

Priority Queues: Binary Heaps

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Data Structures
Data Structures and Algorithms

Outline

- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

Definition

Binary max-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at least the values of its children.

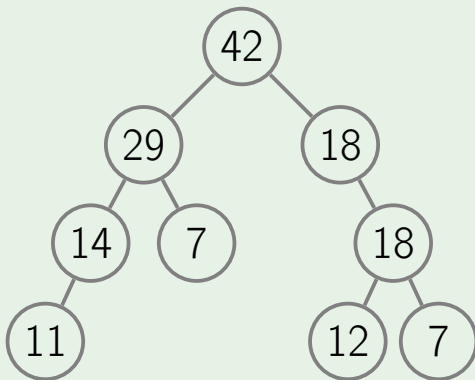
Definition

Binary max-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at least the values of its children.

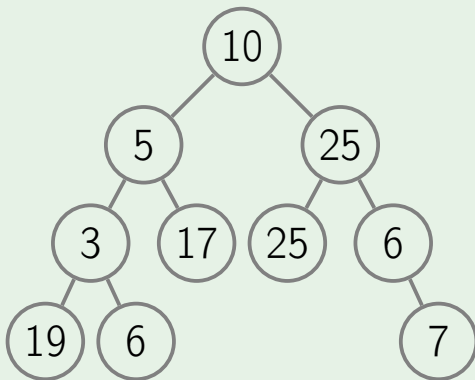
In other words

For each edge of the tree, the value of the parent is at least the value of the child.

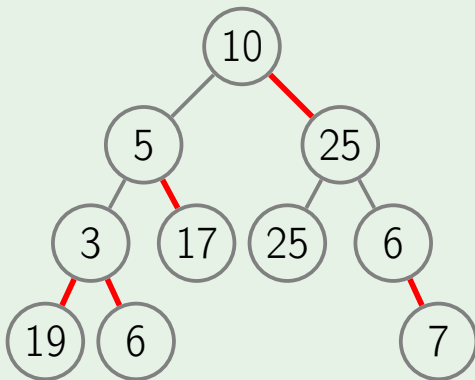
Example: heap



Example: not a heap



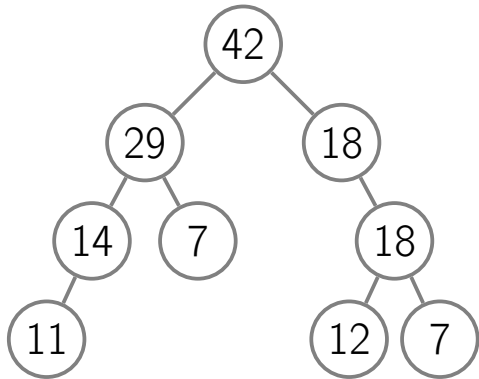
Example: not a heap



Outline

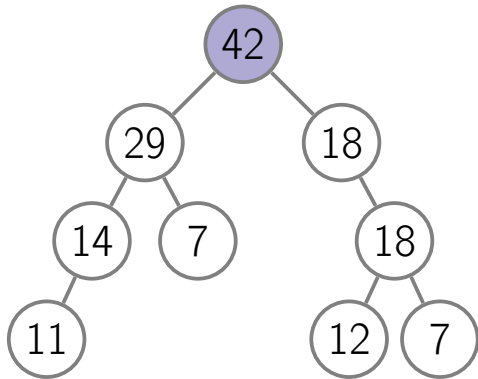
- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

GetMax



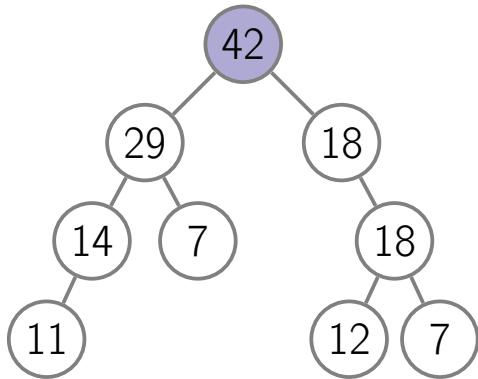
GetMax

return the root
value



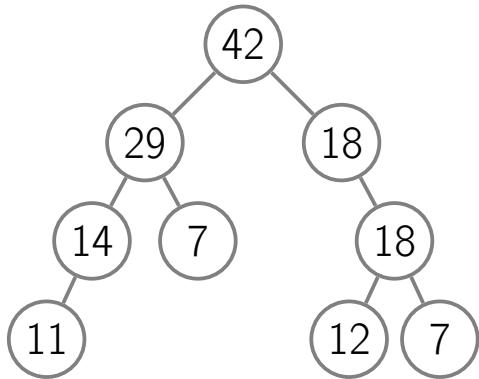
GetMax

return the root
value



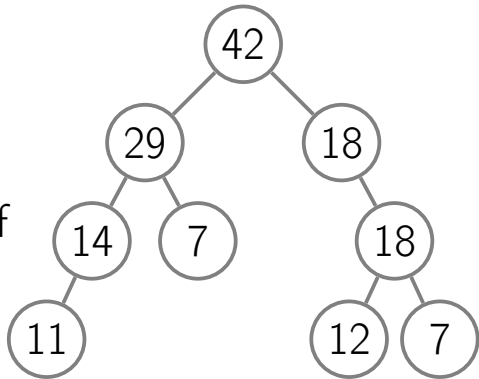
running time: $O(1)$

Insert



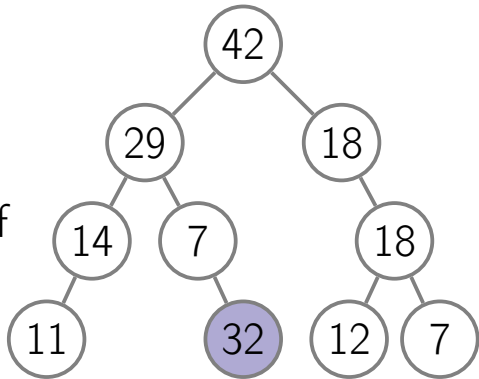
Insert

attach a new
node to any leaf



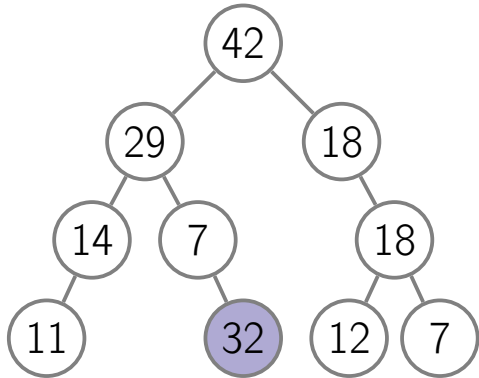
Insert

attach a new
node to any leaf



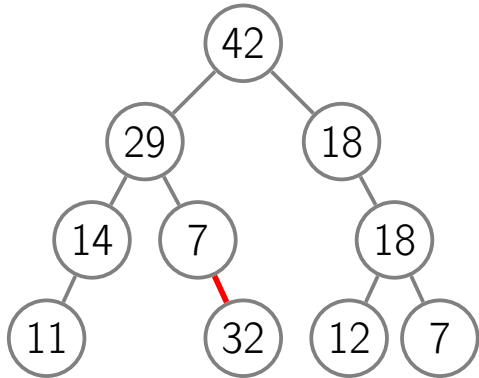
Insert

this may violate
the heap prop-
erty



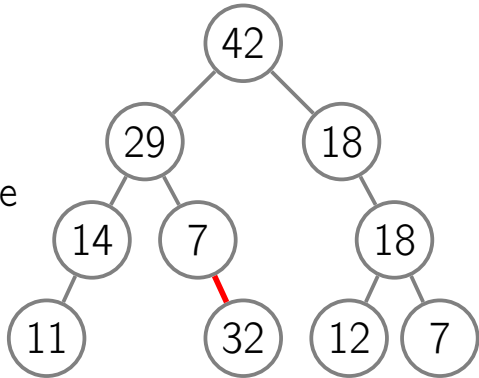
Insert

this may violate
the heap prop-
erty



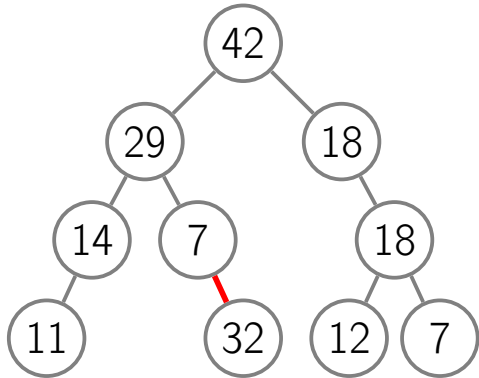
Insert

to fix this, we
let the new node
sift up

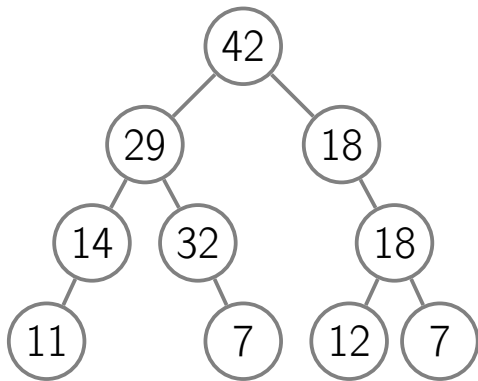


SiftUp

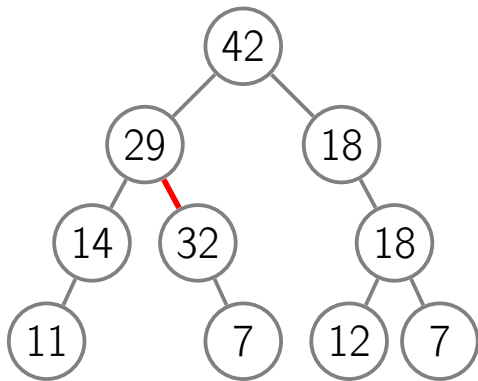
for this, we
swap the prob-
lematic node
with its parent
until the prop-
erty is satisfied



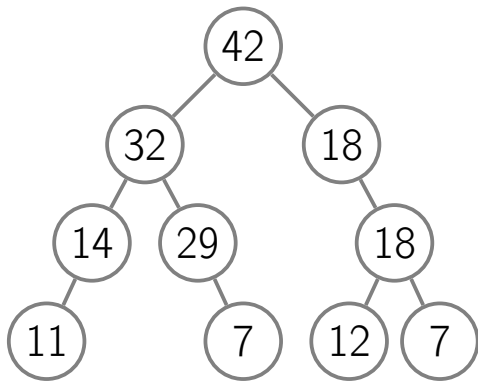
SiftUp



SiftUp

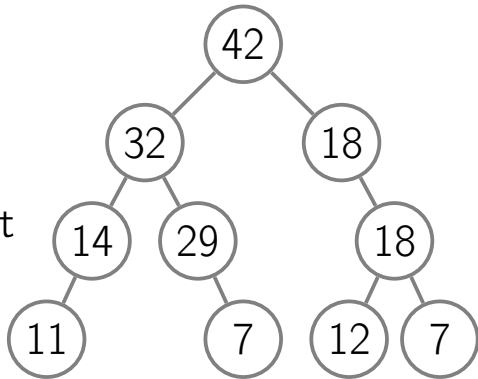


SiftUp



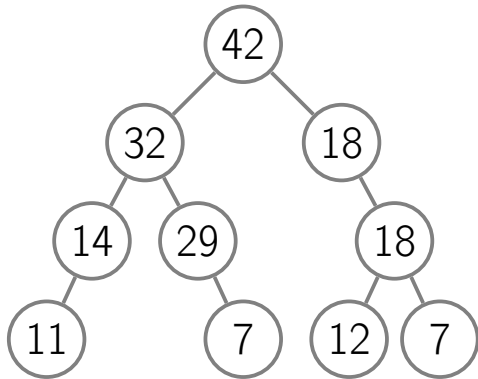
SiftUp

invariant: heap
property is vio-
lated on at most
one edge

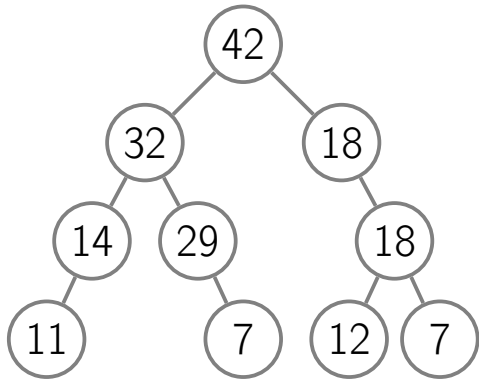


SiftUp

this edge gets
closer to the
root while sift-
ing up

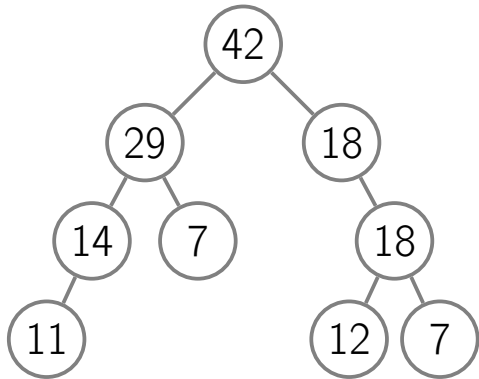


SiftUp



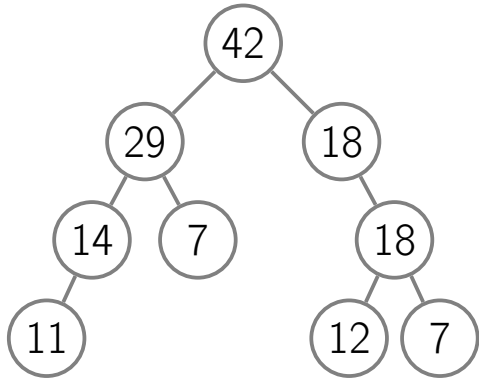
running time: $O(\text{tree height})$

ExtractMax



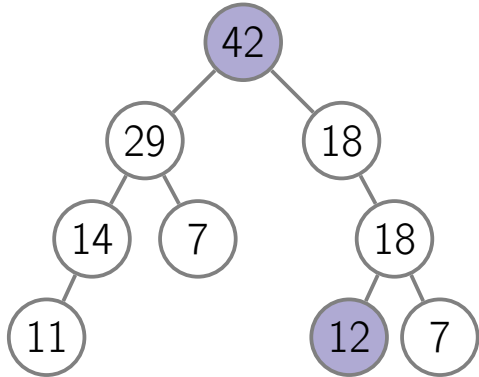
ExtractMax

replace the root
with any leaf



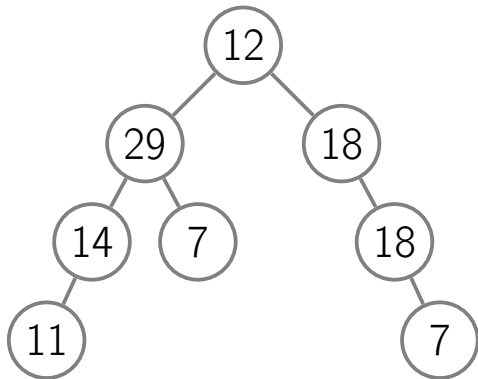
ExtractMax

replace the root
with any leaf



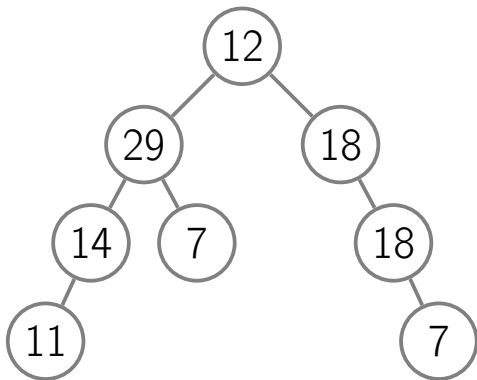
ExtractMax

replace the root
with any leaf



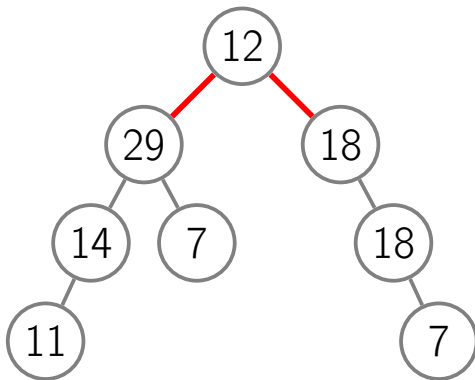
ExtractMax

again, this may
violate the heap
property



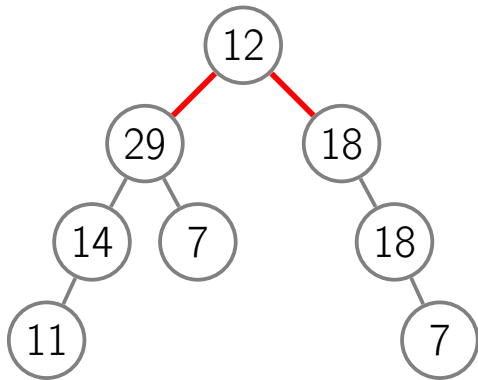
ExtractMax

again, this may
violate the heap
property



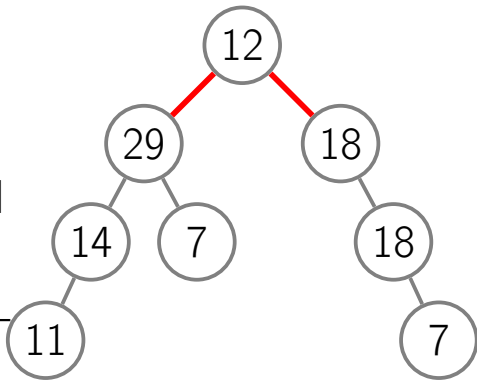
ExtractMax

to fix it, we let
the problematic
node sift down

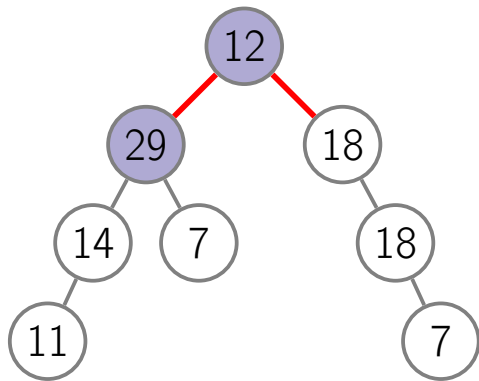


SiftDown

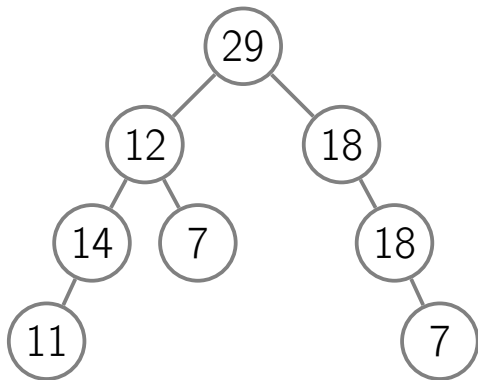
for this, we
swap the prob-
lematic node
with larger child
until the heap
property is satis-
fied



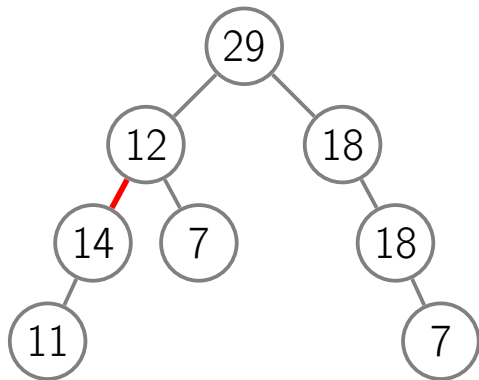
SiftDown



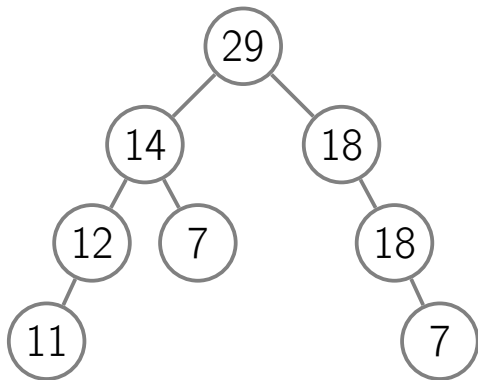
SiftDown



SiftDown

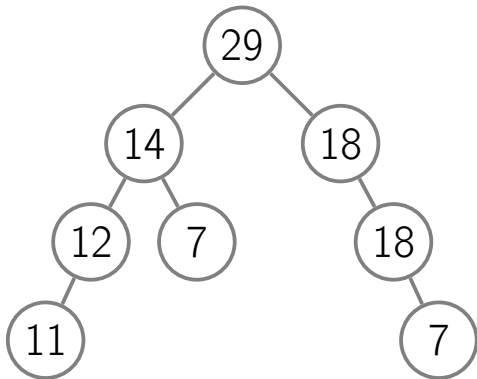


SiftDown

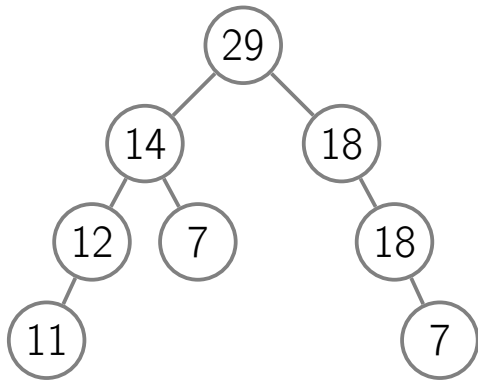


SiftDown

we swap with
the larger child
which automatically fixes one
of the two bad
edges

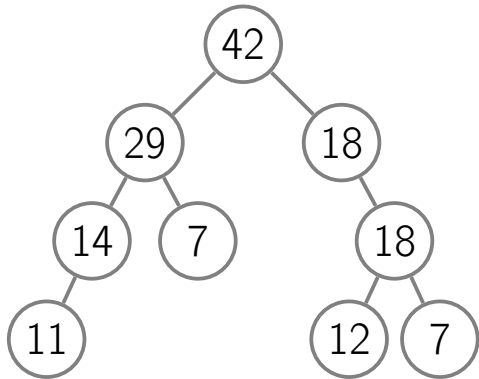


SiftDown



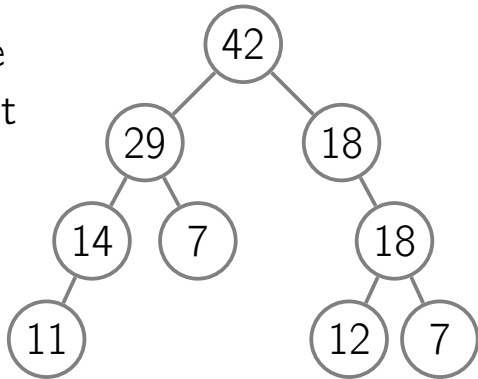
running time: $O(\text{tree height})$

ChangePriority



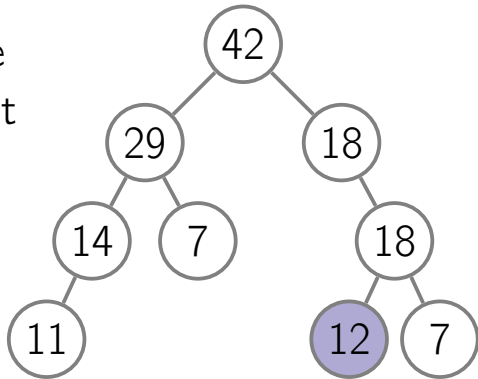
ChangePriority

change the priority and let the changed element sift up or down depending on whether its priority decreased or increased



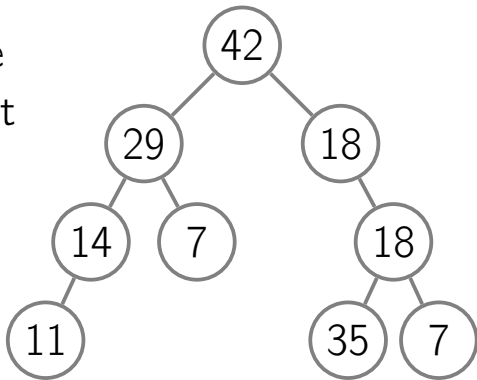
ChangePriority

change the priority and let the changed element sift up or down depending on whether its priority decreased or increased

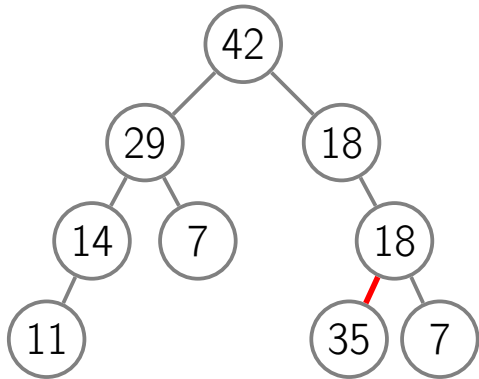


ChangePriority

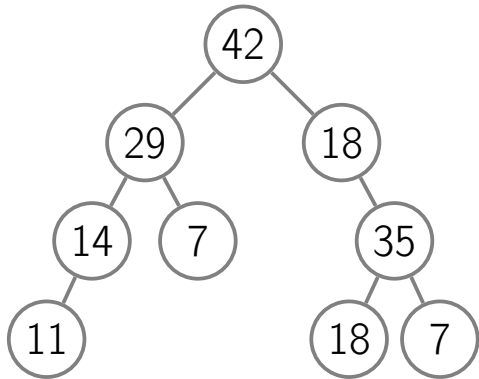
change the priority and let the changed element sift up or down depending on whether its priority decreased or increased



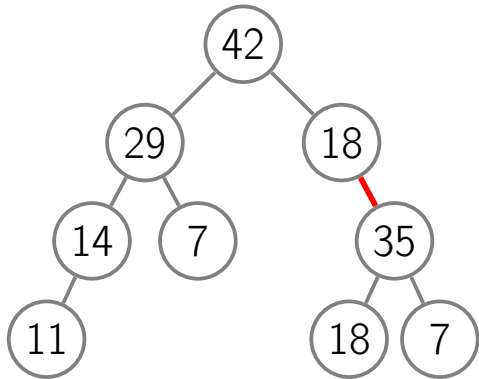
ChangePriority



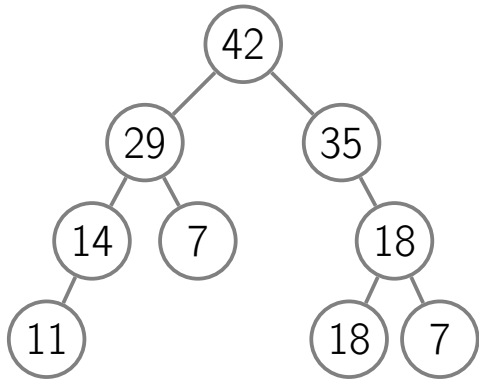
ChangePriority



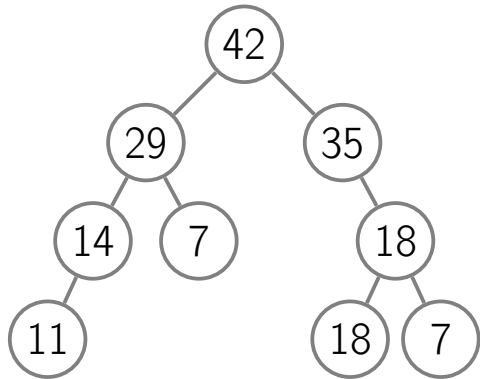
ChangePriority



ChangePriority

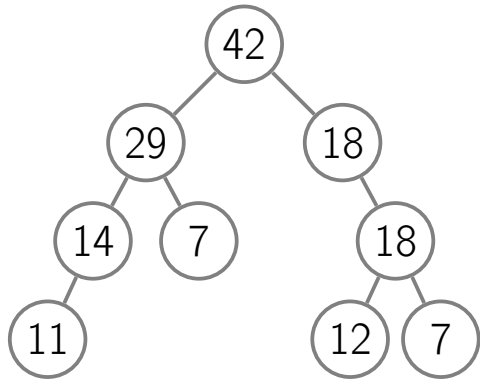


ChangePriority



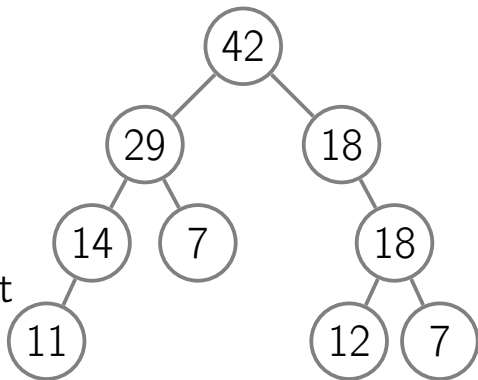
running time: $O(\text{tree height})$

Remove

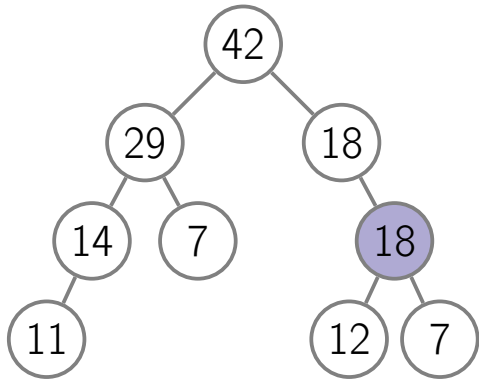


Remove

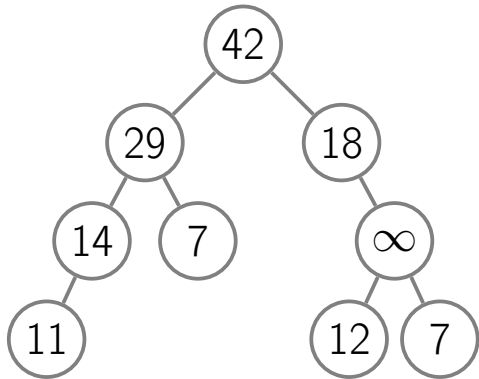
change the priority of the element to ∞ ,
let it sift up,
and then extract maximum



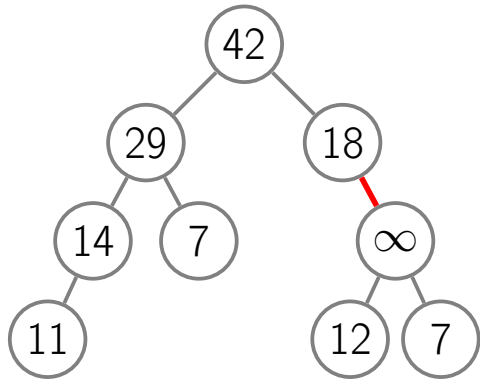
Remove



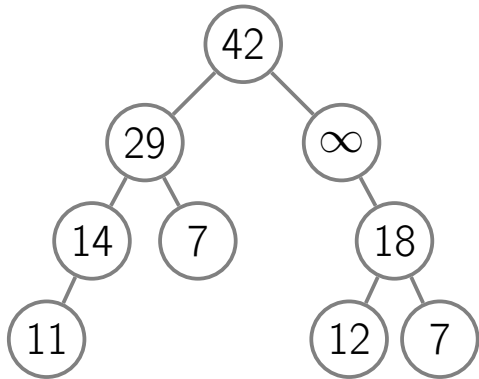
Remove



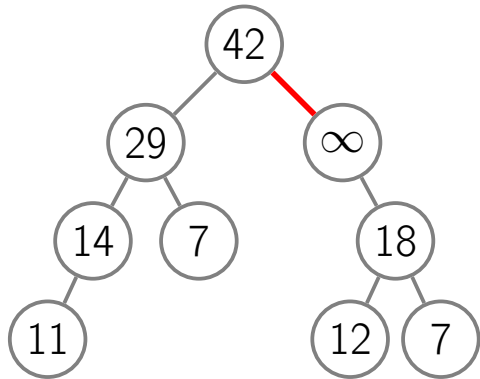
Remove



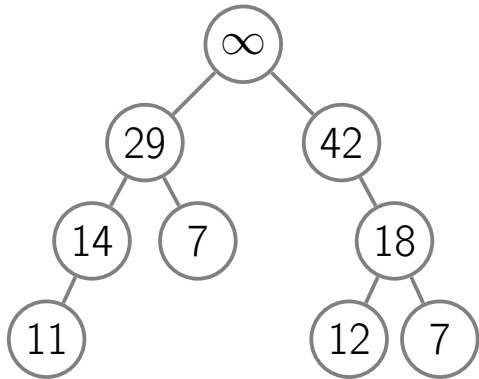
Remove



Remove

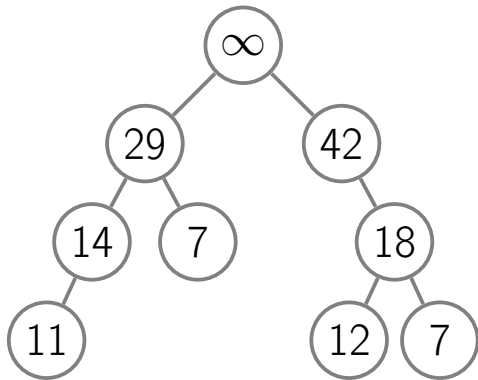


Remove

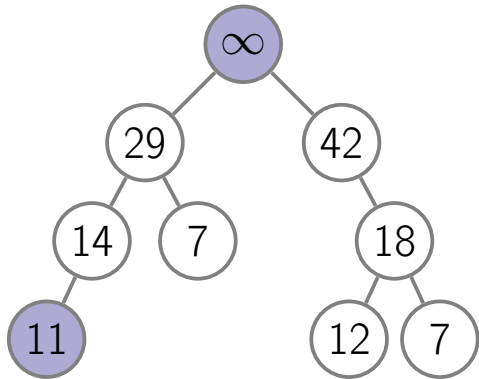


Remove

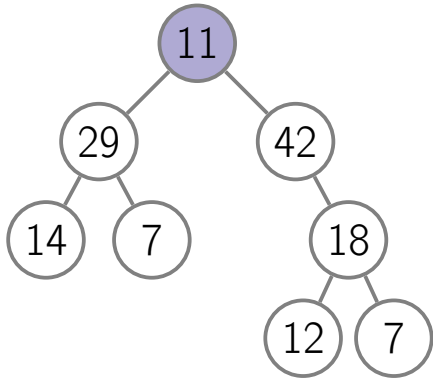
now, call
ExtractMax()



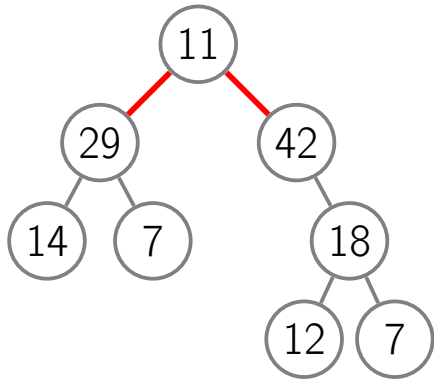
Remove



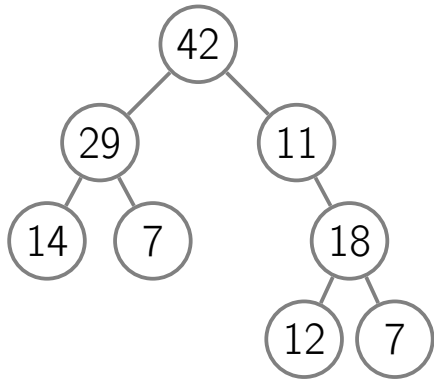
Remove



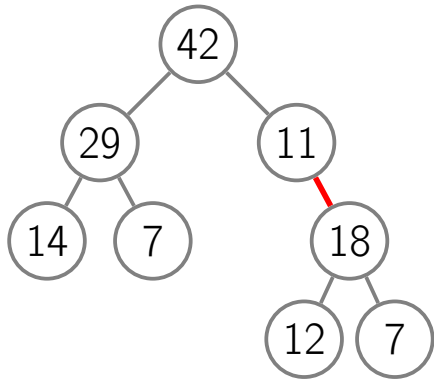
Remove



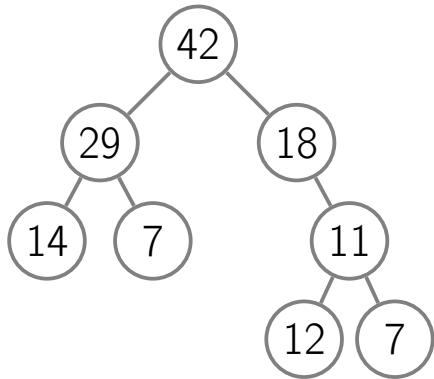
Remove



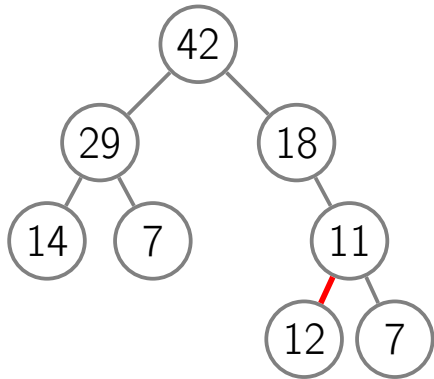
Remove



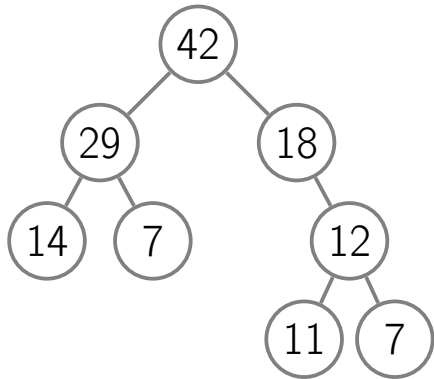
Remove



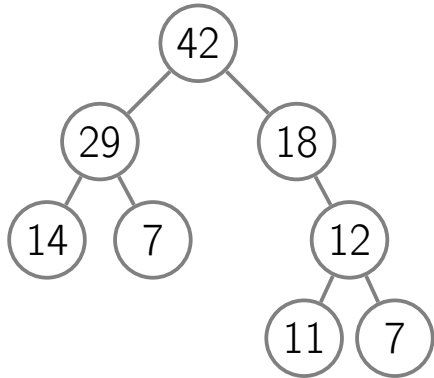
Remove



Remove



Remove



running time: $O(\text{tree height})$

Summary

- GetMax works in time $O(1)$, all other operations work in time $O(\text{tree height})$

Summary

- GetMax works in time $O(1)$, all other operations work in time $O(\text{tree height})$
- we definitely want a tree to be shallow

Outline

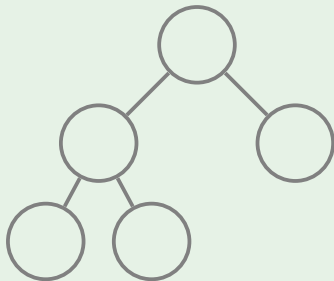
- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

How to Keep a Tree Shallow?

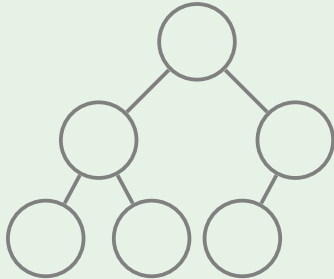
Definition

A binary tree is **complete** if all its levels are filled except possibly the last one which is filled from left to right.

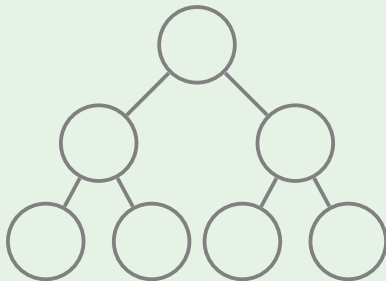
Example: complete binary tree



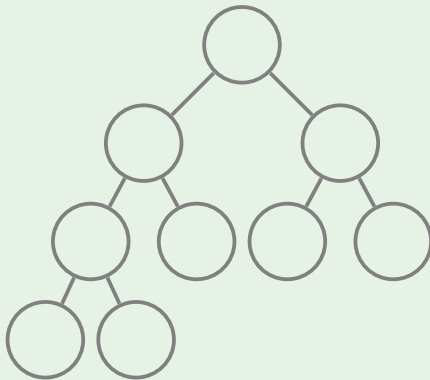
Example: complete binary tree



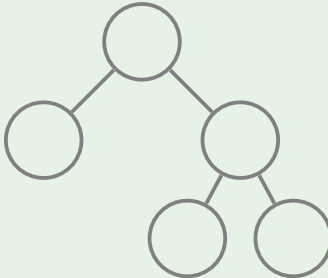
Example: complete binary tree



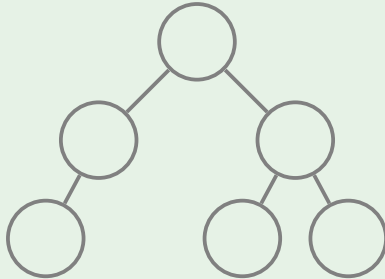
Example: complete binary tree



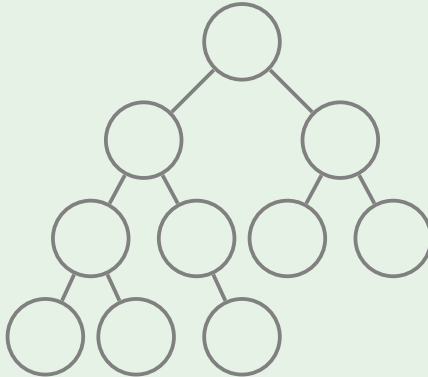
Example: **not** complete binary tree



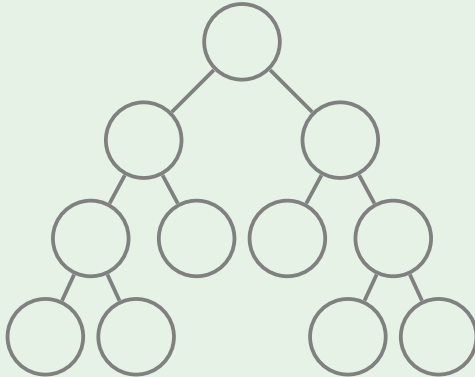
Example: **not** complete binary tree



Example: **not** complete binary tree



Example: **not** complete binary tree



First Advantage: Low Height

Lemma

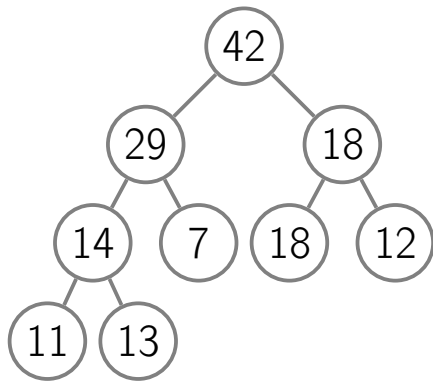
A complete binary tree with n nodes has height at most $O(\log n)$.

Proof

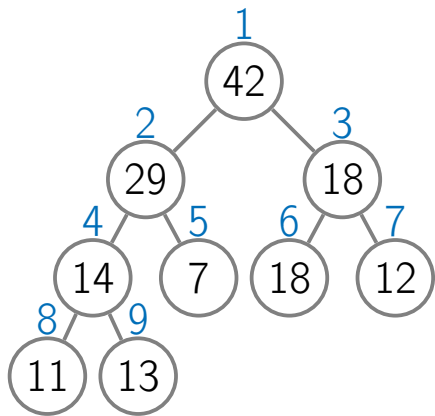
- Complete the last level to get a full binary tree on $n' \geq n$ nodes and the same number of levels ℓ .
- Note that $n' \leq 2n$.
- Then $n' = 2^\ell - 1$ and hence
$$\ell = \log_2(n' + 1) \leq \log_2(2n + 1) = O(\log n).$$



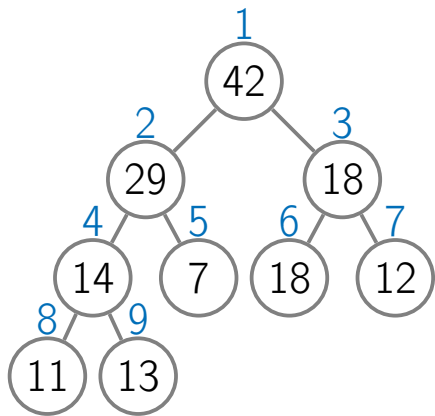
Second Advantage: Store as Array



Second Advantage: Store as Array



Second Advantage: Store as Array

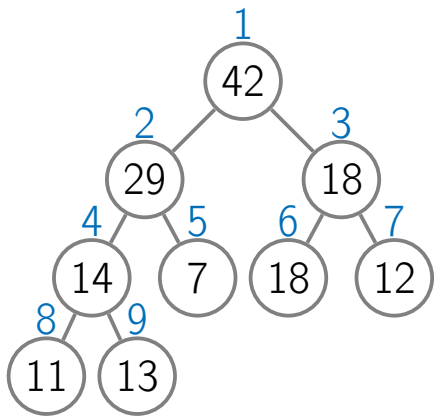


$$\text{parent}(i) = \lfloor \frac{i}{2} \rfloor$$

$$\text{leftchild}(i) = 2i$$

$$\text{rightchild}(i) = 2i + 1$$

Second Advantage: Store as Array



$$\text{parent}(i) = \lfloor \frac{i}{2} \rfloor$$

$$\text{leftchild}(i) = 2i$$

$$\text{rightchild}(i) = 2i + 1$$

1	2	3	4	5	6	7	8	9
42	29	18	14	7	18	12	11	13

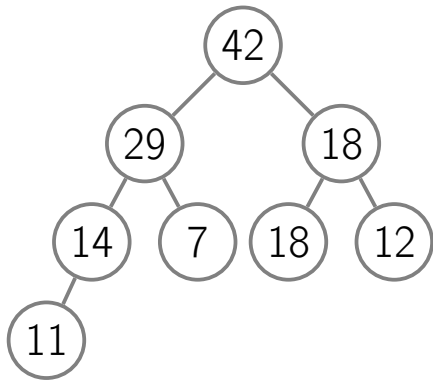
- What do we pay for these advantages?

- What do we pay for these advantages?
- We need to keep the tree complete.

- What do we pay for these advantages?
- We need to keep the tree complete.
- Which binary heap operations modify the shape of the tree?

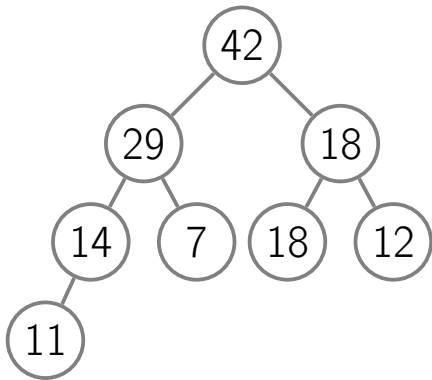
- What do we pay for these advantages?
- We need to keep the tree complete.
- Which binary heap operations modify the shape of the tree?
- Only Insert and ExtractMax (Remove changes the shape by calling ExtractMax).

Keeping the Tree Complete



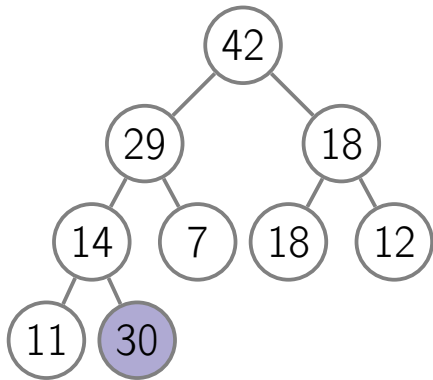
Keeping the Tree Complete

to insert an element, insert it as a leaf in the leftmost vacant position in the last level and let it sift up



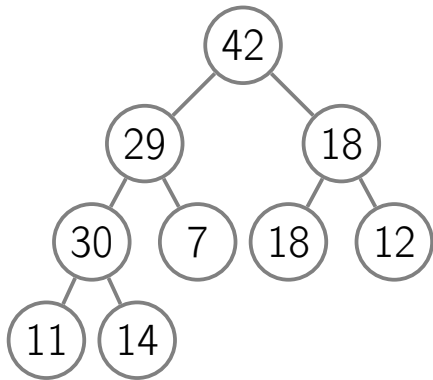
Keeping the Tree Complete

to insert an element, insert it as a leaf in the **leftmost vacant position in the last level** and let it sift up



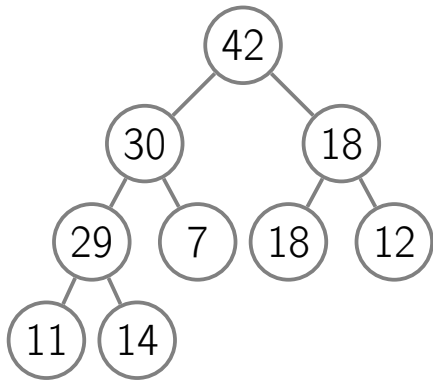
Keeping the Tree Complete

to insert an element, insert it as a leaf in the **leftmost vacant position in the last level** and let it sift up



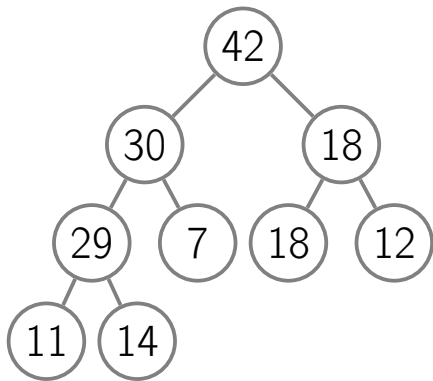
Keeping the Tree Complete

to insert an element, insert it as a leaf in the **leftmost vacant position in the last level** and let it sift up



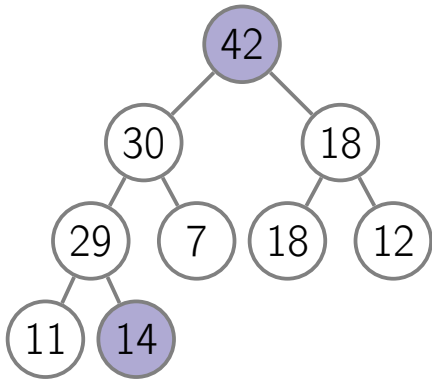
Keeping the Tree Complete

to extract the maximum value,
replace the root
by **the last leaf**
and let it sift
down



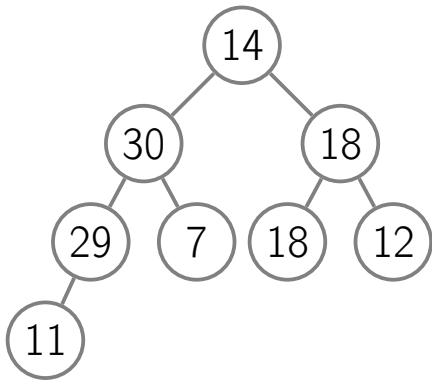
Keeping the Tree Complete

to extract the maximum value, replace the root by **the last leaf** and let it sift down



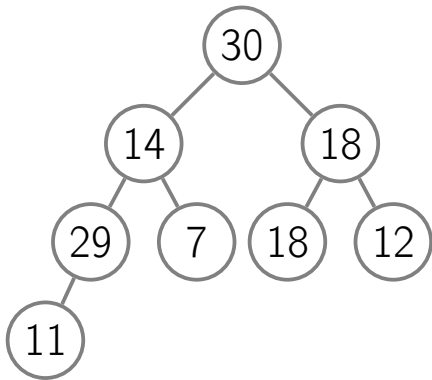
Keeping the Tree Complete

to extract the maximum value,
replace the root
by **the last leaf**
and let it sift
down



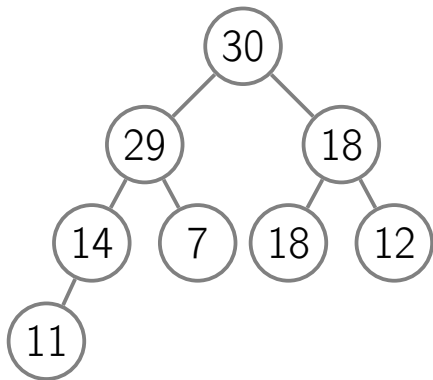
Keeping the Tree Complete

to extract the maximum value, replace the root by **the last leaf** and let it sift down



Keeping the Tree Complete

to extract the maximum value,
replace the root
by **the last leaf**
and let it sift
down



Outline

- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

General Setting

- *maxSize* is the maximum number of elements in the heap

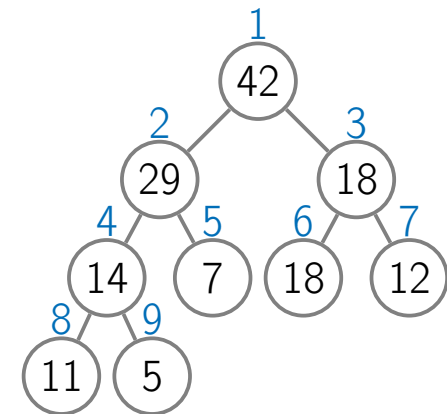
General Setting

- *maxSize* is the maximum number of elements in the heap
- *size* is the size of the heap

General Setting

- *maxSize* is the maximum number of elements in the heap
- *size* is the size of the heap
- $H[1 \dots \textit{maxSize}]$ is an array of length *maxSize* where the heap occupies the first *size* elements

Example



size = 9

maxSize = 13

	1	2	3	4	5	6	7	8	9	10	11	12	13
<i>H</i>	42	29	18	14	7	18	12	11	5	30	29	2	8

Parent(i)

return $\lfloor \frac{i}{2} \rfloor$

LeftChild(i)

return $2i$

RightChild(i)

return $2i + 1$

SiftUp(i)

```
while  $i > 1$  and  $H[\text{Parent}(i)] < H[i]$ :  
    swap  $H[\text{Parent}(i)]$  and  $H[i]$   
     $i \leftarrow \text{Parent}(i)$ 
```


SiftDown(*i*)

$maxIndex \leftarrow i$

$\ell \leftarrow \text{LeftChild}(i)$

if $\ell \leq size$ and $H[\ell] > H[maxIndex]$:

$maxIndex \leftarrow \ell$

$r \leftarrow \text{RightChild}(i)$

if $r \leq size$ and $H[r] > H[maxIndex]$:

$maxIndex \leftarrow r$

if $i \neq maxIndex$:

swap $H[i]$ and $H[maxIndex]$

SiftDown($maxIndex$)

Insert(p)

```
if  $size = maxSize$ :  
    return ERROR  
 $size \leftarrow size + 1$   
 $H[size] \leftarrow p$   
SiftUp( $size$ )
```

ExtractMax()

```
result  $\leftarrow H[1]$   
H[1]  $\leftarrow H[size]$   
size  $\leftarrow size - 1$   
SiftDown(1)  
return result
```

Remove(i)

$H[i] \leftarrow \infty$

SiftUp(i)

ExtractMax()

ChangePriority(i, p)

$oldp \leftarrow H[i]$

$H[i] \leftarrow p$

if $p > oldp$:

 SiftUp(i)

else:

 SiftDown(i)

Summary

The resulting implementation is

- **fast**: all operations work in time $O(\log n)$ (GetMax even works in $O(1)$)

Summary

The resulting implementation is

- **fast**: all operations work in time $O(\log n)$ (GetMax even works in $O(1)$)
- **space efficient**: we store an array of priorities; parent-child connections are not stored, but are computed on the fly

Summary

The resulting implementation is

- **fast**: all operations work in time $O(\log n)$ (GetMax even works in $O(1)$)
- **space efficient**: we store an array of priorities; parent-child connections are not stored, but are computed on the fly
- **easy to implement**: all operations are implemented in just a few lines of code

Outline

- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

Sort Using Priority Queues

HeapSort($A[1 \dots n]$)

create an empty priority queue

for i from 1 to n :

 Insert($A[i]$)

for i from n downto 1:

$A[i] \leftarrow \text{ExtractMax}()$

- The resulting algorithm is comparison-based and has running time $O(n \log n)$ (hence, asymptotically optimal!).

- The resulting algorithm is comparison-based and has running time $O(n \log n)$ (hence, asymptotically optimal!).
- Natural generalization of selection sort: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure.

- The resulting algorithm is comparison-based and has running time $O(n \log n)$ (hence, asymptotically optimal!).
- Natural generalization of selection sort: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure.
- Not in-place: uses additional space to store the priority queue.

This lesson

In-place heap sort algorithm. For this, we will first turn a given array into a heap by permuting its elements.

Turn Array into a Heap

BuildHeap($A[1 \dots n]$)

$size \leftarrow n$

for i from $\lfloor n/2 \rfloor$ downto 1:

 SiftDown(i)

- We repair the heap property going from bottom to top.

- We repair the heap property going from bottom to top.
- Initially, the heap property is satisfied in all the leaves (i.e., subtrees of depth 0).

- We repair the heap property going from bottom to top.
- Initially, the heap property is satisfied in all the leaves (i.e., subtrees of depth 0).
- We then start repairing the heap property in all subtrees of depth 1.

- We repair the heap property going from bottom to top.
- Initially, the heap property is satisfied in all the leaves (i.e., subtrees of depth 0).
- We then start repairing the heap property in all subtrees of depth 1.
- When we reach the root, the heap property is satisfied in the whole tree.

- We repair the heap property going from bottom to top.
- Initially, the heap property is satisfied in all the leaves (i.e., subtrees of depth 0).
- We then start repairing the heap property in all subtrees of depth 1.
- When we reach the root, the heap property is satisfied in the whole tree.
- [Online visualization](#)

- We repair the heap property going from bottom to top.
- Initially, the heap property is satisfied in all the leaves (i.e., subtrees of depth 0).
- We then start repairing the heap property in all subtrees of depth 1.
- When we reach the root, the heap property is satisfied in the whole tree.
- [Online visualization](#)
- Running time: $O(n \log n)$

In-place Heap Sort

HeapSort($A[1 \dots n]$)

BuildHeap(A) $\{size = n\}$

repeat $(n - 1)$ times:

 swap $A[1]$ and $A[size]$

$size \leftarrow size - 1$

 SiftDown(1)

Building Running Time

- The running time of BuildHeap is $O(n \log n)$ since we call SiftDown for $O(n)$ nodes.

Building Running Time

- The running time of BuildHeap is $O(n \log n)$ since we call SiftDown for $O(n)$ nodes.
- If a node is already close to the leaves, then sifting it down is fast.

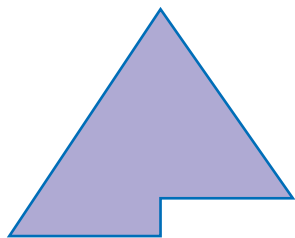
Building Running Time

- The running time of BuildHeap is $O(n \log n)$ since we call SiftDown for $O(n)$ nodes.
- If a node is already close to the leaves, then sifting it down is fast.
- We have many such nodes!

Building Running Time

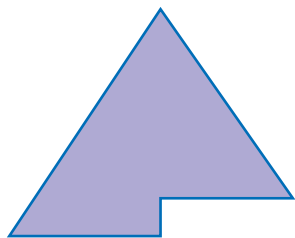
- The running time of BuildHeap is $O(n \log n)$ since we call SiftDown for $O(n)$ nodes.
- If a node is already close to the leaves, then sifting it down is fast.
- We have many such nodes!
- Was our estimate of the running time of BuildHeap too pessimistic?

Building Running Time



# nodes	$T(\text{SiftDown})$
1	$\log_2 n$
2	
\vdots	\vdots
$\leq n/4$	2
$\leq n/2$	1

Building Running Time



# nodes	$T(\text{SiftDown})$
1	$\log_2 n$
2	
\vdots	\vdots
$\leq n/4$	2
$\leq n/2$	1

$$\begin{aligned} T(\text{BuildHeap}) &\leq \frac{n}{2} \cdot 1 + \frac{n}{4} \cdot 2 + \frac{n}{8} \cdot 3 + \dots \\ &\leq n \cdot \sum_{i=1}^{\infty} \frac{i}{2^i} = 2n \end{aligned}$$

Estimating the Sum

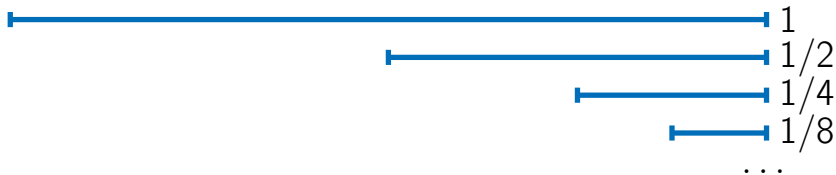


$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

Estimating the Sum



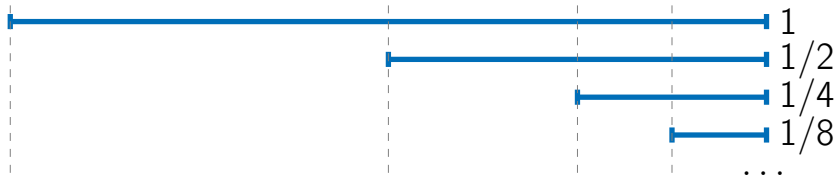
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$



Estimating the Sum



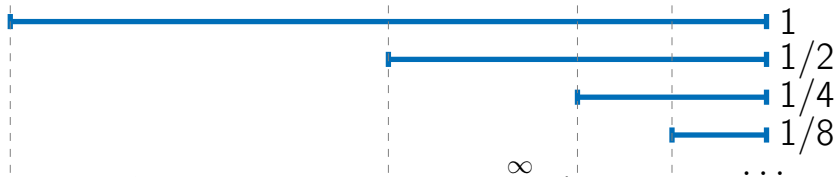
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$



Estimating the Sum



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$



$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots = \sum_{k=1}^{\infty} \frac{k}{2^k} = 2$$

Partial sorting

Input: An array $A[1 \dots n]$, an integer $1 \leq k \leq n$.

Output: The last k elements of a sorted version of A .

Partial sorting

Input: An array $A[1 \dots n]$, an integer $1 \leq k \leq n$.

Output: The last k elements of a sorted version of A .

Can be solved in $O(n)$ if $k = O(\frac{n}{\log n})!$

PartialSorting($A[1 \dots n], k$)

BuildHeap(A)

for i from 1 to k :

 ExtractMax()

PartialSorting($A[1 \dots n], k$)

BuildHeap(A)

for i from 1 to k :

 ExtractMax()

Running time: $O(n + k \log n)$

Summary

Heap sort is a time and space efficient comparison-based algorithm: has running time $O(n \log n)$, uses no additional space.

Outline

- 1 Binary Trees
- 2 Basic Operations
- 3 Complete Binary Trees
- 4 Pseudocode
- 5 Heap Sort
- 6 Final Remarks

0-based Arrays

Parent(i)

return $\lfloor \frac{i-1}{2} \rfloor$

LeftChild(i)

return $2i + 1$

RightChild(i)

return $2i + 2$

Binary Min-Heap

Definition

Binary **min**-heap is a binary tree (each node has zero, one, or two children) where the value of each node is **at most** the values of its children.

Can be implemented similarly.

d -ary Heap

- In a d -ary heap nodes on all levels except for possibly the last one have exactly d children.

d -ary Heap

- In a d -ary heap nodes on all levels except for possibly the last one have exactly d children.
- The height of such a tree is about $\log_d n$.

d -ary Heap

- In a d -ary heap nodes on all levels except for possibly the last one have exactly d children.
- The height of such a tree is about $\log_d n$.
- The running time of SiftUp is $O(\log_d n)$.

d -ary Heap

- In a d -ary heap nodes on all levels except for possibly the last one have exactly d children.
- The height of such a tree is about $\log_d n$.
- The running time of SiftUp is $O(\log_d n)$.
- The running time of SiftDown is $O(d \log_d n)$: on each level, we find the largest value among d children.

Summary

- Priority queue supports two main operations: Insert and ExtractMax.

Summary

- Priority queue supports two main operations: Insert and ExtractMax.
- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$).

Summary

- Priority queue supports two main operations: Insert and ExtractMax.
- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$).
- Binary heap gives an implementation where both operations take $O(\log n)$ time.

Summary

- Priority queue supports two main operations: Insert and ExtractMax.
- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$).
- Binary heap gives an implementation where both operations take $O(\log n)$ time.
- Can be made also space efficient.