

Disjoint Sets: Efficient Implementations

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Data Structures
Data Structures and Algorithms

Outline

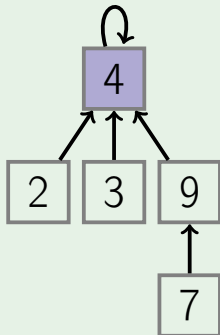
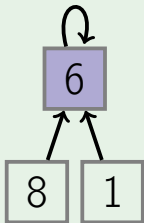
- 1 Trees
- 2 Union by Rank
- 3 Path Compression
- 4 Analysis

- Represent each set as a rooted tree

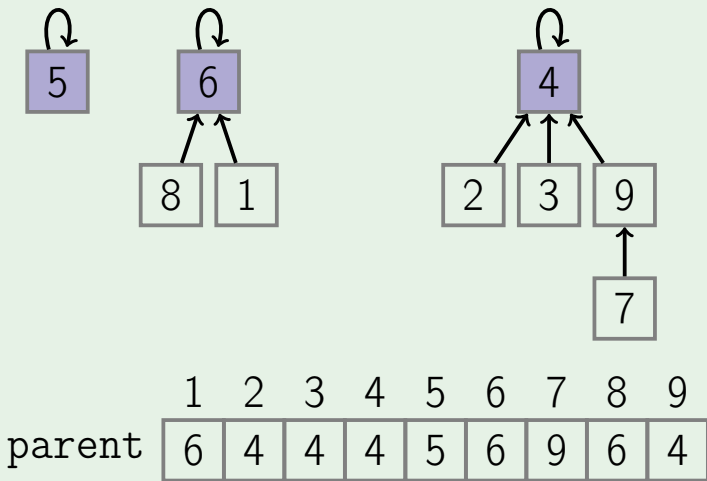
- Represent each set as a rooted tree
- ID of a set is the root of the tree

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- ID of a set is the root of the tree
- Use array $\text{parent}[1 \dots n]$: $\text{parent}[i]$ is the parent of i , or i if it is the root

Example



Example



MakeSet(i)

parent[i] $\leftarrow i$

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Running time: $O(1)$

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Running time: $O(1)$

Find(i)

while $i \neq \text{parent}[i]$:

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return i

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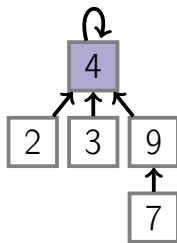
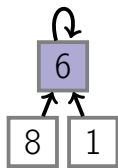
Running time: $O(\text{tree height})$

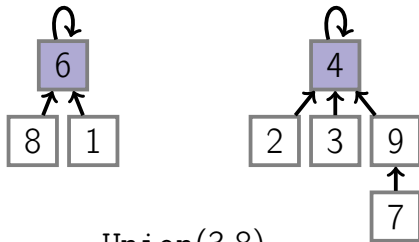
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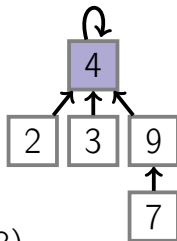
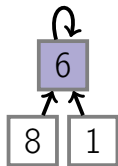
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- Hang one of the trees under the root of the other one
- Which one to hang?
- A shorter one, since we would like to keep the trees shallow

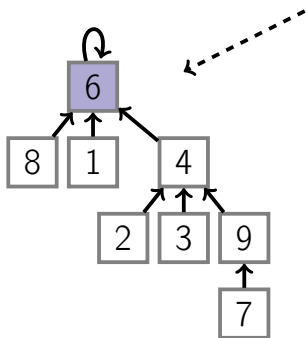


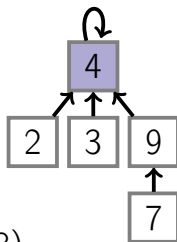
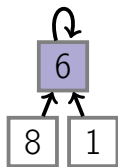


Union(3,8)

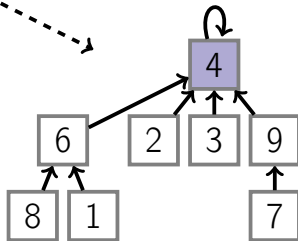
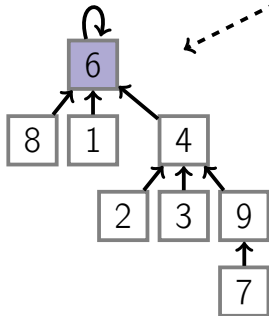


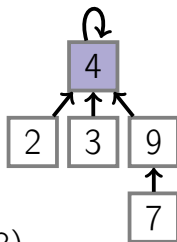
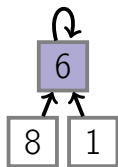
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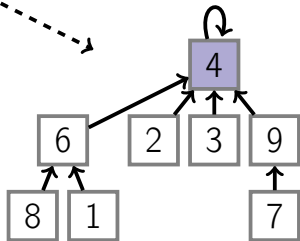
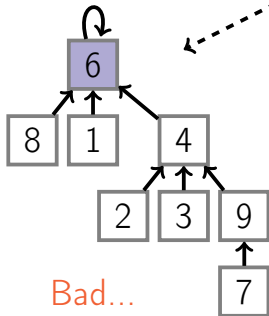


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- 1 Trees
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- (The reason we call it `rank`, but not `height` will become clear later)
- Hanging a shorter tree under a taller one is called a **union by rank heuristic**

MakeSet(i)

```
parent[ $i$ ]  $\leftarrow i$   
rank[ $i$ ]  $\leftarrow 0$ 
```

Find(i)

```
while  $i \neq$  parent[ $i$ ]:  
     $i \leftarrow$  parent[ $i$ ]  
return  $i$ 
```

Union(i, j)

```
 $i\_id \leftarrow \text{Find}(i)$   
 $j\_id \leftarrow \text{Find}(j)$   
if  $i\_id = j\_id$ :  
    return  
if  $\text{rank}[i\_id] > \text{rank}[j\_id]$ :  
     $\text{parent}[j\_id] \leftarrow i\_id$   
else:  
     $\text{parent}[i\_id] \leftarrow j\_id$   
    if  $\text{rank}[i\_id] = \text{rank}[j\_id]$ :  
         $\text{rank}[j\_id] \leftarrow \text{rank}[j\_id] + 1$ 
```

Example

Query:

	1	2	3	4	5	6
parent						
rank						

Example

Query:

MakeSet(1)

MakeSet(2)

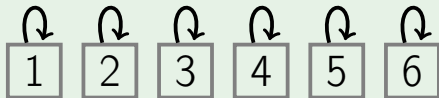
...

MakeSet(6)

	1	2	3	4	5	6
parent						
rank						

Example

Query:

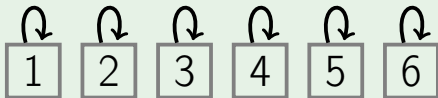


	1	2	3	4	5	6
parent	1	2	3	4	5	6
rank	0	0	0	0	0	0

Example

Query:

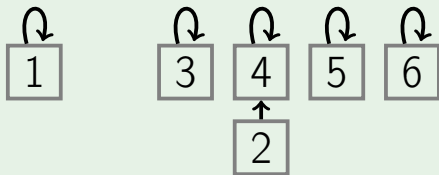
Union(2, 4)



	1	2	3	4	5	6
parent	1	2	3	4	5	6
rank	0	0	0	0	0	0

Example

Query:

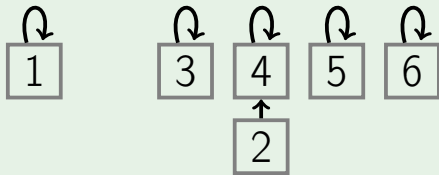


	1	2	3	4	5	6
parent	1	4	3	4	5	6
rank	0	0	0	1	0	0

Example

Query:

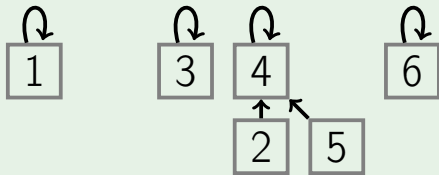
Union(5, 2)



	1	2	3	4	5	6
parent	1	4	3	4	5	6
rank	0	0	0	1	0	0

Example

Query:

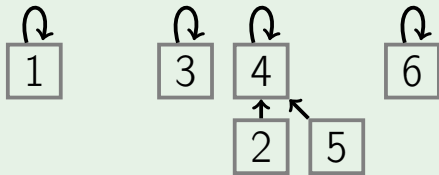


	1	2	3	4	5	6
parent	1	4	3	4	4	6
rank	0	0	0	1	0	0

Example

Query:

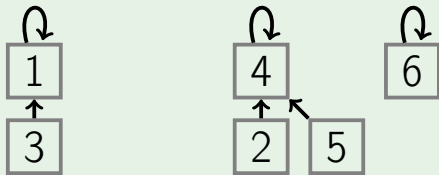
Union(3, 1)



	1	2	3	4	5	6
parent	1	4	3	4	4	6
rank	0	0	0	1	0	0

Example

Query:

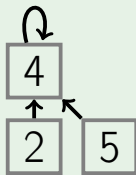
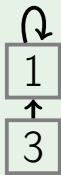


	1	2	3	4	5	6
parent	1	4	1	4	4	6
rank	1	0	0	1	0	0

Example

Query:

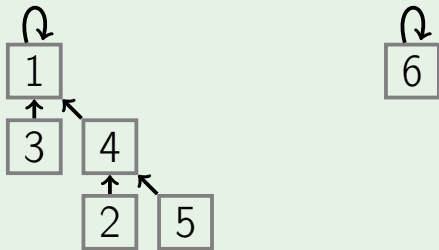
Union(2, 3)



	1	2	3	4	5	6
parent	1	4	1	4	4	6
rank	1	0	0	1	0	0

Example

Query:

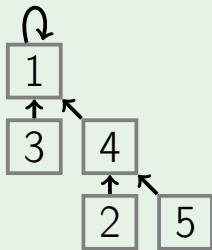


	1	2	3	4	5	6
parent	1	4	1	1	4	6
rank	2	0	0	1	0	0

Example

Query:

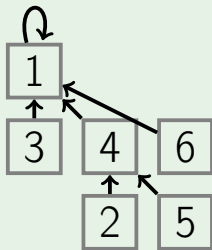
Union(2, 6)



	1	2	3	4	5	6
parent	1	4	1	1	4	6
rank	2	0	0	1	0	0

Example

Query:



	1	2	3	4	5	6
parent	1	4	1	1	4	1
rank	2	0	0	1	0	0

Important property: for any node i , $\text{rank}[i]$ is equal to the height of the tree rooted at i

Lemma

The height of any tree in the forest is at most $\log_2 n$.

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Follows from the following lemma.

Lemma

Any tree of height k in the forest has at least 2^k nodes.

Proof

Induction on k .

- Base: initially, a tree has height 0 and one node: $2^0 = 1$.
- Step: a tree of height k results from merging two trees of height $k - 1$. By induction hypothesis, each of two trees has at least 2^{k-1} nodes, hence the resulting tree contains at least 2^k nodes.



Summary

The union by rank heuristic guarantees that Union and Find work in time $O(\log n)$.

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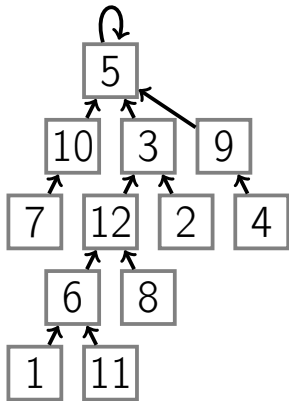
Next part

We'll discover another heuristic that improves the running time to nearly constant!

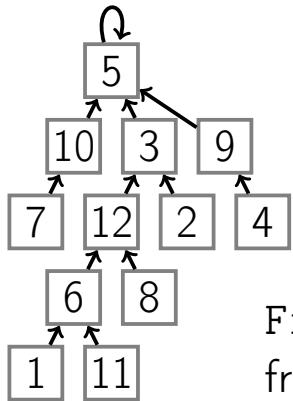
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Path Compression: Intuition

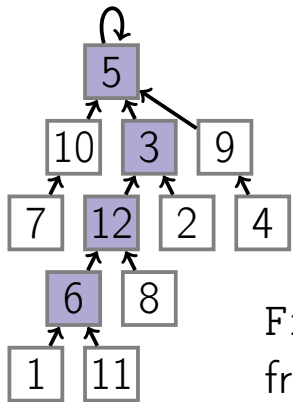


Path Compression: Intuition



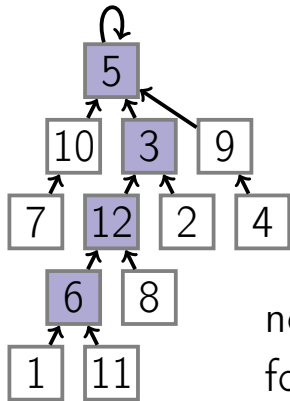
Find(6) traverses the path
from 6 to the root

Path Compression: Intuition



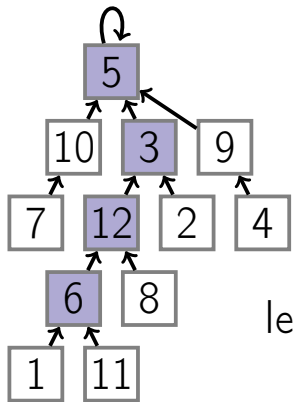
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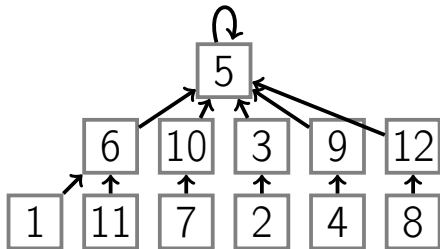
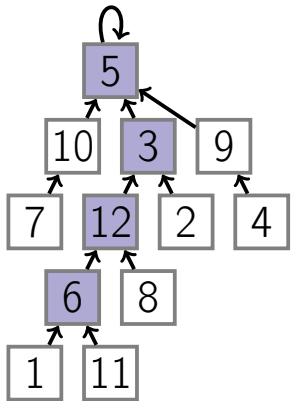
not only it finds the root
for 6, it does so for all the
nodes on this path

Path Compression: Intuition

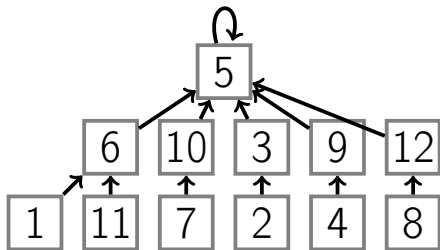
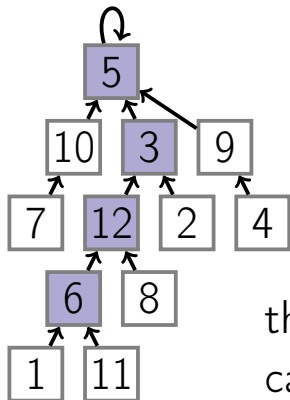


let's not lose this useful info

Path Compression: Intuition



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the resulting heuristic is called **path compression**

Find(i)

```
if  $i \neq \text{parent}[i]$ :  
     $\text{parent}[i] \leftarrow \text{Find}(\text{parent}[i])$   
return  $\text{parent}[i]$ 
```

Definition

The **iterated logarithm** of n , $\log^* n$, is the number of times the logarithm function needs to be applied to n before the result is less or equal than 1:

$$\log^* n = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \log^*(\log n) & \text{if } n > 1 \end{cases}$$

Example

n	$\log^* n$
$n = 1$	0
$n = 2$	1
$n \in \{3, 4\}$	2
$n \in \{5, 6, \dots, 16\}$	3
$n \in \{17, \dots, 65536\}$	4
$n \in \{65537, \dots, 2^{65536}\}$	5

Lemma

Assume that initially the data structure is empty. We make a sequence of m operations including n calls to MakeSet. Then the total running time is $O(m \log^* n)$.

In other words

The amortized time of a single operation is $O(\log^* n)$.

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Nearly constant!

For practical values of n , $\log^* n \leq 5$.

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- ③ Path Compression
- ④ Analysis

Goal

Prove that when both union by rank heuristic and path compression heuristic are used, the average running time of each operation is nearly constant.

Height \leq Rank

- When using path compression, $\text{rank}[i]$ is no longer equal to the height of the subtree rooted at i

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- Still, the height of the subtree rooted at i is at most $\text{rank}[i]$

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- When using path compression, $\text{rank}[i]$ is no longer equal to the height of the subtree rooted at i
- Still, the height of the subtree rooted at i is at most $\text{rank}[i]$
- And it is still true that a root node of rank k has at least 2^k nodes in its subtree: a root node is not affected by path compression

Important Properties

- 1 There are at most $\frac{n}{2^k}$ nodes of rank k

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- 1 There are at most $\frac{n}{2^k}$ nodes of rank k
- 2 For any node i ,
 $\text{rank}[i] < \text{rank}[\text{parent}[i]]$
- 3 Once an internal node, always an internal node

$T(\text{all calls to Find}) =$

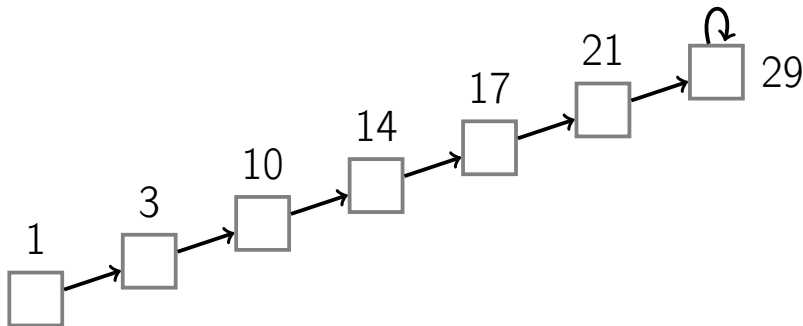
$\#(i \rightarrow j) =$

$\#(i \rightarrow j: j \text{ is a root}) +$

$\#(i \rightarrow j: \log^*(\text{rank}[i]) < \log^*(\text{rank}[j])) +$

$\#(i \rightarrow j: \log^*(\text{rank}[i]) = \log^*(\text{rank}[j]))$

$$\begin{aligned}
 T(\text{all calls to Find}) = \\
 \#(i \rightarrow j) = \\
 \#(i \rightarrow j: j \text{ is a root}) + \\
 \#(i \rightarrow j: \log^*(\text{rank}[i]) < \log^*(\text{rank}[j])) + \\
 \#(i \rightarrow j: \log^*(\text{rank}[i]) = \log^*(\text{rank}[j]))
 \end{aligned}$$



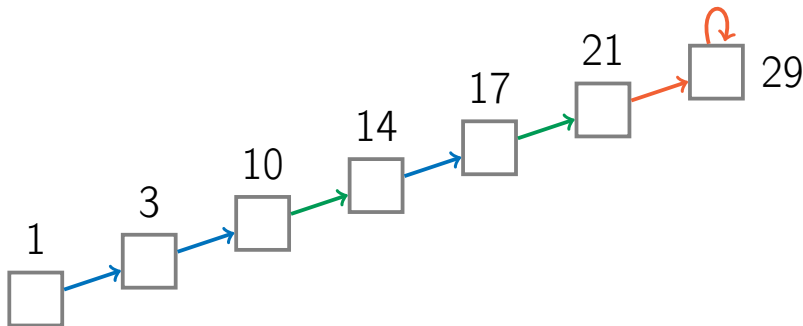
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Claim

$$\#(i \rightarrow j: j \text{ is a root}) \leq O(m)$$

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Proof

There are at most m calls to Find.



Claim

$$\begin{aligned} \#(i \rightarrow j: \log^*(\text{rank}[i]) < \log^*(\text{rank}[j])) \\ \leq O(m \log^* n) \end{aligned}$$

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Proof

There are at most $\log^* n$ different values for $\log^*(\text{rank})$. □

Claim

$$\#(i \rightarrow j: \log^*(\text{rank}[i]) = \log^*(\text{rank}[j])) \leq O(n \log^* n)$$

Proof

- assume $\text{rank}[i] \in \{k + 1, \dots, 2^k\}$

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- after a call to $\text{Find}(i)$, the node i is adopted by a new parent of strictly larger rank
- after at most 2^k calls to $\text{Find}(i)$, the parent of i will have rank from a different interval

Proof (Continued)

- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k + 1, \dots, 2^k\}$

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- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k + 1, \dots, 2^k\}$
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Proof (Continued)

- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k + 1, \dots, 2^k\}$
- each of them contributes at most 2^k
- the contribution of all the nodes with rank from this interval is at most $O(n)$
- the number of different intervals is $\log^* n$

Proof (Continued)

- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k + 1, \dots, 2^k\}$
 - each of them contributes at most 2^k
 - the contribution of all the nodes with rank from this interval is at most $O(n)$
 - the number of different intervals is $\log^* n$
 - thus, the contribution of all nodes is $O(n \log^* n)$
- 

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- Use the root of the set as its ID
- Union by rank heuristic: hang a shorter tree under the root of a taller one
- Path compression heuristic: when finding the root of a tree for a particular node, reattach each node from the traversed path to the root
- Amortized running time: $O(\log^* n)$
(constant for practical values of n)