Graphs – Introduction (theory)

Graphs versus networks

Mostly a terminology difference:

- graphs in the mathematical world
- networks elsewhere

I will mostly say *graphs* (this is a math course) but might oscillate

Beware: language is not consistent, so make sure you read the definitions at the start of whatever source you are using

Graphs vs digraphs vs multigraphs vs multidigraphs vs ...

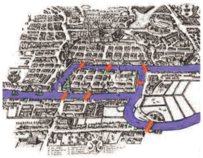
Name-wise and notation-wise, this domain is a mess. We see definitions later, but as far as possible, in these notes:

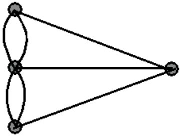
- The vertex set V is essentially the only constant in what follows.
- An undirected graph is denoted $G = (V, E)$, where E are the edges.
- An undirected multigraph is denoted $G_M = (V, E)$. We will not be using these much.
- A directed graph (or digraph) is denoted $G = (V, A)$, where A are the arcs.
- A directed multigraph (or multidigraph) is denoted $G_M = (V, A)$.
- Any of the above is called a graph and is denoted $G = (V, X)$, when we seek generality.

And just to confuse the whole thing more: we often say *graph* for *unoriented graph*.

The bridges of Königsberg

A "real life" problem from Euler: is it possible to cross the seven bridges in a continuous walk without recrossing any of them?



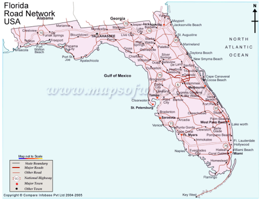


Mathematical problem

Is it possible to find a trail containing all edges of the graph?

Finding a cycle with all vertices

A salesperson must visit a couple of cities for their job. Is it possible for them to plan a round trip using highways enabling him to visit each specified city exactly once?



- vertices correspond to cities
- two vertices are connected iff a highway connects the corresponding cities and does not pass through any other city.

Mathematical problem

Is it possible to find a cycle containing all graph vertices?

How far is it to drive through n cities?

What is the minimal length of driving needed to drive through n cities?

- vertices correspond to the cities
- all cities are connected; each edge has a value assigned to it

Mathematical problem

What is the minimal spanning tree associated to the graph?

Binary relation

Definition 1

- A binary relation is an arbitrary association of elements of one set with elements of another (maybe the same) set.
- A binary relation over the sets X and Y is defined as a subset of the Cartesian product $X \times Y = \{(x, y) | x \in X, y \in Y\}$.
- $(x, y) \in R$ is read "x is $R-$related to y" and is denoted xRy.
- If $(x, y) \notin R$, we write not xRy.

Definition 2 (Properties of binary relations)

A binary relation R over a set X is

- **Reflexive** if $\forall x \in X$, xRx.
- **Irreflexive** if there does not exist $x \in X$ such that xRx.
- **Symmetric** if $xRy \Rightarrow yRx$.
- **Asymmetric** if $xRy \Rightarrow$ not yRx.
- **Antisymmetric** if xRy and $yRx \Rightarrow x = y$.
- **Transitive** if xRy and $yRz \Rightarrow xRz$.
- **Total** (or **complete**) if $\forall x, y \in X$, xRy or yRx.

Definition 3 (Equivalence relation)

A relation which is reflexive, symmetric and transitive is called an **equivalence relation**.

Definition 4 (Partial order)

A relation which is reflexive, antisymmetric and transitive is called a **partial order**.

Graph

Intuitively: a graph is a set of points, and a set of relations between the points

The points are called the *vertices* of the graph and the relations are the *edges* of the graph

We can also think of the relations as being one directional, in which case the relations are the *arcs* of the digraph (a contraction of “directed graph“)

Graph, vertex and edge

Definition 5 (Graph)

An undirected graph is a pair $G = (V, E)$ of sets such that

- V is a set of points: $V = \{v_1, v_2, v_3, .., v_p\}$
- E is a set of 2-element subsets of V: $E = \{\{v_i, v_j\}, \{v_i, v_k\}, \ldots, \{v_n, v_p\}\}$, also noted $E = \{v_i v_j, v_i v_k, \ldots, v_n v_p\}$.

Definition 6 (Vertex)

The elements of V are the vertices (or nodes, or points) of the graph G. V is the vertex set of the graph G, also noted $V(G)$.

Definition 7 (Edge)

The elements of E are the edges (or lines) of the graph G. E is the edge set of the graph G, also noted $E(G)$.

Order and Size

Definition 8 (Order of a graph)

The number of vertices in G is the order of G. Using the notation $|V(G)|$ for the *cardinality* of $V(G)$,

$$|V(G)| = \text{order of G.}$$

Definition 9 (Size of a graph)

The number of edges in G is the size of G,

$$|E(G)| = \text{size of G.}$$

- A graph having order p and size q is called a $(p, q)-$graph.
- A graph is finite if $|V(G)| < \infty$.

Adjacent - Incident

Definition 10 (Incident)

- If $e = uv \in E(G)$, then u and v are each incident with e.
- The two vertices incident with an edge are its ends.

Definition 11 (Adjacent)

- Two vertices u and v are adjacent in a graph G if $uv \in E(G)$.
- If uv and uw are distinct edges (i.e. $v \neq w$) of a graph G, then uv and uw are adjacent edges.

Definition 12 (Multiple edge)

Multiple edges are two or more edges connecting the same two vertices within a multigraph.

Definition 13 (Loop)

A **loop** is an edge with both the same ends; *e.g.* $\{u, u\}$ is a loop.

Definition 14 (Simple graph)

A **simple graph** is a graph which contains no loops or multiple edges.

Definition 15 (Multigraph)

A **multigraph** is a graph which can contain multiple edges or loops.

Graph and binary relations

A (simple) graph G can be defined in term of a vertex set V and an irreflexive and symmetric binary relation over V.
R is symmetric if $(u, v) \in R \Rightarrow (v, u) \in R$ (or, in other words, $uRv \implies vRu$).
Hence, $\{(u, v), (v, u)\} \in E(G)$ ($\{(u, v), (v, u)\}$ is an edge).
The set of edges $E(G)$ is the set of symmetric pairs in R.

Definition 16 (Degree of a vertex)

Let v be a vertex of $G = (V, E)$.

- The number of edges of G incident with v is the **degree** of v in G.
- The number of edges of G at v is the **degree** of v in G.
- The degree of v in G is noted $d_G(v)$ or $deg_G(v)$.

Theorem 17

Let G be a $(p, q)-$graph with vertices $v_1, \ldots, v_p$, then

$$\sum_{i=1}^{p} d_G(v_i) = 2q.$$

Definition 18 (Odd vertex)

A vertex is an odd vertex is its degree is odd.

Definition 19 (Even vertex)

A vertex is called even vertex is its degree is even.

Theorem 20

Every graph contains an even number of odd vertices.

Regular graph

Definition 21 (Regular graph)

If all the vertices of G have the same degree k, then the graph G is $k-$regular.

Definition 22 (Complete graph)

A graph is complete if every two of its vertices are adjacent.

Definition 23 (n-clique)

A simple, complete graph on n vertices is called an n-**clique** and is often denoted K_n

Note that a complete graph of order p is $(p-1)-$regular

Bipartite graph

Definition 24 (Bipartite graph)

A graph $G = (V, E)$ is called **bipartite** if it is possible to partition the vertex set of G into two subsets, say V_1 and V_2 so that every edge of G joins a vertex of V_1 with a vertex of V_2 and no vertex joins another vertex of its own set.

Definition 25 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets X_1 and X_2, s.t. no two vertices in the same set are adjacent. This graph may be written $G = (X_1, X_2, U)$

Complete Bipartite graph

Definition 26 (Complete bipartite graph)

A bipartite graph in which every two vertices from the 2 different partitions are adjacent is called a complete bipartite graph.

Definition 27 (Complete bipartite graph)

If $\forall x_1 \in X_1, \forall x_2 \in X$, we have $m_G(x_1, x_2) \geq 1$, then $G = (X_1, X_2, U)$ is a **complete bipartite** graph

A simple, complete bipartite graph with $|X_1| = p$ and $|X_2| = q$ is often denoted $K_{p,q}$

Some specific graphs

Definition 28 (Tree)

Any connected graph that has no cycles is called a tree.

Definition 29 (Cycle C_n)

For $n \geq 3$, the cycle, denoted C_n, is a connected graph of order n that is a cycle on n vertices.

Definition 30 (Path P_n)

The path, P_n, is a connected graph that consists of $n \geq 2$ vertices and $n - 1$ edges. Two vertices of P_n have degree 1 and the rest are of degree 2.

Definition 31 (Star S_n)

The star of order n is the complete bipartite graph $K_{1,n-1}$ (1 vertex of degree $n - 1$, and $n - 1$ vertices of degree 1).

Isomorphic graphs

Definition 32 (Isomorphic graphs)

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. G_1 and G_2 are **isomorphic** if there exists an isomorphism ϕ from G_1 to G_2, that is defined as an injective mapping $\phi : V(G_1) \to V(G_2)$ such that two vertices u_1 and v_1 are adjacent in G_1 if and only if the vertices $\phi(u_1)$ and $\phi(v_1)$ are adjacent in G_2.

Definition 33 (Another formulation of isomorphic graphs)

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs; G_1 and G_2 are isomorphic if there exists an isomorphism ϕ from G_1 to G_2, defined as an injective mapping $\phi : V(G_1) \to V(G_2)$ such that $\{u_1, v_1\} \in E(G_1)$ if and only if $\{\phi(u_1), \phi(v_1)\} \in E(G_2)$ for all $u_1, v_1 \in V(G_1)$.

If ϕ is an isomorphism from G_1 to G_2, then the inverse mapping ϕ^{-1} from $V(G_2)$ to $V(G_1)$ also satisfies the definition of an isomorphism.
As a consequence, if G_1 and G_2 are isomorphic graphs, then

- G_1 is isomorphic to G_2
- G_2 is isomorphic to G_1

Theorem 34

The relation "is isomorphic to" is an equivalence relation on the set of all graphs.

Theorem 35

If G_1 and G_2 are isomorphic graphs, then the degrees of vertices of G_1 are exactly the degrees of vertices of G_2.

Subgraph

Definition 36 (Subgraph)

Let $G = (V(G), E(G))$ be a graph. A graph $H = (V(H), E(H))$ is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 37 (Another definition of subgraph)

If a graph F is isomorphic to a subgraph H of G, then F is also called a subgraph of G.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs.

Definition 38 (Union of G_1 and G_2)

$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

Definition 39 (Intersection of G_1 and G_2)

$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$

Definition 40 (Disjoint graphs)

If $G_1 \cap G_2 = (\emptyset, \emptyset) = \emptyset$ (empty graph) then G_1 and G_2 are disjoint.

Definition 41 (Complement of G_1)

The complement $\bar{G}_1$ of G_1 is the graph on V_1, with the edge set $E(\bar{G}_1) = [V_1]^2 \setminus E_1$ ($e \in E(\bar{G}_1)$ iff $e \notin E_1$).

Walk

Let $G = (V, E)$ be a graph.

Definition 42 (Walk)

A **walk** in a graph G is a non-empty alternating sequence $v_0 e_0 v_1 e_1 v_2 \dots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. This walk begins with v_0 and ends with v_k.

Definition 43 (Length of a walk)

The length of a walk is equal to the number of edges in the walk.

Definition 44 (Closed walk)

If $v_0 = v_k$, the walk is closed.

Trail and path

Let $G = (V, E)$ be a graph.

Definition 45 (Trail)

If the edges in the walk are all distinct, it defines a trail in G.

Definition 46 (Path)

If the vertices in the walk are all distinct, it defines a path in G.

The sets of vertices and edges determined by a trail is a subgraph.

Distance between two vertices

Let $G = (V, E)$ be a graph.

Definition 47 (Distance between two vertices)

The distance $d(u, v)$ in G between two vertices u and v is the length of the shortest path linking u and v in G. If no such path exists, we assume $d(u, v) = \infty$.

Circuit and cycle

Definition 48 (Circuit)

A trail linking u to v, containing at least 3 edges and in which $u = v$, is a circuit.

Definition 49 (Cycle)

A circuit which does not repeat any vertices (except the first and the last) is a cycle (or **simple circuit**).

Definition 50 (Length of a cycle)

The length of a cycle is its number of edges.

Definition 51 (Eulerian trail)

A trail containing all the vertices and edges of a multigraph M is called a Eulerian trail of M.

Definition 52 (Traversable graph)

If a graph G has a Eulerian trail, then G is called a traversable graph.

Definition 53 (Eulerian circuit)

A circuit containing all the vertices and edges of a multigraph M is called a Eulerian circuit of M.

Definition 54 (Eulerian graph)

A graph (resp. multigraph) containing an Eulerian circuit is called a Eulerian graph (resp. multigraph).

Theorem 55

A multigraph M is traversable if and only if M is connected and has exactly two odd vertices. Furthermore, any Eulerian trail of M begins at one of the odd vertices and ends at the other odd vertex.

Theorem 56

A multigraph M is Eulerian if and only if M is connected and every vertex of M is even.

Fleury's algorithm to find a Eulerian circuit

(for a connected graph with no odd vertices)

- Pick any vertex as a starting point.
- Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to.
- Continue until you return to your starting point.

RESULT: a Eulerian circuit

Fleury's algorithm to find a Eulerian trail

(for a connected graph with exactly 2 odd vertices)

- Start at one of the odd vertices.
- Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to.
- Continue until every edge has been traveled.

RESULT: a Eulerian trail

Definition 57 (Hamiltonian path)

A path containing all vertices of a graph G is called a Hamiltonian path of G.

Definition 58 (Traceable graph)

If a graph G has an Hamiltonian path, then G is called a traceable graph.

Definition 59 (Hamiltonian cycle)

A cycle containing all vertices of a graph G is called a Hamiltonian cycle of G.

Definition 60 (Hamiltonian graph)

A graph containing a Hamiltonian cycle is called a Hamiltonian graph.

Theorem 61 (Dirac's theorem)

If G is a graph of order $p \geq 3$ such that $deg(v) \geq p/2$ for every vertex v of G, then G is Hamiltonian.

Theorem 62 (Ore's theorem)

If G is a graph of order $p \geq 3$ such that for all distinct nonadjacent vertices u and v of G,

$$deg(u) + deg(v) \geq p,$$

then G is Hamiltonian.

Connected vertices and graph, components

Definition 63 (Connected vertices)

Two vertices u and v in a graph G are **connected** if $u = v$, or if $u \neq v$ and there exists a path in G that links u and v.

Definition 64 (Connected graph)

A graph is **connected** if every two vertices of G are connected; otherwise, G is **disconnected**.

A necessary condition for connectedness

Theorem 65

A connected graph on p vertices has at least $p - 1$ edges.

In other words, a connected graph G of order p has $\text{size}(G) \geq p - 1$.

Connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a path in G connecting x and y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 66 (Connected component of a graph)

The classes of the equivalence relation $\equiv$ partition X into connected sub-graphs of G called connected components (or **components** for short) of G.

A connected subgraph H of a graph G is a component of G if H is not contained in any connected subgraph of G having more vertices or edges than H.

Vertex deletion & cut vertices

Definition 67 (Vertex deletion)

If v is a vertex of G, the graph $G - v$ is the graph formed from G by removing v and all edges incident with v.

Definition 68 (Cut-vertices)

If v is a cut-vertex of a connected graph G, then $G - v$ is disconnected.

Edge deletion & bridges

Definition 69 (Edge deletion)

If e is an edge of G, the graph $G - e$ is the graph formed from G by removing e from G.

Definition 70 (Bridge)

An edge e in a connected graph G is called a bridge if $G - e$ is disconnected.

Theorem 71

Let G be a connected graph. An edge e of G is a bridge of G if and only if e does not lie on any cycle of G.

Planar graph

Definition 72 (Planar graph)

A graph is planar if it can be drawn in the plane with no crossing edges. Otherwise it is nonplanar.

Definition 73 (Plane graph)

A plane graph is a graph that is drawn in the plane with no crossing edges. (This is only possible if the graph is planar.)

Let G be a plane graph.

- the connected parts of the plane are called **regions**.
- vertices and edges that are incident with a region R make up a **boundary** of R.

Theorem 74 (Euler's formula)

Let G be a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2.$$

Corollary 75

Let G be a plane graph with p vertices, q edges, r regions, and k connected components, then

$$p - q + r = k + 1.$$

Theorem 76

Let G be a connected planar graph with p vertices and q edges, where $p \geq 3$, then

$$q \leq 3p - 6.$$

(a maximal connected planar graph with p vertices has $q = 3p - 6$ edges.)

Corollary 77

If G is a planar graph, then $\delta(G) \leq 5$, where $\delta(G)$ is the minimal degree of G. (every planar graph contains a vertex of degree less than 6)

Two well-known non-planar graphs

$K_{3,3}$ is non-planar.
K_5 is non-planar.
As a matter of fact, a graph having a subgraph that is $K_{3,3}$ or K_5 is a sufficient condition for non-planarity, as we see later.

Definition 78 (Subdivision of G)

Given a graph G, a subdivision of G is a graph that can be obtained by inserting any number of vertices of degree 2 along a edges of G.

Theorem 79 (Kuratowski Theorem)

A graph G i planar if and only if it contains no subgraph isomorphic to K_5 or $K_{3,3}$ or any subdivision of K_5 or $K_{3,3}$.

Note: If a graph G is nonplanar and G is a subgraph of G', then G' is also nonplanar.

Definition 80 (Coloring of a graph G)

- assignment elements (colors) of some sets to vertices of G.
- one color to each vertex so that adjacent vertices are assigned to different colors.

(A coloring of a graph G is an assignment of colors to the vertices of G such that adjacent vertices have different colors.)

Definition 81 ($n-$coloring of G)

A $n-$coloring is a coloring of G using n colors.

Definition 82 ($n-$colorable)

G is $n-$colorable if there exists a coloring of G which uses n colors.

Definition 83 (Chromatic number)

The chromatic number of a graph G is the minimal value n for which an $n-$coloring of G exists. $\chi(G) =$ chromatic number.

Theorem 84 (Some properties)

- $\chi(G) = 1$ *if and only if* G *have no edges.*
- *If* $G = K_{n,m}$*, then* $\chi(G) = 2$*.*
- *If* $G = K_n$*, then* $\chi(G) = n$*.*
- *For any graph* G*,*

  $$\chi(G) \leq 1 + \Delta(G)$$

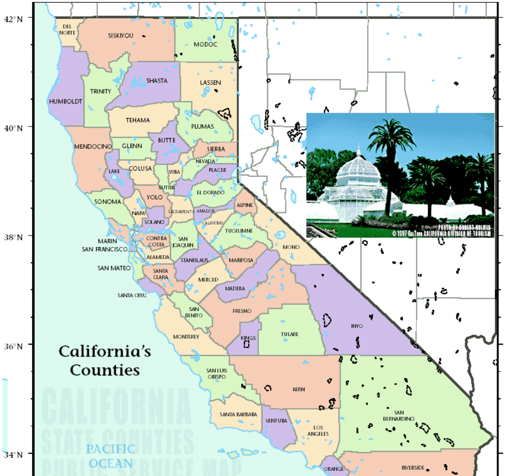
  where $\Delta(G)$ *is the maximum degree of* G*.*
- *If* G *is a planar graph, then* $\chi(G) \leq 4$*.*
- *If* G *is a planar graph, then* $\chi(G) \leq 5$*.*

Welch-Powell algorithm for coloring a graph G

1. Order the vertices of G in decreasing degree. (Such an ordering may not be unique since some vertices may have the same degree.)
2. Use one color to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this color.
3. Start again at the top of the list and repeat the process painting previously unpainted vertices using a second color.
4. Continue repeating with additional colors until all the vertices have been painted.

“Real life” problem

What is the minimal number of colors to color all counties in the map? (two adjacent counties must have different colors)



“Real life” problem

What is the minimal number of colors to color all counties in the map? (two adjacent counties must have different colors) Mathematical representation:

- vertices correspond to the counties
- two vertices are linked iff the two counties are adjacent.

Mathematical problem

What is the chromatic number of the graph associated to the map?

Definitions

Definition 85 (Digraph)

A directed graph (or digraph) is a pair $G = (V, A)$ of sets such that

- V is a set of points: $V = \{v_1, v_2, v_3, .., v_p\}$
- A is a set of ordered pairs of V: $A = \{(v_i, v_j), (v_i, v_k), \ldots, (v_n, v_p)\}$, also noted $A = \{v_i v_j, v_i v_k, \ldots, v_n v_p\}$.

Definition 86 (Vertex)

The elements of V are the vertices of the digraph G. V is the vertex set of the digraph G, also noted $V(G)$.

Definition 87 (Arc)

The elements of A are the arcs (directed edges) of the digraph G. A is the arc set of the digraph G, also noted $A(G)$.

Digraph and binary relation

A digraph D can be defined in term of a vertex set V and an irreflexive relation R over V.
The defining relation R of the digraph G need not be symmetric.

Directed network

Definition 88 (Directed network)

A directed network is a digraph together with a function f,

$$f : A \to \mathbb{R},$$

which maps the arc set A into the set of real number. The value of the arc $uv \in A$ is $f(uv)$.

Loops & Multiple arcs

Definition 89 (Loop)

A **loop** is an arc with both the same ends; *e.g.* (u, u) is a loop.

Definition 90 (Multiple arcs)

Multiple arcs (or multi-arcs) are two or more arcs connecting the same two vertices.

Multidigraph/Digraph

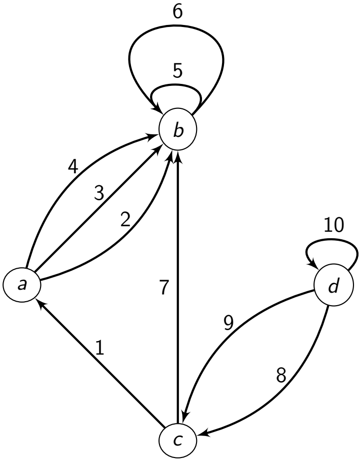
Definition 91 (Multidigraph)

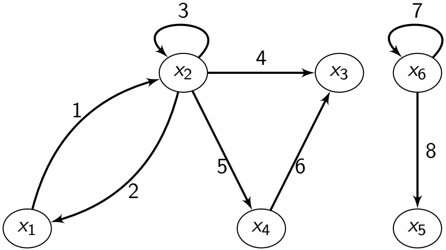
A multidigraph is a digraph which allows repetition of arcs or loops.

Definition 92 (Digraph)

In a digraph, no more than one arc can join any pair of vertices.

Examples





Let $G = (V, A)$ be a digraph.

Definition 93 (Arc endpoints)

For an arc $u = (x, y)$, vertex x is the initial endpoint, and vertex y is the terminal endpoint

Definition 94 (Predecessor - Successor)

If $(u, v) \in A(G)$ is an arc of G, then

- u is a predecessor of v,
- v is a successor of u.

Definition 95 (Neighbours of a vertex)

Let $x \in V$ be a vertex. The neighbours of x is the set $\Gamma(x) = \Gamma_G^+(x) \cup \Gamma_G^-(x)$, where $\Gamma_G^+(x)$ and $\Gamma_G^-(x)$ are, respectively, the set of successors and predecessors of v.

Sources and sinks

Definition 96 (Directed away - Directed towards)

If $a = (u, v) \in A(G)$ is an arc of G, then

- the arc a is said to be directed away from u,
- the arc a is said to be directed towards v.

Definition 97 (Source - Sink)

- Any vertex which has no arcs directed towards it is a source.
- Any vertex which has no arcs directed away from it is a sink.

Adjacent arcs

Definition 98 (Adjacent arcs)

Two arcs are **adjacent** if they have at least one endpoint in common.

Arc incident out of a subset of arcs

Definition 99 (Arc incident out of $A \subset X$)

If the initial endpoint of an arc u belongs to A, and if the terminal endpoint of arc u does not belong to A, then u is said to be incident out of A, and we write $u \in \omega^+(A)$. Similarly, we define an arc incident into A, and the set $\omega^-(A)$. Finally, the set of arcs incident to A is denoted

$$\omega(A) = \omega^+(A) \cup \omega^-(A).$$

Definition 100 (Symmetric graph)

If $m_G^+(x, y) = m_G^-(x, y)$ for all $x, y \in X$, the graph G is **symmetric**. A 1-graph $G = (X, U)$ is symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \in U$$

Definition 101 (Anti-symmetric graph)

If for each pair $(x, y) \in X \times X$,

$$m_G^+(x, y) + m_G^-(x, y) \leq 1$$

then the graph G is **anti-symmetric**. A 1-graph $G = (X, U)$ is anti-symmetric if, and only if,

$$(x, y) \in U \implies (y, x) \notin U$$

An anti-symmetric 1-graph without its direction is a simple graph

Definition 102 (Subgraph of G generated by $A \subset X$)

The **subgraph** of G generated by A is the graph with A as its vertex set and with all the arcs in G that have both their endpoints in A. If $G = (X, \Gamma)$ is a 1-graph, then the subgraph generated by A is the 1-graph $G_A = (A, \Gamma_A)$ where

$$\Gamma_A(x) = \Gamma(x) \cap A \qquad (x \in A)$$

Definition 103 (Partial graph of G generated by $V \subset U$)

The graph (X, V) whose vertex set is X and whose arc set is V. In other words, it is graph G without the arcs $U - V$

Definition 104 (Partial subgraph of G)

A partial subgraph of G is the subgraph of a partial graph of G

Degree

Let v be a vertex of a digraph $G = (V, A)$.

Definition 105 (Outdegree of a vertex)

The number of arcs directed away from a vertex v, in a digraph is called the outdegree of v and is written $d^+(v)$ or $outdeg(v)$.

Definition 106 (Indegree of a vertex)

The number of arcs directed towards a vertex v, in a digraph is called the indegree of v and is written $d^-(v)$ or $indeg(v)$.

Definition 107 (Degree)

For any vertex v in a digraph, the **degree** of v is defined as $d(v) = d^+(v) + d^-(v)$.

Theorem 108

For any (di)graph, the sum of the degrees of the vertices equals twice the number of edges (arcs).

Corollary 109

In any (di)graph, the sum of the degrees of the vertices is a nonnegative even integer.

Theorem 110

If G is a digraph with a vertex set $V(G) = \{v_1, \ldots, v_p\}$ and having q arcs then

$$\sum_{i=1}^{p} d^+(v_i) = \sum_{i=1}^{p} d^-(v_i) = q.$$

Definition 111 (Regular digraph)

A digraph G is $r-$regular if $indeg(v) = outdeg(v) = r$ for each vertex v of G.

Walks

Let $G = (V, A)$ be a digraph.

Definition 112 (Directed walk)

A directed walk in a digraph G is a non-empty alternating sequence $v_0 a_0 v_1 a_1 v_2 \ldots a_{k-1} v_k$ of vertices and arcs in G such that $a_i = (v_i, v_{i+1})$ for all $i < k$. This walk begins with v_0 and ends with v_k.

Definition 113 (Length of a directed walk)

The length of a directed walk is equal to the number of arcs in the directed walk.

Definition 114 (Closed walk)

If $v_0 = v_k$, the walk is closed.

Trails

Let $G = (V, A)$ be a digraph.

Definition 115 (Directed trail)

A directed walk in G in which all arcs are distinct is a directed trail in G.

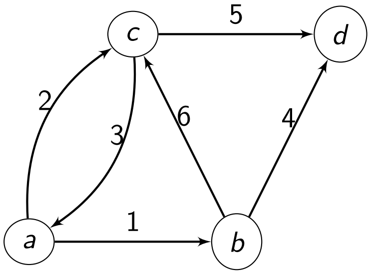
Definition 116 (Directed path)

A directed walk in G in which all vertices are distinct is a directed path in G.

Definition 117 (Directed cycle)

A closed walk is a directed cycle if it contains at least three vertices and all its vertices are distinct except for $v_0 = v_k$.

Examples of directed cycles



Cycles:

- $\boldsymbol{\mu}^1 = (1, 6, 2) = [abca]$
- $\boldsymbol{\mu}^2 = (1, 6, 3) = [abca]$
- $\boldsymbol{\mu}^3 = (2, 3) = [aca]$
- $\boldsymbol{\mu}^4 = (1, 4, 5, 2) = [abdca]$
- $\boldsymbol{\mu}^5 = (6, 5, 4) = [acdb]$
- $\boldsymbol{\mu}^6 = (1, 4, 5, 3) = [abdca]$

Given a cycle μ, denote μ^+ the set of all arcs in μ that are in the direction that the cycle is traversed and μ^- the set of all the other arcs in μ

Number the arcs in G as $1, 2, \dots, m$, then the cycle μ is the vector

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$$

where

$$\mu_i = \begin{cases} 0 & \text{if } i \notin \mu^+ \cup \mu^- \\ +1 & \text{if } i \in \mu^+ \\ -1 & \text{if } i \in \mu^- \end{cases}$$

Cocycles

Let $A \subset X$ be nonempty and denote $\omega^+(A)$ the set of arcs that have only their initial endpoint in A and $\omega^-(A)$ the set of arcs that have only their terminal endpoint in A. Let

$$\omega(A) = \omega^+(A) \cup \omega^-(A)$$

A **cocycle** is a nonempty set of arcs of the form $\omega(A)$, partitioned into two sets $\omega^+(A)$ and $\omega^-(A)$

An **elementary cocycle** is the set of arcs joining two connected subgraphs A_1 and A_2 s.t.

- $A_1, A_2 \neq \emptyset$
- $A_1 \cap A_2 = \emptyset$
- $A_1 \cup A_2 = C$, with C a connected component of the graph

A colouring lemma

Lemma 118 (Arc colouring Lemma)

Consider G with arcs $1, \ldots, m$. Colour arc 1 black and arbitrarily colour the remaining arcs red, black or green. Then exactly one of the following holds true:

1. *there is an elementary cycle containing arc 1 and only red and black arcs with the property that all black arcs in the cycle have the same direction*
2. *there is an elementary cocycle containing arc 1 and only green and black arcs, with the property that all black arcs in the cocycle have the same direction*

Independent cycles and cycle bases

Consider cycles $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \ldots, \boldsymbol{\mu}^k$. The cycles are **independent** if

$$c_1\boldsymbol{\mu}^1 + c_2\boldsymbol{\mu}^2 + \cdots + c_k\boldsymbol{\mu}^k = \mathbf{0}$$

$$\iff c_1 = c_2 = \cdots = c_k = 0$$

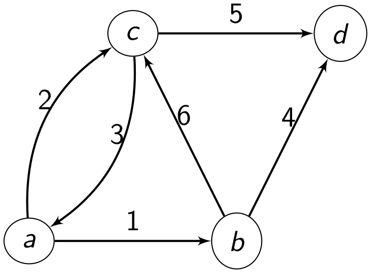
A **cycle basis** is an independent set $\{\boldsymbol{\mu}^1, \boldsymbol{\mu}^2, \ldots, \boldsymbol{\mu}^k\}$ of cycles such that any cycle $\boldsymbol{\mu}$ can be written as

$$\boldsymbol{\mu} = c_1\boldsymbol{\mu}^1 + c_2\boldsymbol{\mu}^2 + \cdots + c_k\boldsymbol{\mu}^k$$

for $c_1, \ldots, c_k \in \mathbb{R}$

The constant k is the **cyclomatic number** of G, denoted $\nu(G)$

Example



Elementary cycles:

- $\boldsymbol{\mu}^1 = (1, 6, 2) = [abca]$
- $\boldsymbol{\mu}^2 = (1, 6, 3) = [abca]$
- $\boldsymbol{\mu}^3 = (2, 3) = [aca]$
- $\boldsymbol{\mu}^4 = (1, 4, 5, 2) = [abdca]$
- $\boldsymbol{\mu}^5 = (6, 5, 4) = [acdb]$
- $\boldsymbol{\mu}^6 = (1, 4, 5, 3) = [abdca]$

We have $\boldsymbol{\mu}^1 - \boldsymbol{\mu}^2 + \boldsymbol{\mu}^3 = \mathbf{0}$

An important result

Theorem 119

Let G be a graph with n vertices, m arcs and p connected components. Then the cyclomatic number of G is

$$\nu(G) = m - n + p$$

Definitions

Definition 120 (Underlying graph)

Given a digraph, the undirected graph with each arc replaced by an edge is called the underlying graph.

Definition 121 (Weakly connected digraph)

If the underlying graph is a connected graph, then the digraph is weakly connected.

Definition 122 (Strongly connected digraph)

A digraph G is strongly connected if for every two distinct vertices u and v of G, there exists a directed path from u to v.

Definition 123 (Disconnected digraph)

A digraph is said to be **disconnected** if it is not weakly connected.

Strong connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a directed path in G from x to y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 124 (Connected component of a graph)

Sets of the form

$$A(x_0) = \{x : x \in X, x \equiv x_0\}$$

are equivalence classes. They partition X into strongly connected sub-digraphs of G called strongly connected components (or **strong components**) of G.

A strong component in G is a maximal strongly connected subdigraph of G.

Let $G = (V, A)$ be a digraph.

Theorem 125

Properties

- *If a digraph is strongly connected, it has only one strongly connected component.*
- *The strongly connected components partition the vertices in the digraph, with every vertex in exactly one strongly connected component.*

Algorithm for determining strongly connected components in $G = (V, A)$

- Determine the strongly connected component $C(v)$ containing the vertex v; if $V - C(v)$ is non-empty, re-do the same operation on the sub-digraph $G' = (V - C(v), A')$.
- To determine $C(v)$, the strongly connected component containing v: let v be a vertex of a digraph , which is not already in any strongly connected component.
  1. Mark the vertex v with $\pm$
  2. Mark with $+$ all successors (not already marked with $+$) of a vertex marked with $+$
  3. Mark with $-$ all predecessors (not already marked with $-$) of a vertex marked with $-$
  4. Repeat until no more possible marking with $+$ or $-$

  All vertices marked with $\pm$ belong to the same strongly connected component $C(v)$ containing the vertex v.

Condensation of a digraph

Definition 126 (Condensation of a digraph)

The condensation G^* of a digraph G is a digraph having as vertices the strongly connected components (SCC) of G and such that there exists an arc in G^* from a SCC C_i to another SCC C_j if there is an arc in G from some vertex of S_i to a vertex of S_j.

Definition 127 (Articulation set)

For a connected graph, a set A of vertices is called an **articulation set** (or a **cutset**) if the subgraph of G generated by $X - A$ is not connected

Definition 128 (Stable set)

A set S of vertices is called a stable set if no arc joins two distinct vertices in S

Orientation

Definition 129 (Orienting a graph)

Given a connected *graph*, we describe the act of assigning a direction to each edge (edge $\rightarrow$ arc) as orienting the graph.

Definition 130 (Strong orientation)

If the resulting digraph is strongly connected the orientation is called a strong orientation.

Orientable graph

Definition 131 (Orientable graph)

A connected graph G is called **orientable** if it is possible to assign a direction to each edge of G to produce a strongly connected digraph D. (If there exists a strong orientation of a connected graph, then the graph is orientable.)

Theorem 132

A connected graph G is orientable (has a strong orientation) if and only if G contains no bridges; that is every edge is contained in a cycle.

Trees

Definition 133 (Forest, trees and branches)

- A connected graph with no cycle is a tree.
- A tree is a connected acyclic graph, its edges are called branches.
- A graph (connected or not) without any cycle is a forest. Each component is a tree. (A forest is a graph whose connected components are trees)

Theorem 134 (Properties)

- *Every edge of a tree is a bridge (the deletion of any edge of a tree diconnects it)*
- *Given two vertices u and v of a tree, there is an unique path linking u to v.*
- *A tree with p vertices and q edges satisfies $q = p - 1$. Thus, a tree is minimally connected.*

Spanning tree

Definition 135 (Spanning tree)

A **spanning tree** of a connected graph G is a subgraph of G that contains all the vertices of G and is a tree.

A graph may have many spanning trees.

Minimal spanning tree

Definition 136 (Value of a spanning tree)

The **value of a spanning tree** T of order p is

$$\sum_{i=1}^{p-1} f(e_i)$$

where f is the function that maps the edge set into the set of real number.

Definition 137 (Minimal spanning tree)

Let G be an undirected network, and let T be a **minimal spanning tree** of G. Then T is a spanning tree whose the value is minimum.

Algorithm to find a minimal spanning tree

Let $G = (V(G), E(G))$ be an undirected network, and let T be a minimal spanning tree.

1. sort the edges of G in increasing order by value
2. $T = (V(G), \emptyset)$
3. for each edge e in sorted order if the endpoints of e are disconnected in T add e to T

Minimal connector problem

- Model: a graph G such that edges represent all possible connections, and each edge has a positive value which represents its cost; an undirected network G
- Solution: a minimal spanning tree T of G
  - a spanning tree of G is a subgraph of G that contains all the vertices of G and is a tree.
  - the cost of the spanning tree is the sum of values of the edges of T
  - a spanning tree such that no other spanning tree has a smaller cost is a minimmal spanning tree.

Theorem 138 (Characterisation of trees)

$H = (X, U)$ a graph of order $|X| = n > 2$. The following are equivalent and all characterise a tree :

1. *H connected and has no cycles*
2. *H has $n-1$ arcs and no cycles*
3. *H connected and has exactly $n-1$ arcs*
4. *H has no cycles, and if an arc is added to H, exactly one cycle is created*
5. *H connected, and if any arc is removed, the remaining graph is not connected*
6. *Every pair of vertices of H is connected by one and only one chain*

Proof of Theorem ??

We make abundant use of Theorem **??**

[**1** $\implies$ **2**] Let p be the number of connected components, m the number of arcs and $\nu(H)$ the cyclomatic number. Since H connected, $p = 1$. Since H has no cycles,
$\nu(H) = m - n + p = 0$
$\implies m = n - p = n - 1$

[**2** $\implies$ **3**] Assume H has no cycles ($\nu(H) = 0$) and has $n - 1$ arcs ($m = n - 1$). Then, since

$$\nu(H) = m - n + p$$

$p = \nu(H) - m + n = 0 - (n - 1) - n = 1$, i.e., H is connected

Proof of Theorem ?? (cont.)

[**3** $\implies$ **4**] Assume H connected ($p = 1$) and contains exactly $n - 1$ arcs ($m = n - 1$). Then

$$\nu(H) = m - n + p = (n - 1) - n + 1 = 0$$

and H has no cycles

Now add an arc, i.e., suppose $m = n$. Then $\nu(H) = m - n + p = n - n + 1 = 1$ and there is one cycle in the new graph

Proof of Theorem ?? (cont.)

[**4** $\implies$ **5**] Assume H has no cycles ($\nu(H) = 0$) and that addition of an arc to H creates exactly one cycle

Suppose H not connected. Then there are two vertices, say a and b, that are not connected and adding the arc (a, b) does not create a cycle, a contradiction with "addition of an arc to H creates exactly one cycle"
$\implies p = 1$. Since $\nu(H) = 0$, this implies that $m = n - 1$

Now suppose we remove an arc. We obtain graph H' with

$$m' = n' - 2 \quad \text{and} \quad \nu(H') = 0$$

So

$$p' = \nu(H') - m' + n' = 2$$

$\implies H'$ not connected

Proof of Theorem ?? (cont.)

[**5** $\implies$ **6**] Assume H connected and if any arc is removed, the remaining graph is not connected

Any vertices $a, b \in X$ are connected by a chain (since H connected). That chain is unique: suppose there is a second chain connecting a to b; then removing an arc from that chain does not disconnect the graph, since there is still the original chain connecting a and b

[**6** $\implies$ **1**] Assume every pair of vertices of H is connected by one and only one chain Now assume H has a cycle. Then at least one pair of vertices would be connected by two distinct chains, a contradiction $\square$

Definition 139 (Pendant vertex)

A vertex is **pendant** if it is adjacent to exactly one other vertex

Theorem 140

A tree of order $n \geq 2$ has at least two pendant vertices

Proof: Suppose H tree of order $n \geq 2$ with 0 or 1 pendant vertices

Consider a traveller traversing the graph edges starting from a pendant vertex (if there is one) or anywhere if there is no pendant vertex

If he does not allow himself to use same edge twice, he cannot go to the same vertex twice (H has no cycle)

If he arrives at vertex x, he can depart x using a new edge (x is not a pendant vertex as there are 0 or 1 pendant vertex and if there is 1, that's where he started)

So the trip continues without end, which is impossible as H finite □

Theorem 141

A graph $G = (X, U)$ has a partial graph that is a tree $\iff$ G connected

Recall that a partial graph is a graph generated by a subset of the arcs (Definition **??** slide **??**)

Proof: Suppose G not connected. Then no partial graph of G is connected $\implies$ there is no partial graph of G that is a tree [we want to show $P \iff Q$, we start with $\wedge Q \implies \wedge P$ (which $\iff$ $P \implies Q$)]

Suppose G connected. Look for an arc whose removal does not disconnect G

- if there is none, G is a tree by Theorem **??**(5)
- if there is, remove it and look for another one, etc. When no more arcs can be removed, the remaining graph is a tree with vertex set X $\square$

Spanning tree

The procedure in the proof of Theorem **??** gives a **spanning tree**

Can also build a spanning tree as follows:

- Consider any arc u_0
- Find arc u_1 that does not form a cycle with u_0
- Find arc u_2 that does not form a cycle with $\{u_0, u_1\}$
- Continue
- When you cannot continue anymore, you have a spanning tree

Theorem 142

G connected graph with ≥ 1 arc. TFAE

1. *G strongly connected*
2. *Every arc lies on a circuit*
3. *G contains no cocircuits*

Proof: [**1** $\implies$ **2**] (x, y) an arc of G; there is a path from y to x (G strongly connected), so arc (x, y) is contained in a circuit of G
[**2** $\implies$ **3**] Suppose G has a cocircuit containing arc (x, y); then G cannot have a circuit containing this arc by the Arc Colouring Lemma (Lemma **??** slide **??**) with all arcs coloured black. This contradicts (2)
[**3** $\implies$ **1**] Assume G connected graph without cocircuits, but G not strongly connected. Since G not strongly connected, it has > 1 strongly connected component. Since G connected, there exist two distinct strongly connected components that are joined by an arc (a, b). Arc (a, b) is not contained in any circuit because otherwise a and b would be in the same strongly connected component. By Lemma **??**, arc (a, b) is contained in some co- circuit. This contradicts (3) $\square$

Theorem 143

G graph with ≥ 1 arc. TFAE

1. *G is a graph without circuits*
2. *Each arc is contained in a cocircuit*

Theorem 144

If G is a strongly connected graph of order n, then G has a cycle basis of $\nu(G)$ circuits

Definition 145 (Node, anti-node, branch)

$G = (X, U)$ strongly connected without loops and > 1 vertex. For each $x \in X$, there is a path from it and a path to it so x has at least 2 incident arcs. Specifically,

- $x \in X$ with > 2 incident arcs is a **node**
- $x \in X$ with 2 incident arcs is an **anti-node**

A path whose only nodes are its endpoints is a **branch**

Definition 146 (Minimally connected graph)

G is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

A minimally connected graph is 1-graph without loops

Definition 147 (Contraction)

$G = (X, U)$. The **contraction** of the set $A \subset X$ of vertices consists in replacing A by a single vertex a and replacing each arc into (resp. out of) A by an arc with same index into (resp. out of) a

Theorem 148

G minimally connected, $A \subset X$ generating a strongly connected subgraph of G. Then the contraction of A gives a minimally connected graph

Proof: First, show that contraction of A yields a 1-graph. If this were not the case, there would exist $x \notin A$ and $a, a' \in A$ s.t. $(x, a), (x, a') \in U$ (or, with $(a, x), (a', x) \in U$ but this would not change the proof). If one of these arcs is removed, the graph remains strongly connected. Thus, G is not minimally connected, a contradiction

Now show that the contraction of A yields a graph G' that is minimally connected. Clearly, G' strongly connected. If an arc u is removed, the remaining graph is not strongly connected, since the graph $(X, U - \{u\})$ not strongly connected $\square$

Theorem 149

G a minimally connected graph, G' be the minimally connected graph obtained by the contraction of an elementary circuit of G. Then

$$\nu(G) = \nu(G') + 1$$

Theorem 150

G minimally connected of order $n \geq 2 \implies G$ has ≥ 2 anti-nodes

Theorem 151

$G = (X, U)$. Then the graph C' obtained by contracting each strongly connected component of G contains no circuits

Arborescences

Definition 152 (Root)

Vertex $a \in X$ in $G = (X, U)$ is a **root** if all vertices of G can be reached by paths *starting* from a

Not all graphs have roots

Definition 153 (Quasi-strong connectedness)

G is **quasi-strongly connected** if $\forall x, y \in X$, exists $z \in X$ (denoted $z(x, y)$ to emphasize dependence on x, y) from which there is a path to x and a path to y

Strongly connected $\implies$ quasi-strongly connected (take $z(x, y) = x$); converse not true
Quasi-strongly connected $\implies$ connected

Arborescence

Definition 154 (Arborescence)

An **arborescence** is a tree that has a root

Lemma 155

$G = (X, U)$ has a root $\iff$ G quasi-strongly connected

Theorem 156

H graph of order $n > 1$. TFAE (and all characterise an arborescence)

1. *H quasi-strongly connected without cycles*
2. *H quasi-strongly connected with $n-1$ arcs*
3. *H tree having a root a*
4. *$\exists a \in X$ s.t. all other vertices are connected with a by 1 and only 1 path from a*
5. *H quasi-strongly connected and loses quasi-strong connectedness if any arc is removed*
6. *H quasi-strongly connected and $\exists a \in X$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

7. *H has no cycles and $\exists a \in X$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

Theorem 157

G has a partial graph that is an arborescence $\iff$ G quasi-strongly connected

Theorem 158

$G = (X, E)$ simple connected graph and $x_1 \in X$. It is possible to direct all edges of E so that the resulting graph $G_0 = (X, U)$ has a spanning tree H s.t.

1. *H is an arborescence with root x_1*
2. *The cycles associated with H are circuits*
3. *The only elementary circuits of G_0 are the cycles associated with H*

Counting trees

Proposition 159

X a set with n distinct objects, $n_1, \ldots, n_p$ nonnegative integers s.t. $n_1 + \cdots + n_p = n$. The number of ways to place the n objects into p boxes $X_1, \ldots, X_p$ containing $n_1, \ldots, n_p$ objects respectively is

$$\binom{n}{n_1, \ldots, n_p} = \frac{n!}{n_1! \cdots n_p!}$$

Proposition 160 (Multinomial formula)

Let $a_1, \ldots, a_p \in \mathbb{R}$ be p real numbers, then

$$(a_1 + \cdots + a_p)^n = \sum_{n_1, \ldots, n_p \geq 0} \binom{n}{n_1, \ldots, n_p} (a_1)^{n_1} \cdots (a_p)^{n_p}$$

Theorem 161

Denote $T(n; d_1, \ldots, d_n)$ the number of distinct trees H with vertices $x_1, \ldots, x_n$ and with degrees $d_H(x_1) = d_1, \ldots, d_H(x_n) = d_n$. Then

$$T(n; d_1, \ldots, d_n) = \binom{n-2}{d_1 - 1, \ldots, d_n - 1}$$

Theorem 162

The number of different trees with vertices $x_1, \ldots, x_n$ is n^{n-2}

There is a whole industry of similar results (as well as for arborescences), but we will stop here. The main point is that we are talking about a large number of possibilities..

Matrices associated to a graph/digraph

There are multiple matrices associated to a graph/digraph.
The branch of graph theory that studies the properties of matrices derived from graphs and uses of these matrices in determining graph properties is *spectral graph theory*.
Graphs greatly simplify some problems in linear algebra and vice versa.

Adjacency matrix (undirected case)

Let $G = (V, E)$ be a graph of order p and size q, with vertices $v_1, v_2, \ldots, v_p$ and edges $e_1, e_2, \ldots, e_q$.

Definition 163 (Adjacency matrix)

The adjacency matrix is

$$M_A = M_A(G) = [m_{ij}]$$

is a $p \times p$ matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 164 (Adjacency matrix and degree)

The sum of the entries in row i of the adjacency matrix is the degree of v_i in the graph.

Definition 165 (Multiplicity of a pair)

The **multiplicity** of a pair x, y is the number $m_G^+(x, y)$ of arcs with initial endpoint x and terminal endpoint y. Let

$$m_G^-(x, y) = m_G^+(y, x)$$
$$m_G(x, y) = m_G^+(x, y) + m_G^-(x, y)$$

If $x \neq y$, then $m_G(x, y)$ is number of arcs with both x and y as endpoints. If $x = y$, then $m_G(x, y)$ equals twice the number of loops attached to vertex x. If $A, B \subset X$, $A \neq B$, let

$$m_G^+(A, B) = \{u : u \in U, u = (x, y), x \in A, y \in B\}$$
$$m_G(A, B) = m_G^+(A, B) + m_G^+(A, B)$$

Adjacency matrix of a multigraph

Definition 166 (Matrix associated with G)

If G has vertices $x_1, x_2, \ldots, x_n$, then the **matrix associated** with G is

$$a_{ij} = m_G^+(x_i, x_j)$$

Definition 167 (Adjacency matrix)

The matrix $a_{ij} + a_{ji}$ is the **adjacency matrix** associated with G

Adjacency matrix

Let $D = (V, A)$ be a digraph of order p with vertices denoted by $v_1, v_2, \ldots, v_p$.

Definition 168 (Adjacency matrix)

The adjacency matrix $M = M(D) = [m_{ij}]$ is a $p \times p$ matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if arc } v_i v_j \in A \\ 0 & \text{otherwise} \end{cases}$$

Theorem 169 (Properties)

- *M is not necessarily symmetric.*
- *The sum of any column of M is equal to the number of arcs directed towards v_j*
- *The sum of the entries in row i is equal to the number of arcs directed away from vertex v_i.*
- *The $(i, j)-$entry of M^n is equal to the number of walks of length n from vertex v_i to v_j.*

Incidence matrix

Let $D = (V, A)$ be a digraph of order p, and size q, with vertices denoted by $v_1, v_2, \ldots, v_p$, and arcs denoted $a_1, a_2, \ldots, a_q$.

Definition 170

Definition The incidence matrix $B = B(D) = [b_{ij}]$ is a $p \times q$ matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if arc } a_j \text{ is directed away from a vertex } v_i \\ -1 & \text{if arc } a_j \text{ is directed towards a vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$

Adjacency matrix

We have already seen adjacency matrices, let us recall the definition here

Definition 171 (Adjacency matrix)

G a 1-graph, then the **adjacency matrix** $A = [a_{ij}]$ is defined as follows

$$a_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \in U \\ 0 & \text{otherwise} \end{cases}$$

We often write $A(G)$ and, reciprocally, if A is an adjacency matrix, $G(A)$ the corresponding graph

G undirected $\implies A(G)$ symmetric

$A(G)$ has nonzero diagonal entries if G is not simple

Adjacency matrix (multigraph case)

Definition 172 (Adjacency matrix of a multigraph)

G an ℓ-graph, then the adjacency matrix $M_A = [m_{ij}]$ is defined as follows

$$m_{ij} = \begin{cases} k & \text{if arc there are } k \text{ arcs } (i,j) \in U \\ 0 & \text{otherwise} \end{cases}$$

with $k \leq \ell$

G undirected $\implies$ $M_A(G)$ symmetric

$M_A(G)$ has nonzero diagonal entries if G is not simple.

Weighted adjacency matrices

Sometimes, adjacency matrices (typically for 1-graphs) have real entries, usually positive

This means that the arcs/edges have been given a weight

Theorem 173 (Number of walks of length n)

Let A be the adjacency matrix of a graph $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then the $(i,j)-$entry of A^n, $n \geq 1$, is the number of different walks linking v_i to v_j of length n in G.

(two walks of the same length are equal if their edges occur in exactly the same order)
Example: let A be the adjacency matrix of a graph $G = (V(G), E(G))$.

- the $(i,i)-$entry of A^2 is equal to the degree of v_i.
- the $(i,i)-$entry of A^3 is equal to twice the number of C_3 containing v_i.

Incidence matrix

Let $G = (V, E)$ be a graph of order p, and size q, with vertices denoted by $v_1, v_2, \ldots, v_p$, and edges denoted $e_1, e_2, \ldots, e_q$.

Definition 174 (Incidence matrix)

The incidence matrix is

$$B = B(G) = [b_{ij}]$$

is that $p \times q$ matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 175 (Incidence matrix and degrees)

The sum of the entries in row i of the incidence matrix is the degree of v_i in the graph.

Incidence matrix

Definition 176 (Incidence matrix – Undirected case)

$G = (X, E)$ an undirected graph of order n with p edges. The **incidence** matrix of G is an $n \times p$ matrix with vertices as rows and edges as columns and where $B = [b_{ij}]$ satisfies

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is incident to vertex } i \\ 0 & \text{otherwise} \end{cases}$$

Definition 177 (Incidence matrix – Directed case)

$G = (X, U)$ a directed graph of order n with p arcs. The **incidence** matrix of G is an $n \times p$ matrix with vertices as rows and edges as columns and where $B = [b_{ij}]$ satisfies

$$b_{ij} = \begin{cases} 1 & \text{if arc } j \text{ ``enters'' vertex } i \\ -1 & \text{if arc } j \text{ ``leaves'' vertex } i \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of a graph

We will come back to this later, but for now..

Definition 178 (Spectrum of a graph)

The **spectrum** of a graph G is the spectrum (set of eigenvalues) of its associated adjacency matrix $A(G)$

This is regardless of the type of adjacency matrix or graph

Degree matrix

Definition 179 (Degree matrix)

The **degree** matrix $D = [d_{ij}]$ for G is a $n \times n$ diagonal matrix defined as

$$d_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

In an undirected graph, this means that each loop increases the degree of a vertex by two

In a directed graph, the term "degree" may refer either to indegree (the number of incoming edges at each vertex) or outdegree (the number of outgoing edges at each vertex)

Laplacian matrix

Definition 180

$G = (X, U)$ a simple graph with n vertices. The **Laplacian** matrix is

$$L = D(G) - A(G)$$

where $D(G)$ is the degree matrix and $A(G)$ is the adjacency matrix

G simple graph $\implies$ $A(G)$ only contains 1 or 0 and its diagonal elements are all 0
For directed graphs, either the indegree or outdegree is used, depending on the application
Elements of L are given by

$$\ell_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Distance matrix

Let G be a graph of order p with vertices denoted by $v_1, v_2, \ldots, v_p$.

Definition 181 (Distance matrix)

The distance matrix $DIST = DIST(D) = [d_{ij}]$ is a $p \times p$ matrix in which

$$d_{ij} = dist(v_i, v_j).$$

Note $d_{ii} = 0$ for $i = 1, \cdots, p$.

Definition 182

Properties

- M is not necessarily symmetric.
- The sum of any column of M is equal to the number of arcs directed towards v_j
- The sum of the entries in row i is equal to the number of arcs directed away from vertex v_i.
- The $(i, j)-$entry of M^n is equal to the number of walks of length n from vertex v_i

Counting paths

To count paths between vertices x and y in a graph, we use the adjacency matrix

Theorem 183

G a graph and $A(G)$ its adjacency matrix. Denote $P = [p_{ij}]$ the matrix $P = A^k$. Then p_{ij} is the number of distinct paths of length k from i to j in G

This provides an interesting connection with linear algebra

Definition 184 (Irreducible matrix)

A matrix $A \in \mathcal{M}_n$ is **reducible** if $\exists P \in \mathcal{M}_n$, permutation matrix, s.t. $P^T AP$ can be written in block triangular form. If no such P exists, A is **irreducible**

Theorem 185

A irreducible $\iff$ $G(A)$ strongly connected

Theorem 186

Let A be the adjacency matrix of a graph G on p vertices. A graph G on p vertices is connected if and only if

$$I + A + A^2 + \cdots + A^{p-1} = C$$

has no zero entries.

Theorem 187

Let M be the adjacency matrix of a digraph D on p vertices. A digraph D on p vertices is strongly connected if and only if

$$I + M + M^2 + \cdots + M^{p-1} = C$$

has no zero entries.

Perron-Frobenius theorem

$A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ **nonnegative** if $a_{ij} \geq 0$ $\forall i, j = 1, \ldots, n$; $\mathbf{v} \in \mathbb{R}^n$ nonnegative if $v_i \geq 0$ $\forall i = 1, \ldots, n$. **Spectral radius** of A

$$\rho(A) = \max_{\lambda \in \mathrm{Sp}(A)} \{|\lambda|\}$$

$\mathrm{Sp}(A)$ the **spectrum** of A

Theorem 188 (PF – Nonnegative case)

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$. *Then* $\exists \mathbf{v} \geq \mathbf{0}$ *s.t.*

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

Theorem 189 (PF – Irreducible case)

Let $0 \leq A \in \mathcal{M}_n(\mathbb{R})$ *irreducible. Then* $\exists \mathbf{v} > \mathbf{0}$ *s.t.*

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

Primitive matrices

Definition 190

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$ **primitive** (with **primitivity index** $k \in \mathbb{N}_+^*$) if $\exists k \in \mathbb{N}_+^*$ s.t.

$$A^k > 0,$$

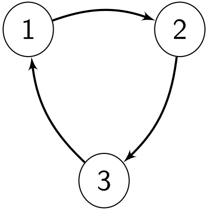
with k the smallest integer for which this is true. A **imprimitive** if it is not primitive

A primitive $\implies$ A irreducible; converse false

Theorem 191

$A \in \mathcal{M}_n(\mathbb{R})$ irreducible and $\exists i = 1, \ldots, n$ s.t. $a_{ii} > 0 \implies A$ primitive

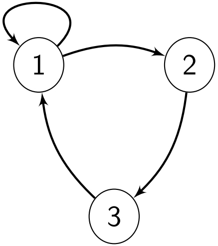
Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d = \gcd$ of all the lengths of closed walks in $G(A)$



Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walks in $G(A)$ (lengths): $1 \to 1$ (3), $2 \to 2$ (3), $2 \to 2$ (3) $\implies$ gcd $= 3$ $\implies$ $d = 3$ (here, all eigenvalues have modulus 1)



$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walk $1 \to 1$ has length 1 $\implies$ gcd of lengths of closed walks is 1 $\implies$ A primitive

Theorem 192

$0 \leq A \in \mathcal{M}_n$. *A primitive* $\implies A^k > 0$ *for some* $0 < k \leq (n-1)n^n$

Theorem 193

$A \geq 0$ *primtive. Suppose the shortest simple directed cycle in* $G(A)$ *has length* s, *then primitivity index is* $\leq n + s(n-1)$

Theorem 194

$0 \leq A \in \mathcal{M}_n$ *primitive* $\iff A^{n^2-2n+2} > 0$

Theorem 195

$0 \leq A \in \mathcal{M}_n$ *irreducible. A has d positive entries on the diagonal* $\implies$ *primitivity index* $\leq 2n - d - 1$

Theorem 196

$0 \leq A \in \mathcal{M}_n$, $\lambda_P = \rho(A)$ the Perron root of A, $\mathbf{v}_P$ and $\mathbf{w}_P$ the corresponding right and left Perron vectors of A, respectively, d the index of imprimitivity of A (with $d = 1$ when A is primitive) and $\lambda_j \in \sigma(A)$ the spectrum of A, with $j = 2, \ldots, n$ unless otherwise specified (assuming $\lambda_1 = \lambda_P$)

NONNEGATIVE

REDUCIBLE

- $\lambda_P \geq 0$
- $\mathbf{w}_P \geq 0$
- $\mathbf{v}_P \geq 0$
- $\lambda_P \geq |\lambda_j|$

IRREDUCIBLE

IMPRIMITIVE

- $\lambda_P > 0$
- $\mathbf{w}_P > 0$
- $\mathbf{v}_P > 0$
- $\lambda_P = |\lambda_j|$, $j = 2, \ldots, d$
- $\lambda_P > |\lambda_j|$, $j > d$

PRIMITIVE

- $\lambda_P > 0$
- $\mathbf{w}_P > 0$
- $\mathbf{v}_P > 0$
- $\lambda_P > |\lambda_j|$, $j \neq P$

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