UNIT - I
LINEAR ALGEBRA - I

Topic Learning Objectives:
Upon Completion of this unit, students will be able to:

- Understand the concept of vector spaces and subspaces, basis and dimension of these vector spaces and subspaces.
- Find the four fundamental subspaces w.r.t. a given matrix, find the rank and nullity w.r.t. the given matrix and hence verify the rank-nullity theorem.
- Study Linear transformations, with projection, rotation and reflection matrices as special cases.
- Obtain the matrix representation, Kernel and image of a linear transformation.

Introduction:

This unit deals with the vectors, which are functions arising in engineering systems. We study the linear combination of these vectors that is the combination of addition of two vectors and multiplication of a vector by a scalar. In applications of linear algebra, subspaces usually arise in one of two ways-the set of all solutions to a system of homogeneous linear equations or as the set of all linear combinations of certain specified vectors. We study null spaces and column spaces which describe these situations. Next we study the transformation of vectors from one vector space to another, whose applications can be seen in computer graphics and signal processing.

Vector Spaces:

Let F be a field, V be a non-empty set. For every ordered pair $\alpha, \beta \in V$, let there be defined uniquely a sum $\alpha + \beta$ and for every $\alpha \in V$, and $c \ \ in \ F$ a scalar product $c \cdot \alpha \in V$. The set V is called a vector space over the field F, if the following axioms are satisfied for every $\alpha, \beta, \gamma \in V$ and for every $c, c' \in F$.
(i) $\alpha + \beta \in V$,
(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$,
(iii) Identity element w.r.t addition exists. i.e, $\exists \ e \in V$ s.t. $\alpha + e = e + \alpha = \alpha$,
(iv) Inverse element w.r.t addition exists.i.e, $\exists \ \alpha^{-1} \in V$ s.t. $\alpha + \alpha^{-1} = e = \alpha^{-1} + \alpha$,
(v) $\alpha + \beta = \beta + \alpha$,
(vi) $c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta$,
(vii) $(c + c') \cdot \alpha = c \cdot \alpha + c' \cdot \alpha$,
(viii) $(c \cdot c') \cdot \alpha = c \cdot (c' \cdot \alpha)$
(ix) $1 \cdot \alpha = \alpha$, $\forall \alpha \in V$, where 1 is the unit element of F.

Examples:

Problem 1 *Let F be a field and n be a positive integer. Let $V_n(F)$ be the set of all ordered n tupples of the elements of the field F. i.e, $V_n(F) = \{(x_1, x_2, \ldots, x_n) | x_i \in F\}$. Define addition and scalar multiplication as below:*
(a) $\alpha + \beta = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
(b) $c \cdot \alpha = c.(x_1, x_2, \ldots, x_n) = (c.x_1, c \cdot x_2, \ldots, c.x_n)$, $\forall c \in F$.

Solution: (i) $\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in V_n(F)$

$$\begin{aligned}
(ii) \quad (\alpha+\beta)+\gamma &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) + (z_1, z_2, \ldots, z_n)\\
&= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \ldots, (x_n + y_n) + z_n)\\
&= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n))\\
&= (x_1, x_2, \ldots, x_n) + (y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n)\\
&= \alpha + (\beta + \gamma)
\end{aligned}$$

$$\begin{aligned}
(iii) \quad \alpha + 0 &= (x_1, x_2, \ldots, x_n) + (0, 0, \ldots, 0)\\
&= (x_1, x_2, \ldots, x_n)\\
&= (0, 0, \ldots, 0) + (x_1, x_2, \ldots, x_n)\\
&= 0 + \alpha
\end{aligned}$$

$\therefore (0, 0, \ldots, 0)$ is the identity.

$$\begin{aligned}
(iv) \quad \alpha + (-\alpha) &= (x_1, x_2, \ldots, x_n) + (-x_1, -x_2, \ldots, -x_n)\\
&= (x_1 - x_1, x_2 - x_2, \ldots, x_n - x_n)\\
&= (0, 0, \ldots, 0)\\
\therefore -\alpha &= (-x_1, -x_2, \ldots, -x_n) \text{ is the additive inverse of } \alpha = (x_1, x_2, \ldots, x_n)
\end{aligned}$$

$$\begin{aligned}
(v) \quad \alpha + \beta &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)\\
&= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\\
&= (y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n)\\
&= (y_1, y_2, \ldots, y_n) + (x_1, x_2, \ldots, x_n)\\
&= \beta + \alpha
\end{aligned}$$

$$\begin{aligned}
(vi) \quad c \cdot (\alpha + \beta) &= c \cdot (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\\
&= (c \cdot (x_1 + y_1), c \cdot (x_2 + y_2), \ldots, c \cdot (x_n + y_n))\\
&= (c \cdot x_1, c \cdot x_2, \ldots, c \cdot x_n) + (c \cdot y_1, c \cdot y_2, \ldots, c \cdot y_n)\\
&= c \cdot (x_1, x_2, \ldots, x_n) + c \cdot (y_1, y_2, \ldots, y_n)\\
&= c \cdot \alpha + c \cdot \beta
\end{aligned}$$

$$\begin{aligned}
(vii) \quad (c + c') \cdot \alpha &= (c + c') \cdot (x_1, x_2, \ldots, x_n)\\
&= ((c + c')x_1, (c + c')x_2, \ldots, (c + c')x_n)\\
&= (cx_1 + c'x_1, cx_2 + c'x_2, \ldots, cx_n + c'x_n)\\
&= (c \cdot x_1, c \cdot x_2, \ldots, c \cdot x_n) + (c' \cdot x_1, c' \cdot x_2, \ldots, c' \cdot x_n)\\
&= c \cdot (x_1, x_2, \ldots, x_n) + c' \cdot (x_1, x_2, \ldots, x_n)\\
&= c \cdot \alpha + c' \cdot \alpha
\end{aligned}$$

$$\begin{aligned}
(viii) \quad (c \cdot c') \cdot \alpha &= (c \cdot c') \cdot (x_1, x_2, \ldots, x_n)\\
&= ((c \cdot c') \cdot x_1, (c \cdot c') \cdot x_2, \ldots, (c \cdot c') \cdot x_n)\\
&= (c \cdot (c' \cdot x_1), c \cdot (c' \cdot x_2), \ldots, c \cdot (c' \cdot x_n))\\
&= c \cdot (c' \cdot x_1, c' \cdot x_2, \ldots, c' \cdot x_n)\\
&= c \cdot (c' \cdot (x_1, x_2, \ldots, x_n)\\
&= c \cdot (c' \cdot \alpha)
\end{aligned}$$

$$\begin{aligned}(ix) \quad 1\cdot\alpha &= 1\cdot(x_1,x_2,\dots,x_n)\\ &= (1\cdot x_1, 1\cdot x_2,\dots,1\cdot x_n)\\ &= (x_1,x_2,\dots,x_n)\\ &= \alpha\end{aligned}$$

Thus $V_n(F)$ is a vector space over the field F.

Note:

1. With $F = \mathbb{R}, V_1(\mathbb{R}), V_2(\mathbb{R}), V_3(\mathbb{R})$ are all vector spaces. They are also denoted as $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ respectively. The elements of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ are real numbers, plane vectors and space vectors respectively.

2. If $F = \mathbb{R}, V_n(\mathbb{R})$ is denoted as $\mathbb{R}^n$. If $F = \mathbb{C}, V_n(\mathbb{C})$ is denoted as $\mathbb{C}^n$.

Problem 2 *Show that $V = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of all rationals, is a vector space under usual addition and scalar multiplication.*

Solution:
Let $\alpha = a_1 + b_1\sqrt{2}$, $\beta = a_2 + b_2\sqrt{2}$, $\gamma = a_3 + b_3\sqrt{2} \in V$ and $c, c' \in \mathbb{Q}$
(i) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) \in V$
(ii) $(\alpha + \beta) + \gamma = ((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) + (a_3 + b_3\sqrt{2}) = \alpha + (\beta + \gamma)$
(iii) 0 is the additive identity, as $0 + \alpha = \alpha = \alpha + 0$.
(iv) $-\alpha = -a_1 - b_1\sqrt{2}$ is the additive inverse of α as $\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$.
(v) $\alpha + \beta = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \beta + \alpha$.
(vi) $c\cdot(\alpha + \beta) = c\cdot((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) = c\cdot\alpha + c\cdot\beta$.
(vii) $(c + c')\alpha = (c + c')(a_1 + b_1\sqrt{2}) = c\cdot\alpha + c'\alpha$.
(viii) $(c\cdot c')\alpha = (c\cdot c')(a_1 + b_1\sqrt{2}) = c\cdot(c'\cdot\alpha)$.
(ix)$1\cdot\alpha = 1\cdot(a_1 + b_1\sqrt{2}) = a_1 + b_1\sqrt{2} = \alpha$.
Thus V is a vector space over $\mathbb{Q}$.

3. Let V be the set of all polynomials of degree $\leq n$, with coefficients in the field F, together with zero polynomial. Then show that V is a vector space under addition of polynomials and scalar multiplication of polynomials with the scalar $c \in F$ defined by $c(a_0 + a_1x + \cdots + a_nx^n) = ca_0 + ca_1x + \cdots + ca_nx^n$.
Solution:
(i) Sum of polynomials is again a polynomial
(ii) Sum of polynomials will be associative.
(iii) 0 is the additive identity.
(iv) If $\alpha = a_0 + a_1x + \cdots + a_nx^n$ then $-\alpha = -a_0 - a_1x - \cdots - a_nx^n$ is the additive inverse.
(v) Sum of polynomials is commutative.
(vi) $c\cdot(\alpha + \beta) = c\cdot\alpha + c\cdot\beta$ will hold.
(vii) $(c + c')\cdot\alpha = c\cdot\alpha + c'\cdot\alpha$ will hold.
(viii) $(c\cdot c')\cdot\alpha = c\cdot(c'\cdot\alpha)$ will hold.
(ix) $1\cdot\alpha = 1\cdot(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1x + \dots a_nx^n = \alpha$.
Thus V is a vector space over F.

4. Let $V = \{\begin{bmatrix} x & y \\ -y & x \end{bmatrix} | x, y \in \mathbb{C}\}$ under usual addition and scalar multiplication, with field $\mathbb{C}$ of complex numbers. Show that V is a vector space.
Solution:

Let $\alpha = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}, \beta = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}, \gamma = \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} \in V$
and $c_1 = a_1 + b_1 i, c_2 = a_2 + b_2 i \in \mathbb{C}$.
(i) $\alpha + \beta \in V$,
(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$,
(iii) $\alpha + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \alpha$.
$\therefore \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity.
(iv) $-\alpha = \begin{bmatrix} -x_1 & -y_1 \\ y_1 & -x_1 \end{bmatrix}$ is the additive inverse.
(v) $\alpha + \beta = \beta + \alpha$ will hold.
(vi) $c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta$ will hold.
(vii) $(c + c') \cdot \alpha = c \cdot \alpha + c' \cdot \alpha$ will hold.
(viii) $(c \cdot c') \cdot \alpha = c \cdot (c' \cdot \alpha)$ will hold.
(ix) $1 \cdot \alpha = 1 \cdot \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = \alpha$.
Thus V is a vector space over $\mathbb{C}$.

5. Let $\mathbb{R}^+$ be the set of all positive integers. Define the operations of addition and scalar multiplication as below:
$\alpha + \beta = \alpha\beta, \forall\ \alpha, \beta \in \mathbb{R}^+$
$c \cdot \alpha = \alpha^c$, $\alpha \in \mathbb{R}^+$ and $c \in \mathbb{R}$
Show that $\mathbb{R}^+$ is a vector space over the real field.
Solution:
(i) $\alpha + \beta = \alpha\beta \in \mathbb{R}^+$.
(ii) $(\alpha + \beta) + \gamma = (\alpha\beta) + \gamma = (\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha + \beta\gamma = \alpha + (\beta + \gamma)$.
(iii) $\alpha + 1 = \alpha \cdot 1 = \alpha = 1 \cdot \alpha = 1 + \alpha$.
(iv) $\alpha + \frac{1}{\alpha} = \alpha \cdot \frac{1}{\alpha} = 1 = \frac{1}{\alpha} \cdot \alpha = \frac{1}{\alpha} + \alpha$.
$\therefore \frac{1}{\alpha}$ is the additive inverse of α.
(v) $\alpha + \beta = \alpha \cdot \beta = \beta \cdot \alpha = \beta + \alpha$.
(vi) $c \cdot (\alpha + \beta) = c \cdot (\alpha\beta) = (\alpha\beta)^c = \alpha^c\beta^c = \alpha^c + \beta^c = c \cdot \alpha + c \cdot \beta$.
(vii) $(c + c')\alpha = \alpha^{(c+c')} = \alpha^c\alpha^{c'} = \alpha^c + \alpha^{c'} = c \cdot \alpha + c' \cdot \alpha$.
(viii) $(c \cdot c') \cdot \alpha = \alpha^{(c \cdot c')} = \alpha^{(c' \cdot c)} = (\alpha^{c'})^c = c(\alpha^{c'}) = c \cdot (c'\alpha)$
(ix) $1 \cdot \alpha = \alpha^1 = \alpha$, where 1 is the unit element of $\mathbb{R}^+$.

Exercise:
1. If $\mathbb{R}$ is the field of real numbers and V is the set of vectors in a plane, further if addition of vectors is the internal binary composition in V and the multiplication of elements of $\mathbb{R}$ with those of V as the external composition, prove that $V(\mathbb{R})$ is a vector space.
2. Let $\mathbb{R}$ be the field of real numbers and let P_3 be the set of all polynomials of degree at most 3, over the field $\mathbb{R}$. Prove that P_3 is a vector space over the field $\mathbb{R}$.

Subspaces:
A non empty subset W of a vector space V over a field F is called a subspace of V, if W is itself a vector space over F, under the same operations of addition and scalar multiplication as defined in V.
Note:

(i) The set $\{0\}$ consisting of zero vector of V is a subspace of V.
(ii) The whole vector space V, itself is a subspace of V.
These two subspaces are called trivial or improper subspaces of V.
Any subspace W of V different from $\{0\}$ and V is called a proper subspace of V.

Theorem: A non empty subset W of a vector space V over a field F is a subspace of V, if and only if (i) $\forall \alpha, \beta \in W$, $\alpha + \beta \in W$ (ii) $\forall c \in F, \alpha \in W, c \cdot \alpha \in W$.

Examples:
1. Verify whether $W = \{f(x)|2f(0) = f(1)\}$ over $0 \leq x \leq 1$, is a subspace of $V = \{\text{all functions}\}$ over the field $\mathbb{R}$.
Solution:
Let $f_1, f_2 \in W$.
Thus $2f_1(0) = f_1(1)$ and $2f_2(0) = f_2(1)$
Consider, $2(f_1 + f_2)(0) = 2[f_1(0) + f_2(0)]$
$= 2f_1(0) + 2f_2(0)$
$= f_1(1) + f_2(1)$
$= (f_1 + f_2)(1)$
Thus, $f_1 + f_2 \in W$. $i.e., W$ is closed under vector addition.
Consider, $2(cf_1)(0) = (2c)f_1(0)$
$= c.2f_1(0)$
$= c.f_1(1)$
$= (cf_1)(1)$.
Thus $cf_1 \in W$ $i.e., W$ is closed under scalar multiplication.
Hence W is a subspace.

2. Is the subset $W = \{(x_1, x_2, x_3)|x_1^2 + x_2^2 + x_3^3 \leq 1\}$ of $V_3(\mathbb{R})$ a subspace of $V_3(\mathbb{R})$?
Solution:
Let $\alpha = (1, 0, 0)$, where $1^2 + 0^2 + 0^2 = 1$ and $\beta = (0, 1, 0)$, where $0^2 + 1^2 + 0^2 = 1$ be two vectors in W.
Consider $\alpha + \beta = (1, 0, 0) + (0, 1, 0) = (1, 1, 0)$, where, $1^2 + 1^2 + 0^2 = 2 \nleq 1$.
Hence $\alpha + \beta \notin W$. $\therefore W$ is not a subspace.

3. Show that the subset $W = \{(x_1, x_2, x_3)|x_1 + x_2 + x_3 = 0\}$ of the vector space $V_3(\mathbb{R})$ is a subspace of $V_3(\mathbb{R})$.
Solution:
Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3)$ be any two elements of W.
Then, $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$.
Consider, $c_1\alpha + c_2\beta = c_1(x_1, x_2, x_3) + c_2(y_1, y_2, y_3)$
$= (c_1x_1, c_1x_2, c_1x_3) + (c_2y_1, c_2y_2, c_2y_3)$
$= (c_1x_1 + c_2y_1, c_1x_2 + c_2y_2, c_1x_3 + c_2y_3)$
To show that $c_1\alpha + c_2\beta \in W$, we have to show that the sum of the components of $c_1\alpha + c_2\beta$ is zero.
$\therefore$ consider $c_1x_1 + c_2y_1 + c_1x_2 + c_2y_2 + c_1x_3 + c_2y_3$
$= c_1(x_1 + x_2 + x_3) + c_2(y_1 + y_2 + y_3)$
$= c_1 \cdot 0 + c_2 \cdot 0$
$= 0$
$\therefore c_1\alpha + c_2\beta \in W$, hence W ia a subspace of $V_3(\mathbb{R})$.

4. Verify whether $W = \{$ Polynomial of degreee three $\}$ defined on $0 \leq x \leq 1$ is a subspace of the vector space $V = \{$ all polynomials $\}$ over $\mathbb{R}$.

Solution:

The set of all polynomials of degree three is not a subspace, as the sum of two polynomials of degree three need not be of degree three.

$\because f_1(x) = 3x^3 - 4x^2 + 2x + 1, f_2(x) = -3x^3 + 3x^2 + 2x + 5$

$\implies f_1(x) + f_2(x) = -x^2 + 4x + 6$ which is not a polynomial of degree three.

Hence W is not a subspace of V

5. Let $V = \mathbb{R}^3$, the vector space of all ordered triplets of real numbers, over the field of real numbers. Show that the subset $W = \{(x, 0, 0) | x \in \mathbb{R}\}$ is a subspace of $V = \mathbb{R}^3$.

Solution:

The element $\mathbf{0} = (0, 0, 0) \in W$.

Thus W is non-empty.

Let $\alpha_1 = (x_1, 0, 0)$ and $\alpha_2 + (x_2, 0, 0)$ be any two elements of W.

Then $\alpha_1 + \alpha_2 = (x_1, 0, 0) + (x_2, 0, 0) = (x_1 + x_2, 0, 0) \in W$.

$\therefore W$ is closed under addition.

Again, for any scalar $c \in \mathbb{R}$,

$c \cdot \alpha_1 = c \cdot (x_1, 0, 0) = (cx_1, 0, 0) \in W$

$\therefore W$ is closed under scalar multiplication.

Hence W is a subspace of $\mathbb{R}^3$.

Exercise:

1. Let H $= \{a - 3b, b - a, a, b\}$ where a and b are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$.

2. Let V be a vector space of all 2 x 2 matrices over reals. Determine whether W is a sub-space of V or not, where

(i) W consists of all matrices with non-zero determinant.

(ii) W consists of all matrices A such that $A^2 = A$.

(iii) W $= \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in \mathbb{R}\}$.

Answer: (i) W is not a subspace. (ii) W is not a subspace. (iii) W is a subspace.

Linear Combination

Let V be a vector space over the field F and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any n vectors of V. The vector of the form, $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$, where $c_1, c_2, \ldots, c_n \in F$, is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Linear Span of S

Let S be a non empty subset of a vector space $V(F)$. The set of all linear combinations of finite number of elements of S is called the linear span of S and is denoted by $L[S]$.

i.e, $L[S] = \{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n | c_i \in F, \alpha_i \in S, i = 1, 2, \ldots, n$ and n is any positive integer$\}$

If $\alpha \in L[S]$, then α is of the form, $\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$, for some scalars $c_1, c_2, \ldots, c_n \in F$.

Theorem: Let S be a non-empty subset of a vector space $V[F]$. Then

(i) $L[S]$ is a subspace of V

(ii) $S \subseteq L[S]$

(iii) $L[S]$ is the smallest subspace of V containing S.

Linear Dependence and Independence

A set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly dependent if

there exists scalars $c_1, c_2, \ldots, c_n \in F$, not all zero such that $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$.
A set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of vectors of a vector space $V[F]$ is said to be linearly independent if $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$.

Examples:

1. Show that the vectors $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots 0), \ldots e_n = (0, 0, 0, \ldots, 1)$ of the vector space $V_n(\mathbb{R})$ are linearly independent.

Solution:

Let $c_1, c_2, \ldots, c_n \in \mathbb{R}$
Consider $c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0$
$\implies c_1(1, 0, 0, \ldots, 0) + c_2(0, 1, 0, \ldots 0) + \cdots + c_n(0, 0, 0, \ldots, 1) = (0, 0, 0, \ldots, 0)$
$\implies (c_1, c_2, \ldots, c_n) = (0, 0, 0, \ldots, 0)$
$\implies c_1 = 0, c_2 = 0, \ldots, c_n = 0$
i.e., $e_1, e_2, e_3, \ldots, e_n$ are linearly independent.

2. Show that the set $S = \{(1, 0, 1), (1, 1, 0), (-1, 0, -1)\}$ is linearly dependent in $V_3(\mathbb{R})$.

Solution:

Consider $c_1(1, 0, 1) + c_2(1, 1, 0) + c_3(-1, 0, -1) = (0, 0, 0)$
$\implies (c_1 + c_2 - c_3, c_2, c_1 - c_3) = (0, 0, 0)$
$\implies c_1 + c_2 - c_3 = 0, c_2 = 0, c_1 - c_3 = 0$
$\implies c_1 = 1, c_2 = 0, c_3 = 1$
Thus there exists, not all zeros, scalars, such that $c_1(1, 0, 1) + c_2(1, 1, 0) + c_3(-1, 0, -1) = (0, 0, 0)$
$\therefore S$ is linearly dependent.

Note:

1.The set $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ of vectors of the vector space $V_3(\mathbb{R})$ is linearly dependent iff $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$
2. Two vectors $\alpha, \beta \in V_2(\mathbb{R})$ are linearly dependent iff $\alpha = k\beta$ for some non zero $k \in \mathbb{R}$
3. A set of vectors of V, containing the zero vector is linearly dependent.
4. The set consisting of a single vector α of V is linearly independent iff $\alpha \neq 0$.

Exercise:

1. Determine if the following sets of vectors are linearly independent or linearly dependent.
$v_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
Answer: Linearly independent.
2. Check for the linear independence/dependence of the column vectors
$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$
Answer: Linearly independent.
3. Let V be a vector space of real valued derivable functions on $(0, \infty)$, then show that the set $S = \{x^2e^x, xe^x, (x^2 + x - 1)e^x\}$ is linearly independent.
4. In a P_2 let $v_1 = 2t^2 + t + 2$; $v_2 = t^2 - 2t$; $v_3 = 5t^2 - 5t + 2$; $v_4 = -t^2 - 3t - 2$. Determine if the vector $u = t^2 + t + 2$ belongs to the span$\{v_1, v_2, v_3, v_4\}$. (Here P_2 is the vector space of all polynomials of degree at the most n over the field of real numbers).

Answer: Yes, $u = t^2 + t + 2$ belongs to the span$\{v_1, v_2, v_3, v_4\}$.

Basis

A subset B of a vector space $V[F]$ is called a basis of V if
(i) B is a linearly independent set
(ii) $L[B] = V$
i.e., a basis of a vector space $V[F]$ is linearly independent subset which spans the whole space.

Note: The zero vector cannot be an element of a basis of a vector space because a set of vectors with zero vector is always linearly dependent.

Examples:

1. Show that the vectors $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots 0), \ldots e_n = (0, 0, 0, \ldots, 1)$ of the vector space $V_n(\mathbb{R})$ form a basis of $V_n(\mathbb{R})$.

Solution:

Consider $S = \{e_1, e_2, \ldots, e_n\}$
$c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0$
$\implies c_1(1, 0, \ldots, 0) + c_2(0, 1, \ldots, 0) + \cdots + c_n(0, 0, \ldots, 1) = (0, 0, \ldots, 0)$
$\implies (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$
$c_1 = 0, c_2 = 0, \ldots, c_n = 0$
Hence S is linearly independent.
Further, any vector $(x_1, x_2, \ldots, x_n) \in V_n(\mathbb{R})$ can be expressed as a linear combination of the elements of S, as $(x_1, x_2, \ldots, x_n) = x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Hence $L[S] = V_n(\mathbb{R})$
$\therefore S$ is a basis of $V_n(\mathbb{R})$.

Standard Basis

The basis $S = \{e_1, e_2, \ldots, e_n\}$ of the vector space $V_n(\mathbb{R})$ is called the standard basis.
example: The vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ of $V_3(\mathbb{R})$ form a basis of $V_3(\mathbb{R})$, and is called the standard basis.

2. Show that the set $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a basis of the vector space $V_3(\mathbb{R})$.

Solution:

Let $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$
Consider, $c_1(1, 1, 0) + c_2(1, 0, 1) + c_3(0, 1, 1) = (0, 0, 0)$
$\implies (c_1 + c_2, c + 1 + c_3, c_2 + c_3) = (0, 0, 0)$
$\implies c_1 + c_2 = 0, c_1 + c_3 = 0, c_2 + c_3 = 0$
$\implies c_1 = 0, c_2 = 0, c_3 = 0$
$\therefore B$ is linearly independent.
Let $(x_1, x_2, x_3) \in V_3(\mathbb{R})$ be arbitrary.
Let $c_1, c_2, c_3 \in \mathbb{R}$, such that
$(x_1, x_2, x_3) = c_1(1, 1, 0) + c_2(1, 0, 1) + c_3(0, 1, 1)$
$(x_1, x_2, x_3) = (c_1 + c_2, c_1 + c_3, c_2 + c_3)$
$\implies x_1 = c_1 + c_2, x_2 = c_1 + c_3, x_3 = c_2 + c_3$
$\implies c_1 = \dfrac{x_1 + x_2 - x_3}{2}, c_2 = \dfrac{x_1 - x_2 + x_3}{2}, c_3 = \dfrac{-x_1 + x_2 + x_3}{2}$
$\therefore (x_1, x_2, x_3) = \dfrac{x_1 + x_2 - x_3}{2}(1, 1, 0) + \dfrac{x_1 - x_2 + x_3}{2}(1, 0, 1) + \dfrac{-x_1 + x_2 + x_3}{2}(0, 1, 1)$
$\therefore L[B] = V_3(\mathbb{R})$

Exercise:

1. Let W be a subspace of $\mathbb{R}^4$ spanned by the vectors $\alpha_1 = (1, 2, 2, 1)$; $\alpha_2 = (0, 2, 0, 1)$; $\alpha_3 = (-2, 0, -4, 3)$. Prove that $\alpha_1, \alpha_2, \alpha_3$ form a basis for W.

2. Find the basis for the subspace spanned by the vectors
$v_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$, $v_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$, $v_4 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T$, $v_5 = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T$, $v_6 = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T$.

Answer: Basis= $\{v_1, v_2, v_3\}$.

3. Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vector as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Answer: $e_1 = 0.7\alpha_1 + 0.3\alpha_2 + 0.2\alpha_3$, $e_2 = -0.2\alpha_1 + 0.2\alpha_2 - 0.2\alpha_3$, $e_1 = -0.3\alpha_1 + 0.3\alpha_2 + 0.2\alpha_3$.

Finite dimensional space

A vector space $V[F]$ is said to be a finite dimensional space if it has a basis with finite number of elements.

Dimension of a vector space V

The dimension of a finite dimensional vector space V over F is the number of elements in any basis of V and is denoted by $d[V]$.
example: $V_n(\mathbb{R})$ is a n dimensional space. $V_3(\mathbb{R})$ is a 3 dimensional space.

Note:

(i) Any two bases of a finite dimensional vector space V have the same finite number of elements.
(ii) A vector space which is not finitely generated may be called an infinite dimensional space.
(iii) In an n dimensional vector space $V(F)$
(a) any $n + 1$ elements of V are linearly dependent.
(b) no set of $n - 1$ elements can span V.
(iv) In an n dimensional vector space $V(F)$ any set of n linearly independent vectors is a basis.
(v) Any linearly independent set of elememts of a finite dimensional vector space V is a part of a basis.
(vi) For n vectors of $n-$dimensional vector space V, to be a basis, it is sufficient that they span V or that they are Linearly independent.

Examples:

1. Let $A = \{(1, -2, 5), (2, 3, 1)\}$ be a linearly independent subset of $V_3(\mathbb{R})$. Extend this to a basis of $V_3(\mathbb{R})$.

Solution:

Let $\alpha_1 = (1, -2, 5), \alpha_2 = (2, 3, 1)$
Let S be the subspace spanned by $\{\alpha_1, \alpha_2\}$
$\therefore S = \{c_1\alpha_1 + c_2\alpha_2 | c_1, c_2 \in \mathbb{R}\}$
$c_1\alpha_1 + c_2\alpha_2 = c_1(1, -2, 5) + c_2(2, 3, 1)$
$= (c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2)$
$\therefore S = \{(c_1 + 2c_2, -2c_1 + 3c_2, 5c_1 + c_2) | c_1, c_2 \in \mathbb{R}\}$
Chose a vector of $V_3(\mathbb{R})$, outside of S.
$(1, 0, 0) \notin S$

$\therefore$ the set $A = \{(1,-2,5),(2,3,1),(1,0,0)\}$ is a basis of $V_3(\mathbb{R})$.

2. Given two linearly independent vectors $(2,1,4,3)\&(2,1,2,0)$, find a basis of $V_4(\mathbb{R})$ that includes these two vectors.

Solution:
$\alpha_1 = (2,1,4,3), \alpha_2 = (2,1,2,0)$
$S = \{c_1\alpha_1 + c_2\alpha_2 | c_1, c_2 \in \mathbb{R}\}$
Choose $\alpha_3 = (1,0,0,0)$ & $\alpha_4 = (0,1,0,0) \notin S$
$\therefore \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis of $V_4(\mathbb{R})$.

The non-zero rows of a row-reduced echelon form of a matrix are linearly independent.

3. Test the following set of vectors for linear dependence in $V_3(\mathbb{R})$.
$\{(1,0,1),(0,2,2),(3,7,1)\}$. Do they form a basis?

Solution:
Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 7 & 1 \end{bmatrix}$
$|A| = 1(2-14) - 0(0-6) + 1(0-6) = -18 \neq 0$.
Therefore the given set in linearly independent.
Any three vectors in $V_3(\mathbb{R})$ which are linearly independent is a basis of $V_3(\mathbb{R})$.

4. Does the set $S = \{(1,2,3),(3,1,0),(-2,1,3)\}$ form a basis of $\mathbb{R}^3$.

Solution:
Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix}$
$|A| = 1(3-0) - 2(9+0) + 3(3+2) = 0$.
$\therefore S$ is linearly dependent and hence is not a basis of $\mathbb{R}^3$.

5. Show that the vectors $(1,1,2,4),(2,-1,-5,2),(1,-1,-4,0),(2,1,1,6)$ are linearly dependent in $\mathbb{R}^4$ and extract a linearly independent subset. Also find the dimension and a basis of the subspace spanned by them.

Solution:
Consider $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$
$R_2 = R_2 - 2R_1; R_3 = R_3 - R_1; R_4 = R_4 - 2R_1 \implies$
$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$
$R_2 = R_2 \div (-3) \implies$
$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$
$R_3 = R_3 + 2R_2; R_4 = R_4 + R_2 \implies$

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The final matrix is in echelon form and the rank of A is 2. Therefore the given vectors are linearly dependent.
The corresponding non-zero rows of the initial matrix are $(1, 1, 2, 4)\,\&(2, -1, -5, 2)$, which are linearly independent.
The dimension of the subspace spanned by the vectors is 2. These two vectors form a basis of the subspace.

6. Let S be the subspace of $\mathbb{R}^3$ defined by $S = \{(a, b, c)|a + b + c = 0\}$. Find a basis and dimension of S.
Solution:
$S \neq \mathbb{R}^3$ $[\because (1, 2, 3) \in \mathbb{R}^3\text{but}(1, 2, 3) \notin S]$
$\alpha = (1, 0, -1)\,\&\,\beta = (1, -1, 0) \in S$ and further they are independent.
$\therefore d[S] = 2$ and hence $\{\alpha, \beta\}$ forms a basis of S.

7. Show that the field $\mathbb{C}$ of complex numbers is a vector space over the field $\mathbb{R}$ of reals. What is its dimension?
Solution:
$\mathbb{C} = \{a + ib|a, b \in \mathbb{R}\}$
$\mathbb{C}$ is closed under $'+'$.
$\mathbb{C}$ is associative under $'+'$.
$0 + i0$ is the identity w.r.t $'+'$.
$-a - ib$ is the inverse of $a + ib$.
$\mathbb{C}$ is commutative.
Hence $(\mathbb{C}, +)$ is an abelian group.
$c \cdot ((a_1 + ib_1) + (a_2 + ib_2)) = c \cdot (a_1 + ib_1) + c \cdot (a_2 + ib_2) \in \mathbb{C}$
$(c_1 + c_2) \cdot (a_1 + ib_1) = c_1 \cdot (a_1 + ib_1) + c_2 \cdot (a_1 + ib_1) \in \mathbb{C}$
$(c \cdot c^{'}) \cdot (a_1 + ib_1) = c \cdot (c^{'} \cdot (a_1 + ib_1)) \in \mathbb{C}$
$'1'$ is the unity
Therefore $\mathbb{C}$ is a vector space over $\mathbb{R}$
Let $\alpha \in \mathbb{C}, \alpha = a + ib \quad \text{s.t.} \quad a, b \in \mathbb{R}$
$\therefore \alpha = 1 \cdot a + i \cdot b = a \cdot 1 + b \cdot i$
i.e., every element of $\mathbb{C}$ is a linear combination of the elements $1, i$. That is $\{1, i\}$ generates $\mathbb{C}$.
Further $c_1 \cdot 1 + c_2 \cdot i = 0 \implies c_1 = 0\&c_2 = 0$
$\therefore \{1, i\}$ is linearly independent.
$\therefore \{1, i\}$ is a basis of $\mathbb{C}$ and $d[\mathbb{C}] = 2$.

8. Let V be the vector space of 2×2 symmetric matrices over the field F. Show that $d[V] = 3$.
Solution:
Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in V, a, b, c \in F$.
Set $a = 1, b = 0, c = 0; a = 0, b = 1, c = 0; a = 0, b = 0, c = 1$
We get three matrices $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
We shall show that these elements of V form a basis.

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in V$ be arbitrary.
Then, $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Thus $\{E_1, E_2, E_3\}$ generates V.
Suppose $c_1 E_1 + c_2 E_2 + c_3 E_3 = 0,\ c_1, c_2, c_3 \in F$
$\implies c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\implies \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies c_1 = c_2 = c_3 = 0.$
$\therefore \{E_1, E_2, E_3\}$ is linearly independent.
Hence $\{E_1, E_2, E_3\}$ is a basis of V and $d[V] = 3$.

9. Find the basis and dimension of the subspace spanned by the subset $S = \{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}\}$ of the vector space of all 2×2 matrices over $\mathbb{R}$.

Solution:
Let $\alpha, \beta, \gamma, \delta$ are the matrices of S.
Then the coordinates of $\alpha, \beta, \gamma, \delta$ w.r.t standard basis are
$(1, -5, -4, 2), (1, 1, -1, 5), (2, -4, -5, 7), (1, -7, -5, 1)$.

Consider $\begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$

$R_2 = R_2 - R_1, R_3 = R_3 - 2R_1, R_4 = R_4 - R_1 \implies$

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

$R_3 = R_3 - R_2, R_4 = 3R_4 + R_2 \implies$

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The final matrix has two non-zero rows.
$\therefore$ d(subspace) $= 2$.
Further the matrices corresponding to the non-zero rows in the final matrix are $\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}$.

10. In a vector space $V_3(\mathbb{R})$, let $\alpha = (1, 2, 1), \beta = (3, 1, 5)\ \&\ \gamma = (-1, 3, -3)$. Show that the subspace spanned by $\{\alpha, \beta\}\ \&\ \{\alpha, \beta, \gamma\}$ are the same.

Solution:
Consider, $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 5 \\ -1 & 3 & -3 \end{vmatrix} = 1(-3-15) - 2(-9+5) + 1(9+1) = -18+8+10 = 0$$

$\therefore \{\alpha, \beta, \gamma\}$ is linearly dependent.
Let $\gamma = c_1\alpha + c_2\beta$
$(-1, 3, -3) = c_1(1, 2, 1) + c_2(3, 1, 5)$
$\implies (-1, 3, -3) = (c_1 + 3c_2, 2c_1 + c_2, c_1 + 5c_2)$
$\implies c_1 + 3c_2 = -1, 2c_1 + c_2 = 3, c_1 + 5c_2 = -3$
Solving these equations, we get $c_1 = 2, c_2 = -1$.
$\therefore \gamma \in$ subspace spanned by $\{\alpha, \beta\}$.
$\therefore$ the subspace spanned by $\{\alpha, \beta\}$ & $\{\alpha, \beta, \gamma\}$ are the same.

Exercise:

1. Let V be a space of 2 x 2 matrices over $\mathbb{R}$ and let W be the sub-space generated by $\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$. Show that (i) $\{\begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\}$ forms a basis set. (ii) $dimW = 2$.

Four Fundamental Subspaces

Null Space

The null space of a $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogrneous equation $Ax = 0$.

Theorem:

The null space of a $m \times n$ matrix A, is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Examples:

1. Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

Determine if u belongs to the null space of A.

Solution:

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore u$ is in Nul A.

2. Find a spanning set for the null space of the matrix $\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Solution:

Consider $Ax = 0$.
Reducing A to echelon form

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix},$$

$3R_2 + R_1$, $3R_3 + 2R_1 \implies$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{bmatrix}$$

$R_2 \div 5, R_3 \div 13 \implies$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\implies \begin{cases} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 2x_2 - \frac{1}{3}x_3 + \frac{1}{3}x_4 - \frac{7}{3}x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$

$\implies x_1 = 2x_2 - \frac{1}{3}(-2x_4 + 2x_5) + \frac{1}{3}x_4 - \frac{7}{3}x_5$

$\implies x_1 = 2x_2 + x_4 - 3x_5$, with x_2, x_4, x_5 as free variables.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$= x_2u + x_4v + x_5w.$

Every linear combination of u, v, w is an element of Nul(A). Thus $\{u, v, w\}$ is a spanning set for Nul(A).

Column Space:

The Column space of an $m \times n$ matrix A, written as Col(A), is the set of all linear combinations of the columns of A.

If $A = [a_1, a_2, \ldots, a_n]$, then Col$(A) = \text{span}\{a_1, a_2, \ldots, a_n\}$.

Theorem: The column space of a $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Examples:

1. Find a matrix A such that $W = \text{Col}(A)$.

$$W = \{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R}\}$$

Solution:

W can be written as

$$W = \{a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R}\} = \text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Using the vectors in the spanning set as the columns of A, we get $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Then $W = \text{Col}(A)$ as desired.

Note: The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ iff the equation $Ax = b$ has a solution for each b in $\mathbb{R}^m$.

2. Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

(a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

(b) If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

Solution:

(a) m = 3, Col(A) is a subspace of $\mathbb{R}^m$, where $m = 3$. (b) $n = 4$, Null$A$ is a subspace of $\mathbb{R}^n$, where $n = 4$.

3. Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, find a non-zero vector in Col(A) and a non-zero vector in Null$A$.

Solution:

Any column of A belongs to Col(A). eg. $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \in \text{Col}(A)$.

Consider $Ax = 0$

$A = \begin{bmatrix} 2 & 4 & -2 & 1 & :0 \\ -2 & -5 & 7 & 3 & :0 \\ 3 & 7 & -8 & 6 & :0 \end{bmatrix}$

$R_2 + R_1; 2R_3 - 3R_1 \implies A = \begin{bmatrix} 2 & 4 & -2 & 1 & :0 \\ 0 & -1 & 5 & 4 & :0 \\ 0 & 2 & -10 & 9 & :0 \end{bmatrix}$

$R_3 + 2R_2 \implies A = \begin{bmatrix} 2 & 4 & -2 & 1 & :0 \\ 0 & -1 & 5 & 4 & :0 \\ 0 & 0 & 0 & 17 & :0 \end{bmatrix}$

$\implies 2x_1 + 4x_2 - 2x_3 + x_4 = 0; -x_2 + 5x_3 + 4x_4 = 0; 17x_4 = 0$

$\implies x_4 = 0, x_2 = 5x_3, x_1 = -9x_3 \implies x_3$ is a free variable.

Let $x_3 = 1$, then $x_1 = -9, x_2 = 5, x_4 = 0$. The vector $x = (-9, 5, 1, 0) \in \text{Nul}(A)$.

4. With $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$,

(a) Determine if u is in Nul(A). Could u be in Col(A)? (b) Determine if v is in Col(A). Could v be in Nul(A)?

solution:

(a) Consider $Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\therefore u \notin \text{Nul}(A)$.

$\because u$ has 4 entries and Col(A) is subspace of $\mathbb{R}^3$, $u \notin \text{Col}(A)$.

(b) Consider $[A \quad v]$

$\begin{bmatrix} 2 & 4 & -2 & 1 & :3 \\ -2 & -5 & 7 & 3 & :-1 \\ 3 & 7 & -8 & 6 & :3 \end{bmatrix}$

$R_2 + R_1; 2R_3 - 3R_1 \implies$

$\begin{bmatrix} 2 & 4 & -2 & 1 & :3 \\ 0 & -1 & 5 & 4 & :2 \\ 0 & 2 & -10 & 9 & :-6 \end{bmatrix}$

$R_3 + 2R_2 \implies$

$\begin{bmatrix} 2 & 4 & -2 & 1 & :3 \\ 0 & -1 & 5 & 4 & :2 \\ 0 & 0 & 0 & 17 & :0 \end{bmatrix}$

$\implies Ax = v$ is consistent.

$\therefore v$ is in Col(A).

$\because v$ has 3 entries and Nul(A) is a subspace of $\mathbb{R}^4$, $v \notin \text{Nul}(A)$.

Row Space:

If A is an $m \times n$ matrix, each row of A has n entries and this can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted by Row$A$. Each row has n entries, so Row$A$ is a subspace of $\mathbb{R}^n$. Since the rows of A are identified with the columns of A^T, we could also write Col(A^T) in place of Row$A$.

Examples:

1. Let $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5-3 & \end{bmatrix}$.

Solution:

$r_1 = (-2, -5, 8, 0, 17)$
$r_2 = (1, 3, -5, 1, 5)$
$r_3 = (3, 11, -19, 7, 1)$
$r_4 = (1, 7, -13, 5, -3)$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{r_1, r_2, r_3, r_4\}$. That is Row$A$ = Span$\{r_1, r_2, r_3, r_4\}$.

Theorem:

If two matrices A and B are equivalent, then their row spaces are the same. If B is in echelon form, then non-zero rows of B form a basis for the row space of A as well as for that of B.

2. Find the basis for the row space of

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Solution:

To find the basis for the row space, reduce A to echelon form.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5-3 & \end{bmatrix}$$

$R_1 \longleftrightarrow R_2 \implies$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5-3 & \end{bmatrix}$$

$R_2 + 2R_1, R_3 - 3R_1, R_4 - R_1 \implies$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix}$$

$R_3 - 2R_2, R_4 - 4R_2 \implies$

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix}$$

$R_3 \longleftrightarrow R_4 \implies$

$$B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of B form a basis for the row space of A(as well as for the row space of B).

Basis for Row$A = \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

3. Find the basis for the row space of

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

Solution:

$R_2 + R_1, R_3 - 2R_1, R_4 + R_1 \implies$

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix}$$

$R_3 + R_2, R_4 - 3R_2 \implies$

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$R_3 \longleftrightarrow R_4 \implies A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of B form the basis for the rowspace of A.

Row$A = \{(2, -3, 6, 2, 5), (0, 0, 3, -1, 1), (0, 0, 0, 1, 3)\}$

Left null space

The left null space of an $m \times n$ matrix A written as Nul(A^T), is the set of all solutions to the homogeneous equation $A^T y = 0$.

Theorem

The left null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$. Equivalently, the set of all solutions to a system $A^T y = 0$ n homogeneous linear equations in m unknowns is a subspace of $\mathbb{R}^m$.

Examples:

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Determine if v belongs to the left null space of A.

Solution:

$$A^T v = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+3 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore v$ is in Nul(A^T).

2. Find a spanning set for the left null space of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution:

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\implies y_1 = 0$, y_2 is a free variable.

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_2 \end{bmatrix} = y_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = y_2 u.$

$\therefore$ every linear combination of u is an element of $\text{Nul}(A^T)$. Thus $\{u\}$ is a spanning set of $\text{Nul}(A^T)$.

Example:

1. Find the dimension and basis for the four fundamental subspaces of the matrix $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$.

Solution:

$R_2 - 2R_1, R_3 + R_1 \implies$

$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$

$R_3 - 2R_2 \implies$

$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \div 3 \implies$

$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$Ax = 0 \implies x_1 + 3x_2 + 3x_3 + 2x_4 = 0, 3x_3 + 3x_4 = 0$

$\implies x_3 = -x_4, x_1 = 3x_2 + x_4$

x_2, x_4 are free variables.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\therefore \{(-3, 1, 0, 0), (1, 0, -1, 1)\}$ forms a basis of $\text{Nul}(A)$ and $\dim(\text{Nul}(A)) = 2$.

$\{(1, 2, -1), (3, 9, 3)\}$ forms a basis of $\text{Col}(A)$ and $\dim(\text{Col}(A)) = 2$.

$\{(1, 3, 3, 2), (2, 6, 9, 7)\}$ forms a basis of $\text{Row}A$ and $\dim(\text{Row})A = 2$.

$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$

$R_2 - 3R_1, R_3 - 3R_1, R_4 - 2R_1 \implies$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \longleftrightarrow R_3 \implies$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \div 3 \implies \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^T y = 0 \implies$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies y_1 + 2y_2 - y_3 = 0, y_2 + 2y_3 = 0 \implies y_2 = -2y_3, y_1 = -2y_2 + y_3 = 4y_3 + y_3 = 5y_3$

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5y_3 \\ -2y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

$\therefore \{(5, -2, 1)\}$ forms a basis of $\text{Nul}(A^T)$ and $\dim\text{Nul}(A^T) = 1$.

The dimension of the column space of A, is called the rank of A.

Since Row$A$ is the same as $\text{Col}(A^T)$, the dimension of the row space of A is the rank of A^T.

The dimension of the null space is called the nullity of A.

Rank-Nullity Theorem:

For an $m \times n$ matrix A, rank A + nullity A = n.

Examples:

1. If A is a 7×9 matrix with two dimensional null space, what is the rank of A?

Solution:

rank + nullity = 9 $\implies$ rank + 2 = 9 $\implies$ rank = 7.

2. Could a 6×9 matrix have a two-dimensional null space?

Solution:

rank + nullity = 9 $\implies$ rank + 2 = 9 $\implies$ rank = 7, which contradicts that basis of $\text{Col}(A)$ is a subspace of $\mathbb{R}^6$

$\therefore$ a 6×9 matrix cannot have a two-dimensional null space.

Exercise:

1. Determine if $u = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in $\text{Nul}(A)$, where $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$

Answer: Yes, u is in $\text{Nul}(A)$.

2. Determine if $v = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in $\text{Nul}(A)$, where $A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}$

Answer: Yes, v is in $\text{Nul}(A)$.

3. Find an explicit description of $\text{Nul}(A)$, by listing vectors that span the null space.

(i) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$, (ii) $A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Answer: (i) $\text{Nul}(A) = \{c_1u_1 + c_2u_2\}$, where $u_1 = (7, -4, 1, 0), u_2 = (-6, 2, 0, 1)$.

(ii) $\text{Nul}(A) = \{c_1u_1 + c_2u_2 + c_3u_3\}$, where $u_1 = (-6, 2, 1, 0, 0), u_2 = (8, -1, 0, 1, 0), u_3 = (-1, 0, 0, 0, 1)$.

4. Find A such that the given set is $\text{Col}(A)$.

(i) $A = \left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r,s,t \quad \text{real} \right\}$, (ii) $A = \left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} : b,c,d \quad \text{real} \right\}$.

Answer: (i) $r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix}$, (ii) $b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$

5. Find (a) k such that $\text{Nul}(A)$ is a subspace of $\mathbb{R}^k$, (b) k such that $\text{Col}(A)$ is a subspace of $\mathbb{R}^k$

(i) $\begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$, (ii) $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$, (iii) $\begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$, (iv) $\begin{bmatrix} 1 & -3 & 9 & 0 & -5 \end{bmatrix}$

Answer: (i) (a) $k=2$, (b) $k=4$, (ii) (a) $k=3$, (b) $k=4$, (iii) (a) $k=5$, (b) $k=2$, (iv) (a) $k=5$, (b) $k=1$.

6. With (i) $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$, (ii) $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$, find a non-zero vector in $\text{Nul}(A)$ and a non-zero vector in $\text{Col}(A)$.

Answer: (i) $(3,1) \in \text{Nul}(A)$, $(2,-1,-4,3) \in \text{Col}(A)$.
(ii) $c_1(7,-4,1,0) + c_2(-6,2,0,1) \in \text{Nul}(A)$, $c_1(1,0) + c_2(3,1) \in \text{Col}(A)$.

7. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if u is in $\text{Col}(A)$. Is u in $\text{Nul}(A)$?

Solution: $u \in \text{Col}(A)$, $u \in \text{Nul}(A)$.

8. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if v is in $\text{Col}(A)$. Is v in $\text{Nul}(A)$?

Answer: $v \in \text{Col}(A)$, $v \in \text{Nul}(A)$.

9. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x+2y+z=0$.

Answer: Basis $= \{(-2,1,0),(-1,0,1)\}$.

10. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y=5x$.

Answer: Basis $= \{(1,5)\}$.

11. Find the basis for $\text{Nul}(A)$ and $\text{Col}(A)$. Assume that A is row equivalent to B.

(i) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Answer: (i) Basis for $\text{Nul}(A) = \{(-6,-5/2,1,0),(-5,-3/2,0,1)\}$ and basis for $\text{Col}(A) = \{(-2,2,-3),(4,-6,8)\}$.
(ii) Basis for $\text{Nul}(A) = \{(-2,1,0,0,0),(-4,0,7/5,1,0)\}$ and
basis for $\text{Col}(A) = \{(1,2,1,3),(-5,-5,0,-5),(-3,2,5,-2)\}$.

12. Find the basis for the row space and column space of the matrix

$$U = \begin{bmatrix} 1 & 5 & -2 & 3 & 5 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer: Basis for Row$A = \{(1,5,-2,3,5),(0,0,1,-1,0),(0,0,0,0,1)\}$,
basis for Col$(A) = \{(1,0,0,0),(-2,1,0,0),(5,0,1,0)\}$.

13. Determine a basis for the null space and left null space of the matrix,

$$A = \begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}.$$

Answer: Basis of Nul$(A) = \{(-1,-1/2,-2,1,0,0),(0,-1/2,0,0,1,0),(1,1,5,0,0,1)\}$,
basis of A$^T = \{\}$.

14. Find the bases for the null spaces of the matrices

(i) $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$, (ii) $A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$.

Answer: (i) Basis of Nul$(A) = \{(3,5,1,0),(-2,-4,0,1)\}$,
(ii) Basis of Nul$(A) = \{(5,4,1,0,0),(-7,-6,0,3,1)\}$

15. Find a bases for the four fundamental sub-spaces of $A = \begin{bmatrix} -1 & 2 & -1 & 5 & 6 \\ 4 & -4 & -4 & -12 & -8 \\ 2 & 0 & -6 & -2 & 4 \\ -3 & 1 & 7 & -2 & 12 \end{bmatrix}$.

Answer: Basis of Nul$(A) = \{(3,2,1,0,0),(0,-8,0,2,1)\}$,
Basis of Row$A = \{(-1,2,-1,5,6),(4,-4,-4,-12,-8),(-3,1,7,-2,12)\}$,
Basis of Col$(A) = \{(-1,4,2,-3),(2,-4,0,1),(5,-12,-2,-2)\}$,
Basis of Nul$(A^T) = \{(-2,-1,1,0)\}$.

Transformation:

Consider the matrix equation $Ax = b$, where A is an $m \times n$ matrix, x ia an $n \times 1$ matrix and b is an $m \times 1$ matrix.

In other words x is a vector in $\mathbb{R}^n$ and b is a vector in $\mathbb{R}^m$.

Solving the equation $Ax = b$ amounts to finding all vectors x in $\mathbb{R}^n$ that are transformed into the vector b in $\mathbb{R}^m$ under the action of multiplication by A.

The correspondence from x to Ax is a function from one set of vectors to another.

This concept generalizes the common notation of a function as a rule that transforms one real number into another.

A transformation(or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T and $\mathbb{R}^m$ is called the co domain of T. The notation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the co domain is $\mathbb{R}^m$.

For x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the image of x. The set of all images $T(x)$ is called the range of T.

Matrix Transformations:

For each x in $\mathbb{R}^n$, $T(x)$ is computed as Ax, where A is an $m \times n$ matrix. It is also denoted by the matrix transformation $x \longrightarrow Ax$. Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and codomain of T is $\mathbb{R}^m$ when each column of A has m entries.

The range of T is the set of all linear combinations of the columns of A, because each image $T(x)$ is of the form Ax.

Example:

1. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T(x) = Ax$, so that $T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$.

(a) Find $T(u)$, the image of u under the transformation T.
(b) Find an x in $\mathbb{R}^2$ whose image under T is b.
(c) Is there more than one x whose image under T is b?
(d) Determine if c is in the range of the transformation T.

Solution:

(a) $T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$.

(b) $T(x) = b \implies Ax = b \implies \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, which can be written in the matrix form as $\begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & -5 \end{bmatrix}$

Reducing to echelon form as below:

$R_2 = R_2 - 3R_1; R_3 = R_3 + R_1 \implies$
$\begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & -2 \end{bmatrix}$

$R_3 = 14R_3 - 4R_2 \implies$
$\begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 0 \end{bmatrix}$

$\implies x_1 - 3x_2 = 3; 14x_2 = -7 \implies x_2 = -1/2, x_1 = 3/2$. Hence $x = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$.

(c) From (b) we can see that, the vector x is unique.

(d) Let $T(x) = c \implies Ax = c \implies \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \implies \begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & 5 \end{bmatrix}$

$R_2 = R_2 - 3R_1; R_3 = R_3 + R_1 \implies$
$\begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & 8 \end{bmatrix}$

$R_3 = 14R_3 - 4R_2 \implies$
$\begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 140 \end{bmatrix}$

Third row shows that $0 = 140$(which is invalid). Hence the system is inconsistent.
Hence c is not in the range of the transformation.

2. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(x) = Ax$. Find the images under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$.

Solution:

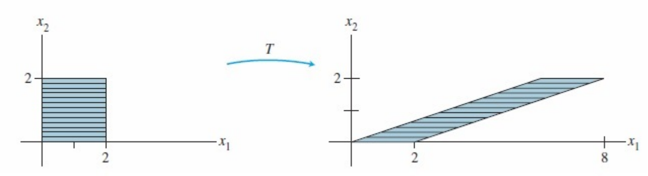
$T(u) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$T(v) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}.$

3. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. Then the transformation $T(x) = Ax$, transforms the square with vertices $(0,0), (2,0), (2,2), (0,2)$ to

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$

Sketching the above transformation, we see

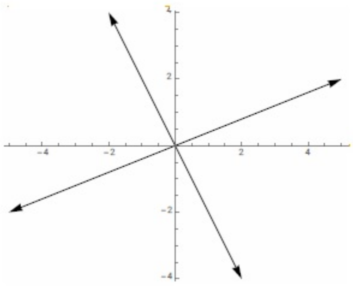


that the square has been transformed to a parallelogram(In other words, the square has been stretched to a parallelogram, keeping the base intact).

4. Let $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

(i) With $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

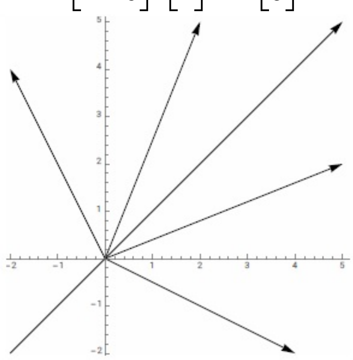
$Au = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$, $Av = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$



Reflected about origin.

(ii) With $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

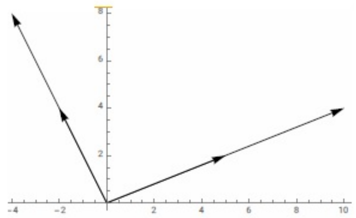
$Au = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $Av = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$



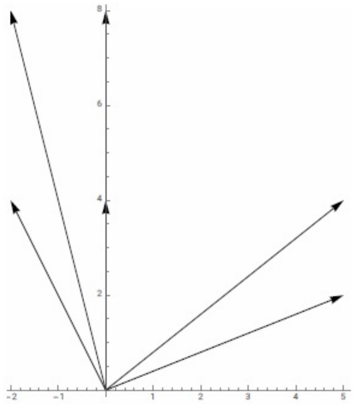
Reflected about the line $y = x$.

(iii) With $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,

$Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$, $Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$



Stretches the vector.
(iv) With $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$,
$Au = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $Av = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$



Rotates about origin and projects onto y axis.
If A is an $m \times n$ matrix, then the transformation $x \longrightarrow Ax$ has the properties $A(u + v) = Au + Av$ and $A(cu) = cA(u)$ for all u, v in $\mathbb{R}^n$ and all scalars c.
These properties, written in function notation, identify the most important class of transformations in Linear Algebra.

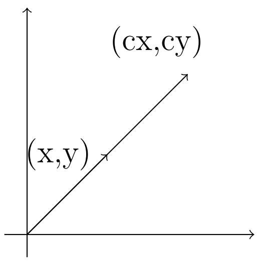
Linear transformation:
Let U and V be two vector spaces over the same field F. The mapping $T : U \longrightarrow V$ is said to be a linear transformation, if
(i) $T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \forall \quad \alpha, \beta \in U$
(ii) $T(c.\alpha) = c.T(\alpha) \quad \forall \quad c \in F, \alpha \in U$.
A Linear transformation $T : U \longrightarrow V$ is also called a linear map on U.
Note:
(i) Every matrix transformation is a linear transformation.
(ii) Linear transformations preserve the operations of vector addition and scalar multiplication.
Theorem:
A mapping $T : U \longrightarrow V$ from the vector space $U(F)$ into $V(F)$ is a linear transformation iff $T(c_1\alpha + c_2\beta) = c_1T(\alpha) + c_2T(\beta) \quad \forall \quad c_1, c_2 \in F \quad \text{and} \quad \alpha, \beta \in U$.

Stretch, Rotation, Reflection, Projection
Suppose x is an n-dimensional vector. When A multiplies x, it transforms that vector into a new vector Ax. This happens at every point x of the n-dimensional space $\mathbb{R}^n$. The whole space is transformed or 'mapped into itself" by the matrix A.

Stretch:

A multiple of the identity matrix A = cI = $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ stretches every vector by the same factor c. The whole space expands or contracts(or goes through the origin and out the opposite side when c is negative).

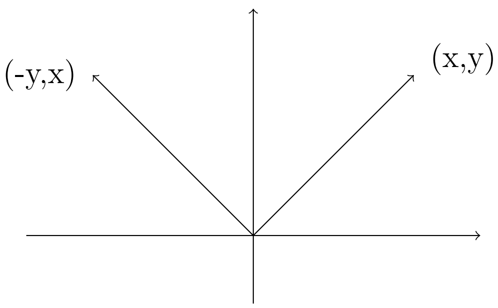
$\alpha = (x, y)$, then $A\alpha = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} = (cx, cy)$



Rotation:

A rotation matrix turns the whole space around the origin.

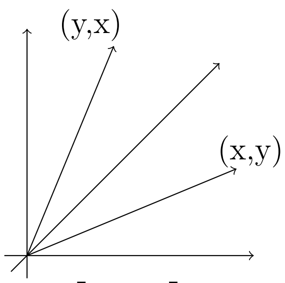
$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ turns all vectors through 90° transforming every point (x, y) to $(-y, x)$



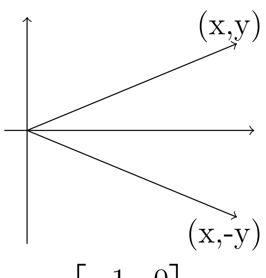
Reflection:

A reflection matrix transforms every vector into its image on the opposite side of a mirror.

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ gives the reflection through $y = x$.



$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ gives the reflection through $x-$axis.



$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ gives the reflection through $y-$axis.

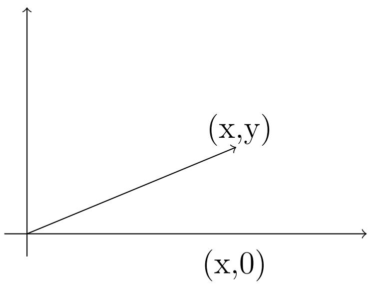
$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ gives the reflection through $y = -x$.

$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ gives the reflection through the origin.

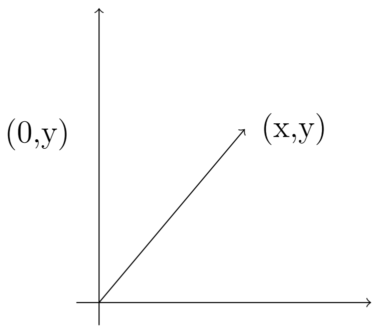
Projection:

A projection matrix takes the whole space onto a lower-dimensional subspace(not invertible).

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ transforms each vector (x, y) in the plane to the nearest point $(x, 0)$ on the horizontal axis.



$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ transforms each vector (x, y) in the plane to the nearest point $(0, y)$ on the vertical axis.



Rotation Q, Projection P, Reflection H

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Rotation about 90°

$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Projection onto the $x-$axis.

$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Reflection about 45°

Rotation:

The family of rotations can be represented in matrix form as

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$Q_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

Rotation in backwards through θ, $\quad Q_{-\theta} = [Q_\theta]^T$

$$Q_\theta \cdot Q_{-\theta} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Q_\theta . Q_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = Q_{2\theta}$$

$$Q_\theta^{-1} = \frac{1}{c^2+s^2}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = Q_\theta^T$$

$$Q_\theta \cdot Q_\phi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

$Q_\theta \cdot Q_\phi = Q_{\theta+\phi}$ If $\phi = -\theta$ then $Q_\theta \cdot Q_{-\theta} = Q_{\theta-\theta} = Q_0 = I$. Inverse exists.

Projections:

The family of rotations can be represented in matrix form as

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$|P| = c^2s^2 - c^2s^2 = 0 \implies$ inverse doesn't exist.

$$P^2 = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = P$$

Projection twice on the $\theta-$line is the same as projecting once on $\theta-$line.
Points on the $y-$axis is projected to $(0,0)$
Points on $\theta-$ line is projected onto itself.

Reflection:

The family of reflection can be represented in matrix form as

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$H^2 = (2P-I)^2 = 4P^2 - 4P + I = 4P - 4P + I (\because P^2 = P) \implies H^2 = I$
Two reflections bring back the original. Reflection of Reflection = Original.

$$H^{-1} = H, H = 2P - I \implies 2P = H + I$$

.

Problems:

1. If T is a mapping from $V_3(\mathbb{R})$ into $V_3(\mathbb{R})$ defined by $T(x_1,x_2,x_3) = (0,x_2,x_3)$. Show that T is a linear transformation.

Solution:

Let $\alpha = (x_1,x_2,x_3), \beta = (y_1,y_2,y_3) \in V_3(\mathbb{R})$
Consider, $T(\alpha+\beta) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$
$= T(x_1+y_1, x_2+y_2, x_3+y_3)$
$= T(0, x_2+y_2, x_3+y_3)$
$= (0,x_2,x_3) + (0,y_2,y_3)$
$= T(\alpha) + T(\beta)$
Consider, $T(c\alpha) = T(c(x_1,x_2,x_3))$
$= T(cx_1,cx_2,cx_3)$
$= (0,cx_2,cx_3)$
$= c(0,x_2,x_3)$
$= cT(x_1,x_2,x_3)$
$= cT(\alpha)$
$\therefore T$ is a linear transformation.

2. Find the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(1,1) = (0,1,2), T(-1,1) = (2,1,0)$.
Solution:
$\{(1,1), (-1,1)\}$ forms a basis of $\mathbb{R}^2$.
Let $\alpha = (x,y) \in \mathbb{R}^2$
$(x,y) = c_1(1,1) + c_2(-1,1) \implies (x,y) = (c_1 - c_2, c_1 + c_2)$
$\implies x = c_1 - c_2, y = c_1 + c_2 \implies c_1 = \dfrac{x+y}{2}, c_2 = \dfrac{y-x}{2}$
$\therefore (x,y) = \dfrac{x+y}{2}(1,1) + \dfrac{y-x}{2}(-1,1)$
$\therefore$ the required transformation is
$T(x,y) = \dfrac{x+y}{2}(0,1,2) + \dfrac{y-x}{2}(2,1,0)$
$T(x,y) = (y-x, y, x+y)$

3. Is there a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which $T(2,2) = (4,-6)$ and $T(5,5) = (2,-3)$?
Solution:
The vectors $(2,2)$ and $(5,5) \in \mathbb{R}^2$ are linearly dependent as $(5,5) = \dfrac{5}{2}(2,2)$
If T is linear, $T(5,5) = T(\dfrac{5}{2}(2,2)) = \dfrac{5}{2}T(2,2) = \dfrac{5}{2}(4,-6) = (10,-15)$
But $T(5,5) = (2,-3) \neq (10,-15)$.
$\therefore$ a linear map with the given data doesn't exist.

4. Let $M(\mathbb{R})$ be the vector space of all 2×2 matrices over $\mathbb{R}$ and B be a fixed non-zero element of $M(\mathbb{R})$. Show that the mapping $T : M(\mathbb{R}) \longrightarrow M(\mathbb{R})$ defined by $T(A) = AB - BA, \forall A \in M(\mathbb{R})$ is a linear map.
Solution:
Let A and $C \in M(\mathbb{R})$ be arbitrary.
Consider, $T(A+C) = (A+C)B - B(A+C) = AB + CB - BA - BC = AB - BA + CB - BC = T(A) + T(C)$
Let $c \in \mathbb{R}$ be any scalar.
Consider, $T(c \cdot A) = (c \cdot A)B - B(c \cdot A) = c \cdot (AB - BA) = c \cdot T(A)$
$\therefore T$ is a linear transformation

5. If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that $T(1,0) = (1,1)$ & $T(0,1) = (-1,2)$, show that T maps the square with vertices $(0,0), (1,0), (1,1), (0,1)$ into a parallelogram.
Solution:
$\{(1,0), (0,1)\}$ forms a basis of $\mathbb{R}^2$
$(x,y) = x(1,0) + y(0,1)$
$\therefore T(x,y) = xT(1,0) + yT(0,1) = x(1,1) + y(-1,2) = (x-y, x+2y)$
Now, $T(0,0) = (0,0) = A$, $T(1,0) = (1,1) = B$, $T(1,1) = (0,3) = C$, $T(0,1) = (-1,2) = D$.
To show that A, B, C, D are vertices of a parallelogram, we shall show that the diagonals AC and BD bisect each other.
Midpoint of $AC = (0, \dfrac{3}{2})$, Midpoint of $BD = (0, \dfrac{3}{2})$
Diagonals bisect each other. Hence $ABCD$ is a parallelogram.

6. If $T : V_1(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$ is defined by $T(x) = (x, x^2, x^3)$, verify whether T is linear or not.

Solution:
Let $x, y \in V_1(\mathbb{R})$
Consider, $T(x+y) = (x+y, (x+y)^2, (x+y)^3)$
$T(x) + T(y) = (x, x^2, x^3) + (y, y^2, y^3)) = (x+y, x^2+y^2, x^3+y^3)$
We can see that, $T(x+y) \neq T(x) + T(y)$
$\therefore T$ is not a linear transformation.

7. Find the linear transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (1,1)$ and $f(0,1) = (-1,2)$.
Soln: Let $(x,y) \in \mathbb{R}^2$. Then $(x,y) = x(1,0) + y(0,1)$
Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = xf(1,0) + yf(0,1) = x(1,1) + y(-1,2)$
Hence $f(x,y) = (x-y, x+2y) \quad \forall \quad (x,y) \in \mathbb{R}^2$

8. Find the linear transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,1) = (0,1)$ and $f(-1,1) = (3,2)$.
Solution:
$(1,1), (-1,1)$ of $\mathbb{R}^2$ forms a basis of $\mathbb{R}^2$
Let $\alpha = (x,y) \in \mathbb{R}^2$ be arbitrary.
Let$(x,y) = c_1(1,1) + c_2(-1,1) = (c_1 - c_2, c_1 + c_2)$
$\implies c_1 - c_2 = x, c_1 + c_2 = y \implies c_1 = \dfrac{x+y}{2}, c_2 = \dfrac{y-x}{2}$
$\therefore (x,y) = \dfrac{x+y}{2}(1,1) + \dfrac{y-x}{2}(-1,1)$
Hence the required transformation is $f(x,y) = \dfrac{x+y}{2}f(1,1) + \dfrac{y-x}{2}f(-1,1)$
$= \dfrac{x+y}{2}(0,1) + \dfrac{y-x}{2}(3,2)$
$\therefore f(x,y) = (\frac{3y-3x}{2}, \frac{3y-x}{2})$.

9. $f : V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$ is defined by $f(x,y,z) = (x+y, y+z)$, show that f is a linear transformation.
Solution:
Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ be any two elements of $V_3(\mathbb{R})$.
Consider, $f(\alpha + \beta) = f(x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$= (x_1 + x_2 + y_1 + y_2, y_1 + y_2 + z_1 + z_2) = ((x_1 + y_1) + (x_2 + y_2), (y_1 + z_1) + (y_2 + z_2))$
$= (x_1 + y_1, y_1 + z_1) + (x_2 + y_2, y_2 + z_2) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2)$
$\therefore f(\alpha + \beta) = f(\alpha) + f(\beta)$.
Consider, $f(c.\alpha) = f(cx_1, cy_2, cz_2)$
$= (cx_1 + cy_1, cy_1 + cz_2) = c(x_1 + y_1, y_1 + z_1) = cf(x_1, y_1, z_1)$
$\therefore f(c.\alpha) = cf(\alpha)$
Hence f is a linear transformation.

10. If T is a mapping from $V_2(\mathbb{R})$ into $V_2(\mathbb{R})$ defined by $T(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, show that T is a linear transformation.
Solution:
Let $\alpha = (x_1, x_2), \beta = (y_1, y_2) \in V_2(\mathbb{R})$
Consider, $T(\alpha + \beta) = T(x_1 + y_1, x_2 + y_2)$
$= ((x_1 + y_2)\cos\theta - (x_2 + y_2)\sin\theta, (x_1 + y_2)\sin\theta - (x_2 + y_2)\cos\theta)$
$= ((x_1\cos\theta - x_2\sin\theta) + (y_1\cos\theta - y_2\sin\theta), (x_1\sin\theta + x_2\cos\theta) + (y_1\sin\theta + y_2\cos\theta))$
$= (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta) + (y_1\cos\theta - y_2\sin\theta, y_1\sin\theta + y_2\cos\theta)$

$= T(x_1, x_2) + T(y_1, y_2) \therefore T(\alpha + \beta) = T(\alpha) + T(\beta)$
Consider, $T(c.\alpha) = T(cx_1, cx_2)$
$= (cx_1 \cos\theta - cx_2 \sin\theta, cx_1 \sin\theta + cx_2 \cos\theta) = c(x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$
$= cT(x_1, x_2)$
$\therefore T(c.\alpha) = cT(\alpha)$
Hence T is a linear transformation.

Range and Kernel of a Linear transformation
Definition:
Let $T : V \longrightarrow W$ be a linear transformation. The range of T is the set $R(T) = \{T(\alpha)|\alpha \in V\}$
Definition:
Let $T : V \longrightarrow W$ be a linear transformation. The kernel(or null space) of T is the set $N(T) = \{\alpha \in V|T(\alpha) = 0\}$, where 0 is the zero vector of W.

Note:
(i) For the identity map $I : V \longrightarrow V$ the range is the entire space V and the kernel is the zero subspace.
(ii) For the zero linear map $T : V \longrightarrow W$ defined by $T(\alpha) = 0 \forall \alpha \in V$, the range $R(T) = \{0\}$ = zero space of V and the null space $N(T) = V$.

Theorem
Let $T : V \longrightarrow W$ be a linear transformation.
Then (a) $R(T)$ is a subspace of W.
(b) $N(T)$ is a subspace of V.
(c) T is one-one iff $N(T) = \{0\}$, where 0 is the zero vector of W.

Theorem
Let $T : V \longrightarrow W$ be a linear transformation. The dimension of the range space $R(T)$ is called the rank of the linear transformation T and is denoted by $r(T)$. The dimension of the null space $N(T)$ is called the nullity of the linear transformation T and is denoted by $n(T)$.

Theorem
Let $T : V \longrightarrow W$ be a linear transformation. If the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ generate V, then the vectors $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ generates $R(T)$.
Theorem(Rank-Nullity theorem)
Let $T : V \longrightarrow W$ be a linear transformation and V be a finite dimensional vector space. Then $r(T) + n(T) = d[V] (d[R(T)] + d[N(T)] = d[V])$

Problems:
1. Let $T : V \longrightarrow W$ be a linear transformation defined by $T(x, y, z) = (x + y, x - y, 2x + z)$. Find the range, null space, rank, nullity and hence verify the rank-nullity theorem.
Solution:

$$T(e_1) = T(1, 0, 0) = (1, 1, 2) = \alpha_1$$
$$T(e_2) = T(0, 1, 0) = (1, -1, 0) = \alpha_2$$
$$T(e_3) = T(0, 0, 1) = (0, 0, 1) = \alpha_3$$

$\{\alpha_1, \alpha_2, \alpha_3\}$ generates $R(T)$

Consider $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $|A| = -2 \neq 0$
$\therefore \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent. Thus it is a basis of $R(T)$.
$d[R(T)] = 3$.
Let $\alpha \in R(T)$
Then

$$\alpha = c_1(\alpha_1) + c_2(\alpha_2) + c_3(\alpha_3)$$
$$= c_1(1, 1, 2) + c_2(1, -1, 0) + c_3(0, 0, 1)$$
$$= (c_1 + c_2, c_1 - c_2, 2c_1 + c_3)$$

$\therefore R(T) = \{(c_1 + c_2, c_1 - c_2, 2c_1 + c_3) | c_1, c_2, c_3 \in \mathbb{R}\}$
Suppose

$$T(x, y, z) = (0, 0, 0)$$
$$\implies (x + y, x - y, 2x + z) = (0, 0, 0)$$
$$\implies x + y = 0, x - y = 0, 2x + z = 0$$
$$\implies x = 0, y = 0, z = 0$$
$$\therefore N(T) = \{(0, 0, 0)\}$$
$$d[N(T)] = 0$$

rank + nullity $= 3 + 0 = 3 = d[V_3(\mathbb{R})]$.

Exercise:

1. Find the range space, null space, rank and nullity of T and verify the rank-nullity theorem for, $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by $T(x, y, z) = (x + y, x - y, 2x + z)$.
Answer: Range space $= \{(1, 1, 2), (1, -1, 0), (0, 0, 1)\}$, rank $= 3$, Null space $= \{(0, 0, 0)\}$, nullity $= 0$. rank + nullity $= 3 + 0 = 3 =$ dimension of domain.
2. Find the range space, null space, rank and nullity of T and verify the rank-nullity theorem for, $T : V_3(\mathbb{R}) \longrightarrow V_4(\mathbb{R})$, defined by $T(e_1) = (0, 1, 0, 2), T(e_2) = (0, 1, 1, 0), T(e_3) = (0, 1, -1, 4)$.
Answer: Range space $= \{(0, 1, 0, 2), (0, 1, 1, 0)\}$, rank $= 2$, Null space $= \{(-2, 1, 1)\}$, nullity $= 1$. rank + nullity $= 2 + 1 = 3 =$ dimension of domain.