18.01 Problem Set Solutions
Problem Set 4

Niranjan Krishna
realniranjankrishna@gmail.com

September 11, 2020

Link to Problem Set : Problem Set 4

1. (a) According to the Mean Value theorem

$$\frac{f(x) - f(0)}{x - 0} = f'(x)$$

$$f(x) = f(0) + f'(x)x$$

Since $f(0) = 0$, we can rewrite this as

$$f(x) = f'(x)x$$

From the questions we know that $x \geq 0$ and $f'(x) \geq 0$.

So $f'(x)x \geq 0$
So $f(x) \geq 0$

(b)

$$f(x) = x - \ln(x+1)$$

Here $x \geq 0$

$$f'(x) = \frac{d}{dx}x - \frac{d}{dx}\ln(x+1)$$

$$= 1 - \frac{1}{x+1}$$

$$\frac{x}{x+1}$$

Here $\frac{x}{x+1} < 0$ when

$$x < (x+1) \cdot 0 = x < 0$$

This is not possible as $x \geq 0$ So the derivative is always positive. By the previous theorem, this means that $f(x) \geq 0$ Therfore

$$x - \ln(x + 1 \geq 0$$

$$x \geq \ln(x + 1)$$

which is the same as saying

$$\ln(x + 1) \leq x$$

(c) $f(x) = \ln(1 + x) - x + \frac{x^2}{2}$

$$f(0) = \ln(1 + 0) - 0 + \frac{0^2}{2} = 0$$

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - 1 + x \\ &= \frac{1 - (1 + x)}{1 + x} + x \\ &= \frac{1 - (1 + x) + x(x + 1)}{1 + x} + \\ &= \frac{1 - 1 - x + x^2 + x}{1 + x} \\ &= \frac{x^2}{1 + x} \end{aligned}$$

$\frac{x^2}{x+1} < 0$ whenever

$$x^2 < (x + 1) \cdot 0 = 0$$

But x^2 cannot be less than zero. Hence $\frac{x^2}{x+1} \geq 0$
Hence by the aforementioned theorem $f(x) \geq 0$

$f(x) = -\ln(1 + x) + x - \frac{x^2}{2} + \frac{x^3}{3}$

$$f(0) = -\ln(1 + 0) + 0 - \frac{0^2}{2} + \frac{0^3}{3} = 0$$

$$\begin{aligned} f'(x) &= \frac{1}{1+x} + 1 - x + x^2 \\ &= \frac{-1 + (1 + x) - x(x + 1) + x^2(x + 1)}{1 + x} \\ &= \frac{x^3}{x + 1} \end{aligned}$$

$\frac{x^3}{x+1} < 0$ whenever

$$\frac{x^3}{x + 1} < 0$$

$$x^3 < -0$$

But x^3 is only negative when x is negative. Since $x \geq 0$ that's not possible So by $f'(x) \geq 0$. Hence $f(x) \geq 0$

(d)

$$ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{2n+1}}{2n+1} = \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}, x \geq 0, n \geq 0$$

$$ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^{2n}}{2n} = \sum_{k=1}^{2n} (-1)^{k+1} \frac{x^k}{k}, x \geq 0, n \geq 1$$

(e) Let $u = -x$, then $x = -u$. So

$$\ln(1-u) \leq -u, 0 \leq u < 1$$

$$f(x) = -u - \ln(1-u)$$

$$f'(x) = -1 - (\frac{-1}{1-u})$$

$$= \frac{1}{1-u} - 1$$

$$= \frac{1 - 1(1-u)}{1-u}$$

$$= \frac{u}{1-u}$$

$\frac{u}{1-u} < 0$ whenever

$$u < (1-u) \cdot 0$$

$$u < 0$$

Since $0 \leq u < 1$ that is not possible. Hence $f(x)$ is positive Therefore

$$-u - \ln(1-u) \geq 0$$

$$-u \geq \ln(1-u)$$

$$x \geq \ln(1+x)$$

$$\ln(1+x) \leq x$$

2. (a) (From the textbook)

(b) $\int \tan x \sec^2 x = \frac{1}{2} \tan^2 x$
Let $u = \tan x$,$\frac{du}{dx} = \sec^2 x, du = \sec^2 x dx$

$$\int udu = \frac{u^2}{2} + C = \frac{tan^2 x}{2} + C$$

$\int \tan x \sec^2 x = \frac{1}{2} \tan^2 x$
Let $u = \sec x$,$\frac{du}{dx} = \sec x \tan x, du = \sec x \tan x dx$

$$\int udu = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C$$

3. (From the textbook)
4. (From the textbook)
5. First we need to find the interval to take the Riemann Sum.

$$Interval = \frac{b-a}{n}$$
$$= \frac{1-0}{n} = \frac{1}{n}$$

So the sum is equal to

$$\sum_{i=0}^{n-1} e^{\frac{i}{n}} \frac{1}{n}$$
$$= \frac{1}{n} \sum_{i=0}^{n-1} e^{\frac{i}{n}}$$
$$= \frac{1}{n} \sum_{i=0}^{n-1} e^{(\frac{1}{n})^i}$$

Using the formula for geometric series, we get

$$= \frac{1}{n} \frac{1-(e^{\frac{1}{n}})^n}{1-e^{\frac{1}{n}}}$$
$$= \frac{1-e}{n(1-e^{\frac{1}{n}})}$$

Next we should take the limit as n approaches infinity to complete the sum

$$\lim_{n\to\infty} \frac{1-e}{n(1-e^{\frac{1}{n}})}$$

Using the linear approximation for $e^x = 1 + x$ at x near 0.
This is used because as $n \to \infty$,$\frac{1}{n} \to 0$ Therefore

$$= \lim_{n\to\infty} \frac{1-e}{n(1-(1+\frac{1}{n}))}$$
$$= \lim_{n\to\infty} \frac{1-e}{n(1-(\frac{n+1}{n}))}$$
$$= \lim_{n\to\infty} \frac{1-e}{n(\frac{n-(n+1)}{n}))}$$
$$= \lim_{n\to\infty} \frac{1-e}{n(\frac{-1}{n})}$$
$$= \lim_{n\to\infty} \frac{1-e}{-1}$$
$$= \lim_{n\to\infty} e-1$$
$$= e-1$$

6. (Did it on paper. Too much latex even for me. Might do it someday :))