

Basic Data Structures: Dynamic Arrays and Amortized Analysis

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Data Structures
Data Structures and Algorithms

Outline

- 1 Dynamic Arrays
- 2 Amortized Analysis—Aggregate Method
- 3 Amortized Analysis—Banker's Method
- 4 Amortized Analysis—Physicist's Method

Problem: static arrays are static!

```
int my_array[100];
```

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Semi-solution: dynamically-allocated arrays:

```
int *my_array = new int[size];
```

Problem: might not know max size when allocating an array

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All problems in computer science can be solved by another level of indirection.

Solution: *dynamic arrays* (also known as *resizable arrays*)

Idea: store a pointer to a dynamically allocated array, and replace it with a newly-allocated array as needed.

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

*must be constant time

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- $\text{Remove}(i)$: Removes element at location i
- $\text{Size}()$: the number of elements

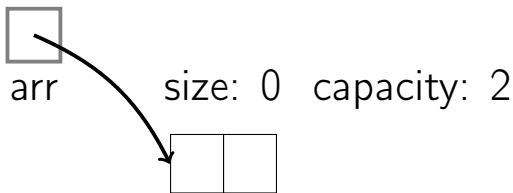
*must be constant time

Implementation

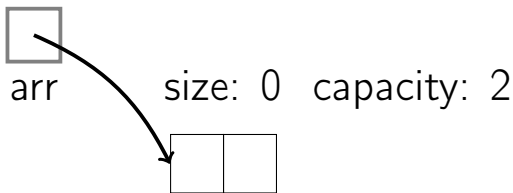
Store:

- `arr`: dynamically-allocated array
- `capacity`: size of the dynamically-allocated array
- `size`: number of elements currently in the array

Dynamic Array Resizing

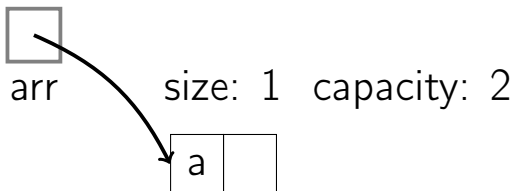


Dynamic Array Resizing



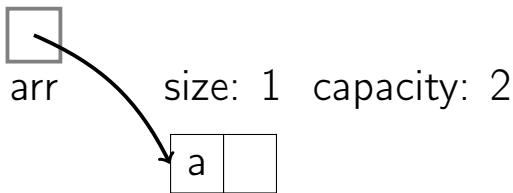
PushBack(a)

Dynamic Array Resizing

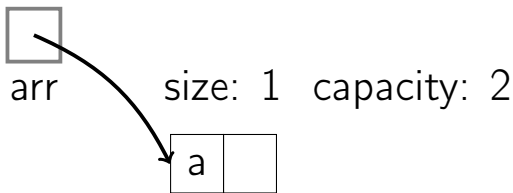


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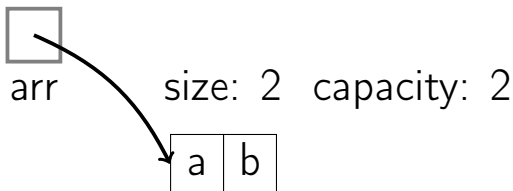


Dynamic Array Resizing



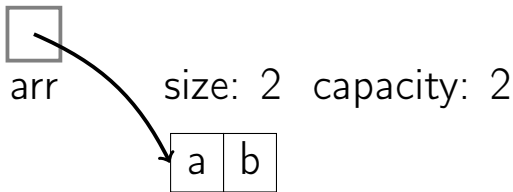
PushBack(b)

Dynamic Array Resizing

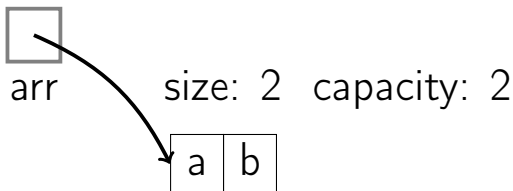


`PushBack(b)`

Dynamic Array Resizing

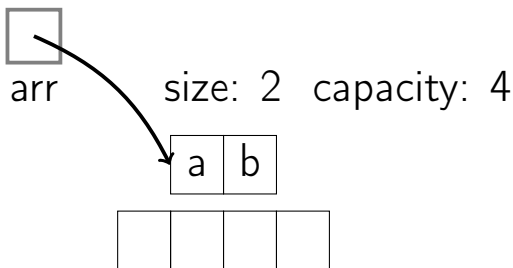


Dynamic Array Resizing



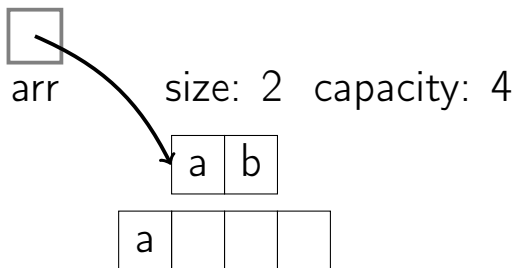
PushBack(c)

Dynamic Array Resizing



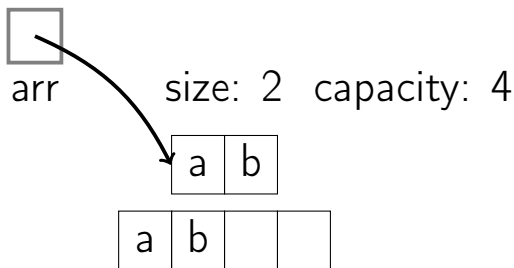
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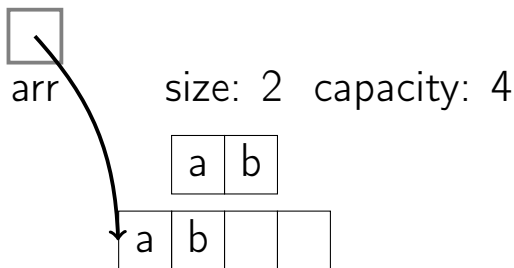
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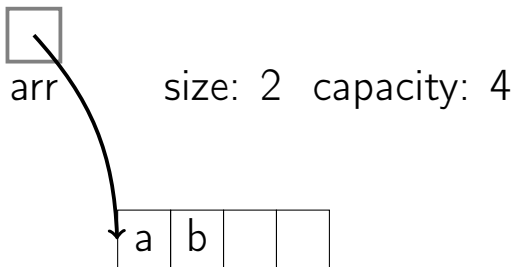
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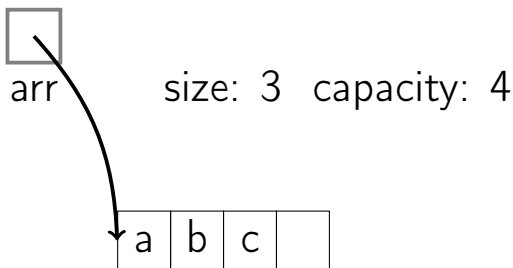
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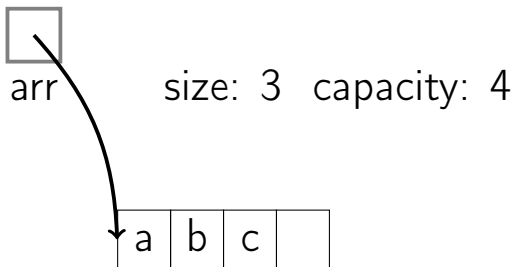
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Dynamic Array Resizing

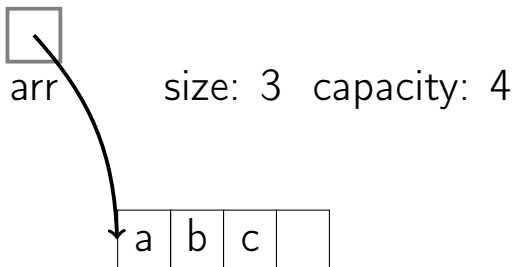


PushBack(c)

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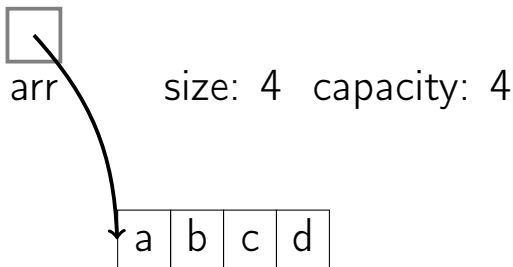


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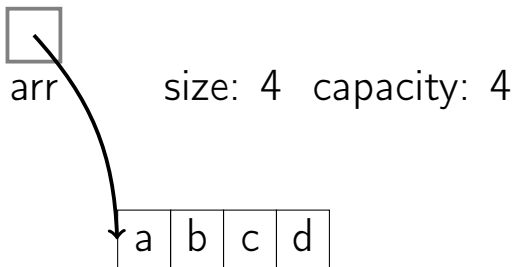
PushBack(d)

Dynamic Array Resizing



`PushBack(d)`

Dynamic Array Resizing

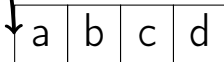


Dynamic Array Resizing



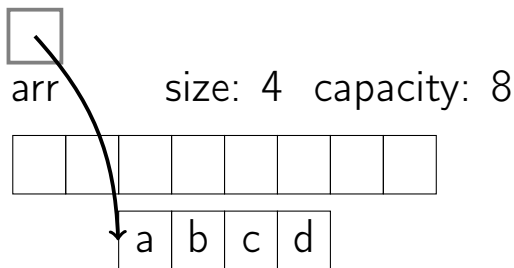
arr

size: 4 capacity: 4



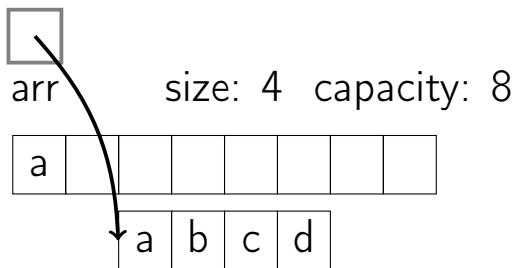
PushBack(e)

Dynamic Array Resizing



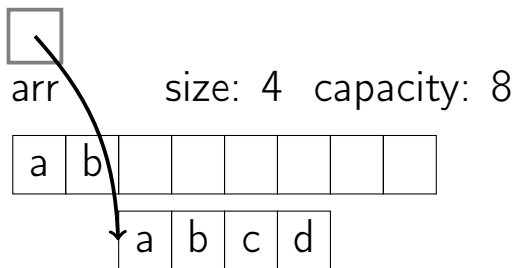
PushBack(e)

Dynamic Array Resizing



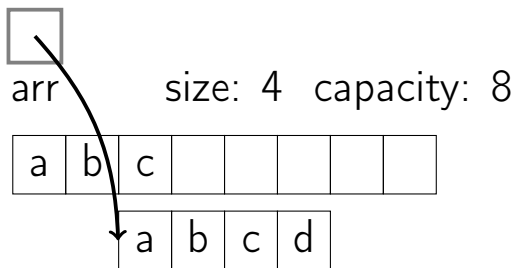
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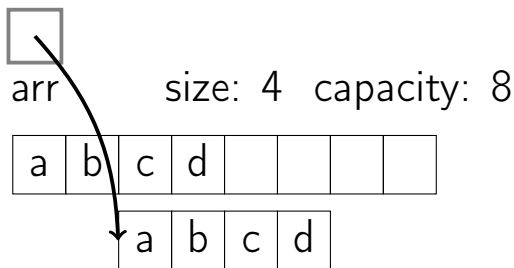
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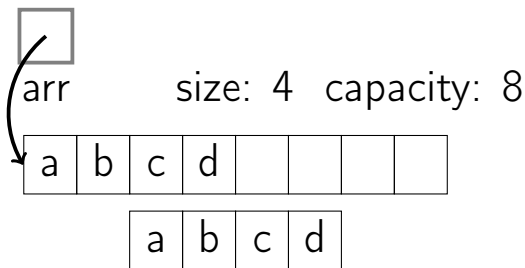
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Dynamic Array Resizing



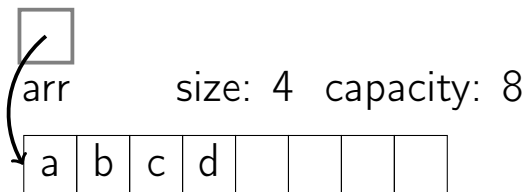
PushBack(e)

Dynamic Array Resizing



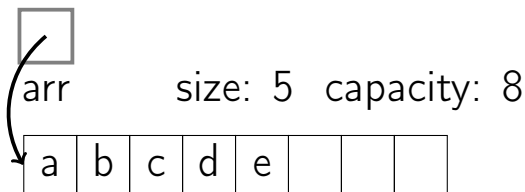
PushBack(e)

Dynamic Array Resizing



PushBack(e)

Dynamic Array Resizing



`PushBack(e)`

Get(*i*)

```
if  $i < 0$  or  $i \geq size$ :
```

```
    ERROR: index out of range
```

```
return arr[i]
```

Set(*i*, *val*)

if $i < 0$ or $i \geq \text{size}$:

 ERROR: index out of range

$\text{arr}[i] = \text{val}$

PushBack(*val*)

```
if size = capacity:  
    allocate new_arr[ $2 \times \textit{capacity}$ ]  
    for i from 0 to size - 1:  
        new_arr[i]  $\leftarrow$  arr[i]  
    free arr  
    arr  $\leftarrow$  new_arr; capacity  $\leftarrow 2 \times \textit{capacity}$   
arr[size]  $\leftarrow$  val  
size  $\leftarrow$  size + 1
```

Remove(*i*)

if $i < 0$ or $i \geq \text{size}$:

 ERROR: index out of range

for j from i to $\text{size} - 2$:

$\text{arr}[j] \leftarrow \text{arr}[j + 1]$

$\text{size} \leftarrow \text{size} - 1$

```
Size()
```

```
return size
```

Common Implementations

- C++: `vector`
- Java: `ArrayList`
- Python: `list` (the only kind of array)

Runtimes

Get(i) | $O(1)$

Runtimes

Get(i)	$O(1)$
Set(i, val)	$O(1)$

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PushBack(val)	$O(n)$

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Remove(i)	$O(n)$
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- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted—at most half.

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- ① Dynamic Arrays
- ② Amortized Analysis—Aggregate Method
- ③ Amortized Analysis—Banker's Method
- ④ Amortized Analysis—Physicist's Method

Sometimes, looking at the individual worst-case may be too severe. We may want to know the total worst-case cost for a sequence of operations.

Dynamic Array

We only resize every so often.

Many $O(1)$ operations are followed by an $O(n)$ operations.

What is the total cost of inserting many elements?

Definition

Amortized cost: Given a sequence of n operations, the amortized cost is:

$$\frac{\text{Cost}(n \text{ operations})}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

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Let $c_i =$ cost of i 'th insertion.

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Let $c_i = \text{cost of } i\text{'th insertion.}$

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \end{cases}$$

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Dynamic array: n calls to PushBack

Let $c_i =$ cost of i 'th insertion.

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$$\frac{\sum_{i=1}^n c_i}{n}$$

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$$\frac{\sum_{i=1}^n c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n}$$

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- Use the tokens to pay for expensive operations.

Like an amortizing loan.

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Charge 3 for each insertion: 1 token is the raw cost for insertion.

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- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.

Banker's Method

Dynamic array: n calls to PushBack

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- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.
- Place one token on the newly-inserted element, and one token $\frac{\text{capacity}}{2}$ elements prior.

Dynamic Array Resizing



arr

size: 0 capacity: 0

Dynamic Array Resizing



arr

size: 0 capacity: 0

PushBack(a)

Dynamic Array Resizing



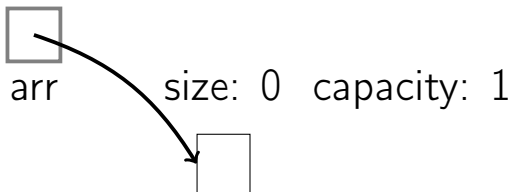
arr

size: 0 capacity: 1



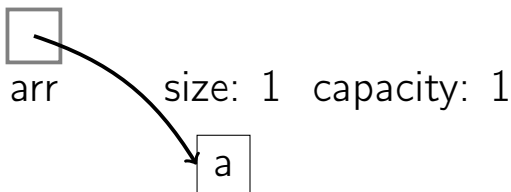
PushBack(a)

Dynamic Array Resizing



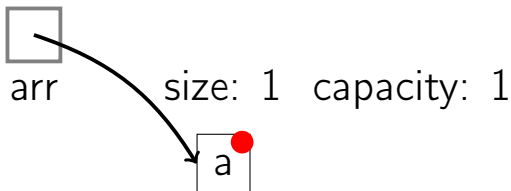
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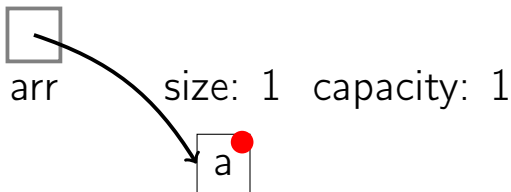
PushBack(a)

Dynamic Array Resizing

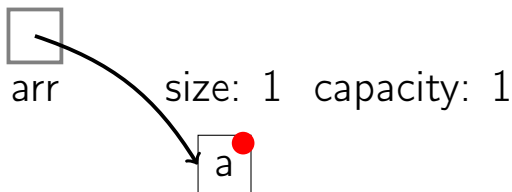


PushBack(a)

Dynamic Array Resizing

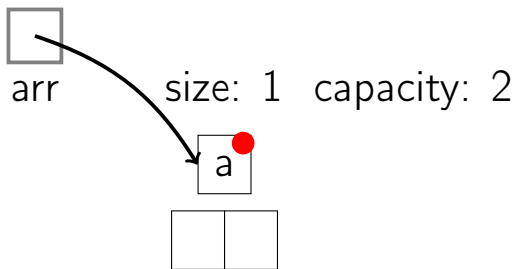


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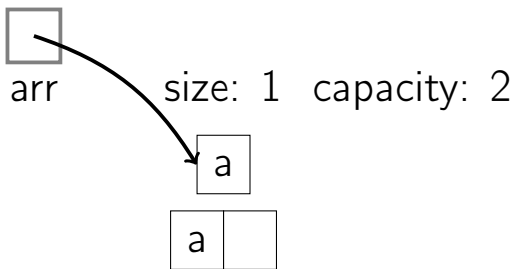
PushBack(b)

Dynamic Array Resizing



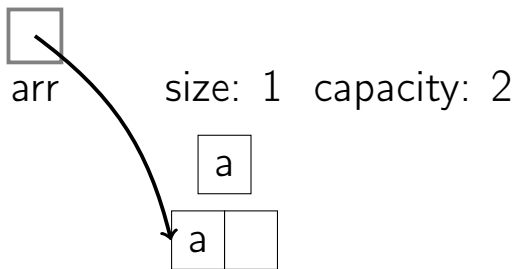
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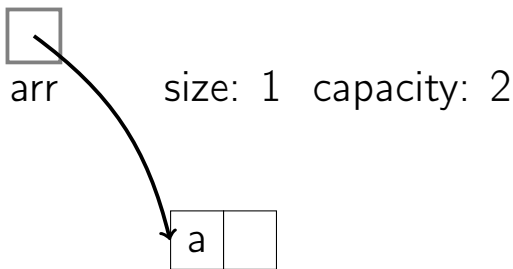
PushBack(b)

Dynamic Array Resizing



`PushBack(b)`

Dynamic Array Resizing



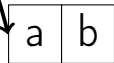
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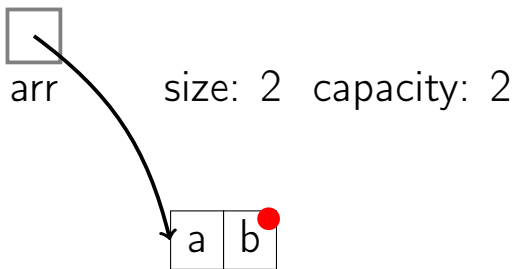
arr

size: 2 capacity: 2



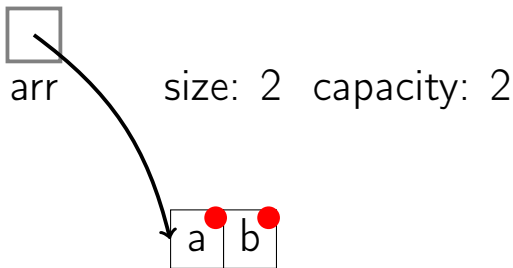
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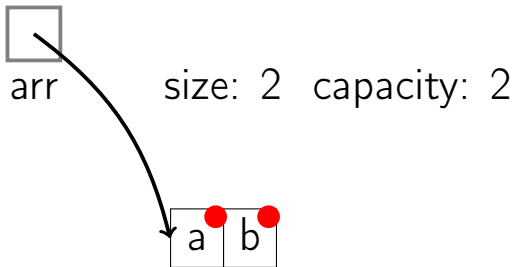
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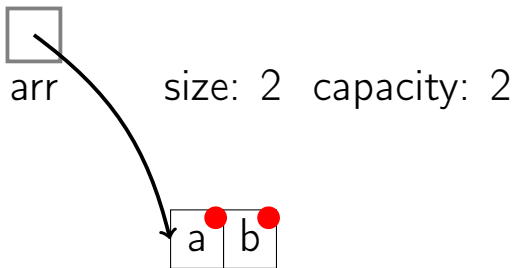


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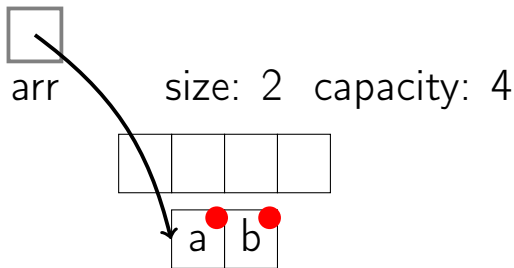


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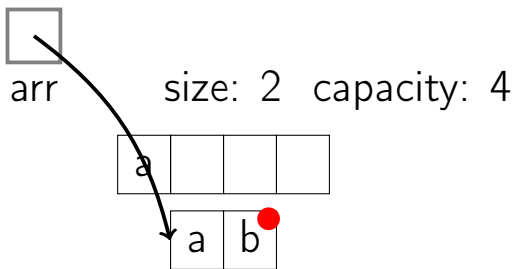
PushBack(c)

Dynamic Array Resizing



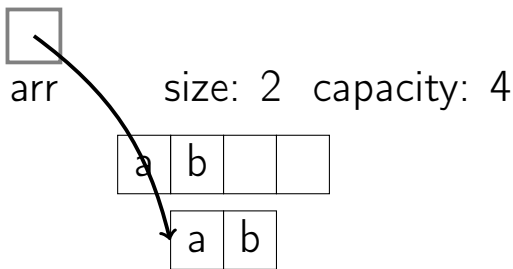
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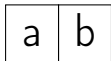
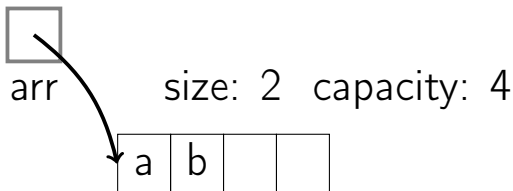
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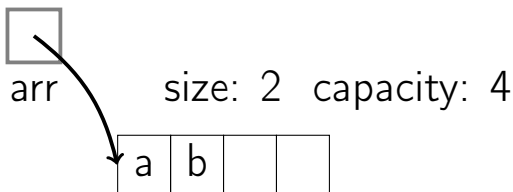
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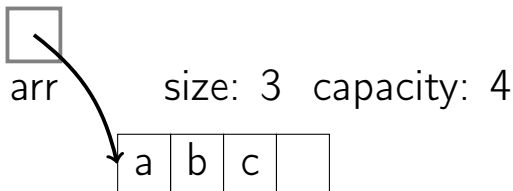
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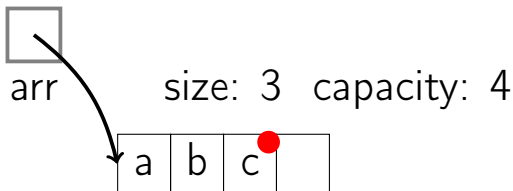
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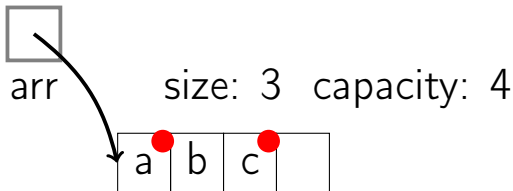
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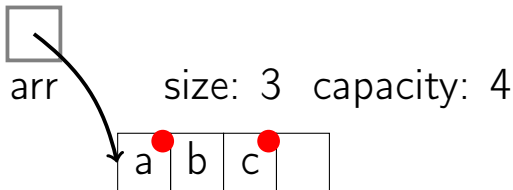
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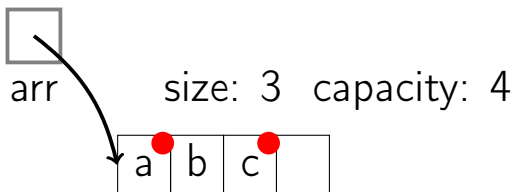


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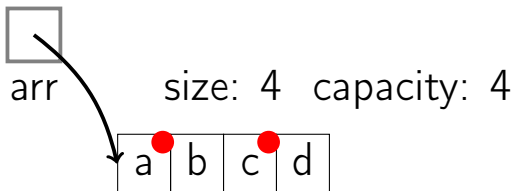


Dynamic Array Resizing



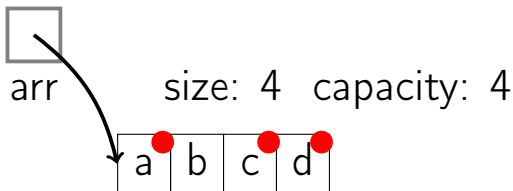
`PushBack(d)`

Dynamic Array Resizing



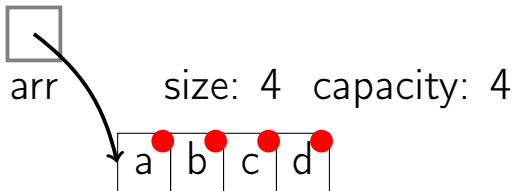
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Dynamic Array Resizing



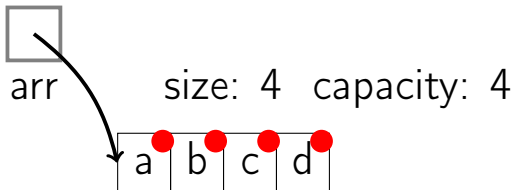
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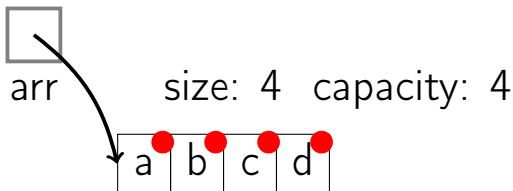


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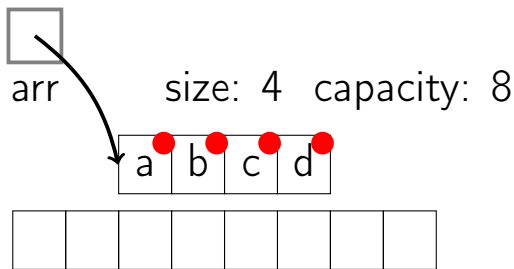


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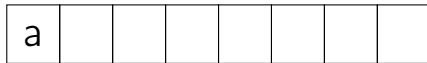
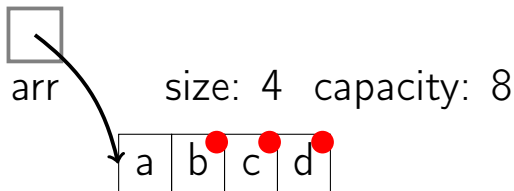
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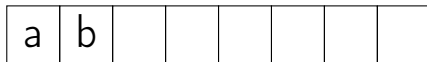
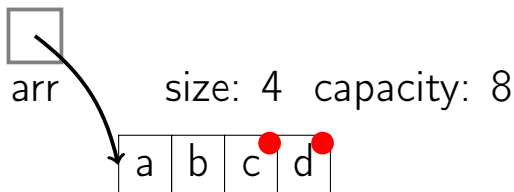
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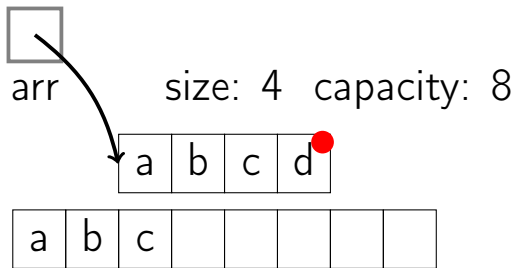
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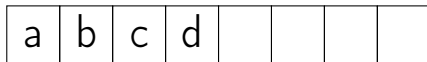
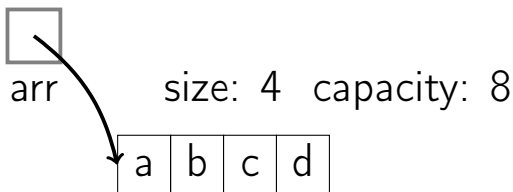
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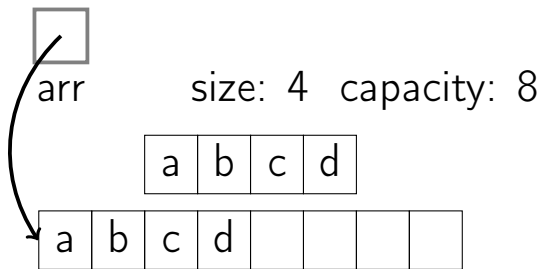
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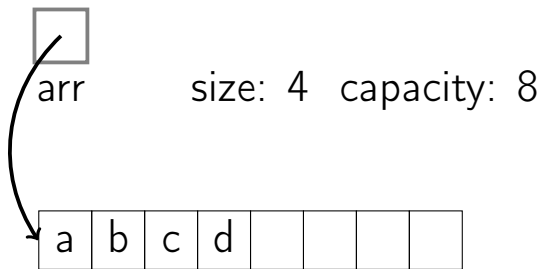
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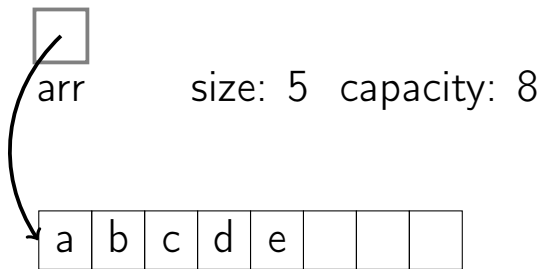
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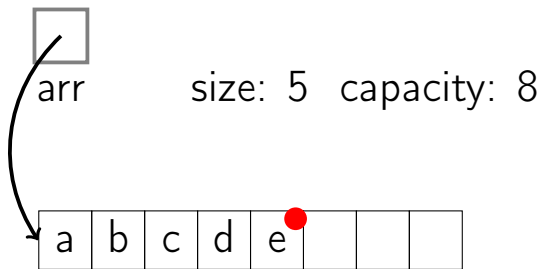
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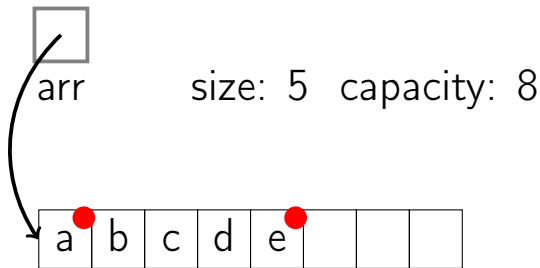
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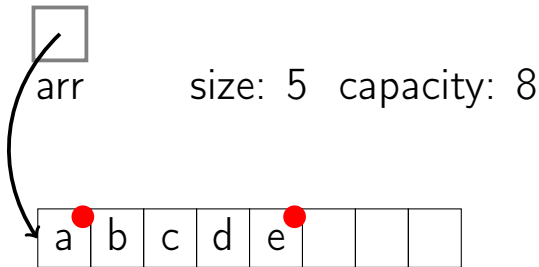
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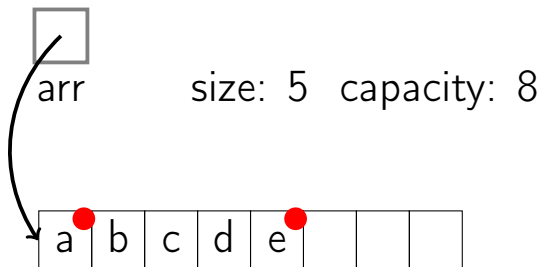


PushBack(e)

Dynamic Array Resizing



Dynamic Array Resizing



$O(1)$ amortized cost for each PushBack

Banker's Method

Dynamic array: n calls to PushBack

Charge 3 for each insertion. 1 coin is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the coin that's present on each element that needs to move.
- Place one coin on the newly-inserted element, and one coin $\frac{\text{capacity}}{2}$ elements prior.

Outline

- ① Dynamic Arrays
- ② Amortized Analysis—Aggregate Method
- ③ Amortized Analysis—Banker's Method
- ④ Amortized Analysis—Physicist's Method

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$$\begin{aligned} & \sum_{i=1}^n (c_i + \Phi(h_i) - \Phi(h_{i-1})) \\ &= c_1 + \Phi(h_1) - \Phi(h_0) + \\ & \quad c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\ & \quad c_n + \Phi(h_n) - \Phi(h_{n-1}) \end{aligned}$$

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Dynamic array: n calls to PushBack

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Dynamic array: n calls to PushBack

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- $\Phi(h_i) = 2 \times \textit{size} - \textit{capacity} > 0$
(since $\textit{size} > \frac{\textit{capacity}}{2}$)

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

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$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= (\text{size}_i) + 2 - k \\ &= (k+1) + 2 - k \\ &= 3 \end{aligned}$$

Alternatives to Doubling the Array Size

We could use some different growth factor (1.5, 2.5, etc.).

Could we use a constant amount?

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

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$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \end{cases}$$

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- Three ways to do analysis:
 - Aggregate method (brute-force sum)
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 - Physicist's method (potential function, Φ)
- Nothing changes in the code: runtime analysis only.