

18.01 Problem Set Solutions

Problem Set 3

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Link to Problem Set : [Problem Set 3](#)

1. (a) According to the Mean Value theorem

$$\frac{f(x) - f(0)}{x - 0} = f'(x)$$

$$f(x) = f(0) + f'(x)x$$

Since $f(0) = 0$, we can rewrite this as

$$f(x) = f'(x)x$$

From the questions we know that $x \geq 0$ and $f'(x) \geq 0$.

$$\text{So } f'(x)x \geq 0$$

$$\text{So } f(x) \geq 0$$

(b)

$$f(x) = x - \ln(x + 1)$$

Here $x \geq 0$

$$\begin{aligned} f'(x) &= \frac{d}{dx}x - \frac{d}{dx}\ln(x + 1) \\ &= 1 - \frac{1}{x + 1} \\ &\quad \frac{x}{x + 1} \end{aligned}$$

Here $\frac{x}{x+1} < 0$ when

$$x < (x + 1) \cdot 0 = x < 0$$

This is not possible as $x \geq 0$ So the derivative is always positive.
 By the previous theorem, this means that $f(x) \geq 0$ Therefore

$$x - \ln(x+1) \geq 0$$

$$x \geq \ln(x+1)$$

which is the same as saying

$$\ln(x+1) \leq x$$

$$(c) \ f(x) = \ln(1+x) - x + \frac{x^2}{2}$$

$$f(0) = \ln(1+0) - 0 + \frac{0^2}{2} = 0$$

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - 1 + x \\ &= \frac{1 - (1+x)}{1+x} + x \\ &= \frac{1 - (1+x) + x(x+1)}{1+x} + \\ &= \frac{1 - 1 - x + x^2 + x}{1+x} \\ &= \frac{x^2}{1+x} \end{aligned}$$

$$\frac{x^2}{x+1} < 0 \text{ whenever}$$

$$x^2 < (x+1) \cdot 0 = 0$$

But x^2 cannot be less than zero. Hence $\frac{x^2}{x+1} \geq 0$
 Hence by the aforementioned theorem $f(x) \geq 0$

$$f(x) = -\ln(1+x) + x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$f(0) = -\ln(1+0) + 0 - \frac{0^2}{2} + \frac{0^3}{3} = 0$$

$$\begin{aligned} f'(x) &= \frac{1}{1+x} + 1 - x + x^2 \\ &= \frac{-1 + (1+x) - x(x+1) + x^2(x+1)}{1+x} \\ &= \frac{x^3}{x+1} \end{aligned}$$

$$\frac{x^3}{x+1} < 0 \text{ whenever}$$

$$\frac{x^3}{x+1} < 0$$

$$x^3 < -0$$

But x^3 is only negative when x is negative. Since $x \geq 0$ that's not possible So by $f'(x) \geq 0$. Hence $f(x) \geq 0$

(d)

$$\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{2n+1}}{2n+1} = \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{x^k}{k}, x \geq 0, n \geq 0$$

$$\ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{2n}}{2n} = \sum_{k=1}^{2n} (-1)^{k+1} \frac{x^k}{k}, x \geq 0, n \geq 1$$

(e) Let $u = -x$, then $x = -u$. So

$$\ln(1-u) \leq -u, 0 \leq u < 1$$

$$f(x) = -u - \ln(1-u)$$

$$f'(x) = -1 - \left(\frac{-1}{1-u}\right)$$

$$= \frac{1}{1-u} - 1$$

$$= \frac{1 - 1(1-u)}{1-u}$$

$$= \frac{u}{1-u}$$

$$\frac{u}{1-u} < 0 \text{ whenever}$$

$$u < (1-u) \cdot 0$$

$$u < 0$$

Since $0 \leq u < 1$ that is not possible. Hence $f(x)$ is positive Therefore

$$-u - \ln(1-u) \geq 0$$

$$-u \geq \ln(1-u)$$

$$x \geq \ln(1+x)$$

$$\ln(1+x) \leq x$$

2. (a) (From the textbook)

$$(b) \int \tan x \sec^2 x = \frac{1}{2} \tan^2 x$$

$$\text{Let } u = \tan x, \frac{du}{dx} = \sec^2 x, du = \sec^2 x dx$$

$$\int u du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$$

$$\int \tan x \sec^2 x = \frac{1}{2} \tan^2 x$$

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$$\int u du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C$$

3. (From the textbook)
4. (From the textbook)
5. First we need to find the interval to take the Riemann Sum.

$$\begin{aligned} \text{Interval} &= \frac{b-a}{n} \\ &= \frac{1-0}{n} = \frac{1}{n} \end{aligned}$$

So the sum is equal to

$$\begin{aligned} &\sum_{i=0}^{n-1} e^{\frac{i}{n}} \frac{1}{n} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} e^{\frac{i}{n}} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} e^{(\frac{1}{n})^i} \end{aligned}$$

Using the formula for geometric series, we get

$$\begin{aligned} &= \frac{1}{n} \frac{1 - (e^{\frac{1}{n}})^n}{1 - e^{\frac{1}{n}}} \\ &= \frac{1 - e}{n(1 - e^{\frac{1}{n}})} \end{aligned}$$

Next we should take the limit as n approaches infinity to complete the sum

$$\lim_{n \rightarrow \infty} \frac{1 - e}{n(1 - e^{\frac{1}{n}})}$$

Using the linear approximation for $e^x = 1 + x$ at x near 0.

This is used because as $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$ Therefore

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{1 - e}{n(1 - (1 + \frac{1}{n}))} \\ &= \lim_{n \rightarrow \infty} \frac{1 - e}{n(1 - (\frac{n+1}{n}))} \\ &= \lim_{n \rightarrow \infty} \frac{1 - e}{n(\frac{n-(n+1)}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{1 - e}{n(\frac{-1}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{1 - e}{-1} \\ &= \lim_{n \rightarrow \infty} e - 1 \\ &= e - 1 \end{aligned}$$

6. (Did it on paper. Too much latex even for me. Might do it someday :))