Gaussian Kernel function:-

$x' = y$

$$K(x,y) = e^{-\beta \|x-y\|^2}$$

expand this

$$K(x,y) = e^{-\beta(x^2+y^2-2xy)}$$

$$= e^{-\beta x^2} \cdot e^{-\beta y^2} \cdot e^{2xy\beta} \longrightarrow ①$$

Taylor series for exponential function is

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

using taylor series we expand the term: $e^{2xy\beta}$.

$z = 2xy\beta$.

$$e^{2xy\beta} = \sum_{k=0}^{\infty} \frac{2xy\beta^k}{k!} = 1 + 2xy\beta + \frac{(2xy\beta)^2}{2!} + \frac{(2xy\beta)^3}{3!} + \cdots$$

seperate x and y terms.

$$\cancel{= 1 + \sqrt{2x\beta} \cdot \sqrt{2\beta y} +}$$

$$= 1 + (\sqrt{2\beta} \cdot x) \cdot (\sqrt{2\beta} \cdot y) + \frac{(2\beta x^2)^{1/2}}{\sqrt{2!}} \cdot \frac{(2\beta y^2)^{1/2}}{\sqrt{2!}} + \cdots$$

from ① → write x and y terms seperately.

$$K(x,y) = e^{-\beta x^2}\left(1, \sqrt{2\beta}x, \frac{(2\beta x^2)^{1/2}}{\sqrt{2!}}, \ldots\right) \cdot e^{-\beta y^2}$$

$$\left(1, \sqrt{2\beta}y, \frac{(2\beta y^2)^{1/2}}{\sqrt{2!}} \ldots\right)$$

can be writen as dot product of $\phi(x)^T \cdot \phi(y)$.

$$K(x,y) = \phi(x)^T \cdot \phi(y)$$

Polynomial Kernel functions:

$$K(\bar{x}, \bar{x}') = (1 + \vec{x} \cdot \vec{x}')^p$$

Lets assume $p = 1$

$$K(\bar{x}, \bar{x}') = (1 + \vec{x} \cdot \vec{x}')$$

as we know ~~x^T~~

$$x^T \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 \dots$$

as you can see in the above equation $(1 + \vec{x} \cdot \vec{x}')$ that is just a dot product $(\vec{x} \cdot \vec{x}')$ and we are adding a constant to it resulting in scalar.

Lets assume $p = 2$.

$$K(\bar{x} \cdot \bar{x}') = (1 + \bar{x} \cdot \bar{x}')^2$$

$$= 1 + (\bar{x} \cdot x')^2 \cdot 2(\bar{x} \cdot x')^2$$

two terms are dot products

$\Rightarrow$ the expression resulting in a scalar quantity.

hence polynomial kernel is also a dot product

Hyperbolic Tangent or Sigmoid :-

$$K(\vec{x}, \vec{x}') = \tanh(\vec{x} \cdot \vec{x}' + \delta)$$

This is similar to polynomial function only difference is we are applying tangent (tanh) to the dot product.

$\bar{x}, \vec{x}'$ the dot product of these two are $x^T x'$.

$\tanh(\bar{x} \cdot x' + \delta)$

the above tanh will map $(\bar{x} \cdot \vec{x}' + \delta)$ to the range of $(-1, 1)$ which is a scalar value resultant of a dotproduct.

→ Hence we can conclude that sigmoid kernel is also a non linear transformation of the dot product.