5. Suppose X and Y are continuously distributed with joint pdf f given as

$$f(x,y) = \begin{cases} \frac{k}{\sqrt{xy}} & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of k.

$$1 = \int_0^1 \int_0^1 \frac{k}{\sqrt{xy}}\, dx\, dy$$

$$\int x^{-\frac{1}{2}}\, dx = 2x^{\frac{1}{2}}$$

$$= \int_0^1 k \int_0^1 \frac{1}{x^{\frac{1}{2}} y^{\frac{1}{2}}}\, dx\, dy$$

$$\int_0^1 \frac{1}{y^{\frac{1}{2}}} k \int_0^1 \frac{1}{x^{\frac{1}{2}}}\, dx\, dy$$

$$\int_0^1 \frac{1}{y^{\frac{1}{2}}} k\, 2 \left[\sqrt{x} \right]_0^1 dy$$

$$\int_0^1 \frac{1}{y^{\frac{1}{2}}} k \cdot 2\, dy$$

$$2k \left[2 \cdot y^{\frac{1}{2}} \right]_0^1$$

$$4k(1) = 1$$

$$\boxed{\frac{1}{4}}$$

6. In the previous exercise, what is the probability that $Y \ge X$?

$$\int_0^1 \int_x^1 \frac{1}{4\sqrt{xy}}\, dy\, dx$$

$$\int_0^1 \frac{1}{4\sqrt{x}} \int_x^1 y^{-\frac{1}{2}}\, dy\, dx$$

$$2\left[\sqrt{y} - \sqrt{x} \right]^1$$

$$\int_0^1 \frac{1}{4} \cdot \frac{1}{\sqrt{x}} \cdot \left[2 - 2\sqrt{x} \right] dx$$

$$\int_0^1 \frac{1}{4\sqrt{x}} \cdot \left(2 - 2\sqrt{x} \right) dx$$

$$\int_0^1 \frac{1}{2\sqrt{x}} - \frac{1}{2x}\, dx$$

$$\frac{1}{2} \int_0^1 x^{-\frac{1}{2}} - x^{-1}\, dx$$

$$\frac{1}{2} \left[2x^{\frac{1}{2}} - \ln x \right]_0^1$$

$$\frac{1}{2} \left(2\sqrt{1} - \ln 1 \right) = 1$$

$$2 - 0 \quad 2 \quad \frac{1}{2}$$

$$\boxed{\frac{1}{2}}$$