4. Suppose X and Y are continuously distributed with joint pdf

$$f(x,y) = \begin{cases} kxy^2 & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where k is an unknown constant.

(a) Find the value of the constant k.

$$1 = \int_0^4 \int_0^2 kxy^2 \, dx \, dy$$

$$1 = \int_0^4 \tfrac{1}{2} k y^2 \left(x^2\right)_0^2 dy$$

$$1 = \int_0^4 2ky^2 \, dy$$

$$\tfrac{2}{3}k\left(y^3\right)_0^4$$

$$1 = \frac{128}{3}k = \frac{3}{128}$$

$\boxed{3/128}$

(b) Find the probability that $0 \le X \le 1$.

$$P(0 \le X \le 1) = \int_0^1 \int_0^4 \frac{3}{128} xy^2 \, dy \, dx$$

$$= \frac{3}{128}x\left[\tfrac{1}{3}y^3\right]_0^4$$

$$\int_0^1 \frac{1}{128}x\left[64\right] dx$$

$$\int_0^1 x \, dx \; \tfrac{1}{2}\left[x^2\right]_0^1 = \tfrac{1}{4}$$

$\boxed{\tfrac{1}{4} = 25\%}$

(c) Calculate the marginal probability distributions.

$$\int_0^4 \frac{3}{128} xy^2 \, dy = \tfrac{1}{2}x$$

$$\int_0^2 \frac{3}{128} xy^2 \, dx = \frac{3}{64}y^2$$

$f(x) \{ \tfrac{1}{2}x$ if $0 < x < 2$

$f(y) \{ \tfrac{3}{64}y^2$ if $0 < y < 4$

$\boxed{\tfrac{1}{2}x, \; \tfrac{3}{64}y^2}$

(d) Are X and Y independent?

$$f(x,y) = \frac{3}{128}xy^2$$

$$\left(\tfrac{1}{2}x\right)\left(\tfrac{3}{64}y^2\right)$$

$$= \frac{3}{128}xy^2 = \frac{3}{128}xy^2 \checkmark$$

$\boxed{\text{independent}}$

(e) Find the expected value of $X + Y$.

$$E(X+Y) = E(X) + E(Y)$$

$$E(X+Y) = \tfrac{4}{3} + 3$$

$$= \tfrac{13}{3}$$

$\boxed{\tfrac{13}{3}}$

6)