Attitude Estimation

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1 Equations for Forward and Measurement Models

The attitude dynamics of a rigid body can be described as

$$\dot{q} = \Omega(\omega)q \tag{1}$$

where

$$\Omega(\omega) = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{2}$$

where ω_1 is the roll rate, ω_2 is the pitch rate and ω_3 is the yaw rate.
We define the state as

$$x = \begin{bmatrix} q \\ b \end{bmatrix} \tag{3}$$

where q is the quaternion estimate of the attitude and b is the estimate of bias on the rate gyros.
The system is described by

$$\begin{aligned} \dot{x} &= f(x, \omega) \\ y &= h(x) \end{aligned} \tag{4}$$

where

$$f(x, \omega) = \begin{bmatrix} \Omega(\omega - b)q \\ 0 \end{bmatrix}, \quad h(x) = eul(q) \tag{5}$$

And $eul(q)$ is defined below:

$$eul(q) = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \arctan(\frac{2(q_2 q_3 + q_0 q 1)}{1 - 2(q_1^2 + q_2^2)} \\ \\ \arcsin(-2(q_1 q_3 - q_0 q_2)) \\ \\ \arctan \frac{2(q_1 q_2 + q_0 q 3)}{1 - 2(q_2^2 + q_3^2)} \end{bmatrix} \tag{6}$$

where ϕ is the roll angle, θ is the pitch angle and ψ is the yaw or heading angle.

2 Updated EKF Equations

State and Covariance Propagation Equations

$$
\begin{aligned}
F_k &= (I_{7\times 7} + A_k * T_s) \\
G_k &= (I_{7\times 7} * T_s + A_k * T_s^2/2) \\
G_{k_w} &= G_k * B \\
Q_k &= 1\text{e}{-3} * \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1\text{e}{-2} & & \\ & & & & & 1\text{e}{-2} & \\ & & & & & & 1\text{e}{-2} \end{pmatrix} \\
Q_{k_w} &= G_{k_w} * Q_k * G_{k_w}^T \\
x_{k+1|k} &= x_{k|k} + T_s f(x_{k|k}, w_k) \\
P_{k+1|k} &= F_k * P_{k|k} * F_k^T + Q_{k_w}
\end{aligned} \tag{7}
$$

where T_s is the integration time step size, B is the matrix that controls whether noise enters all the states. Note that Q_k can be tuned.

Formulation of A_k the jacobian of $f(x, w)$

$$A_k = \frac{\partial f}{\partial x} = \begin{bmatrix} A_{11} & A_{12} \\ 0^{3x4} & 0^{3x3} \end{bmatrix} \tag{8}$$

$$A_{11} = \frac{\partial \{\Omega(w-b)q\}}{\partial q} = \Omega(w-b) \tag{9}$$

$$A_{12} = \frac{\partial \{\Omega(w-b)q\}}{\partial b} = \frac{1}{2} \begin{bmatrix} q_1 & q_2 & q_3 \\ -q_0 & q_3 & -q_2 \\ -q_3 & -q_0 & q_1 \\ q_2 & -q_1 & -q_0 \end{bmatrix} \tag{10}$$

State and Covariance Update Equations

$$
\begin{aligned}
R_{k+1} &= \frac{I_{3x3}}{T_s} \\
S_{k+1} &= (H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1}) \\
K_{k+1} &= P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} \\
x_{k+1|k+1} &= x_{k+1|k} + K_{k+1}(y_{k+1} - h(x_{k+1|k}) \\
P_{k+1|k+1} &= (I_{7x7} - K_{k+1} H_{k+1}) P_{k+1|k} (I_{7x7} - K_{k+1} H_{k+1})^T + K_{k+1} R_{k+1} K_{k+1}^T
\end{aligned} \tag{11}
$$

Formulation of H_k the jacobian of $h(x)$

$$H_k = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial \phi}{\partial q} & 0^{3x1} \\ \frac{\partial \theta}{\partial q} & 0^{3x1} \\ \frac{\partial \psi}{\partial q} & 0^{3x1} \end{bmatrix} \tag{12}$$

$$\frac{\partial \phi}{\partial q} = \frac{2c}{a^2 + c^2} \begin{bmatrix} q1 \\ q_0 + 2q_1 a \\ q_3 + 2q_2 a \\ q_2 \end{bmatrix}^T \tag{13}$$

$$a = 2(q_0 q_1 + q_2 q_3) \tag{14}$$

$$c = 1 - 2(q_1^2 + q_2^2) \tag{15}$$

$$\frac{\partial \theta}{\partial q} = \frac{2}{\sqrt{1 - \gamma^2}} \begin{bmatrix} q_2 \\ -q_3 \\ q_0 \\ -q_1 \end{bmatrix}^T \tag{16}$$

$$\gamma = -2(q_1 q_3 - q_0 q_2) \tag{17}$$

$$\frac{\partial \psi}{\partial q} = \frac{2\beta}{\alpha^2 + \beta^2} \begin{bmatrix} q_3 \\ q_2 \\ q_1 + 2q_2\alpha \\ q_0 + 2q_3\alpha \end{bmatrix}^T \tag{18}$$

$$\alpha = 2(q_1 q_2 + q_0 q_3) \tag{19}$$

$$\beta = 1 - 2(q_2^2 + q_3^2) \tag{20}$$

3 Equations to Derive a Measurement

The procedure for obtaining an attitude measurement using a low-cost GPS receiver and three axis accelerometers is addressed below. To make a measurement of attitude the following steps are followed

- Obtain three consecutive GPS position measurements
- Difference the GPS measurements to obtain two velocity measurements
- Average the velocity measurements to give average velocity over two time steps.
- Calculate the heading ψ from velocity:

$$\psi = \arctan\left(\frac{\dot{Y}}{\dot{X}}\right) \tag{21}$$

- Difference the GPS calculated velocities to obtain a GPS acceleration measurement, $\rightarrow a_{GPS}$

- Average the accelerometer over the same time period that the GPS velocity is calculated, $\rightarrow$ a
- Calculate the roll ϕ and pitch θ using the accelerometer and GPS acceleration rotated by ψ using the below equations.

$$r_x = -(cos(\psi)a_{GPS_x} + sin(\psi)a_{GPS_y}) \tag{22}$$

$$r_y = -(-sin(\psi)a_{GPS_x} + cos(\psi)a_{GPS_y}) \tag{23}$$

$$r_z = g - a_{GPS_z} \tag{24}$$

$$\sigma_\theta = \frac{r_x a_x + r_z\sqrt{r_x^2 + r_z^2 - a_x^2}}{r_x^2 + r_z^2} \tag{25}$$

$$\theta = \arctan\left(\frac{\sigma_\theta r_x - a_x}{\sigma_\theta r_z}\right) \tag{26}$$

$$r_\theta = r_x sin(\theta) + r_z cos(\theta) \tag{27}$$

$$\sigma_\phi = \frac{r_y a_y + r_\theta\sqrt{r_x^2 + r_\theta^2 - a_y^2}}{r_y^2 + r_\theta^2} \tag{28}$$

$$\phi = \arctan\left(\frac{\sigma_\phi r_y - a_y}{\sigma_\phi r_\theta}\right) \tag{29}$$

$$\tag{30}$$

Where r is a reference acceleration vector in the ECEF frame, a is the acclerometer measurement and a_{GPS} is the GPS acceleration measurement.