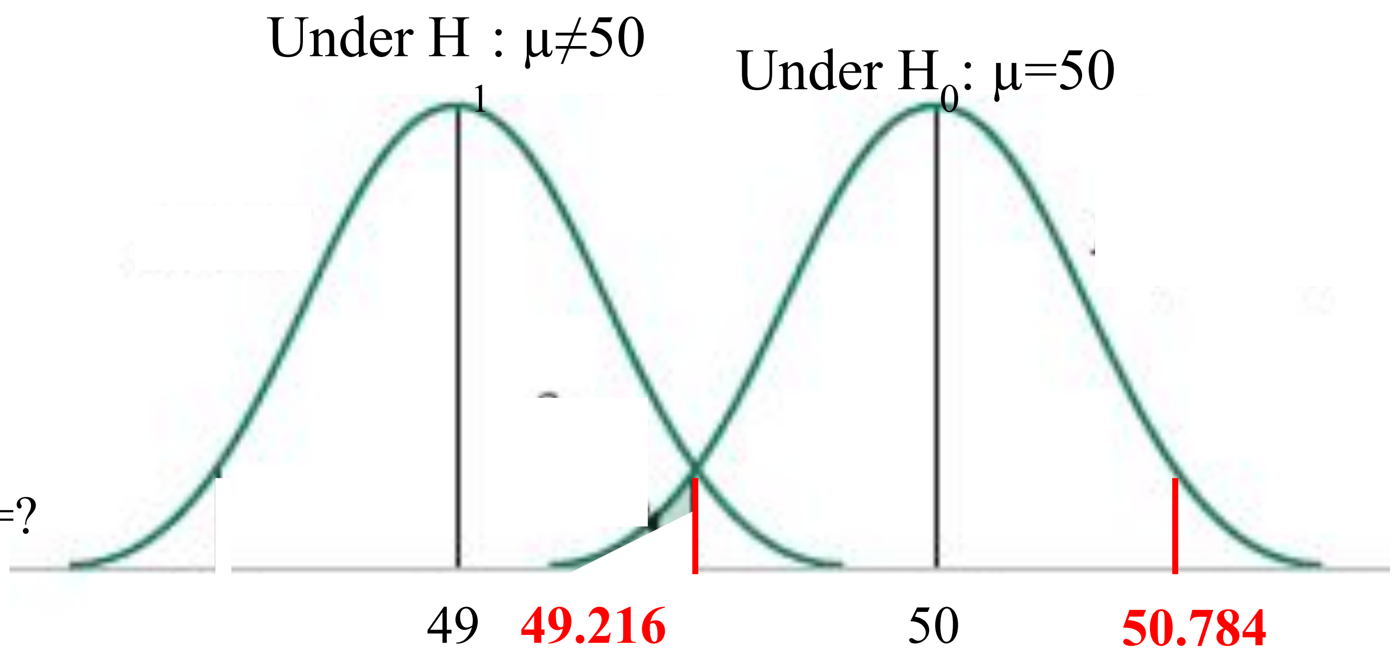
Critical points are:
$50 \pm z_{0.025}\sigma/\sqrt{n}$
$= 50 \pm 1.96*2/5$
49.216 and 50.784



9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

H_0: $\mu=50$
H_1: $\mu\neq 50$
$\sigma=2$
$\alpha=0.05$
$n=25$
If true $\mu=49$, $\beta=?$



$$\beta = P(49.216 \le \bar{X} \le 50.784 \quad \text{when } \mu = 49)$$

$$\beta = P\left(\frac{49.216 - 49}{2/\sqrt{25}} \le Z \le \frac{50.784 - 49}{2/\sqrt{25}} \right) \Rightarrow z_\beta = z_{0.295} = 0.54$$

$$\beta = P(0.54 \le Z \le 4.46) = 0.295$$

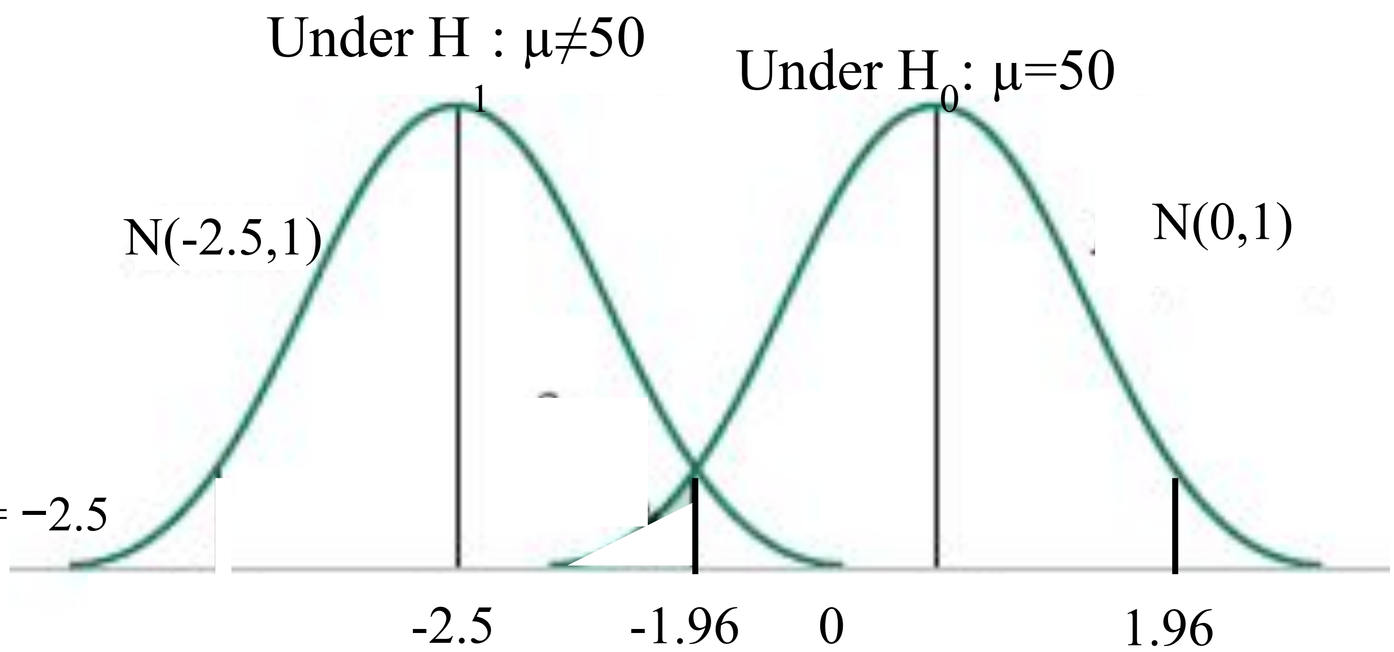
9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

Standardize the normal graphs

$\delta = -1$

$$\frac{\delta\sqrt{n}}{-\varphi*} = \frac{\sqrt{25}}{2} = -2.5$$



With standard normal graph:

$\beta = P(-1.96 \leq \bar{X} \leq 1.96$ when $\mu = -2.5)$

$$\beta = P\left(\frac{-1.96+2.5}{1} \leq Z \leq \frac{1.96+2.5}{1}\right)$$

$\beta = P(0.54 \leq Z \leq 4.46) = 0.295$

With the formula:

$$\beta = \Phi\left(1.96 - \frac{-1*\sqrt{25}}{2}\right) - \Phi\left(-1.96 - \frac{-1*\sqrt{25}}{2}\right)$$

$\beta = \Phi(4.46) - \Phi(0.54)$

$= 0.295$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

Consider the rocket propellant problem of Example 9-2. Suppose that the analyst wishes to design the test so that if the true mean burning rate differs from 50 centimeters per second by as much as 1 centimeter per second, the test will detect this (i.e., reject H_0: $\mu = 50$) with a high probability, say 0.90. Now, we note that $\sigma = 2$, $\delta = 51 - 50 = 1$, $\alpha = 0.05$, and $\beta = 0.10$. Since $z_{\alpha/2} = z_{0.025} = 1.96$ and $z_\beta = z_{0.10} = 1.28$, the sample size required to detect this departure from H_0: $\mu = 50$ is found by Equation 9-19 as

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} = \frac{(1.96 + 1.28)^2 2^2}{(1)^2} \simeq 42$$

The approximation is good here, since $\Phi(-z_{\alpha/2} - \delta\sqrt{n}/\sigma) = \Phi(-1.96 - (1)\sqrt{42}/2) = \Phi(-5.20) \simeq 0$, which is small relative to β.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Using Operating Characteristic Curves

When performing sample size or type II error calculations, it is sometimes more convenient to use the **operating characteristic (OC) curves** in Appendix Charts VII*a* and VII*b*. These curves plot β as calculated from Equation 9-17 against a parameter d for various sample sizes n. Curves are provided for both $\alpha = 0.05$ and $\alpha = 0.01$. The parameter d is defined as

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma} \tag{9-21}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-4

Consider the propellant problem in Example 9-2. Suppose that the analyst is concerned about the probability of type II error if the true mean burning rate is $\mu = 51$ centimeters per second. We may use the operating characteristic curves to find β. Note that $\delta = 51 - 50 = 1$, $n = 25$, $\sigma = 2$, and $\alpha = 0.05$. Then using Equation 9-21 gives

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma} = \frac{1}{2}$$

and from Appendix Chart VII*a*, with $n = 25$, we find that $\beta = 0.30$. That is, if the true mean burning rate is $\mu = 51$ centimeters per second, there is approximately a 30% chance that this will not be detected by the test with $n = 25$.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.3 Large Sample Test

If the distribution of the population is not known, but **n>40** sample standard deviation “s” can be substituted for “σ” and test procedures in Section 9.2 are valid.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.1 Hypothesis Tests on the Mean

One-Sample *t*-Test

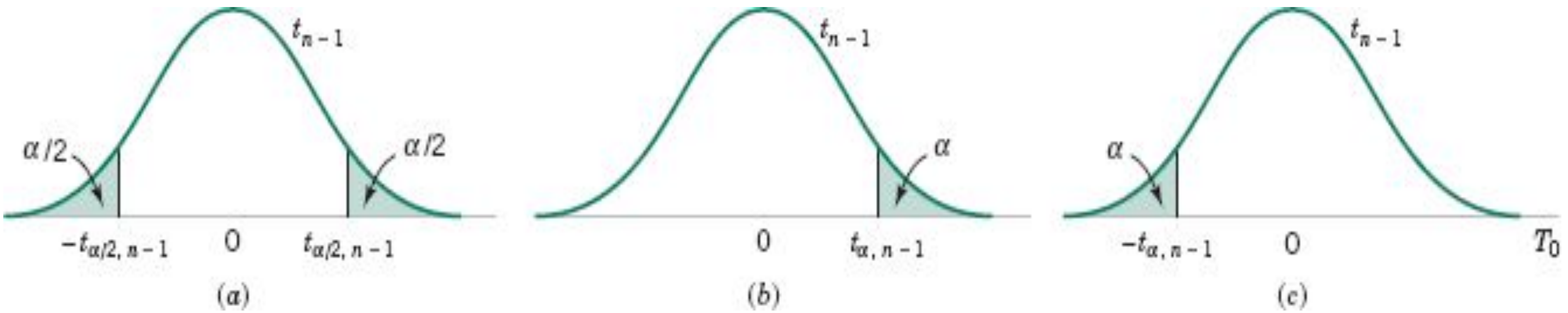
Null hypothesis: $H_0: \mu = \mu_0$

Test statistic: $$T_0 = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

Alternative hypothesis	Rejection criteria
$H_1: \mu \neq \mu_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1: \mu > \mu_0$	$t_0 > t_{\alpha,n-1}$
$H_1: \mu < \mu_0$	$t_0 < -t_{\alpha,n-1}$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.1 Hypothesis Tests on the Mean



The reference distribution for H_0: $\mu = \mu_0$ with critical region for (a) H_1: $\mu \neq \mu_0$, (b) H_1: $\mu > \mu_0$, and (c) H_1: $\mu < \mu_0$.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

The increased availability of light materials with high strength has revolutionized the design and manufacture of golf clubs, particularly drivers. Clubs with hollow heads and very thin faces can result in much longer tee shots, especially for players of modest skills. This is due partly to the "spring-like effect" that the thin face imparts to the ball. Firing a golf ball at the head of the club and measuring the ratio of the outgoing velocity of the ball to the incoming velocity can quantify this spring-like effect. The ratio of velocities is called the coefficient of restitution of the club. An experiment was performed in which 15 drivers produced by a particular club maker were selected at random and their coefficients of restitution measured. In the experiment the golf balls were fired from an air cannon so that the incoming velocity and spin rate of the ball could be precisely controlled. It is of interest to determine if there is evidence (with $\alpha = 0.05$) to support a claim that the mean coefficient of restitution exceeds 0.82. The observations follow:

0.8411	0.8191	0.8182	0.8125	0.8750
0.8580	0.8532	0.8483	0.8276	0.7983
0.8042	0.8730	0.8282	0.8359	0.8660

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

The sample mean and sample standard deviation are $\bar{x} = 0.83725$ and $s = 0.02456$. The normal probability plot of the data in Fig. 9-9 supports the assumption that the coefficient of restitution is normally distributed. Since the objective of the experimenter is to demonstrate that the mean coefficient of restitution exceeds 0.82, a one-sided alternative hypothesis is appropriate.

The solution using the eight-step procedure for hypothesis testing is as follows:

1. The parameter of interest is the mean coefficient of restitution, μ.
2. H_0: $\mu = 0.82$
3. H_1: $\mu > 0.82$. We want to reject H_0 if the mean coefficient of restitution exceeds 0.82.
4. $\alpha = 0.05$
5. The test statistic is

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

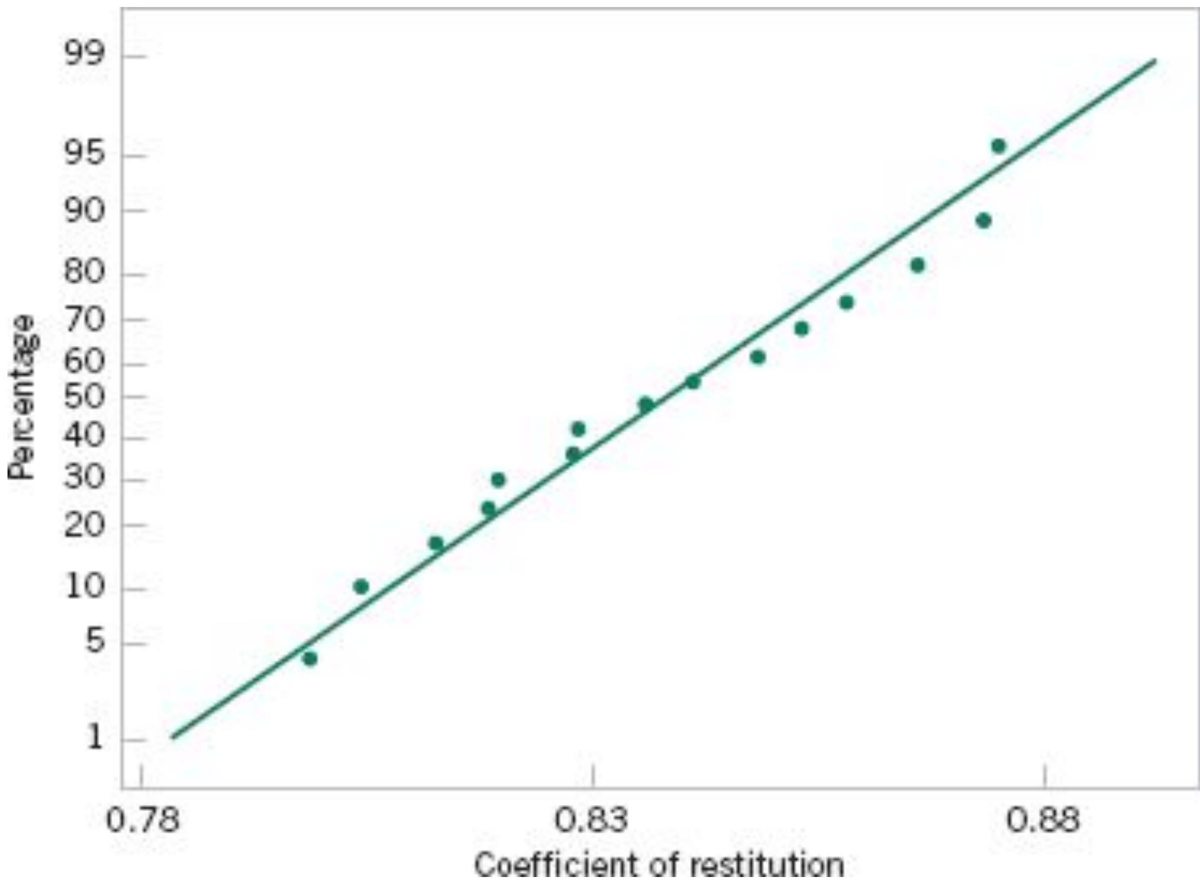
6. Reject H_0 if $t_0 > t_{0.05,14} = 1.761$

Single sided (upper)
$\alpha = 0.05$
$t_{0.05,14} = 1.761$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

Normal probability plot of the coefficient of restitution data from Example 9-6.



9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

7. Computations: Since $\bar{x} = 0.83725$, $s = 0.02456$, $\mu_0 = 0.82$, and $n = 15$, we have

$$t_0 = \frac{0.83725 - 0.82}{0.02456/\sqrt{15}} = 2.72$$

8. Conclusions: Since $t_0 = 2.72 > 1.761$, we reject H_0 and conclude at the 0.05 level of significance that the mean coefficient of restitution exceeds 0.82.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.2 *P*-value for a *t*-Test

The *P*-value for a *t*-test is just the smallest level of significance at which the null hypothesis would be rejected.

To illustrate, consider the *t*-test based on 14 degrees of freedom in Example 9-6. The relevant critical values from Appendix Table IV are as follows:

Critical Value:	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
Tail Area:	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005

$t_0 =$ 2.72, this is between two tabulated values, 2.624 and 2.977. Therefore, the *P*-value must be between 0.01 and 0.005. (upper sided) : 0.005<*P*-value<0.01.

Suppose $t_0 =$ 2.72 for a two-sided test, then

0.005*2<*P*-value<0.01*2 0.01<*P*-value<0.02

Worked out problems

If t_0 is negative, Since the *t*-distribution is symmetric, we can look up the probability for the positive *t*

For lower sided hypothesis, subtract the probability values from 1.

For the hypothesis test H0 : μ = 7 against H1 : μ ≠ 7 with variance unknown and n = 20, approximate the P-value for each of the following test statistics.

(a) t0 = 2.05 (b) t0 =−1. 84 (two sided)

1) $n = 20$

degree of freedom $= 19$

$t_0 = \underline{2.05}$

look in table 19^{th} row and column values ~~that~~ ~~to~~ ~~before~~ where 2.05 lies between,

$\Rightarrow 2.05$ lies between 0.05 & 0.025

$\therefore 0.025 < p < 0.05$

for two sided,

$0.025 \times 2 < p < 0.05 \times 2$

$0.05 < p < 0.1$

$n = 20$

degree $- 19$

$t_0 = -1.84$

Since 't' distribution is ~~...~~ symmetric,

Consider 1.84, use t-table

1.84 lies between 0.05 & 0.025

$\therefore 0.025 \times 2 < p < 0.05 \times 2$

$\therefore 0.05 < p < 0.1$

Worked out problems

For the hypothesis test H0 : μ = 10 against H1 : μ > 10 with variance unknown and n = 15, approximate the P-value for each of the following test statistics. (a) t0 = 2.05 (b) t0 =− 1.84 (upper sided hypothesis)

$n = 15$
degree = 14
$t_0 = 2.05$

$H_0 : \mu_1 = 10$
$H_1 : \mu_2 \geq 10$

2.05 lies between
$0.025 < p < 0.05$

$n = \cancel{20}\ 15$ $H_0 : \mu_0 = 10$
$H_1 > 10$
degree = 14
$t_0 = -1.84$

Same as previous problem,

$\therefore 0.025 < p < 0.05$

Since t_0 is −ve and hypothesis is upper sided, subtract the prob from 1

$1 - 0.025 < p < 1 - 0.05$
$0.975 < p < 0.95$

Rearrange,

$0.95 < p < 0.975$

Worked out problems

For the hypothesis test H0 : μ = 5 against H1 : μ < 5 with variance unknown and n = 12, approximate the P-value for each of the following test statistics.
(a) t0 = 2.05 (b) t0 =−1.84 (lower sided)

$n = 20$
degree $= 19$
$t_0 = 2.05$
$H_0: \mu = 5$
$H_1: \mu < 5$

2.05 lies between
$0.025 < p < 0.05$
Since t_0 is +ve, it indicates the result in opposite direction,
so the range is
$1 - 0.025 < p < 1 - 0.05$
$0.975 < p < 0.95$
rearranging,
$0.95 < p < 0.975$

$n = 20$
degree $= 19$
$t_0 = \cancel{18}\ -1.84$
$\cancel{\Rightarrow 0.05 < p < 0.1}$
$\Rightarrow 0.025 < p < 0.05$
Since t_0 is −ve, and we look for lower sided prob, keep the prob. as such.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Suppose that we wish to test the hypothesis that the variance of a normal population σ^2 equals a specified value, say σ_0^2, or equivalently, that the standard deviation σ is equal to σ_0. Let X_1, $X_2, \ldots, X_n$ be a random sample of n observations from this population. To test

$$H_0\colon \sigma^2 = \sigma_0^2$$

$$H_1\colon \sigma^2 \neq \sigma_0^2$$

we will use the **test statistic:**

$$X_0^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

the null hypothesis $H_0\colon \sigma^2 = \sigma_0^2$ would be rejected if

$$\chi_0^2 > \chi_{\alpha/2,n-1}^2 \qquad \text{or if} \quad \chi_0^2 < \chi_{1-\alpha/2,n-1}^2$$

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

The same test statistic is used for one-sided alternative hypotheses. For the one-sided hypothesis

$$H_0: \sigma^2 = \sigma_0^2$$
$$H_1: \sigma^2 > \sigma_0^2$$

we would reject H_0 if $\chi_0^2 > \chi^2_{\alpha,n-1}$, whereas for the other one-sided hypothesis

$$H_0: \sigma^2 = \sigma_0^2$$
$$H_1: \sigma^2 < \sigma_0^2$$

we would reject H_0 if $\chi_0^2 < \chi^2_{1-\alpha,n-1}$.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

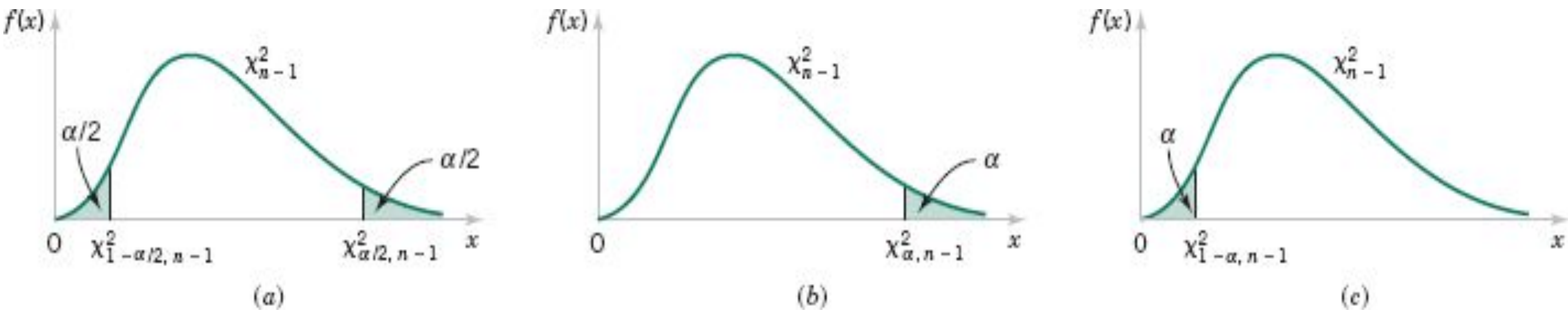


Figure 9-11 Reference distribution for the test of H_0: $\sigma^2 = \sigma_0^2$ with critical region values for (a) H_1: $\sigma^2 \neq \sigma_0^2$, (b) H_1: $\sigma^2 > \sigma_0^2$, and (c) H_1: $\sigma^2 < \sigma_0^2$.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume exceeds 0.01 (fluid ounces)^2, an unacceptable proportion of bottles will be underfilled or overfilled. Is there evidence in the sample data to suggest that the manufacturer has a problem with underfilled or overfilled bottles? Use $\alpha = 0.05$, and assume that fill volume has a normal distribution.

Using the eight-step procedure results in the following:

1. The parameter of interest is the population variance σ^2.
2. $\boxed{H_0\colon \sigma^2 = 0.01}$
3. $\boxed{H_1\colon \sigma^2 > 0.01}$
4. $\alpha = 0.05$
5. The test statistic is

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-8

6. Reject H_0 if $\chi_0^2 > \chi^2_{0.05,19} = 30.14$.
7. Computations:

$$\chi_0^2 = \frac{19(0.0153)}{0.01} = 29.07$$

8. Conclusions: Since $\chi_0^2 = 29.07 < \chi^2_{0.05,19} = 30.14$, we conclude that there is no strong evidence that the variance of fill volume exceeds 0.01 (fluid ounces)^2.

Non parametric test procedures

hypothesis-testing and confidence interval procedures parametric, as they are based on a particular parametric family of distributions, say normal.

i. THE WILCOXON SIGNED-RANK TEST :

- Null Hypothesis : H0 : $\mu = \mu_0$.
- Assume that X1 , X2 , …, Xn is a random sample from a continuous and symmetric distribution with mean (and median) μ.
- Compute the differences Xi – μ0 , i = 1, 2, …, n.
- Rank the absolute differences X ,i , , ,n i − μ = . . . 0 1 2 in ascending order, and then give the ranks the signs of their corresponding differences.
- Let W+ be the sum of the positive ranks and W– be the absolute value of the sum of the negative ranks, and let W = min(W+ , W−).
- Look for the Critical Values for the Wilcoxon Signed-Rank Test from the table (the table provides significance levels of α = 0.10, α = 0.05, α = 0.02 and α = 0.01 for the two-sided test)

Non parametric test procedures

- If the alternative hypothesis is H1 : μ≠μ , then if the observed value of
- the statistic $w \leq w\alpha$ * , the null hypothesis H0 : μ=μ is rejected. Table (Wilcoxon Signed-Rank Test table (1 sample) provides significance levels of $\alpha = 0.10$, $\alpha = 0.05$, $\alpha = 0.02$ and $\alpha = 0.01$ for the two-sided test
- **For one-sided tests, if the alternative is H1 : $\mu > \mu_0$, reject H0 : $\mu=\mu_0$ if $w^- \leq w_\alpha$ * and if the alternative is H1 : $\mu < \mu_0$, reject H_0 : μ=μ if $w^+ \leq w_\alpha$ * . The significance levels for one sided tests provided in the Table are α = 0.05, 0.025, 0.01, and 0.005.**

Wilcoxon Signed-Rank Test example

Illustrate the Wilcoxon signed rank test by applying it to the propellant shear strength data from the following table. Assume that the underlying distribution is a continuous symmetric distribution. The results of testing 20 randomly selected motors are shown in Table 9-5. We would like to test the hypothesis that the median shear strength is 2000 psi, using $\alpha = 0.05$.

Observation i	Shear Strength x_i
1	2158.70
2	1678.15
3	2316.00
4	2061.30
5	2207.50
6	1708.30
7	1784.70
8	2575.10
9	2357.90
10	2256.70
11	2165.20
12	2399.55
13	1779.80
14	2336.75
15	1765.30
16	2053.50
17	2414.40
18	2200.50
19	2654.20
20	1753.70

The eight-step procedure is applied as follows:

1. **Parameter of interest:** The parameter of interest is the mean (or median) of the distribution of propellant shear strength.

2. **Null hypothesis:** $H_0 : \mu = 2000$ psi

3. **Alternative hypothesis:** $H_0 : \mu \neq 2000$ psi

4. α =0.05

5. Test Statistics : W = min(W+ , W−)

6. **Reject H_0 if:** We will reject H_0 if $w \leq w^*_{0.05} = 52$ from Wilcoxon Signed-Rank Test table (1 sample)5.

7. Computations: The signed ranks from the given table are shown in the following display: The sum of the positive ranks is w+ = (1 + 2 + 3 + 4 + 5 + 6 + 11 + 13 + 15 + 16 + 17 + 18 + 19 + 20) = 150, and the sum of the absolute values of the negative ranks is w- = (7 + 8 + 9 + 10 + 12 + 14) = 60. Therefore, .

w = min (150 60)= 60 (next slide)

The eight-step procedure is applied as follows:

7. Computations : $\alpha = 0.05$

Observation	Difference $x_i - 2000$	Signed Rank
16	+53.50	+1
4	+61.30	+2
1	+158.70	+3
11	+165.20	+4
18	+200.50	+5
5	+207.50	+6
7	−215.30	−7
13	−220.20	−8
15	−234.70	−9
20	−246.30	−10
10	+256.70	+11
6	−291.70	−12
3	+316.00	+13
2	−321.85	−14
14	+336.75	+15
9	+357.90	+16
12	+399.55	+17
17	+414.40	+18
8	+575.10	+19
19	+654.20	+20

8. Conclusions: Because w = 60 is not less than or equal to the critical value w0.05 = 52, we cannot reject the null hypothesis that the mean (or median, because the population is assumed to be symmetric) shear strength is 2000 psi

Sign test

Step 1: Define the Parameter of Interest
The parameter of interest is the median ($\tilde{\mu}$) of the population (or the median difference if comparing paired samples).
Step 2: State the Hypotheses
For a two-tailed test:

$$H_0 : \tilde{\mu} = \mu_0$$

$$H_1 : \tilde{\mu} \neq \mu_0$$

- For a **one-tailed test:**

$$H_1 : \tilde{\mu} > \mu_0 \quad \text{or} \quad \tilde{\mu} < \mu_0$$

where μ_0 is the hypothesized median.

Step 3: Compute the Differences and Assign Signs
For each observation x_i:

$$d_i = x_i - \mu_0$$

Then assign:
- "+" if $d_i > 0$
- "–" if $d_i < 0$
- Ignore if $d_i = 0$

Let:
- n= number of non-zero differences
- R^+ =number of "+" signs
- R^- =number of "–" signs

Sing test

Step 4: Define the Test Statistic

$$R^+ = number\ of\ positive\ signs$$

Under H_0,

$$R^+ \sim Binomial(n,p=0.5)$$

since each observation has an equal chance of being above or below the median.

Step 5: Determine the Rejection Region

•Choose the **significance level (α)**.

•For a **two-tailed test**, reject H_0if the number of "+" signs is too **small** or too **large**.

•For a **one-tailed test**, reject H_0if $R^+$is **too large** (right-tailed) or **too small** (left-tailed).

Step 6: Compute the P-Value

The **P-value** is the probability of obtaining a result as extreme (or more extreme) than the observed one under H_0.

•For a **two-tailed test:**

$$P = 2 \times P(R^+ \geq observed)$$

Using the **binomial formula:**

$$P(R^+ = r) = \binom{n}{r}(0.5)^n$$

Then sum probabilities from the observed value to n (or symmetrically for the lower tail).

Sign test

Step 7: Make the DecisionIf $P \leq \alpha$, reject $H0$.If $P > \alpha$ fail to reject $H0$ .Step 8: State the Conclusion

If H_0 is rejected → conclude that the **median differs** from the hypothesized value.
If H_0 is not rejected → conclude that there is **no significant evidence** that the median differs.

The sign test example

A researcher wants to test whether a new teaching method changes the median test score of students compared to the traditional method (which has a median score of 75). Six students are randomly selected, and their scores after using the new method are:

Student	Score
1	78
2	72
3	80
4	77
5	74
6	79

Use the **Sign Test** at **α = 0.05** to test if the **median differs from 75**.

Solution :

Step 1: Define hypotheses

$$H_0: \tilde{\mu} = 75$$
$$H_1: \tilde{\mu} \neq 75$$

This is a **two-tailed test**.

Step 2: Compute differences and assign signs

The sign test example

Student	Score	($d_i = x_i - 75$)	Sign
1	78	+3	+
2	72	-3	–
3	80	+5	+
4	77	+2	+
5	74	-1	–
6	79	+4	+

Step 3: Count positive and negative signs

$R^+ = 4$) (**Number of "+" signs**)

$R^- = 2$(**number of "–" signs**)

$n = 6$

Step 4: Test statistic

Under H_0 ,$R^+ \sim$ *Binomial(n=6, p=0.5)*

Step 5: Determine rejection region

For a **two-tailed test** with $n = 6$ and $\alpha = 0.05$,
We reject H_0if the number of pluses is **too small (≤1)** or **too large (≥5)**.

Step 6: Compute P-value

We calculate:

$$P = 2P(R^+ \geq 4)$$

Since the distribution is symmetric, we use:

$$P(R^+ \geq 4) = P(R^+ = 4) + P(R^+ = 5) + P(R^+ = 6)$$

Using the **binomial formula:**

The sign test example

Using the **binomial formula:**

$$P(R^+ = r) = \binom{6}{r}(0.5)^6$$

r	$\binom{6}{r}$	**Probability**
4	15	15 × (0.5)^6 = 15/64 = 0.2344
5	6	6/64 = 0.0938
6	1	1/64 = 0.0156

$$P(R^+ \geq 4) = 0.2344 + 0.0938 + 0.0156 = 0.3438$$

$$P = 2 \times 0.3438 = 0.6876$$

Step 7: Decision

$$P = 0.6876 > 0.05$$

Fail to reject H_0

There is **no significant evidence** that the median test score differs from 75.

Solve :

1. The titanium content in an aircraft-grade alloy is an important determinant of strength. A sample of 20 test coupons reveals the following titanium content (in percent):
8.32, 8.05, 8.93, 8.65, 8.25, 8.46, 8.52, 8.35, 8.36, 8.41, 8.42, 8.30, 8.71, 8.75, 8.60, 8.83, 8.50, 8.38, 8.29, 8.46. The median titanium content should be 8.5%.

(a) Use the Wilcoxon signed-rank test with $\alpha = 0.05$ to investigate this hypothesis.

2. An inspector are measured the diameter of a ball bearing using a new type of caliper. The results were as follows (in mm): 0.265, 0.263, 0.266, 0.267, 0.267, 0.265, 0.267,0.267, 0.265, 0.268, 0.268, and 0.263.

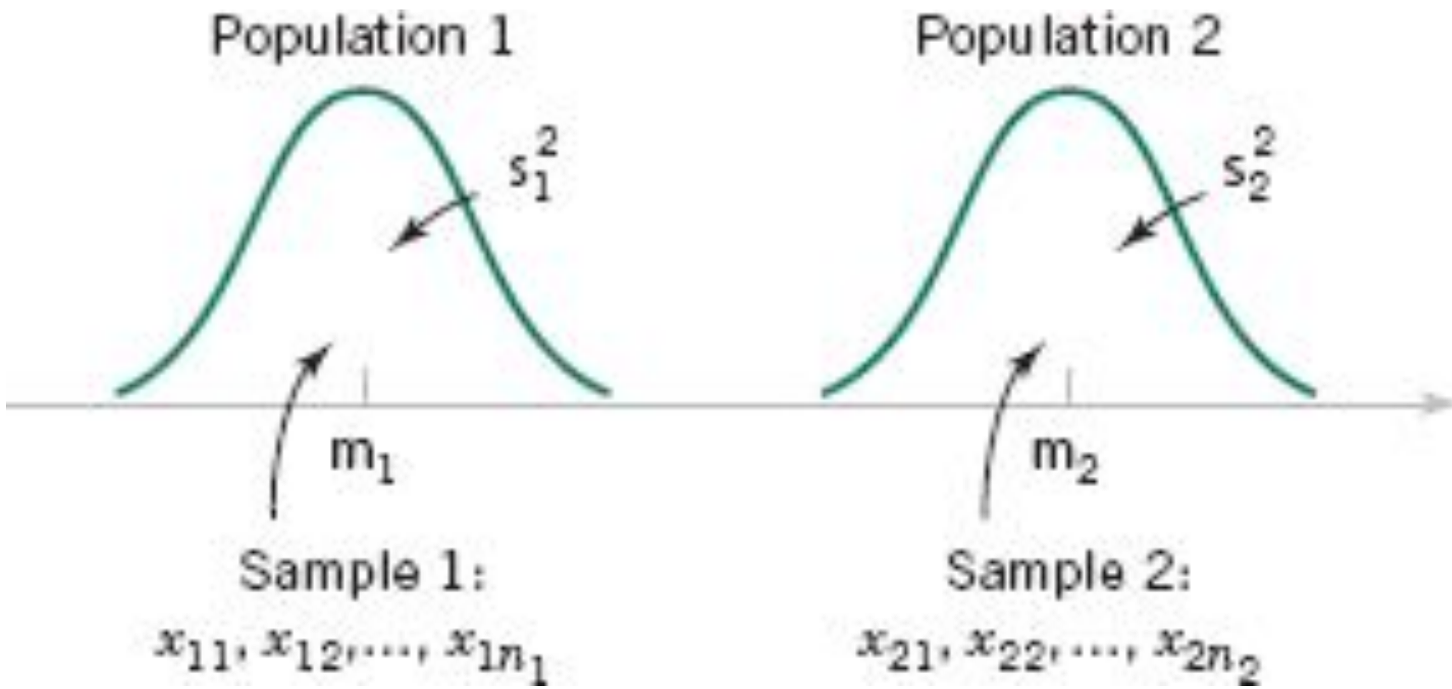
(a) Use the Wilcoxon signed-rank test to evaluate the claim that the mean ball diameter is 0.265 mm. Use $\alpha = 0.05$.

Chapter 10

Statistical Inference for Two Samples

- Inference on the difference in means of two normal distributions, variances known
- Inference on the difference in means of two normal distributions, variances unknown
- Paired t-test
- Inference on the variances of two normal distributions
- Inference on two population proportions

10-1 Introduction



Two independent populations.

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Assumptions

1. $X_{11}, X_{12}, \ldots, X_{1n_1}$ is a random sample from population 1.
2. $X_{21}, X_{22}, \ldots, X_{2n_2}$ is a random sample from population 2.
3. The two populations represented by X_1 and X_2 are independent.
4. Both populations are normal.

$$E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu_1 - \mu_2$$

$$V(\overline{X}_1 - \overline{X}_2) = V(\overline{X}_1) + V(\overline{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

The quantity

$$Z = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \tag{10-1}$$

has a $N(0, 1)$ distribution.

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

10-1.1 Hypothesis Tests for a Difference in Means, Variances Known

Null hypothesis: $H_0\colon \mu_1 - \mu_2 = \Delta_0$

Test statistic:
$$Z_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \qquad (10\text{-}2)$$

Alternative Hypotheses	Rejection Criterion
$H_1\colon \mu_1 - \mu_2 \neq \Delta_0$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon \mu_1 - \mu_2 > \Delta_0$	$z_0 > z_\alpha$
$H_1\colon \mu_1 - \mu_2 < \Delta_0$	$z_0 < -z_\alpha$

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-1

A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using $\alpha = 0.05$?

We apply the eight-step procedure to this problem as follows:

1. The quantity of interest is the difference in mean drying times, $\mu_1 - \mu_2$, and $\Delta_0 = 0$.
2. $H_0: \mu_1 - \mu_2 = 0$, or $H_0: \mu_1 = \mu_2$.
3. $H_1: \mu_1 > \mu_2$. We want to reject H_0 if the new ingredient reduces mean drying time.
4. $\alpha = 0.05$

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-1

5. The test statistic is

$$z_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

where $\sigma_1^2 = \sigma_2^2 = (8)^2 = 64$ and $n_1 = n_2 = 10$.

6. Reject H_0: $\mu_1 = \mu_2$ if $z_0 > 1.645 = z_{0.05}$.

7. Computations: Since $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, the test statistic is

$$\boxed{z_0 = \frac{121 - 112}{\sqrt{\dfrac{(8)^2}{10} + \dfrac{(8)^2}{10}}} = 2.52}$$

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-1

8. Conclusion: Since $z_0 = 2.52 > 1.645$, we reject H_0: $\mu_1 = \mu_2$ at the $\alpha = 0.05$ level and conclude that adding the new ingredient to the paint significantly reduces the drying time. Alternatively, we can find the P-value for this test as

$$P\text{-value} = 1 - \Phi(2.52) = 0.0059$$

Therefore, H_0: $\mu_1 = \mu_2$ would be rejected at any significance level $\alpha \geq 0.0059$.

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

10-1.3 Confidence Interval on a Difference in Means, Variances Known

Definition

If $\bar{x}_1$ and $\bar{x}_2$ are the means of independent random samples of sizes n_1 and n_2 from two independent normal populations with known variances σ_1^2 and σ_2^2, respectively, a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10\text{-}7)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-4

Tensile strength tests were performed on two different grades of aluminum spars used in manufacturing the wing of a commercial transport aircraft. From past experience with the spar manufacturing process and the testing procedure, the standard deviations of tensile strengths are assumed to be known. The data obtained are as follows: $n_1 = 10$, $\bar{x}_1 = 87.6$, $\sigma_1 = 1$, $n_2 = 12$, $\bar{x}_2 = 74.5$, and $\sigma_2 = 1.5$. If μ_1 and μ_2 denote the true mean tensile strengths for the two grades of spars, we may find a 90% confidence interval on the difference in mean strength $\mu_1 - \mu_2$ as follows:

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$87.6 - 74.5 - 1.645\sqrt{\frac{(1)^2}{10} + \frac{(1.5)^3}{12}} \le \mu_1 - \mu_2 \le 87.6 - 74.5 + 1.645\sqrt{\frac{(1^2)}{10} + \frac{(1.5)^2}{12}}$$

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Example 10-4

Therefore, the 90% confidence interval on the difference in mean tensile strength (in kilograms per square millimeter) is

$$12.22 \leq \mu_1 - \mu_2 \leq 13.98 \text{ (in kilograms per square millimeter)}$$

Notice that the confidence interval does not include zero, implying that the mean strength of aluminum grade 1 (μ_1) exceeds the mean strength of aluminum grade 2 (μ_2). In fact, we can state that we are 90% confident that the mean tensile strength of aluminum grade 1 exceeds that of aluminum grade 2 by between 12.22 and 13.98 kilograms per square millimeter.

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

Choice of Sample Size

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (\sigma_1^2 + \sigma_2^2) \qquad (10\text{-}8)$$

The required sample size so that the error in estimating $\mu_1 - \mu_2$ by $\bar{x}_1 - \bar{x}_2$ will be less than E at $100(1-\alpha)\%$ confidence.

10-1 Inference for a Difference in Means of Two Normal Distributions, Variances Known

One-Sided Confidence Bounds

Upper Confidence Bound

$$\mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad (10\text{-}9)$$

Lower Confidence Bound

$$\bar{x}_1 - \bar{x}_2 - z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \qquad (10\text{-}10)$$

Solve :

10-1. Consider the hypothesis test $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$ with known variances $\sigma_1 = 10$ and $\sigma_2 = 5$. Suppose that sample sizes $n_1 = 10$ and $n_2 = 15$ and that $\overline{x}_1 = 4.7$ and $\overline{x}_2 = 7.8$. Use $\alpha = 0.05$.

(a) Test the hypothesis and find the P-value.

Hint.

, the two-tailed p-value is calculated as:

$$\text{p-value} = 2 \times (1 - P(|Z|))$$

Solution :

Given:

- Variances: $\sigma_1^2 = 10$, $\sigma_2^2 = 5$
- Sample sizes: $n_1 = 10$, $n_2 = 15$
- Sample means: $\bar{x}_1 = 4.7$, $\bar{x}_2 = 7.8$
- Significance level: $\alpha = 0.05$

$$Z = \frac{4.7 - 7.8}{1.1547} = \frac{-3.1}{1.1547} \approx -2.68$$

$P(Z)$ for $|Z| = 2.68$ is approximately 0.9964.

The two-sided p-value is:

$$\text{p-value} = 2 \times (1 - 0.9964) \approx 2 \times 0.0036 = 0.00726$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.1 Hypotheses Tests for a Difference in Means, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

We wish to test:

$$H_0\colon \mu_1 - \mu_2 = \Delta_0$$
$$H_1\colon \mu_1 - \mu_2 \neq \Delta_0$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.1 Hypotheses Tests for a Difference in Means, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

combine S_1^2 and S_2^2 to form an estimator of σ^2

The pooled estimator of σ^2:

The pooled estimator of σ^2, denoted by S_p^2, is defined by

$$\boxed{S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}} \qquad (10\text{-}12)$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.1 Hypotheses Tests for a Difference in Means, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Given the assumptions of this section, the quantity

$$T = \frac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad (10\text{-}13)$$

has a t distribution with $\boxed{n_1 + n_2 - 2}$ degrees of freedom.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Definition: The Two-Sample or Pooled *t*-Test

Null hypothesis: $H_0\colon \mu_1 - \mu_2 = \Delta_0$

Test statistic:

$$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{10-14}$$

Alternative Hypothesis	Rejection Criterion
$H_1\colon \mu_1 - \mu_2 \neq \Delta_0$	$t_0 > t_{\alpha/2, n_1+n_2-2}$ or $t_0 < -t_{\alpha/2, n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 > \Delta_0$	$t_0 > t_{\alpha, n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 < \Delta_0$	$t_0 < -t_{\alpha, n_1+n_2-2}$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently in use, but catalyst 2 is acceptable. Since catalyst 2 is cheaper, it should be adopted, providing it does not change the process yield. A test is run in the pilot plant and results in the data shown in Table 10-1. Is there any difference between the mean yields? Use $\alpha = 0.05$, and assume equal variances.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

Table 10-1 Catalyst Yield Data, Example 10-5

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

The solution using the eight-step hypothesis-testing procedure is as follows:

1. The parameters of interest are μ_1 and μ_2, the mean process yield using catalysts 1 and 2, respectively, and we want to know if $\mu_1 - \mu_2 = 0$.
2. H_0: $\mu_1 - \mu_2 = 0$ or H_0: $\mu_1 = \mu_2$
3. H_1: $\mu_1 \neq \mu_2$
4. $\alpha = 0.05$
5. The test statistic is

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

6. Reject H_0 if $t_0 > t_{0.025,14} = 2.145$ or if $t_0 < -t_{0.025,14} = -2.145$.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-5

7. Computations: From Table 10-1 we have $\bar{x}_1 = 92.255$, $s_1 = 2.39$, $n_1 = 8$, $\bar{x}_2 = 92.733$, $s_2 = 2.98$, and $n_2 = 8$. Therefore

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(7)(2.39)^2 + 7(2.98)^2}{8 + 8 - 2} = 7.30$$

$$s_p = \sqrt{7.30} = 2.70$$

and

$$\boxed{t_0 = \frac{\bar{x}_1 - \bar{x}_2}{2.70\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{92.255 - 92.733}{2.70\sqrt{\frac{1}{8} + \frac{1}{8}}} = -0.35}$$

8. Conclusions: Since $-2.145 < t_0 = -0.35 < 2.145$, the null hypothesis cannot be rejected. That is, at the 0.05 level of significance, we do not have strong evidence to conclude that catalyst 2 results in a mean yield that differs from the mean yield when catalyst 1 is used.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.1 Hypotheses Tests for a Difference in Means, Variances Unknown

Case 2: $\sigma_1^2 \neq \sigma_2^2$

$$T_0^* = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} \qquad (10\text{-}15)$$

is distributed approximately as *t* distribution with degrees of freedom ν given by

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.1 Hypotheses Tests for a Difference in Means, Variances Unknown

Case 2: $\sigma_1^2 \neq \sigma_2^2$

$$v = \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{(S_1^2/n_1)^2}{n_1 - 1} + \dfrac{(S_2^2/n_2)^2}{n_2 - 1}} \qquad (10\text{-}16)$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6

Arsenic concentration in public drinking water supplies is a potential health risk. An article in the *Arizona Republic* (Sunday, May 27, 2001) reported drinking water arsenic concentrations in parts per billion (ppb) for 10 methropolitan Phoenix communities and 10 communities in rural Arizona. The data follow:

Metro Phoenix ($\overline{x}_1 = 12.5, s_1 = 7.63$)	Rural Arizona ($\overline{x}_2 = 27.5, s_2 = 15.3$)
Phoenix, 3	Rimrock, 48
Chandler, 7	Goodyear, 44
Gilbert, 25	New River, 40
Glendale, 10	Apachie Junction, 38
Mesa, 15	Buckeye, 33
Paradise Valley, 6	Nogales, 21
Peoria, 12	Black Canyon City, 20
Scottsdale, 25	Sedona, 12
Tempe, 15	Payson, 1
Sun City, 7	Casa Grande, 18

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

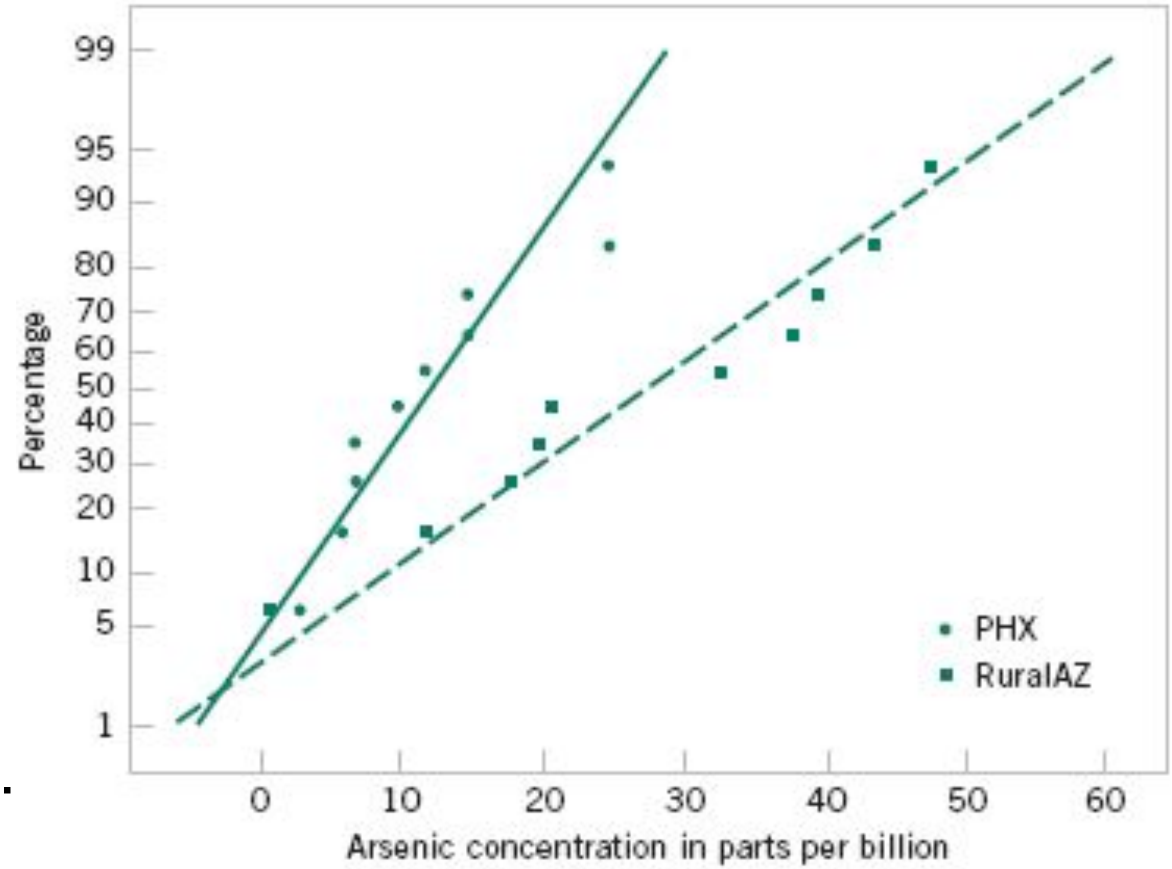


Figure 10-2 Normal probability plot of the arsenic concentration data from Example 10-6.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

We wish to determine it there is any difference in mean arsenic concentrations between metropolitan Phoenix communities and communities in rural Arizona. Figure 10-3 shows a normal probability plot for the two samples of arsenic concentration. The assumption of normality appears quite reasonable, but since the slopes of the two straight lines are very different, it is unlikely that the population variances are the same.

Applying the eight-step procedure gives the following:

1. The parameters of interest are the mean arsenic concentrations for the two geographic regions, say μ_1 and μ_2, and we are interested in determining whether $\mu_1 - \mu_2 = 0$.
2. H_0: $\mu_1 - \mu_2 = 0$ or H_0: $\mu_1 = \mu_2$
3. H_1: $\mu_1 \neq \mu_2$
4. $\alpha = 0.05$ (say)

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

5. The test statistic is

$$t_0^* = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

6. The degrees of freedom on t_0^* are found from Equation 10-16 as

$$v = \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}} = \frac{\left[\dfrac{(7.63)^2}{10} + \dfrac{(15.3)^2}{10}\right]^2}{\dfrac{[(7.63)^2/10]^2}{9} + \dfrac{[(15.3)^2/10]^2}{9}} = 13.2 \simeq 13$$

Therefore, using $\alpha = 0.05$, we would reject H_0: $\mu_1 = \mu_2$ if $t_0^* > t_{0.025,13} = 2.160$ or if $t_0^* < -t_{0.025,13} = -2.160$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Example 10-6 (Continued)

7. Computations: Using the sample data we find

$$\boxed{t_0^* = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{12.5 - 27.5}{\sqrt{\dfrac{(7.63)^2}{10} + \dfrac{(15.3)^2}{10}}} = -2.77}$$

8. Conclusions: Because $t_0^* = -2.77 < t_{0.025,13} = -2.160$, we reject the null hypothesis. Therefore, there is evidence to conclude that mean arsenic concentration in the drinking water in rural Arizona is different from the mean arsenic concentration in metropolitan Phoenix drinking water. Furthermore, the mean arsenic concentration is higher in rural Arizona communities. The P-value for this test is approximately $P = 0.016$.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.3 Confidence Interval on the Difference in Means, Variance Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

If $\bar{x}_1, \bar{x}_2, s_1^2$ and s_2^2 are the sample means and variances of two random samples of sizes n_1 and n_2, respectively, from two independent normal populations with unknown but equal variances, then a **100(1 − α)% confidence interval on the difference in means $\mu_1 - \mu_2$ is**

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2, n_1+n_2-2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$\leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2, n_1+n_2-2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \qquad (10\text{-}19)$$

where $s_p = \sqrt{[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/(n_1 + n_2 - 2)}$ is the pooled estimate of the common population standard deviation, and $t_{\alpha/2, n_1+n_2-2}$ is the upper $\alpha/2$ percentage point of the t distribution with $n_1 + n_2 - 2$ degrees of freedom.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances <u>Unknown</u>

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

An article in the journal *Hazardous Waste and Hazardous Materials* (Vol. 6, 1989) reported the results of an analysis of the weight of calcium in standard cement and cement doped with lead. Reduced levels of calcium would indicate that the hydration mechanism in the cement is blocked and would allow water to attack various locations in the cement structure. Ten samples of standard cement had an average weight percent calcium of $\bar{x}_1 = 90.0$, with a sample standard deviation of $s_1 = 5.0$, while 15 samples of the lead-doped cement had an average weight percent calcium of $\bar{x}_2 = 87.0$, with a sample standard deviation of $s_2 = 4.0$.

We will assume that weight percent calcium is normally distributed and find a 95% confidence interval on the difference in means, $\mu_1 - \mu_2$, for the two types of cement. Furthermore, we will assume that both normal populations have the same standard deviation.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Example 10-8 (Continued)

The pooled estimate of the common standard deviation is found using Equation 10-12 as follows:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{9(5.0)^2 + 14(4.0)^2}{10 + 15 - 2}$$

$$= 19.52$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Example 10-8 (Continued)

Therefore, the pooled standard deviation estimate is $s_p = \sqrt{19.52} = 4.4$. The 95% confidence interval is found using Equation 10-19:

$$\bar{x}_1 - \bar{x}_2 - t_{0.025,23}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{0.025,23}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

or upon substituting the sample values and using $t_{0.025,23} = 2.069$,

$$90.0 - 87.0 - 2.069(4.4)\sqrt{\frac{1}{10} + \frac{1}{15}} \le \mu_1 - \mu_2$$

$$\le 90.0 - 87.0 + 2.069(4.4)\sqrt{\frac{1}{10} + \frac{1}{15}}$$

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Example 10-8 (Continued)

which reduces to

$$-0.72 \leq \mu_1 - \mu_2 \leq 6.72$$

Notice that the 95% confidence interval includes zero; therefore, at this level of confidence we cannot conclude that there is a difference in the means. Put another way, there is no evidence that doping the cement with lead affected the mean weight percent of calcium; therefore, we cannot claim that the presence of lead affects this aspect of the hydration mechanism at the 95% level of confidence.

10-2 Inference for a Difference in Means of Two Normal Distributions, Variances Unknown

10-2.3 Confidence Interval on the Difference in Means, Variance Unknown

Case 2: $1^{2}\ 2,\ ^{2}$

If $\bar{x}_1$, $\bar{x}_2$, s_1^2, and s_2^2 are the means and variances of two random samples of sizes n_1 and n_2, respectively, from two independent normal populations with unknown and unequal variances, an approximate $100(1-\alpha)\%$ confidence interval on the difference in means $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \qquad (10\text{-}20)$$

where v is given by Equation 10-16 and $t_{\alpha/2,v}$ is the upper $\alpha/2$ percentage point of the t distribution with v degrees of freedom.

Non parametric testing the Difference in Two Means

Distributions of $X1$ and $X2$ are continuous The **Wilcoxon rank-sum test** can be used to test the hypothesis

$$H_0: \mu_1 = \mu_2 \qquad\qquad H_1: \mu_1 \neq \mu_2$$

Arrange all $n1 + n2$ observations in ascending order of magnitude and assign ranks to them. If two or more observations are tied (identical), use the mean of the ranks that would have been assigned if the observations differed
Let $W1$ be the sum of the ranks in the smaller sample (1), and define $W2$ to be the sum of the ranks in the other sample. Then

$$W_2 = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} - W_1$$

If the sums of the ranks differ greatly, conclude that the means are not equal.

The null H_0:µ1 = µ2 is rejected in favor of $H1$:µ1 < µ2, if either of the observed values $w1$ or $w2$ is less than or equal to the tabulated critical value w_α

Non parametric testing the Difference in Two Means

one-sided alternatives.

If the alternative is $H1:\mu1 < \mu2$, reject $H0$ if $w_1 \leq w_\alpha$; for $H_1:\mu_1 > \mu_2$, reject H_0 if $w_2 \leq w_\alpha$.

For these one-sided tests, the tabulated critical values w_α correspond to levels of significance of $\alpha = 0.025$ and $\alpha = 0.005$.

Non parametric testing the Difference in Two Means

Example :

The mean axial stress in tensile members used in an aircraft structure is being studied. Two alloys are being investigated. Alloy 1 is a traditional material, and alloy 2 is a new aluminum lithium alloy that is much lighter than the standard material. Ten specimens of each alloy type are tested, and the axial stress is measured. The sample data are assembled in Table below: Using $\alpha = 0.05$, we wish to test the hypothesis that the means of the two stress distributions are identical.

Alloy 1		Alloy 2	
3238 psi	3254 psi	3261 psi	3248 psi
3195	3229	3187	3215
3246	3225	3209	3226
3190	3217	3212	3240
3204	3241	3258	3234

Non parametric testing the Difference in Two Means

Apply the eight step hypothesis-testing procedure to this problem:

1. Parameter of interest: The parameters of interest are the means of the two distributions of axial stress.

2. Null hypothesis: $H0$: μ1 = μ2

3. Alternative hypothesis: $H1$: μ1 ≠ μ2

4. α = 0.05

5. Test statistic: We will use the Wilcoxon rank-sum test statistic

$$W_2 = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} - W_1$$

6. **Reject H_0 if:** for α = 0.05 and $n1 = n2 = 10$, Wilcoxon rank-sum gives the critical value as $w0.05 = 78$. If either w_1 or w_2 is less than or equal to $w0.05 = 78$, we will reject H_0: μ1 = μ2.

Non parametric testing the Difference in Two Means

7. **Computations**

Alloy Number	Axial Stress	Rank
2	3187 psi	1
1	3190	2
1	3195	3
1	3204	4
2	3209	5
2	3212	6
2	3215	7
1	3217	8
1	3225	9
2	3226	10
1	3229	11
2	3234	12
1	3238	13
2	3240	14
1	3241	15
1	3246	16
2	3248	17
1	3254	18
2	3258	19
2	3261	20

Non parametric testing the Difference in Two Means

The sum of the ranks for alloy 1 is
$w1 = 2 + 3 + 4 + 8 + 9 + 11 + 13 + 15 + 16 + 18 = 99$

and for alloy 2

$$w_2 = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} - w_1 = \frac{(10+10)(10+10+1)}{2} - 99 = 111$$

8. **Conclusion:** Because neither w_1 nor w_2 is less than or equal to $w0.05 = 78$, we cannot reject the null hypothesis that both alloys exhibit the same mean axial stress.

Non parametric testing the Difference in Two Means – another example

One of the authors travels regularly to Seattle, Washington. He uses either Delta or Alaska airline. Flight delays are sometimes unavoidable, but he would be willing to give most of his business to the airline with the best on-time arrival record. The number of minutes that his flight arrived late for the last six trips on each airline follows. Is there evidence that either airline has superior on-time arrival performance? Use **α** = 0.01 and the Wilcoxon rank-sum test

Delta:	13, 10, 1, –4, 0, 9 (minutes late)
Alaska:	15, 8, 3, –1, –2, 4 (minutes late)

Solution :

Apply the ei

1. Parameter of interest: The parameters of interest are the means of the two distributions of axial stress.

2. Null hypothesis: $H0$: $\mu 1 = \mu 2$

3. Alternative hypothesis: $H1$: $\mu 1 \neq \mu 2$

4. **α** = 0.01

5. Test statistic: We will use the Wilcoxon rank-sum test statistic

Non parametric testing the Difference in Two Means – another example

6. **Reject H_0 if: for $\alpha = 0.01$ and $n1 = n2 = 10$,** Wilcoxon rank-sum **gives the critical value as $w0.01=71$. If either w_1 or w_2 is less than or equal to $w0.01$, we will reject H_0: $\mu 1 = \mu 2$.**

7. i. Combine the data for both airlines and rank them in ascending order, assigning average ranks for any tied observations.

Value	Airline	Rank
–4	Delta	1
–2	Alaska	2
–1	Alaska	3
0	Delta	4
1	Delta	5
3	Alaska	6
4	Alaska	7
8	Alaska	8
9	Delta	9
10	Delta	10
13	Delta	11
15	Alaska	12

Non parametric testing the Difference in Two Means – another example

7. ii. Calculate the Rank Sums:

Since both samples have size, we can take anyone as $w1$

- Rank sum for Delta: 1+4+5+9+10+11 =**40**
- iii. Then calculate

$$w_2 = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} - w_1$$

Therefore, w2= ((6+6)(6+6+1) /2)-40=**38**

8. **Conclusion: Because both *w*1 and *w*2 are less than or equal to *w*0.01 = 71, we reject the null hypothesis.**

10-2 Inference on the Variances of Two Normal Distributions

Suppose that two independent normal populations are of interest when the population means and variances $\mu_1, \sigma_1^2, \mu_2,$ and σ_2^2, are unknown.

We wish to test the hypotheses

$$H_0: \sigma_1^2 = \sigma_2^2$$
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

The development of a test procedure for these hypotheses requires a new probability distribution, the F distribution. The random variable F is defined to be the ratio of two independent chi-square random variables, each divided by its number of degrees of freedom. That is,

$$F = \frac{W/u}{Y/v}$$

where W and Y are independent chi-square random variables with u and v degrees of freedom, respectively.

10-2 Inference on the Variances of Two Normal Distributions

The mean and variance of the F distribution are $\mu = v/(v-2)$ for $v > 2$, and

$$\sigma^2 = \frac{2v^2(u+v-2)}{u(v-2)^2(v-4)}, \qquad v > 4$$

HYPOTHESIS TESTS ON THE RATIO OF TWO VARIANCES

Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ be a random sample from a normal population with mean μ_1 and variance σ_1^2, and let $X_{21}, X_{22}, \ldots, X_{2n_2}$ be a random sample from a second normal population with mean μ_2 and variance σ_2^2. Assume that both normal populations are independent. Let S_1^2 and S_2^2 be the sample variances. Then the ratio

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

has an F distribution with $n_1 - 1$ numerator degrees of freedom and $n_2 - 1$ denominator degrees of freedom.

10-2 Inference on the Variances of Two Normal Distributions

Null hypothesis: $H_0: \sigma_1^2 = \sigma_2^2$

Test statistic: $$F_0 = \frac{S_1^2}{S_2^2} \tag{10-31}$$

Alternative Hypotheses	Rejection Criterion
$H_1: \sigma_1^2 \neq \sigma_2^2$	$f_0 > f_{\alpha/2, n_1-1, n_2-1}$ or $f_0 < f_{1-\alpha/2, n_1-1, n_2-1}$
$H_1: \sigma_1^2 > \sigma_2^2$	$f_0 > f_{\alpha, n_1-1, n_2-1}$
$H_1: \sigma_1^2 < \sigma_2^2$	$f_0 < f_{1-\alpha, n_1-1, n_2-1}$

The lower-tailed percentage points $f_{1-\alpha, u, v}$ can be found as follows.

$$f_{1-\alpha,u,v} = \frac{1}{f_{\alpha,v,u}}$$

For example, to find the lower-tailed percentage point $f_{0.95,5,10}$, note that

$$f_{0.95,5,10} = \frac{1}{f_{0.05,10,5}} = \frac{1}{4.74} = 0.211$$

10-2 Inference on the Variances of Two Normal Distributions

Oxide layers on semiconductor wafers are etched in a mixture of gases to achieve the proper thickness. The variability in the thickness of these oxide layers is a critical characteristic of the wafer, and low variability is desirable for subsequent processing steps. Two different mixtures of gases are being studied to determine whether one is superior in reducing the variability of the oxide thickness. Sixteen wafers are etched in each gas. The sample standard deviations of oxide thickness are $s1 = 1.96$ angstroms and $s2 = 2.13$ angstroms, respectively. Is there any evidence to indicate that either gas is preferable? Use a fixed-level test with $\alpha = 0.05$.

Solution :

The eight-step hypothesis-testing procedure may be applied to this problem as follows:

1. **Parameter of interest:** The parameters of interest are the variances of oxide thickness σ_1^2 and σ_2^2. We will assume that oxide thickness is a normal random variable for both gas mixtures.
2. **Null hypothesis:** $H_0\text{: } \sigma_1^2 = \sigma_2^2$
3. **Alternative hypothesis:** $H_1\text{: } \sigma_1^2 \neq \sigma_2^2$
4. $\alpha = 0.05$

10-2 Inference on the Variances of Two Normal Distributions

5, **Test statistic:** The test statistic is given

$$f_0 = \frac{s_1^2}{s_2^2}$$

6. **Reject H_0 if:** Because $n_1 = n_2 = 16$ and $\alpha = 0.05$, we will reject $H_0: \sigma_1^2 = \sigma_2^2$ if $f_0 > f_{0.025,15,15} = 2.86$ or if $f_0 < f_{0.975,15,1}$
$= 1/f_{0.025,15,15} = 1/2.86 = 0.35$. Refer to Figure 10-6(a).

7 **Computations:** Because $s_1^2 = (1.96)^2 = 3.84$ and $s_2^2 = (2.13)^2 = 4.54$, the test statistic is

$$f_0 = \frac{s_1^2}{s_2^2} = \frac{3.84}{4.54} = 0.85$$

8 **Conclusion:** Because $f_{0.975,15,15} = 0.35 < 0.85 < f_{0.025,15,15} = 2.86$, we cannot reject the null hypothesis H_0: $\sigma_1^2 = \sigma_2^2$ a
the 0.05 level of significance.

11.3 Hypothesis tests on simple linear regression

In simple linear regression, hypothesis testing is used to determine whether the relationship between the independent variable (X) and the dependent variable (Y) is statistically significant. This involves testing hypotheses for the regression coefficients.

The expected value of Y for each value of x is $E(Y|x) = \beta_0 + \beta_1 x$

Y, can be described by the model $Y = \beta_0 + \beta_1 x + \epsilon$

The least squares estimators of β_0 and β_1, say, $\hat{\beta}_0$ and $\hat{\beta}_1$

11.3 Hypothesis tests on simple linear regression

The **least squares estimates** of the intercept and slope in the simple linear regression model are

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{11-7}$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - \frac{\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} \tag{11-8}$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$.

11.3 Hypothesis tests on simple linear regression

Given data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, let

$$S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}$$

$$S_{xy} = \sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}$$

Substitute Sxx, Sxy in 11-18, Hence,

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

11.3 Hypothesis tests on simple linear regression

For Slope

The appropriate hypotheses for the slope are

$$H_0\!: \beta_1 = \beta_{1,0} \qquad\qquad H_1\!: \beta_1 \neq \beta_{1,0}$$

Test Statistic for the Slope is

$$\boxed{T_0 = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\hat{\sigma}^2 / S_{xx}}}}$$

follows the t distribution with $n - 2$ degrees of freedom under $H_0\!: \beta_1 = \beta_{1,0}$. We would reject $H_0\!:\beta_1 = \beta_{1,0}$ if

$$|t_0| > t_{\alpha/2,n-2}$$

11.3 Hypothesis tests on simple linear regression

For intercept

$$H_0: \beta_0 = \beta_{0,0} \qquad H_1: \beta_0 \neq \beta_{0,0}$$

Test Statistic for the Slope is

$$T_0 = \frac{\hat{\beta}_0 - \beta_{0,0}}{\sqrt{\hat{\sigma}^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right]}} = \frac{\hat{\beta}_0 - \beta_{0,0}}{se\left(\hat{\beta}_0\right)}$$

and reject the null hypothesis if the computed value of this test statistic, t0, is such that

$$|t_0| > t_{\alpha/2, n-2}$$

11.3 Hypothesis tests on simple linear regression – an example

We will test for significance of regression using the model for the oxygen purity data shown in the table : α = 0.01.

Observation Number	Hydrocarbon Level $x(\%)$	Purity $y(\%)$
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	89.54
15	1.11	89.85
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20	0.95	87.33

The hypotheses are

$$H_0: \beta_1 = 0 \qquad H_1: \beta_1 \neq 0$$

$$t_{0.005,18} = 2.88,$$

$$\hat{\beta}_1 = 14.947 \quad n = 20, \quad S_{xx} = 0.68088, \quad \hat{\sigma}^2 = 1.18$$

T-statistics :

$$t_0 = \frac{\hat{\beta}_1}{\sqrt{\hat{\sigma}^2 / S_{xx}}} = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)} = \frac{14.947}{\sqrt{1.18/0.68088}} = 11.35$$

Because the reference value of t is $t_{0.005,18} = 2.88,$ the value of the test statistic is very far into the critical region, implying that H0: β1 = 0 should be rejected.

Unit V- Chapter 9

Book Back problems

Topic : Hypothesis testing

Formula:

$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$ → **significance level**

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}) \qquad (9\text{-}4)$$

Problems:

1. State whether each of the following situations is a correctly stated hypothesis testing problem and why.

   (a) H_0: $\mu = 25$, H_1: $\mu \neq 25$ → Correctly stated hypothesis

   (b) H_0: $\sigma > 10$, H_1: $\sigma = 10$ → **Incorrectly stated hypothesis**: No.
   The null hypothesis (H_0) should typically represent a statement of "no effect" or "no difference," and it is usually a statement of equality (e.g., $\sigma = 10$).

   (c) H_0: $\ddot{X} = 50$, H_1: $\ddot{X} \neq 50$ → **Incorrectly stated hypothesis**: No.
   The null hypothesis (H0) should refer to a population parameter (e.g., the population mean, population proportion, or population standard deviation), not a sample statistic ($\ddot{X}$).

   (d) H_0: $p = 0.1$, H_1: $p = 0.5$ → **Incorrectly stated hypothesis**: No.
   This setup is problematic because both hypotheses state specific values for the population proportion (p). Having two specific values for p in the null and alternative hypotheses (0.1 vs. 0.5) is not a standard way to frame hypothesis tests.

   (e) H_0 : $s = 30$, H_1 : $s > 30$ → **Incorrectly stated hypothesis**: No.
   The null hypothesis (H0) should refer to a population parameter (e.g., the population mean, population proportion, or population standard deviation), not a sample statistic (s).

2. A textile fiber manufacturer is investigating a new drapery yarn, which the company claims has a mean thread elongation of 12 kilograms with a standard deviation of 0.5 kilograms. The company wishes to test the hypothesis H0: $\mu = 12$ against H1: $\mu < 12$, using a random sample of four specimens.

   (a) What is the type I error probability if the critical region is defined as $x < 11.5$. kilograms?
   (b) Find β for the case in which the true mean elongation is 11.25 kilograms.

(a) Population mean (μ) = 12 kg

Population standard deviation (σ) = 0.5 kg

Sample size (n) = 4

Critical region defined as ($\ddot{x}$<11.5kg)

The standard error of the mean (SE) is given by:
SE=σ/sqrt(n) = 0.5/(sqrt(4)) = 0.25

Calculate the Z-score for the Critical Value
We need to find the Z-score that corresponds to the sample mean ($\ddot{x}$=11.5) under the null hypothesis.

$$Z = \frac{\bar{x} - \mu}{SE}$$

Substitute the values:

$$Z = \frac{11.5 - 12}{0.25} = \frac{-0.5}{0.25} = -2$$

Find the Type I Error Probability (α)
The type I error probability is the probability that the sample mean falls in the critical region $\ddot{x}$<11.5 when μ=12. This corresponds to the probability of $Z<-2$ under the standard normal distribution.
Using a standard normal distribution table or statistical software, we find:

$$P(Z<-2) \approx 0.0228$$

(b) Find β for the case in which the true mean elongation is 11.25 kilograms.
Population mean (μ) = 12 kg

Population standard deviation (σ) = 0.5 kg

Sample size (n) = 4

Critical region defined as ($\ddot{x}$<11.5kg)

True mean (μ_{true}) = 11.25 kg

Calculate the Z-score for the True Mean: To find β, we need to calculate the Z-score for $\ddot{x}$=11.5 when μ_{true}=11.25

$$Z = \frac{\bar{x} - \mu_{true}}{SE}$$

Substitute the values:

$$Z = \frac{11.5 - 11.25}{0.25} = \frac{0.25}{0.25} = 1$$

Find the Probability of Type II Error (β): The type II error probability is the probability of $\ddot{x}$>11.5 when μ_{true}=11.25. This corresponds to $P(Z>1)$ for the standard normal distribution.

Using a standard normal distribution table :

$$P(Z>1)\approx 1-0.8413=0.1587$$

3. The heat evolved in calories per gram of a cement mixture is approximately normally distributed. The mean is thought to be 100, and the standard deviation is 2. You wish to test $H0:\mu = 100$ versus $H1:\mu \neq 100$ with a sample of $n = 9$ specimens.
   (a) If the acceptance region is defined as $98.5 \leq \ddot{x} \leq 101.5$, find the type I error probability α. (b) Find β for the case in which the true mean heat evolved is 103

(a). **Given Data**:

- Population mean (μ) = 100
- Population standard deviation (σ) = 2
- Sample size (n) = 9
- Acceptance region: $98.5 \leq \ddot{x} \leq 101.5$

Calculate the Standard Error of the Mean (SEM):

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{2}{\sqrt{9}} = \frac{2}{3} \approx 0.667$$

Calculate the Z-scores for the Acceptance Region Boundaries:

For the lower bound (=98.5):

$$Z = \frac{98.5-100}{0.667} \approx \frac{-1.5}{0.667} \approx -2.25$$

For the upper bound ($\ddot{x}$ =101.5):

$$Z = \frac{101.5-100}{0.667} \approx \frac{1.5}{0.667} \approx 2.25$$

Find the Type I Error Probability (α): The type I error probability is the probability of observing a sample mean outside the acceptance region. This is represented by:

$$\alpha=P(Z<-2.25)+P(Z>2.25)$$

Using a standard normal distribution table,

$P(Z<-2.25)\approx 0.0122 \text{and} P(Z>2.25)\approx 0.0122$

Therefore:

$\alpha=0.0122+0.0122=0.0244$

Therefore , The type I error probability (α\alphaα) is approximately **0.0244** or **2.44%**.

A. ONE SAMPLE

Tests on the Mean of a Normal Distribution, Variance Known / variances unknown and test on standard deviation/ variance:

1. Tests on the Mean of a Normal Distribution, Variance Known

a. Test Procedure :

1. From the problem context, identify the parameter of interest.
2. State the null hypothesis, H_0.
3. Specify an appropriate alternative hypothesis, H_1.
4. Choose a significance level, α
5. Determine an appropriate test statistic.
6. State the rejection region for the statistic (based on test statistic or p-value).
7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
8. Decide whether or not H_0 should be rejected and report that in the problem context.

b. Formula :

Test statistics :

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \tag{9-8}$$

c. Tests on Mean of a Normal Distribution, Variance Known can be conducted by two approaches

i. Approach 1 :Hypothesis Tests on the Mean

1. Reject H_0 if the observed value of the test statistic z_0 is either: (two sided)

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

$$z_0 > z_{\alpha/2} \text{ or } z_0 < -z_{\alpha/2}$$

Fail to reject H_0 if

$-z_{\alpha}/2 < z_0 < z_{\alpha}/2$

2. One sided (upper)

$$H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0$$

Reject H_0 : if $z_0 > z_\alpha$

Fail to reject H_0 if : $z_\alpha < z_0$

3. One sided (lower)

$$H_0: \mu = \mu_0$$
$$H_1: \mu < \mu_0$$

Reject H_0 : if $z_0 < -z_\alpha$

Fail to reject H_0 if : $-z_\alpha < z_0$

ii. Approach 2 :P-value Tests on the Mean **(p-value is the smallest level of significance at which you would be willing to reject the null hypothesis)**

$$P = \begin{cases} 2[1 - \Phi(|z_0|)] & \text{for a two-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for a upper-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu < \mu_0 \end{cases} \quad (9\text{-}15)$$

Problems :

1. A hypothesis will be used to test that a population mean equals 7 against the alternative that the population mean does not equal 7 with known variance σ. What are the critical
   values for the test statistic Z_0 for the following significance levels?
   (a) 0.01 (b) 0.05 (c) 0.10.
   (a) **Significance Level: α=0.01**

- For a two-tailed test, split α into two tails: α/2=0.005
  The corresponding critical values for Z are found using the standard normal table (z-table) : $Z_0=\pm 2.57$
  (b) **Significance Level: α=0.05**
- For a two-tailed test, split α into two tails: α/2=0.025
  The corresponding critical values for Z are found using the standard normal table (z-table) : $Z_0=\pm 1.96$
- For a two-tailed test, split α into two tails: α/2=0.05
  The corresponding critical values for Z are found using the standard normal table (z-table) : $Z_0=\pm 1.64$

2. A hypothesis will be used to test that a population mean equals 10 against the alternative that the population mean is more than 10 with known variance σ. What is the critical value for the test statistic Z0 for the following significance levels?
   (a) 0.01 (b) 0.05 (c) 0.10

- For a one-tailed test, α: 0.01
  The corresponding critical values for Z are found using the standard normal table (z-table) : $Z_0=2.33$
- For a one-tailed test, α: 0.05

The corresponding critical values for Z are found using the standard normal table (z-table) : Z_0=2.33

- For a one-tailed test, α: 1.64

  The corresponding critical values for Z are found using the standard normal table (z-table) : Z_0=1.28

3. For the hypothesis test $H_0 : \mu = 7$ against $H_1: \mu \neq 7$ and variance known, calculate the P-value for each of the following test statistics.

(a) z_0 =2.05 (b) z_0 =−1.84 (c) z_0 = 0.4

To calculate the p-value for a two-tailed hypothesis test $H_0 : \mu = 7$ vs. $H_1 : \mu \neq 7$, the p-value is calculated as:

$$p\text{-value} = 2 \times P(Z > |z_0|)$$

(a)

Find the area to the right of $z_0 = 2.05$ in the standard normal distribution:

$$P(Z > 2.05) = 1 - P(Z \leq 2.05)$$

Using the standard normal table, $P(Z \leq 2.05) = 0.9798$.

Therefore:

$$P(Z > 2.05) = 1 - 0.9798 = 0.0202$$

(b)

Find the area to the right of $z_0 = -1.84$. Using symmetry of the standard normal distribution:

$$P(Z > -1.84) = P(Z < 1.84)$$

From the standard normal table, $P(Z \leq 1.84) = 0.9671$.

Therefore:

$$P(Z > -1.84) = 1 - 0.9671 = 0.0329$$

Since this is a two-tailed test:

$$p\text{-value} = 2 \times 0.0329 = 0.0658$$

(c)

$$z_0 = 0.4$$

Find the area to the right of $z_0 = 0.4$:

$$P(Z > 0.4) = 1 - P(Z \leq 0.4)$$

From the standard normal table, $P(Z \leq 0.4) = 0.6554$.

Therefore:

$$P(Z > 0.4) = 1 - 0.6554 = 0.3446$$

Since this is a two-tailed test:

$$p\text{-value} = 2 \times 0.3446 = 0.6892$$

4. For the hypothesis test H0 : μ = 10 against H1:μ >10 and variance known, calculate the P-value for each of the following test statistics.
(a) z0 =2.05 (b) z_0 =− 1.84 (c) z0 = 0.4

For the hypothesis test $H_0: \mu = 10$ vs. $H_1: \mu > 10$ with known variance, this is a **one-tailed test**. The p-value is calculated as:

$$p\text{-value} = P(Z > z_0)$$

(a)

Find the area to the right of $z_0 = 2.05$ in the standard normal distribution:

$$P(Z > 2.05) = 1 - P(Z \leq 2.05)$$

Using the standard normal table, $P(Z \leq 2.05) = 0.9798$.

Therefore:

$$P(Z > 2.05) = 1 - 0.9798 = 0.0202$$

p-value = 0.0202

(b)

Find the area to the right of $z_0 = -1.84$:

$$P(Z > -1.84)$$

Using the standard normal table, $P(Z \leq -1.84) = 0.0329$.

Therefore:

$$P(Z > -1.84) = 1 - 0.0329 = 0.9671$$

p-value = 0.9671

(c)

Find the area to the right of $z_0 = 0.4$:

$$P(Z > 0.4) = 1 - P(Z \leq 0.4)$$

From the standard normal table, $P(Z \leq 0.4) = 0.6554$.

Therefore:

$$P(Z > 0.4) = 1 - 0.6554 = 0.3446$$

p-value = 0.3446

(5) For the hypothesis test H0 : μ = 5 against H < 1:μ 5 and variance known, calculate the P-value for each of the following test statistics. (a) z0 =2.05

For the hypothesis test $H_0 : \mu = 5$ vs. $H_1 : \mu < 5$ with known variance, this is a **one-tailed test** (lower tail). The p-value is calculated as:

$$p\text{-value} = P(Z < z_0)$$

Find $P(Z < 2.05)$ using the standard normal table:

$$P(Z \leq 2.05) = 0.9798$$

Since this is a lower-tailed test, the p-value is:

$$p\text{-value} = P(Z < 2.05) = 0.9798$$

(6) The mean water temperature downstream from a discharge pipe at a power plant cooling tower should be no more than 100°F. Past experience has indicated that the standard deviation of temperature is 2°F. The water temperature is measured on nine randomly chosen days, and the average temperature is found to be 98°F.

(a) Is there evidence that the water temperature is acceptable at α = 0.05?

(b) What is the P-value for this test?

(c) What is the probability of accepting the null hypothesis at α = 0.05 if the water has a true mean temperature of 104°F?

Given :

Null Hypothesis: $H_0 : \mu = 100$ (mean temperature is 100°F).

Alternative Hypothesis: $H_1 : \mu < 100$ (mean temperature is less than 100°F).

Significance Level: $\alpha = 0.05$.

Standard Deviation: $\sigma = 2$°F (known).

Sample Size: $n = 9$.

Sample Mean: $\bar{x} = 98$°F.

(a) Is there evidence that the water temperature is acceptable at α=0.05?

Use eight step procedure

i. The parameter of interest is the **population mean temperature** (μ) of the water in the cooling tower

ii. State the null hypothesis $H_0 : \mu=100$

iii. State the alternate hypothesis $H_1 : \mu<100$

iv. The significance level $\alpha=0.05$

v. Determine an appropriate test statistic

$$Z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

vi. State the rejection criteria : Reject H_0 if $Z_0 \leq Z_{critical}$

From the standard normal distribution table, the critical value for α=0.05 is: $Z_{critical}$=−1.645. So, reject Reject H_0 if $Z_0 \leq -1.645$

vii. Compute sample quantities and the test statistic

$$Z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{98 - 100}{2/\sqrt{9}} = \frac{-2}{2/3} = -3$$

viii. Decide whether or not to reject H_0

Compare Z_0 to Z_{critical}:

$$Z_0 = -3 \quad \text{and} \quad Z_{\text{critical}} = -1.645$$

Since $Z_0 \leq Z_{\text{critical}}$, we **reject** H_0.

Conclusion: At α=0.05, there is sufficient evidence to conclude that the mean water temperature is significantly less than 100°F, indicating that the water temperature is acceptable.

(b) To calculate P-value for this test

Critical values : $Z_0 \leq -1.645$ and calculated Z_0 is -3 from (a)
p-value is (P(Z)) → P(Z< -3) that is, we want to test lower sided.
From Z table.
P(Z<-3) =0.001350
Since 0.001350 is less than α (0.001350<0.05), we reject NULL hypothesis

2. Tests on the Mean of a Normal Distribution, Variance Unknown (t-test)

Follow the 8-step procedure as mentioned in the previous section.

Formula :

Approach 1 :Hypothesis Tests on the Mean

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $T_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}}$

Alternative hypothesis	Rejection criteria
$H_1\colon \mu \neq \mu_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1\colon \mu > \mu_0$	$t_0 > t_{\alpha,n-1}$
$H_1\colon \mu < \mu_0$	$t_0 < -t_{\alpha,n-1}$

Approach 2 :P-value Tests on the Mean with unknown variance:

For two-sided: find the range of p-value from t-table, then multiply 2 with both lower and upper bound.

Upper sided : for +ve t_0, the p-value obtained from t-table will be the answer. For -ve, subtract the obtained -p-value from1.

Lower sided : for -ve t_0, the p-value obtained from t-table will be the answer. For +ve, subtract the obtained -p-value from1.

Method to look for -p-value in t table:

For example, for t_0 = 2.72, this is between two tabulated values, 2.624 and 2.977 in t table. Therefore, the P-value must be between 0.01 and 0.005. : 0.005<P-value<0.01.

Critical Value:	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
Tail Area:	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005

Suppose t_0 =2.72 for a two-sided test, then

0.005*2<*P*-value<0.01*2 → 0.01<*P*-value<0.02

Problems:

1. A hypothesis will be used to test that a population mean equals 7 against the alternative that the population mean does not equal 7 with unknown variance. What are the critical values for the test statistic T_0 for the following significance levels and sample sizes?

(a) α = 0.01 and *n* = 20

(b) α = 0.05 and *n* = 12

(c) α = 0.10 and *n* = 15

(a) **α=0.01 and n=20:**

- Degrees of freedom: df=n−1=20−1=19.
- Significance level in one tail: α/2=0.01/2=0.005
- From the t-distribution table for df=19 :

  $t_{0.005,19}$=2.861

  critical values are : $\pm t_{0.005,19}$=±2.861.

(b) **α=0.05 and n=12:**

- Degrees of freedom: df=n−1=12-1=11.
- Significance level in one tail: α/2=0.05/2=0.025
- From the t-distribution table for df=11 :

  $t_{0.025,11}$=2.201

  critical values are : $\pm t_{0.025,11}$=±2.201.

(c) α = 0.10 and *n* = 15

Degrees of freedom: df=n−1=15−1=14.

Significance level in one tail: α/2=0.10/2=0.05.

From the t-distribution table for df=14:

t0.05,14=2.145.

Critical values are: ±t0.05,14=±2.145.

2. A hypothesis will be used to test that a population mean equals 10 against the alternative that the population mean is greater than 10 with unknown variance. What is the critical value for the test statistic *T*0 for the following significance levels?

(a) $\alpha = 0.01$ and $n = 20$

(b) $\alpha = 0.05$ and $n = 12$

(c) $\alpha = 0.10$ and $n = 15$

In a one-tailed test, the critical value corresponds to the significance level (α) in **one tail** of the t-distribution.

Locate $t_{\alpha,df}$ for the given α and df

The critical value is $t_{\alpha,df}$, such that: $P(T \geq t_{\alpha,df}) = \alpha$.

(a) α=0.01 and n=20:
Degrees of freedom: $df = n-1 = 20-1 = 19$
Significance level: $\alpha = 0.01$
From the t-distribution table for df=19
$t_{0.01,19} = 2.539$
The critical value is:
$t_0 = 2.539$.

similarly do for (b) and (c)

3. A hypothesis will be used to test that a population mean equals 5 against the alternative that the population mean is less than 5 with unknown variance. What is the critical value for the test statistic T_0 for the following significance levels?
(a) $\alpha = 0.01$ and $n = 20$
(b) $\alpha = 0.05$ and $n = 12$
(c) $\alpha = 0.10$ and $n = 15$

From a t-table or software, locate $t_{\alpha,df}$ for the given α and df. Note that the t-value for the **left tail** will be negative, so we report it as $-t_{\alpha,df}$.
(a) **α=0.01 and n=20:**
Degrees of freedom: $df = n-1 = 20-1 = 19$
Significance level: $\alpha = 0.01$
From the t-distribution table for df=19
$t_{0.01,19} = 2.539$
The critical value is:
$t_0 = -2.539$.

similarly do for (b) and (c)

4. For the hypothesis test H0: μ = 7 against H1: μ ≠ 7 with variance unknown and n = 20, approximate the P-value for each of the following test statistics.
(a) t0 = 2.05 (b) t0 = − 1.84.

1) $n = 20$

degree of freedom $= 19$

$t_0 = \underline{2.05}$

look in table 19th row and column values ~~that~~ ~~to~~ ~~before~~ where 2.05 lies between,

$\Rightarrow$ 2.05 lies between 0.05 & 0.025

$\therefore 0.025 < p < 0.05$

for two sided,

$0.025 \times 2 < p < 0.05 \times 2$

$0.05 < p < 0.1$

$n = 20$

degree $= 19$

$t_0 = -1.84$

Since 't' distribution is ~~[illegible]~~ symmetric,

Consider 1.84, use t-table

1.84 lies between 0.05 & 0.025

$\therefore 0.025 \times 2 < p < 0.05 \times 2$

$\therefore 0.05 < p < 0.1$

5. For the hypothesis test H0: μ = 5 against H1: μ < 5 with variance unknown and n = 12, approximate the P-value for each of the following test statistics.
(a) t0 = 2.05 (b) t0 = − 1.84

$n = 20$

degree $= 19$

$t_0 = 2.05$

$H_0: \mu = 5$

$H_1: \mu < 5$

2.05 lies between

$0.025 < P < 0.05$

Since t_0 is +ve, it indicates the result in opposite direction,

So the range is

$1 - 0.025 < P < 1 - 0.05$

$0.975 < P < 0.95$

rearranging,

$0.95 < P < 0.975$

$n = 20$

degree $= 19$

$t_0 = \cancel{1.8}\ -1.84$

~~$\Rightarrow 0.05 < p < 0.1$~~

$\Rightarrow 0.025 < p < 0.05$

Since t_0 is −ve, and we look for lower sided prob, keep the prob. as such.

6. For the hypothesis test H0: $\mu = 10$ against H1: $\mu > 10$ with variance unknown and $n = 15$, approximate the P-value for each of the following test statistics.
(a) $t_0 = 2.05$ (b) $t_0 = -1.84$

$n = 15$
degrees = 14
$t_0 = 2.05$
$H_0: \mu_1 = 10$
$H_1: \mu_0 > 10$
2.05 lies between
$0.025 < p < 0.05$

$n = \cancel{20}\ 15$ $\quad H_0: \mu_0 = 10$
degree = 14 $\quad H_1 > 10$
$t_0 = -1.84$
Same as previous problem,
$\therefore 0.025 < p < 0.05$

Since t_0 is $-$ve and hypothesis is upper sided, subtract the prob from 1

$1 - 0.025 < p < 1 - 0.05$
$0.975 < p < 0.95$
Rearrange,
$0.95 < p < 0.975$

7. The results of a study that measured the body weight (in grams) for guinea pigs at birth is given below:
   421.0 452.6 456.1 494.6 373.8 90.5 110.7 96.4 81.7 102.4 241.0 296.0 317.0 290.9 256.5 447.8 687.6 705.7 879.0 88.8 296.0 273.0 268.0 227.5 279.3 258.5 296.0.
   a) Test the hypothesis that mean body weight is 300 grams. Use $\alpha = 0.05$.
   b) What is the smallest level of significance at which you would be willing to reject the null hypothesis?
   (a) apply the eight step procedure
   1 : The parameter of interest is the population mean body weight of guinea pigs at birth (μ).
   2. The null hypothesis is: $H_0{:}\mu=300$
   3. The alternative hypothesis is: $H_1{:}\mu\neq300$
   4. The level of significance is chosen as: $\alpha=0.05$
   5. Test statistic for a one-sample t-test is:

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

6. The rejection region is based on a two-tailed t-test with df=n−1=27−1=26

The critical t-values are:

$t_{0.025,26}=\pm 2.056$

Reject H_0 if: $t_0<-2.056$ and $t_0 > 2.056$

7. Compute Necessary Quantities and Test Statistic

7.1: Sample Mean ($\bar{x}$)

$$\bar{x} = \frac{\sum x_i}{n} = 325.50 \text{ grams.}$$

7.2: Sample Standard Deviation (s)

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = 198.79 \text{ grams.}$$

7.3: Compute Test Statistic (t_0)

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{325.50 - 300}{198.79/\sqrt{27}} = 0.666.$$

8. **Conclusion : Decide Whether or Not to Reject H_0**

The test statistic is $t_0=0.666$

The critical t-values are ±2.056 .

Since −2.056≤0.666≤2.056 we **fail to reject the null hypothesis**.

b) What is the smallest level of significance at which you would be willing to reject the null hypothesis?

This means we need to compute p-value to decide whether H_0 will be rejected or not

$t_0=0.666$ from (a)

look in the t table ,

for degree 26, 0.666 lies between 0.256 and 0.684. The corresponding values at the top is 0.40 and 0.25.

24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690

So, p-value lies between 0.40 and 0.25

for a two-sided test, then multiply both bounds by 2 ➔

0.25*2 <= P <=0.40*2 → 0.50<=P<=0.80.

Since P> α, (=0.05), **we fail to reject.H_0.**

Both (a) and (b) give same result.

Tests on the Variance and Standard Deviation of a Normal Distribution

3. **Tests on the Variance and Standard Deviation of a Normal Distribution (chi-square test)**

Follow the 8-step procedure as mentioned in the previous section.

Formula :

Approach 1 :Hypothesis Tests on the Mean

Null hypothesis: $H_0: \sigma^2 = \sigma^2_0$

Test statistic: $\chi^2_0 = \dfrac{(n-1)S^2}{\sigma^2_0}$

Alternative Hypothesis	Rejection Criteria
$H_1: \sigma^2 \neq \sigma^2_0$	$\chi^2_0 > \chi^2_{\alpha/2,n-1}$ or $\chi^2_0 < \chi^2_{1-\alpha/2,n-1}$
$H_1: \sigma^2 > \sigma^2_0$	$\chi^2_0 > \chi^2_{\alpha,n-1}$
$H_1: \sigma^2 < \sigma^2_0$	$\chi^2_0 < \chi^2_{1-\alpha,n-1}$

Approach 2 :P-value Tests on the Variance and Standard deviation

From χ^2 table, locate the row that corresponds to degree of freedom and find the range within which χ^2_0 falls. And then, find the corresponding α value that will be the range of p-value.

1. Consider the test of H0 :σ^2=7 against :$\sigma^2 \neq 7$. What are the critical values for the test statistic χ^2_0 for the following significance levels and sample sizes?
   (a) α = 0.01 and n = 20
   (b) α = 0.05 and n = 12
   (a) **α=0.01, n=20:**
   - Degrees of freedom: df=n−1=19
   - α/2=0.01/2=0.005
   - critical values:
     - Upper critical value: $\chi^2_{0.995,19}$
     - Lower critical value: $\chi^2_{0.005,19}$

   (B) **α=0.05, n=12:**
   - Degrees of freedom: df=n−1=11
   - α/2=0.05/2=0.025
   - critical values:
     - Upper critical value: $\chi^2_{0.025,11}$
     - Lower critical value: $\chi^2_{0.975,11}$

2. Consider the test of H0 :σ_2=10 against :σ_2>10. What are the critical values for the test statistic χ^2_0 for the following significance levels and sample sizes?
   (a) α = 0.01 and n = 20
   (b) α = 0.05 and n = 12
   (a) **α=0.01, n=20:**
   - Degrees of freedom: df=n−1=19

- α=0.01
- critical values:
  - Upper critical value: $\chi^2_{0.01,19}$

(b) **α=0.05, n=12:**

- Degrees of freedom: df=n−1=11
- α=0.05
- critical values:
  - Upper critical value: $\chi^2_{0.05,11}$

3. Consider the test of H0 :σ_2=5 against :σ_2<5. What are the critical values for the test statistic χ^2_0 for the following significance levels and sample sizes?
   (a) α = 0.01 and n = 20
   (b) α = 0.05 and n = 12
   (a) **α=0.01, n=20:**
   - Degrees of freedom: df=n−1=19
   - α=0.01
   - critical values:
     - lower critical value: $\chi^2_{0.05,19}$

   (b) **α=0.05, n=12:**
   - Degrees of freedom: df=n−1=11
   - α=0.05
   - critical values:
     - lower critical value: $\chi^2_{0.95,11}$

4. Consider the hypothesis test of H0:σ^2 = 7 against H1:$\sigma^2 \neq 7$. Approximate the P-value for each of the following test statistics.
   (a) χ^2_0= 25.2 and n = 20
   (b) χ^2_0 = 15.2 and n = 12

   (a) χ^2_0= 25.2 and n = 20, degree of freedom = 19,
   25.2 falls between 18.34 and 27.2. The corresponding α is 0.100 and 0.500.
   Therefore p-value is → 0.100 <=p<= 0.500.

   (b) χ^2_0= 15.2 and n = 12, degree of freedom = 11,
   15.2 falls between 10.34 and 17.2. The corresponding α is 0.100 and 0.500.
   Therefore p-value is → 0.100 <=p<= 0.500.

5. There were 17 players, and the sample standard deviation of performance was 0.09 seconds. (a) Is there strong evidence to conclude that the standard deviation of performance time exceeds the historical value of 0.75 seconds? Use α = 0.05.
   Eight steps:
   1. The parameter of interest is the **population variance** (σ^2), specifically testing whether it exceeds the historical value of $\sigma^2=0.75^2$

2. The null hypothesis is: $H0:\sigma^2=0.75^2$
3. The alternate hypothesis is: $H1:\sigma^2>0.75^2$
4. $\alpha=0.05$.
5. The test statistic for testing the variance is:

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

6. The rejection region for the test is in the upper tail of the chi-square distribution. We reject H0 if: $\chi_0^2>\chi_{\alpha,df}$. that is $\chi_0^2>\chi_{0.05,16}$ and $\chi_{0.05,16} = 26.3$
7. Compute any necessary sample quantities and the test statistic.

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{16 \times 0.09^2}{0.75^2}.$$

$$\chi_0^2 = \frac{16 \times 0.0081}{0.5625} = \frac{0.1296}{0.5625} \approx 0.2304.$$

8. Conclusion : Decide whether or not H0 should be rejected. Compare the test statistic to the critical value:
   $\chi_{0.05,16} = 26.3$ and $\chi_0^2=0.2304$
   since 0.2304 is much less than 26.3, that is $\chi_0^2<\chi_{0.05,16,}$ we fail to reject H0

4. Nonparametric test procedures

4.1 Wilcoxon Signed-Rank Test

Example 1 : The titanium content in an aircraft-grade alloy is an important determinant of strength. A sample of 20 test coupons reveals the following titanium content (in percent):

8.32, 8.05, 8.93, 8.65, 8.25, 8.46, 8.52, 8.35, 8.36, 8.41, 8.42, 8.30, 8.71, 8.75, 8.60, 8.83, 8.50, 8.38, 8.29, 8.46. The median titanium content should be 8.5%.

Use the Wilcoxon signed-rank test with $\alpha = 0.05$ to investigate this hypothesis.

Solution :

1. The parameter of interest is the **population mean** (or median),
2. The null hypothesis is: $H0:\mu =8.5$
3. The alternate hypothesis is: $H1:\mu >8.5$
4. $\alpha=0.05$.
5. The test statistic for testing the mean is $W = \min(W+ , W-)$
6. Reject H_0 if $W <=W_{0.05}$, $W_{0.05=}$ 52 (from Wilcoxon Signed Rank test table)
7. Computations:

Data	Difference (Xi-mean)	Absolute value	Rank
8.5	0	0	-
8.52	0.02	0.02	1
8.46	-0.04	0.04	-2.5
8.46	-0.04	0.04	-2.5
8.42	-0.08	0.08	-4
8.41	-0.09	0.09	-5
8.6	0.1	0.1	6
8.38	-0.12	0.12	-7
8.36	-0.14	0.14	-8
8.65	0.15	0.15	9.5
8.35	-0.15	0.15	-9.5
8.32	-0.18	0.18	-11
8.3	-0.2	0.2	-12
8.71	0.21	0.21	13.5
8.29	-0.21	0.21	-13.5
8.25	-0.25	0.25	-15.5
8.75	0.25	0.25	15.5
8.83	0.33	0.33	16
8.93	0.43	0.43	17
8.05	-0.45	0.45	-18

The sum of the positive ranks is w+ = 1+6+9.5+13.5+15.5+16+17=78.5

The sum of the negative ranks is w+ = 2.5+2.5+4+5+7+8+9.5+11+12+13.5+15.5+18 =108.5

W = min (78.5, 108.5)= 78.5

8. Conclusions: Because W = 78.5 is not less than or equal to the critical value w0.05 = 52, we fail to reject the null hypothesis

B. TWO SAMPLES TEST

Three types of tests

- **Inference on the Difference in Means of two Normal Distributions, Variances known**
- **Inference on the Difference in Means of two Normal Distributions, Variances Unknown**
- **Inference on the Variances of Two Normal Distributions**

Inference on the Difference in Means of two Normal Distributions, Variances known

Approach 1 :HYPOTHESIS TESTS ON THE **DIFFERENCE IN MEANS**, VARIANCES KNOWN

Formula :

Null hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$

Test statistic:
$$Z_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad (10\text{-}2)$$

Alternative Hypotheses	Rejection Criterion
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	$z_0 < -z_\alpha$

Approach 2 :**P-value Tests on** THE DIFFERENCE IN **the Mean**

$$P = \begin{cases} 2[1 - \Phi(|z_0|)] & \text{for a two-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for a upper-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu < \mu_0 \end{cases} \qquad (9\text{-}15)$$

Confidence Interval on the Difference in Means, Variances Known

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Problems :

1. Consider the hypothesis test $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$ with known variances $\sigma_1 = 10$ and $\sigma_2 = 5$. Suppose that sample sizes n1 = 10 and n2 = 15 and that $\ddot{x}_1 = 4.7$ and $\ddot{x}_2 = 7.8$ Use α= . 0 05. (a) Test the hypothesis and find the P-value.
    1. The parameter of interest is the difference in means
    2. The null hypothesis is: $H_0 : \mu_1 = \mu_2$
    3. The alternate hypothesis is: $H_1 : \mu_1 \neq \mu_2$
    4. α=0.05.
    5. The test statistic for testing the variance is:

$$Z_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(or)

$$z_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

6. The rejection region for the test is

$$z_0 > z_{\alpha/2} \text{ or } z_0 < -z_{\alpha/2}$$

α=0.05, α/2=0.025

CI % = 95%

Therefore, $z_{\alpha/2}=1.96$

The critical values are $-1.96 <= z_0 <= 1.96$

7. Compute any necessary sample quantities and the test statistic.

$Z_0=$

$$\frac{4.7 - 7.8}{\sqrt{\frac{10^2}{10} + \frac{5^2}{15}}}$$

$$= \frac{4.7 - 7.8}{3.42} = \frac{-3.1}{3.42} \approx -0.906$$

8. Conclusion : The computed z=−0.906 does not fall in the rejection region (−1.96 to 1.96). Therefore, we fail to reject the null hypothesis.

(b) p-value is, for a two-tailed test, the P-value is: =2(1-P(|Z|))

Using a standard normal table, the cumulative probability for $z_0=0.906$ is approximately 0.8176. so,

P=2·(1−0.8176)=2·0.1824=0.3648

Since the P-value (0.3648) is greater than α=0.05, we fail to reject the null hypothesis.

2. Consider the hypothesis test $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 < \mu_2$ with known variances $\sigma_1 = 10$ and $\sigma_2 = 5$. Suppose that sample sizes n1 = 10 and n2 = 15 and that $\ddot{x}_1 = 14.2$ and $\ddot{x}_2 = 19.7$. Use α= . 0 05. (a) Test the hypothesis and find the P-value.

   Do as in Problem (1) (Above)
   Hint :

   Given:

   - Variances: $\sigma_1^2 = 10$, $\sigma_2^2 = 5$
   - Sample sizes: $n_1 = 10$, $n_2 = 15$
   - Sample means: $\bar{x}_1 = 4.7$, $\bar{x}_2 = 7.8$
   - Significance level: $\alpha = 0.05$

   $$Z = \frac{4.7 - 7.8}{1.1547} = \frac{-3.1}{1.1547} \approx -2.68$$

   $P(Z)$ for $|Z| = 2.68$ is approximately 0.9964.

   The two-sided p-value is:

   $$\text{p-value} = 2 \times (1 - 0.9964) \approx 2 \times 0.0036 = 0.00726$$

3. Two machines are used for filling plastic bottles with a net volume of 16.0 ounces. The fill volume can be assumed to be normal with standard deviation $\sigma_1 = 0.020$. and $\sigma_2 = 0.025$ ounces. A member of the quality engineering staff suspects that both machines fill to the same mean net volume, whether or not this volume is 16.0 ounces. A random sample of 10 bottles is taken from the output of each machine.
   (a) Do you think the engineer is correct? Use α = 0.05. What is the P-value for this test?
   (b) Calculate a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

   (a) Do as in problem (1) (ALL EIGHT STEPS to BE WRITTEN)
   Hints:

   Null Hypothesis (H_0): The mean fill volumes of Machine 1 and Machine 2 are the same:

   $\mu_1 = \mu_2$

   Alternative Hypothesis (H_a): The mean fill volumes of Machine 1 and Machine 2 are different:

   $\mu_1 \neq \mu_2$

   $$z = \frac{16.016 - 16.006}{0.0101} = \frac{0.010}{0.0101} = 0.494$$

   For a two-tailed test, the P-value is:

   $$P = 2 \cdot (1 - \Phi(|z|))$$

$$\Phi(0.494) \approx 0.6893$$

$$P = 2 \cdot (1 - 0.6893) = 2 \cdot 0.3107 = 0.621$$

Since $P = 0.621 > 0.05$, we fail to reject the null hypothesis.

(b) Calculate a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

From the z-table, $z_{0.025} = 1.96$. Substituting the values:

$$\text{Lower limit} = (16.016 - 16.006) - 1.96 \cdot 0.0101 = 0.010 - 0.0198 = -0.0148$$

$$\text{Upper limit} = (16.016 - 16.006) + 1.96 \cdot 0.0101 = 0.010 + 0.0198 = 0.0248$$

The 95% confidence interval is:

$$(-0.0148, 0.0248)$$

Inference on the Difference in Means of two Normal Distributions, Variances Unknown

2 cases :

$\sigma_1^2 = \sigma_2^2 = \sigma^2$ and $\sigma_1^2 \neq \sigma_2^2$

a. Case 1:

$\sigma^2_1 = \sigma^2_2$

The pooled estimator of σ^2, denoted by S_p^2, is defined by

$$\boxed{S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$$

Null hypothesis: $H_0\colon \mu_1 - \mu_2 = \Delta_0$

Test statistic:
$$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{10-14}$$

Alternative Hypothesis	Rejection Criterion
$H_1\colon \mu_1 - \mu_2 \neq \Delta_0$	$t_0 > t_{\alpha/2, n_1+n_2-2}$ or $t_0 < -t_{\alpha/2, n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 > \Delta_0$	$t_0 > t_{\alpha, n_1+n_2-2}$
$H_1\colon \mu_1 - \mu_2 < \Delta_0$	$t_0 < -t_{\alpha, n_1+n_2-2}$

Degree of freedom = n1+n2-2

Confidence Interval on the Difference in Means, Variances UnKnown:

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2, n_1+n_2-2}\, s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2, n_1+n_2-2}\, s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Problems:

1.

Consider the hypothesis test H0 : µ1 = µ2 against H2 : µ1 ≠ µ2 . Suppose that sample sizes are n1 = 15 and n2 = 15, that $\ddot{x}_1$ =4.7 and $\ddot{x}_2$ = 7.8, and that s_1^2= 4 and s_2^2= 6.25. Assume that $\sigma^2_1 = \sigma^2_2$ and that the data are drawn from normal distributions. Use α= . 0 05.

(a) Test the hypothesis and find the P-value.

Solution :

Eight step procedure:

1. Parameter of interest : μ_1 is equal to μ_2
2. Null hypothesis :

$$H_0 : \mu_1 = \mu_2$$

3. Alternate hypothesis

$$H_a : \mu_1 \neq \mu_2$$

4. α=0.05

5. Test Statistic

$$S_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$ and $$T_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

df=n1+n2−2=15+15−2=28

6. Rejection Criteria : reject H_0 if $t_{0.025,28} < t_0$(-2.048) and $t_{0.025,28} > t_0$(2.048)

7. Compute the Test Statistic

Pooled variance:

Substitute $n_1 = 15$, $n_2 = 15$, $s_1 = 4$, and $s_2 = 6.25$:

$$s_p^2 = \frac{(15-1)(4^2) + (15-1)(6.25^2)}{15+15-2} = \frac{14 \cdot 16 + 14 \cdot 39.0625}{28} = \frac{224 + 546.875}{28} = \frac{770.875}{28} \approx 27.53$$

Test Statistic: t_0

Substitute $\bar{x}_1 = 4.7$, $\bar{x}_2 = 7.8$, $s_p^2 \approx 27.53$, $n_1 = 15$, $n_2 = 15$:

$$t = \frac{4.7 - 7.8}{\sqrt{27.53\left(\frac{1}{15} + \frac{1}{15}\right)}} = \frac{-3.1}{\sqrt{27.53 \cdot \frac{2}{15}}} = \frac{-3.1}{\sqrt{3.6707}} = \frac{-3.1}{1.916} \approx -1.617$$

8. Conclusion t_0=-1.617, which lies between -2.048 and 2.048, we fail to reject H_0 (null) hypothesis

To find the p-value:

Because $|t_0|$ = 1.617, we find from t table that $t_{0.10,28}$ =1.313 and $t_{0.05,28}$= 1.701. Therefore, because 1.313 < 1.617 < 1.701 we conclude that lower and upper bounds on the *P*-value are 0.05*2 < *P* < 0.10*2 that is 0.10<p<0.20. Therefore, because the *P*-value exceeds α = 0.05, we fail to reject the null hypothesis.

2. Consider the hypothesis test H0 : $\mu1 = \mu2$ against H1 : $\mu1 \neq \mu2$. Suppose that sample sizes n1 = 10 and n2 = 10, that $\ddot{x}_1$ =7.8 and $\ddot{x}_2$ = 5.6, and s_1^2= 4 and s_2^2= 9. Assume that $\sigma^2_1 = \sigma^2_2$ and that the data are drawn from normal distributions. Use α= . 0 05.

(a) Test the hypothesis and find the *P*-value.

Do By Yourself

Problems on Confidence Interval on the Difference in Means, Variances Unknown

1. The diameter of steel rods manufactured on two different extrusion machines is being investigated. Two random samples of sizes $n1 = 15$ and $n1 = 17$ are selected, and the sample means and sample variances are $\ddot{X}_1 = 8.73$, and $\ddot{X}_2 = 8.68$, and $s_1^2 = 0.35$ and $s_2^2 = 0.40$, respectively. Assume that $\sigma^2_1 = \sigma^2_2$ and that the data are drawn from a normal distribution.
   (a) Is there evidence to support the claim that the two machines produce rods with different mean diameters? Use $\alpha = 0.05$ in arriving at this conclusion. Find the *P*-value.
   (b) Construct a 95% confidence interval for the difference in mean rod diameter. Interpret this interval.

1. Parameter of interest : μ_1 is equal to μ_2
2. Null hypothesis :

$$H_0 : \mu_1 = \mu_2$$

3. Alternate hypothesis

$$H_a : \mu_1 \neq \mu_2$$

4. α=0.05

5. Test Statistic

$$S_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$ and $$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

df=n1+n2−2=15+17−2=30

6. Rejection Criteria : reject H_0 if $t_{0.025,30} < t_0$ (-2.042) and $t_{0.025,30} > t_0$ (2.042)

7. Compute the Test Statistic

Pooled variance:

Given $n_1 = 15$, $n_2 = 17$, $s_1^2 = 0.35$, and $s_2^2 = 0.40$:

$$s_p^2 = \frac{(15-1)(0.35)+(17-1)(0.40)}{15+17-2} = \frac{14(0.35)+16(0.40)}{30} = \frac{4.9+6.4}{30} = \frac{11.3}{30} \approx 0.3767$$

Test Statistic: t_0

Substitute $\bar{x}_1 = 8.73$, $\bar{x}_2 = 8.68$, $s_p^2 \approx 0.3767$, $n_1 = 15$, $n_2 = 17$:

$$t = \frac{8.73-8.68}{\sqrt{0.3767\left(\frac{1}{15}+\frac{1}{17}\right)}} = \frac{0.05}{\sqrt{0.3767\,(0.0667+0.0588)}} = \frac{0.05}{\sqrt{0.3767\cdot 0.1255}} = \frac{0.05}{\sqrt{0.0473}} = \frac{0.05}{0.2175} \approx 0.230$$

8. Conclusion t_0=0.230, which lies between -2.042 and 2.042, we fail to reject H_0 (null) hypothesis

To find the p-value:

Because $|t_0|$ = 0.230, we find from t table that $t_{0.40,30}$ = (0.256) which is > t_0 and α value corresponding to 0.256 is 0.40. Since t_0 = 0.230 the α value will be greater than 0.40*2=0.80. Therefore, p-value will be greater than 0.80 and exceeds α = 0.05. we fail to reject H_0 (null) hypothesis

(b) Construct a 95% confidence interval for the difference in mean rod diameter. Interpret this interval

CI is calculated as :

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,n_1+n_2-2} s_p \sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,n_1+n_2-2}\ s_p \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

(or)

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2,df} \cdot \sqrt{s_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

Solution:

$$\sqrt{s_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} = \sqrt{0.3767\left(\frac{1}{15}+\frac{1}{17}\right)} = \sqrt{0.3767 \cdot (0.0667+0.0588)} = \sqrt{0.3767 \cdot 0.1255} = \sqrt{0.0473} \approx 0.2175$$

$\mathbf{t_{\alpha/2,df} = t_{0.05/2,30} = t_{0.025,30} = 2.042}$

2.042 * 0.2175≈0.444

$\ddot{X}_1 - \ddot{X}_2$=8.73−8.68=0.05

CI: 0.05±0.444

Lower bound: 0.05−0.444=−0.394

Upper bound: 0.05+0.444=0.494

Thus, the 95% confidence interval is:

(−0.394,0.494)

b. Case 2:

$\sigma^2_1 \neq \sigma^2_2$

Formula :

Test statistics :

$$T_0^* = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Degrees of freedom ν given by,

$$v = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

Degree of freedom = n1+n2-2

Confidence Interval on the Difference in Means, Variances UnKnown:

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Problem:

Two companies manufacture a rubber material intended for use in an automotive application. The part will be subjected to abrasive wear in the field application, so you decide to compare the material produced by each company in a test. Twenty-five samples of material from each company are tested in an abrasion test, and the amount of wear after 1000 cycles is observed. For company 1, the sample mean and standard deviation of wear are ẍ1 = 20 milligrams/1000 cycles and s_1 = 2 milligrams/1000 cycles, and for company 2, you obtain ẍ2 = 15 milligrams/1000 cycles and s_2 = 8 milligrams/1000 cycles.

(a) Do the data support the claim that the two companies produce material with different mean wear? Use $\alpha = 0.05$, and assume that each population is normally distributed but that their variances are not equal. What is the P-value for this test?

(b) Construct confidence intervals that will address the questions in part (a).

Solution:

1. Parameter of interest : μ_1 is equal to μ_2

2. Null hypothesis :

$$H_0 : \mu_1 = \mu_2$$

3. Alternate hypothesis

$$H_a : \mu_1 \neq \mu_2$$

4. α=0.05

5. Test Statistic

$$T_0^* = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

- ẍ1=20, ẍ2=15 are the sample means.
- s1=2 and s2=8 are the sample standard deviations.
- n1=25 and n2=25 are the sample sizes.
- Degrees of freedom (v) is given as

$$v = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

=

$$\frac{(0.16 + 2.56)^2}{\frac{0.16^2}{24} + \frac{2.56^2}{24}} = \frac{(2.72)^2}{\frac{0.0256}{24} + \frac{6.5536}{24}} = \frac{7.3984}{0.001067 + 0.273067} = \frac{7.3984}{0.274134} \approx 27.0$$

6. Rejection Criteria : reject H_0 if $t_{0\ .025,27} < t_0(-2.052)$ and $t_{0\ .025,27} > t_0(2.052)$

7. Compute the Test Statistic

Test Statistic:

$$\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{2^2}{25} + \frac{8^2}{25}} = \sqrt{\frac{4}{25} + \frac{64}{25}} = \sqrt{\frac{68}{25}} = \sqrt{2.72} \approx 1.649$$

$$t_0 = \frac{20 - 15}{1.649} = \frac{5}{1.649} \approx 3.03$$

8. Conclusion t_0=3.03, which does not lie between -2.052 and 2.052, we reject H_0 (null) hypothesis

To find the p-value:

Because $|t_0|$ = 3.03, we find from t table that $t_{0.0025,27}$ = 3.057 and $t_{0\ .005,27}$= 2.771. Therefore, because 2.771 < 3,03 < 3.057 we conclude that lower and upper bounds on the *P*-value are 0.0025*2 < *P* < 0.005*2 that is 0.0050<p<0.010. Therefore, because the *P*-value is less than $\alpha = 0.05$, we reject the null hypothesis.

(b) Construct confidence intervals that will address the questions in part (a).

Formula:

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{2^2}{25} + \frac{8^2}{25}} = \sqrt{\frac{4}{25} + \frac{64}{25}} = \sqrt{\frac{68}{25}} = \sqrt{2.72} \approx 1.649$$

$t_{0\ .025,27}$ = 2.052

Therefore,

$$t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = 2.052 \cdot 1.649 \approx 3.384$$

ẍ1-ẍ2=20−15=5

Lower bound: $5 - 3.384 = 1.616$

Upper bound: $5 + 3.384 = 8.384$

Thus, the 95% confidence interval is: $(1.616, 8.384)$

Inference on the Variances of Two Normal Distributions

Formula:

We wish to test the hypotheses :

$$H_0: \sigma_1^2 = \sigma_2^2$$
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

Use F distribution

Null hypothesis: $H_0: \sigma_1^2 = \sigma_2^2$

Test statistic: $F_0 = \dfrac{S_1^2}{S_2^2}$

Alternative Hypotheses	**Rejection Criterion**
$H_1: \sigma_1^2 \neq \sigma_2^2$	$f_0 > f_{\alpha/2, n_1-1, n_2-1}$ or $f_0 < f_{1-\alpha/2, n_1-1, n_2-1}$
$H_1: \sigma_1^2 > \sigma_2^2$	$f_0 > f_{\alpha, n_1-1, n_2-1}$
$H_1: \sigma_1^2 < \sigma_2^2$	$f_0 < f_{1-\alpha, n_1-1, n_2-1}$

The lower-tailed percentage points $f_{1-\alpha, u, v}$ can be found as follows.

$$f_{1-\alpha,u,v} = \frac{1}{f_{\alpha,v,u}}$$

For example, to find the lower-tailed percentage point $f_{0.95,5,10}$, note that

$$f_{0.95,5,10} = \frac{1}{f_{0.05,10,5}} = \frac{1}{4.74} = 0.211$$

Confidence Interval on the Difference in Means, Variances UnKnown:

$$\frac{s_1^2}{s_2^2} f_{1-\alpha/2, n_2-1, n_1-1} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{s_1^2}{s_2^2} f_{\alpha/2, n_2-1, n_1-1}$$

Problems :

1. For an F distribution, find the following: (a) $f_{0.25,5,10}$ (b) $f_{0.10,24,9}$
   $f_{0.25,5,10}=1.59$, $f_{0.10,24,9}=2.28$
2. Consider the hypothesis test H_0: $\sigma^2_1 = \sigma^2_2$ against H_0: $\sigma^2_1 < \sigma^2_2$ respectively. Suppose that the sample sizes are n1 = 5 and n2 = 10, and that $s_1^2 = 23.2$ and $s_2^2 = 28.8$. Use $\alpha = 0.05$.
   (a)Test the hypothesis and
   (b) Explain how the test could be conducted with a confidence interval on σ_1 / σ_2
   (a)
   Solution : Apply the eight step procedure :

1. Parameter of interest : : σ^2_1 and σ^2_2
2. Null hypothesis :

H_0: $\sigma^2_1 = \sigma^2_2$

3. Alternate hypothesis

H_1: $\sigma^2_1 < \sigma^2_2$

4. α=0.05
5. Test Statistic

$$f0 = \frac{s_1^2}{s_2^2}$$

6. Rejection Criteria : reject H_0 if $f_0 < F_{0.95,4,9}$ that is $f_0 <0.1667$

Note: $H_1: \sigma_1^2 < \sigma_2^2$ $\quad f_0 < f_{1-\alpha, n_1-1, n_2-1}$

To find : $F_{0.95,4,9} \rightarrow 1/(F_{0.05,9,4})$ that is 1/6=0.1667

7. Compute the Test Statistic

df1=4, and df2=9

Test Statistic:

$s_1^2 = 23.2$

$s_2^2 = 28.8$

$$f_0 = \frac{23.2}{28.8} \approx 0.8056$$

8. Conclusion f_0=0.8056, which is not less than 0.1667, we fail to reject H_0 (null) hypothesis

(b) confidence interval on σ_1 / σ_2

$$\frac{s_1^2}{s_2^2} f_{1-\alpha/2, n_2-1, n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{s_1^2}{s_2^2} f_{\alpha/2, n_2-1, n_1-1}$$

$s_1^2 = 23.2$

$s_2^2 = 28.8$

$$\frac{s_1^2}{s_2^2} = \frac{23.2}{28.8} \approx 0.8056$$

$f_{1-\alpha/2, n2-1, n1-1} = f_{0.975,9,4} = 1/(f_{0.025,4,9}) = 1/4.72 = 0.2118$

$f_{\alpha/2.,n2-1,n1-1} = f_{0.025,9,4} = 8.90$

$0.8056 * 0.2118 <= \sigma^2_1/\sigma^2_2 <= 0.8056 * 8.90$

Lower bound : 0.1706

Upper bound : 7.169

Non parametric testing the Difference in Two Means : Refer PPT and class notes.

Example 1 :

Steps Explained with an Example

1. Parameter of Interest

Identify the parameter in the problem that you want to make a decision about. For instance, if you're analyzing the average height of students in a university, the parameter of interest might be the **population mean height** (μ).

2. Null Hypothesis, H0H_0H0

State the **null hypothesis**. The null hypothesis typically assumes no effect or no difference. In our example, the null hypothesis might be:

H0:μ=170 cm

This means we assume the average height of students is 170 cm.

3. Alternative Hypothesis, H_1

Specify an **alternative hypothesis** that reflects the claim or what you are testing for. There are three types:

- **Two-tailed test**: H_1:μ≠170(testing for any difference)
- **One-tailed test (greater)**: H_1:μ>170
- **One-tailed test (lesser)**: H_1:μ<170

Let's say we want to test if students are, on average, taller than 170 cm. Our alternative hypothesis would be: (one tailed test)

H1:μ>170

4. Test Statistic

Choose a **test statistic** based on the type of hypothesis and data characteristics. Since we assume the population standard deviation is known, we use the zzz-test:

$$z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

where:

- ẍ is the sample mean,
- μ is the hypothesized population mean (from H0),
- σ is the population standard deviation, and
- n is the sample size.

5. Reject H_0 :

Set the **rejection criteria** based on the significance level (α\alphaα). For a one-tailed test at a 5% significance level (α=0.05):

- Reject H_0 if $z_0>1.645$ **(for a right-tailed test** – upper tail).

6. Computations

Compute sample quantities and substitute them into the test statistic equation. Suppose we have:

- Sample mean $\ddot{x}$ =172
- Population standard deviation σ=5
- Sample size n=30

Then, calculate z_0:

$$z_0 = \frac{172 - 170}{5/\sqrt{30}} = \frac{2}{0.9129} \approx 2.19$$

7. Draw Conclusions

Compare the calculated z_0 to the critical value and make a decision:

- Since 2.19>1.645, we **reject H_0** at the 5% significance level.

Conclusion: We have enough evidence to conclude that the average height of students is greater than 170 cm.

Example 2 :

In the above problem

3. Alternative Hypothesis, H_1

Specify an **alternative hypothesis** that reflects the claim or what you are testing for. There are three types:

- **Two-tailed test**: H_1:μ≠170(testing for any difference)

5. Reject H_0 :

Set the **rejection criteria** based on the significance level (α\alphaα). For a one-tailed test at a 5% significance level (α=0.05):

- Reject H_0 if $|z_0|>1.96$ (for two tailed test).

7. Draw Conclusions

Compare the calculated z_0 to the critical value and make a decision:

- Since 2.19>1.96, we **reject H_0** at the 5% significance level.