Weingarten calculus (an introduction)

G locally compact group

$\exists$ left / right invariant measure — unique up to constant

$\hookrightarrow$ if compact $\hookrightarrow$ both   wlog prob measure

- Haar measure μ_G $\boxed{\mu(G)=1}$

$$\boxed{\mu(x\cdot A)=\mu(A\cdot y)=\mu(A)}$$

$\forall x,y\in G \quad \forall A\subset G$

- original construction / proof non explicit.

- how to compute μ_G explicitly?

$\hookrightarrow$ Weingarten calculus

(1978)

- Today: $G = U_n \subset M_n(\mathbb{C})$

$(G = O_n \subset M_n(\mathbb{R}))$

problem / question:

$$\begin{pmatrix} u_{11}^{(n)} & \cdots & u_{1n}^{(n)} \\ \vdots & & \\ u_{n1}^{(n)} & \cdots & u_{nn}^{(n)} \end{pmatrix} = U \in U_n$$

random $\sim \mu_G$

joint distribution of $(u_{11}^{(n)} \ldots \ldots u_{nn}^{(n)})$?

old results: (approximate)

$$\sqrt{n}\, u_{11}^{(n)} \xrightarrow[n\to+\infty]{\mathcal{L}} N_{\mathbb{C}}(0,1)$$

$$\begin{matrix} \sqrt{n}\, u_{11}^{(n)} & \cdots & \sqrt{n}\, u_{1n}^{(n)} \\ \vdots & & \vdots \\ \sqrt{n}\, u_{n1}^{(n)} & \cdots & \sqrt{n}\, u_{nn}^{(n)} \end{matrix} \xrightarrow[n\to+\infty]{\mathcal{L}} \text{iid complex Gaussian}$$

k^2 (Tiefeng Jiang)

$k \ll \sqrt{n}$

old result (exact) $\leftarrow$ RT

1994 Diaconis–Shashahani

Th: the moments of degree $\leq 2n$ of

$\mathrm{Tr}\,U, \mathrm{Tr}\,\bar{U}$ are

the same as those of

$x, \bar{x}$ where $x \in N_{\mathbb{C}}(0,1)$

$(\mathrm{Tr}\,x = \mathrm{Tr}\,V \times V^n)$

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general question:

$$E\left(u_{i_1 j_1} \cdots u_{i_n j_n} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_n j'_n}\right) = ? \quad (*)$$

Thm (Weingarten [C, Śniady]):

$$\overbrace{(*) = \sum_{\substack{\sigma\in S_n\\ \tau\in S_n}} \underbrace{\delta_{i,i',\sigma}}_{\prod_{l=1}^{k}\delta_{i_l\, i'_{\sigma(l)}}}\ \delta_{j,j',\tau}\ \underbrace{Wg(\sigma\tau^{-1}, n)}_{}}^{\sigma}$$

generalized Kronecker

$\exists$ function of permutations and n s.t.

def of Wg:

$$\sigma \to \left(\begin{array}{c} \tau \\ \downarrow \\ Wg(\sigma\tau^{-1}, n) \end{array} \right) = \sigma \to \left(\begin{array}{c} \tau \\ \downarrow \\ n^{\#\text{loops}(\sigma\tau^{-1})} \end{array} \right)^{-1}$$

index set $> S_k$

$n^k I_{k!}$ $\qquad$ $n^k I_{k!}$

<u>example</u>: $k=2$ $\quad e \quad (1\,2)$

$$e\begin{pmatrix} n^2 & n^1 \end{pmatrix}$$

$(12)\begin{pmatrix} n^1 & n^2 \end{pmatrix}$

$\sigma = (1)(2\,3\,4)(5\,6)$
$\#\mathrm{loop}(\sigma) = 3$

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idea of ~~proof~~ :

$$Z_k := E\left(U^{\otimes k} \otimes \bar{U}^{\otimes k}\right) \in M_n(\mathbb{C})^{\otimes 2k}$$

Fact Z_k is a self-adjoint projection over the fixed points of $U^{\otimes k} \otimes \bar{U}^{\otimes k}$ over $(\mathbb{C}^n)^{\otimes k} \otimes (\bar{\mathbb{C}}^n)^{\otimes k}$

$Wg(e) = \frac{1}{n}$ $\qquad (n)^{-1} = (n^{-1})$

$Wg((12)) = \frac{-1}{(n-1)n(n+1)}$ $\qquad \begin{pmatrix} n^2 & n \\ n & n^2 \end{pmatrix}^{-1} = \begin{pmatrix} \cdot & ? \\ \cdot & \cdot \end{pmatrix}$

$Wg((123)) = \frac{2}{\quad}$

$(n-2)(n-1)n(n+1)(n+2)$

$$\mathrm{Wg}((1234)) = \frac{-5}{(n-3)\cdots\cdots(n+3)} \qquad (-1)^{k+1} \qquad C_{k-1} \text{ Catalan} \qquad (C.03)$$

+ if $\sigma = c_1 \cdots c_\ell$ cycle decomposition

$$\boxed{\mathrm{Wg}(\sigma, n) = \mathrm{Wg}(c_1)\cdots\mathrm{Wg}(c_\ell)\,(1+O(n^{-2}))}$$

[W. Matsumoto (2018 ALEA): this asymptotic is uniform for $k \ll n^{1/9}$

(C. Śniady 2007(?) Ann IHP)