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The Mathematics Behind Digital Sound

Digital sound – it's all around us. In our communications, radios and a million more places besides. To a lot of us, modern life would be unthinkable without the fruits of recorded sound and media, but the mathematical methodology used to reach this point has taken centuries of development, and even longer to reach the point of mass consumption. While there are many techniques used in these processes, one of the most significant, in one form or another, is the Fourier transform.

> *"Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them" – Jean-Baptiste Joseph Fourier (1768 – 1830)*

As it sometimes is with great discoveries, Joseph Fourier developed the theory of Fourier series and the Fourier transform while looking for a solution to something else[1]. Though the connection may not seem obvious, he was trying to, and successfully did, solve the heat equation (that describes the rate of change of temperature at a point in space) and realised that the solution could be represented as a sum of sine and cosine functions – what he went on to theorise as a Fourier series[2].

What Fourier had the revolutionary insight to consider was that you can get arbitrarily close to the real form of the function by adding multiple sine and cosine waves of increasingly large frequencies multiplied by relative factors, here a_n and b_n. Mathematically, it looks like this[3]:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)].$$

This is essentially a sum of superposed sine and cosine functions, with each additional term increasing the accuracy of the approximation compared to the original function.

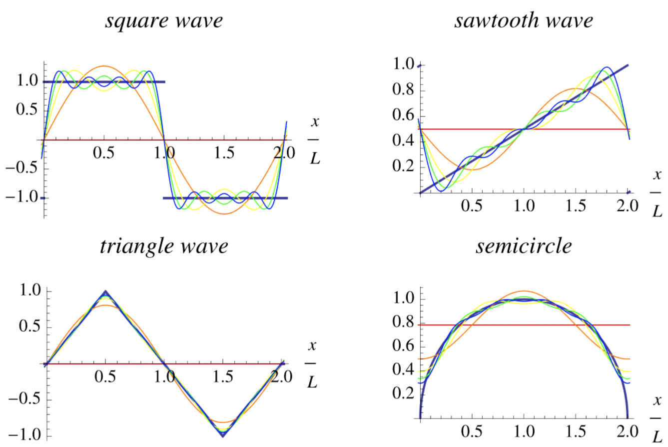


Fig. 1: Graphs showing Fourier series being used on several unfriendly functions.

[1] (2012). In I. Stewart, 17 Equations That Changed The World (pp. 149-163). Profile Books Ltd.
[2] Ibid.
[3] Weisstein, E. W. (n.d.). *Fourier Series*. Retrieved from MathWorld - A Wolfram Web Resource: https://mathworld.wolfram.com/FourierSeries.html

The process, now known as Fourier analysis, is quite remarkable, indeed winning an award from the Paris Institute in 1811[4], despite causing some controversy for not being sufficiently rigorous. Fortunately, with input from later figures, such as Riemann, we now know Fourier to have been right in spirit. However, while he originally claimed that this approximation could work for any function, this was later revised to a large, but not universal, set of functions due to irregular behaviour at discontinuities with some – perhaps a classic case of overconfidence. This should not, though, devalue his discoveries as they are impressive nonetheless.

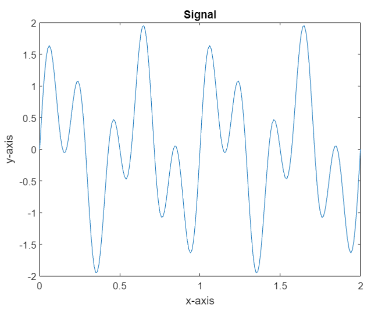
From the generalised Fourier series, the Fourier transform can be derived, which turns out as, for a function[5] $f(x)$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,\mathrm{d}x.$$

There is also the inverse Fourier transform, which reverses the operation:

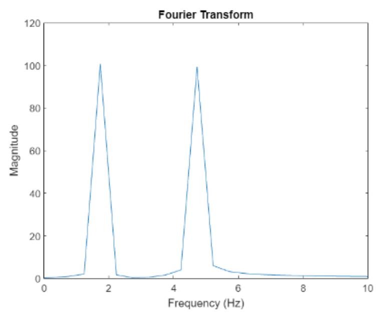
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\,\mathrm{d}\xi.$$

Here, $\hat{f}(\xi)$ is the Fourier transform of $f(x)$, which, for an input frequency ξ, describes its prevalence in the original function; we can sort of see where it comes from, as the sums from the Fourier series have become infinite sums expressed over increasingly small intervals (hence the integration involved) and mapped onto the complex plane using Euler's identity for greater ease, though the actual derivation is of course more complicated than this. The resulting function, as mentioned, describes the strength of the various frequencies in $f(x)$, which when plotted or analysed, helps identify the dominant frequencies from the original function as shown[6].



< Fig. 2: Graph showing superposed sine waves, representing sound wave of overlapping 2Hz and 5Hz.

Fig 3: > Fourier transform of the signal first observed in Fig. 2.



To help demonstrate, we have two overlapping signals in figure two[7], which are initially unrecognisable as any particular frequencies. However, as soon as we apply the Fourier

[4] (2012). In I. Stewart, 17 Equations That Changed The World (pp. 149-163). Profile Books Ltd.
[5] Weisstein, E. W. (n.d.). *Fourier Transform*. Retrieved from MathWorld - A Wolfram Web Resource: https://mathworld.wolfram.com/FourierTransform.html
[6] Sanderson, G. (n.d.). *Visual Introduction to the Fourier Transform*. Retrieved from 3Blue1Brown: https://www.3blue1brown.com/lessons/fourier-transforms
[7] MathWorks. (n.d.). Fourier Transforms. Retrieved from MathWorks MATLAB: https://uk.mathworks.com/help/matlab/math/fourier-transforms.html (also see Appendix 1)

transform (see figure three), the signal reveals itself as being composed of 2Hz and 5Hz, which is accurate as it agrees with the originally plotted function of $y = \sin(2 \times 2\pi x) + \sin(5 \times 2\pi x)$. This isn't just useful for identifying one or two overlaps of fairly simple functions like we have here, but also works with any number of overlapping functions and provides relative weightings of the frequencies too!

At this point, you may be asking yourself, this is all great mathematics, but how does this fascinating equation apply to me? Not to worry though, for there are a multitude of ways, including signal analysis and digital sound processing, which, for all intents and purposes, are ubiquitous in the modern world.

Significantly, the Fourier transform is used extensively in the analysis and processing of the various frequencies found in audio and video signals, such as those from an audio recording. For example, imagine you are on a phone call with a friend who has a Labrador and they blow a dog whistle that is beyond the range of human hearing. It is impossible for a human to hear this owing to the limitations of our feeble ears, so there is no point in the phones sending this data over a call, as it would increase the traffic being sent over an already congested network. While the data recorded at this high a range is bound to be minute compared to the main parts of the call in any case, likely just 1-2% of the total call data, multiplying this factor over an entire population's calls could add up to millions of pounds worth of higher operating costs for a network, pushing up costs for everybody, for no good reason.

To fix this with the Fourier transform, the process first requires a discrete version of the function, which happens to be a by-product of how microphones record audio (taking a sample of the amplitude of the sound detected at regular intervals), so that it can utilise the 'fast Fourier transform' algorithm – designed by Cooley and Tukey in 1965[8]. Once this is applied, the device can then modify the result of the transform to remove any unnecessary frequencies, e.g. anything below 20Hz and above 20,000Hz (the range of human hearing), before running the inverse Fourier transform on this[9]. This produces a final recording of audibly identical quality to the original but with a lower file size; this is sometimes known as 'lossy' compression. In fact, we can go even further with this – there is a much smaller range of frequencies at which humans are accustomed to hearing and speaking at and anything which doesn't fit can also be removed from the transmission (anything outside [300Hz, 3,400Hz], for example), albeit at the cost of potentially being noticed. Note that lossy compression can also use other methods to reduce file size.

Following this initial step, other techniques are often applied in conjunction with the above as to optimise our calls' use of network bandwidth further whilst keeping the quality

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[8] Cooley, J. W., & Tukey, J. W. (1965). An algorithm for the machine calculation of complex Fourier series. *Mathematics of Computation 19*, 297-301.

[9] Gupta, A. (2013, Janurary). Fourier Transform and Its Application in Cell Phones. *International Journal of Scientific and Research Publications, 3*(1), 1386.

of audio adequately intelligible. One of the most interesting of these methods is known as multiplexing[10], a process done by a multiplexer device. This is when your communications are transmitted on the same physical link, such as a copper telephone line or fibre optic cable, as other communications. First, the frequency range of your transmission is confined (sometimes using a Fourier transform), then applying vertical transformations to the frequency range and layering this with other similarly transformed signals while on the shared physical link. The process is then reversed on the recipient's end using a demultiplexer – if you've ever wondered how a single telecoms cabinet can handle your entire area's connections, this is how. Coincidentally, all of these layers of concurrent compression also contribute to ruining hold music[11] as, despite the algorithms being suitable for voices, the rich and varied sound of musical instruments does not respond well to all of these concatenated chains of audio compression.

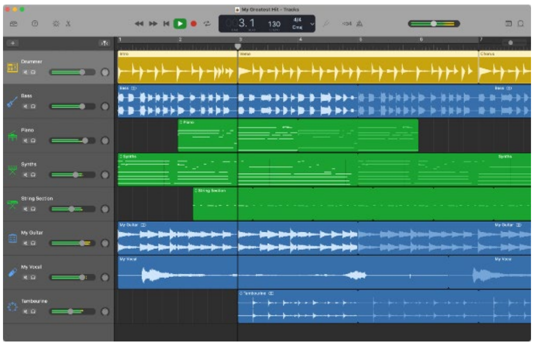

Fig. 4: Apple GarageBand in use (image from Apple)

In a similar vein, the Fourier transform can also be used by artists and creators to fine tune any recordings, remixes or sound effects they might be working on. This comes in the form of sound editing software, such as Apple's GarageBand, which allows specific portions of audio to be isolated, modified and reinserted in order to create the effects desired or remove an unfortunate contamination of audio[12], such as a cough or thud. In terms of the Fourier transform, this is done by first identifying the frequency of the unwanted sound, isolating the relevant portion of audio, removing the sound's frequency from the Fourier transform and then performing the inverse function on this result - a fairly straightforward process to understand now that we can recognise the mathematical principles at play. Such techniques can also be automated by the so-called filters that you might find bundled in audio software or the operating system of your phone, applying the same techniques to add in or remove specified frequencies from your audio. For example, a voice isolation filter may first analyse the typical frequencies of your voice with machine learning, and then use Fourier transforms to remove anything unlikely to be you. Alternatively, a static filter might try to emulate phone static by adding in additional random frequencies from across the spectrum over the original audio; the limit here really just comes down to a creator's imagination.

[10] Stewart, J. M. (2013, August 22). What Is Multiplexing? Retrieved from Skillsoft: https://www.globalknowledge.com/us-en/resources/resource-library/articles/what-is-multiplexing/
[11] If unfamiliar, this is the recorded music played by a company when they have placed a phone call on hold
[12] Chapagain, A. (2019). Sound Editing using Fourier Transform. doi:10.13140/RG.2.2.13640.16645

Going back to Fourier's quote, the range of applications mentioned above are, in my opinion, a practical demonstration of the 'diverse phenomena' he mentions, united by the Fourier transform, this case's 'secret analogy'. Indeed, what I have discussed is by no means exhaustive, with the Fourier transform finding further applications in audio as well as elsewhere within the digital sphere and beyond, such as in image and video processing, cryptography and in the fluid dynamics used to predict the weather or design aeroplanes. In conclusion then, while this piece is intended to help you realise how momentous and influential Fourier's discoveries were and continue to be, I would also like to encourage you to try to identify other areas of life where these concepts appear, or even those from a different equation or theorem, and to truly appreciate the sheer number of ways that mathematics governs human existence.

References:

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Appendix:

1. MATLAB code used to generate Fig. 2 and 3.

```
Ts = 1/100;
t = 0:Ts:2;
x = 2*sin(2*pi*5*t) + sin(2*pi*2*t);
plot(t,x)
xlabel('x-axis')
ylabel('y-axis')
title('Signal');

y = fft(x);
fs = 1/Ts;
f = (0:length(y)-1)*fs/length(y);

n = length(x);
fshift = (-n/2:n/2-1)*(fs/n);
yshift = fftshift(y);
plot(fshift, abs(yshift))
xlim([0 10])
xlabel('Frequency (Hz)')
ylabel('Magnitude')
title('Fourier Transform');
```