

# Page rank (and Markov chains)

# What makes an important webpage?

In days of yore, the web was a small thing

Alta Vista was the search engine of choice

Google started in 1998, based on an algorithm (PageRank) described in a paper of Page, Brin, Motwani and Winograd

# Overview

Give each page a rating (of its importance), a recursively defined measure whereby a page becomes important if important pages link to it

Recursive definition: the importance of a page refers back to the importance of other pages that link to it

**Random surfer** model: a random surfer on the web follows links from page to page. Page rank  $\simeq \mathbb{P}$  random surfer lands on a particular page. Popular page  $\implies$  higher probability to go there

Example of a Markov chain

# Markov chain

A Markov chain is a *stochastic process* in which the evolution through time depends only on the current state of the system (we say the process is *memoryless*)

Markov chains are an interesting combination of matrix theory and graph theory

They form the theoretical foundation for Hidden Markov processes or Markov Chain Monte Carlo (MCMC) methods, are used in ML

Conduct an experiment with a set of  $r$  possible outcomes

$$S = \{S_1, \dots, S_r\}$$

Experiment repeated  $n$  times (with  $n$  large, potentially infinite)

System has *no memory*: the next state depends only on the present state

Probability of  $S_j$  occurring on the next step, given that  $S_i$  occurred on the last step, is

$$p_{ij} = p(S_j|S_i)$$

Suppose that  $S_i$  is the current state, then one of  $S_1, \dots, S_r$  must be the next state; so

$$p_{i1} + p_{i2} + \dots + p_{ir} = 1, \quad 1 \leq i \leq r$$

(Some of the  $p_{ij}$  can be zero, all that is needed is that  $\sum_{j=1}^r p_{ij} = 1$  for all  $i$ )

## Definition 1

An experiment with finite number of possible outcomes  $S_1, \dots, S_r$  is repeated. The sequence of outcomes is a **Markov chain** if there is a set of  $r^2$  numbers  $\{p_{ij}\}$  such that the conditional probability of outcome  $S_j$  on any experiment given outcome  $S_i$  on the previous experiment is  $p_{ij}$ , i.e., for  $1 \leq i, j \leq r$ ,  $n = 1, \dots$ ,

$$p_{ij} = \Pr(S_j \text{ on experiment } n+1 \mid S_i \text{ on experiment } n)$$

Outcomes  $S_1, \dots, S_r$  are **states** and  $p_{ij}$  are **transition probabilities**.  $P = [p_{ij}]$  the **transition matrix**

The matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix}$$

has

- ▶ nonnegative entries,  $p_{ij} \geq 0$
- ▶ entries less than 1,  $p_{ij} \leq 1$
- ▶ row sum 1, which we write

$$\sum_{j=1}^r p_{ij} = 1, \quad i = 1, \dots, r$$

or, using the notation  $\mathbb{1}^T = (1, \dots, 1)$ ,

$$P\mathbb{1} = \mathbb{1}$$

# (super simple) Mendelian genetics

A *phenotypic trait* (eye colour, hair colour, etc.) is determined by a specific pair of alleles, each of which may be two types, say G and g

Each individual can have

- ▶ GG combination (*dominant*)
- ▶ Gg or gG, considered equivalent genetically (*hybrid*)
- ▶ gg combination (*recessive*)

Individuals bearing GG or gg alleles are *homozygotes*, hybrids with Gg alleles are called *heterozygotes*

GG and gg combinations lead to different phenotypes, Gg combination leads to expressing the same phenotype as individuals bearing a GG combination, hence the name dominant given to GG



In sexual reproduction, offspring inherit one allele of the pair from each parent

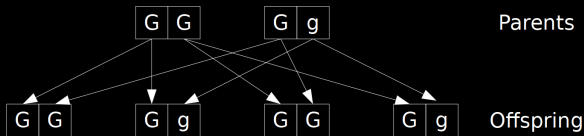
Alleles inherited from each parent are selected at random, independently of each other

This determines probability of occurrence of each type of offspring. The offspring

- ▶ of two GG parents must be GG
- ▶ of two gg parents must be gg
- ▶ of one GG and one gg parent must be Gg
- ▶ other cases must be examined in more detail

# GG and Gg parents

Suppose one parent GG and the other Gg



		Parent 1	
		G	G
Parent 2	G	GG	GG
	g	Gg	Gg

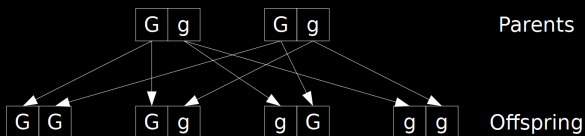
To determine  $\mathbb{P}$  that offspring is of a certain type, count number of outcomes of each type (GG and Gg) and divide by 4

$\implies$  offspring have probability

- ▶ 1/2 of being GG
- ▶ 1/2 of being Gg

# Gg and Gg parents

Both parents are hybrid

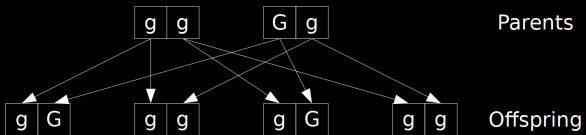


⇒ offspring have probability

- ▶ 1/4 of being GG
- ▶ 1/2 of being Gg
- ▶ 1/4 of being gg

# gg and Gg parents

Recessive and hybrid parents



⇒ offspring have probability

- ▶ 1/2 of being Gg
- ▶ 1/2 of being gg

# General case

$p_i(n)$ : probability that state  $S_i$  occurs on the  $n^{th}$  repetition of the experiment,  $1 \leq i \leq r$

Since one the states  $S_i$  must occur on the  $n^{th}$  repetition

$$p_1(n) + p_2(n) + \cdots + p_r(n) = 1$$

$p_i(n+1)$ : probability that state  $S_i$ ,  $1 \leq i \leq r$ , occurs on  $(n+1)^{th}$  repetition of the experiment

$r$  ways to be in state  $S_i$  at step  $n+1$ :

1. Step  $n$  is  $S_1$ . Probability of getting  $S_1$  on  $n^{th}$  step is  $p_1(n)$ , and probability of having  $S_i$  after  $S_1$  is  $p_{1i}$ . Therefore  $P(S_i|S_1) = p_{1i}p_1(n)$
2. We get  $S_2$  on step  $n$  and  $S_i$  on step  $(n+1)$ . Then  $P(S_i|S_2) = p_{2i}p_2(n)$
- ...
- $r$ . Probability of occurrence of  $S_i$  at step  $n+1$  if  $S_r$  at step  $n$  is  $P(S_i|S_r) = p_{ri}p_r(n)$

$$\begin{aligned}\implies p_i(n+1) &= P(S_i|S_1) + \cdots + P(S_i|S_r) \\ &= p_{1i}p_1(n) + \cdots + p_{ri}p_r(n)\end{aligned}$$

Therefore,

$$\begin{aligned}p_1(n+1) &= p_{11}p_1(n) + p_{21}p_2(n) + \cdots + p_{r1}p_r(n) \\ &\vdots \\ p_r(n+1) &= p_{1r}p_1(n) + p_{2r}p_2(n) + \cdots + p_{rr}p_r(n)\end{aligned}$$

In matrix form

$$p(n+1) = p(n)P, \quad n = 1, 2, 3, \dots$$

where  $p(n) = (p_1(n), p_2(n), \dots, p_r(n))$  is a (row) probability vector and  $P = (p_{ij})$  is a  $r \times r$  transition matrix,

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix}$$

So

$$\begin{aligned} & (p_1(n+1), \dots, p_r(n+1)) \\ &= (p_1(n), \dots, p_r(n)) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix} \end{aligned}$$

Easy to check that this gives the same expression as before

# Stochastic matrices

## Definition 2 (Stochastic matrix)

The nonnegative  $r \times r$  matrix  $M$  is **stochastic** if  $\sum_{j=1}^r a_{ij} = 1$  for all  $i = 1, 2, \dots, r$

If each row sum and each column sum equals one, the matrix is **doubly stochastic**

## Theorem 3

*Let  $M$  be a stochastic matrix. Then all eigenvalues  $\lambda$  of  $M$  are such that  $|\lambda| \leq 1$ . Furthermore,  $\lambda = 1$  is an eigenvalue of  $M$*



# Case of an irreducible matrix

## Definition 4 ((Ir)reducible matrix)

$A \in \mathcal{M}_n(\mathbb{C})$  **reducible** if exists permutation matrix  $P \in \mathcal{M}_n(\mathbb{R})$  and  $r \in N = \{1, \dots, n\}$  s.t.

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where  $A_{11} \in \mathcal{M}_r$ ,  $A_{22} \in \mathcal{M}_{n-r}$

If there is no such  $P$ , then  $A$  **irreducible**

## Theorem 5

$A \in \mathcal{M}_n(\mathbb{C})$ . Then  $A$  irreducible  $\iff \mathcal{G}(A)$  strongly connected

## Theorem 6

$A \in \mathcal{M}_n$  stochastic matrix. Then  $\rho(A) = 1$  is an eigenvalue with multiplicity 1 and  $|\lambda| < 1$  for all  $\lambda \in \text{Sp}(A)$

# Long time behaviour

Let  $p(0)$  be the initial distribution (row) vector. Then

$$\begin{aligned}p(1) &= p(0)P \\p(2) &= p(1)P \\&= (p(0)P)P \\&= p(0)P^2\end{aligned}$$

Iterating, we get, for any  $n$ ,

$$p(n) = p(0)P^n$$

Therefore,

$$\begin{aligned}\lim_{n \rightarrow +\infty} p(n) &= \lim_{n \rightarrow +\infty} p(0)P^n \\&= p(0) \lim_{n \rightarrow +\infty} P^n\end{aligned}$$

if this limit exists

$$\lim_{n \rightarrow +\infty} p(n) = p(0) \lim_{n \rightarrow +\infty} P^n$$

Does the limit exist?

### Theorem 7

*If  $M, N$  are nonsingular stochastic matrices, then  $MN$  is a stochastic matrix*

So the product of any number of stochastic matrices is also stochastic

### Corollary 8

*If  $M$  is a nonsingular stochastic matrix, then for any  $k \in \mathbb{N}$ ,  $M^k$  is a stochastic matrix*

# Remember this?

## Corollary 9

$A \in \mathcal{M}_n$  column stochastic and

$$0 \leq v(t) = (v_1(t), \dots, v_n(t))^T \in \mathbb{R}^n$$

Consider the sequence

$$v(t+1) = Av(t)$$

with initial condition  $v(0)$  s.t.

$$\|v(0)\| = K \in \mathbb{R}_+ \setminus \{0\}$$

Then for all  $t = 1, 2, \dots$

$$\|v(t)\| = K$$

# Regular Markov chains

## Definition 10 (Regular Markov chain)

A **regular** Markov chain has  $P^k$  (entry-wise) positive for some integer  $k > 0$ , i.e.,  $P^k$  has only positive entries

## Definition 11 (Primitive matrix)

A nonnegative matrix  $M$  is **primitive** if, and only if, there is an integer  $k > 0$  such that  $M^k$  is positive.

## Theorem 12

*Markov chain regular  $\iff$  transition matrix  $P$  primitive*

# More on primitive matrices

## Definition 13

$\mathcal{M}_n(\mathbb{R}) \ni A \geq 0$  is **primitive** (with **primitivity index**  $k \in \mathbb{N}_+^*$ ) if  $\exists k \in \mathbb{N}_+^*$  s.t.

$$A^k > 0$$

with  $k$  the smallest integer for which this is true.  $A$  **imprimitive** if it is not primitive

Primitivity  $\implies$  irreducibility; the converse is not true

## Theorem 14

*Sufficient condition for primitivity is irreducibility with at least one positive diagonal entry*

Here  $d$  is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as  $\lambda_p = \rho(A)$ ). If  $d = 1$ , then  $A$  is primitive. We have that  $d = \gcd$  of all the lengths of closed walks in  $G(A)$ .

# Behaviour of a regular MC

## Theorem 15

*If  $P$  is the transition matrix of a regular Markov chain, then*

- 1. the powers  $P^n$  approach a stochastic matrix  $W$*
- 2. each row of  $W$  is the same (row) vector  $w = (w_1, \dots, w_r)$*
- 3. the components of  $w$  are positive*

So if the Markov chain is regular

$$\lim_{n \rightarrow +\infty} p(n) = p(0) \quad \lim_{n \rightarrow +\infty} P^n = p(0)W$$

# Computing $W$

If  $p(n)$  converges, then  $p(n+1) = p(n)P$ , so  $w$  is a **fixed point** of the system. Write

$$wP = w$$

and solve for  $w$ , i.e., find  $w$  as left eigenvector corresponding to the eigenvalue 1 or as (right) eigenvector associated to eigenvalue 1 for the transpose of  $P$

$$P^T w^T = w^T$$

$w$  might have to be normalized (you want a probability vector). Check that the norm  $\|w\|$  defined by

$$\|w\| = w_1 + \cdots + w_r$$

is equal to one. If not, use

$$\frac{w}{\|w\|}$$



# Back to genetics

Suppose we want to understand what it means to have hybrid individuals in the population

Investigate this using a process of continued matings

- ▶ Start with an individual of known or unknown genetic character (dominant, hybrid or recessive) and mate it with a hybrid
- ▶ Assume that the mating results in at least one offspring; choose one of the offspring at random and mate it with a hybrid
- ▶ Repeat this process through a number of generations

What can we expect in terms of the genetic composition of the population after a while?

⇒ consider MC with states GG, Gg and gg

3 states:  $S_1 = GG$ ,  $S_2 = Gg$  and  $S_3 = gg$ ; we use  $GG$ ,  $Gg$  and  $gg$  as well to name the states

$\nearrow$	$GG$	$Gg$	$gg$
$GG$	0.5	0.5	0
$Gg$	0.25	0.5	0.25
$gg$	0	0.5	0.5

The transition probabilities are thus

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

so

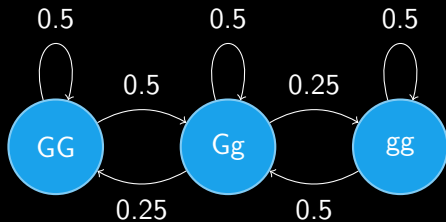
$$P^2 = \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{pmatrix}$$

$\implies P$  primitive  $\implies$  Markov chain regular

### Theorem 16

*$M$  primitive if the associated connection graph is strongly connected and there is at least one positive entry on the diagonal of  $M$*

This is checked directly on the transition graph



Compute left eigenvector associated to 1

$$(w_1, w_2, w_3) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (w_1, w_2, w_3)$$

$$\begin{aligned} \frac{1}{2}w_1 + \frac{1}{4}w_2 &= w_1 \\ \frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3 &= w_2 \\ \frac{1}{4}w_2 + \frac{1}{2}w_3 &= w_3 \end{aligned}$$

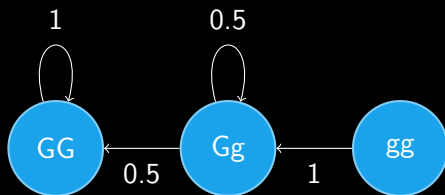
So  $w_1 = w_2/2$ ,  $w_3 = w_2/2$  and thus

$$\frac{1}{4}w_2 + \frac{1}{2}w_2 + \frac{1}{4}w_2 = w_2,$$

that is,  $w_2 = w_2$ , i.e.,  $w_2$  can take any value

$$\implies w = \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right)$$

Suppose now the same type of experiment, but mate each new generation with a GG individual instead of a Gg individual



$\nearrow$	GG	Gg	gg
GG	1	0	0
Gg	0.5	0.5	0
gg	0	1	0

and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- ▶ leave gg after 1 iteration and can never return
- ▶ when we leave Gg, we can never return
- ▶ we can never leave GG when we get there

# Absorbing Markov chains

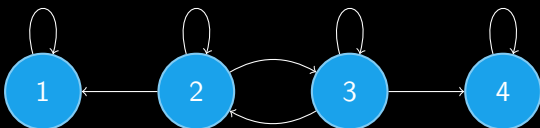
## Definition 17 (Absorbing state)

A state  $S_i$  in a Markov chain is **absorbing** if whenever it occurs on the  $n^{th}$  generation of the experiment, it then occurs on every subsequent step. In other words,  $S_i$  is absorbing if  $p_{ii} = 1$  and  $p_{ij} = 0$  for  $i \neq j$

## Definition 18 (Absorbing chain)

A Markov chain is **absorbing** if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state. In an absorbing Markov chain, a state that is not absorbing is called **transient**

Suppose we have a chain like the following



1. Does the process eventually reach an absorbing state?
2. What is the average number of steps spent in a transient state, if starting in a transient state?
3. What is the average number of steps before entering an absorbing state?
4. What is the probability of being absorbed by a given absorbing state, when there are more than one, when starting in a given transient state?



The answer to the first question ( “Does the process eventually reach an absorbing state?” ) is given by the following result

### Theorem 19

*In an absorbing Markov chain, the probability of reaching an absorbing state is 1*

To answer the other questions, write the transition matrix in **standard** form

For an absorbing chain with  $k$  absorbing states and  $r - k$  transient states, write transition matrix as

$$P = \begin{pmatrix} \mathbb{I}_k & \mathbf{0} \\ R & Q \end{pmatrix}$$

with following meaning

	Absorbing states	Transient states
Absorbing states	$\mathbb{I}_k$	$\mathbf{0}$
Transient states	$R$	$Q$

with  $\mathbb{I}_k$  the  $k \times k$  identity matrix,  $\mathbf{0}$  an  $k \times (r - k)$  matrix of zeros,  $R$  an  $(r - k) \times k$  matrix and  $Q$  an  $(r - k) \times (r - k)$  matrix. The matrix  $\mathbb{I}_{r-k} - Q$  is invertible. Let

- ▶  $N = (\mathbb{I}_{r-k} - Q)^{-1}$  the **fundamental matrix** of the MC
- ▶  $T_i$  sum of the entries on row  $i$  of  $N$
- ▶  $B = NR$

## Answers to our remaining questions:

2.  $N_{ij}$  average number of times the process is in the  $j$ th transient state if it starts in the  $i$ th transient state
3.  $T_i$  average number of steps before the process enters an absorbing state if it starts in the  $i$ th transient state
4.  $B_{ij}$  probability of eventually entering the  $j$ th absorbing state if the process starts in the  $i$ th transient state

## Back to the genetic example

The matrix is already in standard form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_1 & \mathbf{0} \\ R & Q \end{pmatrix}$$

with  $\mathbb{I}_1 = 1$ ,  $\mathbf{0} = (0 \ 0)$  and

$$R = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad Q = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

We have

$$\mathbb{I}_2 - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}$$

so

$$N = (\mathbb{I}_2 - Q)^{-1} = 2 \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

Then

$$T = N\mathbb{1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$B = NR = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2.  $N_{ij}$  average number of times the process is in the  $j$ th transient state if it starts in the  $i$ th transient state

$$N = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

3.  $T_i$  average number of steps before the process enters an absorbing state if it starts in the  $i$ th transient state

$$T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

4.  $B_{ij}$  probability of eventually entering the  $j$ th absorbing state if the process starts in the  $i$ th transient state

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$