3. Karnaugh Map

Standard forms for expressing Logic Functions

There are two standard forms inwhich logic functions can be expressed: (1) Sum-of-products form and (2) Product-of-sums form.

1.Sum-of-products (SOP) form

In the sum-of products form, the logic function is written as a simple sum of terms. For example,

(i) consider the function $L = (\overline{w} + x\,y)\,(x + yz)$

$$\begin{aligned} \text{Expanding, } L &= (\overline{w} + xy)\,x + (\overline{w} + x\,y)\,yz \\ &= \overline{w}x + xxy + \overline{w}yz + xyyz \\ &= \overline{w}x + xy + \overline{w}yz + xyz \end{aligned}$$

This is the desired sum of products form.

(ii) Consider another example in which $L = (\overline{A\,B} + C)\ \overline{D}$

Expanding, $L = (\overline{A\,B})\,D + C\,\overline{D}$

This is the desired sum-of-products form.

(iii) Let us now find the logic equation in sum-of-products form for L as described by the truth table:

x	y	z	L
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

We see that $L = 1$ for the conditions of rows 3,4,5,7 and 8. Consider row 3; $L = 1$ if $x = 0$ AND $y = 1$ AND $z = 0$. These three combinations can be combined into one expression, $\overline{x}\,y\,\overline{z} = 1$.

For row 4, $L = 1$ only if $x = 0, y = 1$ and $z = 1$. $\therefore \overline{x}\,y\,z = 1$

We note that either row 3 or row 4 leads to $L = 1$.

Similarly, rows 5,7 and 8 leads to the terms $x\,\overline{y}\,\overline{z}$, $x\,y\,\overline{z}$ and $x\,y\,z$ respectively. If any one of these is 1 then L will be 1. Thus, they are simply OR ed. Hence the final logic equation of L is

$$L = \overline{x}\,y\,\overline{z} + \overline{x}\,y\,z + x\,\overline{y}\,\overline{z} + x\,y\,\overline{z} + x\,y\,z \qquad \text{.............(1)}$$

This is in the sum-of-products form. Each individual term is called 'minterm'. The minterm will consist of all variables or their complements. On simplification equation (1) becames

$$L = y + x\,\overline{y}\,\overline{z} \qquad \text{.............(2)}$$

2. *Product-of-sums (POS) form*

The product-of-sums form consists of a product of terms in which each terms consists of a sum of all or part of the variables.

1. For example, consider the function (as before)

$$L = (\overline{w} + x\,y)\,(x + y\,z)$$

The terms $\overline{w} + x\,y$ and $x + y\,z$ can be converted by the use of the theorem

$$A + (B\,.\,C) = (A + B)\,.\,(A + C)$$

$$\therefore\ L = (\overline{w} + x)\,.\,(\overline{w} + y)\,.\,(x + y)\,.\,(x + z).$$

This is the product-of-sums form of the given expression.

2. Consider the truth table given above. The logic function L can be expressed in the product of sums form by noting the rows for which $L = 0$. Rows 1, 2 and 6 give $L = 0$. For each of these rows, a sum of terms is formed. If a variable has the value 1, it is complemented; while if the variable has the value 0, it is kept unchanged. Thus, for row 1, we have the sum $x + y + z$ for $L = 0$. i.e., $L = x + y + z$ for row 1.

Similarly, for row 2 the sum is $(x + y + \overline{z})$ and for row 6, it is $(\overline{x} + y + \overline{z})$ for $L = 0$. (Because this is the only way to get a logical sum of 0 for the given input conditions.)

$$L = \overline{x} + y + \overline{z} = \overline{1} + 0 + \overline{1} = 0 + 0 + 0 = 0.$$

Hence the three terms must be ANDed together to get the equation for L in the product of sums as

$$L = (x+y+z).(x+\overline{y}+\overline{z}).(x+\overline{y}+z) \qquad \text{......................(3)}$$

Each of the three terms is called a 'maxterm'. The maxterm will consist of all variables or their complements. On simplification, equation (3) becomes

$$L = y + x\overline{y}\,\overline{z} \qquad \text{......................(4)}$$

Equations (2) and (4) are the same.

Boolean Equations (1) and (3) are given in different forms. But they perform the same logic function, which is described by the truth table.

Significance of the two forms of expressions

When a logic function is expressed in the sum-of-products form, the circuit will consist of as many AND gates as the number of terms in the expression (giving us the product) followed by one OR gate (giving us the sum). i.e., all sum-or-products equation will lead to two level AND - OR structure in the circuit.

Similarly, when we begin with the product of sums expression the resulting circuit will be a two-level OR-AND structure.

In the former case, the circuit can be redrawn by replacing all gates by NAND gates only in order to implement the given logic function. In the latter case,in order to implement the given logic function it should be expressed as the product - of - sum form. This can be implemented by OR - AND structure. Here again the circuit can be redrawn by replacing all the gates by NOR gates only. Thus implementation of a logic function (when expressed in one of the two standard forms) is made easier with the use of NAND gates only or NOR gates only.

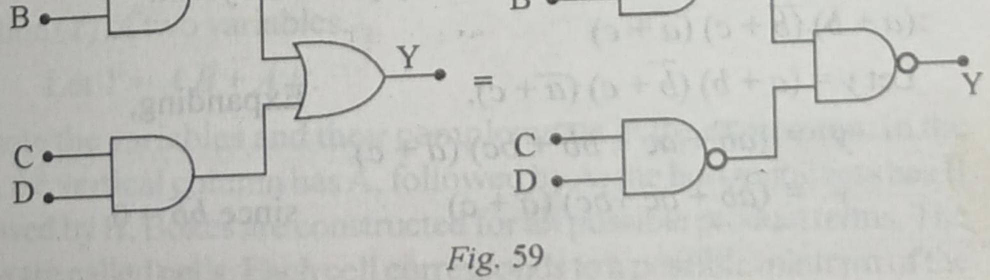


Fig. 59

The universal building blocks-NAND, NOR - with large number of input terminals (unlike other gates) are commercially available in I.C. form for use.

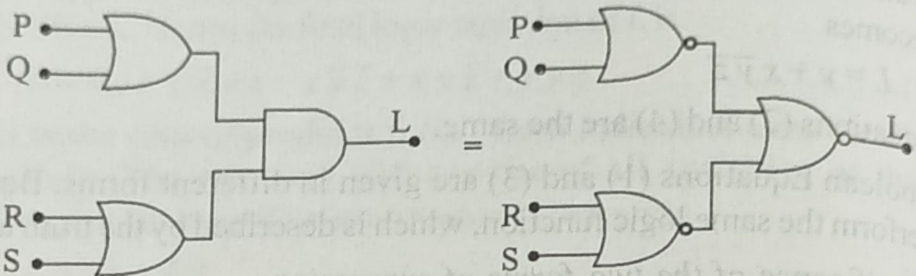


Fig. 60

Figure above shows the AND-OR circuit followed by the corresponding circuit with NAND only. Also the OR-AND circuit is followed by the corresponding circuit with NOR only.

Example

1. *Obtain the sum-of-products for the logic function*

$$f = (A + C + D\,C)\,(B + B\,C + D)$$

Given, $f = (A + C + D\ \ C)\,(B + B\ \ C + D)$. Expanding,

$$f = AB + ABC + AD + BC + BC + CD + BCD + BCD + DC$$

$$= AB + ABC + AD + BC + CD + BCD$$

$$= AB\,(1 + C) + AD + BC + CD\,(1 + B)$$

$$\therefore f = AB + AD + BC + CD \qquad \text{since } 1 + C = 1;\ 1 + B = 1.$$

This is in the sum-of-products form.

2. *Convert the following into sum-of-products form:*

$(a + b)\,(\overline{b} + c)\,(\overline{a} + c)$

Let $y = (a + b)\,(\overline{b} + c)\,(\overline{a} + c)$. Expanding,

$$y = (a\overline{b} + ac + b\overline{b} + bc)\,(\overline{a} + c)$$

$$y = (a\overline{b} + ac + bc)\,(\overline{a} + c) \qquad \text{since } b\overline{b} = 0$$

$$= a\bar{b}\bar{a} + a\bar{b}c + ac\bar{a} + acc + bc\bar{a} + bcc$$
$$= a\bar{b}c + ac + bc\bar{a} + bc \qquad \text{since } \bar{a}a = 0$$
$$= ac(\bar{b}+1) + bc(\bar{a}+1)$$
$$y = ac + bc \qquad \text{since } 1 + \bar{b} = 1$$

This is the sum-of-products form.

Karnaugh Map

While implementing a given logic function, it is advisable to use the minimum hardware (circuitry). For this, the logic function is to be brought to the simplest form possible. This can be done using Boolean Algebra. Another technique, known as Karnaugh map, provides a systematic method for simplifying and manipulating Boolean expression.

The Karnaugh map is a visual display of the fundamental products, present in the sum-of products (SOP) form or product-of sums (POS) form of a given switching function (Boolean function).

The K-map is a matrix of cells for expressing a logic function and it is used to obtain graphically a minimal algebraic expression for the function. The map will contain 2^n cells for an n-variable function.

By the use of the Karnaugh map procedure, one can implement a given switching function (Boolean expression) with a minimum number of gates. This is achieved by minimising the number of terms as well as the terms themselves, in the switching function.

Two Variable map

We shall start by drawing the Karnaugh map for the switching function (Y) of two variables.

Let $Y = A\,\overline{B} + A\,B$.

We note the variables and their complements in the expression. In the map, the vertical column has $\overline{A}$, followed by A; the horizontal row has $\overline{B}$ followed by B. Boxes are constructed for all possible product terms. The boxes are called cells. Each cell corresponds to a possible minterm of the function of two variables. Looking at the expression, a 1 is written in the appropriate cell corresponding to each minterm in the expression. In all other cells, a 0 has to be written. Most often, 0 may be omitted in the cells. Thus we have the following map.

The 1 - cells are adjacent horizontally. The adjacent cells are encircled. Such encircled cells are called sub-cube. These cells corresponds to minterms that are the same in A but different in B. These two terms can be combined into a single term, by eliminating the variable that changes form (we retain the term, remaing unchanged). Thus, the sub-cube in the map gives the simplified Boolean expression

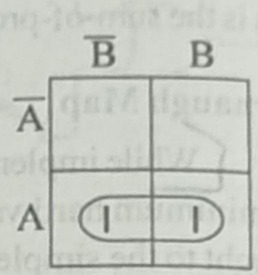


Fig.61

$$Y = A.$$

To verify this, consider the given sum-of-products form of Y.

$$Y = A\,\overline{B} + A\,B = A\,(\overline{B} + B) = A \qquad \text{since } \overline{B} + B = 1$$

Example:

Simplify the logic function $Y = A\,\overline{B} + A\,B + \overline{A}\,B$ using Karnaugh map and hence implement Y.

The given logic function is of two variables. The variables are A and B. The variables and the complements are listed along the side and top of the Karnaugh map as shown in figure 62.

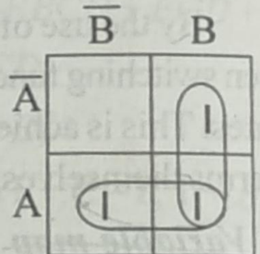


Fig.62

On inspecting the given logic function Y, we find that there is no term as $\overline{A}\,\overline{B}$ (for upper left cell). So the upper left cell is left blank. The term $A\,\overline{B}$ is represented as 1 in the lower left cell. Similarly the term $A\,B$ and $\overline{A}\,B$ are represented in the lower right and upper right cells respectively.

The adjacent cells having 1 are encircled as pairs. There is a horizondal pair and a vertical pair. For the horizontal sub-cube the simplified expression is A (since A remains the same for the sub cube). For the vertical sub-cube the simplified expression is B. (since B remains the same for the sub-cube in this case).

Hence the simplified Boolean expression for Y using K-map is

$$\boxed{Y=A+B}$$

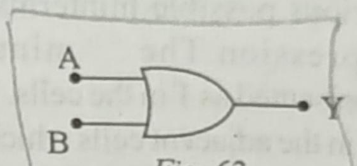


Fig. 63

The logic function Y can be easily implemented using OR gate as shown in figure 63.

Three variable map

Consider the switching function

$$Y=\overline{A}\,B\,\overline{C}+A\,B\,\overline{C}+A\,B\,C$$

There are three min-terms. Each minterm is represented as 1 in the appropriate cell in the Karnaugh map. The map consists of cells meant for various possible minterms.

	$\overline{B}\,\overline{C}$	$\overline{B}\,C$	$B\,C$	$B\,\overline{C}$
$\overline{A}$				1
A			1	1

Fig. 64

Of the three variables, B and C or combination with their complements are grouped and written progressively as $\overline{B}\,\overline{C}$, $\overline{B}\,C$, $B\,C$ and $B\,\overline{C}$ on top of the map. The variable A and its complement $\overline{A}$ are written on the left side of the map as shown. ($\overline{A}$ first; A next).

Now enter 1 for the term $\overline{A}\,B\,\overline{C}$, $A\,B\,\overline{C}$ and $A\,B\,C$. The adjacent cells with 1 are encircled to form sub-cubes. There is a horizontal sub cube; we retain the variables that remain unchanged. Therefore, the simplified expression is $A\,B$.

Similarly, for the vertical sub-cube, the simplified expression is $B\,\overline{C}$.

$\therefore$ The resultant Boolean expression for the given logic function is

$$Y=A\,B+B\,\overline{C}$$

Example:

1. ***Simplify the switching function given below using Karnaugh map***

$$Y=\overline{A}\,B\,\overline{C}+\overline{A}\,B\,C+A\,B\,\overline{C}+A\,B\,C$$

The Karnaugh map is constructed with cells meant for various possible minterms in the given expression. The minterms are represented as 1 in the cells. There are four 1-s in the adjacent cells which form a quad. A quad is a group of four 1-s that are horizontally and vertically adjacent. In the quad, the numeral that remains unchanged is found to be B. $\therefore$ The simplified expression for the given switching function is

	$\overline{B}\,\overline{C}$	$\overline{B}\,C$	$B\,C$	$B\,\overline{C}$
$\overline{A}$			1	1
A			1	1

Fig. 65

$$Y=B$$

2. ***Simplify the given logic function using Karnaugh map:***

$$Y=\overline{A}\,\overline{B}\,\overline{C}+A\,\overline{B}\,C+A\,B\,C+\overline{A}\,B\,\overline{C}$$

The Karnaugh map with three variables is constructed. Enter 1 for each minterm in the given expression. The adjacent 1 -s are encircled to form sub-cube. The first sub-cube is for constant A. The value of the pair is AC.

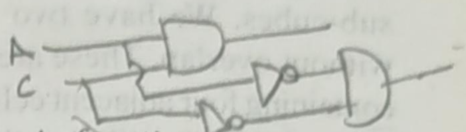
Consider the values of upper left and upper right cells in the $\overline{A}$ row of the Karnuagh map. It can be considered that the right and the left edges of the Karnaugh map touch. This is called rolling the map. A sub-cube is thus formed along the $\overline{A}$ row. The value of this sub-cube is $\overline{A}\,\overline{C}$.

	$\overline{B}\,\overline{C}$	$\overline{B}\,C$	$B\,C$	$B\,\overline{C}$
$\overline{A}$	1			1
A		1	1	

Fig. 66

∴ The simplified expression for the given switching function is

$$Y = A\,C + \overline{A}\,\overline{C}$$



Problem

1. *Using Karnuagh map, simplify the logic function*
$Y = \overline{X}\,\overline{Y}\,\overline{Z} + \overline{X}\,\overline{Y}\,Z + X\,\overline{Y}\,\overline{Z} + X\,\overline{Y}\,Z$ [*Ans* : $L = \overline{Y}$]

A C AC $\overline{A}$

Four variable map

Given that $Y = \overline{A}\,\overline{B}\,\overline{C}\,D + \overline{A}\,\overline{B}\,C\,D + \overline{A}\,B\,C\,\overline{D} + A\,B\,C\,\overline{D}$. To obtain the Karnaugh map we take combinations of two numerals (A, B) and their complements on left side of the Karnaugh map as $\overline{A}\,\overline{B}$, $\overline{A}\,B$, AB, $A\,\overline{B}$ in order and the combinations of the other two numerals (C, D and their complements) as $\overline{C}\,\overline{D}$, $\overline{C}\,D$, $C\,D$, $C\,\overline{D}$ taken in order on the top of the Karnaugh map. Each minterm is represented as 1 in appropriate cell. Encricling the cells with 1, we get two sub-cube as marked.

	$\overline{C}\,\overline{D}$	$\overline{C}\,D$	$C\,D$	$C\,\overline{D}$
$\overline{A}\,\overline{B}$		1	1	
$\overline{A}\,B$				1
$A\,B$				1
$A\,\overline{B}$				

Fig. 67

The value of the first sub-cube is $\overline{A}\,\overline{B}\,D$

The value of the second sub-cube is $C\,\overline{D}\,B$

The simplified expression for Y is $Y = \overline{A}\,\overline{B}\,D + C\,\overline{D}\,B$

Example

1. *Draw the Karnaugh map for simplifying the switching function*

$$Y = \overline{A}\,B\,\overline{C}\,\overline{D} + A\,B\,\overline{C}\,\overline{D} + \overline{A}\,B\,\overline{C}\,D + A\,B\,\overline{C}\,D + \overline{A}\,\overline{B}\,C\,D + \overline{A}\,B\,C\,D + A\,B\,C\,D + A\,\overline{B}\,C\,D$$

The adjacent 1-cells are encircled to get sub-cubes. We have two sub-cubes without overlap. These are (i) a quad containing four adjacent cells covering $\overline{A}\,B$ and A B row. (ii) Another quad in the $C\,D$ column.

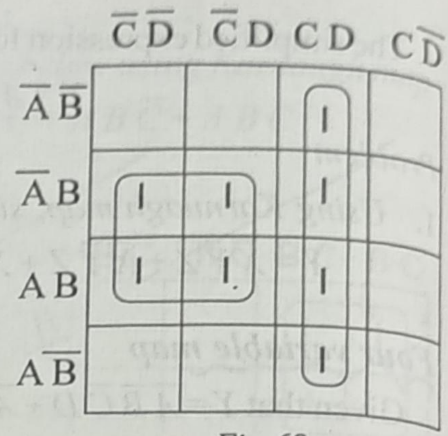


Fig. 68

A quad always shows two literals remaining unchanged. In quad (i) the unchanging literals are $B\,\overline{C}$. In quad (ii), $C\,D$ remains unchanged.

$\therefore\ Y = B\,\overline{C} + C\,D$

2. *What is the simplified Boolean expression for the given switching function? Use Karnaugh map.*

$$Y = \overline{A}\,B\,C\,D + A\,B\,\overline{C}\,\overline{D} + A\,B\,\overline{C}\,D + A\,B\,C\,D + A\,B\,C\,\overline{D} + A\,\overline{B}\,\overline{C}\,D + A\,\overline{B}\,C\,D + A\,\overline{B}\,C\,\overline{D}$$

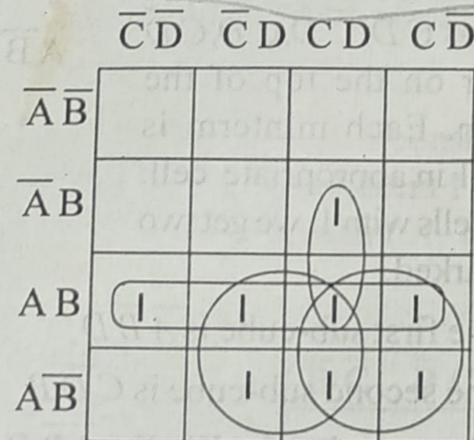


Fig. 69

There are three quads and one sub-cube. They are:

quad (i) with four 1 -s in the $A\,B$ row

quad (ii) in the $C\,D$ and $C\,\overline{D}$ column

quad (iii) in the $\overline{C}\,D$ and $C\,D$ column

Pair (i) along $C\,D$ column in $\overline{A}\ B$ and $A\,B$ row quad (i) gives two literals remaining unchanged. They are $A\,B$.

quad (ii) gives $A\,C$

quad (iii) gives $A\,D$

pair (i) gives $C\,D\,B$

By ORing these products, we get the simplified expression in sum-of-product form as

$$Y = A\,B + A\,C + A\,D + C\,D\,B.$$

Implicant: A product term in a sum-of-products expression is called an implicant. An implicant that cannot be fully enclosed by another implicant on a Karnaugh map is known as prime implicant. A prime implicant that encloses one or more 1-cells and cannot be enclosed by any other prime implicants is known as essential prime implicant.

3. *For the given Karnaugh map, obtain the simplified Boolean expression.*

	$\overline{C}\,\overline{D}$	$\overline{C}\,D$	$C\,D$	$C\,\overline{D}$
$\overline{A}\,\overline{B}$				
$\overline{A}\,B$		1		
$A\,B$	1	1	1	1
$A\,\overline{B}$	1	1	1	1

Fig. 70

There are eight adjacent cells, having 1. These form an octet (sub-cube). Also there is a single cell having a 1. The octet has a value A (an octet always leaves one literal unchanged).

Method of addressing a cell in K-map

1. Two variable map

The two variable map is shown in the figure 72. There are four cells in it to accommodate for minterms. Let A and B be the two variables in a given logic expression. The possible minterms are $\overline{A}\,\overline{B}$, $\overline{A}\,B$, $A\,B$, $A\,\overline{B}$.

These terms can be accommodated in appropriate boxes (cells) in the Karnaugh map. Putting 1s for non-barred terms and 0's for bar terms, the possible cell-designations are 00,01,11,10. (Think that these are binary numbers). 0,1,3,2 are the corresponding decimals. m_0, m_1, m_3, m_2 are the Karnâugh map cells.

	$\overline{B}$	B
$\overline{A}$		
A		

m_0	m_1
m_2	m_3

Fig. 72

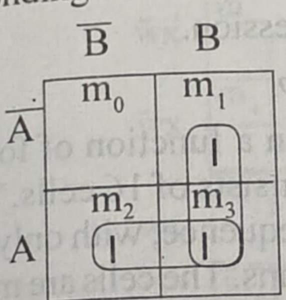
The map is redrawn, assigning the labels m_0, m_1, m_2 and m_3 order as shown in the figure 72. Here m_0 represents the cell $\overline{A}\,\overline{B}$ and m_1 represents the cell $\overline{A}\,B$, m_2 and m_3 represent the cells $A\,\overline{B}$ and $A\,B$ respectively. The given logic function can be written as $f = \Sigma m\,(0, 1, 2, 3)$.

Example:

1. *Place on a truth table the given function by entering it in a Karnaugh map and simplify* $f = \Sigma m\,(1, 2, 3)$

Since the given function has cell - designation upto 3, there will be a two variable, 4 cell K-map. The cells are m_0, m_1, m_2 and m_3. Of these, m_1, m_2, and m_3, (as given in the problem) are marked 1 in the map.

There are two sub-cubes. The horizontal sub-cube yields A and the vertical sub-cube yields B remaining unchanged. $f = A + B$. The corresponding truth table is as shown in figure 73.



A	B	f
0	1	1
1	0	1
1	1	1
0	0	0

Fig. 73

2. *Addressing three variable map*

There are 8 minterms of three binary value ($2^3 = 8$) $\therefore$ The K-map consists of 8 cells. The minterms are arranged not in a binary sequence but in a sequence similar to the reflected code. According to the reflected code, the cells are named as shown in the figure 74.

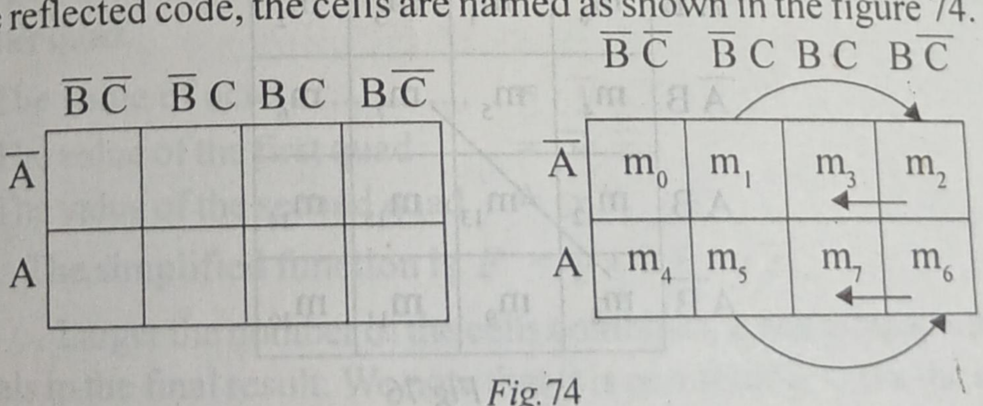


Fig. 74

Example

Obtain the simplified expression in sum-of-products for the Boolean expression $f(A, B, C) = \Sigma m\,(2, 3, 6, 7)$.

The given function is of three numerals A, B, C. The function is given in the reflected sequence form by the decimal numbers representing the cells of the Karnaugh map. i.e., the minterms of the function f in the SOP form are given by their decimal numbers for the cells, ie., the cells 2, 3, 6 and 7 are marked 1. We get a quad, in which the numeral remaining unchanged is B.

	$\overline{B}\,\overline{C}$	$\overline{B}C$	BC	$B\overline{C}$
$\overline{A}$	0	1	3 1	2 1
A	4	5	7 1	1 6

Fig. 75

$$\therefore f = B.$$

This is the logic function for the given expression.

3. *Addressing four binary variable map*

There are 16 possible minterms in a function of four binary variables. $2^4 = 16$. The Karnaugh map consists of 16 cells. The rows and columns are numbered in a reflected sequence, with only one digit changing value between two adjacent columns. The cells are marked m_0, m_1, m_3, and m_2 in the first row (reflected sequence), m_4, m_5, m_7, and m_6 in the second row. After filling m_7, we jump to the fourth row left and write m_8, m_9, m_{11}, and m_{10}. Then the third row cells are named as m_{12}, m_{13}, m_{15}, and m_{14}.

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	m_0	m_1	m_3	m_2
$\overline{A}B$	m_4	m_5	m_7	m_6
AB	m_{12}	m_{13}	m_{15}	m_{14}
$A\overline{B}$	m_8	m_9	m_{11}	m_{10}

Fig. 76

For example, the cell m_5 represents the minterm $A\,B\,C\,D\,(0101_2 = 5)$; the cell 15 represents the miterm $A\,B\,C\,D\,(1\,1\,1\,1 = 15)$ and so on.

Problem

Express the Boolean function $F = A + \overline{B}C$ into a sum of minterms and also give the minterm designation of the function.

[*Ans*: $F = \Sigma m\,(1, 4, 5, 6, 7)$]

Example

1. *Using Karnaugh map, simplify the Boolean Function* $F(w, x, y, z) = \Sigma\, m\,(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$.

The Karnaugh map is obtained by marking 1 in the cells corresponding to the minterms given in the problem.

	$\overline{y}\overline{z}$	$\overline{y}z$	yz	$y\overline{z}$
$\overline{w}\overline{x}$	m_0 1	m_1 1	m_3	m_2 1
$\overline{w}x$	m_4 1	m_5 1	m_7	m_6 1
wx	m_{12} 1	m_{13} 1	m_{15}	m_{14} 1
$w\overline{x}$	m_8 1	m_9 1	m_{11}	m_{10}

1
2
4
6
8

Fig. 77

The circled eight sub-cubes produce one octet. By rolling m_0 and m_4 on the left with m_2 and m_6 on the right, we get a quad. Similarly, by roling of m_4 and m_{12} on the left side with m_6, and m_{14} on the right, we get another quad.

The value of octet................. $= \overline{y}$

The value of the first quad $= \overline{w}\,\overline{z}$

The value of the second quad $= x\,\overline{z}$

$\therefore$ The simplified function is $F = \overline{y} + \overline{w}\,\overline{z} + x\,\overline{z}$

Larger the number of the cells combines, lesser is the number of literals in the final result. We note that it is permissible to use the same cell more than once in Karnaugh map. For example, m_0, m_4, m_{12}, and m cells are used more than once.

2. *Minimise the following function* $f = \Sigma m(0, 1, 2, 3, 11, 12, 14, 15)$.

The minterms given in the problem are marked as 1 in the corresponding cells of the Karnaugh map. Since 15 is the maximum number given, a four variable Karnaugh map is constructed, where $2^4 = 16$ cells would be available.

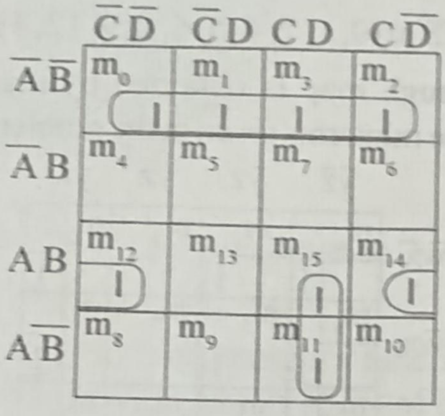


Fig. 78

The four 1 s is the $\overline{A}\,\overline{B}$ row form a quad. m_{14}, and m_{12} form a pair, by rolling, m_{11} and m_{15} form sub-cube of a pair.

For the quad, the value is $\overline{A}\,\overline{B}$. For the other two pairs, the values are $A B \overline{D}$ and $C D A$ respectively. $\therefore$ The simplified expression is

$$f = \overline{A}\,\overline{B} + A B \overline{D} + A C D$$

This can be readily implemented using logic gates.

3. *Using Karnaugh map, prove the following.*

$$A \overline{C}\,\overline{D} + A \overline{B} D + A C \overline{D} = A(\overline{B} + \overline{D}).$$

The given expression consists of four literals. *A, B, C, D.* Therefore, a four variable map is constructed. There are three terms and each one is to be expressed as 1 in the appropriate cell of the K - map.

In a four variable map a cell represents a minterm with four literals. But the first term in the given expression is $A \overline{C}\,\overline{D}$ and it has only three literals. To make it a four literal term, the $A\overline{C}\,\overline{D}$ term is replaced by $AB\overline{C}\,\overline{D}+A\overline{B}\,\overline{C}\,\overline{D}$. There is no B term in original expression. Hence, a four literal term with B and another four literal term with its complement $\overline{B}$ are ORed in lieu of $A\,\overline{C}\,\overline{D}$. Also, we know that

$$A\overline{C}\,\overline{D}=A\overline{C}\,\overline{D}(B+\overline{B})=AB\overline{C}\,\overline{D}+A\overline{B}\,\overline{C}\,\overline{D} \text{ since } B+\overline{B}=1$$

Similarly, $A\overline{B}D = A\overline{B}CD+A\overline{B}\,\overline{C}D$

and $AC\overline{D}=ABC\overline{D}+A\overline{B}C\overline{D}$

Thus, the given expression is

$$Y=AB\overline{C}\,\overline{D}+A\overline{B}\,\overline{C}\,\overline{D}+A\overline{B}CD+A\overline{B}\,\overline{C}D+ABC\overline{D}+A\overline{B}C\overline{D}$$

Now, each minterm is represented as 1 in K-map

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$				
$\overline{A}B$				
AB	1			1
$A\overline{B}$	1	1	1	1

Fig. 79

With four 1 s in the $A\,\overline{B}$ row, a quad is formed. Another quad is formed with $A\,B$ and $A\,\overline{B}$ rows on the left most and those on the right most cells after rolling

the value of the first quad $= A\overline{B}$

The value of the quad by rolling $= A\,\overline{D}$

$\therefore$ The simplified function is $Y = A\overline{B}+A\overline{D}$

i.e., $Y = A(\overline{B}+\overline{D})$

Summary

From what has been learnt so far, it is seen that

(i) Karnaugh map is a matrix of cells for expressing a logic function and it is used to obtain graphically a minimal algebraic expression for the function. The map contains 2^n cells for an n-variable case.

(ii) A literal is a logical variable in complemented or un-complemented form. A term containing literals corresponding to all the variables in AND ed form is known as a minterm. A term containing literals corresponding to all the variables in ORed form is known as maxterm.

(iii) In a Karnaugh map, two adjacent cells each having 1 form a pair. Four adjacent 1 cells form a quad. Eight adjacent 1-cells form an octet. A pair yields a simplified term of three literals. A quad produces a term of two literals as the result. An octet gives a term of simply one literal.

Preparation of truth table from the Karnaugh map

(Using min-term representation for each cell in the map)

Consider a Karnaugh map for three variables (A, B, C)

	$\overline{B}\,\overline{C}$	$\overline{B}\,C$	$B\,C$	$B\,\overline{C}$
$\overline{A}$	m_0	m_1	m_3	m_2
A	m_4	m_5	m_7	m_6

A \ BC	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Fig. 80

The characteristic of each cell can be represented by the binary digit corresponding to the literals of that cell. For example the cell m_5 corresponds to row 1 and column 01. When these numbers are combined (row-column wise) we get a binary number 101. This binary number has the decimal equivalent of 5. This justifies the labelling of the cell as m_5.

The labelling procedure is true for any cell in the Karnaugh map.

$m_5 \Rightarrow 101.$

Expressing this interms of the variables A, B, C.

$$m_5 \Rightarrow A\,\overline{B}\,C$$

For example, the minterms for m_2, m_4, m_7 can be written as

$$m_2 \Rightarrow 010 \Rightarrow \overline{A}\,B\,\overline{C}$$

$$m_4 \Rightarrow 100 \Rightarrow A\,\overline{B}\,\overline{C}$$

$$m_7 \Rightarrow 111 \Rightarrow A\,B\,C$$

Similarly in a four -variable K-map the cell m_0, m_1, m_2, m_3, m_4 can be written as 0000,0001,0010, 0011, 0100 respectively. Also, m_9, m_{12}, m_{15}, are

$$m_9 \Rightarrow 1001 \Rightarrow A\,\overline{B}\,\overline{C}\,D$$

$$m_{12} \Rightarrow 1100 \Rightarrow A\,B\,\overline{C}\,\overline{D}$$

$$m_{15} \Rightarrow 1111 \Rightarrow A\,B\,C\,D$$

Given a switching function in the SOP form, one can write the binary equivalent for each minterm. (whether it is in m form or literal form) and prepare the truth-table accordingly.

Example

Prepare truth table for the function

$f(A, B, C) = \Sigma m\,(0, 2, 4, 5, 6)$ and obtain the simplified function.

The 1-cells in the K-map are given as m_0, m_2, m_4, m_5 and m_6. They have literal equivalent as

$$m_0 \Rightarrow 000 \Rightarrow \overline{A}\,\overline{B}\,\overline{C}$$

$$m_2 \Rightarrow 010 \Rightarrow \overline{A}\,B\,\overline{C}$$

$$m_4 \Rightarrow 100 \Rightarrow A\,\overline{B}\,\overline{C}$$

$$m_5 \Rightarrow 101 \Rightarrow A\,\overline{B}\,C$$

$$m_6 \Rightarrow 110 \Rightarrow A\,B\,\overline{C}$$

$\therefore$ The truth table is as follows:

A	*B*	*C*	*f*
0	0	0	1
0	1	0	1
1	0	0	1
1	0	1	1
1	1	0	1

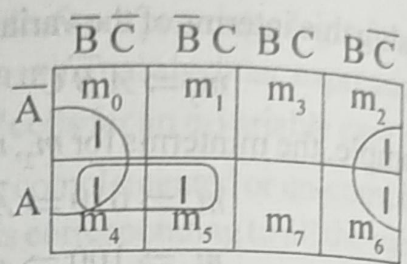


Fig. 81

The K-map with three variables can be now prepared.

The adjacent 1-cells after rolling give a quad. The value of the quad is $\overline{C}$. Also form a sub-cube whose values is $A\,\overline{B}$.

$\therefore$ The simplified expression is $f = \overline{C} + A\,\overline{B}$.

Don't care conditions

While preparing the Karnaugh map, we care about each cell, making the entry 1 in cells, for each min-term in the given expression of the function, *f*. All other cells may be maked 0 (or left empty)

There are times, when this is not the case. There may be cases in which certain combinations of input variables do not occur. Also, for some functions, the outputs corresponding to certain combinations of input variables may not matter. In such situations, we have the freedom to assume a 0 or 1 as output for each of these input combinations.

i.e., We don't care whether the output is 1 or 0 for these input combinations without affecting the desired logic operations. Such output value is called don't care state and it denoted by the symbol X. The X mark in a cell may be assumed to be a 1 or 0 so as to minimise the function using Karnaugh map method. If an X mark is present adjacent to a 1, the X mark may be taken as 1 so as to enlarge the sub-cube (for completing a pair, quad or octet) for minimising the given function.

Thus don't care condition is a logic net work in which output is independent of the state of the input.

Example

Minimise the switching function, characterised by a Karnaugh map given below.

We have to choose each X mark as 0 or 1 so that the overall switching function is minimised. Larger the sub-cube, fewer will be the variables required to represent the function. In the Karnaugh map, we have two 1-cells and four don't care conditions (X). If two X marks, adjacent to each 1 be taken as 1, then we can form a quad. So the two X marks are taken as 1. Further enlargement of sub-cube (quad or pair) is not possible. Therefore the other two X marks are treated as 0.

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$				X
$\overline{A}B$		1	X	
AB		X	1	
$A\overline{B}$	X			

Fig. 82

$\therefore$ The value of the quad $= B\,D$

$\therefore$ The simplified expression of the expression of the function is $f = B\,D$

On the other hand, if the don't care conditions are not used, the function will be $f = \overline{A}B\overline{C}D + ABCD$, which is not in simplified form and hence not desirable.

Product-of-sums simplification(Karnaugh map using max-terms)

To simplify a switching function (f) given in sum-of-products form, the minterms of the function are represented as 1 in the corresponding cells of the Karnaugh map. The blank cells have been marked with 0's. These are min-terms, not appearing in the function f.

The adjacent 0-cells are encircled to form sub-cubes such as pairs quads or octet. With the value of these sub-cubes, the (simplified form) sum of products of these sub-cubes is found as $\overline{f}$. This is nothing but the complement of the function f. By using De Morgan's laws, the complement f is found. The result will be in the product-of-sums form (POS).

Consider the product of sums expression $(a + b + \overline{c} + \overline{d})(\overline{a} + \overline{b} + c + \overline{d})$. For writing this in Max term representation, we replace all the bared numbers by 1 and all the un-bared numbers by 0. The decimal equivalent of the binary so formed, gives the Max-term representation. The binary words are 0011, 0101. Hence the decimal equivalents are 3, 5.

$\therefore$ The max term representation is $F = \pi\, M\,(3,5)$

Example

1. *Using Karnaugh map, simplify the switching function*

$$F = (a+b+c).\,(a+b+\overline{c}).\,(\overline{a}+b+c).\,(\overline{a}+b+\overline{c}).$$

The given expression is in the product-of-sums form. Each term is a max term. We replace all bared letter by 1 and all un-barred letters by 0.

The max-term designators are obtained as follows:

$$(a+b+c) = 000 = m_0$$
$$(a+b+\overline{c}) = 001 = m_1$$
$$(\overline{a}+b+c) = 100 = m_4$$
$$(\overline{a}+b+\overline{c}) = 101 = m_5$$

The max-term representation of the given expression is

$F = \pi\, M\,(0,1,4,5)$. Here π stands for product.

The function has three variables. We prepare a 3 variable K-map.

	$\overline{b}\,\overline{c}$	$\overline{b}\,c$	$b\,c$	$b\,\overline{c}$
$\overline{a}$	m_0 0	m_1 0	m_3	m_2
a	m_4 0	m_5 0	m_7	m_6

Fig. 83

The m_0, m_1, m_4, m_5 cells are marked 0 and the map is simplified. We get a quad. The variable that remains unchanged in the quad is $\overline{b}$. This gives the complement of the switching function F.

$\therefore\ \overline{F} = \overline{b}$ or $F = b$.

2. *Simplify the following Boolean function in product-of-sums using Karnaugh map.*

$F(A, B, C, D) = \Sigma m\ (0, 1, 2, 5, 8, 9, 10).$

The Karnaugh map for the function F is prepared by marking 1 in the cells $m_0, m_1, m_2, m_5, m_8, m_9$ and m_{10}. The other cells are marked 0.

	$\overline{C}\,\overline{D}$	$\overline{C}\,D$	$C\,D$	$C\,\overline{D}$
$\overline{A}\,\overline{B}$	m_0 1	m_1 1	m_3 0	m_2 1
$\overline{A}\,B$	m_4 0	m_5 1	m_7 0	m_6 0
$A\,B$	m_{12} 0	m_{13} 0	m_{15} 0	m_{14} 0
$A\,\overline{B}$	m_8 1	m_9 1	m_{11} 0	m_{10} 1

Fig. 84

The max-terms are denoted as 0 in map. Forming sub-cubes with adjacent 0, we get three quads, whose values are $A\,B$, $C\,D$, $B\,\overline{D}$. The sum of these would give the complement of F.

$$\therefore\ \overline{F} = A\,B + C\,D + B\,\overline{D}$$

Applying De Morgan's theorem, we take the complement as

$$\overline{\overline{F}} = \overline{A\,B + CD + B\,\overline{D}}$$

$$\therefore\ F = \overline{A\,B}\,.\,\overline{C\,D}\,.\,\overline{B\,\overline{D}}$$

$$F = (\overline{A} + \overline{B})\,(\overline{C} + \overline{D})\,(\overline{B} + D)$$

Thus the minimised function F is obtained in the product-of-sums form (POS)

3. *Given the truth table, obtain the simplified switching function in the product-of-sums form, using Karnaugh map.*

Using minterms, the function F is expressed as

$$F(x, y, z) = \overline{x}\,\overline{y}\,z + \overline{x}\,y\,z + x\,\overline{y}\,\overline{z} + x\,y\,\overline{z}.$$

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Counting the rows from top to bottom as $m_0, m_1, m_2, m_3, \ldots\ldots m_7$,we find $F = 1$ for $m_1, m_3, m_4,$ and m_6. So, the function can be expressed as

$F(x, y, z) = \Sigma\, m\,(1,3,4,6)$.

In product of Max-terms, the same function can be expressed as $F(x, y, z) = \pi M\,(0,2,5,7)$ since $F = 0$ for m_0, m_2, m_5 and m_7 rows.

The Karnaugh map of this function is drawn. The 1-s for represent the min-term and 0-s represent the max-term. One can simplify this function, by marking the 1-s each minterm for which the function is a 1. The remaining cells are marked with 0 s.

If, on the other hand, the product of max term is initialy taken, one can mark 0 in those cella listed in the function. The remaining cells are then marked as 1. Once 1-s and 0-s are marked, the function can be simplified is one of the standard forms.

	$\bar{y}\bar{z}$	$\bar{y}z$	yz	$y\bar{z}$
$\bar{x}$	0	1	1	0
x	1	0	0	1

Fig. 85

For the sum-of-products, we combine the 1-s to get $F = \bar{x}z + x\bar{z}$.

For the product-of-sums we combine the 0-s to get the simplified complement function as $\overline{F} = xz + \overline{x}\,\overline{z}$

Taking the complement of this, we get

$$\overline{\overline{F}} = \overline{xz + \overline{x}\,\overline{z}} = \overline{xz}\,.\,\overline{\overline{x}\,\overline{z}} = (\overline{x} + \overline{z})\,.\,(\overline{\overline{x}} + \overline{\overline{z}})$$

$$F = (\overline{x} + \overline{z})\,.\,(x + z)$$

This is the required product-of-sums form (POS)

4. *Design a circuit by following the K-map simplification procedure and draw it, using NAND only for the function*

$$Z = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,C + A\,B\,\overline{C} + A\,B\,C\,.$$

The given function has 4 minterms. they can be entered in the Karnaugh map. The cell in the K-map have representation as

	001,	101,	110,	111
The decimals are	1,	5,	6,	7
The representations :	m_1,	m_5,	m_6,	m_7

The K-map is prepared and these cells are marked 1. Encircling adjacent 1s, we find two pairs. The first pair has value. $\overline{B}\,C$. The second pair has value: $A\,B$. $\therefore$ The simplified function is-

$$Z = A\,B + \overline{B}\,C$$

$$\overline{Z} = \overline{A\,B + \overline{B}\,C}$$

$$= \overline{A\,B}\,.\,\overline{\overline{B}\,C}$$

$$\overline{\overline{Z}} = \overline{\overline{A\,B}\,.\,\overline{\overline{B}\,C}}$$

$$Z = \overline{\overline{A\,B}\,.\,\overline{\overline{B}\,C}}$$

	$\overline{B}\,\overline{C}$	$\overline{B}\,C$	$B\,C$	$B\,\overline{C}$
$\overline{A}$	m_0	m_1 1	m_3	m_2
A	m_4	m_5 1	m_7 1	m_6 1

Fig. 86

The function can be implemented using NAND gates only as shown in

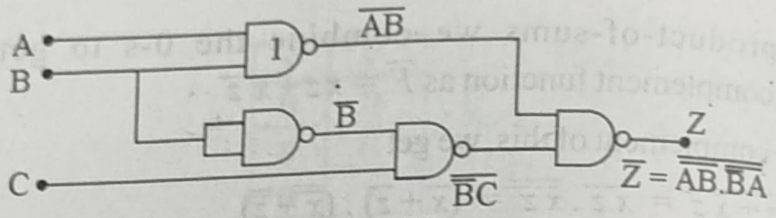


Fig. 87

5. *Design a circuit by following K- map simplification procedure and draw it using NAND only for the function* $\overline{Z} = \overline{A} B C + A \overline{B} C + A B \overline{C} + A B C$ [*Ans* : $Z = \overline{\overline{B C} . \overline{A B} . \overline{A C}}$]

6. *Using Karnaugh map, simplify the expression*

$$Y = \overline{A}\,\overline{B} C D + \overline{A} B C \overline{D} + \overline{A} B C D + A B \overline{C}\,\overline{D} + A B C \overline{D} + A B \overline{C} D + A B C D$$

and design a combinational network for the simplified expression.

Given : $Y = \overline{A}\,\overline{B} C D + \overline{A} B C \overline{D} + \overline{A} B C D + A B \overline{C}\,\overline{D} + A B C \overline{D} + A B \overline{C} D + A B C D$

Writing the binary representation for each minterm and the minterm designator, the terms are

	0011,	0110,	0111,	1100,	1110,	1101,	1111
i.e.,	3	6	7	12	14	13	15

	$\overline{C}\,\overline{D}$	$\overline{C} D$	$C D$	$C \overline{D}$
$\overline{A}\,\overline{B}$	m_0	m_1	m_3 1	m_2
$\overline{A} B$	m_4	m_5	m_7 1	m_6 1
$A B$	m_{12} 1	m_{13} 1	m_{15} 1	m_{14} 1
$A \overline{B}$	m_8	m_9	m_{11}	m_{10}

Fig. 88

$\therefore$ The cells in the K-map, having 1s are $m_3, m_6, m_7, m_{12}, m_{14} m_{13}$ and m_{15}

The value of the quad in $A\,B$ row is $A\,B$. The value of the pair in $C\,D$ column is $\overline{A}\,C\,D$. The value of the pair in $\overline{A}\,B$ row is $\overline{A}\,B\,C$.

$\therefore Y = A\,B + \overline{A}\,C\,D + \overline{A}\,B\,C$

The logic diagram to implement this is as given in figure 89.

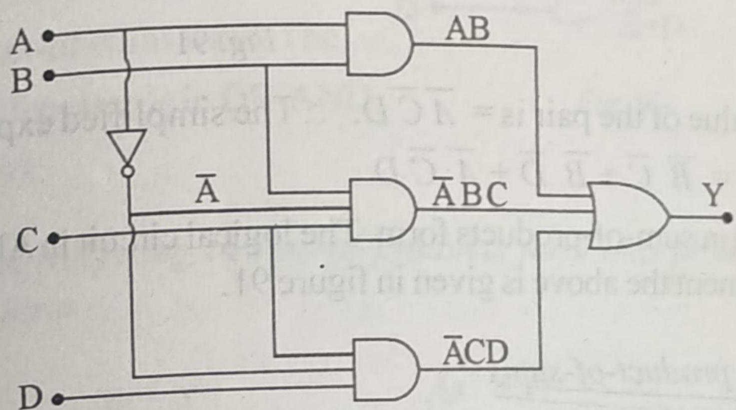


Fig. 89

7. Simplify the following Boolean function in (a) sum-of-products and (b) product-of-sum forms.

$$F(A, B, C, D) = \Sigma m\,(0, 1, 2, 5, 8, 9, 10)$$

(a) In sum-of-products $F(A, B, C, D) = \Sigma m\,(0,1,2,5,8,9,10)$. The minterms given in the problem are marked 1 in appropriate cells as m_0, m_1, m_2, m_5, etc.

	$\overline{C}\,\overline{D}$	$\overline{C}\,D$	$C\,D$	$C\,\overline{D}$
$\overline{A}\,\overline{B}$	m_0 1	m_1 1	m_3	m_2 1
$\overline{A}\,B$	m_4	m_5 1	m_7	m_6
$A\,B$	m_{12}	m_{13}	m_{15}	m_{14}
	1	1		1

Fig. 90

The numerals 1 in the cells m_0, m_1 and m_8, m_9 form a quad by rolling. The numeral that remains unchanged in the quad $= \overline{B}\,\overline{C}$. The 1 s in the cells m_0, m_2 and m_8, m_{10} form a quad by rolling. The numeral tha remains unchanged in the quad $= \overline{B}\,\overline{D}$. The adjecent 1s in m_1 and m_5 form a pair.

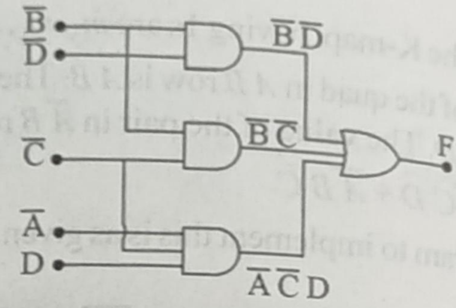


Fig. 91

The value of the pair is $= \overline{A}\,\overline{C}D$. $\therefore$ The simplified expression for F is

$$F = \overline{B}\,\overline{C} + \overline{B}\,\overline{D} + \overline{A}\,\overline{C}D$$

This is in sum-of-products form. The logical circuit in AND-OR form to implement the above is given in figure 91.

(b) In product-of-sums

$$F = (A, B, C, D) = \Sigma m\,(0, 1, 2, 5, 8, 9, 10)$$

The cells corresponding to m_0, m_1, m_2, m_5 etc. have 1 as given in the problem. To get the function F in product-of-sum form, we mark 0's in the remaining cells. These are the minterms not included in function F. $\therefore$ The simplified expression denotes the complement of F (i.e., $\overline{F}$).

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	m_0 1	m_1 1	m_3 0	m_2 1
$\overline{A}B$	m_4 0	m_5 1	m_7 0	m_6 0
AB	m_{12} 0	m_{13} 0	m_{15} 0	m_{14} 0
$A\overline{B}$	m_8 1	m_9 1	m_{11} 0	m_{10} 1

Fig. 92

The 0's in the CD column form a quad

The value of the quad is $= CD$

The 0's in the $A\,B$ row form a quad. The value of the quad is $= A\,B$

The 0's in m_4, m_{12}, and m_6, m_{14} cells form a quad by rolling.

The value of the quad is $= B\,\overline{D}$

The simplified function is given by

$$\overline{F} = C\,D + A\,B + B\,\overline{D}$$

Taking complement,

$$\overline{\overline{F}} = F = \overline{C\,D + A\,B + B\,\overline{D}}.$$

$$F = \overline{C\,D}\,.\,\overline{A\,B}\,.\,\overline{B\,\overline{D}}.$$

$$F = (\overline{C} + \overline{D})\,(\overline{A} + \overline{B})\,(\overline{B} + D)$$

This is in product-of-sums form. The topology of the function is in OR-AND, given in figure 93.

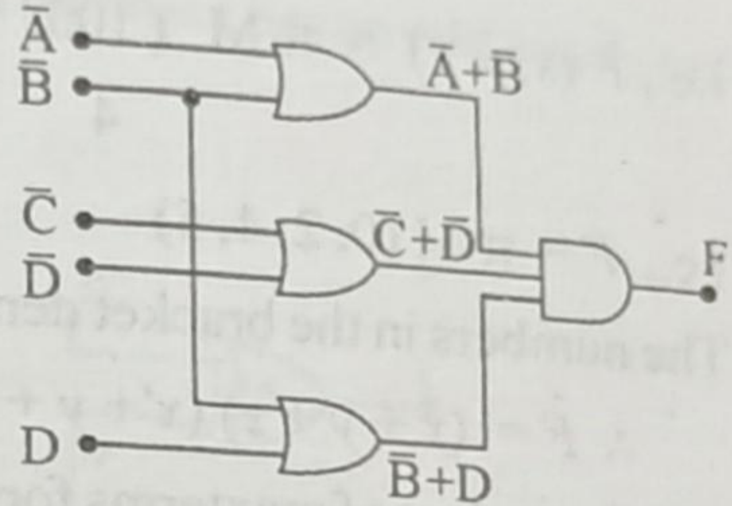


Fig. 93