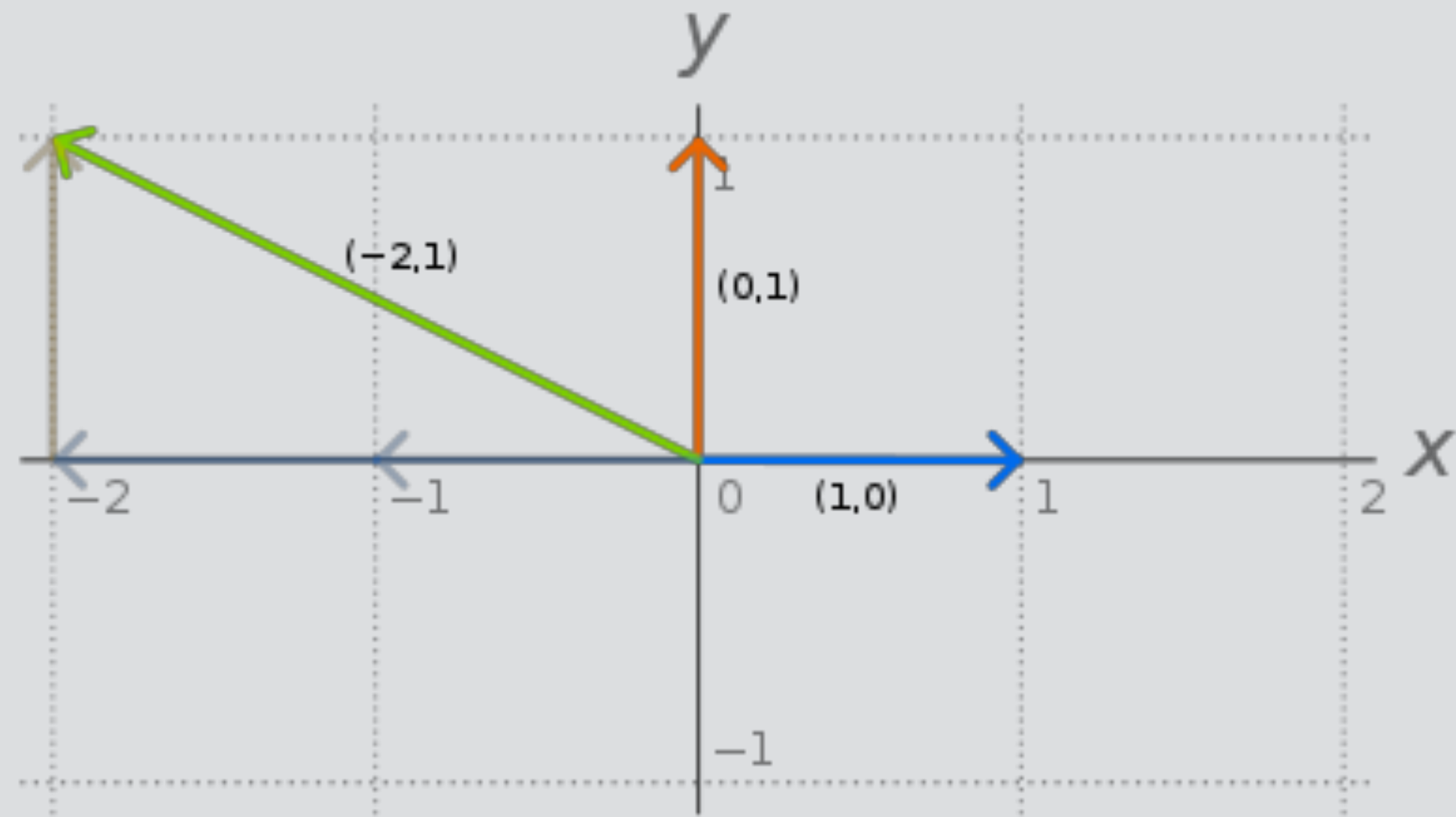


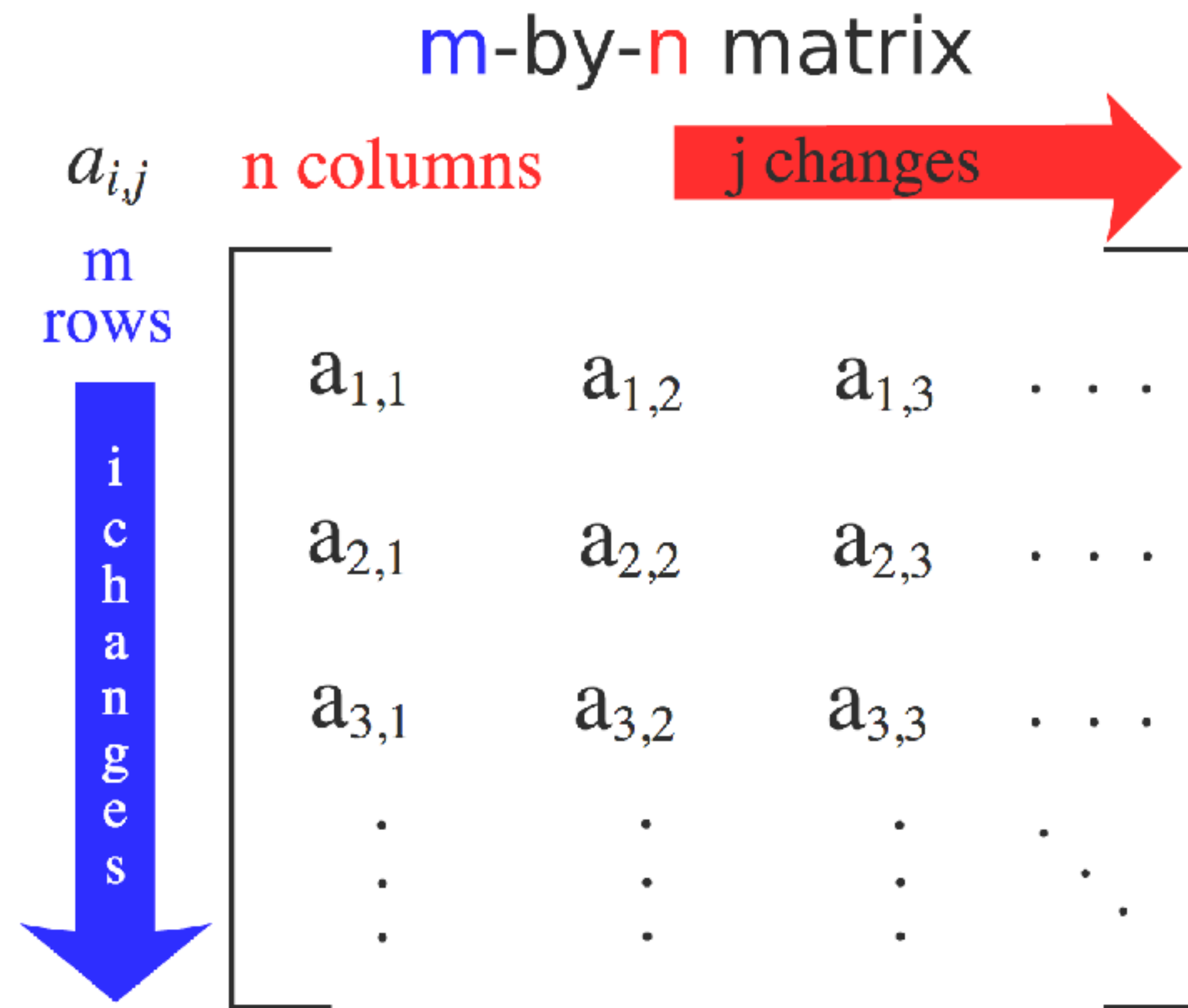
# Linear Algebra Foundations Part 1



CS 3113

**Matrix refresher.**

# A matrix.



A 2x3 matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{bmatrix}$$

A 3x3 matrix.

$$\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

**Matrix operations.**

**Matrix addition.**

To add two matrices, add their corresponding entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} + \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A+J & B+K & C+L \\ D+M & E+N & F+O \\ G+P & H+Q & I+R \end{bmatrix}$$



**Matrix subtraction.**

To subtract two matrices, subtract their corresponding entries.

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} - \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} = \begin{bmatrix} A-J & B-K & C-L \\ D-M & E-N & F-O \\ G-P & H-Q & I-R \end{bmatrix}$$

**Matrix addition and subtraction can only  
happen  
with matrices that are the same size!**

**Transpose of a matrix.**

Transpose of a matrix is a matrix whose columns are the rows of the original matrix (and its rows are the columns).

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$$

**M**

$$\begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

**M<sup>T</sup>**

**Matrix/scalar multiplication.**

Multiply each entry of the matrix by the scalar.

$$S \times \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} S \times A & S \times B & S \times C \\ S \times D & S \times E & S \times F \\ S \times G & S \times H & S \times I \end{bmatrix}$$

**Matrix/matrix multiplication.**



You can only multiply two matrices  
if the number of columns of the first matrix  
equals the number of rows of the second.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = ?$$

It results in a matrix that is number of rows of first matrix by number of columns of second matrix.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

For each row, find dot product with each column.

The diagram illustrates the dot product of a row and a column. On the left, a 2x3 matrix is shown with elements A, B, C in the first row and D, E, F in the second row. A red arrow points from the first row to a red box containing A, B, and C. On the right, a 3x2 matrix is shown with elements J, M, P in the first column and K, N, Q in the second column. A red arrow points from the first column to a red box containing J, M, and P. An equals sign follows the matrices.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

The diagram shows the calculation of the dot product for the first row and first column. A large red bracket on the left contains the expression  $A \times J + B \times M + C \times P$ . A large red bracket on the right is also shown.

$$\left[ A \times J + B \times M + C \times P \right]$$

For each row, find dot product with each column.

The diagram illustrates the dot product of a row and a column. On the left, a 2x3 matrix is shown with elements A, B, C in the first row and D, E, F in the second row. A red arrow points to the first row, which is highlighted with a red background. To the right of this matrix is a 3x2 matrix with elements J, M, P in the first column and K, N, Q in the second column. A red arrow points to the second column, which is highlighted with a red background. An equals sign follows the matrices.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

The diagram shows the resulting row vector from the dot products. It is a 1x2 matrix with two entries. The first entry is the dot product of the first row and first column:  $A \times J + B \times M + C \times P$ . The second entry is the dot product of the first row and second column:  $A \times K + B \times N + C \times Q$ .

$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \end{bmatrix}$$

For each row, find dot product with each column.

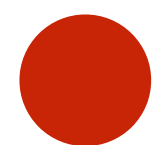
$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K \\ M & N \\ P & Q \end{bmatrix} =$$

$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P & \end{bmatrix}$$

For each row, find dot product with each column.

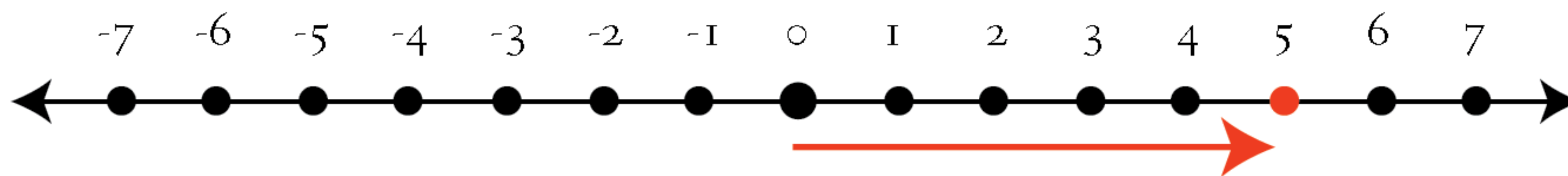
$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} J & K & L \\ M & N & O \\ P & Q & R \end{bmatrix} =$$

$$\begin{bmatrix} A \times J + B \times M + C \times P & A \times K + B \times N + C \times Q \\ D \times J + E \times M + F \times P & D \times K + E \times N + F \times Q \end{bmatrix}$$



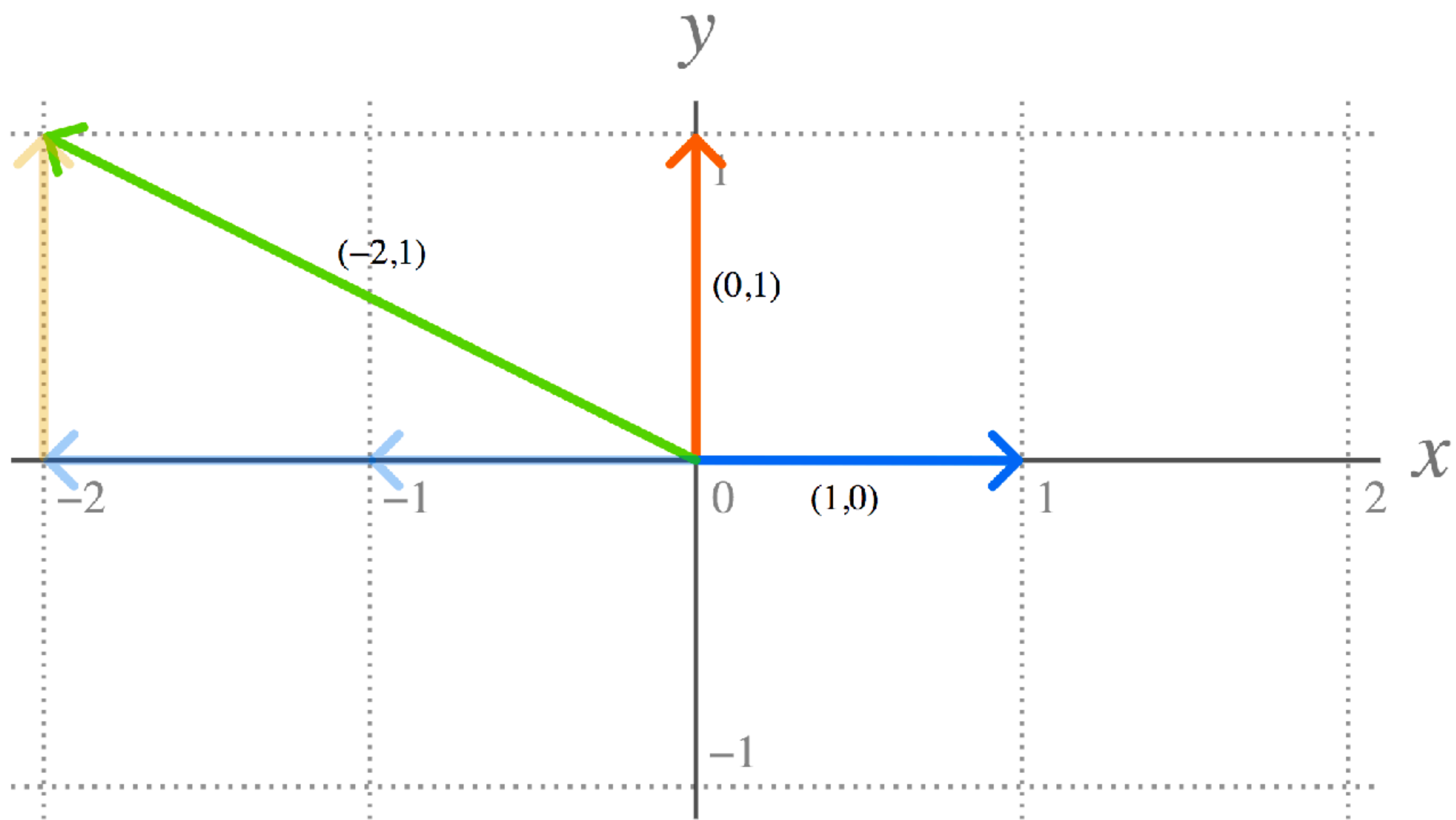
# Vectors



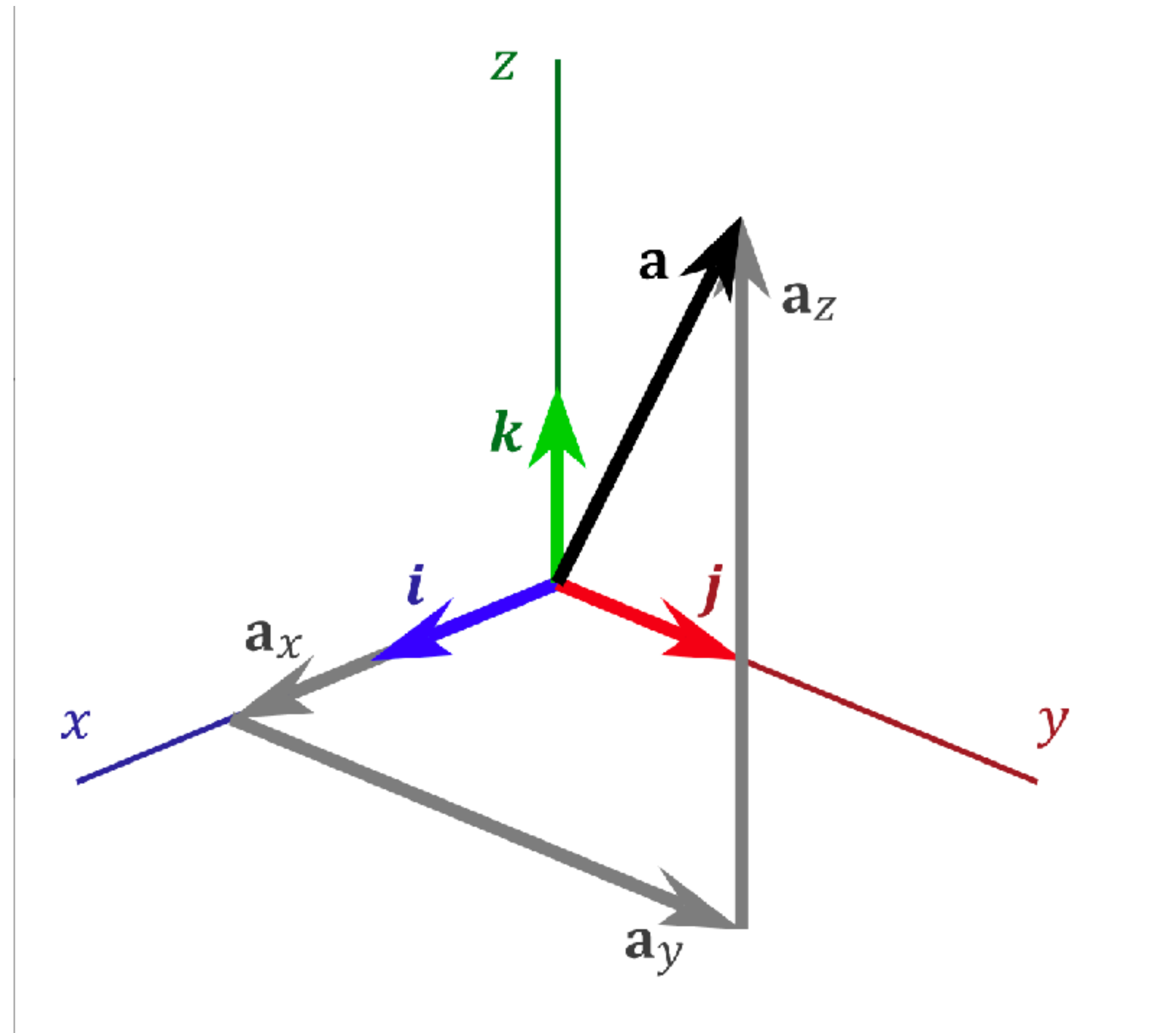


# Basis Vectors

# Vectors in $N$ dimensions



# Basis vectors.



$$\vec{v} = \sum_i c_i \vec{b}_i.$$

$$\vec{v} = \sum_i c_i \vec{b}_i = \left[ \begin{array}{ccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array} \right] \left[ \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right].$$

# Basis vectors as matrices

# Transformation matrices



# Linear transformations

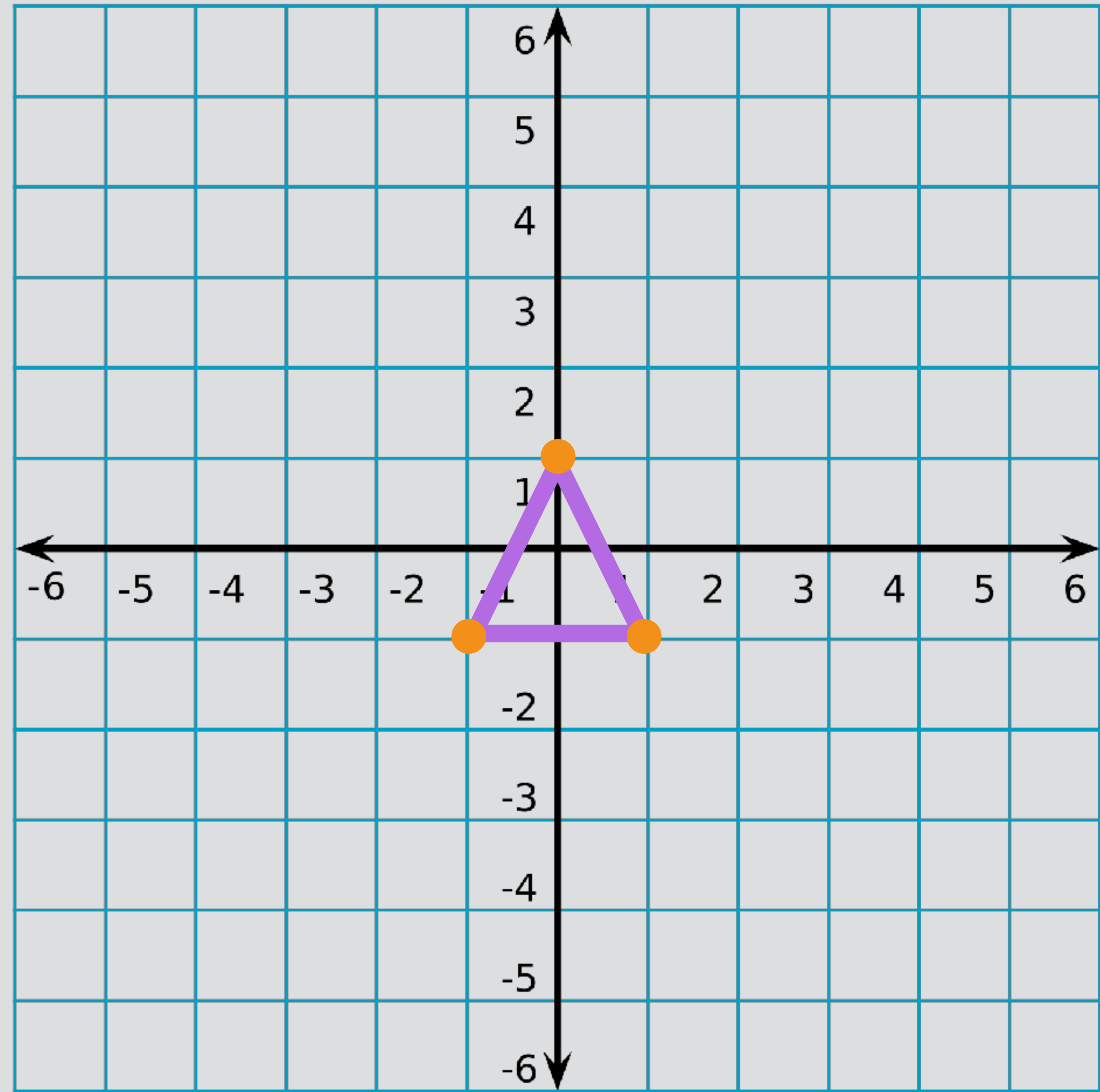
Scale

# Scale

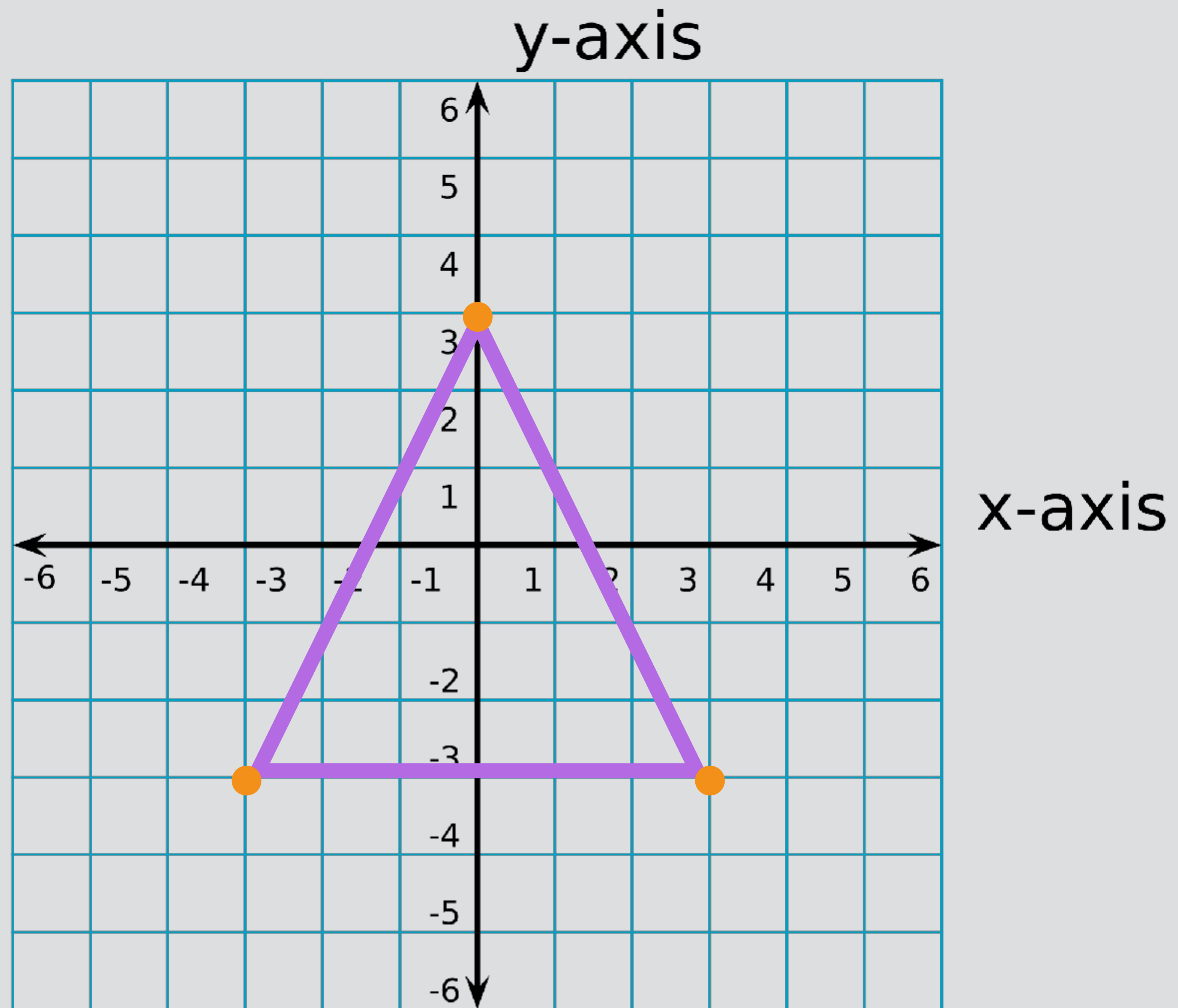
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} AX + BY \\ CX + DY \end{bmatrix}$$

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} sxX + 0Y \\ 0X + syY \end{bmatrix}$$

y-axis



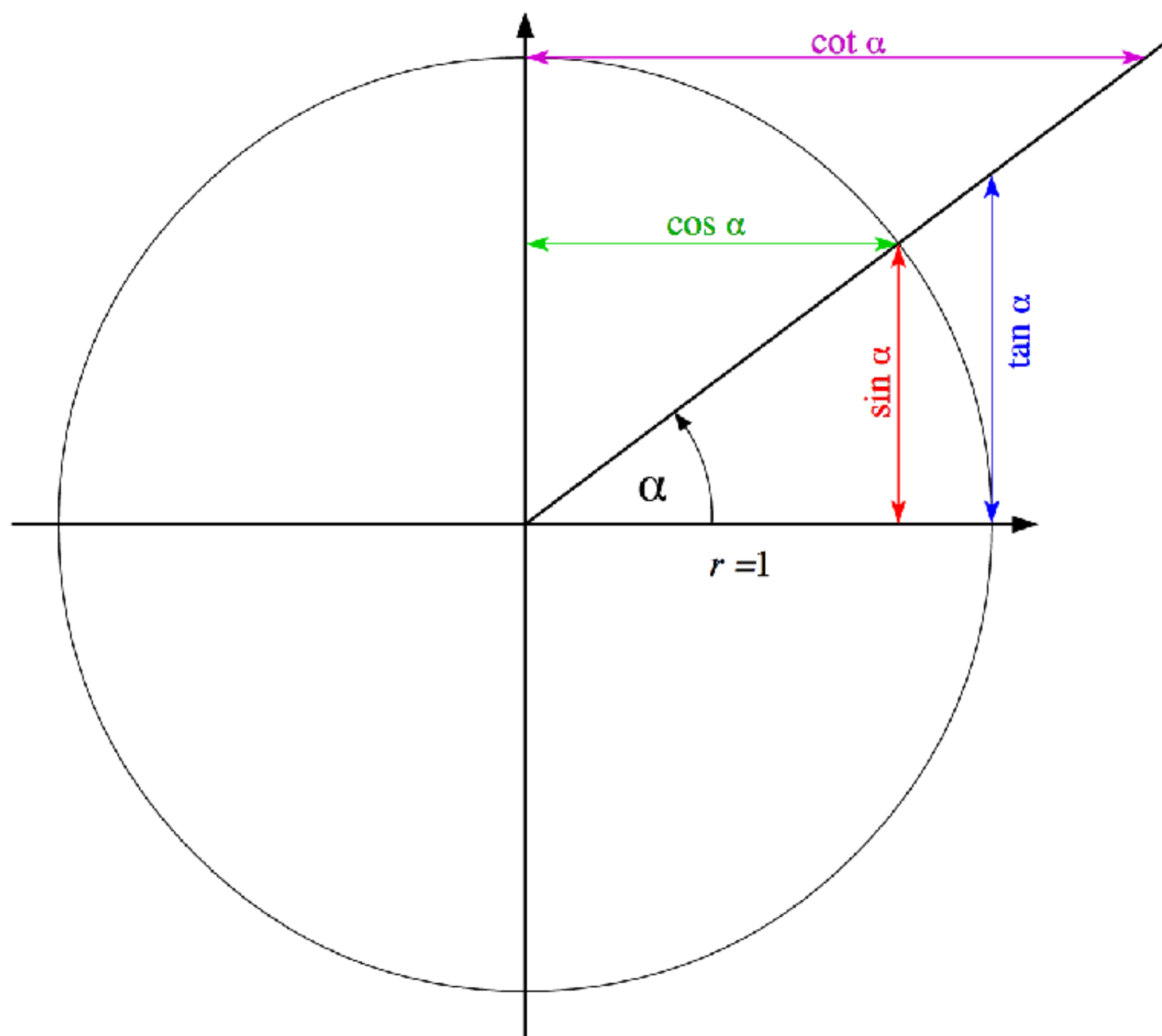
x-axis



# Rotation

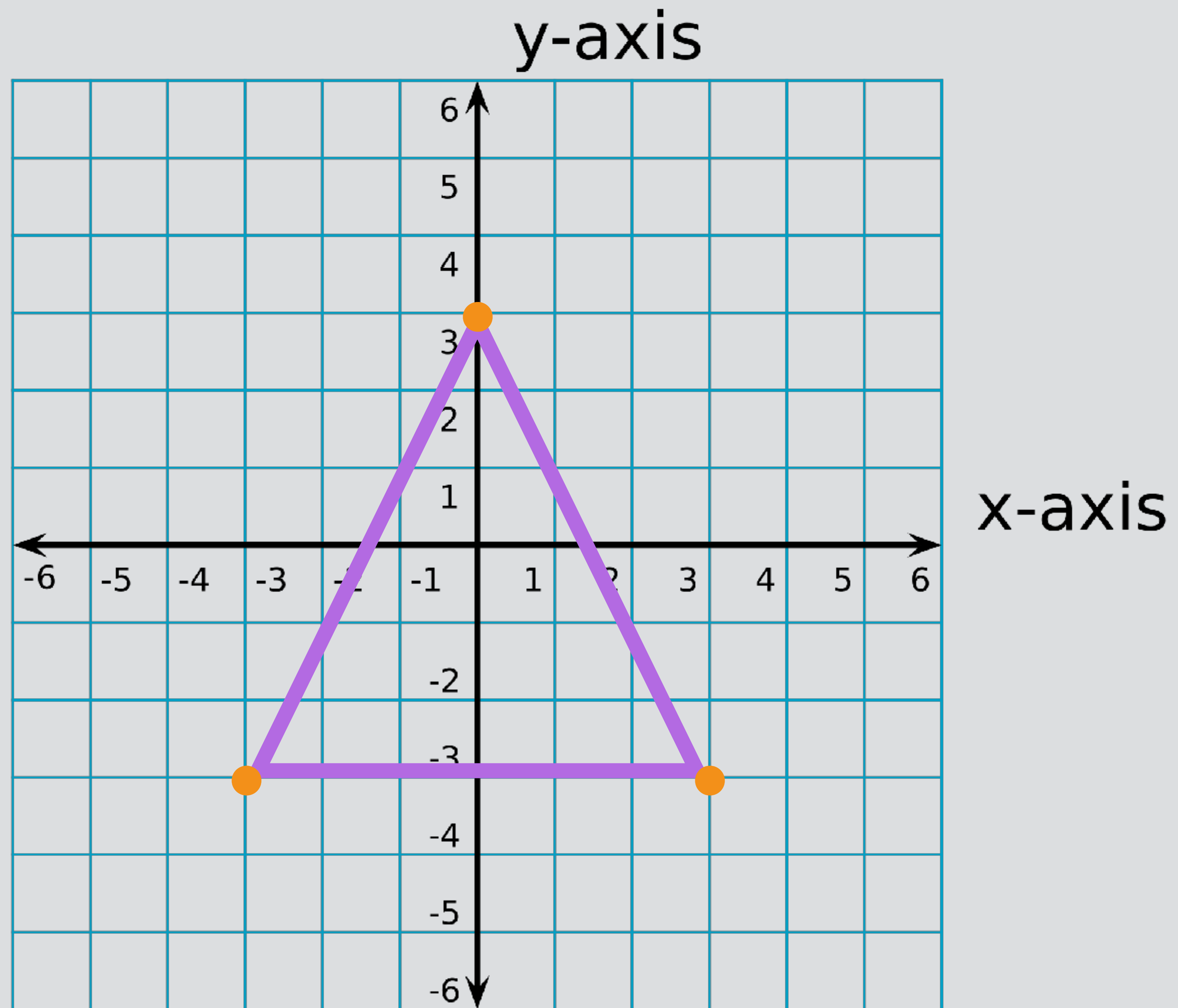
# Rotation

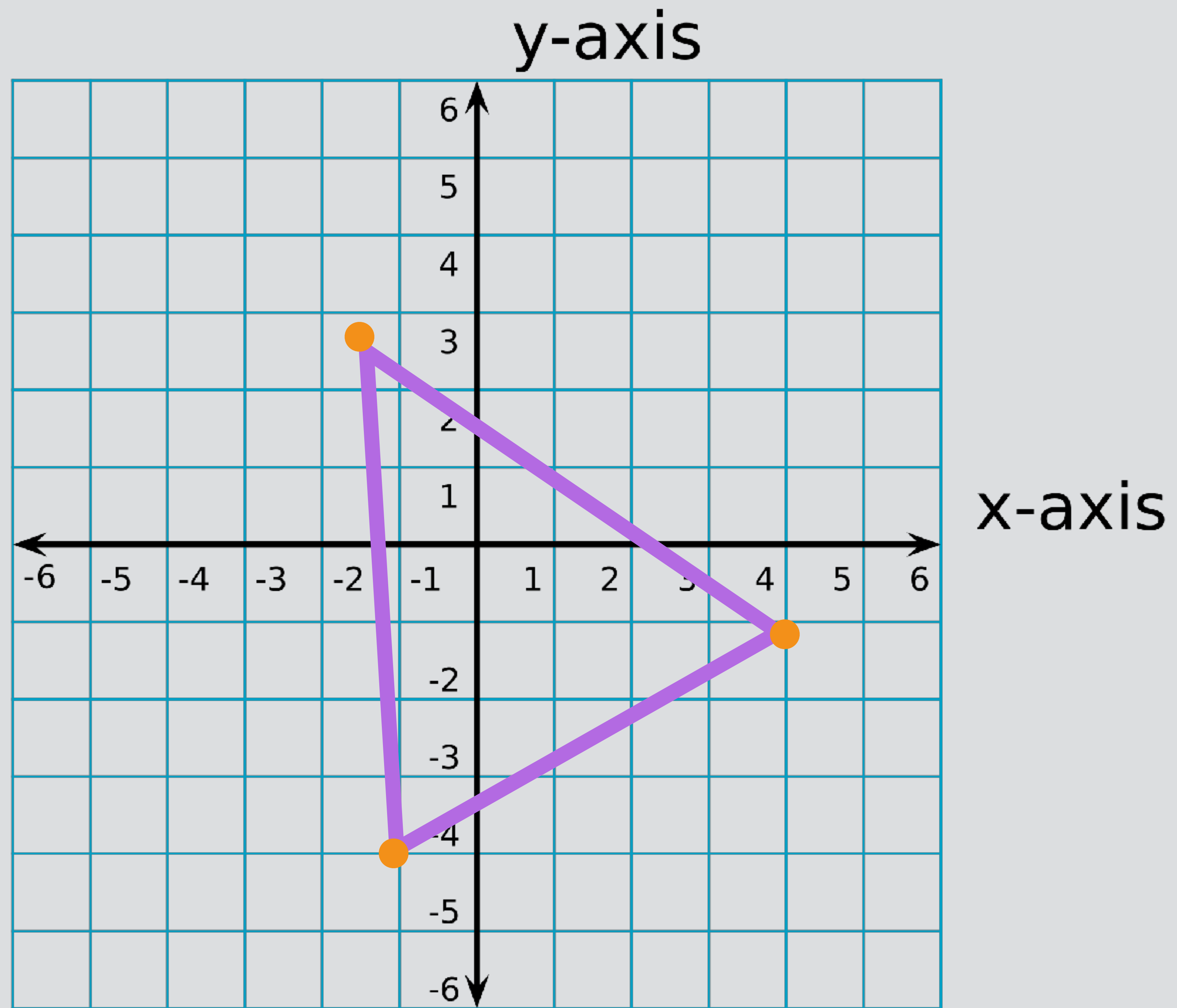
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta X + -\sin\theta Y \\ \sin\theta X + \cos\theta Y \end{bmatrix}$$



$$\begin{bmatrix} \cos \theta X + -\sin \theta Y \\ \sin \theta X + \cos \theta Y \end{bmatrix}$$







# Homogeneous coordinates

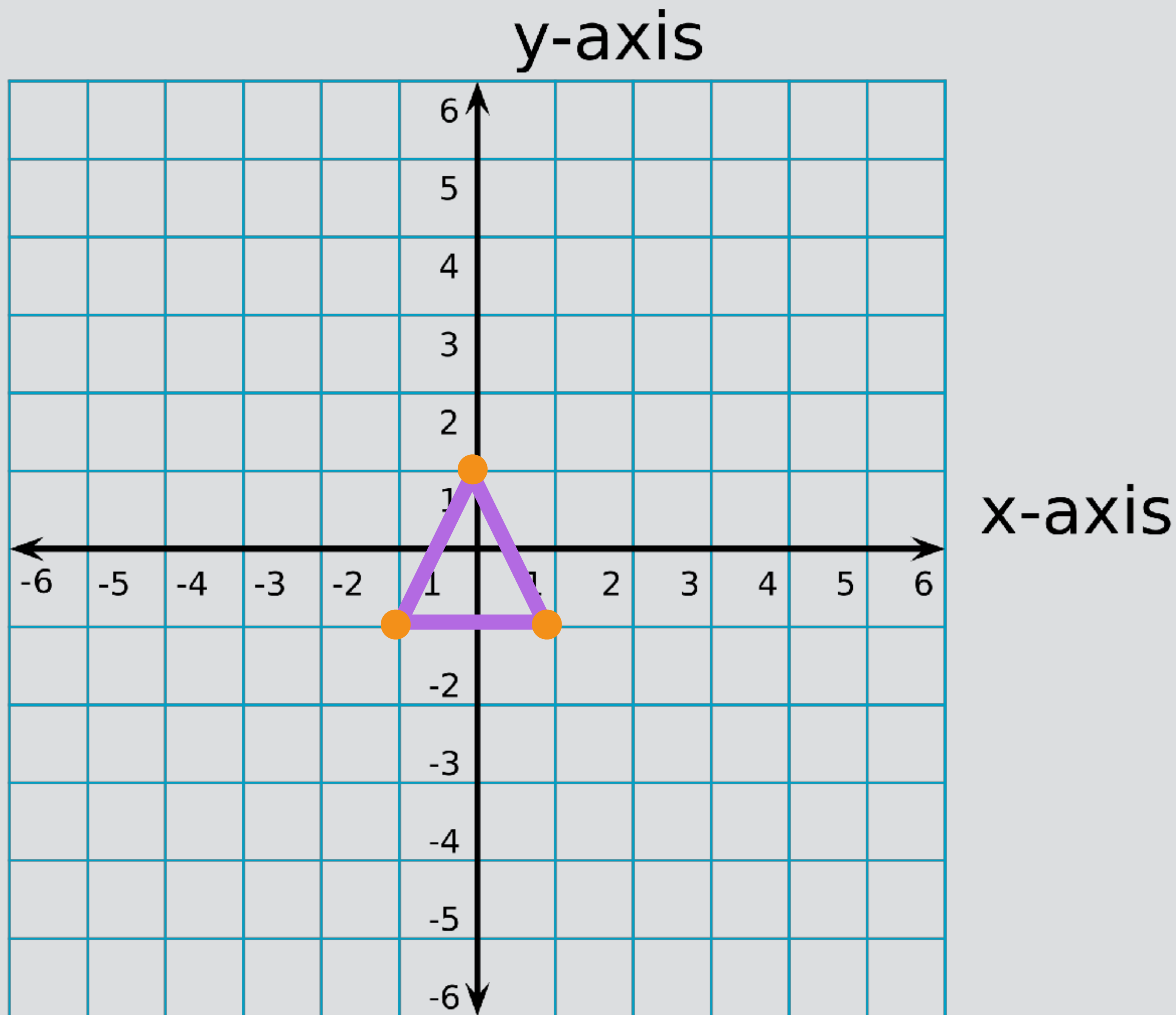
# Affine transformations

**An affine transformation matrix combines a linear transformation matrix with a translation using homogeneous coordinates.**

## 2D Affine Identity

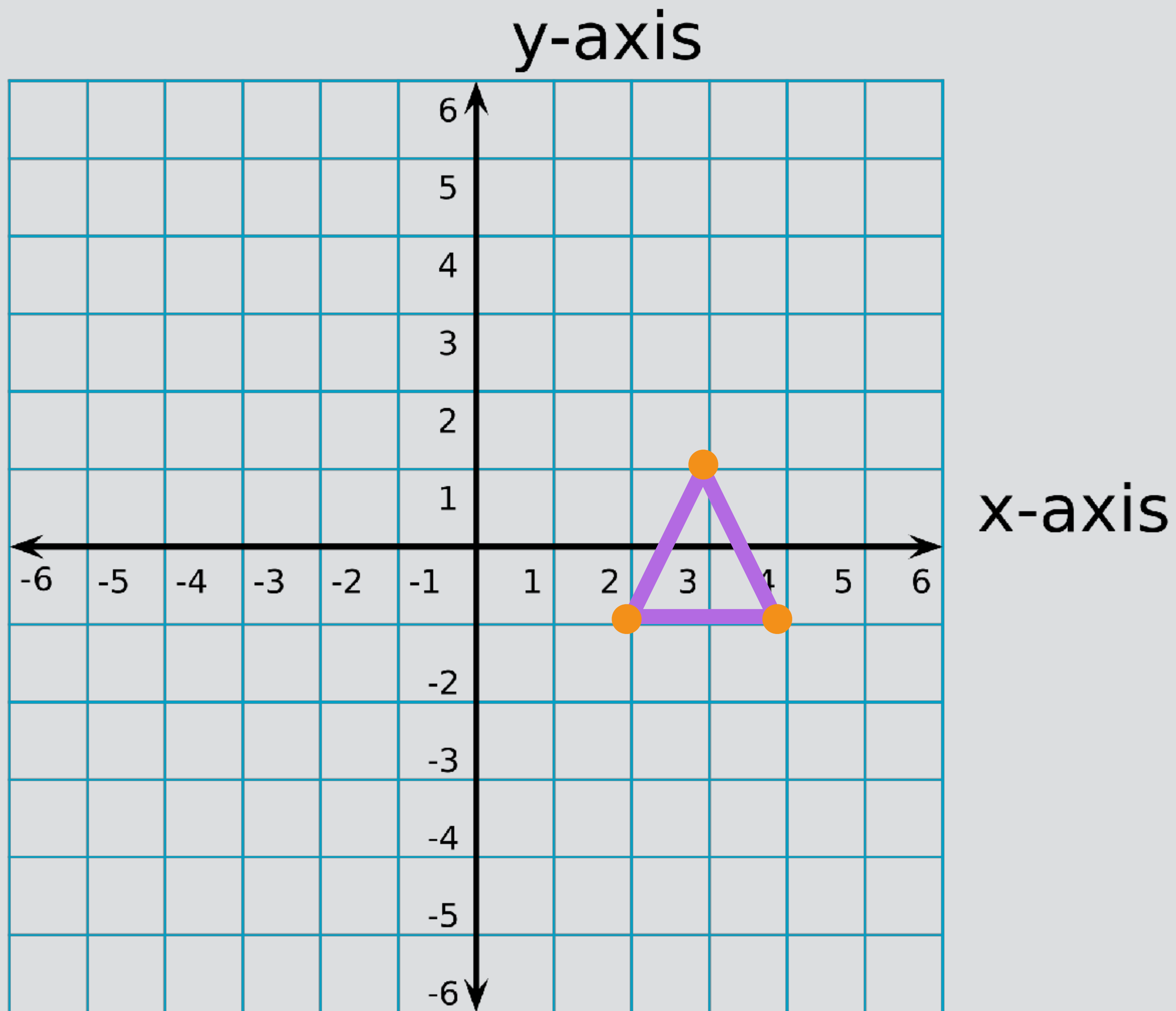
Linear part

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$



## 2D Translation

$$\begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$



**The 3D affine transformation matrix.**



**Identity matrix.**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Scale matrix.

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Z-axis **rotation** matrix.

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Translate matrix.

$$\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Building a single matrix.**