18.01 Problem Set Solutions
Problem Set 1

Niranjan Krishna
realniranjankrishna@gmail.com

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Link to Problem Set : Problem Set 1

1.

$$\frac{x-1}{x+1} = \frac{x-1}{x+1} \cdot \frac{x-1}{x-1}$$

$$= \frac{x^2-2x+1}{x^2-1} = \frac{x^2-1}{x^2+1} - \frac{2x}{x^2-1}$$

Here the first expression is an even function and the second is an odd function.

2.

(a)

(b)

3. The point where the ant starts to see the tower will be a tangent line from the tower to the parabola. We already know one point of the tangent line. That is $(0, 1100)$. Since 1000 is the height of the mound and 100 is the height of the tower the y coordinate is equal to 1100. The x coordinate is found by finding at what x does y have the maximum height. This is done by computing the derivative and setting it to 0.

$$\frac{d}{dx}1000 - x^2 = -2x$$

$$-2x = 0$$

$$x = 0$$

We also have found that the slope of the parabola is $-2x$ using this method. Now we need to find the equation of of the point where the tangent and the parabola intercept. Since the tangent line intercepts the parabola it's slope must be the slope of the parabola. So the equation of the tangent line becomes

$$y = -2x \cdot x + 1100$$
$$= -2x^2 + 1100$$

Next, the point of interception must have the y coordinates of the parabola and the tangent line to be equal. So

$$1000 - x^2 = -2x^2 + 1100$$

Subtracting 1000 from both sides

$$-x^2 = -2x^2 + 100$$

Addin $2x^2$to the left side

$$x_2 = 100$$
$$x = \pm 10$$

The ant can climb up from either side and see the tower when distance in the x direction is 10 and the distance in the y direction is

$$y = 1000 - x^2 = 1000 - 100 = 900$$

4. (This is the 21st question from section 3.1 of the Calculus with Analytic Geometry textbook.There are conflicting versions so I won't be doing this one)

5. (a) Avg Rate of change = $\frac{f(x_2)-f(x_1)}{x_2-x_1}$

$$= \frac{f(5) - f(0)}{5 - 0}$$
$$= \frac{\frac{(10-5)^2}{5} - \frac{(10-0)^2}{5})}{5}$$
$$= -3$$

Water drains at 3 litres per minute in the first five minutes

(b) Rate of change at t seconds can be found by taking the derivative of the function and plugging value of t. The derivative is

$$\frac{d}{dx}\frac{(10-t)^2}{5} = \frac{2t-20}{5}$$

Plugging 5 in place of t, we get

$$\frac{2 \cdot 5 - 20}{5} = \frac{-10}{5} = -5$$

6. (From textbook)

7. (a) $\frac{d}{dx}(u \cdot v \cdot w) = \frac{d}{dx}((u \cdot v) \cdot w)$

$$= \frac{d}{dx}(u \cdot v) \cdot w + \frac{d}{dx}(w) \cdot (u \cdot v)$$

$=$

$$= (\frac{d}{dx}(u) \cdot v + \frac{d}{dx}(v) \cdot u) \cdot w + \frac{d}{dx}(w) \cdot (u \cdot v)$$

$$= \frac{d}{dx}u \cdot v \cdot w + \frac{d}{dx}v \cdot u \cdot w + \frac{d}{dx}w \cdot u \cdot v$$

(b)

$$\frac{d}{dx}u_1 \cdot u_2 \dots \cdot u_n = \frac{d}{dx}u_1 \cdot (u_2 \cdot u_3 \dots \cdot u_n) + \frac{d}{dx}u_2 \cdot (u_1 \cdot u_3 \dots \cdot u_n) + \frac{d}{dx}u_n \cdot (u_1 \cdot u_2 \dots \cdot u_{n-1})$$

Let's prove this using induction. For $n = 1$,the equation is true.

$$\frac{d}{dx}u_1 = \frac{d}{dx}u_1$$

Let's assume it is true for n equations. Then for n+1

$$\frac{d}{dx}u_1 \cdot u_2 \dots \cdot u_n \cdot u_{n+1} = \frac{d}{dx}(u_1 \cdot u_2 \dots \cdot u_n) \cdot u_{n+1} + \frac{d}{dx}u_{n+1} \cdot (u_1 \cdot u_2 \dots \cdot u_n)$$

$$= \frac{d}{dx}u_1 \cdot (u_2 \cdot u_3 \dots \cdot c_n \cdot u_{n+1}) + \frac{d}{dx}u_2 \cdot (u_1 \cdot u_3 \dots \cdot c_n \cdot u_{n+1}) + \frac{d}{dx}u_n \cdot (u_1 \cdot u_2 \dots \cdot c_{n-1} \cdot u_{n+1}) +$$
$$\frac{d}{dx}u_{n+1} \cdot (u_1 \cdot u_2 \dots \cdot u_n)$$

which proves our assertion