UNIVERSITY OF LINCOLN

SCHOOL OF MATHEMATICS AND PHYSICS

Analysis of Shapes in Nature Using Advanced Mathematical and Computing Techniques

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April 25, 2024

Scientific Report in the 3rd Year Module MTH3009 Mathematics Project

Abstract

Placeholder abstract text

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1 Introduction

2 Theoretical Background

2.1 The Elliptic Fourier Transform

The Fourier transform is a way of finding constituent frequencies within a function on the real domain. The Fourier transform extends the concept of the Fourier series (which openerates on a bounded interval) to the real domain. That is to say, the Fourier Series operations on a periodic function with period P and decomposes it into a series of sinusoidal waves, while the Fourier transform does the same but for any complex valued function $f(x)$.

[Get references from book from differential equations module]

The elliptic Fourier transform extends this concept to a finite series of points on the (x, y) plane. This is useful for practical applications in computing because one of the most typical ways to model a two-dimensional curve in a computer program is just that, a finite series of points on the (x, y) plane.

The way that the elliptic Fourier transform works is by taking said series of points and assuming that they are connected by straight lines, and modelling x and y coordinates as periodic functions of a parameter t, such that $x = x(t)$ and $y = y(t)$. Importantly, $x(t)$ and $y(t)$ have the same period T.

2.1.1 The Fourier Series

The Fourier series (1) lets you take a single repeating unit of a periodic function and use this to generate an infinite sum of sinusoidal functions that converge to it. This is useful because trigonmetric functions have mathematical properties that not all functions posess, allowing you to work with them nicely in more situations.

$$s(x) \sim A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{2\pi n x}{P}\right) + B_n \sin\left(\frac{2\pi n x}{P}\right)\right) \tag{1}$$

The $\sim$ symbol indicates that the series doesn't strictly converge in all cases.

The coefficients A_0, A_n and B_n are given by (2).

$$\begin{aligned} A_0 &= \frac{1}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} s(x)\,dx \\ A_n &= \frac{1}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} s(x)\cos\left(\frac{2\pi n x}{P}\right)dx \\ B_n &= \frac{1}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} s(x)\sin\left(\frac{2\pi n x}{P}\right)dx \end{aligned} \tag{2}$$

Here A_0 is a constant, and A_n and B_n are functions of n. In fact, since both the denominator of the fraction and the range of the integral for A_0 are the same, P, A_0 is simply the average around which the function oscillates.

A good intuition for why the Fourier series works is that the integral of the product of a function and a sinusoid is a measure of how well that function *matches* the sinusoid. For example, the sin function perfectly matches itself since $\sin^2 x$ has a positive average, while the sin and cos functions are a perfect anti-match since $\sin x \cos x$ simplifies to $\frac{\sin 2x}{2}$ which averages around 0.

2.1.2 Exponential Form

The concept of the Fourier series can actually be extended to the complex numbers $\mathbb{C}$.

This isn't terribly surprising, since there is a nice connection between the complex numbers and trigonometric functions given by Euler's formula (3).

$$e^{ix} = \cos x + i \sin x \tag{3}$$

Thus if we take A_n and B_n from (2) we can let a new variable

$$c_n = A_n + iB_n$$

and we can get

$$s(x) \sim A_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{2\pi n x}{P}} \tag{4}$$

2.1.3 The Fourier Transform

The transform function ($S(t)$) for frequency f is given by (5).

$$S(t) = \int_{-\infty}^{\infty} s(t) \cdot \mathrm{e}^{-i2\pi f t} dt \tag{5}$$

2.1.4 Elliptical Fourier Analysis

The seminal paper on this topic is [1].

For:

$$\begin{aligned} x(t) &= A_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T} \right] \\ y(t) &= C_0 + \sum_{n=1}^{\infty} \left[c_n \cos \frac{2n\pi t}{T} + d_n \sin \frac{2n\pi t}{T} \right] \end{aligned} \tag{6}$$

This paper gives the coefficients (for a discrete set of points that form a closed con-

tour):

$$
\begin{aligned}
a_n &= \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\left[\cos\frac{2n\pi t_p}{T}-\cos\frac{2n\pi t_{p-1}}{T}\right]\\
b_n &= \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\left[\sin\frac{2n\pi t_p}{T}-\sin\frac{2n\pi t_{p-1}}{T}\right]\\
c_n &= \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\left[\cos\frac{2n\pi t_p}{T}-\cos\frac{2n\pi t_{p-1}}{T}\right]\\
d_n &= \frac{T}{2n^2\pi^2}\sum_{p=1}^{K}\left[\sin\frac{2n\pi t_p}{T}-\sin\frac{2n\pi t_{p-1}}{T}\right]
\end{aligned}
\tag{7}
$$

It also gives the centres:

$$
\begin{aligned}
A_0 &= \frac{1}{T}\sum_{p=1}^{K}\left[\frac{\Delta x_p}{2\Delta t_p}\left(t_p^2-t_{p-1}^2\right)+\xi_p\left(t_p-t_{p-1}\right)\right]\\
C_0 &= \frac{1}{T}\sum_{p=1}^{K}\left[\frac{\Delta y_p}{2\Delta t_p}\left(t_p^2-t_{p-1}^2\right)+\delta_p\left(t_p-t_{p-1}\right)\right]
\end{aligned}
\tag{8}
$$

Where:

$$
\begin{aligned}
\xi_p &= \sum_{j=1}^{p-1}\Delta x_j-\frac{\Delta x_p}{\Delta t_p}\sum_{j=1}^{p-1}\Delta t_j\\
\delta_p &= \sum_{j=1}^{p-1}\Delta y_j-\frac{\Delta y_p}{\Delta t_p}\sum_{j=1}^{p-1}\Delta t_j
\end{aligned}
\tag{9}
$$

You can see this implemented in graph_area.cpp.

2.2 Processing the Images

In order to find shapes in our images for the purposes of analysis, we will need to detect edges. For this we will use the Canny Edge detection method, developed by John F. Canny.

2.2.1 Convolution Kernels

The most basic aspect of edge detection is the convolution kernel. This is an $M\times M:\{M=2n+1, n\in\mathbb{N}\}$ matrix (an odd sided square matrix). Odd side lengths allow the kernel to be centered at each pixel. If we let the function $f(x,y)$ represent the original image, $g(x,y)$ represent the convolved image and ω represent the kernel, then:

$$
g(x,y)=\sum_{i=-n}^{n}\sum_{j=-n}^{n}\omega(i,j)f(x-i,y-j) \tag{10}
$$

Notice that the coordinates of the kernel are not 1 to M as with a traditional matrix, but rather $-n$ to n.

The effect of this on an image will be to make each pixel of a convolved image a function of the surrounding pixels. The simplest convolution matrix is the identity convolution matrix, (11).

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

More examples include edge detection kernels like (12) and (13).

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \tag{13}$$

2.2.2 Active Contours

2.2.3 Canny Edge Detection

3 Results

3.1 Data Considered

A set of images of leaves were taken from [2, Fig. 111-8].

Figure 1: Cordate

Figure 2: Cuneate

Figure 3: Deltoid

Figure 4: Elliptic

Figure 5: Hastate

Figure 6: Lanceolate

Figure 7: Linear

Figure 8: Obcordate

Figure 9: Obdeltoid

Figure 10: Oblanceolate

Figure 11: Oblong

Figure 12: Obovate

Figure 13: Orbiculate

Figure 14: Ovate

Figure 15: Peltate

Figure 16: Reniform

Figure 17: Rhombic

Figure 18: Sagitate

Figure 19: Spathulate

Figure 20: Subulate

3.2 Image Processing

These images were then processed using the MATLAB® script **contour.m** in order to produce comma–separated value (.csv) files with a series of coordinates representing the contour. This script is slightly inspired by work done previously in the coursework for the *Image Processing* module (**CMP3108**).

The images were first converted to grayscale and then black and white.

```
leaf_image = im2gray(leaf_image);
leaf_image = imbinarize(leaf_image);
...
leaf_image = 255 * leaf_image;
```

As an example, figures (21), (22) and (23) show what three of the leaf types looked like after this process.

Figure 21: Reniform (binarised)

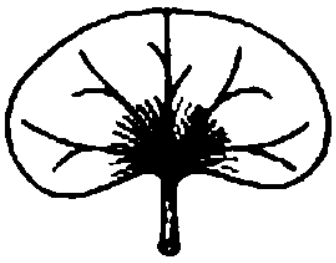


Figure 22: Spathulate (binarised)

Figure 23: Subulate (binarised)

Next, an active contour was applied, creating a binary image deliniating the regions for inside and outside the leaf.

```
leaf_contour = activecontour(leaf_image, contour_mask, 1000, "edge");
```

Figures (24), (25) and (26) show what these three leaf types looked like after this process.

Figure 24: Reniform (after active contour)

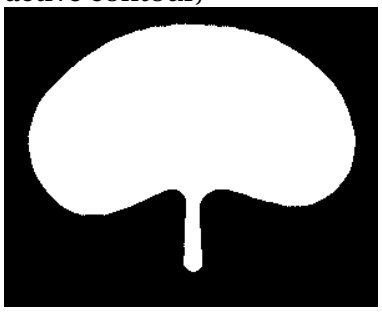
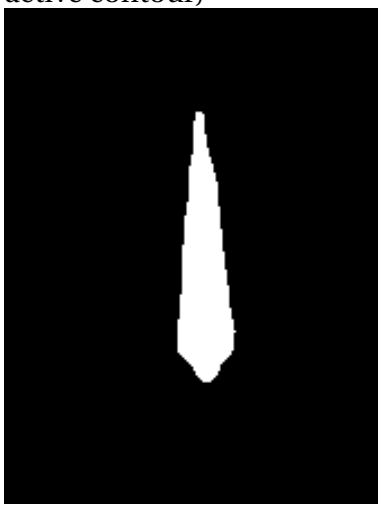


Figure 25: Spathulate (after active contour)

Figure 26: Subulate (after active contour)



The final step is extracting a contour from these binary images and write these contours to csv files. This was done using MATLAB®'s built-in `bwboundaries` function.

```
leaf_boundary = bwboundaries(leaf_contour, "noholes");
...
leaf_boundary = leaf_boundary{1};
writematrix(leaf_boundary, strcat("csv/", leaf_type, ".csv"));
```

After this process is completed the code produces a graph of these values for manual checking purposes.

```
plot(leaf_boundary(:,2), -leaf_boundary(:,1));
axis equal;
```

This actually proved invaluable since some of the images were initially broken, with the contours containing unwanted invaginations. This turned out to be caused by the initial boundary of the active contour being too thick and encroaching on the leaves themselves. The solution was to lower the thickness of the initial contour, by lowering the value of `contour_border_thickness` to 15 pixels.

```
contour_border_thickness = 15;
contour_mask( ...
    contour_border_thickness:end-contour_border_thickness, ...
    contour_border_thickness:end-contour_border_thickness ...
) = 1;
```

For examples of these graphs, see figures (27), (28) and (29).

Figure 27: Reniform (contour graph)

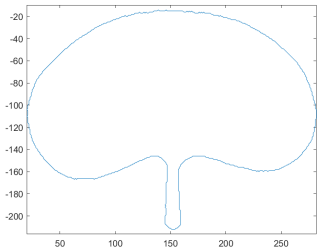


Figure 28: Spathulate (contour graph)

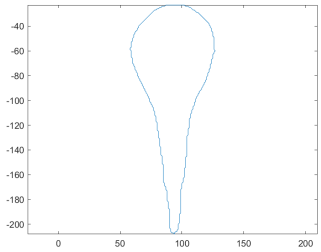
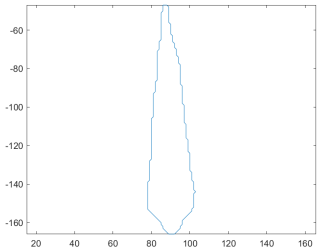


Figure 29: Subulate (contour graph)



3.3 Elliptic Fourier Analysis

4 Conclusion

5 Acknowledgements

I'd like to start by thanking paleontologists and science communicators David Moscato and Will Harris of the *Common Descent* podcast [3] for sparking an interest in the life sciences that led me to choosing this particular project subject.

I'd also like to thank my friends and family for supporting me throughout the project, especially my friends Tom and Ruby for giving some much needed last-minute encouragement.

I'd also like to extend appreciation for my personal tutor, Bart Vorselaars, for helping me with some personal issues in a way that took some stress off during the later stages of this project.

Finally, and perhaps most importantly, I'd like to thank my project supervisor Danilo Roccatano for assisting me effectively throughout the project and helping manage time and prioritise tasks to meet the deadline near the end.

6 Bibliography

References

[1] F. P. Kuhl and C. R. Giardina, "Elliptic fourier features of a closed contour," *Computer Graphics and Image Processing*, vol. 18, no. 3, pp. 236–258, 1982.

[2] L. Benson, *Plant Classification*. Heath, 1957.

[3] D. Moscato and W. Harris, "The common descent podcast." `https://commondescentpodcast.com/`, 2024.

[4] T. C. J. Smith, "Thatchapthere/uni-project." `https://github.com/ThatChapThere/uni-project`, 2024.

7 Appendix

All code used in this project can be found hosted on my personal GitHub [4].

contour.m

```
leaf_types = ["cordate" "cuneate" "deltoid" "elliptic" "hastate" ...
    "lanceolate" "linear" "obcordate" "obdeltoid" "oblanceolate" ...
    "oblong" "obovate" "orbiculate" "ovate" "peltate" "reniform" ...
    "rhombic" "sagitate" "spathulate" "subulate"
];

for leaf_type = leaf_types
    leaf_image = imread(strcat("original/", leaf_type, '.png'));

    % Convert image to grayscale and binarise
    leaf_image = im2gray(leaf_image);
    leaf_image = imbinarize(leaf_image);

    % Binarisation makes the values 1 and 0,
    % multiplying by 255 makes it a standard black and white image
    leaf_image = 255 * leaf_image;

    % Write the binarised image to a file
    imwrite(leaf_image, strcat("binarised/", leaf_type, ".png"));

    % Create a mask for and active contour,
    % this is a black binary image with a white border
    contour_mask = zeros(size(leaf_image));
    contour_border_thickness = 15;
    contour_mask( ...
        contour_border_thickness:end-contour_border_thickness, ...
        contour_border_thickness:end-contour_border_thickness ...
    ) = 1;

    % Apply an active contour to the binarised leaf image,
    % and save the result
    leaf_contour = activecontour(leaf_image, contour_mask, 1000, "edge");
    imwrite(leaf_contour, strcat("contour_mask/", leaf_type, ".png"));
```

```
    % Find the boundary of the image as a set of points
    leaf_boundary = bwboundaries(leaf_contour, "noholes");

    % Convert the boundary from a cell to a matrix,
    % and save to a csv file
    leaf_boundary = leaf_boundary{1};
    writematrix(leaf_boundary, strcat("csv/", leaf_type, ".csv"));

    % Save the contour to an image so that it can be checked for any issues
    plot(leaf_boundary(:,2), -leaf_boundary(:,1));
    axis equal;
    saveas(gcf, strcat("contour/", leaf_type, ".png"))
end
```

graph_area.cpp

```
#include <cairomm/context.h>
#include <cmath>
#include <numbers>
#include <algorithm>
#include <iostream>
#include "graph_area.h"
#include "elliptic_fourier_analyser.h"
#include "elliptic_fourier_curve.h"

using std::sin;
using std::cos;
using std::min;
using std::max;
using std::numbers::pi;
using std::cout;
using std::endl;
using std::isnan;
using std::vector;
using std::array;

GraphArea::GraphArea()
{
```

```
    set_draw_func(sigc::mem_fun(*this, &GraphArea::draw));
}

GraphArea::~GraphArea()
{
}

// This is a method of an area and the only information
// we have about said area is its width and height
void GraphArea::draw(
    const Cairo::RefPtr<Cairo::Context>& cr,
    const int width,
    const int height
) {
    vector<array<double, 2>> coords = {
        {-0.5, 0},
        {-0.5, 0.5},
        {0.5, 0.5},
        {0.5, 0},
        {0.375, -0.125},
        {0.375, -0.5},
        {0.125, -0.5},
        {0.125, -0.375},
        {0, -0.5},
    };
    draw_discrete(cr, width, height, coords);
    EllipticFourierAnalyser analyser(coords);
    for(int i = 1; i <= 10; i++)
    {
        auto curve = analyser.analyse(i);
        auto curve_shape = curve.get_shape(1000);
        draw_continuous(cr, width, height, curve_shape);
    }
}

void GraphArea::draw_discrete(
    const Cairo::RefPtr<Cairo::Context>& cr,
    const int width,
```

```
    const int height,
    vector<array<double, 2>> coords
) {
    cr->set_source_rgb(0, 0, 0); // black
    int cx = width / 2;
    int cy = height / 2;
    int r = min(width, height) / 2;
    cr->move_to(cx + coords[0][0] * r, cy + coords[0][1] * r);
    double x, y;
    for(int i = 0; i < coords.size(); i++) {
        x = coords[i][0];
        y = coords[i][1];
        cr->line_to(cx + x * r, cy + y * r);
    }
    cr->line_to(cx + coords[0][0] * r, cy + coords[0][1] * r);
    cr->stroke();
    cr->set_source_rgb(1, 0.5, 0); // orange
    for(int i = 0; i < coords.size(); i++) {
        x = coords[i][0];
        y = coords[i][1];
        cr->arc(cx + x * r, cy + y * r, 10,
                0, 2 * pi);
        cr->fill();
    }
}

void GraphArea::draw_continuous(
    const Cairo::RefPtr<Cairo::Context>& cr,
    const int width,
    const int height,
    vector<array<double, 2>> coords
) {
    cr->set_source_rgb(0, 0, 1); // blue
    int cx = width / 2;
    int cy = height / 2;
    int r = min(width, height) / 2;
    cr->move_to(cx + coords[0][0] * r, cy + coords[0][1] * r);
    double x, y;
```

```
    for(int i = 1; i < coords.size(); i++) {
        x = coords[i][0];
        y = coords[i][1];
        if(isnan(x) || isnan(y)) continue;
        cr->line_to(cx + x * r, cy + y * r);
    }
    cr->stroke();
}
```