Binary Search Trees: Split and Merge

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Data Structures
Data Structures and Algorithms

Learning Objectives

- Implement merging and splitting of AVL trees.
- Analyze the runtime of these operations.

New Operations

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New Operations

Another useful feature of binary search trees is the ability to recombine them in interesting ways. We discuss two new operations:

- Merge Combines two binary search trees into a single one.
- Split Breaks one binary search tree into two.

Outline

1. Merge

2. Split

Merge

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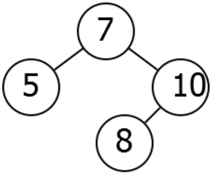
Merge

Input: Roots R_1 and R_2 of trees with all keys in R_1's tree smaller than those in R_2's

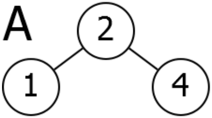
Output: The root of a new tree with all the elements of both trees

Problem

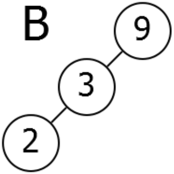
Which tree can be merged with the given one?



A



B



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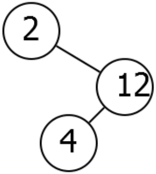

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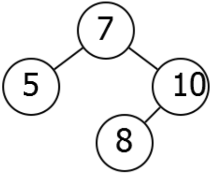

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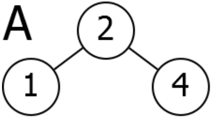

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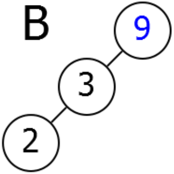

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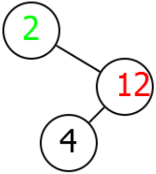
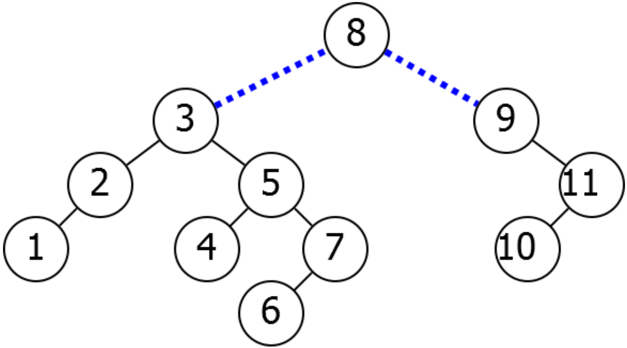


Extra Root

Easy if you have an extra node to add as root.



Implementation

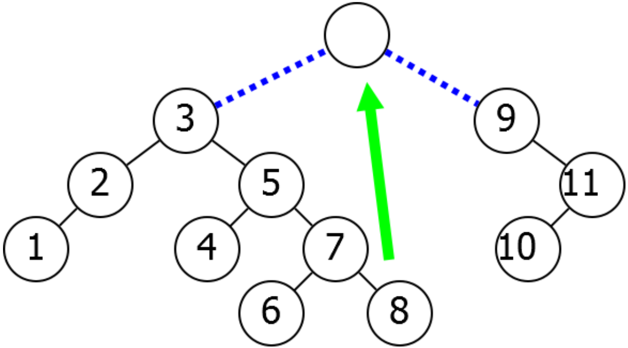
`MergeWithRoot`(R_1, R_2, T)

T.`Left` $\leftarrow R_1$
T.`Right` $\leftarrow R_2$
R_1.`Parent` $\leftarrow T$
R_2.`Parent` $\leftarrow T$
`return` T

Time $O(1)$.

Get Root

Get new root by removing largest element of left subtree.



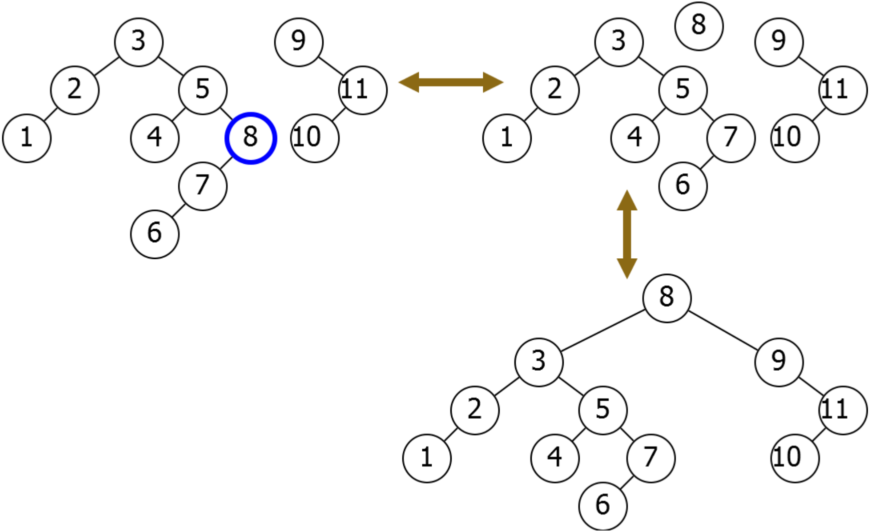
Merge

Merge(R_1, R_2)

$T \leftarrow$ Find(∞, R_1)
Delete(T)
MergeWithRoot(R_1, R_2, T)
return T

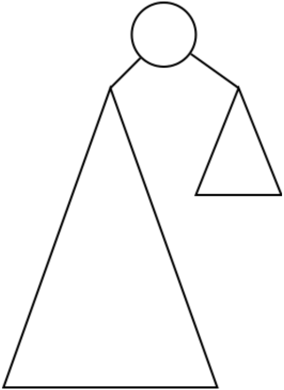
Time $O(h)$.

Merge



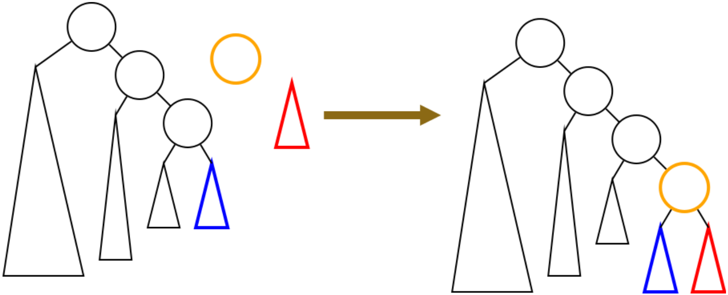
Balance

Unfortunately, this merge does not preserve balance properties.



Idea

Go down side of tree until merge with subtree of same height.



Implementation

$\texttt{AVLTreeMergeWithRoot}(R_1, R_2, T)$

if $|R_1.\texttt{Height} - R_2.\texttt{Height}| \leq 1$:
  $\texttt{MergeWithRoot}(R_1, R_2, T)$
  $T.\texttt{Ht} \leftarrow \max(R_1.\texttt{Height}, R_2.\texttt{Height}) + 1$
  return T

Implementation (continued)

$\texttt{AVLTreeMergeWithRoot}(R_1, R_2, T)$

```
else if R₁.Height > R₂.Height:
    R' ← AVLTreeMWR(R₁.Right, R₂, T)
    R₁.Right ← R'
    R'.Parent ← R₁
    Rebalance(R₁)
    return root
else if R₁.Height < R₂.Height:
    ...
```

Analysis

- Each step changes height difference by 1 or 2.
- Eventually within 1.
- Time $O(|R_1.\mathtt{Height} - R_2.\mathtt{Height}| + 1)$.

Outline

1. Merge
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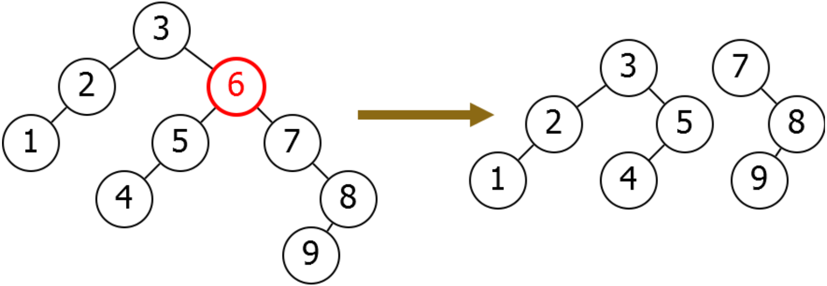
Split

Break tree into two trees:



Formal Definition

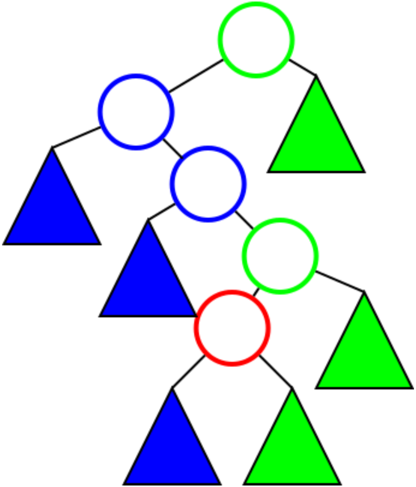
Split

Input: Root R of a tree, key x

Output: Two trees, one with elements $\leq x$, one with elements $> x$.

Idea

Search for x, merge subtrees.



Implementation

$\texttt{Split}(R, x)$

```
if R = null:
    return (null, null)
if x ≤ R.Key:
    (R1, R2) ← Split(R.Left, x)
    R3 ← MergeWithRoot(R2, R.Right, R)
    return (R1, R3)
if x > R.Key:
    ...
```

AVL Trees

- Using `AVLMergeWithRoot` maintains balance.
- Time $= \sum O(|h_i - h_{i+1}| + 1) = O(h_{max}) = O(\log(n))$.

Conclusion

Summary

- `Merge` combines trees.
- `Split` turns one tree into two.
- Both can be implemented in $O(\log(n))$ time for AVL trees.