
CHAPTER 1

METRIC SPACES

1 Fundamental Definitions n' Stuff

Definition 1.1: Metric Space

A set X along with a function $d : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ called distance, is said to be a metric space if:

1. $d(x, y) = 0 \iff x = y$ (Positivity)
2. $d(x, y) = d(y, x), \forall x, y \in X$ (Symmetric)
3. $\forall x, y, z \in X$ we have, $d(x, y) \leq d(x, z) + d(y, z)$ (Triangle Inequality)

Example : \mathbb{R}^n as a metric space

Note that \mathbb{R}^n , the set of all n-tuples of \mathbb{R} , is a metric space with

$$d(\vec{x}, \vec{y}) := |\vec{x} - \vec{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Definition 1.2: Open and Closed Balls around x in X

Open ball is defined as:

$$B_r(x) := \{y \in X : d(y, x) < r\}$$

Closed ball is defined as:

$$B_{[r]}(x) := \{y \in X : d(y, x) \leq r\}$$

Definition 1.3: Convexity

A set S in \mathbb{R}^n is said to be convex if $\forall x, y \in S, t \in [0, 1], x + t(y - x) \in S$

Example : Open and closed balls in \mathbb{R}^n are convex

Consider $B_r(x) := \{z \in \mathbb{R}^n : |z - x| < r\}$. Consider arbitrary p and q in $B_r(x)$. We have that $d(p, x) < r$ and $d(q, x) < r$. Consider $p + t(q - p)$ and consider $d(p + t(q - p), x) = |p + t(q - p) - x| = |tq + (1 - t)p - x| = |tq - tx + (1 - t)p - (1 - t)x| \leq t|q - x| + (1 - t)|p - x| = td(q, x) + (1 - t)d(p, x) < r$. Replacing $<$ with \leq in the above proves the result for closed balls. ■

Definition 1.4: Sequences in Metric Spaces

A sequence $\{x_n : x_n \in X\}$ is a mapping from the naturals to X , where order is implicit. We say a sequence in a metric space X is convergent to $x \in X$ if:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, n \geq n_0)(d(x_n, x) < \varepsilon)$$

Definition 1.5: Cauchy Sequence in X

A sequence $\{x_n\}$ in X is said to be cauchy if

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n, m \in \mathbb{N})(n, m \geq n_0 \implies d(x_n, x_m) < \varepsilon)$$

Theorem 1.6

Convergence \implies cauchy

Proof for Theorem.

Say x_n converges to x in the metric space.

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n, m \geq n_0)(d(x_n, x) < \varepsilon, d(x_m, x) < \varepsilon)$$

which implies (from triangle inequality)

$$(\forall \varepsilon > 0)(\exists n_0)(\forall n, m \geq n_0)(d(x_n, x) < \varepsilon, d(x_m, x) < \varepsilon \implies d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < 2\varepsilon)$$

Which is the cauchy condition. ■

1.1 Inner products, Norms and some common metrics

Definition 1.7: Inner Product

Let \mathbb{F} be either \mathbb{C} or \mathbb{R} , and V a vector space over \mathbb{F} . An **inner product** on V is a function that assigns to each ordered pair of vectors $v, u \in V$ a scalar $\langle v|u \rangle \in \mathbb{F}$ so that the following holds:

1. $\langle v + cw|u \rangle = \langle v|u \rangle + c\langle w|u \rangle$
2. $\langle u|v \rangle = \overline{\langle v|u \rangle}$ where $\bar{\cdot}$ is just complex conjugation
3. $\langle u|u \rangle > 0$ if $u \neq 0$

A space $V(\mathbb{F}), \langle \cdot \rangle$ is called an *inner product space*.

Definition 1.8: Norm (From the inner product)

Given an inner product space $V(\mathbb{F}), \langle \cdot \rangle$, we define the *norm* or *length* of a vector x , given by $\|x\|$ as $\sqrt{\langle x|x \rangle}$

Example : p -norm is a metric

The p -norm is defined on \mathbb{R}^n as:

$$\|\vec{x}\|_p := (|x_1|^p + |x_2|^p \cdots |x_n|^p)^{1/p} = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

We say $d_p(\vec{x}, \vec{y})$, the distance between \vec{x} and \vec{y} to be $\|x - y\|_p = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$. From this definition, unique 0 distance property and reflexivity property are obvious. Consider $d_p(x, z)$ and $d_p(z, y)$. $d_p(x, y) = (\sum_{i=1}^k |(x_i - z_i) + (z_i - y_i)|^p)^{1/p}$ whence from Minkowski we see that

$$\left(\sum_{i=1}^k |(x_i - z_i) + (z_i - y_i)|^p \right)^{1/p} \leq \left(\sum_{i=1}^k |x_i - z_i|^p \right)^{1/p} + \left(\sum_{i=1}^k |z_i - y_i|^p \right)^{1/p} = d_p(x, z) + d_p(y, z)$$

Therefore, p -norm is a metric ■

Example : The Max Norm

Suppose $x \in \mathbb{R}^n$, the max norm $\|x\|_\infty$ is defined as:

$$\|\vec{x}\|_\infty := \max_{1 \leq i \leq n} |x_i|$$

Clearly, $d_\infty(x, y) = 0$ if and only if $x = y$. Moreover, from the properties of $|\cdot|$, it is reflexive. Consider $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$, $d_\infty(x, z) = \max\{|x_1 - z_1|, |x_2 - z_2|, \dots, |x_n - z_n|\}$ and $d_\infty(y, z) = \max\{|y_1 - z_1|, |y_2 - z_2|, \dots, |y_n - z_n|\}$.

$$d_\infty(x, y) = \max_{1 \leq i \leq n} \{|(x_i - z_i) + (z_i - y_i)|\} \leq \max_{1 \leq i \leq n} \{|(x_i - z_i)|\} + \max_{1 \leq i \leq n} \{|(y_i - z_i)|\} = d_\infty(x, z) + d_\infty(z, y)$$

Hence the max norm is a metric. ■

Example : Norm on function spaces

Let X be a non empty set. Define $\mathfrak{B}(X)$ as the space of all bounded, real functions. Then $\|f\|_\infty := \sup_{x \in X} |f(x)|$ defines a norm on $\mathfrak{B}(X)$ since $|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| = \|f\|_\infty + \|g\|_\infty$ which gives us $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ ■

Example : the $\frac{d}{1+d}$ metric

Say X, d is a metric space, then $g(x, y) = \frac{d(x, y)}{1+d(x, y)}$ is a metric.

Proof. exercise to the reader □

Example : Constructing a metric from an injective function

Let Z be a set and (X, d) a metric space. Let $f : Z \rightarrow X$ be an injective function. Define $\delta(x, y) = d(f(x), f(y))$. This defines a metric on Z and makes (Z, δ) a metric space since $\delta(x, x) = d(f(x), f(x)) = 0$, $\delta(x, y) = 0 \implies d(f(x), f(y)) = 0$ which gives $x = y$, and $\delta(x, y) = d(f(x), f(y)) \leq d(f(x), f(z)) + d(f(z), f(y)) = \delta(x, z) + \delta(y, z)$ which gives triangle. ■

Example : The discrete metric

Let X be any set whatsoever. Define the metric

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

This is a metric due to positive definiteness and symmetry. The triangle inequality is, also, not hard to see. ■

Example : Space of all bounded sequences

Let X be the set of all bounded sequences of reals, i.e, the set of all x such that $x = (x_1, x_2, \dots) = \{x_n\}_{n=1}^\infty$ with $\sup_{i=1}^\infty (\{x_i\}) < \infty$. If $\{x_i\}$ and $\{y_i\}$ are sequences (i.e, elements in X) let the distance between two sequences be defined by

$$d(x, y) := \sup_{i=1}^\infty |x_i - y_i|$$

Clearly, it is positive definite and symmetric. Consider three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X . $d(x, y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$, $d(x, z) = \sup\{|x_i - z_i| : i \in \mathbb{N}\}$ and $d(y, z) = \sup\{|y_i - z_i| : i \in \mathbb{N}\}$. Consider $d(x, z) = \sup\{|x_i - z_i| = |x_i - y_i + y_i - z_i| : i \in \mathbb{N}\} \leq \sup\{|x_i - y_i| + |y_i - z_i| : i \in \mathbb{N}\} \implies \leq \sup\{|x_i - y_i| : i \in \mathbb{N}\} + \sup\{|y_i - z_i| : i \in \mathbb{N}\}$ which proves the triangle. ■

Basics on Continuity (Read definition of sequence in metric space, and limit points)

Definition 1.9: Continuous functions

A function $f : X \rightarrow Y$ that maps a metric space X, d_x to a metric space Y, d_y is said to be continuous at $a \in X$ (which is a limit point of X) if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in X)(d_x(p, a) < \delta \implies d_y(f(p), f(a)) < \varepsilon)$$

Theorem 1.10: Sequential Criteria for continuity

Let X, d_x and Y, d_y be metric spaces and $f : X \rightarrow Y$. Let a be a limit point of X . Then f is continuous at a if and only if for every sequence $(q_n) \in X$, $q_n \neq a$ with $\lim(q_n) = a$, we have $\lim(f(q_n)) = f(a)$

Proof for Theorem.

\implies) We have that for every $\varepsilon > 0$, there exists a $\delta > 0$ so that for all $z \in X$ such that $z \in B_\delta(a)$, we have $f(z) \in B_\varepsilon(f(a))$. Consider an arbitrary sequence in X that is so that $(q_n) \in X$, $q_n \neq a$ and $\lim(q_n) = a$. Which reads $\forall \delta > 0, \exists n_0 \in \mathbb{N}$ so that $\forall n \geq n_0$, we have $q_n \in B_\delta(a)$. Let $\varepsilon > 0$ be arbitrary. From the definition of continuity, there exists a corresponding $\delta > 0$ so that $\forall z \in X$, $z \in B_\delta(a)$ would imply $f(z) \in B_\varepsilon(f(a))$. For this δ_ε , there exists an $n_0(\delta)(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0$, $q_n \in B_\delta(a)$ which would imply $f(q_n) \in B_\varepsilon(f(a))$. This means that $\forall \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that $\forall n \geq n_0$, we have $f(q_n) \in B_\varepsilon(f(a))$ which means that $\lim(f(q_n)) = f(a)$.

\impliedby) Suppose that every sequence $q_n \in X$ so that $q_n \neq a$ with $\lim(q_n) = a$, we have that $\lim(f(q_n)) = f(a)$ but that the function f is *not* continuous. i.e, $\exists \varepsilon > 0$ so that $\forall \delta > 0, \exists q_\delta \in X$ so that $d_x(q_\delta, a) < \delta$ but $d_y(f(q_\delta), f(a)) \geq \varepsilon$. Choose $\delta = 1$ and get the corresponding $x_1 \in B_1(a)$ for which we have $d_y(f(x_1), f(a)) \geq \varepsilon$. Choose $\delta = 1/2$ and get $x_2 \in X$ so that $d_x(x_2, a) < 1/2$ but $d_y(f(x_2), f(a)) \geq \varepsilon$. As such, keep going with $\delta = 1/n$ to generate a sequence $x_n \in X$ so that $d(x_n, a) < 1/n$ but $d(f(x_n), f(a)) \geq \varepsilon$. Note that x_n converges by definition, to a , but there exists an $\varepsilon > 0$ so that no matter what n_0 we take, there exists some $n \geq n_0$ so that $d_y(f(x_n), f(a)) \geq \varepsilon$ which means that the sequence $f(x_n)$ does not converge to $f(a)$, which contradicts hypothesis. ■

Corollary 1.11

Composition of continuous functions (at a point a) is finally a continuous function

Proof for Corollary.

Suppose X, d_x, Y, d_y and Z, d_z are metric spaces with $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ two continuous functions. $g \circ f : X \rightarrow Z$. Consider a sequence $(x_n) \in X$ that converges to a . From sequential criteria, $f(x_n) \rightarrow f(a)$. We have $f(x_n)$, a sequence in Y , that converges

to $f(a)$. From the continuity of g , we note that $g \circ (f(x_n))$ converges to $g \circ (f(a))$, which ultimately tells that for every sequence $\{x_n\}$ in X that converges to a , $g \circ f(x_n)$ is a sequence in Z that converges to $g \circ f(a)$ whence we are done ■

Example : Dirichlet Function

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

This function is discontinuous at every point of \mathbb{R} . Bear in mind that to show that a function is discontinuous at a point z , we need only show that *one* sequence in A which converges to z is such that $\lim(f(x_n)) \neq f(\lim(x_n))$. Consider $a \in \mathbb{R}$ an irrational point. Consider the rational sequence x_n that converges to a via density theorem. $f(x_n) = 1$ for every $n \in \mathbb{N}$ which means $\lim f(x_n) = 1$. But $f(\lim(x_n)) = 0$. Hence, discontinuous at $a \in \mathbb{R}$, irrational. In fact, the same argument can be re-used to show that f is irrational at every point. ■

Example : Thomae Function

We define the Thomae function as follows: Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/n & \text{if } x \text{ is rational, with } x = m/n, \gcd(m, n) = 1 \end{cases}$$

This function is discontinuous at rational points quite obviously since, consider a rational number m/n where $f(m/n) = 1/n > 0$. Consider an irrational sequence in \mathbb{R}^+ converging to m/n . Clearly, $f(x_n) = 0$ for every $n \in \mathbb{N}$ whence, it is clear.

Consider z an irrational point where $f(z) = 0$. Consider a $\varepsilon > 0$ and an n_0 so that $1/n_0 < \varepsilon$. Consider the subinterval $(z - \gamma, z + \gamma)$ for some γ . We note that in this subinterval, only finite rational points have their denominator smaller than n_0 (because $(z - \gamma)n < m < (z + \gamma)n$ and this interval has size $2n\gamma$. For a fixed n and γ , m can pick values only from this interval. If $n < n_0$, then the interval size is smaller than $2n_0(\gamma)$. If we make γ small enough, m would be restricted to just one value. say, m_0 . With this γ (and m_0 fixed) the only values n can take that are less than n_0 are certainly finite now since there are only finite ns smaller than n_0 . Summarising, for a given ε , we choose an n_0 so that $1/n_0 < \varepsilon$, where we concluded that for small enough γ , only finite rational points in the interval $(z - \gamma, z + \gamma)$ have their denominator smaller than n_0 . Now, we can eliminate all these rational points by picking an appropriate δ so that no rational points with denominator smaller than n_0 exist in $(z - \delta, z + \delta)$. Therefore, for a given ε ($\exists n_0 : 1/n_0 < \varepsilon$) there exists δ so that either a point q in $(z - \delta, z + \delta)$ is irrational, whence $f(q) = 0 < \varepsilon$, or it has its denominator larger than n_0 , which means $f(q) = 1/n < 1/n_0 < \varepsilon$ which leads to $|f(q) - f(z)| = |f(q)| = f(q) < \varepsilon$ which is the " $\varepsilon - \delta$ " criterion for Continuity. Hence, at all irrational points z , the Thomae function is continuous. ■

Now we deal with, primarily, continuous functions on closed, bounded intervals of the kind $I = [a, b] \subset \mathbb{R}$.

Theorem 1.12: Boundedness Theorem

Continuous functions on closed bounded intervals are bounded

Proof for Theorem.

Suppose $f : \mathbb{I} \rightarrow \mathbb{R}$ is actually unbounded, i.e., for any $M \in \mathbb{R}$ we take, there exists an $x_M \in \mathbb{I}$ so that $f(x_M) > M$. Let $M_1 = 1$ and get the corresponding x_1 so that $f(x_1) > 1$. Do the same for $M_n = n$ to get x_n so that $f(x_n) > n$. This sequence $f(x_n)$ is divergent. But the sequence $\{x_n\}$ is bounded, hence from Bolzano, there is a subsequence x_{n_k} that is convergent, say to p . But $f(x_{n_k})$ is a subsequence of $f(x_n)$, which means it diverges. But according to continuity, $f(x_{n_k})$ converges to $f(p)$, which is a contradiction. Hence, f is bounded. ■

Theorem 1.13: Maxima-Minima Theorem

Let $f : \mathbb{I} \rightarrow \mathbb{R}$ be a continuous function on closed bounded interval \mathbb{I} . Then f attains its maxima and minima.

Proof for Theorem.

We know that $f(\mathbb{I})$ is actually bounded, which means it has supremum and an infimum U and L . Suppose it *does not* attain bounds, i.e., (talking about upper bound) $\forall x \in \mathbb{I}$, $f(x) < U$. Consider $U - 1$. There exists $f(x_1)$ so that $U - 1 < f(x_1) < U$. Choose x_2 likewise so that $U - 1/2 < f(x_2) < U$. Keep going as such, to find x_n so that $U - 1/n < f(x_n) < U$. This means from squeeze theorem that $f(x_n)$ converges to U . Note that x_n that we have collected is a sequence that is in \mathbb{I} , a bounded, closed interval. Hence, it has a convergent subsequence $x_{n_k} \rightarrow x$. Also, $f(x_{n_k})$ is a subsequence of $f(x_n)$ which means $f(x_{n_k}) \rightarrow U$. But from continuity, $f(x_{n_k}) \rightarrow f(x)$, which implies $f(x) = U$, where $x \in \mathbb{I}$. This means that the function actually *does* attain upper bound (similar argument for the lower bound can be performed). ■

Theorem 1.14: Location of roots theorem

Let $f : \mathbb{I} \rightarrow \mathbb{R}$ be a continuous function on a closed bounded interval \mathbb{I} . Let $a < b$ with $f(a) < 0$ and $f(b) > 0$ (or the other way around). Then there exists $c \in (a, b)$ so that $f(c) = 0$

Proof for Theorem.

Let $a_0 = a$ and $b_0 = b$ with $a < b$. a_0, b_0 forms an interval of size $\xi = b_0 - a_0$. Look at $f(a_0 + b_0/2)$. Is it greater than 0? If so, make $a_1 = a_0$, and $b_1 = a + b/2$, a new interval a_1, b_1 of size $\xi/2$. This new interval obeys the property that $f(a_1) < 0$ and $f(b_1) > 0$. Is $f(a_0 + b_0/2) < 0$? If so, make $b_1 = b_0$ and $a_1 = a_0 + b_0/2$ to make a_1, b_1 a new interval of size $\xi/2$. Again, $f(a_1) < 0$ and $f(b_1) > 0$. Suppose you have made the n -th interval a_n, b_n so that $f(a_n) < 0$ and $f(b_n) > 0$ of size $\xi/(2^n)$. Make a_{n+1}, b_{n+1} by asking similar questions as above. We thus have a sequence of nested intervals $[a_n, b_n]$, for which $f(a_n) < 0$ and

$f(b_n) > 0$. From Nested interval theorem, we have $\gamma \in \cap_n [a_n, b_n]$, moreover, since the size of these intervals are converging to 0, this is a unique point in the intersection. $a_n \rightarrow \gamma$ and $b_n \rightarrow \gamma$. From continuity, then, we have $f(a_n) \rightarrow f(\gamma) \leq 0$ and $f(b_n) \rightarrow f(\gamma) \geq 0$ which means that $f(\gamma) = 0$. ■

Theorem 1.15: Bolzano's IVT

Let \mathbb{I} be a closed, bounded interval and let $f : \mathbb{I} \rightarrow \mathbb{R}$ be a continuous function. If $a, b \in \mathbb{I}$ with $k \in \mathbb{R}$ satisfying $f(a) < k < f(b)$, then there is a point c in \mathbb{I} so that $f(c) = k$

Proof for Theorem.

Let $a < b$ and define $g(x) = f(x) - k$. This is a continuous mapping on \mathbb{I} . We then have $g(a) < 0 < g(b)$ whence from location or roots theorem, we find $c \in \mathbb{I}$ so that $g(c) = 0$ or $f(c) = k$. ■

Theorem 1.16: Preservation of intervals Theorem

Let \mathbb{I} be a closed, bounded interval. Let $f : \mathbb{I} \rightarrow \mathbb{R}$ be a continuous map. Then, $f(\mathbb{I})$ is also a closed bounded interval.

Proof for Theorem.

Consider $f(\mathbb{I}) := \{z \in \mathbb{R} : \exists x \in \mathbb{I} \text{ such that } f(x) = z\}$. We know that continuous functions attain bounds, i.e, $f(\mathbb{I})$ has a maxima and a minima that will (eventually) be the end points of our closed bounded interval. If the function is constant, we are done. Consider a point $\min(f(\mathbb{I})) < z < \max(f(\mathbb{I}))$. By Bolzano's IVT, we know that a pre-image exists for z . This means that z is in the image. We are, therefore, done. The requisite interval is $[\min(f), \max(f)]$. ■

Definition 1.17: Uniform Continuity

A function $f : A \rightarrow \mathbb{R}$ is said to be uniformly continuous on A if for every $\varepsilon > 0$, $\exists \delta > 0$ so that for all $x, u \in A$ such that $0 < |x - u| < \delta$, we have $|f(x) - f(u)| < \varepsilon$.

We see that this definition is similar to the definition of continuity on A except for the fact that δ is independent of the x , the point at which we speak of continuity.

Theorem 1.18

Let f be a function defined on $A \subseteq \mathbb{R}$. The following are equivalent:

1. f is not uniformly continuous on A
2. $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x_\delta, u_\delta \in A$ so that $0 < |x_\delta - u_\delta| < \delta$ but $|f(x_\delta) - f(u_\delta)| \geq \varepsilon$
3. $\exists \varepsilon$ and two sequences x_n and y_n so that $\lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$

Proof for Theorem.

(1) \implies (2)) Obvious negation of the definition

(2) \implies (3)) Choose $\delta = 1$, and get corresponding x_1, u_1 so that $0 < |x_1 - u_1| < 1$ and $|f(x_1) - f(u_1)| \geq \varepsilon$. Choose $\delta = 1/2$ and get the corresponding x_2 and u_2 so that $0 < |x_2 - u_2| < 1/2$ and $|f(x_2) - f(u_2)| \geq \varepsilon$. Keep going as such to get x_n, u_n so that $0 < |x_n - u_n| < 1/n \forall n \in \mathbb{N}$ and $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$. This means that $\lim(x_n - u_n) = 0$ but $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$.

(3) \implies (2)) $\forall \delta > 0$, $\exists n_0(\delta)$ so that $\forall n \geq n_0$, we have $0 < |x_n - u_n| < \delta$ but $|f(x_n) - f(u_n)| \geq \varepsilon$, choose one of these x_n -s for n greater than $n_0(\delta)$ as our x_δ (and likewise for u_δ). We are then done. ■

Theorem 1.19: Uniform Continuity Theorem

Let \mathbb{I} be a closed bounded interval and let $f : \mathbb{I} \rightarrow \mathbb{R}$ be continuous on \mathbb{I} . Then f is uniformly continuous on I

Proof for Theorem.

Suppose f is not uniformly continuous on \mathbb{I} . From the previous *non uniform* criteria, we have that $\exists \varepsilon$ and two sequences in \mathbb{I} , x_n, u_n so that $\lim(x_n - u_n) = 0$ (or $\lim(x_n) = \lim(u_n)$) but $|f(x_n) - f(u_n)| \geq \varepsilon$ for every $n \in \mathbb{N}$. Since x_n and u_n are sequences in \mathbb{I} which is closed and bounded, x_n has a subsequence x_{n_k} converging to x . Since $\lim(x_n - u_n) = 0$, $\lim(x_{n_k} - u_{n_k}) = 0$ Which means that $\lim(u_{n_k}) = x$ as well. We have $x_{n_k} \rightarrow x$ and $u_{n_k} \rightarrow x$, which means from continuity that $f(x_{n_k}) \rightarrow f(x)$ and $f(u_{n_k}) \rightarrow f(x)$, whence we find that $\lim(f(x_{n_k}) - f(u_{n_k})) = 0$ necessarily. But this contradicts the assumption that f is not uniform continuous. Hence, we are done. ■

Sequences of Functions

Given a set $A \subset \mathbb{R}$, we primarily work with a (countably) infinite collection of functions $\{f_n\}$ with $f_n : A \rightarrow \mathbb{R}$. We generate a sequence of numbers by evaluating f_n at a fixed point $x \in A$. It could be, then, that $\{f_n(x)\}$, treated as a sequence of numbers in \mathbb{R} , either converges or does not. For a subset $A_0 \subseteq A$ (which can possibly be empty), $f_n(x)$ converges

for every $x \in A_0$. If this happens, there is a uniquely determined value that we would like to call " $f(x)$ ". Thus, there arises naturally a function $f : A_0 \rightarrow \mathbb{R}$ that we call the "limit" of $\{f_n\}$.

Definition 1.20: Convergence of sequence of functions

Let $\{f_n\}$ be a sequence of functions defined on $A \subseteq \mathbb{R}$. Let $A_0 \subset A$, and let $f : A_0 \rightarrow \mathbb{R}$. We say the sequence (f_n) converges to f **pointwise** if for each $x \in A_0$, The sequence (of real numbers) $f_n(x)$ converges to $f(x)$.

In other words, $\{f_n : A \rightarrow \mathbb{R}\}$ converges to f pointwise if for every $x \in A_0$, $\forall \varepsilon > 0$, $\exists n_0(x, \varepsilon) \in \mathbb{N}$ such that $\forall n \geq n_0(x, \varepsilon)$, we have $|f_n(x) - f(x)| < \varepsilon$

If we remove the dependence of n_0 above on the x , and leave it just to depend on ε , we arrive at the definition of *Uniform Convergence*.

Definition 1.21: Uniform Convergence of a sequence of functions

Let $\{f_n\}$ be a sequence of functions defined on $A \subset \mathbb{R}$. We say $\{f_n\}$ converges **uniformly** to $f : A_0 \rightarrow \mathbb{R}$ if $\forall x \in A_0$, $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0(\varepsilon)$, $|f_n(x) - f(x)| < \varepsilon$.

Since the dependence of n on x is non-existent, we can rewrite the above definition to read: $\{f_n\}$ converges **uniformly** to $f : A_0 \rightarrow \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0(\varepsilon)$, $\forall x \in A_0$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem 1.22: Non uniform convergence criteria

A sequence of functions $\{f_n\}$ defined on A **does not** converge **uniformly** to $f : A_0 \rightarrow \mathbb{R}$ if and only if $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}$, $\exists n_k \geq k$ and $x_k \in A$ so that $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon$, which means for some $\varepsilon > 0$, there exists a subsequence f_{n_k} of f_n and a sequence in x_k in A_0 so that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon \quad \forall k \in \mathbb{N}$$

Proof for Theorem.

■ Obvious ■

Definition 1.23: The Uniform Norm

Say $A \subseteq \mathbb{R}$ and $\phi : A \rightarrow \mathbb{R}$. We say ϕ is bounded on A if $\phi(A)$, the set, is bounded in \mathbb{R} . If it is bounded, we can define what is **the uniform norm of ϕ on A** by:

$$\|\phi\|_A := \sup\{|\phi(x)| : x \in A\}$$

It follows that $\|\phi\|_A \leq \varepsilon \iff |\phi(x)| \leq \varepsilon, \forall x \in A$

Lemma 1.24

A sequence f_n of bounded functions on A converges uniformly to $f : A \rightarrow \mathbb{R}$ if and only if $\|f_n - f\|_A \rightarrow 0$

Proof for Lemma

\implies) Say $f_n \Rightarrow f$, this means that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ so that $\forall n \geq n_0, \forall x \in A$, we have $|(f_n - f)(x)| < \varepsilon$, which means that for every $n \geq n_0$, we have $\|(f_n - f)\|_A \leq \varepsilon$ which concludes the forward direction.

\impliedby) Say $\|f_n - f\|_A \rightarrow 0$, which means that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ so that $\forall n \geq n_0$, we have $\|f_n - f\|_A < \varepsilon$ which means $\sup\{|(f_n - f)(x)| : x \in A\} < \varepsilon$ which means $\forall x \in A, |f_n(x) - f(x)| < \varepsilon$ whence the back implication is also done. ■

Theorem 1.25: Cauchy criteria for uniform convergence

A sequence f_n of bounded functions on A is uniformly convergent to $f : A \rightarrow \mathbb{R}$ (a bounded function) if and only if for every $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ so that $\forall m, n \geq n_0, \|f_m - f_n\|_A \leq \varepsilon$

Proof for Theorem.

\implies) Say $f_n \Rightarrow f$. This means that $\forall \varepsilon > 0, \exists n_0$ so that $\forall n, m \geq n_0$, we have, for every $x \in A$, $|f_n(x) - f(x)| < \varepsilon/2$ and $|f(x) - f_m(x)| < \varepsilon/2$. Simply add them. We then have $|(f_m - f_n)(x)| < \varepsilon$ for every $x \in A$. In terms of the uniform norm. we see that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ so that for all $m, n \geq n_0$, we have $\|f_m - f_n\|_A \leq \varepsilon$.

Suppose that $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ so that $\forall m, n \geq n_0, \|f_m - f_n\|_A \leq \varepsilon$ which means that for every $x \in A$, we have $|f_m(x) - f_n(x)| < \varepsilon$. This means that for every x , $f_n(x)$ is a cauchy sequence, hence convergent to some $f(x)$. More can be said for the sequence at each x . Fix some $x \in A$. We have that $\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n, m \geq n_0$, we have $|f_n(x) - f_m(x)| < \varepsilon$ which means that $\forall n \geq n_0, |f_n(x) - f(x)| < \varepsilon$ which means that, all in all, $\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}$ so that $\forall n \geq n_0, \forall x \in A$, we have $|f_n(x) - f(x)| < \varepsilon$ which is

the uniform convergence criteria. ■

l_p Space**Definition 1.26: l_p -Space**

Define l_p as the space of all sequences $(a_n)_{n=1}^{\infty}$ such that $\sum_{i=1}^{\infty} |a_i|^p$ exists. **The set of all sequences in Real line that are p summable**

It is easily seen that l_p is a vector space over \mathbb{R} . We define the metric on l_p as $d_p(\{x\}, \{y\}) = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p}$. Positive definiteness is clear since the sum would not be 0 if even one index is different. Symmetry is obvious, and triangle:

$$d(x, z) = (\sum_{i=1}^{\infty} |x_i - z_i|^p)^{1/p} \leq (\sum_{i=1}^{\infty} |x_i - y_i + y_i - z_i|^p)^{1/p}$$

which gives

$$\leq (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p} + (\sum_{i=1}^{\infty} |y_i - z_i|^p)^{1/p} = d(x, y) + d(y, z)$$

(This must be obvious from the infinite case of minkowski inequality).

Example : Continuous functions on a closed bounded interval

$C[a, b]$ is a metric space under the metric $d(f, g) := \sup\{|f(x) - g(x)| : x \in [a, b]\}$. Positive definiteness comes from $|\cdot|$. Symmetry is aswell obvious. Look at $d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\} = \sup\{|f(x) - h(x) + h(x) - g(x)| : x \in [a, b]\}$ which gives $\leq \sup\{|f(x) - h(x)| + |h(x) - g(x)| : x \in [a, b]\} \leq \sup\{|f(x) - h(x)| : x \in [a, b]\} + \sup\{|h(x) - g(x)| : x \in [a, b]\}$ which gives us our desired $d(f, g) \leq d(f, h) + d(h, g)$ ■

Example : Space of all sequences in \mathbb{R}

We consider the space of all sequences in \mathbb{R} with the metric $d(\{x\}, \{y\}) := \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{(1 + |x_i - y_i|)^{2^i}}$. This is quite well defined since the series is bounded by $\sum 1/2^i$. Positive definiteness is clear since the sum is of positive numbers. Symmetry is aswell clear since $d(x, y)/1 + d(x, y)$ is a well defined metric as seen before. Consider $d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} (\frac{|x_i - y_i|}{1 + |x_i - y_i|}) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} (\frac{|x_i - z_i|}{1 + |x_i - z_i|}) + \sum_{i=1}^{\infty} \frac{1}{2^i} (\frac{|z_i - y_i|}{1 + |z_i - y_i|}) = d(x, z) + d(z, y)$ (the inequality comes from the metric structure of $d/1 + d$ metric). ■

Definition 1.27: Norm

Given a Linear space $V(\mathbb{R})$, a norm, $\|\cdot\|$, is a function mapping every vector to a real number satisfying the following for every $x, y \in V$ and every $\alpha \in \mathbb{R}$:

1. $\|x\| \geq 0$ and equality of and only if $x = 0$.
2. $\|\alpha x\| = |\alpha| \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle)

Definition 1.28: Psuedo Metrics

Let X be a non-empty set. A psuedometric is a function $d : X \times X \rightarrow \mathbb{R}$ that follows:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$
3. $d(x, y) = d(y, z)$
4. $d(x, y) \leq d(x, z) + d(y, z)$

The part that is different is that, for metric spaces, we require that if the distance is 0, the points are the same. But for psuedometric spaces, the points can be different yet of the same distance.

Theorem 1.29: Coordinate-wise convergence in \mathbb{R}^n with d_p metric

With the metric $d_p(x, y) = (\sum_{i=1}^n (|x_i - y_i|^p))^{1/p}$ on \mathbb{R}^n , convergence implies coordinate wise convergence

Proof for Theorem.

Suppose a sequence \vec{x}_k converges to \vec{x} , i.e, $(x_1^k, x_2^k \cdots x_n^k) \rightarrow (x_1, x_2, \cdots x_n)$. This means that for every $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ so that $\forall k \in \mathbb{N}, k \geq n_0$ would mean that

$$(\sum_{i=1}^n (|x_i^k - x_i|^p))^{1/p} < \varepsilon$$

From here, coordinate wise convergence is obvious.

Suppose that $\lim_{k \rightarrow \infty} x_j^k = x_j$ for every $j \leq n$, i.e, coordinate wise convergence. That means that for a given j , for every $\varepsilon > 0$, there exists $n_0(\varepsilon, j)$ such that $\forall k \in \mathbb{N}, k \geq n_0(\varepsilon, j)$ means that $|x_j^k - x_j| < \frac{\varepsilon}{n^{1/p}}$. If we take the maximum of all the $n_0(\varepsilon, j)$, and sum $(\sum_{i=1}^n |x_j^k - x_j|^p)^{1/p} = \sum_{i=1}^n (\frac{\varepsilon^p}{n})^{1/p} = \varepsilon$, which implies convergence in \mathcal{R}^n . In a similar fashion, we can show that \mathbb{R}^n, d_∞ , the max norm, has equivalence between convergence and coordinate wise convergence ■

Theorem 1.30

\mathbb{R}^n with the metric $d_\infty(x, y) = \max\{|x_j - y_j| : j \leq n\}$ also has an equivalence between convergence and coordinate wise convergence.

Example : Cauchy sequences need not converge on $C[0, 1]$

Consider $X = C[0, 1]$, with the metric $d(f, g) := \int_{[0,1]} |f(x) - g(x)| dx$. Let $f_n(x)$ be a

sequence defined as follows:

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n(x - \frac{1}{2}) + 1, & \text{if } \frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

. Is this sequence cauchy? Yes. Consider $\int_{[0,1]} |f_m(x) - f_n(x)| dx \leq \frac{1}{2}(n + m/(mn))$. For any given ε , we can make m, n large enough to go below it. Suppose that f_n converges to some f , implying that for every $x \in [0, 1]$, we have that $\forall \varepsilon, \exists n_0 \in \mathbb{N}$ so that $\forall n \in \mathbb{N}, n \geq n_0$ implies $d(f_n, f) = \int_{[0,1]} |f_n(x) - f(x)| dx < \varepsilon$. $d(f_n, f) = \int_{[0, \frac{1}{2} - \frac{1}{n}]} |f(x)| dx + \int_{[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]} |f_n(x) - f(x)| dx + \int_{[\frac{1}{2}, 1]} |1 - f(x)| dx$ but since $\lim_{n \rightarrow \infty} (d(f_n, f)) \rightarrow 0$, we require that $\int_{[0, \frac{1}{2} - \frac{1}{n}]} |f(x)| dx = 0$ and $\int_{[\frac{1}{2}, 1]} |1 - f(x)| dx = 0$. Since f is continuous (or supposed to be), we see that $f(x) = 0$ from 0 to $\frac{1}{2}$, but 1 from $\frac{1}{2}$ to 1, which is absurd. ■

Lemma 1.31: A useful lemma comparing norms on \mathbb{R}^n

For \mathbb{R}^n , we have

$$\|\cdot\|_p \leq \|\cdot\|_1 \leq n \|\cdot\|_\infty \leq n \|\cdot\|_p$$

Proof for Lemma

Consider any vector v in \mathbb{R}^n , $v = (x_1, x_2, \dots, x_n)$. We have $\|v\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p} \leq (\sum_{i=1}^n |x_i|) \leq n \max\{x_i : i \leq n\} \leq n (\sum_{i=1}^n |x_i|^p)^{1/p}$ ■

Definition 1.32: Complete Metric Space

A metric space X, d is said to be *complete* if every cauchy sequence in X converges in X

Theorem 1.33

\mathbb{R} with $d(x, y) = |x - y|$ is a complete metric space

Theorem 1.34

\mathbb{R}^n with d_p, d_∞ , is complete.

Proof for Theorem.

Consider a cauchy sequence $\{x^{(k)}\}$ in \mathbb{R}^n , i.e, $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots, x_n^{(k)})$ so that $\forall \varepsilon > 0, \exists m_0(\varepsilon)$ so that $\forall a, b \geq m_0(\varepsilon)$ we have $d(x^{(a)}, x^{(b)}) < \varepsilon$ i.e:

$$|x_i^a - x_i^b| \leq (\sum_{i=1}^n |x_i^{(a)} - x_i^{(b)}|^p)^{1/p} < \varepsilon$$

which implies coordinate wise cauchy, which means coordinate wise convergence. That is

enough to ensure full convergence. This holds for both d_p and d_∞ . ■

Theorem 1.35

ℓ_p space, space of all p -summable sequences in \mathbb{R} , is a complete metric space.

Proof for Theorem.

Let $\{x^k\}$ be a sequence of sequences where $x^{(k)} = (x_1^k, x_2^k \dots)$. The norm here is $d(x^a, x^b) = (\sum_{i=1}^{\infty} |x_i^a - x_i^b|^p)^{1/p}$. $\forall \varepsilon > 0$, $\exists m_0$ so that $\forall a, b \geq m_0$, $d(x^a, x^b) = (\sum_{i=1}^{\infty} |x_i^a - x_i^b|^p)^{1/p} < \varepsilon$. Note that x_i^k is point wise cauchy, hence pointwise convergent to say, $x_i^{(k)} \rightarrow x_i$. Define $x := (x_1, x_2 \dots)$. For a given ε we have n_0 so that

$$(\sum_{i=1}^j |x_i^a - x_i^b|^p)^{1/p} \leq (\sum_{i=1}^{\infty} |x_i^a - x_i^b|^p)^{1/p} < \varepsilon$$

for all $a, b \geq n_0$, (for every j , too). We take the $b \rightarrow \infty$ limit here, to get:

$$(\sum_{i=1}^j |x_i^a - x_i|^p)^{1/p} < \varepsilon$$

for all $a \geq n_0$ and $\forall j$. From monotone convergence theorem, we can take the j limit as well, to arrive at: $\forall \varepsilon$, $\exists n_0$ so that $\forall a \geq n_0$, we have

$$d_p(x^a, x) (\sum_{i=1}^{\infty} |x_i^a - x_i|^p)^{1/p} < \varepsilon$$

which lets us conclude $x^k \rightarrow x$. But the question is, is $x \in \ell_p$? Consider $(\sum_{i=1}^j |x_i|^p)^{1/p} \leq (\sum_{i=1}^j |x_i^a - x_i|^p)^{1/p} + (\sum_{i=1}^j |x_i^a|^p)^{1/p}$ (Minkowski). Both the summands on the right side are convergent (by MCT, and definition, respectively). Hence $x \in \ell_p$, which makes ℓ_p a complete space ■

Theorem 1.36

Set of all bounded sequences on \mathbb{R} with d_∞ norm is complete

Proof for Theorem.

Consider $\{x^k\}$ where $x^k = (x_1^k, x_2^k \dots)$, a sequence that is cauchy. $\forall \varepsilon > 0$, $\exists n_0$ so that $\forall m, n \geq n_0$, we have $|x_i^m - x_i^n| \leq \sup\{|x_i^m - x_i^n| : i \in \mathbb{N}\} < \varepsilon$ (for every $i \in \mathbb{N}$). This ensures coordinate wise convergence. Let $x_i^n \rightarrow x_i$, and let $x = (x_1, x_2 \dots)$ so that $\forall \varepsilon > 0$, $\exists m_0$ so that $\forall n \geq m_0$, we have $|x_i^n - x_i| < \varepsilon$ for all $i \in \mathbb{N}$. This means for all $n \geq m_0$, $\sup\{|x_i^n - x_i| : i \in \mathbb{N}\} < \varepsilon$, which is the condition for convergence. Hence, $x^n \rightarrow x$. Is x a bounded sequence? We require that $|x_i| < M$ for some M , for all $i \in \mathbb{N}$. Fix an ε , and get a corresponding n and fix it so that $|x_i^n - x_i| < \varepsilon$ for all $i \in \mathbb{N}$, we then have: $|x_i| \leq |x_i^n - x_i| + |x_i^n| \leq \varepsilon + |x_i^n| < \text{some fixed number}$ (since $x^n \in$ space of all bounded

sequences $B(\mathbb{N})$, hence we see that $B(\mathbb{N})$ is cauchy complete. ■

Theorem 1.37

Set $C[0, 1]$ of all continuous functions on closed, bounded interval $[0, 1]$ is cauchy complete under the supremum norm.

Proof for Theorem.

Consider $\{f_n\}$ a sequence of continuous functions on $[0, 1]$ that is said to be cauchy. This implies that, $\forall \varepsilon > 0$, $\exists n_0$ so that $\forall n, m \geq n_0$, $\sup\{|f_n(x) - f_m(x)| : x \in [0, 1]\} < \varepsilon$. This means that $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in X$ (beyond a certain n_0). Hence, $f_m(x)$ converges to say $f(x)$. Moreover, note that for every point x , $\forall \varepsilon > 0$, $\exists n_0(\varepsilon)$ (independent of x) so that $|f_n(x) - f(x)| < \varepsilon$ for $n \geq n_0(\varepsilon)$. It is obvious then that $f(x)$ is bounded on $[0, 1]$. We also have that $\forall \varepsilon > 0 \exists n_0(\varepsilon)$ so that $\sup\{|f_n(x) - f(x)| : x \in [0, 1]\} < \varepsilon$, which tells that $f_n \rightarrow f$ (uniformly). Now all that is left to see is that f is continuous.

Let $\varepsilon > 0$ be arbitrary. For every function f_n , there exists $\delta(n)(\varepsilon) > 0$ so that $\forall a, b \in [0, 1]$, $0 < |a - b| < \delta \implies |f_n(a) - f_n(b)| < \varepsilon/4$. We also note that, there exists an n_0 so that for all $m, n > n_0$, we have for every $x \in [0, 1]$, $|f_n(x) - f_m(x)| < \varepsilon/4$ for all $m, n \geq n_0$. Moreover, we also have that there exists $n' \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \varepsilon/4$ for all x , $\forall n \geq n'$.

$|f(a) - f(b)| = |f(a) - f_n(a) + f_n(a) - f_m(a) + f_m(a) - f_m(b) + f_m(b) - f(b)| \leq |f_n(a) - f(a)| + |f_m(a) - f_m(b)| + |f_n(a) - f_m(a)| + |f_m(b) - f(b)|$ There exists δ , minimum of the two δ_n, δ_m that ensures that the first two moduli go lower than $\varepsilon/4$. There exists large enough m, n to ensure the remaining terms go lower than $\varepsilon/4$ as well. All together, we have that $\exists \delta > 0$ so that if $a, b \in [0, 1]$ such that $0 < |a - b| < \delta$, we have $|f(a) - f(b)| < 4(\varepsilon/4) = \varepsilon$. Hence, f is continuous. Therefore, $C[0, 1]$ (and more generally, $C[\mathbb{I}]$) is complete. ■

1.2 More basics- Closures, Relativeness, Interiors and Boundaries

Definition 1.38: Limit Point of a set E

We say p is a limit point of a set E if

$$(\forall \varepsilon > 0)(\exists q_\varepsilon \in E; q_\varepsilon \neq p)(d(q_\varepsilon, p) < \varepsilon)$$

In other words, in every ε -ball around p , there would exist a point q_ε in E , which is different from p .

Theorem 1.39

Every ball / neighbourhood of p which is a limit point of E , would contain infinitely many points q such that $q \in B_\varepsilon(p) \cap E \setminus \{p\}$

Proof for Theorem.

Suppose for some neighbourhood, there only exists finite points q_1, q_2, \dots, q_k such that $q_j \in B_{\varepsilon_0} \cap E \setminus \{p\}$. Let $\delta < \min\{d(p, q_j) : j \in [1, 2, \dots, k]\}$. We then have that, there exists no point $q \in E$ such that its distance from p is less than δ , making p a non-limit point. Absurd. ■

Corollary 1.40

A finite set has no limit points

Theorem 1.41: Recharacterisation of Limit points

A point $p \in X$ is a limit point of $E \subset X$ if and only if there exists a sequence $x_n \in E$, $x_n \neq p \forall n \in \mathbb{N}$ such that $\{x_n\} \rightarrow p$

Proof for Theorem.

\implies) If p is a limit point, around every neighbourhood, there would exist a point $q_\varepsilon \in E$ such that $0 < d(q_\varepsilon, p) < \varepsilon$. Choose $\varepsilon_1 = 1$, and obtain x_1 such that $x_1 \in E$, $x_1 \neq p$ and $0 < d(x_1, p) < 1$. Choose $\varepsilon_2 = \frac{1}{2}(d(x_1, p))$. We find $x_2 \in E$, $x_2 \neq p$ such that $d(x_2, p) < \frac{d(x_1, p)}{2} < \frac{1}{2}$. Continue as such to obtain a sequence that converges to p .

\impliedby) Suppose there is a sequence x_n such that $x_n \neq p \forall n \in \mathbb{N}$ and $\forall \varepsilon, \exists n_0(\varepsilon)$ such that $\forall n \geq n_0$ we have $d(x_n, p) < \varepsilon$ which means for a given ε , there exists a point x_{n_0+1} in E such that it is not equal to p and it is in the ε -ball of p . Hence, p would be a limit point. ■

Definition 1.42: Closed sets in X

A set E is closed in X if every limit point of E is contained in E

Definition 1.43: Equivalent definition of closed sets in X

A set E in X is closed if for every convergent sequence x_n in x such that $\lim(x_n) \neq x_n$ for any n , we have $\lim(x_n) \in E$.

Definition 1.44: Open sets in X

A set E is said to be open if $\forall x \in E, \exists \xi_x > 0$ such that $B_{\xi_x}(x) \subset E$

Theorem 1.45

Every open ball is an open set

Proof for Theorem.

Suppose a is a fixed point in X and $\delta > 0$ is given. $B = B_\delta(a) := \{y \in X : d(y, a) < \delta\}$. Consider arbitrary $z \in B$, for which we have $d(z, a) = t < \delta$. Therefore $\delta - t > 0$. Consider $0 < \xi_z = r < \delta - t$ from Density. Consider an arbitrary x such that $d(x, z) < \xi_z = r < \delta - t$. $d(x, a) \leq d(x, z) + d(z, a) = r + t \leq \delta - t + t = \delta$. We are done. ■

Definition 1.46: Compliment with respect to X

If $E \subseteq X$, we define compliment of E as

$$E^C := \{x \in X : x \notin E\}$$

Definition 1.47: Bounded

A set $E \subset X$ is bounded if \exists a positive number $M > 0$ and $q \in E$ such that $d(x, q) < M \forall x \in E$. i.e, all the points of E gets contained in some ball in X .

Theorem 1.48: De Morgan's Law

Let $\{E_\alpha : \alpha \in A\}$ where A is some arbitrary indexing set represent a collection of sets in X . Then

$$(\cup_\alpha E_\alpha)^C = \cap_\alpha E_\alpha^C$$

Proof for Theorem.

Consider $(\cup_{\alpha} E_{\alpha})^c = \{x \in X : \exists \alpha \in A : x \in E_{\alpha}\}^c = \{x \in X : \forall \alpha \in A : x \notin E_{\alpha}\} = \{x \in X : \forall \alpha \in A : x \in E_{\alpha}^c\} = \cap_{\alpha} E_{\alpha}^c$ ■

Theorem 1.49: The Big Equivalence

$E \subset X$ is open $\iff E^c$ is closed.

Proof for Theorem.

\implies) Suppose that E is open but E^c is not closed. This means that there exists a limit point of E^c that falls in E , i.e, outside E^c . Let this be q . This means for every ε -ball around q , a point of E^c exists. But since E is open and $q \in E$, we have for a particular ε -ball, inside which, no point of E^c resides. Contradiction.

\impliedby) Suppose E is closed but E^c is not open. This means that there is a point in E^c , p , such that for every ε -ball around p , some point in E falls into this ball. But this makes p a limit point of E , which is absurd since E is closed, limit points fall into the sets themselves. ■

Theorem 1.50

For a collection of open sets $\{G_{\alpha} : \alpha \in A\}$, $\cup_{\alpha} G_{\alpha}$ is also an open set.

Proof for Theorem.

Consider $x \in \cup_{\alpha} G_{\alpha}$ which means $\exists \alpha_x \in A$ such that $x \in G_{\alpha_x}$ which means, there would exist an ξ -ball around x that is contained in G_{α_x} which is in turn contained in $\cup_{\alpha} G_{\alpha}$. ■

Corollary 1.51

For any collection of closed sets E_{α} , $\cap_{\alpha} E_{\alpha}$ is also closed.

Proof for Corollary.

$\{E_{\alpha}^c\}$ is a collection of open sets, and $\cup_{\alpha} E_{\alpha}^c$ is an open set, which means $\cup_{\alpha} E_{\alpha}^c = (\cap_{\alpha} E_{\alpha})^c$ is an open set, from which we get that $(\cap_{\alpha} E_{\alpha})$ is a closed set. ■

Theorem 1.52

For any finite collection of open sets $\{E_1, E_2, \dots, E_k\}$, $\cap_{i=1}^k E_i$ is also open.

Proof for Theorem.

Suppose $x \in \cap_{j=1}^k E_j$, which means $\forall j \in [1, k], x \in E_j$. We have $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ such that, the ε_j -ball around x is fully contained in E_j . Choose $0 < \delta < \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$ (the minimum exists by virtue of being a finite set). We see that the δ -ball around x is a subset of every ε_j -ball around x , which means that the δ -ball around x is in every E_j , which proves the theorem. ■

Corollary 1.53

For any finite collection of closed sets $\{G_1, G_2, \dots, G_k\}$ we have $\cup_{j=1}^k G_j$ to be closed

Remark.

In the above theorem and corollary, we require that the collection be finite. The reason is that, we were able to get a minimal ε_j in the proof due to the finiteness of the set. It may not be possible to find a number δ that is both larger than 0 but smaller than a given infinite collection of ε -s. For example, consider the sequence of open sets $(-\frac{1}{n}, \frac{1}{n})$. The infinite intersection of these yields $\{0\}$ which is a closed set by virtue of being finite.

Definition 1.54: Topology

Let X be a set and τ be a family of sets in X . X, τ is called a topology (with elements of τ being called open sets) if:

1. Both ϕ and X are in τ
2. τ is closed under arbitrary unions
3. τ is closed under finite intersections

The complement of an element in τ with respect to X is called a closed set.

Definition 1.55: Closure of a set

Let E' be the set of all limit points of E . Then, the closure of E is :

$$\bar{E} := E \cup E'$$

Theorem 1.56

Closure of a set is closed

Proof for Theorem.

Let p be a limit point of $E \cup E'$. That means that $\forall \varepsilon > 0 \exists q_\varepsilon \in (E \cup E'), q_\varepsilon \neq p$ such that $q_\varepsilon \in B_\varepsilon(p)$. If p is in $E \cup E'$, we are done (especially if p is in E). Suppose p is not in E . $\forall \varepsilon > 0 \exists q_\varepsilon \in (E \cup E'), q_\varepsilon \neq p$ such that $q_\varepsilon \in B_\varepsilon(p)$. If the q_ε we receive falls in E we are ok. Suppose q_ε falls in E' . That means: $\forall \delta > 0, \exists r_\delta \in E, r_\delta \neq q_\varepsilon$ such that $d(r_\delta, q_\varepsilon) < \delta \implies d(r_\delta, p) \leq d(r_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \delta + d(q_\varepsilon, p) < \delta + \varepsilon$. If we choose $\delta_0 < \varepsilon - d(q_\varepsilon, p)$ we get: $d(r_\delta, p) \leq d(r_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \delta + d(q_\varepsilon, p) < \varepsilon$

Summarising we have: $\forall \varepsilon > 0, \exists q_\varepsilon \in E$ or E' where: $q_\varepsilon \in E$ and $q_\varepsilon \in B_\varepsilon(p)$

or

$\exists \delta(\varepsilon) > 0$ such that $\exists r_\delta \in E$ such that $r_\delta \neq p$ and $r_\delta \in B_\varepsilon(p)$. In either case, there would exist a point dependent on ε , in E such that the point itself is different from p , and exists in the ε -ball around p . Hence, we see that p is a limit point of E . Therefore, we see that all the limit points of E either are points of E or points of E' . Hence, \bar{E} is closed. ■

Theorem 1.57

$$\bar{E} = E \iff E \text{ is closed}$$

Proof for Theorem.

\implies) \bar{E} is closed, so E would be too.

\impliedby) if E is closed, $E' \subseteq E \implies E' \cup E = E = \bar{E}$ ■

Theorem 1.58: \bar{E} is the smallest closed set that contains E

If F_α is the collection of all closed sets such that $E \subseteq F_\alpha$, then $\bar{E} \subseteq F_\alpha$ for all α .

Proof for Theorem.

Consider an arbitrary closed set F_α that contains E . It would obviously contain all the limit points of E among other things. Therefore, we can easily see that it contains $E \cup E' = \bar{E}$. ■

Fact 1.59

Topological Definition for Closure: The closure of $A \subseteq X$, denoted by \bar{A} is defined as the smallest closed set that contains A

Lemma 1.60: An equivalent definition for Closure.

An equivalent definition for closure is:

$$\bar{A} := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset\}$$

Proof for Lemma

We see that obviously, if $x \in \bar{A}$, then either it is a point of A , or if not, it happens to be a limit point of A . And the back implication: If q is a point of A or if it is a limit point of A , it obviously falls into \bar{A} . ■

Example : If $E \subseteq \mathbb{R}$ is bounded (and non empty), with $s = \sup(E)$, then $s \in \bar{E}$. If $s \in E$ we are done. If not, then $\forall \varepsilon > 0$, $\exists \varepsilon > \delta(\varepsilon) > 0$, and a point $x_\varepsilon \in E$ such that $s - \varepsilon < s - \delta(\varepsilon) \leq x_\varepsilon < s + \delta < s + \varepsilon$ where $x_\varepsilon \neq s$. Hence, s is a limit point of E and

hence, is a point in the closure.

Definition 1.61: Open Relative

Say $E \subseteq Y \subseteq X$, where X is a metric space. Y is also a metric space. We say E is open relative to Y if $\forall x \in E, \exists \varepsilon > 0$ such that if $y \in Y$ and $y \in B_\varepsilon(x)$ then $y \in E$. Formally:

$$(\forall x \in E)(\exists \varepsilon_x > 0)((y \in Y \cap B_\varepsilon(x)) \implies y \in E)$$

Remark.

A set which is open relative to Y need not be open relative to X . For example, consider \mathbb{R} as a subset of \mathbb{R}^n . An interval in \mathbb{R} is open relative to \mathbb{R} , but it is not open relative to \mathbb{R}^n .

Definition 1.62: Limit point relative

Say $S \subseteq Y \subseteq X$. x is said to be a limit point of S relative to Y if for every ε -ball around x relative to Y (i.e, the set $B_\varepsilon(x) \cap Y$, or $\{y \in Y : d(y, x) < \varepsilon\}$) there exists a point $y \in S$. In technical terms: x is a limit point of

$$(\forall \varepsilon > 0)(\exists y \in Y \cup B_\varepsilon(x))(y \in S)$$

Definition 1.63: Closed relative

A set $S \subset Y \subset X$ is closed relative to Y if every limit point of S relative to Y is in S .

Remark.

Closed relative to Y needn't imply closed relative to X . For example, a convergent sequence of S that converges in X needn't converge in Y . As a result, if this sequence that converges in X (but not in Y) converges outside S , under the subspace topology of Y , this simply does not contribute as a limit point. Whereas, in X this contributes as a limit point, and hence is not closed in X .

Remark.

Note that, we needn't define the concept of "relative limit point" since every limit point relative to Y , is a limit point of S relative to X . But the back direction needn't be true. i.e, you can have x a limit point of S relative to X , but not relative to Y . More concretely, x is such that for every $\varepsilon > 0$, there exists $z \in S$ so that $z \neq x$ and $d(z, x) < \varepsilon$. If z is in X as well as Y , its a limit point of S with respect to both. Issue arises if x is not in Y . Then it does not make sense.

Theorem 1.64

A set $E \subseteq Y \subseteq X$ is open relative to $Y \iff \exists G \subset X$ that is open relative to X , such that $E = G \cap Y$

Proof for Theorem.

\implies) Say E is open relative to Y . This means that $\forall x \in E, \exists \varepsilon_x > 0$ such that if $y \in B_{\varepsilon_x}(x)$ and $y \in Y$, then $y \in E$. Call $G = \cup_{x \in E} B_{\varepsilon_x}(x)$ which is an open set. If $z \in E$, then $z \in G$ obviously, and hence $z \in G \cap Y$. Hence, $E \subseteq G \cap Y$. Consider a point $z \in G \cap Y$ which means z falls in one of the ε -balls around a point of E , and z is in Y . From definition of open relativeness, we see that $z \in E$. Hence, $E = G \cap Y$

\impliedby) Say $E = G \cap Y$ where G is an open set relative to X . Then, for every point in G , there would exist an ε -ball around that point that is completely contained in G . Let $x \in E$ be arbitrary. $\exists \varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset G$. Suppose $y \in Y$ and $y \in B_{\varepsilon_x}(x)$. This would mean that $y \in G \cap Y = E$. Hence, $\forall x \in E \exists \varepsilon > 0$ such that if $y \in B_{\varepsilon}(x)$ and $y \in Y$, then $y \in E$, which is the definition of open relativeness. ■

Fact 1.65

Topological definition of "open-relative": Let $A \subseteq Y \subseteq X$. A is said to be open relative to Y if there exists an open set G , open in X so that $A = G \cap Y$.

Theorem 1.66

A set S is open relative to $Y \subseteq X$ if and only if $Y \setminus S$ is closed relative to Y

Proof for Theorem.

Once we notice that replacing $B_{\varepsilon}(x) \cap Y$ with $B_{\varepsilon}^Y(x)$, i.e, the corresponding ε ball of x with respect to Y , the proof falls in much the same way as the earlier version, done for X . ■

Theorem 1.67

S is closed relative to $Y \subseteq X$ if and only if $S = H \cap Y$ for a closed set H closed in X

Proof for Theorem.

Say S is closed, that means $Y \setminus S = K$ is open relative to Y . That means there exists G open relative to X so that $G \cap Y = K$. $Y \setminus (K) = Y \cap K^C = Y \cap (G \cap Y)^C = Y \cap (G^C \cup Y^C) = (Y \cap G^C) \cap (Y \cup Y^C) = Y \cap G^C$. We have that $Y \setminus K = Y \setminus (Y \setminus S) = S = Y \cap G^C$.

Let $S = H \cap Y$ where H is closed in X . $Y \setminus S = Y \setminus (H \cap Y) = Y \cap (H^C \cup Y^C) = Y \cap H^C$, we done. ■

Theorem 1.68

Suppose $A_1, A_2 \dots, A_n \dots \in X$. Then,

1. If $B_n = \cup_{i=1}^n A_i$, then $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$
2. $B = \cup_{i=1}^\infty A_i$, $\bar{B} \supseteq \cup_{i=1}^\infty \bar{A}_i$ with possibility of strict inequality.

and as a result we have for open sets G_i :

1. $(\cap_{i=1}^n G_i)^\circ = \cap_{i=1}^n (G_i)^\circ$

Proof for Theorem.

Suppose $x \in \cup_{i=1}^n \bar{A}_i$. Which means $x \in \bar{A}_i$ for some i . It is clear that x is either a point of A_i or a limit point of A_i . We have that, whatever maybe the case either x is a point in B_n or a limit point of B_n . Therefore $\bar{B}_n \supset \cup_{i=1}^n \bar{A}_i$.

Suppose x is a point in \bar{B}_n . If it is a point of B_n , we are done. Suppose it is the limit point of B_n , but not a point. Also suppose that x is not a limit point of any $A_i : i = 1 \rightarrow n$. This means that,

$$\exists \varepsilon_1 \text{ such that } \forall q \in A_1, q \neq x, \text{ we have } d(q, x) \geq \varepsilon_1$$

$$\exists \varepsilon_2 \text{ such that } \forall q \in A_2, q \neq x, \text{ we have } d(q, x) \geq \varepsilon_2$$

$$\vdots$$

$$\exists \varepsilon_n \text{ such that } \forall q \in A_n, q \neq x, \text{ we have } d(q, x) \geq \varepsilon_n$$

If we choose $0 < \varepsilon_0 < \min\{\varepsilon_i : i = 1 \rightarrow n\}$, we would have that, for every point q in $A_1 \cup A_2 \dots A_n$, $q \neq x$ (which is needless to say), we have $d(q, x) \geq \varepsilon_0$ which makes x a non-limit point of B_n , which is absurd. Hence we see that if x is point of B_n or a limit point of B_n , then it is a point or the limit point of some A_j .

For a good counterexample, we look to $\mathbb{Q} := \{q : q \text{ is rational}\}$. This set is countable. Let $\{q_1, q_2, \dots\}$ be the enumeration of \mathbb{Q} . Consider $\mathbb{Q} := \cup_{j=1}^\infty \{q_j\}$. The closure of \mathbb{Q} is \mathbb{R} but since these singleton sets are by definition closed, the union of them only gives you \mathbb{Q} .

Proof for topological space:

if $B_n := \cup_{i=1}^n A_i$, is it true in topological spaces (and not just metric spaces) that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$? the direction $\overline{\cup_{i=1}^n A_i} \subseteq \cup_{i=1}^n \bar{A}_i$ can be shown from definition as the smallest closed set that contains B_n . But $\bar{A}_i \subseteq \overline{\cup_{i=1}^n A_i}$ for every i , so if we union them we must get $\cup_{i=1}^n \bar{A}_i \subseteq \overline{\cup_{i=1}^n A_i}$ ■

Definition 1.69: Interior of a Set

Given $S \in X$, the interior \underline{S} is defined as:

$$\underline{S} := \{x \in X : \exists \varepsilon_x > 0 \text{ such that } B_{\varepsilon_x}(x) \subset S\}$$

Theorem 1.70

The Interior is an open set

Proof for Theorem.

Consider $(\underline{S})^C := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap S^C \neq \emptyset\}$. It is possible that x is a point of S^C , if it is in \underline{S}^C . Suppose it is a point of \underline{S}^C but not a point of S^C . From the definition, we see that $\forall \varepsilon > 0, \exists q \in S^C, q \neq x$ such that $d(q, x) < \varepsilon$. This makes x a limit point of S^C , which means, for every x in \underline{S}^C , x is either a point of S^C or a limit point of S^C . Hence,

$$\underline{S}^C \subseteq \bar{S}^C$$

Suppose x is a point of \bar{S}^C . Say it is a point of S^C , then obviously, it is a point of $(\underline{S})^C$. Suppose x is not a point of S^C , but a limit point of S^C . This means $\forall \varepsilon > 0, \exists q \in S^C, q \neq x$ so that $d(q, x) < \varepsilon$. This is precisely the condition for which x is a point of $(\underline{S})^C$. Hence we see $(\underline{S})^C \supseteq (\bar{S}^C)$. Therefore, $(\underline{S})^C = (\bar{S}^C)$. From here we see that \underline{S} is an open set.

Alternate Argument (similar): Consider $(\underline{S})^C := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap S^C \neq \emptyset\}$. Consider a limit point p of $(\underline{S})^C$. $\forall \varepsilon > 0, \exists q_\varepsilon \in \underline{S}^C$ such that $d(q_\varepsilon, p) < \frac{\varepsilon}{2}$. Since q_ε is in $(\underline{S})^C$, we have that: $\forall \delta > 0, \exists r_\delta \in S^C, r_\delta \neq q_\varepsilon$ such that $d(r_\delta, q_\varepsilon) < \delta$

Combining these we have:

$$(\forall \varepsilon > 0)(\exists q_\varepsilon \in \underline{S}^C)(\exists \delta > 0)(\exists q_\delta \in S^C)$$

$$(d(q_\delta, p) \leq d(q_\delta, q_\varepsilon) + d(q_\varepsilon, p) < \frac{\varepsilon}{2} + \delta < \varepsilon)$$

This means p is a limit point of S^C . Hence, \underline{S}^C is closed. ■

Theorem 1.71

$$\underline{S} = S \iff S \text{ is open}$$

Proof for Theorem.

\implies) if $\underline{S} = S$, obviously S is open.

\impliedby) If S is open, then by definition \underline{S} = set of all points in S so that there's an ε -ball of x in S . But that is every point of S . ■

Theorem 1.72

\underline{S} is the largest open set contained in S

Proof for Theorem.

Consider an open subset of S . These are subsets of S whose each point has an ε -ball around it so that the ball is contained in the subset, which is contained in S . So by definition, these points in these subsets are contained in \underline{S} . ■

Theorem 1.73

$$(\underline{S})^C = \overline{(S^C)}$$

"The compliment of the interior is the closure of the compliment"

Proof for Theorem.

Refer to the proof of "Interiors of sets are Open", to see this construction.

Alternate method(slicker):

$\underline{S} \subseteq S \implies S^C \subset (\underline{S})^C$ where $(\underline{S})^C$ is a closed set containing S^C . Since $\overline{S^C}$ is the smallest closed set that contains S^C , we have $\overline{S^C} \subseteq (\underline{S})^C$.

Note that $S^C \subseteq \overline{S^C} \implies (\overline{S^C})^C \subseteq S$ where $(\overline{S^C})^C$ is an open set inside S . Since \underline{S} is the largest open set containing S , we have that $(\overline{S^C})^C \subseteq \underline{S} \implies \underline{S}^C \subseteq \overline{S^C}$. Combining these two set inequalities, we are done. ■

Definition 1.74: Isometry

(X, d_X) is said to be isometric to (Y, d_Y) if there exists a function $\phi : X \rightarrow Y$ referred to as the isometry, such that $d_Y(\phi(x), \phi(y)) = d_X(x, y)$ for all $x, y \in X$

Theorem 1.75

Let X, d_X and Y, d_Y be isometric under the map $\varphi : X \rightarrow Y$. Then the map φ is injective, and its inverse $\varphi^{-1} : \phi(X) \rightarrow X$ is also an isometry.

Proof for Theorem.

Suppose $\varphi(x) = \varphi(y)$. Then $d_Y(\varphi(x), \varphi(y)) = d_X(x, y) = 0 \implies x = y$.

$d_X(\varphi^{-1}(\varphi(a)), \varphi^{-1}(\varphi(b))) = d_X(a, b) = d_Y(\varphi(a), \varphi(b))$ which makes φ^{-1} an isometry from $\phi(X) \rightarrow X$. ■

Theorem 1.76

Let Z be a set and $f : Z \rightarrow (X, d)$ be any injective map from Z to X . Then $\delta(x, y) := d(f(x), f(y))$ would form a metric on Z and make it a metric space, and this would in turn make f an isometry from (Z, δ) to (X, d)

Proof for Theorem.

$\delta(x, y) = d(f(x), f(y))$ would obey all the properties of a metric space, and moreover, $\delta(x, y) = d(f(x), f(y))$ making it by definition an isometry. ■

Definition 1.77: Diam

$$\text{Diam}(A \subseteq X) := \sup\{d(x, y) : x \in A, y \in A\}$$

Definition 1.78: Distance between a point and a set

$$d(x, A) := \inf\{d(x, a) : a \in A\}$$

Theorem 1.79

$$x \in \overline{A} \iff d(x, A) = 0$$

Proof for Theorem.

Obviously, if $d(x, A) = 0$, either $x \in A$ or $\forall \varepsilon > 0, \exists a_\varepsilon \in A, a_\varepsilon \neq x$ such that $d(x, a_\varepsilon) < \varepsilon$, making x a limit point of A . The other direction is just as trivial. ■

Definition 1.80: Isolated point

$x \in X$ is said to be an isolated point of $S \subseteq X$ if $x \in S$ and $d(x, S \setminus \{x\}) \neq 0$ or rather, there is a ball around x such that $B_\varepsilon(x) \setminus \{x\}$ is completely disjoint from S .

Remark.

In some sense, being an isolated point of a set is the opposite of being a limit point. x is a limit point of S if for every $\varepsilon > 0$, there exists a point $s_\varepsilon \in S \setminus \{x\}$ so that $s_\varepsilon \in B_\varepsilon \setminus \{x\}$ or rather for every $\varepsilon > 0$, $B_\varepsilon(x) \setminus \{x\} \cap S \neq \emptyset$. The negation of this statement is that there exists an $\varepsilon > 0$ so that $B_\varepsilon(x) \setminus \{x\} \cap S = \emptyset$.

Definition 1.81: Nearest Point

Let $S \subseteq X$ and $z \in X$. If there exists $s \in S$ so that $\text{dist}(z, S) = d(z, s)$ then s is said to be the *nearest point* of S to z .

Definition 1.82: Boundary Point

A point x in X is said to be a boundary point of $S \subseteq X$ if $\text{dist}(x, S) = \text{dist}(x, S^c) = 0$. Right from the definition, a point x is a boundary point of S if and only if its a boundary point of S^c . The **Boundary** of a set is defined as the set of all boundary points of S denoted $\partial(S)$.

Theorem 1.83

$$\partial(S) = \overline{S} \setminus S^\circ$$

Proof for Theorem.

∂S was defined as the set of all points x so that $\text{dist}(S, x) = \text{dist}(x, S^c) = 0$. By virtue of that, $x \in \partial S$ implies $x \in \overline{A}$. Can $x \in S^\circ$? That would mean that there is an epsilon ball around $x \in S^\circ$ which means that the distance between x and S^c would be non zero. Hence, $\partial S \subseteq \overline{S} \setminus \{S^\circ\}$

Now let $x \in \overline{S} \setminus S^\circ$. Clear that $\text{dist}(x, S) = 0$ Obviously. $\overline{S} \cap (S^\circ)^c = \overline{S} \cap \overline{(S^c)}$. This tells us that x is in $\overline{S^c}$ as well, hence $\text{dist}(x, S^c) = 0$, making $x \in \partial(S)$. This concludes the proof that $\partial(S) = \overline{S} \setminus S^\circ$ ■

Fact 1.84

Boundary of a set is closed ($\partial(S) = \overline{S} \cap \overline{S^c}$).

1.3 Relative closures, interiors and boundary

Definition 1.85: Relative closure ($cls_Z(S)$)

The closure of $S \subset Z \subset X$ relative to Z , denoted as $cls_Z(S)$ is defined either as the set S along with the set of all limit points of S in Z , or equivalently as the smallest closed set, closed relative to Z , that contains S .

Definition 1.86: Relative Interior ($int_Z(S)$)

The interior of $S \subset Z \subset X$ relative to Z , denoted as $int_Z(S)$ is defined either as the set of all points $z \in Z$ so that there exists an $\varepsilon > 0$ ball so that $B_\varepsilon(z) \cap Z$ is fully contained in S , or as the largest open set, open relative to Z , that is inside S .

Theorem 1.87

Let $S \subseteq Z \subseteq X$. Then,

$$cls_Z(S) = cls_X(S) \cap Z$$

Proof for Theorem.

Note that $cls_X(S) \cap Z$ is a closed set, closed relative to Z that contains S . Hence, $cls_Z(S) \subseteq cls_X(S) \cap Z$. $cls_Z(S)$ is the smallest closed set relative to Z that contains S , which means $cls_Z(S) = H \cap Z$ for some closed set H . Note that this closed set must too contain S . Hence, we have $cls_X(S) \subseteq H$ which gives $cls_X(S) \cap Z \subseteq H \cap Z = cls_Z(S)$. Hence we have $cls_Z(S) = cls_X(S) \cap Z$. ■

Theorem 1.88

Let $S \subseteq Y \subseteq X$. Then $int_Y(S) = int_X(S \cup (X \setminus Y)) \cap Y$

Proof for Theorem.

$int_Y(S) = \bigcup_{G \cap Y \subset S} G \cap Y = \left(\bigcup_{G \cap Y \subset S} G \right) \cap Y$ which gives $int_Y(S) = \left(\bigcup_{G \subset S \cup Y^c} G \right) \cap Y = int_X(S \cup Y^c) \cap Y$ ■

1.4 Denseness, topologies and more on open / closed sets

Definition 1.89: Dense sets

A set $A \subseteq X$ is said to be dense in X if $\overline{A} = X$

Theorem 1.90

A set A is dense in X if and only if for every open set $G \in \mathcal{U}$, $A \cap G \neq \emptyset$

Proof for Theorem.

\Rightarrow) Suppose $G_0 \cap A = \emptyset$. That means $A \subseteq G_0^C \neq X$ which means, since G_0^C is open, we have $\bar{A} \subseteq G_0^C \neq X$.

\Leftarrow) if $\bar{A} \subset X$ properly, then $(\bar{A})^C$ is an open set disjoint from A . ■

Question: Suppose metric $\gamma : X \times X \rightarrow \mathbb{R}^+$ induces a topology \mathcal{U} . Can the topology tell us the metric? i.e, can we get back $\gamma : X \times X \rightarrow \mathbb{R}^+$ from the topology \mathcal{U} ?

Theorem 1.91

If \mathcal{U} is a topology arising from $\delta(x, y)$, then the same topology can also be achieved by the metric $d(x, y) = \delta(x, y)/(1 + \delta(x, y))$, as well as $e(x, y) = \delta(x, y)/2$

Proof for Theorem.

Let S be an open r ball in the δ metric. $S_r := \{x \in X : \delta(x, y) < r\}$. Let T_s be the open s ball in the d metric. $T := \{x \in X : \delta(x, y)/(1 + \delta(x, y)) < s\}$. Now let E be the a ball in the e metric. $E_a := \{x \in X : \delta(x, y)/2 < a\}$. Easy to see that $E_{r/2} = S_r$ and $E_r = S_{2r}$. Moreover, $T_{r/(1+r)} = S_r$ and $T_r = S_{r/(1-r)}$. ■

The previous theorem tells us that the answer to the prior question is: no. Different metrics can give rise to the same topologies.

Topologies on Subspaces and Superspaces

Given a topology of X , we would like to classify, or have the structure of the topology generated in a subspace $Y \subset X$.

Theorem 1.92

If \mathcal{U} is the topology of X , then $\mathcal{U} \cap Y$, that is $\{U \cap Y : U \in \mathcal{U}\}$ is the topology of Y .

Proof for Theorem.

We know by definition (and in metric spaces, by virtue of a theorem) that $S \subseteq Y \subseteq X$ is open relative to Y if and only if $S = G \cap Y$ for some open set G relative to X . Let U be an open set in X . That means that $U \cap Y$ is open relative to Y , making it part of the topology. Let K be an open set relative to Y . That means $K = G \cap Y$ for some G open relative to X . Hence, the topology of Y is such that every element in it is of the form $G \cap Y$ for open G in X , and for every open $G \subseteq X$, $G \cap Y$ is part of the topology of Y . Hence, $T = \{U \cap Y : U \in \mathcal{U}\}$ ■

Theorem 1.93

Suppose Z is a metric space and $Z \subseteq X$. Topology of X includes that of Z if and only if Z is open in X .

This essentially means that every set open relative to Z is open relative to X if and only if Z itself is open in X .

Proof for Theorem.

\Leftarrow) Suppose Z is open in X . That means Z is part of the topology of X . Moreover, If $S \subseteq Z \subseteq X$ is open relative to Z (i.e, is part of the topology of Z) then it can be written as $G \cap Z$ which is also again an open set in X . Hence, topology of Z is contained in the topology of X .

\Rightarrow) Suppose the topology of Z is contained in X , that means that every open set relative to Z is actually an open set relative to X . That makes Z an open set relative to X . We are done. ■

Question: Are there non empty metric spaces such that they are open relative to every superspace?

Answer: No. Consider X to be a non empty metric space with distance d . Consider $X \times \{0\} = Y$. It is obvious that Y is bijective to X . Consider $f : X \rightarrow Y$ given by $f(x) = (x, 0)$. Under the obvious metric $k[(x, 0), (y, 0)] = k[f(x), f(y)] = d(x, y)$, we can clearly see that Y, k forms a metric space isometric to X, d . We can therefore identify X, d with Y, k . Now look at $Z = X \times [0, 1]$ with the metric $\delta[(x, r_x), (y, r_y)] := d(x, y) + |r_x - r_y|$. Consider $\text{dist}_Z[(x_0, 0), Z \setminus Y]$. This is $\text{dist}_Z[(x_0, 0), X \times (0, 1]]$ which is 0. Hence, Y is not open wrt Z . Its over.

Definition 1.94: Complete Metric Space

A metric space is said to be complete if X is closed in every metric superspace

Lemma 1.95: Cantor's Intersection Lemma

X is a cauchy complete metric space if and only if for every sequence of nested, non empty closed sets $F_n \subseteq F_{n-1} \subseteq F_{n-2} \cdots$ such that $\text{diam}(F_n) \rightarrow 0$, $\cap_{n \in \mathbb{N}} F_n$ is a singleton.

Proof for Lemma

\Rightarrow) Let X be cauchy complete. Consider $x_1 \in F_1, x_2 \in F_2 \cdots x_n \in F_n$ and so on. We are told $\text{diam}(F_n) \rightarrow 0$, i.e $\forall \varepsilon > 0, \exists n_0$ such that $\forall n \geq n_0, \text{diam}(F_n) < \varepsilon$. This means that $\forall n, m \geq n_0$, we have $d(x_m, x_n) < \varepsilon$. Hence, (x_n) is a cauchy sequence. It converges to x . Note that for any F_k , only finite points of (x_n) lie outside of F_k . Suppose x is not in some F_k . That means it is in F_k^C , an open set. There is a ball around x that is completely

inside F_k^C . Around this ball, though, only finite points of x_n exists. This contradicts the fact that x is a limit of (x_n) . Hence, $x \in F_k$ for all $k \in \mathbb{N}$. Hence $x \in \bigcap_{n \in \mathbb{N}} F_n$. Since $\text{diam}(F_n) \rightarrow 0$, its obvious that two points cannot exist in the final intersection.

\Leftarrow) Let (x_n) be a cauchy sequence. Note that, if $\{x_n : n \in \mathbb{N}\}$ has a limit point, say x , then choose $\varepsilon_1 = 1$ and pick x_{k_1} in this 1-ball. Then, look at the set $\{x_n : n \geq k_1\}$. This set still has x as its limit point, wherein you choose $\varepsilon_2 = 1/2$ and get the corresponding x_{k_2} in the $1/2$ -ball of x . As such keep going to get a subsequence x_{k_j} that converges to x . We know that $\forall \varepsilon > 0, \exists n_0$ so that $\forall n, m \geq n_0, d(x_n, x_m) \leq \varepsilon/2$, more specifically, $\forall n, k_j \geq n_0, d(x_n, x_{k_j}) < \varepsilon/2$. Moreover, $\forall \varepsilon > 0, \exists n'$ so that $\forall k_j \geq n'$ we have $d(x_{k_j}, x) < \varepsilon/2$. Adding these two gives us that $x_n \rightarrow x$. Therefore, we assume that $\{x_n : n \in \mathbb{N}\}$ has no limit point. Let $F_m := \{x_n : n \geq m\}$. Each of these sets, by virtue of having no limit points, are closed. They are, as well, nested and obviously non empty. Moreover, $\text{diam}(F_l) := \sup\{d(x_n, x_m) : m, n \geq l\}$. If $\varepsilon > 0$ is given, there exists a l_0 so that $\forall n, m \geq l_0$, we have $d(x_m, x_n) < \varepsilon$ which means $\sup\{d(x_m, x_n) : n, m \geq l_0\} \leq \varepsilon$. Hence, $\text{diam}(F_n) \rightarrow 0$. Hence, from hypothesis, $\bigcap_{n \in \mathbb{N}} F_n = \{x\}$. $d(x_n, x) \leq \text{diam}(F_n)$, which means that $x_n \rightarrow x$. Hence, cauchy complete. ■

Theorem 1.96

Cauchy completeness \iff completeness

Proof for Theorem.

\implies) Let X be a cauchy complete space. That is, every cauchy sequence in X converges in X . Let $X \subseteq Y$. If (z_n) is a sequence in X , then if it converges, it must converge in X . This is because, if it converges outside of X , that makes (z_n) a cauchy sequence, which by definition must converge in X . Hence, X is closed with respect to Y . Since Y was arbitrary, it is closed in every subspace.

\Leftarrow) Suppose X is closed with respect to every superspace. Suppose (x_n) is a cauchy sequence in X . Consider the completion X' of X , that is, that unique superspace of X such that $\text{cls}_{X'}(X) = X'$ and $X \hookrightarrow X'$ is an isometric embedding. But since X is closed with respect to X' , we have that $X = X'$, and the cauchy sequence in X converges in $X = X'$. Hence X is cauchy complete. ■

An alternate version using an intermediate lemma:

Proof. Let (x_n) be a cauchy sequence in X . Let $x \notin X$. Define $\delta(a, b) = d(a, b)$ if $a, b \in X$. Define $\delta(a, x) = \lim_{n \rightarrow \infty} d(a, x_n)$. This limit exists since $d(a, x_n) \leq d(a, x_m) + d(x_n, x_m) \implies d(a, x_n) - d(a, x_m) \leq d(x_m, x_n)$. The far right can be made arbitrarily small for large values of m, n , hence $d(a, x_n)$ is cauchy, and hence convergent. Now, look at $\delta(x_k, x) = l_k := \lim_{n \rightarrow \infty} d(x_k, x_n)$. For arbitrarily large values of k , by virtue of x_n being a cauchy sequence, we can make l_k arbitrarily small. Hence, $\delta(x_k, x) \rightarrow 0$ meaning, $x_k \rightarrow x$ (under the metric space $X \cup \{x\}, \delta$). All that is left to show is that this function δ is actually a metric. Let $x \notin X$ and $y \in X$. We define $\delta(x, y) := \lim_{n \rightarrow \infty} d(y, x_n)$. $\delta(x, y') := \lim_{n \rightarrow \infty} d(y', x_n)$. Positivity is obvious since we assert that x_n is actually not convergent in X . Let $a, b \in X$. We check

triangle inequality for these. Is $\delta(a, b) = d(a, b) \leq \delta(a, x) + \delta(b, x) = \lim_{n \rightarrow \infty} d(a, x_n) + \lim_{n \rightarrow \infty} d(b, x_n)$? We have the RHS $\lim_{n \rightarrow \infty} (d(a, x_n) + d(b, x_n))$ which would be larger than $d(a, b)$. Now let $a \in X$ and $x = x$. Let $b \in X$. We want to show $\delta(a, x) = \lim_{n \rightarrow \infty} d(a, x_n) \leq \delta(a, b) + \delta(b, x) = d(a, b) + \lim_{n \rightarrow \infty} d(b, x_n)$. In essence we want to show that $\lim_{n \rightarrow \infty} d(a, x_n) \leq \lim_{n \rightarrow \infty} (d(a, b) + d(b, x_n))$. Notice that the right hand side is always larger than $d(a, x_n)$, which is still preserved in the $n \rightarrow \infty$ limit. Hence, triangle inequality is verified. Hence, δ is a metric. \square

Example : *Two disjoint closed subsets X, Y of \mathbb{R} that have $d(X, Y) = 0$*

Consider the sequence of irrational numbers x_1, x_2, \dots that is monotonically increasing and divergent (to $+\infty$). Let this be X . Consider another sequence, likewise monotonic increasing and likewise divergent, y_n of rational numbers obeying the property $d(x_k, y_k) < 1/k$. We are done. \blacksquare

Lemma 1.97

If A is a set in a metric space X , and A is closed, then A can be written as the union of the set of isolated points of A and the set of limit points of A (disjoint union).

Proof for Lemma

That it would be disjoint union is obvious, since isolated points of A are precisely not limit points of A . Since A is closed, A' (derived set) is a subset of A . $A = (A \setminus A') \cup A'$. $(A \setminus A') := \{x \in A : \exists \varepsilon_x \text{ such that } B_\varepsilon(x) \setminus \{x\} \subseteq A^C\}$ is precisely the set of isolated points of A . We are done. \blacksquare

2 Compactness

Definition 2.1: Open Cover

A collection of open sets $G_\alpha \subset X$ is an open cover of a set E if $E \subset \cup_\alpha G_\alpha$

Definition 2.2: Compact Set

A set $E \subset X$ is said to be **Compact** if Every open cover has a finite subcover. i.e, for every collection of open sets G_α , if $E \subseteq \cup_\alpha G_\alpha$, then there would exist a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2} \cdots G_{\alpha_k}\}$ of $\{G_\alpha\}$ such that $E \subseteq \cup_{i=1}^k G_{\alpha_i}$

Remark.

The notion of *Being open* depends largely on the metric space one is talking about. For example, we see that certain sets may be open relative to $Y \subset X$, but not X in itself. This is not the case for compactness though, as shall be seen.

Theorem 2.3: "Compact Relativeness" is conserved.

Definition: We say $E \subseteq Y \subseteq X$ is compact relative to Y if for every open cover G_α open relative to Y we have a finite sub collection G_{α_k} of G_α such that $E \subseteq \cup_{j=1}^k G_{\alpha_j}$.

Theorem: $E \subseteq Y \subseteq X$ is compact relative to $Y \iff E$ is compact relative to X

Proof for Theorem.

\implies) Suppose E is compact relative to Y . This means that, for any collection of sets F_α which are open relative to Y (i.e, $F_\alpha = G_\alpha \cap Y$ where G_α is an open set in X), there exists a finite sub collection $F_{\alpha_1}, F_{\alpha_2} \cdots F_{\alpha_k}$ such that $E \subseteq \cup_{i=1}^k F_{\alpha_i}$. Consider an open cover H_α of E open relative to X . $E \subseteq \cup_\alpha H_\alpha$, but also, $E \subseteq (\cup_\alpha H_\alpha) \cap (Y)$ since E is subset of Y as well. This implies $E \subseteq \cup_\alpha (H_\alpha \cap (Y))$. $\{H_\alpha \cap Y\}$ is an open cover of E open relative to Y which means there would be a finite sub collection $\{H_{\alpha_j} \cap Y : j \in [1, k]\}$ such that $E \subseteq \cup_{j=1}^k (H_{\alpha_j} \cap Y) = (\cup_{j=1}^k H_{\alpha_j}) \cap Y$. Since E is a subset of Y , we then have $E \subseteq (\cup_{j=1}^k H_{\alpha_j})$ which proves that for an arbitrary open cover open relative to X , we have a finite subcover.

\impliedby) Suppose E is open relative to X . Consider an open cover of E open relative to Y , which is $\{F_\alpha\}$. This means that $F_\alpha = G_\alpha \cap Y$ for G_α open relative to X . $E \subseteq \cup_\alpha F_\alpha = (\cup_\alpha G_\alpha) \cap Y$. Since E is a subset of Y , we have $E \subseteq (\cup_\alpha G_\alpha)$. Therefore, there would be a finite subcollection of $\{G_\alpha\}$, $\{G_{\alpha_1}, G_{\alpha_2} \cdots G_{\alpha_k}\}$ such that $E \subseteq \cup_{i=1}^k G_{\alpha_i}$. This means, $E \subseteq \cup_{i=1}^k G_{\alpha_i} \cap Y = \cup_{i=1}^k F_{\alpha_i}$. Hence, for every open cover open relative to Y , there exists a finite subcover.

Remark.

Note that the above theorem "compact-relativeness is conserved" did not use any property of metric spaces, like distances or balls, so this theorem holds in any general Topological space.

Fact 2.4

Every finite set in X is compact

Proof. Consider an open cover G_α for finite set E . This means that, for every point x_1, x_2, \dots, x_k in E , there would exist some $\{\alpha_1, \alpha_2, \dots\}$ collection of " α -s" that is utmost finite, such that $x_j \in G_{\alpha_j}$. Simply take the union of G_{α_j} to get a finite subcover. \square

Theorem 2.5: Alternate definition for compactness

A set E is compact if for every closed collection of sets K_α such that $\cap_\alpha K_\alpha \subset E^c$, we have a finite subcollection $\{K_{\alpha_1}, K_{\alpha_2} \dots K_{\alpha_p}\}$ such that $\cap_{i=1}^p K_{\alpha_i} \subset E^c$

Theorem 2.6

Closed balls in X are closed

Proof for Theorem.

Consider $B_{[\varepsilon]}(p) := \{x \in X : d(x, p) \leq \varepsilon\}$. $B^c = C := \{x \in X, d(x, p) = t_{xp} > \varepsilon\}$. Consider an arbitrary point $x \in C$. We have $d(x, p) = t_{xp} > \varepsilon$. Find, from density, a δ such that $t_{xp} > \delta > \varepsilon$. Let $d(x, y) < \delta - \varepsilon$. We then have from triangle, $d(y, p) \geq d(x, p) - d(x, y) > t_{xp} - (\delta - \varepsilon) > \varepsilon$. Hence, y is also in C . Therefore, C is open, which means B is closed. \blacksquare

Definition 2.7: $diam(A)$

$$diam(A) := \sup\{d(x, y) : x, y \in A\}$$

It is finite if and only if it is bounded. It would be infinite if unbounded.

Lemma 2.8

If x_n is a cauchy sequence and $\{x_n\}$ the set contains a limit point, then the cauchy sequence converges.

Proof for Lemma

$\forall \varepsilon/2 > 0$, $\exists n_0$ so that $\forall n, m \geq n_0$, $d(x_n, x_m) < \varepsilon/2$. Suppose it has a limit point x . Around every $\varepsilon/2$ ball of x , there exists infinite points of $\{x_n\}$. Choose x_1 , and then make

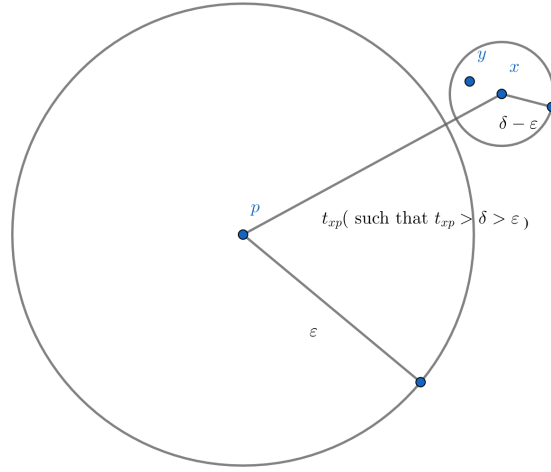


Figure 1.1: Figure for the proof: Closed balls are closed

$\delta < \min\{d(x_1, x)\}$ and get a corresponding x_{n_2} . Then choose $\delta < \min\{x_{n_2}, x_1\}$ and keep going to create a subsequence of $\{x_n\}$ that converges to x . $\forall \varepsilon/2 > 0 \exists n'$ so that $\forall n_k \geq n'$ we have $d(x_{n_k}, x) < \varepsilon/2$. Therefore, $\forall \varepsilon > 0, \exists q = \max n', n_0$ so that $\forall n, n_r \geq q$, we have $d(x_n, x_{n_s}) < \varepsilon/2$ and $d(x_{n_s}, x) < \varepsilon/2$, Adding these two we get the desired result. ■

Theorem 2.9

A metric space X is complete if and only if for every sequence of nested non-empty closed sets $\{F_n : n \in \mathbb{N}\}$ such that $\lim_{n \rightarrow \infty} \text{diam}(F_n) = 0$, there exists one point $x \in \bigcap_{i=1}^{\infty} F_i$

Proof for Theorem.

\Rightarrow) Say X is complete. Every cauchy sequence converges. Consider an arbitrary nested sequence of closed sets so that $F_n \subseteq F_{n-1}$. Choose $x_{n_1} = x_1 \in F_1$. Choose $x_2 \in F_2 \setminus (F_1)$ (if it exists). If not, move on to the next set, and so on. If every set contains only one point, then we are done. If not, we are able to find $x_{n_2} \in F_{n_2}$ such that $x_{n_2} \neq x_{n_1}$. As such create a sequence $x_{n_k} \in F_{n_k}$ where $n_k < n_{k+1}$ (which gives $F_{n_{k+1}} \subset F_{n_k}$). Is x_{n_k} a cauchy sequence? Let $\varepsilon > 0$. Since $\text{diam}(F_{n_k}) \rightarrow 0$, we have that there exists some $k \in \mathbb{N}$ such that for every $n_k \geq k$, we have $0 \leq \text{diam}(F_{n_k}) < \varepsilon$. Let $q, p \geq$ this k . That means x_{n_q} and x_{n_p} are both in F_{n_k} which means $d(x_{n_q}, x_{n_p}) \leq \text{diam}(F_{n_k}) < \varepsilon$ which means this sequence is cauchy, hence convergent. Hence a limit x exists. Is $x \in \bigcap_{i=1}^{\infty} F_i$? If so, we are done. If not, it must exist in $\bigcup_{i=1}^{\infty} F_i^C$. This is a countable union of open sets F_i^C which means it is finally in an open set. Outside any F_j , only finite points of the sequence exists. This immediately contradicts the previous result, since x is a limit point of the sequence, but it is in some F_j^C which contains only finite points of the sequence so it is impossible for a particular (chosen) ε -ball to contain infinite points of $\{x_{n_k}\}$. Hence, we are done. $x \in \bigcap_{i=1}^{\infty} F_i$. Moreover, since $\text{diam}(F_n) \rightarrow 0$, it is trivial to see that only one point survives

the intersection. (If there are two points, we can always make epsilon smaller than their fixed distance).

\Leftarrow) Suppose $\{x_n\}$ is a cauchy sequence, i.e, $\forall \varepsilon > 0 \exists n_0$ so that for all $m, n \geq n_0$, $d(x_n, x_m) < \varepsilon$. Look at $S_m := \{x_n : n \geq m\}$. If any of these sets has a limit point, then we would be done from the previous lemma. So we assume these all don't have a limit point. Hence they are closed, nested sets. Moreover, $\forall \varepsilon > 0, \exists n_0$ so that $\forall n, m \geq n_0$ (or rather, for any two points x and y in S_{n_0}) we have $d(x_n, x_m) < \varepsilon$ (or $d(x, y) < \varepsilon$) which means $\text{diam}(S_{n_0}) < \varepsilon$. This means $\text{diam}(S_n) \rightarrow 0$. This means that $\bigcap_{i=1}^{\infty} S_i = \{x\}$ for some element x . x is in every S_i . Let $\varepsilon > 0$. There exists S_{n_ε} so that $\text{diam}(S_{n_\varepsilon}) < \varepsilon$ which means $d(x_{n_\varepsilon}, x) < \varepsilon$ which means x_{n_ε} forms a convergent subsequence, which means the sequence is finally convergent from previous lemma. ■

Theorem 2.10

Compact sets are closed.

Proof for Theorem.

Method 1:

Let E be compact. Consider a point $p \in E^c$. Let ε_x be the "half" distance between a point $x \in E$ and p . Therefore, $B_{\varepsilon_x}(x)$ is completely outside $B_{\varepsilon_x}(p)$. Consider $\bigcup_{x \in E} B_{\varepsilon_x}(x)$ which is an open cover for E . This means there is a finite subcover

$\{B_{\varepsilon_{x_1}}(x_1), B_{\varepsilon_{x_2}}(x_2), B_{\varepsilon_{x_3}}(x_3) \cdots, B_{\varepsilon_{x_k}}(x_k)\}$ such that $E \subset \bigcup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$. $B_{\varepsilon_{x_i}}(p)$ does not intersect with $B_{\varepsilon_{x_i}}(x_i)$. Therefore, $\bigcap_{i=1}^k B_{\varepsilon_{x_i}}(p)$ does not intersect with any $B_{\varepsilon_{x_i}}(x_i)$ for any i . Hence, it does not intersect with $\bigcup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$ which means $\bigcap_{i=1}^k B_{\varepsilon_{x_i}}(p)$ lies completely outside E . If we choose $\delta < \min\{\varepsilon_{x_1}, \varepsilon_{x_2} \cdots \varepsilon_{x_k}\}$, we would have $B_\delta(p) \subseteq \bigcap_{i=1}^k B_{\varepsilon_{x_i}}(p)$. This means that, for p outside E , there would exist a δ such that the δ -ball around p is fully contained in E^c . This means that E^c is open, hence, E is closed.

Method 2:

Consider E to be compact, i.e, for every closed collection $\{F_\alpha\}$ such that $\bigcap_\alpha F_\alpha \subset E^c$, there exists a finite sub collection $\{F_{\alpha_1}, F_{\alpha_2} \cdots F_{\alpha_k}\}$ such that $\bigcap_{j=1}^k F_{\alpha_j} \subset E^c$. Consider a point p outside E , i.e, in E^c . Notice that $\bigcap_{\varepsilon \in \mathbb{R}^+} B_{[\varepsilon]}(p) = \{p\}$ which is in E^c . This would be a collection of closed sets whose intersection falls completely inside E^c . Hence, there would exist a finite subcollection such that $\bigcap_{j=1}^k B_{[\varepsilon_j]}(p) \subset E^c$ which means there would exist a neighbourhood around p which is completely in E^c . Hence, E^c is open, and E is closed. ■

Fact 2.11

\emptyset and X are both open and closed.

Theorem 2.12

Closed subsets of compact sets are compact

Proof for Theorem.

Consider $K \subset E$ where E is compact and K is closed. K^c is, therefore, open. Consider an arbitrary open cover $\{F_\alpha\}$ for K . since $K \subseteq \cup_\alpha F_\alpha$, and K^c is open, we have $X = \cup_\alpha F_\alpha \cup K^c$ which means $E \subset \cup_\alpha F_\alpha \cup K^c$. Since E is compact, there would exist a finite subcover such that $E \subset \cup_{j=1}^n F_{\alpha_j} \cup K^c$. We then have $K \subset \cup_{j=1}^n F_{\alpha_j} \cup K^c$, which would mean $K \subset \cup_{j=1}^n F_{\alpha_j}$, whence, we see that K is compact. ■

Corollary 2.13

If F is closed, and K is compact, then $F \cap K$ is compact

Fact 2.14

A compact set is bounded

Proof. Consider (WLOG, a non empty compact set E) and an arbitrary point q in X . $B_\varepsilon(x)$ for every $\varepsilon > 0$ forms an open cover for E (since it is basically X). Which means there is a finite subcover, i.e, a number $\varepsilon_0 > 0$ such that $E \subset B_{\varepsilon_0}(p)$ which makes E bounded. □

Theorem 2.15

Finite union of compact sets is compact

Proof for Theorem.

Let K_1, K_2, \dots, K_r be r compact sets. Let $K = \cup_{i=1}^r K_i$. Consider an open cover F_α whose union subsumes K . We have that, for every $i \leq r$, $K_i \subset \cup_\alpha F_\alpha$. Since each K_i is compact, there exists a finite of F_α whose union subsumes K_i . For each $i = 1$, to r , we have a finite subcollection, therefore, taking the union of all these finite subcollections gives us a finite subcollection which subsumes whole of K . Hence K is compact. ■

Remark.

Note that finiteness in the above theorem is important. This is because, each compact may have finite subcollection, but at the end, the union of all these finite collections will be countable, not finite.

Theorem 2.16

If $\{K_\alpha\}$ is a collection of compact sets such that for every finite subcollection $\{K_{\alpha_j} : 1 \leq j \leq k\}$ we have that $\bigcap_{j=1}^k K_{\alpha_j} \neq \emptyset$. Then $\bigcap_\alpha K_\alpha \neq \emptyset$. In pithy words:

"If you have a collection of compact sets for which every finite subcollection's intersection is non-empty, the intersection of the whole collection is non empty"
-Krishna, to Arjuna

Proof for Theorem.

Suppose, on the contrary, let $\bigcap_\alpha K_\alpha = \emptyset$ which means $\bigcup_\alpha K_\alpha^c = X$ which means for every for some α_0 , we have $K_{\alpha_0} \subset \bigcup_\alpha K_\alpha^c$, where $\bigcup_\alpha K_\alpha^c$ is an open cover of K_{α_0} . This implies that there exists a finite subcollection $\{K_{\alpha_1}^c, K_{\alpha_2}^c \cdots K_{\alpha_r}^c\}$ such that $K_{\alpha_0} \subset \bigcup_{j=1}^r K_{\alpha_j}^c \implies \bigcap_{j=1}^r K_{\alpha_j} \subset K_{\alpha_0}^c$. But this means $\bigcap_{j=1}^r K_{\alpha_j} \cap K_{\alpha_0} = \emptyset$, which is absurd since all finite intersection is non empty. ■

Corollary 2.17

If K_1, K_2, \dots is a sequence of non-empty compact sets such that $\cdots K_n \subset K_{n-1} \cdots K_3 \subset K_2 \subset K_1$, then $\bigcap_{i=1}^\infty K_i$ is non empty.

Theorem 2.18: Compactness \implies Limit point Compact

If K is a compact set and E is an infinite subset of K , then E has a limit point in K

Proof for Theorem.

Suppose that E has no limit point in K . Since K is closed, E must have no limit points. Hence, E is closed. Since closed subsets of compact sets are compact, E is compact. If no point of E is a limit point of E , then $\forall x \in E, \exists \varepsilon_x > 0$ such that no point of E apart from x itself falls into the ε_x -ball of x . Consider the open cover $\{B_{\varepsilon_x}(x) : x \in E\}$ of E . This has a finite subcover $\{B_{\varepsilon_{x_1}}(x_1), B_{\varepsilon_{x_2}}(x_2), \dots B_{\varepsilon_{x_l}}(x_l)\}$. We see that $E \subseteq \bigcup_{j=1}^l B_{\varepsilon_{x_j}}(x_j)$. But since for every ε_{x_j} -ball around x_j , no point in E except x_j resides, $\bigcup_{j=1}^l B_{\varepsilon_{x_j}}(x_j)$ will have utmost finite points. Since an infinite set E cannot be the subset of a finite set, we have a contradiction. ■

Definition 2.19: k-cell

a k -cell, E is a set in \mathbb{R}^k such that $E := \{\vec{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^n : a_j \leq x_j \leq b_j \text{ for given } a_j \text{ and } b_j \text{ for every } 1 \leq j \leq k\}$

A k -cell is basically a k dimensional cuboid.

Theorem 2.20 k -cells are closed*Proof for Theorem.*

Consider a k -cell E . Consider a point z not in E , i.e, $\exists j_0$ such that either $z_j < a_j$ or $z_j > b_j$. WLOG, take the case of $z_j < a_j$. Let $0 < \delta < (a_j - z_j)$. Consider a point q in the δ -ball around z . i.e, $d(z, q) < \delta \implies \sqrt{(z_1 - q_1)^2 + (z_2 - q_2)^2 + \dots + (z_k - q_k)^2} < \delta \implies (z_1 - q_1)^2 + (z_2 - q_2)^2 + \dots + (z_k - q_k)^2 < \delta^2 < (a_j - z_j)^2 \implies 0 < (q_j - z_j)^2 < (a_j - z_j)^2 \implies q_j < a_j$. Hence $q \notin E$, which implies there exists, for every x in E^c , a δ for which the δ -ball around x is fully contained in E^c which means E^c is open. This implies E is closed. Same argument applies for the case where $z_j > b_j$. ■

Theorem 2.21Closed intervals in \mathbb{R} are compact*Proof for Theorem.*

Let \mathbb{I} be, WLOG, $[-a, a]$. Suppose it is not compact. i.e, There is an open cover G_α of \mathbb{I} such that there exists no finite subcover. $\forall x \in \mathbb{I}$, $\exists \alpha_x$ such that $x \in G_{\alpha_x}$ and $\exists \varepsilon_x$ such that $B_{\varepsilon_x}(x) \subset G_{\alpha_x}$. $\cup_x B_{\varepsilon_x}(x) \subset \cup_\alpha G_\alpha$ is an open cover for \mathbb{I} . note that, if no finite subcover for G_α exists, then no finite subcover for $B_{\varepsilon_x}(x)$ exists either. So we can safely work with $B_{\varepsilon_x}(x)$. Split the interval into two halves, $[-a, 0]$ and $[0, a]$. One of these intervals is not finitely covered by $B_{\varepsilon_x}(x)$, for if not, the whole thing would be finitely covered. let that interval which is not finitely covered be \mathbb{I}_1 . This interval's size is a . Split this interval into two again. Yet again, one of the halves must not be finitely covered, for if not, \mathbb{I}_1 would be finitely covered, which is contradictory. Let this interval be \mathbb{I}_2 . This is of size $\frac{a}{2}$. Yet again, keep doing this process to obtain a sequence of intervals \mathbb{I}_j , sized $\frac{a}{j}$, which are not finitely covered. These are nested intervals, non empty, and closed. From nested intervals theorem, we see that a point ξ exists in $\cap_{j=1}^\infty \mathbb{I}_j$. ξ is a point in \mathbb{I} , and there is a corresponding ε_ξ . Consider that j_0 for which $\frac{a}{j_0} < \frac{\varepsilon_\xi}{2}$. We know from Archimedean such a j_0 exists. This means that the interval \mathbb{I}_{j_0} containing ξ , sized $\frac{a}{j_0}$, is completely inside the ε_ξ -ball around ξ , which means it is finitely covered. Contradiction. Hence, \mathbb{I} is compact. ■

Corollary 2.22

Since intervals of the form $[-a, a]$ are compact, every closed interval of the form $[a, b]$ is compact since it would be a closed subset of an interval of the form $[-x, x]$.

Generalisation:

Theorem 2.23 n -cells are compact**Proof for Theorem.**

Consider $K := \{\vec{x} \in \mathbb{R}^n : -a \leq x_j \leq a; \forall j \leq n\}$ to be non-compact. There is an open cover G_α of K such that there exists no finite subcover. $\forall x \in K$, $\exists \alpha_x$ such that $x \in G_{\alpha_x}$ and $\exists \varepsilon_x$ such that $B_{\varepsilon_x}(x) \subset G_{\alpha_x}$. $\cup_x B_{\varepsilon_x}(x) \subset \cup_\alpha G_\alpha$ is an open cover for K . Note that, if no finite subcover for G_α exists, then no finite subcover for $B_{\varepsilon_x}(x)$ exists either. So we can safely work with $B_{\varepsilon_x}(x)$. Till here, everything is the same as the 1-d case. Note that here, the n -cell is constructed by taking the cartesian product of n -intervals in \mathbb{R} of the kind $[-a, a]$. Construct 2^n subdivisions of K by halving each interval $[-a, a]$ in the construction of K . The total number of subdivisions we make would be $2 \times 2 \times \cdots \times 2$, n times (simple combinatorial argument: for each i , there exists 2 choices, the two half intervals, for crossing. From $i = 1$, you have 2 choices, likewise, $j = 2, 3, \dots, n$). We assume that atleast one of these 2^n subdivisions are not finitely covered by $\{B_{\varepsilon_x}(x)\}$. We let this one be K_1 , whose each interval size is now a . Subdivide this yet again into 2^n subsets, and assert that one of these subdivisions is not finitely covered. Call this K_2 , whose each interval is of size $\frac{a}{2}$. Construct a sequence of sets K_j , each of whose intervals are sized $\frac{a}{j}$. Each K_j is closed and non empty, hence compact, and are nested. Therefore, $\cap_{j=1}^\infty K_j \neq \emptyset$. Let $p \in \cap_{j=1}^\infty K_j$. For this p , there would exist a ε_p and the corresponding ball $B_{\varepsilon_p}(p)$. We require one of our K_j n -cell to fall into this ε_p ball. Let δ be smaller than $\frac{\varepsilon_p}{2}$. Let p be the centre of the δ -ball. Let p be the centre of the n -cube H in the following construction: Consider w to be the side length of H . We require the diagonal length $w\sqrt{n} = \delta$, which gives us $w = \frac{\delta}{\sqrt{n}}$. Consider H such that each side is the interval $[p_j - \frac{w}{2}, p_j + \frac{w}{2}]$. This would force p to fall in the centre of the n -cube H . This cube is fully contained in the δ ball of p which is contained in the ε_p ball of p . We consider that n cell K_j for which each side $\frac{a}{j} < w$. This can be found, and hence, the this K_j cell is finitely covered by $B_{\varepsilon_p}(p)$, which is absurd. Hence, K is compact. ■

Remark.

We proved the result for k -cells of the kind $[-a, a]^k$, but it is easily generalised by noting that arbitrary k -cells are contained in some k -cell of the above kind. By virtue of being closed, they also are compact.

Theorem 2.24: Heine-Borel

Given a set $E \subset \mathbb{R}^n$, the following are equivalent:

1. E is closed and bounded
2. E is compact

Proof for Theorem.

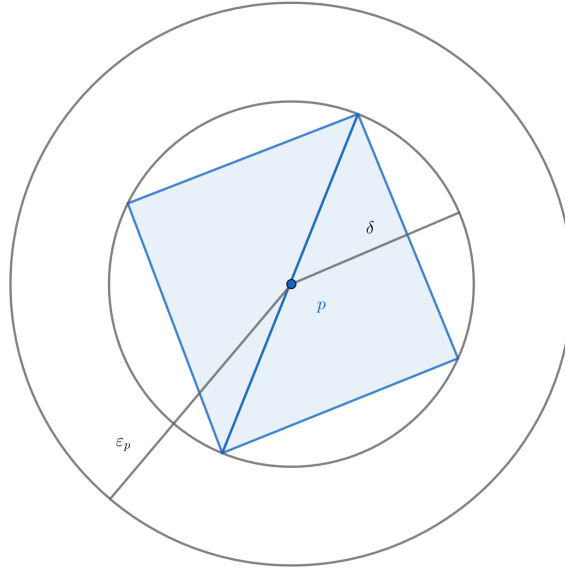


Figure 1.2: Figure for the proof: n -cells are compact. (The ε_p -ball around p , and the n -cell construction)

\Leftarrow) We know that all compact sets are closed.

\Rightarrow) If $E \subset \mathbb{R}^n$ is closed and bounded, it is contained in some n -cell, which is compact. By virtue of being a closed subset of a compact set, E is also compact. ■

Theorem 2.25

If $\{x_n\}$ is a sequence in X convergent to $x \in X$, then the set $\{x_n\}$ has only one limit point, which is x .

Proof for Theorem.

That x is a limit point is clear. We note that $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \in \mathbb{N}, n \geq n_0$ we have $d(x_n, x) < \varepsilon$. i.e., beyond a particular n_0 , every point of $\{X_n\}$ falls in the ε -ball of x . Therefore, only finite points lie outside this ε -ball of x . Suppose it has another limit point y , other than x . Therefore, there would exist a δ such that the δ -ball around y lies completely outside the δ -ball around x . This means that, only finite points of x_n lie in the δ ball of y , making it unviable to be a limit point. ■

Extending Heine Borel we have:

Theorem 2.26: (Extension)

For a subset $E \subset \mathbb{R}^n$, the following are equivalent:

1. E is closed and bounded
2. E is compact
3. every infinite subset K of E has a limit point in E

Proof for Theorem.

(1) \implies (2)) Heine Borel

(2) \implies (3)) Already seen

(3) \implies (1)) Let us assume that E is either not closed, or not bounded. We start by assuming it is not closed. Which means that $\exists q$ outside E such that there exists a sequence in E that converges to q . We take this sequence $\{x_n\}$ as our infinite set, and we see that, from the previous theorem, this has only one limit point q , which lies outside E . Hence, there exists an infinite set $\{x_n\}$ which has no limit point in E .

Suppose that E is unbounded. We then have that, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists y \in E$ such that $d(x, y) > \varepsilon$. Fix some x_0 in X . Choose some y_0 that is a distance $z_{00} = d(y_0, x_0)$ away from x_0 . Look if there is a point y_1 so that its distance from x_0 is more than z_{00} but less than $2(z_{00})$. If it doesn't exist, check for less than $3(z_{00})$. Find some $k_1(z_{00})$ so that distance of y_1 to x_0 is more than z_{00} but less than $k_1(z_{00})$. Same way, for y_2 , find y_2 so that its distance from x_0 is more than $k_1(z_{00})$ but less than some other $k_2(z_{00})$. Inductively, find y_j whose distance is more than $k_{j-1}(z_{00})$ but less than $k_j(z_{00})$. Note that $1 < k_1 < k_2 < \dots$. Hence, for any ε -ball around x_0 , only finite y_j exists in that ball, since there would exist some $k_q(z_{00})$ and $k_{q-1}(z_{00})$ between which ε lies. And inside $k_{q-1}(z_{00})$ ball around x_0 , utmost finite points y_j exists. Hence, x_0 is clearly not a limit point for the set of y_j -s. Consider any other point $a \in X$. For some every ε -ball around x_0 , only finite points exists. For some, perhaps larger δ -ball around a , a chosen ε -ball around x_0 gets subsumed into the δ -ball around a . This implies that only finite points of y_j -s exists in the δ -ball around a as well, making a a non viable limit point. We see that, for this infinite subset $\{y_j\}$ of E , no limit point exists. ■

Remark.

In the previous proof, we note that (3), which is called Limit Point Compactness, implies (1) Closed and Bounded, in any metric space, not just \mathbb{R}^n , as we see in the proof, no property of \mathbb{R}^n was used.

Spoiler Alert: In any metric space, Limit Point Compact \iff Compact

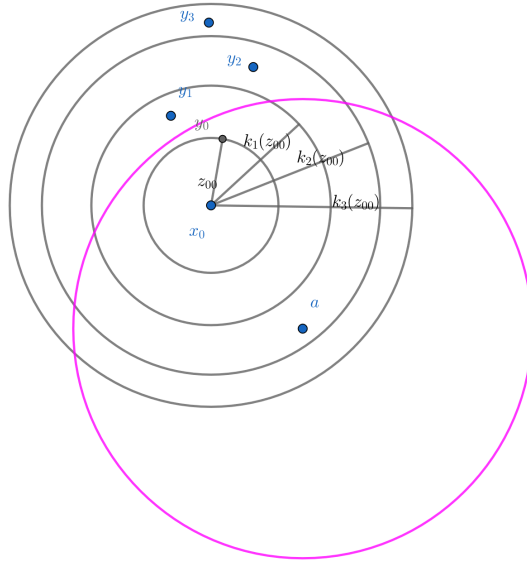


Figure 1.3: Figure for proof: Lim point compact \implies closed+bounded. (The construction of an unbounded sequence)

Theorem 2.27: Weierstrass Theorem

Every Bounded, infinite set in \mathbb{R}^n has a limit point in \mathbb{R}^n .

Proof for Theorem.

If a set is bounded in \mathbb{R}^n , it is the subset of a compact set (i.e, a closed and bounded set). From the previous equivalence, an infinite subset E of a compact set has a limit point in the compact set, which means the bounded, infinite set we have has a limit point in the compact set that contains it, hence, it has a limit point in \mathbb{R}^n . ■

Remark.

The above "Weierstrass Theorem" is just the "Bolzano-Weierstrass" Theorem we saw in sequences. Actually, the "Bolzano-Weierstrass" Theorem is a direct corollary of the more general "Weierstrass Theorem". Let $\{x_n\}$ be any sequence in \mathbb{R} that is bounded. This means that this sequence is the subset of a compact set, hence, has a limit point in \mathbb{R} . This implies, a subsequence of $\{x_n\}$ converges in \mathbb{R} . Hence, every bounded sequence has a convergent subsequence.

Fact 2.28

Let $X = \mathbb{R}^n$. The closure of any open ball is the corresponding closed ball.

Proof. Consider $B := B_\delta(x_0) := \{y \in \mathbb{R}^n : \|y - x_0\| < \delta\}$. Let z_0 be a point on the rim of B , i.e $d(x_0, z_0) = \delta$. Such a point obviously exists. Consider $\vec{\gamma}(t) = t\vec{z}_0 + (1 - t)\vec{x}_0$ with

$t \in (0, 1)$. For every $t \in (0, 1)$, $\vec{\gamma}(t)$ belongs in B . To see this, consider $\|\vec{\gamma}(t) - \vec{x}_0\| = \|t\vec{z}_0 + (1-t)\vec{x}_0 - \vec{x}_0\| = \|t(\vec{z}_0 - \vec{x}_0)\| = t\|\vec{z}_0 - \vec{x}_0\| < t(\delta) < \delta$. Suppose we are given an arbitrary $\eta > 0$. Does there exist a $t \in (0, 1)$ so that $\vec{\gamma}(t)$ belongs in the η -ball of \vec{z}_0 ? i.e, we need a t so that $\|\vec{\gamma}(t) - \vec{z}_0\| = \|t\vec{z}_0 + (1-t)\vec{x}_0 - \vec{z}_0\| = \|(1-t)(\vec{z}_0 - \vec{x}_0)\| < \eta \implies (1-t)\|\vec{z}_0 - \vec{x}_0\| < \eta \implies 1-t < \frac{\eta}{\delta} \implies 1 - \frac{\eta}{\delta} < t < 1$. Such a t exists for every η . Hence, \vec{z}_0 is a limit point of B (by virtue of there existing a sequence of $\vec{\gamma}(t_j)$ that converges to \vec{z}_0). Hence, every point on the rim is a limit point. Moreover, no point w so that $d(w, x_0) > \delta$ is a limit point of B , since there would exist an ε -ball around w so that no point of B falls into it (from openness). Hence, closure of B is the corresponding closed ball, in \mathbb{R}^n .

□

Definition 2.29: Dense sets

A set $A \subseteq X$ is said to be dense in X if every point of x is a limit point of A .

Theorem 2.30

A is dense in X if and only if $\overline{A} = X$ if and only if for every open set U of X , $A \cap U \neq \emptyset$

Proof for Theorem.

\implies) Let A be dense in X . Let U be an open set such that $A \cap U = \emptyset$ which means A is completely contained in U^C which is a closed proper subset of X , which is a contradiction since $\overline{A} \subseteq U^C$ since it is the smallest closed set that contains A . Contradiction.

\impliedby) Suppose A is not dense in X , this means $\overline{A} \subset X$ which means $(\overline{A})^C$ is a non empty open set in X that necessarily does not intersect with A . Contradiction. ■

3 Perfect Sets

Definition 3.1: Perfect Set

A set $E \subset X$ is perfect if every point of E is a limit point of E , and E is closed

Theorem 3.2

Perfect subsets in \mathbb{R}^n are uncountable.

Proof for Theorem.

Suppose E is a perfect set in \mathbb{R}^n but is countable. i.e, it can be enumerated as $E = \{x_1, x_2, \dots\}$.

Choose x_1 , and $\varepsilon_0 = 1$. Let V_0 denote the ε_0 -ball around x_1 . This ball is non empty,

moreover, $\bar{V}_0 \cap E$ (which is the corresponding closed ball of V_0) is non empty, and is compact by virtue of being closed and bounded. Inside, $V_0 \cap E$, there exists infinite points of E , since x_1 is a limit point of E .

Choose an arbitrary point z_1 in V_0 that is not x_1 . Now let $\varepsilon_1 < d(x_1, z_1)$. Let V_1 be the ε_1 -ball around z_1 . Notice the following: z_1 is a limit point of E , hence, there are infinite points of E in V_1 . x_1 is not in \bar{V}_1 . $\bar{V}_1 \cap E$ is closed, bounded and non empty, hence Compact.

Choose a point z_2 in V_1 that is not x_2 , and let $\varepsilon_2 < \min\{\varepsilon_1, d(x_2, z_2)\}$. Let V_2 be the ε_2 -ball around z_2 . Note that, x_2 is not in \bar{V}_2 . Also note yet again that there are infinitely many points of E in V_2 . It is crucial to note now that $\bar{V}_2 \cap E \subset \bar{V}_1 \cap E \subset \bar{V}_0 \cap E$.

Suppose you have already constructed V_k by finding z_k in V_{k-1} that is not x_k and an $\varepsilon_k < \min\{d(z_k, x_k), \varepsilon_{k-1}\}$ such that $x_k \notin \bar{V}_k$, $\bar{V}_k \cap E$ is compact, non empty and $\bar{V}_k \cap E \subset \bar{V}_{k-1} \cap E \dots$.

Now, choose $z_{k+1} \neq x_{k+1}$, inside V_k . Choose $\varepsilon_{k+1} < \min\{d(z_{k+1}, x_{k+1}), \varepsilon_k\}$. Let V_{k+1} be the ε_{k+1} -ball around z_{k+1} . Yet again, we see that $\bar{V}_{k+1} \cap E$ is non empty, x_{k+1} is not in \bar{V}_{k+1} , and $\bar{V}_{k+1} \cap E \subset \bar{V}_k \cap E$. Hence, we have a sequence of non empty, nested compact sets. This implies that $\exists \xi \in E \subset \mathbb{R}^n$ such that $\xi \in \bigcap_{i=1}^{\infty} (\bar{V}_i \cap E)$. Is ξ any one of x_j enumerated? No, because if it was, from the construction, x_j would not belong in V_j . Hence, ξ is not in the enumeration of E . Contradiction. ■

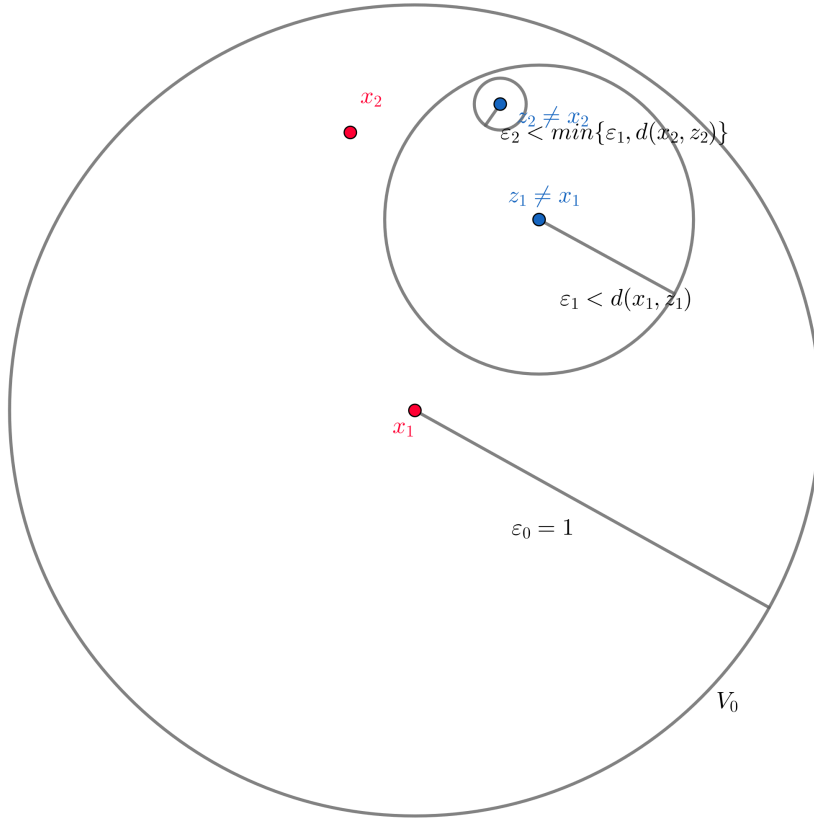


Figure 1.4: Figure: Perfect sets are uncountable. Construction of the nested sequence of compact sets by choosing $z_k \neq x_k \in V_{k-1}$.

Remark.

It is easily seen that, closed intervals in \mathbb{R} are perfect: From density theorem, for every point in I , there would exist a sequence of rationals converging to that point. Moreover, closed intervals in \mathbb{R} are closed since closed balls in metric spaces are closed. Therefore, we see that intervals are uncountable.

3.1 The Cantor Set

The following is the construction of an uncountable, perfect set that contains no intervals: The Cantor Set.

Let $I_0 = [0, 1]$. size of the interval(s) in I_0 is 1, and there are $2^0 = 1$ intervals.

Let $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be constructed by trisecting I_0 and tossing the middle one. Here, we have each interval sized $\frac{1}{3^1}$, and there are $2^1 = 2$ intervals total.

Let $I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ be generated by taking each of the two sub inter-

vals in I_1 , trisecting them, and tossing the middle one, and joining them finally. We have each interval sized $\frac{1}{3^2}$ and there are $2^2 = 4$ intervals total.

Inductively keep making these trisections+tossings to make a sequence of closed, nested intervals (Compact, too) I_k , each containing 2^k intervals each of size $\frac{1}{3^k}$.

Finally, define the Cantor set P as

$$P := \bigcap_{i=1}^{\infty} I_i$$

Note that P is compact since it is the closed subset of a compact set. It is also non empty by virtue of being the intersection of a sequence of nested, non empty, compact sets.

Note that, no interval of the kind $[a, b]$ exists in the Cantor Set. The size of each interval in I_j is $\frac{1}{3^j}$. We can find j so that $\frac{1}{3^j} < b - a \implies \frac{1}{b-a} < 3^j \implies \log_3(\frac{1}{b-a}) < j$. For such I_j , we notice that $[a, b]$ has "inbetween" points that doesn't exist in any of I_j 's intervals. Hence, taking the intersection, these "inbetween" terms don't survive. Hence, no intervals exist.

Theorem 3.3

The Cantor set P is perfect.

Proof for Theorem.

We already know that the Cantor set is closed. We need to show that every point in the cantor set is a limit point. First, observe that, for any I_k , if z is the end point of any of the sub interval of I_k , it survives the ∞ -intersection. This is because, after I_k -s trisection, the end points still stay endpoints. Let ξ be any point in the cantor set, which means it is a point in every I_k . Let $\delta > 0$ be given. Consider the interval $(\xi - \delta, \xi + \delta)$. This interval is sized 2δ . ξ exists in one of the sub intervals of I_k for all $K \geq k_0$ for some k_0 . Choose j so that $\frac{1}{3^j} < \delta$. Then, the interval in I_j containing ξ would fall completely inside $(\xi - \delta, \xi + \delta)$. Choose q as one of the end points of this sub interval of I_j . Therefore, $\forall \xi \in P, \forall \delta > 0, \exists q \in P, q \neq \xi$ so that $q \in (\xi - \delta, \xi + \delta)$. Therefore, every $\xi \in P$ is a limit point of P . Hence, P is perfect. ■

4 Connected Sets

Definition 4.1: Separated Sets

$A \subset X$ and $B \subset X$ are said to be *separated* if $\bar{A} \cap B = \bar{B} \cap A = \phi$, i.e, they are disjoint and no point of one, is the limit point of the other.

Definition 4.2: Connected Set

A set $E \subset X$ is said to be *connected* if it is *not* the union of two non-empty separated sets. In other words, for every "split" of E into two non empty sets, none of them are separated. Even if one split of E is separated, then E is *not connected*.

Example : *Separated \implies Disjoint, but Disjoint $\not\implies$ Separated.*

$[0, 1]$ and $(1, 2)$ are disjoint, but are not connected since a sequence in $(1, 2)$ converges to 1 in $[0, 1]$. ■

Theorem 4.3

$E \subset \mathbb{R}$ is connected $\iff \forall x, y \in E, x < z < y \implies z \in E$.

Proof for Theorem.

\implies) Suppose $\exists x_0, y_0 \in E$ so that $\exists z, x_0 < z < y_0$, but $z \notin E$. Consider $A := (-\infty, z)$ and $B := (z, \infty)$. A and B are seen to be separated, and E is a subset of $A \cup B$, which makes it disconnected.

\impliedby) Suppose that we have E disconnected, which means it is the union of two separated sets A and B that are non-empty. $x_0 \in A$ and $y_0 \in B$. Consider $z(t) = x_0 + t(y_0 - x_0)$ for $t \in [0, 1]$. Note that $z(0) = x_0$ and $z(1) = y_0$.

Conjecture: There exists a $t_B \in (0, 1)$ so that for every $t < t_B$, $z(t)$ does not belong in B . If it is not true, then for every $t \in (0, 1)$, there exists a point $t_B < t$ so that $z(t_B)$ is in B . Choose $t = 1$ to get $z(t_1)$ in B . Choose $t = \frac{t_1}{2}$ to get $z(t_2)$ in B with $t_2 < t_1$ and $t_2 < \frac{1}{2}$. Keep going with $t = \frac{t_{n-1}}{2^{n-1}}$ to get $z(t_n)$ in B with $t_n < t_{n-1}$ and $t_n < \frac{1}{2^{n-1}}$. This gives us a sequence $z(t_k)$ which we can see is monotone decreasing assuming $x_0 < y_0$. This sequence converges to $z(0)$ which is in A which means that there exists a sequence in B , $z(t_n)$ that converges to A . Absurd.

In a similar vein, we can show that there exists $t_A \in (0, 1)$ so that for every $t > t_A$, $z(t)$ is not in A . Consider

$$S_A := \{t \in [0, 1] : z(t) \in A \cap [x_0, y_0]\}$$

and

$$S_B := \{t \in [0, 1] : z(t) \in B \cap [x_0, y_0]\}$$

It is easy to see that S_A and S_B are disjoint. If a sequence in one converges in another, say $t_n \in S_B$ converges to $t_0 \in S_A$. Then $z(t_n) \in B$ by definition, for every n . But then by definition, $z(t_n) = x_0 + t_n(y_0 - x_0) \in B$ such that $\lim(z(t_n)) = x_0 + t_0(y_0 - x_0) \in A$, which means a sequence in B converges in A . Absurd. So S_A and S_B are separated.

Note that, for $t > t_A$, no $z(t)$ is in A . Hence, we see that for every t so that $z(t)$ falls in A , there is an upperbound. Likewise, for every t such that $z(t)$ falls in B , there is a lowerbound. Hence, S_A has a supremum $\sup(S_A)$ and S_B has an infimum $\inf(S_B)$.

At this point, we may as well assume that for every $t < \inf(S_B)$, $t \in S_A$ for if not, what we wanted to prove would get proved. Suppose then, for argument sake, that for every $t > \sup(S_A)$, $t \in S_B$, and likewise, for every $t < \inf(S_B)$, $t \in S_A$. Now then, does $\sup(S_A)$ belong in S_A ? we see that for every $t > \sup(S_A)$, $t \in S_B$ which means we can construct a sequence in S_B using those t s, which converge to $\sup(S_A)$ in S_A . So that is ruled out. So is $\sup(S_A)$ in S_B ? That is not possible either, since S_A is a bounded, infinite set (mainly because supremum isn't in the set), we know that there is a monotone subsequence in S_A converging to $\sup(S_A)$ which is in S_B . We therefore conclude that, there exists a point $t \in (0, 1)$ so that t is neither in S_A , nor in S_B . This translates to there being a point $z = z(t)$, between x_0 and y_0 so that $z(t) \notin A \cup B \implies z \notin E$. We are, therefore, done.

Slicker Argument: Suppose $E = A \cup B$ with $\bar{A} \cap B = \bar{B} \cap A = \emptyset$. Consider $x_0 \in A$ and $y_0 \in B$ and WLOG assume $x_0 < y_0$. Define $z = \sup(A \cap [x_0, y_0])$. There would be a sequence in A that converges to z , by virtue of being the supremum. $z \in \bar{A} \implies z \notin B$. This means $x_0 \leq z < y_0$. If $z \notin A$, we would be done. If $z \in A$, then $z \notin \bar{B}$. Therefore, z is in an open set \bar{B}^C . There would exist an ε_z -ball around z so that it is fully contained outside \bar{B} . Choose $z + \frac{\varepsilon_z}{2}$ as your z' . Note that z' is greater than the supremum of A . We see that z' is not in B , and not in A either. Hence, we are done. ■



Figure 1.5: Figure: Proof for the equivalence for connectedness for sets in \mathbb{R} . A look at $A \cap [x_0, y_0]$ and $B \cap [x_0, y_0]$.

5 Misc Knowledge

Theorem 5.1

1. If A and B are closed, disjoint subsets of X , then A and B are separated.
2. If A and B are open, disjoint sets, then A and B are separated.

Proof for Theorem.

- (1) $A \cap B$ closed implies $A = \overline{A}$ and $B = \overline{B}$ which are disjoint. From here it is obvious.
- (2) A and B are open disjoint sets, then we see that $A \subseteq B^C$ and $B \subseteq A^C$ where A^C and B^C are closed by definition. Since closure is the smallest closed set containing A (and B), we see that $\overline{A} \subseteq B^C$ and $\overline{B} \subseteq A^C$. It is now trivial to see that $\overline{A} \cap \overline{B} \subseteq B^C \cap A^C = \emptyset = \overline{B} \cap \overline{A}$ which is the definition of separated. ■

Corollary 5.2

Let $p \in X$ and $\delta > 0$. Define $A := B_\delta(p)$ and $B = (B_{[\delta]}(p))^C$. A and B are, then, separated.

Proof for Corollary.

Easy to see that they are both open sets that are disjoint. ■

Theorem 5.3

Every connected metric space with atleast two points is uncountable.

Proof for Theorem.

Let a and b be in X . Let $\xi \leq d(a, b)$. Note that, $P = B_\xi(a)$ and $Q = (B_{[\xi]}(a))^C$ are non empty, separated sets (from the previous corollary). If X is not the union of P and Q , then there is a point z_ξ in X so that it is neither in P nor in Q . That means that it is exactly ξ distance away from a . For every $\xi < d(a, b)$, there exists a point z_ξ so that its distance from a is exactly ξ . Therefore, every z_ξ is unique (from positivity property of metric spaces) which means there are uncountable z_ξ -s. ■

Theorem 5.4

If P and Q are connected such that $P \cap Q \neq \emptyset$, then $P \cup Q$ is also connected.

Proof for Theorem.

Suppose $P \cup Q$ is actually not connected. This means $P \cup Q = A \cup B$ for non empty, separated sets A and B . Suppose P is fully contained in A . This means that Q has intersection with A and intersection with B which are non empty. Obviously $Q \subseteq A \cup B$ which means $Q = (A \cap Q) \cup (B \cap Q)$ where $(A \cap Q)$ and $(B \cap Q)$ are separated and non empty. Since Q is connected, this is absurd. Suppose then that P is not fully contained in A . This means that $P \cap A$ and $P \cap B$ is non empty each. This means $P = (P \cap A) \cup (P \cap B)$. From the same reasoning, this is absurd. ■

Lemma 5.5

Given two balls B_1 and B_2 in \mathbb{R}^n that are closed, with $B_1 \cap B_2 = \{z\}$ with $z \in \mathbb{R}^n$, then the interior of $B_1 \cup B_2$, i.e., $\underline{B_1 \cup B_2}$, is $\underline{B_1} \cup \underline{B_2}$, which are their respective open ball counterparts.

Proof for Lemma

We understand that $\underline{B_1} \cup \underline{B_2} \subseteq \underline{B_1 \cup B_2}$. Note that none of the "rim" points of B_1 or B_2 , are in the interior. This would conclude the result. ■

Corollary 5.6

If $A \subset X$ is connected, it needn't be true that \underline{A} is connected.

Proof for Corollary.

Consider the set $B_1 \cup B_2$ from the previous lemma. We note that, its interior is the disjoint union of two non empty open balls. These two sets are separated, which makes $\underline{B_1 \cup B_2}$ a separated set. ■

Theorem 5.7

Let E be the set of all $x \in [0, 1] \subseteq \mathbb{R}$ so that the decimal expansion of x only contains 4 and 7. Then:

1. E is uncountable
2. E is not dense in $[0, 1]$
3. E is compact
4. E is perfect

Proof for Theorem.

(1) E is uncountable via the diagonal argument. If $\{x_1, x_2, \dots\}$ is the enumeration of E , simply take the first decimal place of x_1 and flip it (i.e. to 4 if 7 or vice versa). Likewise for x_2 and so on to get a new decimal expansion that is unlike all $x_1, x_2 \dots x_n \dots$ which is a contradiction.

(2) Obviously, since $0.4 \leq x \leq 0.8$ for any $x \in E$.

(3) We already know E is bounded. Consider E^c . This is the set of all numbers in $[0, 1]$ so that not all points in the decimal expansion is 4 or 7. i.e, there would be a point in the expansion that is neither 4, nor 7. Suppose we take one such arbitrary $x \in E^c$. Let the first non 4, 7 number occur at the j -th place. $0.z_1 z_2 z_3 \dots x_j \dots$. Look for another non 4, 7 after the j -th place (if it exists). If it doesn't exist, then all the numbers after x_j

would be 4 or 7 so safely add $10^{-(j+1)}$ as our ε . This ε range around x would contain only points of E^c . Suppose another point exists that is non 4,7 after j , perhaps at $k > j$ -th index. Then it would look something like: $0.z_1z_2\cdots z_jx_{j+1}x_{j+2}\cdots z_k\cdots$. Here we simply take $\varepsilon = 10^{-(k+1)}$ so that all points in the ε -neighbourhood of x is in E^c . Therefore, E^c is open, which means E is closed. Closed and bounded implies compact in \mathbb{R} .

We saw that E is closed. We need only show that every point of E is a limit point of E . This can be done easily for any ε -ball, using the technique that follows: Choose a k so that $\frac{1}{10^k} < \varepsilon$. Look at the interval $x - \frac{1}{10^k}, x + \frac{1}{10^k}$ that is contained in E . Just find some $n > k$, and flip the 4 to a 7 or vice versa to land in a "different" element from x , yet within the neighbourhood in consideration. Hence, every point is a limit point, making E a perfect set. ■

Theorem 5.8: Existence of a compact set in \mathbb{R} with countable limit points.

Title

Proof for Theorem.

Consider the points $x_1 = 1, x_0 = 0, x_2 = \frac{1}{2}, \dots, x_n = \frac{1}{n} \dots$. Let $x_{11} < x_{12} \dots < x_{1n}$ be a sequence that converges to $x_1 = 1$. Let $x_{21} < x_{22} < x_{23} \dots < x_{2n} \dots$ be a sequence convergent to $x_2 = \frac{1}{2}$, with the added condition that $x_{2j} < x_{1k}$ for every $j, k \in \mathbb{N}$. Likewise for every x_n , create a sequence x_{nk} that converges to x_n . Make sure that $x_{an} < x_{bm}$ if $a > b$, for every m, n . We therefore have:

$$x_{11} < x_{12} \dots \rightarrow x_1$$

$$x_{21} < x_{22} \dots \rightarrow x_2$$

$$\vdots$$

We claim that the set $\Phi = x_0, x_1, x_2, \dots$ along with $x_{11}, x_{12} \dots, x_{21}, x_{22} \dots x_{n1}$ etc. forms a Compact set in \mathbb{R} that has countable limit points.

Suppose that $q \in \mathbb{R} \neq 0$ and $q \neq \frac{1}{j}$ for any j be a limit point of Φ . This means a subsequence z_n in Φ converges to q . Does infinite points of x_{1j} exist in the subsequence z_n convergent to q ? Obviously not, since that would make a subsequence of z_n convergent to one of 1. So only utmost finite elements of x_{1j} are in z_n . Same way one can argue that utmost finite elements of x_{kj} are in z_n for every k . If we establish a subsequence of z_n that converges to 0, then the only limit point of $\{z_n\}$ would be 0 which would mean 0 is where z_n would converge. Enough wishful thinking; Does any point of x_{1j} exist in z_n ? If yes, choose that point. If not, find the next n_1 so that a point in x_{n_1j} is in z_n . Find, then $n_2 > n_1$ so that some point in x_{n_2j} is in z_n . As such keep going, making $n_1 < n_2 < \dots$ and a sequence that is monotone decreasing by construction, that converges to 0. Hence, this is a subsequence in $\{z_n\}$ that converges to 0. If z_n converged to q , then the only limit

point would be q . Hence, $q = 0$. This means that the only limit points of Φ other than $\frac{1}{j}$ is 0. Hence, we have a countable limit point. Hence, Φ is a closed set and bounded obviously, with countable limit points. ■

Theorem 5.9: technique weve already seen

If A and B are separated sets in \mathbb{R}^n (that are non empty), and $\vec{x}_0 \in A$ and $\vec{y}_0 \in B$, define $p(t) = \vec{x}_0 + t(\vec{y}_0 - \vec{x}_0)$ for $t \in [-\infty, \infty]$ and

$$S_A := \{t \in \mathbb{R} : p(t) \in A\}$$

$$S_B := \{t \in \mathbb{R} : p(t) \in B\}$$

Then:

1. S_A and S_B are separated sets
2. $\exists t_0 \in (0, 1)$ so that $t_0 \notin S_A \cup S_B$

Proof for Theorem.

(1) If S_A and S_B weren't disjoint, then obviously A and B wont be. If there is a sequence in S_A converging in S_B or vice-versa, it is easy to see that this would lead to there existing a sequence in A converging to B (or vice-versa). Therefore, S_A and S_B are separated.

(2) Since S_A and S_B are separated and non-empty, there are two points x_0 and y_0 in S_A and S_B respectively. Define $z(l) = x_0 + l(y_0 - x_0)$ for $l \in [0, 1]$. Now, for some l_A , we have that for every $l > l_A$, $z(l) \notin S_A$, the set $G_A := \{l : z(l) \in S_A\}$ is therefore bounded above with supremum u_A . Likewise for some l_B , we have that for every $l < l_B$, $z(l) \notin S_B$, the set $G_B := \{l : z(l) \in S_B\}$ is therefore bounded below with infimum v_B . Note that G_A and G_B are separated sets. We may as well assume that for every $l < l_B$, l is in G_A , or $z(l)$ falls in S_A . If not, we would be done. We now ask: does v_B fall in G_A or G_B ? If it falls in G_B , there exists a sequence in G_A that would converge to v_B , which is absurd. If it falls in G_A , then by virtue of being the infimum of G_B , there is a sequence in G_B converging in G_A . Hence, there exists a point l between x_0 and y_0 $l \notin G_A$ or G_B which means $z(l) \notin S_A$ or S_B . This again means that $p(z(l)) \notin A$ or B . Phew. ■

Corollary 5.10

Every convex set in \mathbb{R}^n is connected.

Proof for Corollary.

If they were not connected, then there would exist sets A and B , non empty, disjoint and separated so that our convex set C would be $A \cup B$. Suppose $x_0 \in A$ and $y_0 \in B$. From the previous theorem we see that, if $z(t) = x_0 + t(y_0 - x_0)$, then there would exist t' so that $z(t') \notin A$ or B , which means it wont be in C . But if x_0 and y_0 are in C , by definition of convexity, $z(t)$ for any t must exist in C . Contradiction. ■

Example : A Pedagogical Example.

If $E := \{q \in \mathbb{Q} : 2 < q^2 < 3\}$ is considered a set in the metric subspace of \mathbb{Q} with the usual distance, then we have:

1. E is closed and bounded (wrt \mathbb{Q} obviously)
2. E is *non compact*.
3. E is open with respect to \mathbb{Q}

Proof. Consider an arbitrary convergent sequence p_n in E . We see that if p_n is contained in E , then $2 < p_n^2 < 3$. If we pass to the limit, we would have $2 \leq q^2 \leq 3$ but the limit q would either not exist in \mathbb{Q} (whence the sequence p_n wouldn't be convergent anyway) or it does, in which case it follows $2 < q^2 < 3$ (since no rational number has its square as 2 or 3). Therefore, we can see that every convergent sequence in E converges in E . Hence, E is closed. Boundedness is obvious.

One way to see that E is non compact is simply by making use of the "conservation" of compactness going from one space to a bigger space or vice-versa. Since E is clearly not compact in \mathbb{R} , it won't be compact in the metric subspace \mathbb{Q} . Another way to see that it is non compact is to consider the following construction: Look at the union of the two split intervals $(-\sqrt{3}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{3})$. Let $h = \sqrt{3} - \sqrt{2}$. We construct it for the right side interval, and simply copy it to the left one. Choose the point $\sqrt{3}$ and a $h/2$ -ball around $\sqrt{3}$ and call it V_0 . This would intersect the right interval at a distance $h/2$ from $\sqrt{3}$. Choose this intersection point and a $h/4$ -ball around this called V_2 . Both V_1 and V_2 together covers $3h/4$ of the interval. Choose this intersection point and let the $h/8$ -ball around this point be V_3 . V_1 , V_2 and V_3 together cover $7h/8$ of the interval. Keep going as such to construct V_n so that $(V_i)_{i=1}^n$ covers $\frac{(n-1)h}{n}$ of the whole interval. As n tends to ∞ , the "coverage" converges to h . This means that for every ε distance away, there exists a finite n_0 so that $V_1 \rightarrow V_{n_0}$ covers up to that ε distance. Every point e in the strict interval would therefore be covered up by some V_k by our construction. But obviously, this "open cover" has no finite subcover, for no finite "coverage" covers all the way till h distance of the interval. Some ε -gap is always left, hence missing points.

E can also be written as the disjoint union of the set A of all p so that $\sqrt{2} < p < \sqrt{3}$ and the set B of all p so that $-\sqrt{3} < p < -\sqrt{2}$. WLOG say $q \in A$. Obviously $\sqrt{2} < q < \sqrt{3}$. Choose $\delta < \min\{\sqrt{3} - q, q - \sqrt{2}\}$ and the δ -ball called V around q . Easy to see that this ball V is fully contained in A . Likewise, it can be shown for B as well. Hence, E is open with respect to \mathbb{Q} . □

Theorem 5.11: Existence of a non-empty, perfect set in \mathbb{R} that contains no rational points

Title

Proof for Theorem.

Definition 5.12: Separability

A metric space X is said to be separable if it has a countable dense subset S .

Example : \mathbb{R}^n is separable

Consider the set of all points $z \in \mathbb{R}^n$ with coordinates in \mathbb{Q} . Basically \mathbb{Q}^n . This would be a countable set, and every point $x \in \mathbb{R}^n$ is a limit point of this set (since coordinate we can have a rational sequence converging to that coordinate)

Definition 5.13: Basis

A collection of open sets $\{V_\alpha : \alpha \in A\}$ is said to be a basis for X , a metric space, if for every point $x \in X$, and for every open set G so that $x \in G$, we have a V_α in the basis so that $x \in V_\alpha \subset G$. In other words, every open set G can be "covered" by a subcollection of $\{V_\alpha\}$

Theorem 5.14

For a metric space X , separability \iff countable basis

Proof for Theorem.

\implies) Suppose X is separable, i.e, has a countably dense set $\{x_1, x_2, \dots, x_n, \dots\}$. Choose all the $1/n$ -balls of each of these points to form a countable set. Choose all the $1/2$ -ball of each for another countable set. As such keep choosing $1/n$ -balls for each x_j in the countably dense set. We claim that this collection of open sets are my basis. Say z is an arbitrary point in X , and G an arbitrary open set containing z . There would exist ε_z^G so that ε_z^G -ball of z would be contained in G . Choose $\delta < \frac{\varepsilon_z^G}{4}$. There would be an n so that $\frac{1}{n} < \delta$. In this $1/n$ ball around z , there would exist a point x_k in the dense set. Choose now, the $1/n$ ball around x_k which contains z , and would be completely contained in the ε_z^G ball, which would be contained in G . Hence, we are done.

\impliedby) Say X has a countable basis $\{V_1, V_2, \dots\}$ that are non empty. Choose $x_1 \in V_1$, $x_2 \in V_2 \dots$. We claim that this set $\{x_1, x_2, \dots\}$ forms our countable dense set. Say a point $z \in X$, choose any arbitrary $\varepsilon > 0$. The ε -ball around z would be an open set, which means a set V_k is inside ε -ball around z . This would mean that x_k would be in the set, which makes z a limit point of this supposed to be dense set $\{x_1, x_2, \dots\}$. We are done.

Theorem 5.15

$$"(\exists\{V_\alpha : \alpha \in A\})(\forall x \in X)(\forall G \subset X : G \text{ open} : x \in G)(\exists \alpha \in A)(x \in V_\alpha \subset G)"$$

$$\iff$$

$$\exists\{V_\alpha : \alpha \in A\}(\forall G \subset X : G \text{ open in } X)(\exists A' \subseteq A)(G = \cup_{\alpha \in A'} V_\alpha)$$

Or in pithy words,

V_α is basis if and only if every open set is the union of a subcollection of $\{V_\alpha\}$

Proof for Theorem.

\implies) Consider an open set G (arbitrary) and an arbitrary point $x \in G$. There exists, for every point x in G a corresponding base V_{α_x} so that $x \in V_{\alpha_x} \subset G$. This means, if we collect all the x in G and its corresponding bases, the union of that basis would give us G .

\impliedby) Suppose for any given $G \subset X$ that is open, there is a subcollection of a (fixed) $\{V_\alpha\}$ so that $G = \cup_{\alpha \in A'} V_{\alpha'}$. Consider an arbitrary point $x \in X$ and an arbitrary $G : x \in G$. This means there is an epsilon ball around x that is fully contained in G . But since open balls are open, there is a subcollection $\{V_{\alpha''}\}$ so that "this ball" = $\cup V_{\alpha''}$. Since $x \in$ this ball, $x \in V_k$ for some k . But this v_k is in G . Hence, we are done. ■

Fact 5.16

$$"(\exists\{V_\alpha : \alpha \in A\})(\forall x \in X)(\forall G \subset X : G \text{ open} : x \in G)(\exists \alpha \in A)(x \in V_\alpha \subset G)"$$

is the same thing as

$$"(\exists\{V_\alpha : \alpha \in A\})(\forall G \subset X : G \text{ open})(\forall x \in X : x \in G)(\exists \alpha \in A)(x \in V_\alpha \subset G)"$$

Definition 5.17

2nd countable

A space X is said to be second countable if it has a countable basis.

Theorem 5.18

Let X be a metric space in which every infinite subset E has a limit point in X . Then X has countable basis (or equivalently is separable).

Proof for Theorem.

Note that no infinite subset of X can be "unbounded" since we have seen before that Limit point compact implies closed and bounded. Consider an arbitrary δ_0 . Choose an initial point x_1 . Is there a point x_2 that is outside the δ -ball around x_1 ? If no, the only δ -ball of x_1 covers the whole space. If yes, then again ask the question, is there a point x_3 that is both outside the δ -ball of x_1 and the δ -ball of x_2 ? If not, then these two balls will cover the whole space, if yes, again ask the question, is there a point x_3 so that it is not in x_1, x_2 and x_3 's δ -balls. Keep going, as such. We claim that it must terminate after a finite number of steps. Suppose not, i.e for every x_n you can find, you can find an x_{n+1} to form an infinite set of x_j such that x_j is not in the δ -ball of any of its preceding elements x_{j-1}, x_{j-2}, \dots . This is true for every x_j . From the hypothesis, this set has a limit point q . Choose $\varepsilon < \delta/4$, within which exists a point x_k in this infinite set. But since no point exists within the δ -ball of x_k apart from itself, no other point exists in the ε -ball around x_k as well. This would contradict the proposition that q is a limit point. Hence, this procedure must terminate in a finite amount of steps. So for a given $\delta = d$, there exists n_δ balls $V_1, V_2, \dots, V_{n_\delta}$ with centres $x_1^d, x_2^d, \dots, x_{n_\delta}^d$. Choose $\delta = 1$, and create a collection of 1-balls (by the procedure mentioned above) around the points $x_1^1, x_2^1, \dots, x_{n_1}^1$. Likewise, for $\delta = 1/2$, find points $x_2^1, x_2^2, \dots, x_{n_2}^2$, and for $\delta = 1/k$, find points $x_2^k, x_2^k, \dots, x_{n_k}^k$. Let z be a point in X . Choose a ε ball around z . Let $1/n < \varepsilon$. There must be some x_j^n so that z is in the $1/n$ -ball of x_j^n . Hence, $\{x_j^k : j \in \mathbb{N}, k \in \mathbb{N}\}$ is the dense set we need. ■

Theorem 5.19

Separable \iff countable basis \iff every open cover of X has a countable subcover

Proof for Theorem.

1 \iff 2) already seen

2 \implies 3) Suppose we have an open cover $\{G_\alpha : \alpha \in A\}$ of X which has a countable basis $\{V_n\}$. Let $x \in X$ be arbitrary, which means it is in some G_α , which is the union of some subfamily of $\{V_n\}$ which means x is in some V_j which is inside G_α . Let $P := \{n \in \mathbb{N} : \exists \alpha_n \text{ such that } V_n \subseteq G_{\alpha_n}\}$. Since every x is in some V_j we have that $X \subseteq \bigcup_{i=1}^\infty V_{n_i} \subseteq \bigcup_{i \in P} G_{\alpha_i}$ which ultimately means that G_{α_i} forms a countable subcover of $\{G_\alpha : \alpha \in A\}$.

3 \implies 1) Suppose every open cover has a countable subcover. Consider $\varepsilon = 1$ and all 1-balls in X . This is an open cover that has a countable subcover of 1-balls centred at $\{x_{11}, x_{12}, \dots\}$. Likewise, consider $1/2$ -balls in X which would have a countable subcollection of $1/2$ -balls centred at $\{x_{21}, x_{22}, x_{23}, \dots\}$. As such keep going for $\varepsilon = 1/n$ to get a countable collection of $1/n$ -balls. Take all these balls (which gives us a countable collection, from axiom of choice) and call this set S . Let $z \in X$. let $\varepsilon > 0$ and $1/n < \varepsilon/4$. Consider the ε -ball around z . For $\delta = 1/n$ we have a countable collection of points $\{x_{n1}, x_{n2}, \dots\}$ whose $1/n$ balls cover the whole space, which means there is a point in S , say x , whose $1/n$ ball contains z . But this ball is in turn in the ε -ball around z , which makes x a point in ε ball of z , hence every point of X is a limit point of S (or a point of S). ■

Theorem 5.20

A compact metric space has a countable base (or equivalently, is separable- has a countable dense set in it).

Proof for Theorem.

Choose $\delta_1 = 1$ and all the δ_1 -balls in X (centred at every point of X). This makes an open cover of X that has a finite subcover. i.e, finite points $x_1^1, x_2^1 \cdots x_{n_1}^1$ whose $\delta_1 = 1$ balls cover the whole space X . Next, choose $\delta_2 = 1/2$ and every δ_2 ball in X . This likewise, would have a finite subcover i.e, δ_2 -balls around $x_1^2, x_2^2 \cdots x_{n_2}^2$. As such, for every $\delta_k = 1/k$, we have points $x_1^k, x_2^k, \cdots x_{n_k}^k$ whose δ_k balls cover the whole space.

That this space is separable is obvious now. Choose $x_j^i : i, j \in \mathcal{N}$. Let z be a point in X . Choose any arbitrary ε . Find an n so that $\frac{1}{n} < \varepsilon$. From the result we have, we can always find a x_j^k in our set so that it is in the $\frac{1}{n}$ ball of z . This means that in every ε ball of z , there exists a point from our stipulated set. Hence, our set is dense in X . ■

Theorem 5.21

If X is a metric space so that every infinite set in X has a limit point in X , then every open cover of X (open relative to X) has a finite subcover. Hence:

"If X is a limit point compact metric space, then X is a compact metric space (compact relative to itself, but it also applies to all spaces it lives in, due to preservation of compactness)"

-Krishna, to Arjuna, at Kurukshetra

Proof for Theorem.

Suppose that X is limit point compact. This means that it has a countable basis. Consider an open cover $\{G_\alpha : \alpha \in A\}$ so that $X = \cup_{\alpha \in A} G_\alpha$. Let $\{V_1, V_2 \cdots\}$ be the countable basis for X . This means that for every open set G and every point $x \in X$ so that $x \in G$, we have an $n \in \mathbb{N}$ so that $x \in V_n \subset G$. Let $P := \{n \in \mathbb{N} : \exists \alpha \in A \text{ such that } V_n \subseteq G_\alpha\}$. Consider any point $x \in X$. For this x , there exists some G_α so that $x \in G_\alpha$ which would imply from the countable bases property that $\exists V_n$ so that $x \in V_n \subseteq G_\alpha$. For every point, there exists a V_n and a corresponding G_α so that $x \in V_n \subset G_\alpha$. Therefore, X is countably "subcovered" by G_{α_j} (those G_α that correspond to the $V_n, n \in P$).

Let this countable subcover be $G_1, G_2 \cdots G_n \cdots$. Suppose to this, there does not exist a finite subcover. i.e,

$$\exists x_1 \in G_1^C$$

,

$$\exists x_2 \in (G_1 \cup G_2)^C = G_1^C \cap G_2^C$$

.

⋮

$$\exists x_n \in (G_1 \cup G_2 \cup G_3 \cup \cdots G_n)^C = \cap_{i=1}^n G_i^C$$

, for every $n \in \mathbb{N}$. Collect all such x_n s to form an infinite set in X . By hypothesis, this has a limit point, call it z . Note that, outside a given G_k^C , only finite points of our stipulated set exists, which are possibly $x_1, x_2 \cdots x_{n-1}$. Suppose there exists a G_j^C so that $z \notin G_j^C$. This means that z is in G_j which is an open set. Therefore, there exists an ε ball that is completely inside G_j , or rather completely outside G_j^C . This means that only finitely many $x_1, x_2 \cdots$ make into this ε -ball of z which is absurd since z is supposedly a limit point. Hence, we cannot have a countable cover that has no finite subcover. Hence, $G_j : j \in \mathbb{N}$ has a finite subcover, which means X is a compact metric space. ■

Remark.

We showed that every limit point compact *metric space* is compact, i.e, every open cover, open relative to X , has a finite subcover. Suppose X is actually a subset of a bigger set H and X inherits the metric from H . From conservation of compact relativeness, we see that X must be compact relative to H as well. Hence, every limit point compact set in X is compact in X .

Definition 5.22: Hausdorff Space

A space X in which two distinct points x and y have disjoint neighbourhoods (open sets) V_x and V_y so that $x \in V_x$ and $y \in V_y$

Theorem 5.23: Equivalent statements pertaining to connectedness

The following are equivalent:

1. only clopen sets in X are X and \emptyset
2. X is not the disjoint union of two non empty open sets
3. X is not the union of disjoint non empty closed sets
4. X is not the union of separated sets $\overline{A} \cap B = \overline{B} \cap A = \emptyset$
5. Every proper non-empty subset of X has a non empty boundary in X
6. X does not have a continuous onto function mapping X to $\{-1, 1\}$ with discrete topology

Proof for Theorem.

1 \implies 2) Say X is the disjoint union of two non empty open sets, $X = A \cup B$ where A and B are open. $X \setminus (A) = B$ which, by definition, is closed. But B by hypothesis is open, hence B is a clopen set in X that is not X or \emptyset .

$2 \implies 1)$ Suppose $Z \subset X$ is a non empty clopen, proper subset of X . $W = X \setminus (Z)$ is also, by definition, clopen. So we have W and Z , disjoint, non empty open sets whose union gives us X .

$1 \implies 3)$ Suppose X is the disjoint union of two closed sets. $X = A \cup B$. $X \setminus (A) = B$ which is open, but also by hypothesis, closed. Hence, there is a clopen set in X .

$3 \implies 1)$ Suppose there exists clopen set $Z \subset X$. Then $W = X \setminus (Z)$ is also clopen by definition. Therefore, W and Z form disjoint closed sets whose union gives us X .

$1 \implies 4)$ Say X is the union of disjoint separated sets A and B so that $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. We then have $B \subseteq (\overline{A})^C$ and likewise $A \subseteq (\overline{B})^C$. $X = A \cup B$. $\overline{(A)} \subseteq X$ and $\overline{B} \subseteq X$ which means that $\overline{A} \cup B = X$. $\overline{A} \cup B \setminus (\overline{A}) = X \setminus (\overline{A}) = B$ tells us that B is a open set. Doing the same to $\overline{B} \cup A = X$ tells us that A , too, is an open set. Hence, X is the disjoint union of two open sets.

$4 \implies 1)$ Say there exists a clopen set $Z \in X$. Consider Z^C which is also clopen, and it is disjoint from Z . This means immediately that $\overline{Z} \cap Z^C = Z \cap Z^C = Z \cap \overline{Z^C} = \emptyset$, hence, X is the union of two separated sets.

$1 \implies 5)$ Suppose there exists a non empty proper subset of X , call it W that has an empty boundary. i.e, $\overline{W} \setminus (W^o) = \emptyset$ which means $\overline{W} \cap (W^o)^C = \emptyset$. $W^C \subseteq (W^o)^C$ which means $\overline{W} \cup (W^o)^C = X$ which makes X the disjoint union of two closed sets

$5 \implies 1)$ Suppose there exists a clopen set W in X . that is neither X nor \emptyset . W^C is, then, also clopen. Boundary of W would be $\overline{W} \setminus (W^o) = W \setminus (W) = \emptyset$. Hence, W is a proper set, non empty such that its boundary is empty.

$1 \implies 6)$ Let f be a continuous onto function from X to the discrete set $\{-1, 1\}$ with the discrete topology. Consider $\alpha = f^{-1}(-1)$ and $\beta = f^{-1}(1)$. Of course, $\alpha \cup \beta = X$. Is α and β disjoint? they must be since $\{-1\}$ and $\{1\}$ are disjoint. Moreover, since f is continuous, and -1 and 1 are open sets in the discrete topology, we have that α and β are open sets. Hence, X is the disjoint union of two open sets.

$6 \implies 1)$ Suppose $X = A \cup B$ where A and B are non empty, disjoint open sets. Define the function $f : X \rightarrow \{-1, 1\}$ given by $f(x) = -1$ if $x \in A$ and $f(x) = 1$ if $x \in B$. $f^{-1}(-1)$ would be A and $f^{-1}(1)$ would be B , which means every open set inverse maps to an open set, making f a continuous onto function from X to the discrete topology $\{-1, 1\}$