Graphs and applications

Graphs versus networks

Mostly a terminology difference:

- graphs in the mathematical world
- networks elsewhere

I will mostly say *graphs* (this is a math course) but might oscillate

I will be basing the theory part on Bergé (Graphs and Hypergraphs, 1973) and Bang-Jensen & Gutin (Digraphs: Theory, Algorithms and Applications, 2009)

Graph theory's vocabulary is not homogeneous. As much as possible, I will point out alternate terms

Graph

Intuitively: a graph is a set of points, and a set of arrows, with each arrow joining one point to another

The points are called the *vertices* of the graph and the arrows are the *arcs* of the graph

Definition 1 (Graph)

A graph G is a pair (X, U), where

1. X a set $\{x_1, x_2, \ldots, x_n\}$ of elements called **vertices**
2. U a family $(u_1, u_2, \ldots, u_m)$ of elements of the Cartesian product $X \times X$, called **arcs**. This family is often be denoted by the set $U = \{1, 2, \ldots, m\}$ of its indices. An element (x, y) of $X \times X$ can appear more than once in this family. A graph in which no element of $X \times X$ appears more than p times is called a p-graph

Definition 2 (Order of a graph)

The number of vertices in a graph is called the **order** of the graph

Definition 3 (Loop)

An arc of G of the form (x, x) is called a **loop**

Definition 4 (Arc endpoints)

For an arc $u = (x, y)$, vertex x is the **initial endpoint**, and vertex y is the **terminal endpoint**

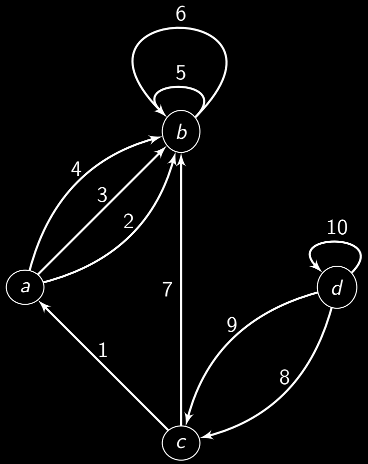
Definition 5 (Successors/Predecessors)

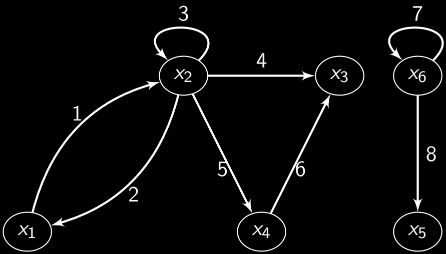
Vertex y is a **successor** (resp. **predecessor**) of vertex x if there is an arc with y as its initial (resp. terminal) endpoint and x as its termina1 (resp. initial) endpoint. The set of all successors (resp. predecessors) of x is denoted $\Gamma_G^+(x)$ (resp. $\Gamma_G^-(x)$)

Definition 6

The set of all neighbours of x is denoted $\Gamma(x) = \Gamma_G^+(x) \cup \Gamma_G^-(x)$

Examples





Directed versus undirected graph

Definition 7 (Edge)

In $G = (X, U)$, each arc $u_i = (x, y)$ determines a continuous line joining x and y. Such a line with no specification of direction is an **edge**, and is denoted by $e_i = [x, y]$. The family $(e_1, e_2, \ldots, e_m)$ of the edges of G is denoted by its set of indices $E = \{1, 2, \ldots, m\}$

Without direction, $G = (X, E)$, we talk of **undirected graph** (or **multigraph**)

An undirected graph in which there is no loops and no more than one edge joins any two vertices is **simple**

In modern texts (e.g., B-J&G), a **digraph** (or directed graph) has no loops

Definition 8 (Adjacent arcs/edges)

Two arcs (or two edges) are **adjacent** if they have at least one endpoint in common

Definition 9 (Multiplicity of a pair)

The **multiplicity** of a pair x, y is the number $m_G^+(x, y)$ of arcs with initial endpoint x and terminal endpoint y. Let

$$m_G^-(x, y) = m_G^+(y, x)$$
$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y)$$

If $x \neq y$, then $m_G(x, y)$ is number of arcs with both x and y as endpoints. If $x = y$, then $m_G(x, y)$ equals twice the number of loops attached to vertex x. If $A, B \subset X$, $A \neq B$, let

$$m_G^+(A, B) = \{u : u \in U, u = (x, y), x \in A, y \in B\}$$
$$m_G(A, B) = m_G^+(A, B) + m_G^+(A, B)$$

Definition 10 (Arc incident to a vertex)

If a vertex x is the initial endpoint of an arc u, which is not a loop, the arc u is **incident out of vertex** x

The number of arcs incident out of x plus the number of loops attached to x is denoted $d_G^+(x)$ and is the **outer demi-degree** of x

An arc **incident into vertex** x and the **inner demi-degree** $d_G^-(x)$ are defined similarly

Definition 11 (Degree)

The **degree** of vertex x is the number of arcs with x as an endpoint, each loop being counted twice. The degree of x is denoted $d_G(x) = d_G^+(x) + d_G^-(x)$

If each vertex has the same degree, the graph is **regular**

Definition 12 (Arc incident out of $A \subset X$)

If the initial endpoint of an arc u belongs to A, and if the terminal endpoint of arc u does not belong to A, then u is said to be incident out of A, and we write $u \in \omega^+(A)$

Similarly, we define an arc incident into A, and the set $\omega^-(A)$

Finally, the set of arcs incident to A is denoted

$$\omega(A) = \omega^+(A) \cup \omega^-(A)$$

Definition 13 (Symmetric graph)

If $m_G^+(x, y) = m_G^-(x, y)$ for all $x, y \in X$, the graph G is **symmetric**. A 1-graph $G = (X, U)$ is symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \in U$$

Definition 14 (Anti-symmetric graph)

If for each pair $(x, y) \in X \times X$,

$$m_G^+(x, y) + m_G^-(x, y) \leq 1$$

then the graph G is **anti-symmetric**. A 1-graph $G = (X, U)$ is anti-symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \notin U$$

An anti-symmetric 1-graph without its direction is a simple graph

Definition 15 (Complete graph)

A graph G is **complete** if

$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y) \geq 1$$

for all $x, y \in X$ such that $x \neq y$

A 1-graph is complete if, and only if,

$$(x, y) \notin U \implies (y, x) \in U$$

Definition 16 (n-clique)

A simple, complete graph on n vertices is called an n-**clique** and is often denoted K_n

Definition 17 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets X_1 and X_2, s.t. no two vertices in the same set are adjacent. This graph may be written $G = (X_1, X_2, U)$

Definition 18 (Complete bipartite graph)

If $\forall x_1 \in X_1, \forall x_2 \in X$, we have $m_G(x_1, x_2) \geq 1$, then $G = (X_1, X_2, U)$ is a **complete bipartite** graph

A simple, complete bipartite graph with $|X_1| = p$ and $|X_2| = q$ is often denoted $K_{p,q}$

Definition 19 (Subgraph of G generated by $A \subset X$)

The **subgraph** of G generated by A is the graph with A as its vertex set and with all the arcs in G that have both their endpoints in A. If $G = (X, \Gamma)$ is a 1-graph, then the subgraph generated by A is the 1-graph $G_A = (A, \Gamma_A)$ where

$$\Gamma_A(x) = \Gamma(x) \cap A \qquad (x \in A)$$

Definition 20 (Partial graph of G generated by $Y \subset U$)

This is the graph (X, V) whose vertex set is X and whose arc set is V. In other words, it is graph G without the arcs $U - V$

Definition 21 (Partial subgraph of G)

A partial subgraph of G is the subgraph of a partial graph of G

Chain

Definition 22 (Chain of length $q > 0$)

A **chain** is a sequence $\mu = (u_1, \ldots, u_q)$ of arcs of G s.t. each arc in the sequence has one endpoint in common with its predecessor in the sequence and its other endpoint in common with its successor in the sequence. The number of arcs in the sequence is the **length** of chain μ. A chain that does not encounter the same vertex twice is **elementary**. A chain that does not use the same arc twice is **simple**

Path

Definition 23 (Path of length $q > 0$)

A **path** of length q is a chain $\mu = (u_1, \ldots, u_i, \ldots, u_q)$ in which the terminal endpoint of arc u_i is the initial endpoint of arc u_{i+1} for all $i < q$. For a 1-graph, a path is completely determined by the sequence of vertices $x_1, x_2, \ldots$ that it encounters. Hence, we often write

$$\begin{aligned} \mu &= ((x_1, x_2), (x_2, x_3), \ldots) \\ &= [x_1, x_2, \ldots, x_k, x_{k+1}] \\ &= \mu[x_1, x_{k+1}] \end{aligned}$$

Vertices x_1 and x_k are the **initial** and **terminal endpoints** of path μ. Similarly, for a simple graph, a chain μ with endpoints x and y is determined by the sequence of its vertices, and we may write

$$\mu = \mu[x, y] = [x, x_1, \ldots, y]$$

Definition 24 (Cycle)

A **cycle** is a chain such that

1. no arc appears twice in the sequence
2. the two endpoints of the chain are the same vertex

Definition 25 (Pseudo-cycle)

A **pseudo-cycle** is a chain $\mu = (u_1, u_2, \ldots, u_q)$ whose two endpoints are the same vertex and whose arcs are not necessarily distinct

Definition 26 (Circuit)

A **circuit** is a cycle $\mu = (u_1, u_2, \ldots, u_q)$ such that for all $i < q$, the terminal endpoint of u_i is the initial endpoint of u_{i+1}

Definition 27 (Connected graph)

A **connected graph** is a graph that contains a chain $\mu[x, y]$ for each pair x, y of distinct vertices

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a chain in G connecting x and y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 28 (Connected component of a graph)

The classes of the equivalence relation $\equiv$ partition X into connected sub-graphs of G called **connected components**

Definition 29 (Articulation set)

For a connected graph, a set A of vertices is called an **articulation set** (or a **cutset**) if the subgraph of G generated by $X - A$ is not connected

Definition 30 (Stable set)

A set S of vertices is called a stable set if no arc joins two distinct vertices in S

Definition 31 (Matrix associated with G)

If G has vertices $x_1, x_2, \ldots, x_n$, then the **matrix associated** with G is

$$a_{ij} = m_G^+(x_i, x_j)$$

Definition 32 (Adjacency matrix)

The matrix $a_{ij} + a_{ji}$ is the **adjacency matrix** associated with G

More on cycles

In a graph $G = (X, U)$, a **cycle** is a sequence of arcs

$$\mu = (u_1, u_2, \ldots, u_q)$$

such that

1. each arc u_k, where $1 < k < q$, has one endpoint in common with the preceding arc u_{k-1}, and the other end point in common with the succeeding arc u_{k+1} (i.e., this sequence is a chain)
2. the sequence does not use the same arc twice
3. the initial vertex and terminal vertex of the chain are the same

An **elementary cycle** is a cycle in which, in addition,

4. no vertex is encountered more than once (except, of course, the initial vertex which is also the terminal vertex)

Given a cycle μ, denote μ^+ the set of all arcs in μ that are in the direction that the cycle is traversed and μ^- the set of all the other arcs in μ

Number the arcs in G as $1, 2, \ldots, m$, then the cycle μ is the vector

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$$

where

$$\mu_i = \begin{cases} 0 & \text{if } i \notin \mu^+ \cup \mu^- \\ +1 & \text{if } i \in \mu^+ \\ -1 & \text{if } i \in \mu^- \end{cases}$$

Cocycles

Let $A \subset X$ be nonempty and denote $\omega^+(A)$ the set of arcs that have only their initial endpoint in A and $\omega^-(A)$ the set of arcs that have only their terminal endpoint in A. Let

$$\omega(A) = \omega^+(A) \cup \omega^-(A)$$

A **cocycle** is a nonempty set of arcs of the form $\omega(A)$, partitioned into two sets $\omega^+(A)$ and $\omega^-(A)$

An **elementary cocycle** is the set of arcs joining two connected subgraphs A_1 and A_2 s.t.

- $A_1, A_2 \neq \emptyset$
- $A_1 \cap A_2 = \emptyset$
- $A_1 \cup A_2 = C$, with C a connected component of the graph

A colouring lemma

Lemma 33 (Arc colouring Lemma)

Consider G with arcs $1, \ldots, m$. Colour arc 1 black and arbitrarily colour the remaining arcs red, black or green. Then exactly one of the following holds true:

1. *there is an elementary cycle containing arc 1 and only red and black arcs with the property that all black arcs in the cycle have the same direction*
2. *there is an elementary cocycle containing arc 1 and only green and black arcs, with the property that all black arcs in the cocycle have the same direction*

Independent cycles and cycle bases

Consider cycles $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \ldots, \boldsymbol{\mu}^k$. The cycles are **independent** if

$$c_1\boldsymbol{\mu}^1 + c_2\boldsymbol{\mu}^2 + \cdots + c_k\boldsymbol{\mu}^k = \mathbf{0} \iff c_1 = c_2 = \cdots = c_k = 0$$

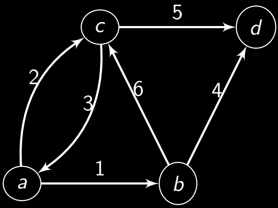
A **cycle basis** is an independent set $\{\boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \ldots, \boldsymbol{\mu}^k\}$ of cycles such that any cycle $\boldsymbol{\mu}$ can be written as

$$\boldsymbol{\mu} = c_1\boldsymbol{\mu}^1 + c_2\boldsymbol{\mu}^2 + \cdots + c_k\boldsymbol{\mu}^k$$

for $c_1, \ldots, c_k \in \mathbb{R}$

The constant k is the **cyclomatic number** of G, denoted $\nu(G)$

Example



Elementary cycles:

- $\boldsymbol{\mu}^1 = (1, 6, 2) = [abca]$
- $\boldsymbol{\mu}^2 = (1, 6, 3) = [abca]$
- $\boldsymbol{\mu}^3 = (2, 3) = [aca]$
- $\boldsymbol{\mu}^4 = (1, 4, 5, 2) = [abdca]$
- $\boldsymbol{\mu}^5 = (6, 5, 4) = [acdb]$
- $\boldsymbol{\mu}^6 = (1, 4, 5, 3) = [abdca]$

We have $\boldsymbol{\mu}^1 - \boldsymbol{\mu}^2 + \boldsymbol{\mu}^3 = \mathbf{0}$

An important result

Theorem 34

Let G be a graph with n vertices, m arcs and p connected components. Then the cyclomatic number of G is

$$\nu(G) = m - n + p$$

Trees and forests

Definition 35 (Tree)

A **tree** is a connected graph without cycles

(A tree is a special kind of 1-graph)

Definition 36 (Forest)

A **forest** is a graph whose connected components are trees

(i.e., a forest is a graph without cycles)

Theorem 37 (Characterisation of trees)

$H = (X, U)$ a graph of order $|X| = n > 2$.
TFAE

1. *H connected and has no cycles*
2. *H has $n - 1$ arcs and no cycles*
3. *H connected and has exactly $n - 1$ arcs*
4. *H has no cycles, and if an arc is added to H, exactly one cycle is created*
5. *H connected, and if any arc is remooed, the remaining graph is not connected*
6. *Every pair of vertices of H is connected by one and only one chain*

Proof of Theorem 37

[$\mathbf{1} \implies \mathbf{2}$] Let p be the number of connected components, m the number of arcs and $v(H)$ the cyclomatic number. Since H connected, $p = 1$. Since H has no cycles,
$v(H) = m - n + p = 0$
$\implies m = n - p = n - 1$

[$\mathbf{2} \implies \mathbf{3}$] Assume H has no cycles ($v(H) = 0$) and has $n - 1$ arcs ($m = n - 1$). Then, since

$$v(H) = m - n + p$$

$p = v(H) - m + n = 0 - (n - 1) - n = 1$, i.e., H is connected

Proof of Theorem 37 (cont.)

[**3** $\implies$ **4**] Assume H connected ($p = 1$) and contains exactly $n-1$ arcs ($m = n-1$). Then

$$v(H) = m - n + p = (n-1) - n + 1 = 0$$

and H has no cycles

Now add an arc, i.e., suppose $m = n$. Then $v(H) = m - n + p = n - n + 1 = 1$ and there is one cycle in the new graph

Proof of Theorem 37 (cont.)

[**4** $\implies$ **5**] Assume H has no cycles ($v(H) = 0$) and that addition of an arc to H creates exactly one cycle

Suppose H not connected. Then there are two vertices, say a and b, that are not connected