Graphs – Introduction (theory)

Graphs versus networks

Mostly a terminology difference:

- graphs in the mathematical world
- networks elsewhere

I will mostly say *graphs* (this is a math course) but might oscillate

I will be basing the theory part on Berge (Graphs and Hypergraphs, 1973) and Bang-Jensen & Gutin (Digraphs: Theory, Algorithms and Applications, 2009)

Graph theory's vocabulary is not homogeneous. As much as possible, I will point out alternate terms

Graph

Intuitively: a graph is a set of points, and a set of arrows, with each arrow joining one point to another

The points are called the *vertices* of the graph and the arrows are the *arcs* of the graph

Definition 1 (Graph)

A graph G is a pair (X, U), where

1. X a set $\{x_1, x_2, \dots, x_n\}$ of elements called **vertices**
2. U a family $(u_1, u_2, \dots, u_m)$ of elements of the Cartesian product $X \times X$, called **arcs**. This family is often be denoted by the set $U = \{1, 2, \dots, m\}$ of its indices. An element (x, y) of $X \times X$ can appear more than once in this family. A graph in which no element of $X \times X$ appears more than p times is called a p-graph

Definition 2 (Order of a graph)

The number of vertices in a graph is called the **order** of the graph

Definition 3 (Loop)

An arc of G of the form (x, x) is called a **loop**

Definition 4 (Arc endpoints)

For an arc $u = (x, y)$, vertex x is the **initial endpoint**, and vertex y is the **terminal endpoint**

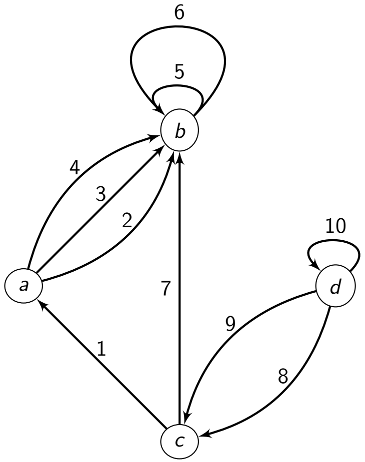
Definition 5 (Successors/Predecessors)

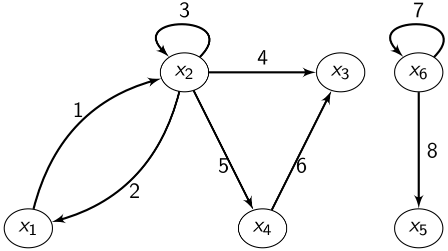
Vertex y is a **successor** (resp. **predecessor**) of vertex x if there is an arc with y as its initial (resp. terminal) endpoint and x as its termina1 (resp. initial) endpoint. The set of all successors (resp. predecessors) of x is denoted $\Gamma_G^+(x)$ (resp. $\Gamma_G^-(x)$)

Definition 6

The set of all neighbours of x is denoted $\Gamma(x) = \Gamma_G^+(x) \cup \Gamma_G^-(x)$

Examples





Directed versus undirected graph

Definition 7 (Edge)

In $G = (X, U)$, each arc $u_i = (x, y)$ determines a continuous line joining x and y. Such a line with no specification of direction is an **edge**, and is denoted by $e_i = [x, y]$. The family $(e_1, e_2, \ldots, e_m)$ of the edges of G is denoted by its set of indices $E = \{1, 2, \ldots, m\}$

Without direction, $G = (X, E)$, we talk of **undirected graph** (or **multigraph**)

An undirected graph in which there is no loops and no more than one edge joins any two vertices is **simple**

In modern texts (e.g., B-J&G), a **digraph** (or directed graph) has no loops

Definition 8 (Adjacent arcs/edges)

Two arcs (or two edges) are **adjacent** if they have at least one endpoint in common

Definition 9 (Multiplicity of a pair)

The **multiplicity** of a pair x, y is the number $m_G^+(x, y)$ of arcs with initial endpoint x and terminal endpoint y. Let

$$m_G^-(x, y) = m_G^+(y, x)$$
$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y)$$

If $x \neq y$, then $m_G(x, y)$ is number of arcs with both x and y as endpoints. If $x = y$, then $m_G(x, y)$ equals twice the number of loops attached to vertex x. If $A, B \subset X$, $A \neq B$, let

$$m_G^+(A, B) = \{u : u \in U, u = (x, y), x \in A, y \in B\}$$
$$m_G(A, B) = m_G^+(A, B) + m_G^+(A, B)$$

Definition 10 (Arc incident to a vertex)

If a vertex x is the initial endpoint of an arc u, which is not a loop, the arc u is **incident out of vertex** x

The number of arcs incident out of x plus the number of loops attached to x is denoted $d_G^+(x)$ and is the **outer demi-degree** of x

An arc **incident into vertex** x and the **inner demi-degree** $d_G^-(x)$ are defined similarly

Definition 11 (Degree)

The **degree** of vertex x is the number of arcs with x as an endpoint, each loop being counted twice. The degree of x is denoted $d_G(x) = d_G^+(x) + d_G^-(x)$

If each vertex has the same degree, the graph is **regular**

Definition 12 (Arc incident out of $A \subset X$)

If the initial endpoint of an arc u belongs to A, and if the terminal endpoint of arc u does not belong to A, then u is said to be incident out of A, and we write $u \in \omega^+(A)$

Similarly, we define an arc incident into A, and the set $\omega^-(A)$

Finally, the set of arcs incident to A is denoted

$$\omega(A) = \omega^+(A) \cup \omega^-(A)$$

Definition 13 (Symmetric graph)

If $m_G^+(x, y) = m_G^-(x, y)$ for all $x, y \in X$, the graph G is **symmetric**. A 1-graph $G = (X, U)$ is symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \in U$$

Definition 14 (Anti-symmetric graph)

If for each pair $(x, y) \in X \times X$,

$$m_G^+(x, y) + m_G^-(x, y) \leq 1$$

then the graph G is **anti-symmetric**. A 1-graph $G = (X, U)$ is anti-symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \notin U$$

An anti-symmetric 1-graph without its direction is a simple graph

Definition 15 (Complete graph)

A graph G is **complete** if

$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y) \geq 1$$

for all $x, y \in X$ such that $x \neq y$

A 1-graph is complete if, and only if,

$$(x, y) \notin U \implies (y, x) \in U$$

Definition 16 (n-clique)

A simple, complete graph on n vertices is called an n-**clique** and is often denoted K_n

Definition 17 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets X_1 and X_2, s.t. no two vertices in the same set are adjacent. This graph may be written $G = (X_1, X_2, U)$

Definition 18 (Complete bipartite graph)

If $\forall x_1 \in X_1, \forall x_2 \in X$, we have $m_G(x_1, x_2) \geq 1$, then $G = (X_1, X_2, U)$ is a **complete bipartite** graph

A simple, complete bipartite graph with $|X_1| = p$ and $|X_2| = q$ is often denoted $K_{p,q}$

Definition 19 (Subgraph of G generated by $A \subset X$)

The **subgraph** of G generated by A is the graph with A as its vertex set and with all the arcs in G that have both their endpoints in A. If $G = (X, \Gamma)$ is a 1-graph, then the subgraph generated by A is the 1-graph $G_A = (A, \Gamma_A)$ where

$$\Gamma_A(x) = \Gamma(x) \cap A \qquad (x \in A)$$

Definition 20 (Partial graph of G generated by $V \subset U$)

The graph (X, V) whose vertex set is X and whose arc set is V. In other words, it is graph G without the arcs $U - V$

Definition 21 (Partial subgraph of G)

A partial subgraph of G is the subgraph of a partial graph of G

Chain

Definition 22 (Chain of length $q > 0$)

A **chain** is a sequence $\mu = (u_1, \ldots, u_q)$ of arcs of G s.t. each arc in the sequence has one endpoint in common with its predecessor in the sequence and its other endpoint in common with its successor in the sequence

The number of arcs in the sequence is the **length** of chain μ

A chain that does not encounter the same vertex twice is **elementary**

A chain that does not use the same arc twice is **simple**

Path

Definition 23 (Path of length $q > 0$)

A **path** of length q is a chain $\mu = (u_1, \ldots, u_i, \ldots, u_q)$ in which the terminal endpoint of arc u_i is the initial endpoint of arc u_{i+1} for all $i < q$. For a 1-graph, a path is completely determined by the sequence of vertices $x_1, x_2, \ldots$ that it encounters. Hence, we often write

$$\begin{aligned}\mu &= ((x_1, x_2), (x_2, x_3), \ldots) \\ &= [x_1, x_2, \ldots, x_k, x_{k+1}] \\ &= \mu[x_1, x_{k+1}]\end{aligned}$$

Vertices x_1 and x_k are the **initial** and **terminal endpoints** of path μ. Similarly, for a simple graph, a chain μ with endpoints x and y is determined by the sequence of its vertices, and we may write

$$\mu = \mu[x, y] = [x, x_1, \ldots, y]$$

Definition 24 (Cycle)

A **cycle** is a chain such that

1. no arc appears twice in the sequence
2. the two endpoints of the chain are the same vertex

Definition 25 (Pseudo-cycle)

A **pseudo-cycle** is a chain $\mu = (u_1, u_2, \ldots, u_q)$ whose two endpoints are the same vertex and whose arcs are not necessarily distinct

Definition 26 (Circuit)

A **circuit** is a cycle $\mu = (u_1, u_2, \ldots, u_q)$ such that for all $i < q$, the terminal endpoint of u_i is the initial endpoint of u_{i+1}

Definition 27 (Connected graph)

A **connected graph** is a graph that contains a chain $\mu[x, y]$ for each pair x, y of distinct vertices

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a chain in G connecting x and y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 28 (Connected component of a graph)

The classes of the equivalence relation $\equiv$ partition X into connected sub-graphs of G called **connected components**

Definition 29 (Articulation set)

For a connected graph, a set A of vertices is called an **articulation set** (or a **cutset**) if the subgraph of G generated by $X - A$ is not connected

Definition 30 (Stable set)

A set S of vertices is called a stable set if no arc joins two distinct vertices in S

Definition 31 (Matrix associated with G)

If G has vertices $x_1, x_2, \ldots, x_n$, then the **matrix associated** with G is

$$a_{ij} = m_G^+(x_i, x_j)$$

Definition 32 (Adjacency matrix)

The matrix $a_{ij} + a_{ji}$ is the **adjacency matrix** associated with G

More on cycles

In a graph $G = (X, U)$, a **cycle** is a sequence of arcs

$$\mu = (u_1, u_2, \dots, u_q)$$

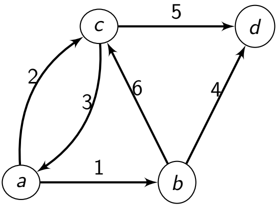
such that

1. each arc u_k, where $1 < k < q$, has one endpoint in common with the preceding arc u_{k-1}, and the other end point in common with the succeeding arc u_{k+1} (i.e., this sequence is a chain)
2. the sequence does not use the same arc twice
3. the initial vertex and terminal vertex of the chain are the same

An **elementary cycle** is a cycle in which, in addition,

4. no vertex is encountered more than once (except, of course, the initial vertex which is also the terminal vertex)

Example



Elementary cycles:

- $\boldsymbol{\mu}^1 = (1, 6, 2) = [abca]$
- $\boldsymbol{\mu}^2 = (1, 6, 3) = [abca]$
- $\boldsymbol{\mu}^3 = (2, 3) = [aca]$
- $\boldsymbol{\mu}^4 = (1, 4, 5, 2) = [abdca]$
- $\boldsymbol{\mu}^5 = (6, 5, 4) = [acdb]$
- $\boldsymbol{\mu}^6 = (1, 4, 5, 3) = [abdca]$

Given a cycle μ, denote μ^+ the set of all arcs in μ that are in the direction that the cycle is traversed and μ^- the set of all the other arcs in μ

Number the arcs in G as $1, 2, \ldots, m$, then the cycle μ is the vector

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$$

where

$$\mu_i = \begin{cases} 0 & \text{if } i \notin \mu^+ \cup \mu^- \\ +1 & \text{if } i \in \mu^+ \\ -1 & \text{if } i \in \mu^- \end{cases}$$

Cocycles

Let $A \subset X$ be nonempty and denote $\omega^+(A)$ the set of arcs that have only their initial endpoint in A and $\omega^-(A)$ the set of arcs that have only their terminal endpoint in A. Let

$$\omega(A) = \omega^+(A) \cup \omega^-(A)$$

A **cocycle** is a nonempty set of arcs of the form $\omega(A)$, partitioned into two sets $\omega^+(A)$ and $\omega^-(A)$

An **elementary cocycle** is the set of arcs joining two connected subgraphs A_1 and A_2 s.t.

- $A_1, A_2 \neq \emptyset$
- $A_1 \cap A_2 = \emptyset$
- $A_1 \cup A_2 = C$, with C a connected component of the graph

A colouring lemma

Lemma 33 (Arc colouring Lemma)

Consider G with arcs $1, \ldots, m$. Colour arc 1 black and arbitrarily colour the remaining arcs red, black or green. Then exactly one of the following holds true:

1. *there is an elementary cycle containing arc 1 and only red and black arcs with the property that all black arcs in the cycle have the same direction*
2. *there is an elementary cocycle containing arc 1 and only green and black arcs, with the property that all black arcs in the cocycle have the same direction*

Independent cycles and cycle bases

Consider cycles $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \ldots, \boldsymbol{\mu}^k$. The cycles are **independent** if

$$c_1\boldsymbol{\mu}^1 + c_2\boldsymbol{\mu}^2 + \cdots + c_k\boldsymbol{\mu}^k = \mathbf{0} \iff c_1 = c_2 = \cdots = c_k = 0$$

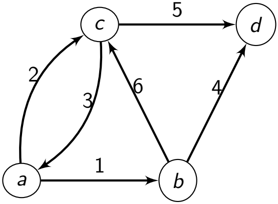
A **cycle basis** is an independent set $\{\boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \ldots, \boldsymbol{\mu}^k\}$ of cycles such that any cycle $\boldsymbol{\mu}$ can be written as

$$\boldsymbol{\mu} = c_1\boldsymbol{\mu}^1 + c_2\boldsymbol{\mu}^2 + \cdots + c_k\boldsymbol{\mu}^k$$

for $c_1, \ldots, c_k \in \mathbb{R}$

The constant k is the **cyclomatic number** of G, denoted $\nu(G)$

Example



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We have $\boldsymbol{\mu}^1 - \boldsymbol{\mu}^2 + \boldsymbol{\mu}^3 = \mathbf{0}$

An important result

Theorem 34

Let G be a graph with n vertices, m arcs and p connected components. Then the cyclomatic number of G is

$$\nu(G) = m - n + p$$

Trees and forests

Definition 35 (Tree)

A **tree** is a connected graph without cycles

(A tree is a special kind of 1-graph)

Definition 36 (Forest)

A **forest** is a graph whose connected components are trees

(i.e., a forest is a graph without cycles)

Theorem 37 (Characterisation of trees)

$H = (X, U)$ a graph of order $|X| = n > 2$. TFAE and all characterise a tree (i.e., H satisfying any of these is a tree):

1. *H connected and has no cycles*
2. *H has $n-1$ arcs and no cycles*
3. *H connected and has exactly $n-1$ arcs*
4. *H has no cycles, and if an arc is added to H, exactly one cycle is created*
5. *H connected, and if any arc is removed, the remaining graph is not connected*
6. *Every pair of vertices of H is connected by one and only one chain*

Proof of Theorem 37

We make abundant use of Theorem 34

[**1** $\implies$ **2**] Let p be the number of connected components, m the number of arcs and $\nu(H)$ the cyclomatic number. Since H connected, $p = 1$. Since H has no cycles,
$\nu(H) = m - n + p = 0$
$\implies m = n - p = n - 1$

[**2** $\implies$ **3**] Assume H has no cycles ($\nu(H) = 0$) and has $n - 1$ arcs ($m = n - 1$). Then, since

$$\nu(H) = m - n + p$$

$p = \nu(H) - m + n = 0 - (n - 1) - n = 1$, i.e., H is connected

Proof of Theorem 37 (cont.)

[**3** $\implies$ **4**] Assume H connected ($p = 1$) and contains exactly $n - 1$ arcs ($m = n - 1$). Then

$$\nu(H) = m - n + p = (n - 1) - n + 1 = 0$$

and H has no cycles

Now add an arc, i.e., suppose $m = n$. Then $\nu(H) = m - n + p = n - n + 1 = 1$ and there is one cycle in the new graph

Proof of Theorem 37 (cont.)

[**4** $\implies$ **5**] Assume H has no cycles ($\nu(H) = 0$) and that addition of an arc to H creates exactly one cycle

Suppose H not connected. Then there are two vertices, say a and b, that are not connected and adding the arc (a, b) does not create a cycle, a contradiction with "addition of an arc to H creates exactly one cycle"
$\implies$ $p = 1$. Since $\nu(H) = 0$, this implies that $m = n - 1$

Now suppose we remove an arc. We obtain graph H' with

$$m' = n' - 2 \quad \text{and} \quad \nu(H') = 0$$

So

$$p' = \nu(H') - m' + n' = 2$$

$\implies$ H' not connected

Proof of Theorem 37 (cont.)

[**5** $\implies$ **6**] Assume H connected and if any arc is removed, the remaining graph is not connected

Any vertices $a, b \in X$ are connected by a chain (since H connected). That chain is unique: suppose there is a second chain connecting a to b; then removing an arc from that chain does not disconnect the graph, since there is still the original chain connecting a and b

[**6** $\implies$ **1**] Assume every pair of vertices of H is connected by one and only one chain
Now assume H has a cycle. Then at least one pair of vertices would be connected by two distinct chains, a contradiction

$\square$

Definition 38 (Pendant vertex)

A vertex is **pendant** if it is adjacent to exactly one other vertex

Theorem 39

A tree of order $n \geq 2$ has at least two pendant vertices

Proof: Suppose H tree of order $n \geq 2$ with 0 or 1 pendant vertices

Consider a traveller traversing the graph edges starting from a pendant vertex (if there is one) or anywhere if there is no pendant vertex

If he does not allow himself to use same edge twice, he cannot go to the same vertex twice (H has no cycle)

If he arrives at vertex x, he can depart x using a new edge (x is not a pendant vertex as there are 0 or 1 pendant vertex and if there is 1, that's where he started)

So the trip continues without end, which is impossible as H finite $\square$

Theorem 40

A graph $G = (X, U)$ *has a partial graph that is a tree* $\iff$ G *connected*

Recall that a partial graph is a graph generated by a subset of the arcs (Definition 20 slide 14)

Proof: Suppose G not connected. Then no partial graph of G is connected $\implies$ there is no partial graph of G that is a tree [we want to show $P \iff Q$, we start with $\wedge Q \implies \wedge P$ (which $\iff P \implies Q$)]

Suppose G connected. Look for an arc whose removal does not disconnect G

- if there is none, G is a tree by Theorem 37(5)
- if there is, remove it and look for another one, etc. When no more arcs can be removed, the remaining graph is a tree with vertex set X $\square$

Spanning tree

The procedure in the proof of Theorem 40 gives a **spanning tree**

Can also build a spanning tree as follows:

- Consider any arc u_0
- Find arc u_1 that does not form a cycle with u_0
- Find arc u_2 that does not form a cycle with $\{u_0, u_1\}$
- Continue
- When you cannot continue anymore, you have a spanning tree

Strongly connected graphs

Recall that paths are defined in Definition 23 (slide 16)

$G = (X, U)$ connected. A **path of length 0** is any sequence $\{x\}$ consisting of a single vertex $x \in X$

For $x, y \in X$, let $x \equiv y$ be the relation "there is a path $\mu_1[x, y]$ from x to y as well as a path $\mu_2[y, x]$ from y to x". This is an equivalence relation (it is reflexive, symmetric and transitive)

Definition 41 (Strong components)

Sets of the form

$$A(x_0) = \{x : x \in X, x \equiv x_0\}$$

are equivalence classes; they partition X and are the **strongly connected components** of G

Definition 42 (Strongly connected graph)

G **strongly connected** if it has a single strong component

Theorem 43

G connected graph with ≥ 1 arc. TFAE

1. *G strongly connected*
2. *Every arc lies on a circuit*
3. *G contains no cocircuits*

Proof: [**1** $\implies$ **2**] (x, y) an arc of G; there is a path from y to x (G strongly connected), so arc (x, y) is contained in a circuit of G
[**2** $\implies$ **3**] Suppose G has a cocircuit containing arc (x, y); then G cannot have a circuit containing this arc by the Arc Colouring Lemma (Lemma 33 slide 24) with all arcs coloured black. This contradicts (2)
[**3** $\implies$ **1**] Assume G connected graph without cocircuits, but G not strongly connected. Since G not strongly connected, it has > 1 strongly connected component. Since G connected, there exist two distinct strongly connected components that are joined by an arc (a, b). Arc (a, b) is not contained in any circuit because otherwise a and b would be in the same strongly connected component. By Lemma 33, arc (a, b) is contained in some cocircuit. This contradicts (3) □

Theorem 44

G graph with ≥ 1 arc. TFAE

1. *G is a graph without circuits*
2. *Each arc is contained in a cocircuit*

Theorem 45

If G is a strongly connected graph of order n, then G has a cycle basis of $\nu(G)$ circuits

Definition 46 (Node, anti-node, branch)

$G = (X, U)$ strongly connected without loops and > 1 vertex. For each $x \in X$, there is a path from it and a path to it so x has at least 2 incident arcs. Specifically,

- $x \in X$ with > 2 incident arcs is a **node**
- $x \in X$ with 2 incident arcs is an **anti-node**

A path whose only nodes are its endpoints is a **branch**

Definition 47 (Minimally connected graph)

G is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

A minimally connected graph is 1-graph without loops

Definition 48 (Contraction)

$G = (X, U)$. The **contraction** of the set $A \subset X$ of vertices consists in replacing A by a single vertex a and replacing each arc into (resp. out of) A by an arc with same index into (resp. out of) a

Theorem 49

G minimally connected, $A \subset X$ generating a strongly connected subgraph of G. Then the contraction of A gives a minimally connected graph

Proof: First, show that contraction of A yields a 1-graph. If this were not the case, there would exist $x \notin A$ and $a, a' \in A$ s.t. $(x, a), (x, a') \in U$ (or, with $(a, x), (a', x) \in U$ but this would not change the proof). If one of these arcs is removed, the graph remains strongly connected. Thus, G is not minimally connected, a contradiction

Now show that the contraction of A yields a graph G' that is minimally connected. Clearly, G' strongly connected. If an arc u is removed, the remaining graph is not strongly connected, since the graph $(X, U - \{u\})$ not strongly connected $\square$

Theorem 50

G a minimally connected graph, G' be the minimally connected graph obtained by the contraction of an elementary circuit of G. Then

$$\nu(G) = \nu(G') + 1$$

Theorem 51

G minimally connected of order $n \geq 2 \implies G$ has ≥ 2 anti-nodes

Theorem 52

$G = (X, U)$. Then the graph C' obtained by contracting each strongly connected component of G contains no circuits

Arborescences

Definition 53 (Root)

Vertex $a \in X$ in $G = (X, U)$ is a **root** if all vertices of G can be reached by paths *starting* from a

Not all graphs have roots

Definition 54 (Quasi-strong connectedness)

G is **quasi-strongly connected** if $\forall x, y \in X$, exists $z \in X$ (denoted $z(x, y)$ to emphasize dependence on x, y) from which there is a path to x and a path to y

Strongly connected $\implies$ quasi-strongly connected (take $z(x, y) = x$); converse not true
Quasi-strongly connected $\implies$ connected

Definition 55 (Arborescence)

An **arborescence** is a tree that has a root

Lemma 56

$G = (X, U)$ has a root $\iff$ G quasi-strongly connected

Theorem 57

H graph of order $n > 1$. TFAE (and all characterise an arborescence)

1. *H quasi-strongly connected without cycles*
2. *H quasi-strongly connected with $n-1$ arcs*
3. *H tree having a root a*
4. *$\exists a \in X$ s.t. all other vertices are connected with a by 1 and only 1 path from a*
5. *H quasi-strongly connected and loses quasi-strong connectedness if any arc is removed*
6. *H quasi-strongly connected and $\exists a \in X$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

7. *H has no cycles and $\exists a \in X$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

Theorem 58

G has a partial graph that is an arborescence $\iff$ G quasi-strongly connected

Theorem 59

$G = (X, E)$ simple connected graph and $x_1 \in X$. It is possible to direct all edges of E so that the resulting graph $G_0 = (X, U)$ has a spanning tree H s.t.

1. *H is an arborescence with root x_1*
2. *The cycles associated with H are circuits*
3. *The only elementary circuits of G_0 are the cycles associated with H*

Counting trees

Proposition 60

X a set with n distinct objects, $n_1, \ldots, n_p$ nonnegative integers s.t. $n_1 + \cdots + n_p = n$. The number of ways to place the n objects into p boxes $X_1, \ldots, X_p$ containing $n_1, \ldots, n_p$ objects respectively is

$$\binom{n}{n_1, \ldots, n_p} = \frac{n!}{n_1! \cdots n_p!}$$

Proposition 61 (Multinomial formula)

Let $a_1, \ldots, a_p \in \mathbb{R}$ be p real numbers, then

$$(a_1 + \cdots + a_p)^n = \sum_{n_1, \ldots, n_p \geq 0} \binom{n}{n_1, \ldots, n_p} (a_1)^{n_1} \cdots (a_p)^{n_p}$$

(Recall the binomial formula, where $p = 2$)

Theorem 62

Denote $T(n; d_1, \ldots, d_n)$ the number of distinct trees H with vertices $x_1, \ldots, x_n$ and with degrees $d_H(x_1) = d_1, \ldots, d_H(x_n) = d_n$. Then

$$T(n; d_1, \ldots, d_n) = \binom{n-2}{d_1 - 1, \ldots, d_n - 1}$$

Theorem 63

The number of different trees with vertices $x_1, \ldots, x_n$ is n^{n-2}

There is a whole industry of similar results (as well as for arborescences), but we will stop here. The main point is that we are talking about a large number of possibilities..

Matrices associated to a graph

There are multiple matrices associated to a graph

We review some of them now

We also review some of their uses

The branch of graph theory that studies the properties of matrices derived from graphs and uses of these matrices in determining graph properties if *spectral graph theory*

Adjacency matrix

We have already seen adjacency matrices, let us recall the definition here

Definition 64 (Adjacency matrix)

G a 1-graph, then the **adjacency matrix** $A = [a_{ij}]$ is defined as follows

$$a_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \in U \\ 0 & \text{otherwise} \end{cases}$$

We often write $A(G)$ and, reciprocally, if A is an adjacency matrix, $G(A)$ the corresponding graph

G undirected $\implies A(G)$ symmetric

$A(G)$ has nonzero diagonal entries if G is not simple

Adjacency matrix – Multigraph case

Definition 65

G an ℓ-graph, then the adjacency matrix $A = [a_{ij}]$ is defined as follows

$$a_{ij} = \begin{cases} k & \text{if arc there are } k \text{ arcs } (i,j) \in U \\ 0 & \text{otherwise} \end{cases}$$

with $k \leq \ell$

G undirected $\implies A(G)$ symmetric

$A(G)$ has nonzero diagonal entries if G is not simple

Weighted adjacency matrices

Sometimes, adjacency matrices (typically for 1-graphs) have real entries, usually positive

This means that the arcs/edges have been given a weight

Spectrum of a graph

We will come back to this later, but for now..

Definition 66 (Spectrum of a graph)

The **spectrum** of a graph G is the spectrum (set of eigenvalues) of its associated adjacency matrix $A(G)$

This is regardless of the type of adjacency matrix or graph

Incidence matrix

Definition 67 (Incidence matrix – Undirected case)

$G = (X, E)$ an undirected graph of order n with p edges. The **incidence** matrix of G is an $n \times p$ matrix with vertices as rows and edges as columns and where $B = [b_{ij}]$ satisfies

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is incident to vertex } i \\ 0 & \text{otherwise} \end{cases}$$

Definition 68 (Incidence matrix – Directed case)

$G = (X, U)$ a directed graph of order n with p arcs. The **incidence** matrix of G is an $n \times p$ matrix with vertices as rows and edges as columns and where $B = [b_{ij}]$ satisfies

$$b_{ij} = \begin{cases} 1 & \text{if arc } j \text{ ``enters'' vertex } i \\ -1 & \text{if arc } j \text{ ``leaves'' vertex } i \\ 0 & \text{otherwise} \end{cases}$$

Degree matrix

Definition 69 (Degree matrix)

The **degree** matrix $D = [d_{ij}]$ for G is a $n \times n$ diagonal matrix defined as

$$d_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

In an undirected graph, this means that each loop increases the degree of a vertex by two

In a directed graph, the term "degree" may refer either to indegree (the number of incoming edges at each vertex) or outdegree (the number of outgoing edges at each vertex)

Laplacian matrix

Definition 70

$G = (X, U)$ a simple graph with n vertices.
The **Laplacian** matrix is

$$L = D(G) - A(G)$$

where $D(G)$ is the degree matrix and $A(G)$ is the adjacency matrix

G simple graph $\implies$ $A(G)$ only contains 1 or 0 and its diagonal elements are all 0

For directed graphs, either the indegree or outdegree is used, depending on the application

Elements of L are given by

$$\ell_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Counting paths

To count paths between vertices x and y in a graph, we use the adjacency matrix

Theorem 71

G a graph and $A(G)$ its adjacency matrix. Denote $P = [p_{ij}]$ the matrix $P = A^k$. Then p_{ij} is the number of distinct paths of length k from i to j in G

This provides an interesting connection with linear algebra

Definition 72 (Irreducible matrix)

A matrix $A \in \mathcal{M}_n$ is **reducible** if $\exists P \in \mathcal{M}_n$, permutation matrix, s.t. P^TAP can be written in block triangular form. If no such P exists, A is **irreducible**

Theorem 73

A irreducible $\iff$ $G(A)$ strongly connected

Perron-Frobenius theorem

Assignment 7 had the following:
$A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ **nonnegative** if $a_{ij} \geq 0$ $\forall i, j = 1, \ldots, n$; $\mathbf{v} \in \mathbb{R}^n$ nonnegative if $v_i \geq 0$ $\forall i = 1, \ldots, n$. **Spectral radius** of A

$$\rho(A) = \max_{\lambda \in \mathrm{Sp}(A)} \{|\lambda|\}$$

$\mathrm{Sp}(A)$ the **spectrum** of A

Theorem 74 (PF – Nonnegative case)

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$. *Then* $\exists \mathbf{v} \geq \mathbf{0}$ *s.t.*

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

Theorem 75 (PF – Irreducible case)

Let $0 \leq A \in \mathcal{M}_n(\mathbb{R})$ *irreducible. Then* $\exists \mathbf{v} > \mathbf{0}$ *s.t.*

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

$\rho(A) > 0$ *and with algebraic multiplicity 1*

No nonnegative eigenvector is associated to any other eigenvalue of A

Primitive matrices

Definition 76

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$ **primitive** (with **primitivity index** $k \in \mathbb{N}_+^*$) if $\exists k \in \mathbb{N}_+^*$ s.t.

$$A^k > 0,$$

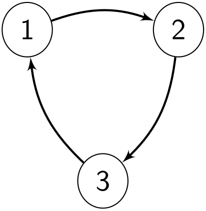
with k the smallest integer for which this is true. A **imprimitive** if it is not primitive

A primitive $\implies$ A irreducible; converse false

Theorem 77

$A \in \mathcal{M}_n(\mathbb{R})$ irreducible and $\exists i = 1, \ldots, n$ s.t. $a_{ii} > 0 \implies A$ primitive

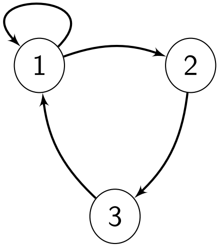
Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d =$ gcd of all the lengths of closed walks in $G(A)$



Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walks in $G(A)$ (lengths): $1 \to 1$ (3), $2 \to 2$ (3), $2 \to 2$ (3) $\implies$ gcd $= 3 \implies$ $d = 3$ (here, all eigenvalues have modulus 1)



$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walk $1 \to 1$ has length 1 $\implies$ gcd of lengths of closed walks is 1 $\implies$ A primitive

Theorem 78

$0 \leq A \in \mathcal{M}_n$. *A primitive* $\implies A^k > 0$ *for some* $0 < k \leq (n-1)n^n$

Theorem 79

$A \geq 0$ *primtive. Suppose the shortest simple directed cycle in* $G(A)$ *has length* s, *then primitivity index is* $\leq n + s(n-1)$

Theorem 80

$0 \leq A \in \mathcal{M}_n$ *primitive* $\iff A^{n^2-2n+2} > 0$

Theorem 81

$0 \leq A \in \mathcal{M}_n$ *irreducible. A has* d *positive entries on the diagonal* $\implies$ *primitivity index* $\leq 2n - d - 1$

Theorem 82

$0 \leq A \in \mathcal{M}_n$, $\lambda_P = \rho(A)$ the Perron root of A, $\mathbf{v}_P$ and $\mathbf{w}_P$ the corresponding right and left Perron vectors of A, respectively, d the index of imprimitivity of A (with $d = 1$ when A is primitive) and $\lambda_j \in \sigma(A)$ the spectrum of A, with $j = 2, \ldots, n$ unless otherwise specified (assuming $\lambda_1 = \lambda_P$)

NONNEGATIVE

REDUCIBLE

- $\lambda_P \geq 0$
- $\mathbf{w}_P \geq 0$
- $\mathbf{v}_P \geq 0$
- $\lambda_P \geq |\lambda_j|$

IRREDUCIBLE

IMPRIMITIVE

- $\lambda_P > 0$
- $\mathbf{w}_P > 0$
- $\mathbf{v}_P > 0$
- $\lambda_P = |\lambda_j|$, $j = 2, \ldots, d$
- $\lambda_P > |\lambda_j|$, $j > d$

PRIMITIVE

- $\lambda_P > 0$
- $\mathbf{w}_P > 0$
- $\mathbf{v}_P > 0$
- $\lambda_P > |\lambda_j|$, $j \neq P$